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# Effective Basepoint-Free Theorem for Semi-Log Canonical Surfaces

by

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## Abstract

This paper proposes a Fujita-type freeness conjecture for semi-log canonical pairs. We prove it for curves and surfaces by using the theory of quasi-log schemes and give some effective very ampleness results for stable surfaces and semi-log canonical Fano surfaces. We also prove an effective freeness for log surfaces.

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## §1. Introduction

We will work over $\mathbb{C}$, the complex number field, throughout this paper. Note that, by the Lefschetz principle, all the results in this paper hold over any algebraically closed field $k$ of characteristic 0.

This paper proposes the following Fujita-type freeness conjecture for projective semi-log canonical pairs.

**Conjecture 1.1** (Fujita-type freeness conjecture for semi-log canonical pairs). *Let $(X, \Delta)$ be an $n$-dimensional projective semi-log canonical pair and let $D$ be a Cartier divisor on $X$. We put $A = D - (K_X + \Delta)$. Assume that*

*(1) $(A^n \cdot X_i) > n^n$ for every irreducible component $X_i$ of $X$, and*

*(2) $(A^d \cdot W) \geq n^d$ for every $d$-dimensional irreducible subvariety $W$ of $X$ for $1 \leq d \leq n-1$.*

*Then the complete linear system $|D|$ is basepoint-free.*

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By [Liu, Corollary 3.5], the complete linear system $|D|$ is basepoint-free if $A^n > \left(\frac{1}{2}n(n+1)\right)^n$ and $(A^d \cdot W) > \left(\frac{1}{2}n(n+1)\right)^d$ hold true in Conjecture 1.1, which is obviously a generalization of Angehrn–Siu's effective freeness (see [AS] and [Fuj1]).

Of course, the above conjecture is a naive generalization of Fujita's celebrated conjecture:

**Conjecture 1.2** (Fujita's freeness conjecture). *Let $X$ be a smooth projective variety with $\dim X = n$ and let $H$ be an ample Cartier divisor on $X$. Then the complete linear system $|K_X + (n+1)H|$ is basepoint-free.*

Now we have the main theorem of this paper:

**Theorem 1.3** (Main theorem; see Theorems 2.1 and 5.1). *Conjecture 1.1 holds true in dimensions 1 and 2.*

We have a corollary of Theorem 1.3:

**Corollary 1.4** (Cf. [LR, Theorem 24]). *Let $(X, \Delta)$ be a stable surface such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $I$ be the smallest positive integer such that $I(K_X + \Delta)$ is Cartier. Then $|mI(K_X + \Delta)|$ is basepoint-free and $3mI(K_X + \Delta)$ is very ample for every $m \geq 4$. If $I \geq 2$, then $|mI(K_X + \Delta)|$ is basepoint-free and $3mI(K_X + \Delta)$ is very ample for every $m \geq 3$. In particular, $12I(K_X + \Delta)$ is always very ample and $9I(K_X + \Delta)$ is very ample if $I \geq 2$.*

Note that a *stable pair* $(X, \Delta)$ is a projective semi-log canonical pair $(X, \Delta)$ such that $K_X + \Delta$ is ample. A *stable surface* is a two-dimensional stable pair. We have a further corollary:

**Corollary 1.5** (Semi-log canonical Fano surfaces). *Let $(X, \Delta)$ be a projective semi-log canonical surface such that $-(K_X + \Delta)$ is an ample $\mathbb{Q}$-divisor. Let $I$ be the smallest positive integer such that $I(K_X+\Delta)$ is Cartier. Then $|-mI(K_X+\Delta)|$ is basepoint-free and $-3mI(K_X+\Delta)$ is very ample for every $m \geq 2$. In particular, $-6I(K_X + \Delta)$ is very ample.*

For log surfaces (see [Fuj3]), the following theorem is a reasonable formulation of the Reider-type freeness theorem. For a related topic, see [Kaw].

**Theorem 1.6** (Effective freeness for log surfaces). *Let $(X, \Delta)$ be a complete irreducible log surface and let $D$ be a Cartier divisor on $X$. We put $A = D - (K_X + \Delta)$. Assume that $A$ is nef, $A^2 > 4$ and $A \cdot C \geq 2$ for every curve $C$ on $X$ such that $x \in C$. Then $\mathcal{O}_X(D)$ has a global section not vanishing at $x$.*

We know that the theory of log surfaces initiated in [Fuj3] now holds in characteristic $p > 0$ (see [FT], [Tan1] and [Tan2]). Therefore, it is natural to propose the following conjecture:

**Conjecture 1.7.** *Theorem 1.6 holds in characteristic $p > 0$.*

Note that the original form of Fujita's freeness conjecture (see Conjecture 1.2) is still open for surfaces in characteristic $p > 0$.

The standard approach to the Fujita-type freeness conjectures is based on the Kawamata–Viehweg vanishing theorem (see [EL]). However, we cannot directly apply the Kawamata–Viehweg vanishing theorem to log canonical pairs and semi-log canonical pairs. Therefore, we will use the theory of quasi-log schemes (see [Fuj4], [Fuj5], [Fuj8], and so on).

We summarize the contents of this paper. In Section 2, we prove Conjecture 1.1 for semi-log canonical curves using the vanishing theorem obtained in [Fuj4]. This section may help the reader to understand more complicated arguments in the subsequent sections. In Section 3, we collect some basic definitions. In Section 4, we quickly recall the theory of quasi-log schemes. Section 5 is the main part of this paper. In this section, we prove Conjecture 1.1 for semi-log canonical surfaces. Section 6 is devoted to the proof of Theorem 1.6, which is an effective freeness for log surfaces. In Section 7, which is independent of the other sections, we prove an effective very ampleness lemma.

For the standard notation and conventions of the minimal model program, see [Fuj2] and [Fuj8]. For the details of semi-log canonical pairs, see [Fuj4]. In this paper, a *scheme* means a separated scheme of finite type over $\mathbb{C}$ and a *variety* means a reduced scheme.

## §2. Semi-log canonical curves

In this section, we prove Conjecture 1.1 in dimension 1 based on [Fuj4]. This section will help the reader to understand the subsequent sections.

**Theorem 2.1.** *Let $(X, \Delta)$ be a projective semi-log canonical curve and let $D$ be a Cartier divisor on $X$. We put $A = D - (K_X + \Delta)$. Assume that $(A \cdot X_i) > 1$ for every irreducible component $X_i$ of $X$. Then the complete linear system $|D|$ is basepoint-free.*

If $(X, \Delta)$ is log canonical (that is, $X$ is normal) in Theorem 2.1, then the statement is obvious. However, Theorem 2.1 seems to be nontrivial when $X$ is not normal.

