Publ. RIMS Kyoto Univ. **53** (2017), 337–347
DOI 10.4171/PRIMS/53-2-5

A Weak Converse Theorem for Degree 2 L-Functions with Conductor 1

by

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Abstract

We show that every normalized function of degree 2 and conductor 1 in the extended Selberg class has real coefficients, and certain invariants agree with those of the L-functions of cusp forms for the full modular group. The result may therefore be regarded as a weak converse theorem in such a general setting.

2010 Mathematics Subject Classification: 11M41, 11F66.
Keywords: L-functions, Selberg class, converse theorems, cusp forms.

§1. Introduction

We briefly recall some well-known properties of the cusp forms of level 1 (i.e., associated with the full modular group Γ) and of their L-functions. Let $\mathcal{S}_w(\Gamma)$ be the complex vector space of holomorphic cusp forms of integral weight w and let a_n denote their Fourier coefficients. Then w is an even integer ≥ 12, and $\mathcal{S}_w(\Gamma)$ has a unique basis $\mathcal{B}$ of normalized (i.e., with $a_1 = 1$) eigenfunctions of the Hecke operators; moreover such eigenforms have $a_n \in \mathbb{R}$. For $f \in \mathcal{B}$, the associated Hecke L-function $L_f(s)$ is entire, has functional equation and Euler product and satisfies the Ramanujan conjecture. Precisely, writing in normalized form

$$a(n) = \frac{a_n}{n^{(w-1)/2}}, \quad L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad \text{and} \quad \Lambda_f(s) = \left(\frac{1}{2\pi}\right)^s \Gamma\left(s + \frac{w-1}{2}\right) L_f(s),$$

Communicated by S. Mochizuki. Received April 2, 2016.

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we have

$$\Lambda_f(s) = i^w \Lambda_f(1-s) \quad \text{and} \quad L_f(s) = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1}, \left(1 - \frac{\overline{\alpha_p}}{p^s}\right)^{-1}, \tag{1}$$

with $|\alpha_p| = 1$; see Serre [17, Chapter VII]. Results of a similar nature hold for the Maass cusp forms; in general, in this case only the weights 0 and 1 matter from the L-functions viewpoint, and there are no forms of odd weight in level 1. In particular, the L-function $L_f(s)$, formed directly with the Fourier–Bessel coefficients a_n of a normalized cusp form f (i.e., $a_1 = 1$ and $a(n) = a_n$) of weight 0 and level 1, is entire and satisfies the functional equation

$$\Lambda_f(s) = (-1)^\varepsilon \Lambda_f(1-s), \tag{2}$$

where

$$\Lambda_f(s) = \left(\frac{1}{\pi}\right)^s \Gamma\left(\frac{s}{2} + \frac{\varepsilon + i\nu}{2}\right) \Gamma\left(\frac{s}{2} + \frac{\varepsilon - i\nu}{2}\right) L_f(s),$$

$\varepsilon = 0$ (resp. $\varepsilon = 1$) if f is even (resp. odd) and $1/4 + \nu^2$, with $\nu \in \mathbb{R}$, is the associated eigenvalue of the Laplacian. Moreover, $a_n \in \mathbb{R}$ if f is an eigenfunction of the Hecke operators. See Duke–Friedlander–Iwaniec [3, Section 8] for a useful explicit account of the analytic side of the Maass forms theory; see also Bump [1, Proposition 1.9.1] for the level 1 case. Since all the above forms are of level 1, their L-functions have degree 2 and conductor 1 (see below for definitions).