*Proof of Theorem 2.1.* We will see that the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(P) \tag{2.1}$$

is surjective for every $P \in X$. Of course, it is sufficient to prove that $H^1(X, \mathcal{I}_P \otimes \mathcal{O}_X(D)) = 0$, where $\mathcal{I}_P$ is the defining ideal sheaf of $P$ on $X$. If $P$ is a zero-dimensional semi-log canonical center of $(X, \Delta)$, then we know that $H^1(X, \mathcal{I}_P \otimes \mathcal{O}_X(D)) = 0$ by [Fuj4, Theorem 1.11]. Therefore, we may assume that $P$ is not a zero-dimensional semi-log canonical center of $(X, \Delta)$. Thus, we see that $X$ is normal, that is, smooth, at $P$ (see, for example, [Fuj4, Corollary 3.5]). We put

$$c = 1 - \mathrm{mult}_P \Delta. \tag{2.2}$$

Then we have $0 < c \leq 1$. We consider $(X, \Delta + cP)$. Then $(X, \Delta + cP)$ is semi-log canonical and $P$ is a zero-dimensional semi-log canonical center of $(X, \Delta + cP)$. Since

$$((D - (K_X + \Delta + cP)) \cdot X_i) > 0 \tag{2.3}$$

for every irreducible component $X_i$ of $X$ by the assumption that $(A \cdot X_i) > 1$ and the fact that $c \leq 1$, we obtain that $H^1(X, \mathcal{I}_P \otimes \mathcal{O}_X(D)) = 0$ (see [Fuj4, Theorem 1.11]). Therefore, we see that $H^1(X, \mathcal{I}_P \otimes \mathcal{O}_X(D)) = 0$ for every $P \in X$. Thus, we have the desired surjection (2.1). □

The above proof of Theorem 2.1 heavily depends on the vanishing theorem for semi-log canonical pairs (see [Fuj4, Theorem 1.11]), which follows from the theory of quasi-log schemes based on the theory of mixed Hodge structures on cohomology with compact support. For the details, see [Fuj4] and [Fuj8]. In dimension 2, we will directly use the framework of quasi-log schemes. Therefore, it is much more difficult than the proof of Theorem 2.1.

## §3. Preliminaries

In this section, we collect some basic definitions.

**Definition 3.1** (Operations for $\mathbb{R}$-divisors). Let $D$ be an $\mathbb{R}$-divisor on an equidimensional variety $X$, that is, $D$ is a finite formal $\mathbb{R}$-linear combination

$$D = \sum_i d_i D_i \tag{3.1}$$

of irreducible reduced subschemes $D_i$ of codimension 1, where $D_i \neq D_j$ for $i \neq j$. We define the *round-up* $\lceil D \rceil = \sum_i \lceil d_i \rceil D_i$ (resp. *round-down* $\lfloor D \rfloor = \sum_i \lfloor d_i \rfloor D_i$), where for every real number $x$, $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \le \lceil x \rceil < x+1$ (resp. $x-1 < \lfloor x \rfloor \le x$). We put

$$D^{<1} = \sum_{d_i<1} d_i D_i \quad \text{and} \quad D^{>1} = \sum_{d_i>1} d_i D_i. \tag{3.2}$$

We call $D$ a *boundary* (resp. *subboundary*) $\mathbb{R}$-divisor if $0 \le d_i \le 1$ (resp. $d_i \le 1$) for every $i$.

**Definition 3.2** (Singularities of pairs). Let $X$ be a normal variety and let $\Delta$ be an $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution such that $\mathrm{Exc}(f) \cup f_*^{-1}\Delta$, where $\mathrm{Exc}(f)$ is the exceptional locus of $f$ and $f_*^{-1}\Delta$ is the strict transform of $\Delta$ on $Y$, has a simple normal crossing support. We can write

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i. \tag{3.3}$$

We say that $(X, \Delta)$ is *sub log canonical* (*sub lc*, for short) if $a_i \ge -1$ for every $i$. We usually write $a_i = a(E_i, X, \Delta)$ and call it the *discrepancy coefficient* of $E_i$ with respect to $(X, \Delta)$. Note that we can define $a(E, X, \Delta)$ for every prime divisor $E$ *over* $X$. If $(X, \Delta)$ is sub log canonical and $\Delta$ is effective, then $(X, \Delta)$ is called *log canonical* (*lc*, for short).

It is well known that there is a largest Zariski open subset $U$ of $X$ such that $(U, \Delta|_U)$ is sub log canonical (see, for example, [Fuj8, Lemma 2.3.10]). If there exist a resolution $f : Y \to X$ and a divisor $E$ on $Y$ such that $a(E, X, \Delta) = -1$ and $f(E) \cap U \ne \emptyset$, then $f(E)$ is called a *log canonical center* (an *lc center*, for short) with respect to $(X, \Delta)$. A closed subset $C$ of $X$ is called a *log canonical stratum* (an *lc stratum*, for short) of $(X, \Delta)$ if and only if $C$ is a log canonical center of $(X, \Delta)$ or $C$ is an irreducible component of $X$. We note that the *non-lc locus* of $(X, \Delta)$, which is denoted by $\mathrm{Nlc}(X, \Delta)$, is $X \setminus U$.

Let $X$ be a normal variety and let $\Delta$ be an effective $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. If $a(E, X, \Delta) > -1$ for every divisor $E$ over $X$, then $(X, \Delta)$ is called *klt*. If $a(E, X, \Delta) > -1$ for every exceptional divisor $E$ over $X$, then $(X, \Delta)$ is called *plt*.

Let us recall the definitions around *semi-log canonical pairs*.

**Definition 3.3** (Semi-log canonical pairs). Let $X$ be an equidimensional variety that satisfies Serre's $S_2$ condition and is normal crossing in codimension 1. Let $\Delta$ be an effective $\mathbb{R}$-divisor whose support does not contain any irreducible components of the conductor of $X$. The pair $(X, \Delta)$ is called a *semi-log canonical pair* (an *slc pair*, for short) if

(1) $K_X + \Delta$ is $\mathbb{R}$-Cartier, and
(2) $(X^\nu, \Theta)$ is log canonical, where $\nu : X^\nu \to X$ is the normalization and $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$, that is, $\Theta$ is the sum of the inverse images of $\Delta$ and the conductor of $X$.

Let $(X, \Delta)$ be a semi-log canonical pair and let $\nu : X^\nu \to X$ be the normalization. We set

$$K_{X^\nu} + \Theta = \nu^*(K_X + \Delta) \tag{3.4}$$

as above. A closed subvariety $W$ of $X$ is called a *semi-log canonical center* (an *slc center*, for short) *with respect to* $(X, \Delta)$ if there exist a resolution of singularities $f : Y \to X^\nu$ and a prime divisor $E$ on $Y$ such that the discrepancy coefficient $a(E, X^\nu, \Theta) = -1$ and $\nu \circ f(E) = W$. A closed subvariety $W$ of $X$ is called a *semi-log canonical stratum* (*slc stratum*, for short) of the pair $(X, \Delta)$ if $W$ is a semi-log canonical center with respect to $(X, \Delta)$ or $W$ is an irreducible component of $X$.

We close this section with the notion of *log surfaces* (see [Fuj3]).

**Definition 3.4** (Log surfaces). Let $X$ be a normal surface and let $\Delta$ be a boundary $\mathbb{R}$-divisor on $X$. Assume that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then the pair $(X, \Delta)$ is called a *log surface*. A log surface $(X, \Delta)$ is not always assumed to be log canonical.

In [Fuj3], we establish the minimal model program for log surfaces in full generality under the assumption that $X$ is $\mathbb{Q}$-factorial or $(X, \Delta)$ has only log canonical singularities. For the theory of log surfaces in characteristic $p > 0$, see [FT], [Tan1] and [Tan2].

## §4. On quasi-log structures

Let us quickly recall the definitions of *globally embedded simple normal crossing pairs* and *quasi-log schemes* for the reader's convenience. For the details, see, for example, [Fuj7] and [Fuj8, Chapters 5 and 6].

**Definition 4.1** (Globally embedded simple normal crossing pairs). Let $Y$ be a simple normal crossing divisor on a smooth variety $M$ and let $D$ be an $\mathbb{R}$-divisor on $M$ such that $\mathrm{Supp}(D+Y)$ is a simple normal crossing divisor on $M$ and that $D$ and $Y$ have no common irreducible components. We put $B_Y = D|_Y$ and consider the pair $(Y, B_Y)$. We call $(Y, B_Y)$ a *globally embedded simple normal crossing pair* and $M$ the *ambient space* of $(Y, B_Y)$. A *stratum* of $(Y, B_Y)$ is the $\nu$-image of a log canonical stratum of $(Y^\nu, \Theta)$, where $\nu : Y^\nu \to Y$ is the normalization and

$K_{Y^\nu} + \Theta = \nu^*(K_Y + B_Y)$, that is, $\Theta$ is the sum of the inverse images of $B_Y$ and the singular locus of $Y$.

In this paper, we adopt the following definition of quasi-log schemes.

**Definition 4.2** (Quasi-log schemes). A *quasi-log scheme* is a scheme $X$ endowed with an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) $\omega$ on $X$, a proper closed subscheme $X_{-\infty} \subset X$ and a finite collection $\{C\}$ of reduced and irreducible subschemes of $X$ such that there is a proper morphism $f : (Y, B_Y) \to X$ from a globally embedded simple normal crossing pair satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor),$$

where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.
(3) The collection of subvarieties $\{C\}$ coincides with the image of $(Y, B_Y)$-strata that are not included in $X_{-\infty}$.

We simply write $[X, \omega]$ to denote the above data

$$\big(X, \omega, f : (Y, B_Y) \to X\big)$$

if there is no risk of confusion. Note that a quasi-log scheme $X$ is the union of $\{C\}$ and $X_{-\infty}$. We also note that $\omega$ is called the *quasi-log canonical class* of $[X, \omega]$, which is defined up to $\mathbb{R}$-linear equivalence. We sometimes simply say that $[X, \omega]$ is a *quasi-log pair*. The subvarieties $C$ are called the *qlc strata* of $[X, \omega]$, $X_{-\infty}$ is called the *non-qlc locus* of $[X, \omega]$ and $f : (Y, B_Y) \to X$ is called a *quasi-log resolution* of $[X, \omega]$. We sometimes use $\mathrm{Nqlc}(X, \omega)$ to denote $X_{-\infty}$. A closed subvariety $C$ of $X$ is called a *qlc center* of $[X, \omega]$ if $C$ is a qlc stratum of $[X, \omega]$ that is not an irreducible component of $X$.

Let $[X, \omega]$ be a quasi-log scheme. Assume that $X_{-\infty} = \emptyset$. Then we sometimes simply say that $[X, \omega]$ is a *qlc pair* or $[X, \omega]$ is a quasi-log scheme with only *quasi-log canonical singularities*.

**Definition 4.3** (Nef and log big divisors for quasi-log schemes). Let $L$ be an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) on a quasi-log pair $[X, \omega]$ and let $\pi : X \to S$ be a proper morphism between schemes. Then $L$ is *nef and log big over $S$ with respect to* $[X, \omega]$ if $L$ is $\pi$-nef and $L|_C$ is $\pi$-big for every qlc stratum $C$ of $[X, \omega]$.

The following theorem is a key result for the theory of quasi-log schemes.

**Theorem 4.4** (Adjunction and a vanishing theorem for quasi-log schemes). *Let $[X, \omega]$ be a quasi-log scheme and let $X'$ be the union of $X_{-\infty}$ with a (possibly empty) union of some qlc strata of $[X,\omega]$. Then we have the following properties.*

- *(i) Assume that $X' \neq X_{-\infty}$. Then $X'$ is a quasi-log scheme with $\omega' = \omega|_{X'}$ and $X'_{-\infty} = X_{-\infty}$. Moreover, the qlc strata of $[X', \omega']$ are exactly the qlc strata of $[X,\omega]$ that are included in $X'$.*
- *(ii) Assume that $\pi : X \to S$ is a proper morphism between schemes. Let $L$ be a Cartier divisor on $X$ such that $L - \omega$ is nef and log big over $S$ with respect to $[X,\omega]$. Then $R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$ for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of $X'$ on $X$.*

For the proof of Theorem 4.4, see, for example, [Fuj5, Theorem 3.8] and [Fuj8, Section 6.3]. We can slightly generalize Theorem 4.4(ii) as follows.

**Theorem 4.5.** *Let $[X,\omega]$, $X'$ and $\pi : X \to S$ be as in Theorem 4.4. Let $L$ be a Cartier divisor on $X$ such that $L - \omega$ is nef over $S$ and that $(L - \omega)|_W$ is big over $S$ for any qlc stratum $W$ of $[X,\omega]$ that is not contained in $X'$. Then $R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$ for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of $X'$ on $X$.*

Theorem 4.5 is obvious by the proof of Theorem 4.4. For a related topic, see [Fuj4, Remark 5.2]. Theorem 4.5 will play a crucial role in the proof of Theorem 1.6 in Section 6.

Finally, we prepare a useful lemma, which is new, for the proof of Theorem 1.3.

**Lemma 4.6.** *Let $[X,\omega]$ be a qlc pair such that $X$ is irreducible. Let $E$ be an effective $\mathbb{R}$-Cartier divisor on $X$. This means that*

$$E = \sum_{i=1}^{k} e_i E_i,$$

*where $E_i$ is an effective Cartier divisor on $X$ and $e_i$ is a positive real number for every $i$. Then we can give a quasi-log structure to $[X, \omega + E]$, that coincides with the original quasi-log structure of $[X,\omega]$ outside* $\operatorname{Supp} E$.

For the details of the quasi-log structure of $[X, \omega + E]$, see the construction in the proof below.

*Proof.* Let $f : (Z, \Delta_Z) \to [X,\omega]$ be a quasi-log resolution, where $(Z, \Delta_Z)$ is a globally embedded simple normal crossing pair. By taking some suitable blow-ups, we may assume that the union of all strata of $(Z, \Delta_Z)$ mapped to $\operatorname{Supp} E$, which is denoted by $Z''$, is the union of some irreducible components of $Z$ (see [Fuj7, Proposition 4.1] and [Fuj8, Section 6.3]). We put $Z' = Z - Z''$ and $K_{Z'} + \Delta_{Z'} = (K_Z+\Delta_Z)|_{Z'}$. We may further assume that $(Z', \Delta_{Z'}+f'^*E)$ is a globally embedded simple normal crossing pair, where $f' = f|_{Z'} : Z' \to X$. By construction, we have a natural inclusion

$$\mathcal{O}_{Z'}(\lceil -(\Delta_{Z'} + f'^*E)^{<1}\rceil - \lfloor(\Delta_{Z'} + f'^*E)^{>1}\rfloor) \subset \mathcal{O}_Z(\lceil -\Delta_Z^{<1}\rceil). \tag{4.1}$$