In [11] we characterized the functions of degree 2 in the Selberg class $\mathcal{S}$ with conductor 1 and a pole at $s = 1$, showing that $F(s) = \zeta(s)^2$ is the only such function. We recall that, roughly, the extended Selberg class $\mathcal{S}^\sharp$ is the class of Dirichlet series satisfying a Riemann-type functional equation, while $\mathcal{S}$ is the subclass of the $F \in \mathcal{S}^\sharp$ satisfying the Ramanujan conjecture and having a general Euler product; again, see below for definitions. In this paper we deal with the general case of functions $F \in \mathcal{S}^\sharp$ with degree 2 and conductor 1, showing that these have some common features with the above L-functions of cusp forms. In particular, we prove that after a natural normalization, the coefficients of such $F(s)$ are real, and certain invariants, including the root number, agree with those coming from (1) and (2). Therefore, our result may be regarded as a weak converse theorem in such a general framework. Moreover, these similarities shed some light on the rather mysterious content of the subclass of the degree 2 functions in $\mathcal{S}^\sharp$, and support the standard conjectures about the degree 2 functions in $\mathcal{S}$.

Now we recall several definitions; we refer to our survey papers [6, 4, 14, 15, 16] for the basic theory of the Selberg classes $\mathcal{S}$ and $\mathcal{S}^\sharp$. Every $F \in \mathcal{S}^\sharp$ is an absolutely convergent Dirichlet series for $\sigma > 1$ whose coefficients are denoted by $a(n)$ and

$(s-1)^m F(s)$ has continuation to $\mathbb{C}$ as an entire function of finite order for some integer $m \geq 0$; moreover, writing $\overline{f}(s) = \overline{f(\overline{s})}$, a functional equation of type

$$\Phi(s) = \omega \overline{\Phi}(1-s) \tag{3}$$

holds, where $|\omega| = 1$ and

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s), \tag{4}$$

with $Q > 0$, $r \geq 0$, $\lambda_j > 0$, $\Re\mu_j \geq 0$. In addition, the functions in the Selberg class $\mathcal{S}$ satisfy the Ramanujan conjecture $a(n) \ll n^{\varepsilon}$ and have a general Euler product

$$F(s) = \prod_p F_p(s) \quad \text{with} \quad \log F_p(s) = \sum_{m=1}^{\infty} \frac{b(p^m)}{p^{ms}}$$

and $b(p^m) \ll p^{\vartheta m}$ for some $\vartheta < 1/2$. The degree d_F, conductor q_F, root number ω_F^* and the first H-invariant $H_F(1)$ (also called the ξ-invariant ξ_F in our previous papers and in the next section) are defined as

$$d_F = 2\sum_{j=1}^{r} \lambda_j, \quad q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j},$$

$$\omega_F^* = \omega \prod_{j=1}^{r} \lambda_j^{-2i\Im\mu_j} \quad \text{and} \quad H_F(1) = 2\sum_{j=1}^{r} (\mu_j - 1/2) = \Re H_F(1) + i d_F \theta_F,$$

say; θ_F is called the internal shift of $F(s)$. All these quantities are invariants of $F(s)$, i.e., they do not depend on the shape (4) of the functional equation (3), which can be modified by means of some formulae of the Γ function. We say that $F \in \mathcal{S}^\sharp$ is *normalized* if $\theta_F = 0$ and its first nonvanishing Dirichlet coefficient equals 1.

Remark. When dealing with $F \in \mathcal{S}^\sharp$ with $d_F = 2$ and $q_F = 1$ we may restrict ourselves, without loss of generality, to normalized functions. Indeed, in such a situation a polar $F(s)$ has $\theta_F = 0$ (see [11, Lemma 4.1]), while an entire $F(s)$ can always be shifted to get $\theta_F = 0$. The second requirement is met by simply dividing by the first nonvanishing coefficient.

Finally, for a normalized $F \in \mathcal{S}^\sharp$ with $d_F = 2$ and $q_F = 1$, we write $H_F(1)$ as

$$H_F(1) = d_F(k_F - 1). \tag{5}$$

The meaning of the invariant k_F, which a priori can take any real value, will be discussed later on.