This is because

$$-\lfloor(\Delta_{Z'} + f'^*E)^{>1}\rfloor \leq -Z''|_{Z'} \tag{4.2}$$

and

$$\mathcal{O}_{Z'}(-Z''|_{Z'}) \subset \mathcal{O}_Z. \tag{4.3}$$

Thus, we have

$$f'_*\mathcal{O}_{Z'}(\lceil -(\Delta_{Z'} + f'^*E)^{<1}\rceil - \lfloor(\Delta_{Z'} + f'^*E)^{>1}\rfloor) \subset f_*\mathcal{O}_Z(\lceil -\Delta_Z^{<1}\rceil) \simeq \mathcal{O}_X. \tag{4.4}$$

By putting

$$\mathcal{I}_{X_{-\infty}} = f'_*\mathcal{O}_{Z'}(\lceil -(\Delta_{Z'} + f'^*E)^{<1}\rceil - \lfloor(\Delta_{Z'} + f'^*E)^{>1}\rfloor), \tag{4.5}$$

$f'\colon (Z', \Delta_{Z'}+f'^*E) \to [X, \omega+E]$ gives a quasi-log structure to $[X, \omega+E]$. By construction, it coincides with the original quasi-log structure of $[X, \omega]$ outside $\operatorname{Supp} E$. □

## §5. Semi-log canonical surfaces

In this section, we prove Conjecture 1.1 for surfaces.

**Theorem 5.1.** *Let $(X, \Delta)$ be a projective semi-log canonical surface and let $D$ be a Cartier divisor on $X$. We put $A = D - (K_X + \Delta)$. Assume that $(A^2 \cdot X_i) > 4$ for every irreducible component $X_i$ of $X$ and that $A \cdot C \geq 2$ for every curve $C$ on $X$. Then the complete linear system $|D|$ is basepoint-free.*

**Remark 5.2.** By assumption and Nakai's ampleness criterion for $\mathbb{R}$-divisors (see [CP]), $A$ is ample in Theorem 5.1. However, we do not use the ampleness of $A$ in the proof of Theorem 5.1.

Our proof of Theorem 5.1 uses the theory of quasi-log schemes.

*Proof.* We will prove that the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(P)$$

is surjective for every $P \in X$.

*Step* 1 (Quasi-log structure). By [Fuj4, Theorem 1.2], we can take a quasi-log resolution $f : (Z, \Delta_Z) \to [X, K_X + \Delta]$. Precisely speaking, $(Z, \Delta_Z)$ is a globally embedded simple normal crossing pair such that $\Delta_Z$ is a subboundary $\mathbb{R}$-divisor on $Z$ with the following properties:

(i) $K_Z + \Delta_Z \sim_{\mathbb{R}} f^*(K_X + \Delta)$.
(ii) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Z(\lceil -\Delta_Z^{<1} \rceil)$ is an isomorphism.
(iii) $\dim Z = 2$.
(iv) $W$ is a semi-log canonical stratum of $(X, \Delta)$ if and only if $W = f(S)$ for some stratum $S$ of $(Z, \Delta_Z)$.

It is worth mentioning that $f : Z \to X$ is not necessarily birational. This step is nothing but [Fuj4, Theorem 1.2].

*Step* 2. Assume that $P$ is a zero-dimensional semi-log canonical center of $(X, \Delta)$. Then $H^i(X, \mathcal{I}_P \otimes \mathcal{O}_X(D)) = 0$ for every $i > 0$, where $\mathcal{I}_P$ is the defining ideal sheaf of $P$ on $X$ (see [Fuj4, Theorem 1.11] and Theorem 4.4). Therefore, the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(P)$$

is surjective.

From now on, we may assume that $P$ is not a zero-dimensional semi-log canonical center of $(X, \Delta)$.

*Step* 3. Assume that there exists a one-dimensional semi-log canonical center $W$ of $(X, \Delta)$ such that $P \in W$. Since $P$ is not a zero-dimensional semi-log canonical center of $(X, \Delta)$, $W$ is normal, that is, smooth, at $P$ by [Fuj4, Corollary 3.5]. By adjunction (see Theorem 4.4), $[W, (K_X + \Delta)|_W]$ has a quasi-log structure with only quasi-log canonical singularities induced by the quasi-log structure $f : (Z, \Delta_Z) \to [X, K_X + \Delta]$ constructed in Step 1. Let $g : (Z', \Delta_{Z'}) \to [W, (K_X + \Delta)|_W]$ be the induced quasi-log resolution. We put

$$c = \sup_{t \geq 0} \left\{ t \,\middle|\, \begin{array}{l} \text{the normalization of } (Z', \Delta_{Z'} + tg^*P) \text{ is} \\ \text{sub log canonical} \end{array} \right\}. \tag{5.1}$$

Then, by [Fuj5, Lemma 3.16], we obtain that $0 < c < 2$. Note that $P$ is a Cartier divisor on $W$. Let us consider $g : (Z', \Delta_{Z'} + cg^*P) \to [W, (K_X + \Delta)|_W + cP]$, which defines a quasi-log structure. Then, by construction, $P$ is a qlc center of $[W, (K_X + \Delta)|_W + cP]$. Moreover, we see that

$$(D|_W - ((K_X + \Delta)|_W + cP)) = (A \cdot W) - c > 0 \tag{5.2}$$

by assumption. Therefore, we obtain that

$$H^i(W, \mathcal{I}_P \otimes \mathcal{O}_W(D)) = 0 \tag{5.3}$$

for every $i > 0$ by Theorem 4.4, where $\mathcal{I}_P$ is the defining ideal sheaf of $P$ on $W$. Thus, the restriction map

$$H^0(W, \mathcal{O}_W(D)) \to \mathcal{O}_W(D) \otimes \mathbb{C}(P) \tag{5.4}$$

is surjective. On the other hand, by Theorem 4.4 again, we have

$$H^i(X, \mathcal{I}_W \otimes \mathcal{O}_X(D)) = 0 \tag{5.5}$$

for every $i > 0$, where $\mathcal{I}_W$ is the defining ideal sheaf of $W$ on $X$. This implies that the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to H^0(W, \mathcal{O}_W(D)) \tag{5.6}$$

is surjective. By combining (5.4) with (5.6), the desired restriction map

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(P) \tag{5.7}$$

is surjective.

Therefore, from now on, we may assume that no one-dimensional semi-log canonical centers of $(X, \Delta)$ contain $P$.

*Step* 4. In this step, we assume that $P$ is a smooth point of $X$. Let $X_0$ be the unique irreducible component of $X$ containing $P$. By adjunction (see Theorem 4.4), $[X_0, (K_X + \Delta)|_{X_0}]$ has a quasi-log structure with only quasi-log canonical singularities induced by the quasi-log structure $f : (Z, \Delta_Z) \to [X, K_X + \Delta]$ constructed in Step 1. By Theorem 4.4,

$$H^i(X, \mathcal{I}_{X_0} \otimes \mathcal{O}_X(D)) = 0 \tag{5.8}$$

for every $i > 0$, where $\mathcal{I}_{X_0}$ is the defining ideal sheaf of $X_0$ on $X$. Therefore, the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to H^0(X_0, \mathcal{O}_{X_0}(D)) \tag{5.9}$$

is surjective. Thus, it is sufficient to prove that the natural restriction map

$$H^0(X_0, \mathcal{O}_{X_0}(D)) \to \mathcal{O}_{X_0}(D) \otimes \mathbb{C}(P) \tag{5.10}$$

is surjective. We put $A_0 = A|_{X_0}$. Since $A_0^2 > 4$, we can find an effective $\mathbb{R}$-Cartier divisor $B$ on $X_0$ such that $\mathrm{mult}_P B > 2$ and that $B \sim_{\mathbb{R}} A_0$. We put

$U = X_0 \setminus \operatorname{Sing} X_0$ and define

$$c = \max\{t \geq 0 \,|\, (U, \Delta|_U + tB|_U) \text{ is log canonical at } P\}. \tag{5.11}$$

Then we obtain that $0 < c < 1$ since $\operatorname{mult}_P B > 2$. By Lemma 4.6, we have a quasi-log structure on $[X_0, (K_X+\Delta)|_{X_0}+cB]$. By construction, there is a qlc center $W$ of $[X_0, (K_X+\Delta)|_{X_0}+cB]$ passing through $P$. Let $X'$ be the union of the non-qlc locus of $[X_0, (K_X+\Delta)|_{X_0}+cB]$ and the minimal qlc center $W_0$ of $[X_0, (K_X+\Delta)|_{X_0}+cB]$ passing through $P$. Note that $D|_{X_0} - ((K_X+\Delta)|_{X_0}+cB) \sim_{\mathbb{R}} (1-c)A_0$. Then, by Theorem 4.4,

$$H^i(X_0, \mathcal{I}_{X'} \otimes \mathcal{O}_{X_0}(D)) = 0 \tag{5.12}$$

for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of $X'$ on $X_0$.

*Case* 1. If $\dim W_0 = 0$, then $P$ is isolated in $\operatorname{Supp} \mathcal{O}_{X_0}/\mathcal{I}_{X'}$. Therefore, the restriction map

$$H^0(X_0, \mathcal{O}_{X_0}(D)) \to \mathcal{O}_{X_0}(D) \otimes \mathbb{C}(P) \tag{5.13}$$

is surjective.

*Case* 2. If $\dim W_0 = 1$, then let us consider the quasi-log structure of $[X', ((K_X+\Delta)|_{X_0}+cB)|_{X'}]$ induced by the quasi-log structure of $[X_0, (K_X+\Delta)|_{X_0}+cB]$ constructed above by Lemma 4.6 (see Theorem 4.4(i)). From now on, we will see that we can take $0 < c' \leq 1$ such that $P$ is a zero-dimensional qlc center of $[X', ((K_X+\Delta)|_{X_0}+cB)|_{X'}+c'P]$ as in Step 3. By assumption, $(U, \Delta|_U + cB|_U)$ is plt in a neighborhood of $P$. We put $\operatorname{mult}_P B = 2+a$ with $a > 0$. We write $\Delta + cB = L + \Delta'$ on $U$, where $L = W_0$ and $L|_U$ is the unique one-dimensional log canonical center of $(U, \Delta|_U + cB|_U)$ passing through $P$. Note that we put $\Delta' = \Delta + cB - L$ on $U$. We put $\operatorname{mult}_P(\Delta+cB) = 1+\delta$ with $\delta \geq 0$, equivalently, $\delta = \operatorname{mult}_P \Delta' \geq 0$. Note that

$$1+\delta = \operatorname{mult}_P(\Delta+cB) = \operatorname{mult}_P \Delta + c(2+a). \tag{5.14}$$

Therefore, we have

$$c = \frac{1+\delta-\alpha}{2+a}, \tag{5.15}$$

where $\alpha = \operatorname{mult}_P \Delta \geq 0$. We also note that

$$\delta \leq \operatorname{mult}_P(\Delta'|_L) < 1. \tag{5.16}$$

Then, we can choose $c' = 1 - \operatorname{mult}_P(\Delta'|_L)$. This is because $(U, \Delta|_U + cB|_U + c'H)$ is log canonical in a neighborhood of $P$ but is not plt at $P$, where $H$ is a general smooth curve passing through $P$.

In this situation, we have

$$
\begin{aligned}
&\deg(D|_L-(K_X+\Delta+cB)|_L-c'P)\\
&\quad\geq\left(1-\frac{1+\delta-\alpha}{2+a}\right)\cdot 2-(1-\delta)\\
&\quad=\frac{1}{2+a}((2+a-1-\delta+\alpha)\cdot 2-(2+a)(1-\delta))\\
&\quad=\frac{1}{2+a}(a+2\alpha+a\delta)\\
&\quad\geq\frac{a}{2+a}>0.
\end{aligned}
\tag{5.17}
$$

Thus, by Theorem 4.4,

$$H^i(X',\mathcal{I}_{X''}\otimes\mathcal{O}_{X'}(D))=0 \tag{5.18}$$

for every $i>0$, where $X''$ is the union of the non-qlc locus of $[X',((K_X+\Delta)|_{X_0}+cB)|_{X'}+c'P]$ and $P$, and $\mathcal{I}_{X''}$ is the defining ideal sheaf of $X''$ on $X'$. Thus, we have

$$H^0(X',\mathcal{O}_{X'}(D))\to\mathcal{O}_{X'}(D)\otimes\mathcal{O}_{X'}/\mathcal{I}_{X''} \tag{5.19}$$

is surjective. Note that $P$ is isolated in $\operatorname{Supp}\mathcal{O}_{X'}/\mathcal{I}_{X''}$. Therefore, we obtain surjections

$$
\begin{aligned}
H^0(X,\mathcal{O}_X(D))&\twoheadrightarrow H^0(X_0,\mathcal{O}_{X_0}(D))\\
&\twoheadrightarrow H^0(X',\mathcal{O}_{X'}(D))\twoheadrightarrow\mathcal{O}_{X'}(D)\otimes\mathbb{C}(P)
\end{aligned}
\tag{5.20}
$$

by (5.9), (5.12) and (5.19). This is the desired surjection.

Finally, we further assume that $P$ is a singular point of $X$.

*Step* 5. Note that $(X,\Delta)$ is klt in a neighborhood of $P$ by assumption. We will reduce the problem to the situation as in Step 4. Let $\pi:Y\to X$ be the minimal resolution of $P$. We put $K_Y+\Delta_Y=\pi^*(K_X+\Delta)$. Since $\operatorname{Bs}|\pi^*D|=\pi^{-1}\operatorname{Bs}|D|$, it is sufficient to prove that $Q\notin\operatorname{Bs}|\pi^*D|$ for some $Q\in\pi^{-1}(P)$. Since $\pi:Y\to X$ is the minimal resolution of $P$, then $f:(Z,\Delta_Z)\to[X,K_X+\Delta]$ factors through $[Y,K_Y+\Delta_Y]$, and $(Z,\Delta_Z)\to[Y,K_Y+\Delta_Y]$ induces a natural quasi-log structure compatible with the original semi-log canonical structure of $(Y,\Delta_Y)$ (see Step 1 and [Fuj4, Theorem 1.2]). We put $Y_0=\pi^{-1}(X_0)$, where $P\in X_0$ as in Step 4. We can take an effective $\mathbb{R}$-Cartier divisor $B'$ on $Y_0$ such that $B'\sim_{\mathbb{R}}(\pi|_{Y_0})^*A_0$, $\operatorname{mult}_Q B'>2$ for some $Q\in\pi^{-1}(P)$, and $B'=(\pi|_{Y_0})^*B$ for some effective $\mathbb{R}$-Cartier divisor $B$

on $X_0$. We put $U' = Y_0 \setminus \operatorname{Sing} Y_0$. We set

$$c = \sup_{t\geq 0} \left\{ t \,\middle|\, \begin{array}{l} (U', (\Delta_Y)|_{U'} + tB'|_{U'}) \text{ is log canonical} \\ \text{at any point of } \pi^{-1}(P) \end{array} \right\}. \tag{5.21}$$