Note that the presence of the conjugate in (3) implies some differences in the structure of the solutions of functional equations (1) (or (2)) and (3). In particular, the solutions in $\mathcal{S}^\sharp$ of (3) form a real vector space, while the L-functions of cusp forms form a complex vector space. Therefore, not every $L_f(s)$ as above belongs to $\mathcal{S}^\sharp$, although the similarities between the two structures are definitely strong; see e.g., Carletti–Monti Bragadin–Perelli [2]. Moreover, such differences vanish if we consider only functions in $\mathcal{S}$ and the $L_f(s)$ associated with eigenforms, since their coefficients are real. Actually, it has been conjectured that the primitive functions of degree 2 in $\mathcal{S}$ coincide with the normalized L-functions associated with holomorphic and nonholomorphic newforms. As a small step in this direction, we prove the following result.

Theorem. *Let $F \in \mathcal{S}^\sharp$ with degree 2 and conductor 1 be normalized. Then*

(i) the Dirichlet coefficients $a(n)$ are real;
(ii) $k_F \in \mathbb{Z}$ and $\omega_F^* = (-1)^{k_F}$.

Note that, a priori, the invariant k_F could be a negative integer. However, we believe that this is never the case, and actually a standard conjecture in Selberg class theory implies that $k_F \geq 0$. Indeed, the strong $\boldsymbol{\lambda}$-conjecture, asserting that every $F \in \mathcal{S}^\sharp$ has a functional equation where all the λ_j in (4) are equal to $1/2$ as in the case of classical L-functions (see [7]), immediately implies that $0 \leq \Re H_F(1) + d_F = d_F k_F$.

We conclude with a brief discussion of the meaning of the invariant k_F in the framework of the L-functions $L_f(s)$ discussed above. In the case of holomorphic modular forms, in view of functional equation (1) the first H-invariant equals $2(w/2-1)$, hence k_F is related to the weight of f:

$$k_F = \frac{w}{2}. \tag{6}$$

For Maass cusp forms the weight is 0, and from (2) we have that the first H-invariant equals $2(\varepsilon-1)$; hence k_F is related to the parity of f:

$$k_F = \varepsilon.$$

Therefore, the data of the functions $F(s)$ in the theorem agree with the corresponding data of the cusp forms of level 1; thus our theorem may be regarded as a weak converse theorem for functions of degree 2 and conductor 1 in the extended Selberg class $\mathcal{S}^\sharp$. Note in particular that the property $k_F \in \mathbb{Z}$, obtained here by purely analytic means, implies, thanks to (6), the well-known algebraic fact that there are no cusp forms of level 1 and odd weight.

We finally remark that, writing $H_F(1)$ as in (5) also for the degree 1 functions in $\mathcal{S}^\sharp$ with $\theta_F = 0$, we have again that k_F coincides with the parity ε since $H_F(1) = \varepsilon - 1$. Indeed, for such functions the allowed Γ-factors are only those of the Dirichlet L-functions associated with even and odd primitive characters (mod q); see [5, Theorem 2]. However, if $q > 1$ the root number is not given by (ii) of our theorem. This agrees with the general expectation that the situation becomes more complicated as soon as we consider L-functions with conductor >1. For instance, in that case there is a wide variety of admissible values for the root numbers of cusp forms of level >1.

§2. Proof of the theorem

The following transformation formula is the basis of the proof of the theorem; we use the standard notation $e(x) = e^{2\pi i x}$.

Lemma. *Let $F \in \mathcal{S}^\sharp$ with $d_F = 2$ be normalized, $\alpha_0 > 0$ and $\alpha_1 \in \mathbb{R}$. Then*

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} e\left(-\alpha_0 n - \alpha_1\sqrt{n}\right) = -\omega_F^* e^{-i(\pi/2)\xi_F} e\left(\frac{\alpha_1^2}{4\alpha_0}\right) (\alpha_0\sqrt{q_F})^{2s-1} \times \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^s} e\left(\frac{n}{\alpha_0 q_F} - \frac{\alpha_1\sqrt{n}}{\alpha_0\sqrt{q_F}}\right) + h(s), \tag{7}$$

where $h(s)$ is holomorphic for $\sigma > 1/2$.

The meaning of (7) is that the difference of the two series, which are absolutely convergent for $\sigma > 1$, is holomorphic for $\sigma > 1/2$. Actually, general results of this type are known (see [10, Theorem 1.1] and [13, Theorem 1]), but in this paper we need a version with explicit constants. We state the lemma under the hypothesis that $F(s)$ is normalized, but in fact we prove it assuming the slightly milder condition $\theta_F = 0$.