Then we have $0 < c < 1$. By adjunction (see Theorem 4.4) and Lemma 4.6, we can consider a quasi-log structure of $[Y_0, (K_Y + \Delta_Y)|_{Y_0} + cB']$. If there is a one-dimensional qlc center $C$ of $[Y_0, (K_Y + \Delta_Y)|_{Y_0} + cB']$ such that

$$(\pi^* D - ((K_Y + \Delta_Y)|_{Y_0} + cB')) \cdot C = (1 - c)(\pi|_{Y_0})^* A_0 \cdot C = 0, \tag{5.22}$$

then we obtain that $C \subset \pi^{-1}(P)$. This means that $P$ is a qlc center of $[X_0, (K_X + \Delta)|_{X_0} + cB]$. In this case, we obtain surjections

$$H^0(X, \mathcal{O}_X(D)) \twoheadrightarrow H^0(X_0, \mathcal{O}_{X_0}(D)) \twoheadrightarrow \mathcal{O}_{X_0}(D) \otimes \mathbb{C}(P) \tag{5.23}$$

as in Case 1 in Step 4 (see (5.9) and (5.13)). Therefore, we may assume that

$$(\pi^* D - ((K_Y + \Delta_Y)|_{Y_0} + cB')) \cdot C > 0 \tag{5.24}$$

for every one-dimensional qlc center $C$ of $[Y_0, (K_Y + \Delta_Y)|_{Y_0} + cB']$. Note that

$$(\pi^* D - (K_Y + \Delta_Y)) \cdot C = (D - (K_X + \Delta)) \cdot \pi_* C = A \cdot \pi_* C \geq 2 \tag{5.25}$$

when $\pi_* C \neq 0$; equivalently, $C$ is not a component of $\pi^{-1}(P)$. Then we can apply the arguments in Step 4 to $[Y_0, (K_Y + \Delta_Y)|_{Y_0} + cB']$ and $\pi^* D$. Thus, we obtain that $Q \notin \operatorname{Bs} |\pi^* D|$ for some $Q \in \pi^{-1}(P)$. This means that $P \notin \operatorname{Bs} |D|$.

Anyway, we obtain that $P \notin \operatorname{Bs} |D|$. □

By Theorem 5.1, we can quickly prove Corollary 1.4 as follows.

*Proof of Corollary 1.4.* We put $D = mI(K_X + \Delta)$ and $A = D - (K_X + \Delta) = (m - 1/I)I(K_X + \Delta)$. Then we obtain that $A \cdot C \geq m - 1/I$ for every curve $C$ on $X$ and that $(A^2 \cdot X_i) \geq (m - 1/I)^2$ for every irreducible component $X_i$ of $X$. By Theorem 5.1, we obtain the desired freeness of $|mI(K_X + \Delta)|$. The very ampleness part follows from Lemma 7.1 below. □

**Remark 5.3.** In Corollary 1.4, $\Delta$ is not necessarily reduced. If $\Delta$ is reduced, then Corollary 1.4 is a special case of [LR, Theorem 24]. We note that $\Delta$ is always assumed to be reduced in [LR].

As a special case of Corollary 1.4, we can recover Kodaira's celebrated result (see [Kod]). We state it explicitly for the reader's convenience.

**Corollary 5.4** (Kodaira). *Let $X$ be a smooth projective surface such that $K_X$ is nef and big. Then $|mK_X|$ is basepoint-free for every $m \geq 4$.*

*Proof of Corollary 5.4.* Apply Corollary 1.4 to the canonical model of $X$. Then we obtain the desired freeness. □

We close this section with the proof of Corollary 1.5.

*Proof of Corollary 1.5.* We put $D = -mI(K_X + \Delta)$ and $A = D - (K_X + \Delta) = -(m + 1/I)I(K_X + \Delta)$. Then we obtain that $A \cdot C \geq m + 1/I$ for every curve $C$ on $X$ and that $(A^2 \cdot X_i) \geq (m + 1/I)^2$ for every irreducible component $X_i$ of $X$. By Theorem 5.1, we obtain the desired freeness of $|-mI(K_X + \Delta)|$. The very ampleness part follows from Lemma 7.1 below. □

## §6. Log surfaces

In this section, we prove Theorem 1.6.

*Proof of Theorem 1.6.* The proof is essentially the same as that of Theorem 5.1. However, there are some technical differences. We will have to use Theorem 4.5 instead of Theorem 4.4(ii). So, we describe it for the reader's convenience.

*Step* 1. We take a resolution of singularities $f : Z \to X$ such that $\operatorname{Supp} f_*^{-1}\Delta \cup \operatorname{Exc}(f)$ is a simple normal crossing divisor on $Z$, where $\operatorname{Exc}(f)$ is the exceptional locus of $f$. We put $K_Z + \Delta_Z = f^*(K_X + \Delta)$. Then, $(Z, \Delta_Z)$ gives a natural quasi-log structure on $[X, K_X + \Delta]$.

*Step* 2. Assume that $(X, \Delta)$ is not log canonical at $x$. We put

$$X' = \operatorname{Nlc}(X, \Delta) \cup \bigcup W, \tag{6.1}$$

where $W$ runs over the one-dimensional log canonical centers of $(X, \Delta)$ such that $A \cdot W = 0$. Then, by Theorem 4.5, we obtain

$$H^i(X, \mathcal{I}_{X'} \otimes \mathcal{O}_X(D)) = 0 \tag{6.2}$$

for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of $X'$. Note that $x$ is isolated in $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}_{X'}$. Therefore, the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(x) \tag{6.3}$$

is surjective. Thus, we obtain $x \notin \operatorname{Bs}|D|$.

From now on, we may assume that $(X, \Delta)$ is log canonical at $x$.

*Step* 3. Assume that $x$ is a zero-dimensional log canonical center of $(X, \Delta)$. We put

$$X' = \mathrm{Nlc}(X, \Delta) \cup \bigcup W \cup \{x\}, \tag{6.4}$$

where $W$ runs over the one-dimensional log canonical centers of $(X, \Delta)$ such that $A \cdot W = 0$. Then, by Theorem 4.5, we obtain

$$H^i(X, \mathcal{I}_{X'} \otimes \mathcal{O}_X(D)) = 0 \tag{6.5}$$

for every $i > 0$. Note that $x$ is isolated in $\mathrm{Supp}\, \mathcal{O}_X/\mathcal{I}_{X'}$. Therefore, we obtain $x \notin \mathrm{Bs}\,|D|$ as in Step 2.

From now on, we may assume that $(X, \Delta)$ is plt at $x$.

*Step* 4. Assume that $(X, \Delta)$ is plt but is not klt at $x$. Let $L$ be the unique one-dimensional log canonical center of $(X, \Delta)$ passing through $x$. We put

$$X' = \mathrm{Nlc}(X, \Delta) \cup \bigcup W \cup L, \tag{6.6}$$

where $W$ runs over the one-dimensional log canonical centers of $(X, \Delta)$ such that $A \cdot W = 0$. By Theorem 4.5, we obtain that

$$H^i(X, \mathcal{I}_{X'} \otimes \mathcal{O}_X(D)) = 0 \tag{6.7}$$

for every $i > 0$, as usual. Therefore, the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to H^0(X', \mathcal{O}_{X'}(D)) \tag{6.8}$$

is surjective. By adjunction (see Theorem 4.4), $[X', (K_X + \Delta)|_{X'}]$ has a quasi-log structure induced by the quasi-log structure $f : (Z, \Delta_Z) \to [X, K_X + \Delta]$ constructed in Step 1. Let $g : (Z', \Delta_{Z'}) \to [X', (K_X + \Delta)|_{X'}]$ be the induced quasi-log resolution. We put

$$c = \sup_{t \geq 0} \left\{ t \,\middle|\, \begin{array}{l} \text{the normalization of } (Z', \Delta_{Z'} + tg^*x) \text{ is sub} \\ \text{log canonical over } X' \setminus \mathrm{Nqlc}((K_X + \Delta)|_{X'}) \end{array} \right\}. \tag{6.9}$$

Then, by [Fuj5, Lemma 3.16], we obtain that $0 < c < 2$. Note that $x$ is a Cartier divisor on $X'$. Let us consider $g : (Z', \Delta_{Z'} + cg^*x) \to [X', (K_X + \Delta)|_{X'} + cx]$, which defines a quasi-log structure. Then, by construction, $x$ is a qlc center of $[X', (K_X + \Delta)|_{X'} + cx]$. Moreover, we see that

$$\deg(D|_L - (K_X + \Delta)|_L - cx) = (A \cdot L) - c > 0 \tag{6.10}$$

by assumption. We put

$$X'' = \mathrm{Nqlc}(X', (K_X + \Delta)|_{X'} + cx) \cup \bigcup W \cup \{x\}, \tag{6.11}$$

where $W$ runs over the one-dimensional qlc centers of $[X', (K_X+\Delta)|_{X'}+cx]$ such that $W \neq L$. Then, by Theorem 4.5, we obtain

$$H^i(X', \mathcal{I}_{X''} \otimes \mathcal{O}_{X'}(D)) = 0 \tag{6.12}$$

for every $i > 0$. Note that $x$ is isolated in $\operatorname{Supp} \mathcal{O}_{X'}/\mathcal{I}_{X''}$. Therefore, the restriction map

$$H^0(X', \mathcal{O}_{X'}(D)) \to \mathcal{O}_{X'}(D) \otimes \mathbb{C}(x) \tag{6.13}$$

is surjective. By combining (6.8) with (6.13), the desired restriction map

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(x) \tag{6.14}$$

is surjective. This means that $x \notin \operatorname{Bs}|D|$.

Thus, from now on, we may assume that $(X, \Delta)$ is klt at $x$.

*Step* 5. In this step, we assume that $x$ is a smooth point of $X$. Since $A^2 > 4$, we can find an effective $\mathbb{R}$-Cartier divisor $B$ on $X$ such that $\operatorname{mult}_x B > 2$ and that $B \sim_{\mathbb{R}} A$. We put

$$c = \max\{t \geq 0 \,|\, (X, \Delta + tB) \text{ is log canonical at } x\}. \tag{6.15}$$

Then we obtain that $0 < c < 1$ since $\operatorname{mult}_x B > 2$. We have a natural quasi-log structure on $[X, K_X+\Delta+cB]$ as in Step 1. By construction, there is a log canonical center of $[X, K_X + \Delta + cB]$ passing through $x$. We put

$$X' = \operatorname{Nlc}(X, \Delta + cB) \cup \bigcup W \cup W_0, \tag{6.16}$$

where $W_0$ is the minimal log canonical center of $(X, \Delta+cB)$ passing through $x$, and $W$ runs over the one-dimensional log canonical centers of $(X, \Delta + cB)$ such that $A \cdot W = 0$. We note that $D - (K_X + \Delta + cB) \sim_{\mathbb{R}} (1 - c)A$. Then, by Theorem 4.5,

$$H^i(X, \mathcal{I}_{X'} \otimes \mathcal{O}_X(D)) = 0 \tag{6.17}$$

for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of $X'$ on $X$.

*Case* 1. If $\dim_x X' = 0$, then $x$ is isolated in $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}_{X'}$. Therefore, the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(x) \tag{6.18}$$

is surjective. Thus, we obtain that $x \notin \operatorname{Bs}|D|$.

*Case* 2. If $\dim_x X' = 1$, then $(X, \Delta + cB)$ is plt at $x$. We write $\Delta + cB = L + \Delta'$, where $L = W_0$ is the unique one-dimensional log canonical center of $(X, \Delta)$ passing through $x$ and $\Delta' = \Delta + cB - L$. We put

$$c' = 1 - \mathrm{mult}_x(\Delta'|_L). \tag{6.19}$$

Then $[X', (K_X + \Delta + cB)|_{X'} + c'x]$ has a quasi-log structure such that $x$ is a qlc center of this quasi-log structure as in the proof of Theorem 5.1, Step 4, Case 2. We put

$$X'' = \mathrm{Nqlc}(X', (K_X + \Delta + cB)|_{X'} + c'x) \cup \bigcup W \cup \{x\}, \tag{6.20}$$

where $W$ runs over the one-dimensional qlc centers of $[X', (K_X + \Delta + cB)|_{X'} + c'x]$ such that $W \neq L$. By (5.17) in the proof of Theorem 5.1, we obtain that

$$\deg(D|_L - (K_X + \Delta + cB)|_L - c'x) > 0. \tag{6.21}$$

Then, by (6.21) and Theorem 4.5,

$$H^i(X', \mathcal{I}_{X''} \otimes \mathcal{O}_{X'}(D)) = 0 \tag{6.22}$$

for every $i > 0$, where $\mathcal{I}_{X''}$ is the defining ideal sheaf of $X''$ on $X'$. Thus, we have

$$H^0(X', \mathcal{O}_{X'}(D)) \to \mathcal{O}_{X'}(D) \otimes \mathcal{O}_{X'}/\mathcal{I}_{X''} \tag{6.23}$$

is surjective. Note that $x$ is isolated in $\mathrm{Supp}\, \mathcal{O}_{X'}/\mathcal{I}_{X''}$. Therefore, we obtain surjections

$$H^0(X, \mathcal{O}_X(D)) \twoheadrightarrow H^0(X', \mathcal{O}_{X'}(D)) \twoheadrightarrow \mathcal{O}_{X'}(D) \otimes \mathbb{C}(x) \tag{6.24}$$

by (6.17) and (6.23). This is the desired surjection.

Finally, we further assume that $x$ is a singular point of $X$.