Proof of the lemma. We follow closely the proofs of [10, Theorem 1.1] and [13, Theorem 1] since, essentially, we need to compute explicitly only the constants appearing in the main term in these theorems. Since the proof of such a Theorem 1.1 is definitely more detailed, but requires $F(s)$ to be entire, at the beginning we follow the proof of the above-quoted Theorem 1, which holds in full generality but is less detailed, and we shift to Theorem 1.1 as soon as the hypothesis that $F(s)$ is regular at $s = 1$ is no longer relevant for our purposes. We shall only briefly describe the required changes; we use the notation in [10].

As already mentioned, we start by following the proof of [13, Theorem 1], till equation (2.10) there, in the special case with $N = 1$, $\kappa_0 = 1$ and $\omega_1 = 1/2$ (see [10, eq. (1.4)]). In view of [13, Lemma 2.1], the sum $\sum_{\emptyset\neq\mathcal{A}\subset\{0,1\}} I_X(s,\mathcal{A})$ contains what in the end will be the main term of the transformation formula, namely the series on the right-hand side of (7). Moreover, the expressions of the terms $I_X(s,\mathcal{A})$ in [13, eq. (2.10)] and in [10, eq. (2.3)], obtained using the functional equation of $F(s)$, are identical, and the integrals contained in such expressions do not depend on the polar structure of $F(s)$ at $s=1$. Since the rest of the proof of the lemma is devoted to the explicit computation of the constants appearing in the main term arising from such integrals, we may now shift to [10, eq. (2.3)] and follow the proof of [10, Theorem 1.1] up to the end.

The first change concerns equation (2.4). Computing explicitly the constant c_0 arising in equation (2.4) after the use of Stirling's formula as in [8, Lemma 2.1], we obtain

$$c_0 = \pi^{r-1/2}2^{r-3/2}\beta^{1/2}\prod_{j=1}^{r}\lambda_j^{-2i\Im\mu_j}; \tag{8}$$

see [10, p. 1409] for the definition of β. Moreover, the explicit value of the constant c_1 in [10, eq. (2.6)] is

$$c_1 = 2^{-r}e^{-i(\pi/2)\xi_F}.$$

Note that the constants c_1 in equations (2.4) and (2.6) are not equal; this is allowed by the notation used in [10, see p. 1407]. Such a situation will be encountered later on as well, and in each case we shall warn the reader. As a consequence, with the notation in [10], the first of the three terms corresponding to $\ell=0$ in equation (2.8) in our case equals

$$\begin{aligned}\frac{1}{2\sqrt{2\pi}}\omega_F^* e^{-i(\pi/2)\xi_F}\beta^{1/2}Q^{1-2s}\sum_{n=1}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}\frac{1}{(2\pi i)^{|\mathcal{A}|}}&\\ \times\int_{\mathcal{L}_{|\mathcal{A}}}\Gamma\left(\frac{3}{2}-2\boldsymbol{x}_{|\mathcal{A}}\right)\left(-\frac{4}{\beta}\right)^{\boldsymbol{x}_{|\mathcal{A}}}G(\boldsymbol{w}_{|\mathcal{A}})\left(\frac{n}{Q^2}\right)^{\boldsymbol{w}_{|\mathcal{A}}}\mathrm{d}\boldsymbol{w}_{|\mathcal{A}}&\\ =\frac{1}{2\sqrt{2\pi}}\omega_F^* e^{-i(\pi/2)\xi_F}\beta^{1/2}Q^{1-2s}\sum_{n=1}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}I_X(s,\mathcal{A},n,0).&\end{aligned} \tag{9}$$

All the remaining terms in equation (2.8), corresponding to $\ell\geq 1$ and to the quantities $J_X(s,\mathcal{A},n,\ell)$ and $h_X(s,\mathcal{A},n,\ell)$, do not contribute to the main term of the transformation formula, so we forget about them.

Now we follow the Mellin transform argument in [10, Section 2.2], applied to the integral $I_X(s,\mathcal{A},n,0)$ in (9). A simple computation shows that the value of the constant c_0 in equation (2.14) (which is different from c_0 in (8)) is

$$c_0 = \frac{1}{2}e^{-i(3\pi/4)}.$$

Therefore, recalling (9), the relevant part of $\sum_{\emptyset\neq\mathcal{A}\subset\{0,1\}} I_X(s,\mathcal{A})$ equals (compare with equation (2.17))

$$\frac{e^{-i(3\pi/4)}}{2\sqrt{2\pi}}\omega_F^* e^{-i(\pi/2)\xi_F}\left(\frac{Q^2\beta}{4}\right)^{1/2-s}\sum_{n=1}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}I_X(s,n,0), \tag{10}$$

where, in our case,

$$I_X(s,n,0)=\int_0^\infty e^{i\sqrt{x}}\left(e^{-\Psi_X(x,n)}e\left(-f\left(\frac{qx}{n},\boldsymbol{\alpha}\right)\right)-1\right)x^{-1/4-s}\,\mathrm{d}x,$$

with

$$\Psi_X(x,n)=\frac{1}{X}\left(\frac{qx}{n}+\left(\frac{qx}{n}\right)^{1/2}\right),\qquad q=q_F(4\pi)^{-2}\quad\text{and}\quad f(\xi,\boldsymbol{\alpha})=\alpha_0\xi+\alpha_1\sqrt{\xi}.$$

Next we follow the saddle point argument in [10, Section 2.3], applied to the integral $I_X(s,n,0)$ in (10). We recall that in our case,

$$\Phi(z,n,\boldsymbol{\alpha})=z^{1/2}-2\pi f\left(\frac{qz}{n},\boldsymbol{\alpha}\right),$$

and let $x_0=x_0(n,\boldsymbol{\alpha})$ be the zero of $\frac{\partial}{\partial z}\Phi(z,n,\boldsymbol{\alpha})$ in [10, Lemma 2.3], i.e., the critical point of $I_X(s,n,0)$; we recall that $x_0>0$. Hence by [10, Lemma 2.4 and eq. (2.29)] the relevant part of the integral $I_X(s,n,0)$ is

$$\gamma x_0^{3/4-s}\int_{-r}^{r}e^{-\Psi_X(w,n)+i\Phi(w,n,\boldsymbol{\alpha})}(1+\gamma\lambda)^{-1/4-s}\,\mathrm{d}\lambda, \tag{11}$$

where

(12)

$$\gamma=\sqrt{2}e^{-i\pi/4},\quad w=x_0(1+\gamma\lambda),\quad r=\frac{\log n}{\sqrt{|R|}}\quad\text{and}\quad R=x_0^2\frac{\partial^2}{\partial z^2}\Phi(z,n,\boldsymbol{\alpha})|_{z=x_0}.$$

Finally we proceed as in [10, Section 2.4] to perform the limit as $X\to\infty$. In view of [10, eqs. (2.38) and (2.39)], for our present purposes such a procedure simply leads to the vanishing of the function $\Psi_X(z,n)$ inside (11). As a consequence, from (10) and (11) we obtain that the main term of the transformation formula is contained inside the quantity

$$\begin{aligned}&-\frac{1}{2\sqrt{\pi}}\omega_F^* e^{-i(\pi/2)\xi_F}\left(\frac{Q^2\beta}{4}\right)^{1/2-s}\\&\qquad\times\sum_{n=n_0}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}\left(x_0^{3/4-s}\int_{-r}^{r}e\left(\frac{1}{2\pi}\Phi(z,n,\boldsymbol{\alpha})\right)(1+\gamma\lambda)^{-1/4-s}\,\mathrm{d}\lambda\right)\\&\quad=\frac{-1}{2\gamma\sqrt{\pi}}\omega_F^* e^{-i(\pi/2)\xi_F}\left(\frac{Q^2\beta}{4}\right)^{1/2-s}\sum_{n=n_0}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}K(s,n),\end{aligned}$$