*Step* 6. Let $\pi : Y \to X$ be the minimal resolution of $x$. We put $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$. Since $\mathrm{Bs}\,|\pi^*D| = \pi^{-1}\,\mathrm{Bs}\,|D|$, it is sufficient to prove that $y \notin \mathrm{Bs}\,|\pi^*D|$ for some $y \in \pi^{-1}(x)$. Since $\pi : Y \to X$ is the minimal resolution of $x$, $f : (Z, \Delta_Z) \to [X, K_X + \Delta]$ factors through $[Y, K_Y + \Delta_Y]$ and $(Z, \Delta_Z) \to [Y, K_Y + \Delta_Y]$ induces a natural quasi-log structure on $[Y, K_Y + \Delta_Y]$. We can take an effective $\mathbb{R}$-Cartier divisor $B'$ on $Y$ such that $B' \sim_{\mathbb{R}} \pi^*A$, $\mathrm{mult}_y B' > 2$ for some $y \in \pi^{-1}(x)$ and $B' = \pi^*B$ for some effective $\mathbb{R}$-Cartier divisor $B$ on $X$. We set

$$c = \sup_{t \geq 0} \left\{ t \,\middle|\, \begin{array}{l} (Y, \Delta_Y + tB') \text{ is log canonical} \\ \text{at any point of } \pi^{-1}(x) \end{array} \right\}. \tag{6.25}$$

Then we have $0 < c < 1$. As in Step 1, we can consider a natural quasi-log structure of $[Y, K_Y + \Delta_Y + cB']$. If there is a one-dimensional qlc center $C$ of $[Y, K_Y + \Delta_Y + cB']$ such that $C \cap \pi^{-1}(x) \neq \emptyset$ and that

$$(\pi^* D - (K_Y + \Delta_Y + cB')) \cdot C = (1-c)\pi^* A \cdot C = 0. \tag{6.26}$$

Then we obtain that $C \subset \pi^{-1}(x)$. This means that $x$ is a qlc center of $[X, K_X + \Delta + cB]$. In this case, we have

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(x) \tag{6.27}$$

is surjective as in Step 5, Case 1. Therefore, we may assume that

$$(\pi^* D - (K_Y + \Delta_Y + cB')) \cdot C > 0 \tag{6.28}$$

for every one-dimensional qlc center $C$ of $[Y, K_Y + \Delta_Y + cB']$ with $C \cap \pi^{-1}(x) \neq \emptyset$. We note that

$$(\pi^* D - (K_Y + \Delta_Y)) \cdot C = (D - (K_X + \Delta)) \cdot \pi_* C = A \cdot \pi_* C \geq 2. \tag{6.29}$$

Then we can apply the arguments in Step 5 to $[Y, K_Y + \Delta_Y + cB']$ and $\pi^* D$. Thus, we obtain that $y \notin \mathrm{Bs}\,|\pi^* D|$ for some $y \in \pi^{-1}(x)$. This means that $x \notin \mathrm{Bs}\,|D|$.

Anyway, we obtain that $x \notin \mathrm{Bs}\,|D|$. □

## §7. Effective very ampleness lemma

In this section, we prove an effective very ampleness lemma. This section is independent of the other sections.

The statement and the proof of [Kol, Lemma 1.2] do not seem to be true as stated. János Kollár and the author think that we need some modifications. So, we prove the following lemma.

**Lemma 7.1.** *Let $(X, \Delta)$ be a projective semi-log canonical pair with $\dim X = n$. Let $D$ be an ample Cartier divisor on $X$ such that $|D|$ is basepoint-free. Assume that $L = D - (K_X + \Delta)$ is nef and log big with respect to $(X, \Delta)$, that is, $L$ is nef and $L|_W$ is big for every slc stratum $W$ of $(X, \Delta)$. Then $(n+1)D$ is very ample.*

We give a detailed proof of Lemma 7.1 for the reader's convenience.

*Proof.* By the vanishing theorem (see [Fuj4, Theorem 1.10]), we obtain that

$$H^i(X, \mathcal{O}_X((n+1-i)D)) = 0 \tag{7.1}$$

for every $i > 0$. Then, by Castelnuovo–Mumford regularity, we see that

$$H^0(X, \mathcal{O}_X(D)) \otimes H^0(X, \mathcal{O}_X(mD)) \to H^0(X, \mathcal{O}_X((m+1)D)) \tag{7.2}$$

is surjective for every $m \geq n+1$ (see, for example, [Kle, Chapter II, Proposition 1]). Therefore, we obtain that

$$\mathrm{Sym}^k H^0(X, \mathcal{O}_X((n+1)D)) \to H^0(X, \mathcal{O}_X(k(n+1)D)) \tag{7.3}$$

is surjective for every $k \geq 1$. We put $A = (n+1)D$ and consider $f = \Phi_{|A|} : X \to Y$. Then there is a very ample Cartier divisor $H$ on $Y$ such that $A \sim f^*H$. By construction and the surjection (7.3), we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^k H^0(Y, \mathcal{O}_Y(H)) & \twoheadrightarrow & \mathrm{Sym}^k H^0(X, \mathcal{O}_X(A)) \\ \downarrow & & \downarrow\!\!\downarrow \\ H^0(Y, \mathcal{O}_Y(kH)) & \hookrightarrow & H^0(X, \mathcal{O}_X(kA)) \end{array} \tag{7.4}$$

for every $k \geq 1$. This implies that $H^0(Y, \mathcal{O}_Y(kH)) \simeq H^0(X, \mathcal{O}_X(kA))$ for every $k \geq 1$. Note that $\mathcal{O}_Y \simeq f_*\mathcal{O}_X$ by

$$0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \delta \to 0 \tag{7.5}$$

and

$$\begin{aligned} 0 &\to H^0(Y, \mathcal{O}_Y(kH)) \to H^0(X, \mathcal{O}_X(kA)) \\ &\to H^0(Y, \delta \otimes \mathcal{O}_Y(kH)) \to H^1(Y, \mathcal{O}_Y(kH)) \to \cdots \end{aligned} \tag{7.6}$$

for $k \gg 0$. By the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle \Phi_{|kA|}} \searrow & \downarrow {\scriptstyle \Phi_{|kH|}} \\ & & \mathbb{P}^N, \end{array} \tag{7.7}$$

where $k$ is a sufficiently large positive integer such that $kA$ and $kH$ are very ample, we obtain that $f$ is an isomorphism. This means that $A = (n+1)D$ is very ample. □

We close this section with a remark on very ampleness for $n$-dimensional stable pairs and semi-log canonical Fano varieties (see [Fuj6]).

**Remark 7.2.** Let $(X, \Delta)$ be a projective semi-log canonical pair with $\dim X = n$.

Assume that $I(K_X + \Delta)$ is an ample Cartier divisor for some positive integer $I$. Then we put $D = I(K_X + \Delta)$, $a = 2$, and apply [Fuj6, Remark 1.3

and Corollary 1.4]. We obtain that $NI(K_X + \Delta)$ is very ample, where $N = (n+1)2^{n+1}(n+1)!(2+n) = 2^{n+1}(n+2)!(n+1)$.

Assume that $-I(K_X+\Delta)$ is an ample Cartier divisor for some positive integer $I$. Then we put $D = -I(K_X + \Delta)$, $a = 1$, and apply [Fuj6, Remark 1.3 and Corollary 1.4]. We obtain that $-NI(K_X + \Delta)$ is very ample, where $N = (n+1)2^{n+1}(n+1)!(1+n) = 2^{n+1}(n+1)^3 n!$.

Our results for surfaces in this paper are much sharper than the above estimates for $n = 2$.

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