with the notation of the above-quoted equation (2.38), where $n_0 \geq 1$ is sufficiently large. Moreover, by [10, Lemma 2.7] the relevant part of such a quantity is

$$\begin{aligned} &-\frac{1}{2}\omega_F^* e^{-i(\pi/2)\xi_F}\left(\frac{Q^2\beta}{4}\right)^{1/2-s}\sum_{n=n_0}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}\left(\frac{x_0^{3/4-s}}{\sqrt{|R|}}e\left(\frac{1}{2\pi}\Phi(x_0,n,\boldsymbol{\alpha})\right)\right) \\ &\qquad = -\frac{1}{2}\omega_F^* e^{-i(\pi/2)\xi_F}\left(\frac{Q^2\beta}{4}\right)^{1/2-s}\sum_{n=n_0}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}M(s,n), \end{aligned} \tag{13}$$

say.

The final step in the proof of the lemma is the explicit computation of the terms $M(s,n)$ in (13). We have

$$\Phi(z,n,\boldsymbol{\alpha}) = \sqrt{z} - 2\pi\left(\alpha_0\frac{qz}{n} + \alpha_1\sqrt{\frac{qz}{n}}\right) = \left(1 - \frac{\alpha_1}{2}\sqrt{\frac{q_F}{n}}\right)\sqrt{z} - \frac{\alpha_0 q_F}{8\pi n}z;$$

hence the zero $x_0 > 0$ of the first z-derivative of $\Phi(z,n,\boldsymbol{\alpha})$ satisfies

$$\sqrt{x_0} = \frac{4\pi n}{\alpha_0 q_F}\left(1 - \frac{\alpha_1}{2}\sqrt{\frac{q_F}{n}}\right).$$

Consequently, in view of (12) we have

$$\Phi(x_0,n,\boldsymbol{\alpha}) = \frac{2\pi n}{\alpha_0 q_F}\left(1 - \frac{\alpha_1}{2}\sqrt{\frac{q_F}{n}}\right)^2 = 2\pi\left(\frac{n}{\alpha_0 q_F} - \frac{\alpha_1\sqrt{n}}{\alpha_0\sqrt{q_F}}\right) + 2\pi\frac{\alpha_1^2}{4\alpha_0}$$

and

$$|R| = \frac{1}{4}\sqrt{x_0}\left(1 - \frac{\alpha_1}{2}\sqrt{\frac{q_F}{n}}\right), \qquad \frac{1}{\sqrt{|R|}} = 2x_0^{-1/4}\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right).$$

Therefore the main term of $M(s,n)$ in (13) is

$$\begin{aligned} &2e\left(\frac{\alpha_1^2}{4\alpha_0}\right)(\sqrt{x_0})^{1-2s}e\left(\frac{n}{\alpha_0 q_F} - \frac{\alpha_1\sqrt{n}}{\alpha_0\sqrt{q_F}}\right) \\ &\qquad = 2e\left(\frac{\alpha_1^2}{4\alpha_0}\right)\left(\frac{4\pi}{\alpha_0 q_F}\right)^{1-2s}\frac{e\left(\frac{n}{\alpha_0 q_F} - \frac{\alpha_1\sqrt{n}}{\alpha_0\sqrt{q_F}}\right)}{n^{2s-1}}\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right). \end{aligned} \tag{14}$$

From (13) and (14) we finally obtain that the main term of the transformation formula is

$$-\omega_F^* e^{-i(\pi/2)\xi_F} e\left(\frac{\alpha_1^2}{4\alpha_0}\right)(\alpha_0\sqrt{q_F})^{2s-1}\sum_{n=n_0}^{\infty}\frac{\overline{a(n)}}{n^s}e\left(\frac{n}{\alpha_0 q_F} - \frac{\alpha_1\sqrt{n}}{\alpha_0\sqrt{q_F}}\right),$$

as required. Moreover, all the terms we neglected in the course of the proof, as well as the missing terms with $n = 1, \ldots, n_0 - 1$ in the above series, contribute to form a function that certainly is holomorphic for $\sigma > 1/2$. The lemma is therefore proved. □

Now we are ready for the proof of the theorem.

Proof of the theorem. For simplicity we write

$$\Omega_F = -\omega_F^* e^{i(\pi/2)\xi_F}; \tag{15}$$

we recall that the standard twist of $F \in \mathcal{S}^\sharp$ with degree 2 is

$$F(s,\alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-\alpha\sqrt{n}) \tag{16}$$

and we refer to [9] and [12] for the properties of $F(s,\alpha)$. Due to the periodicity of the complex exponential we have

$$F(s,\alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-n-\alpha\sqrt{n});$$

hence, recalling that $q_F = 1$, from the lemma with the choice $\alpha_0 = 1$ and $\alpha_1 = \alpha > 0$ we get

$$F(s,\alpha) = \Omega_F e\left(\frac{\alpha^2}{4}\right)\overline{F}(s,\alpha) + h(s), \tag{17}$$

where $h(s)$ is holomorphic for $\sigma > 1/2$ and $\overline{F}(s,\alpha)$ denotes the right-hand side of (16) with $a(n)$ replaced by $\overline{a(n)}$, i.e., the standard twist of the conjugate $\overline{F} \in \mathcal{S}^\sharp$ of $F(s)$.

Next we recall some properties of the standard twist in our special case where $d_F = 2$, $q_F = 1$ and $\theta_F = 0$. From [9, Theorems 1 and 2] (see also [12, Theorem 1]), we have

$$\text{res}_{s=3/4} F(s,\alpha) = c_0(F)\frac{\overline{a(n_\alpha)}}{n_\alpha^{1/4}}, \tag{18}$$

with $c_0(F) \neq 0$ and $n_\alpha = \alpha^2/4$. Now let $\bar{n}$ be the least n with $a(n) \neq 0$, and recall that $a(\bar{n}) = 1$ since $F(s)$ is normalized. Choosing $\alpha = 2\sqrt{\bar{n}}$ and comparing residues at $s = 3/4$ of both sides of (17), and recalling that $\overline{F}(s)$ has degree 2, conductor 1 and $\theta_{\overline{F}} = 0$, from (18) we obtain

$$\frac{c_0(F)}{\bar{n}^{1/4}} = \frac{\Omega_F c_0(\overline{F})}{\bar{n}^{1/4}}$$

since $e\left(\frac{\alpha^2}{4}\right) = e(\bar{n}) = 1$. As a consequence,

$$c_0(F) = \Omega_F c_0(\overline{F}) \tag{19}$$

and hence choosing $\alpha = 2\sqrt{n}$ for arbitrary $n \geq \bar{n}$ and again comparing residues in (17), we have

$$\frac{\overline{a(n)}}{n^{1/4}} = \frac{a(n)}{n^{1/4}}.$$

The first assertion of the theorem then follows.

To prove the other assertions we note that from (19) and the first assertion we obtain

$$\Omega_F = 1. \tag{20}$$

Since in general $\omega_{\bar{F}}^* = \overline{\omega_F^*}$ and $|\omega_F^*| = 1$, in our case we have $\omega_F^* = \overline{\omega_F^*}$ and hence

$$\omega_F^* = \pm 1.$$

Therefore from (15) and (20) we have $e^{i(\pi/2)\xi_F} = \pm 1$; hence ξ_F is an even integer that we write as

$$\xi_F = 2(k-1)$$

for some integer k. Moreover, again from (15) and (20), we obtain

$$\omega_F^* = -e^{-i\pi(k-1)} = (-1)^k,$$

which completes the proof of the theorem. □

Acknowledgements

We wish to thank Philippe Michel for his advice concerning the Maass forms theory. This research was partially supported by Istituto Nazionale di Alta Matematica and by grants PRIN2015 “Number Theory and Arithmetic Geometry” and 2013/11/B/ST1/02799 “Analytic Methods in Arithmetic” of the National Science Centre.

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