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Smoothing Properties and Scattering for the Magnetic Schrödinger and Klein–Gordon Equations in an Exterior Domain with Time-Dependent Perturbations

by

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Abstract

This paper deals with the existence, smoothing properties and scattering of solutions to the magnetic Schrödinger and Klein–Gordon equations in an exterior domain with time-dependent small perturbations. Smoothing properties based on the resolvent estimates will reinforce the abstract scattering theory developed in [8] (K. Mochizuki, in *Proc. 6th ISAAC*, World Scientific Publishing, River Edge, NJ, 2009, 476–485), and our concrete problems are treated in this framework.

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§1. Introduction and results

Let $n \geq 2$ and let Ω be an exterior domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ that is star shaped with respect to the origin 0 (the case $\Omega = \mathbf{R}^n$ is not excluded when $n \geq 3$). We consider in Ω the Schrödinger evolution equation

$$i\,\partial_t u = -\Delta_b u + c(x,t)u \tag{1}$$

and the Klein–Gordon equation

$$\partial_t^2 w - \Delta_b w + m^2 w + b_0(x,t)\,\partial_t w + c(x,t)w = 0, \tag{2}$$

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where $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, Δ_b is the magnetic Laplacian

$$\Delta_b = \nabla_b \cdot \nabla_b = \sum_{j=1}^{n} (\partial_j + ib_j(x))^2$$

with $\partial_j = \partial/\partial x_j$, m is a positive constant, $b_j(x)$ $(j = 1, \dots, n)$ are real-valued smooth functions of $x \in \mathbf{R}^n$ and $c(x,t)$, $b_0(x,t)$ are complex-valued continuous functions of $(x,t) \in \mathbf{R}^n \times \mathbf{R}$. For solutions $u = u(x,t)$ and $w = w(x,t)$ we require the zero Dirichlet conditions

$$u(x,t)|_{\partial\Omega} = 0 \quad \text{and} \quad w(x,t)|_{\partial\Omega} = 0 \tag{3}$$

on the boundary $\partial\Omega$. The function $b(x) = (b_1(x), \cdots, b_n(x))$ represents a magnetic potential. Thus, the magnetic field is defined by its rotation $\nabla \times b(x) = \{\partial_j b_k(x) - \partial_k b_j(x)\}_{j<k}$. We require

$$|\nabla \times b(x)| \le \epsilon_0 (1 + [r])^{-2}, \quad r = |x|. \tag{A1}$$

Here ϵ_0 is a small positive constant and

$$[r] = \begin{cases} r & \text{when } n \ge 3, \\ r(1 + \log r/r_0) & \text{when } n = 2, \end{cases}$$

for a fixed $r_0 > 0$ satisfying $\partial\Omega \subset \{x;\ |x| > r_0\}$. As for the coefficients of the perturbation terms, we require the following:

$$|b_0(x,t)|,\ |c(x,t)| \le \eta(t) + \epsilon_1 (1 + [r])^{-2}, \tag{A2}$$

where $\eta(t)$ is a positive L^1-function of $t \in \mathbf{R}$ and ϵ_1 is a small positive constant.

Equation (1) is considered in the Hilbert space $L^2 = L^2(\Omega)$ with inner product and norm

$$(f,g) = \int_\Omega f(x)\overline{g(x)}\,dx \quad \text{and} \quad \|f\| = \sqrt{(f,f)}.$$

The operator $-\Delta_b$ with domain $C_0^\infty(\Omega)$ is essentially self-adjoint in L^2, and determines the self-adjoint operator L with domain

$$\mathcal{D}(L) = \big\{u(x) \in L^2 \cap H^2_{\mathrm{loc}}(\overline{\Omega});\ -\Delta_b u \in L^2 \text{ and } u|_{\partial\Omega} = 0\big\}. \tag{4}$$

Here $H^k = H^k(\Omega)$ is the usual Sobolev space with norm

$$\|f\|^2_{H^k} = \int_\Omega \sum_{|\alpha| \le k} |\nabla^\alpha f(x)|^2\,dx,$$

$\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex with $|\alpha| = \sum_{j=1}^n |\alpha_j|$, and $H^2_{\mathrm{loc}}(\overline{\Omega})$ is the H^2-space on each compact set of $\overline{\Omega}$.

Then (1) with boundary condition (3) is represented as

$$i\,\partial_t u = Lu + V(t)u \quad \text{in } L^2, \tag{5}$$

where $V(t)u = c(x,t)u$. Moreover, by use of the unitary group of operators $\{e^{-itL};\ t \in \mathbf{R}\}$, (5) with initial data $u(0) = f \in L^2$ reduces to the integral equation

$$u(t) = e^{-itL}f - i\int_0^t e^{-i(t-\tau)L}V(\tau)u(\tau)\,d\tau. \tag{6}$$

Equation (2) is rewritten in the system with the pair $\{w, w_t\}$ (where $w_t = \partial_t w$):

$$\partial_t \begin{pmatrix} w \\ w_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta_b - m^2 & 0 \end{pmatrix}\begin{pmatrix} w \\ w_t \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ c(x,t) & b_0(x,t) \end{pmatrix}\begin{pmatrix} w \\ w_t \end{pmatrix}.$$

It is considered in the energy space $\mathcal{H}_E = H^1_{b,0} \times L^2$, where $H^1_{b,0} = H^1_{b,0}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with norm

$$\|f\|^2_{H^1_{b,0}} = \int_\Omega \{|\nabla_b f(x)|^2 + |f(x)|^2\}\,dx.$$

Thus, the inner product and norm of $\mathcal{H}_E$ are given for $f = (f_1, f_2)$, $g = (g_1, g_2) \in \mathcal{H}_E$ by

$$(f,g)_{\mathcal{H}_E} = \frac{1}{2}\int_\Omega \left\{\nabla_b f_1(x)\overline{\nabla_b g_1(x)} + m^2 f_1(x)\overline{g_1(x)} + f_2(x)\overline{g_2(x)}\right\}\,dx \tag{7}$$

and $\|f\|_{\mathcal{H}_E} = \sqrt{(f,f)_{\mathcal{H}_E}}$, respectively. We define the operator M in $\mathcal{H}_E$ by

$$M = \begin{pmatrix} 0 & i \\ i(\Delta_b - m^2) & 0 \end{pmatrix},$$

with domain

$$\mathcal{D}(M) = \left\{f = \{f_1, f_2\} \in [H^2_{\text{loc}} \cap H^1_{b,0}] \times H^1_{b,0};\ \Delta_b f_1 \in L^2\right\}. \tag{8}$$

Then it forms a self-adjoint operator in $\mathcal{H}_E$, and (2) with boundary condition (3) is represented as

$$i\partial_t u = Mu + V(t)u \quad \text{in } \mathcal{H}_E, \tag{9}$$

where $u = \{w, w_t\}$ and

$$V(t)u = \begin{pmatrix} 0 & 0 \\ -ic(x,t) & -ib_0(x,t) \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Moreover, by use of the unitary group of operators $\{e^{-itM};\ t \in \mathbf{R}\}$ in $\mathcal{H}_E$, (9) with initial data $u = \{w(0), w_t(0)\} = \{f_1, f_2\} \in \mathcal{H}_E$ reduces to the integral equation

$$u(t) = e^{-itM} f - i \int_0^t e^{-i(t-\tau)M} V(\tau) u(\tau)\, d\tau. \tag{10}$$

In this paper we are concerned with some scattering problems of these equations. They are time-dependent perturbations of the unitary groups $\{e^{-itL}\}_{t\in\mathbf{R}}$ in L^2 and $\{e^{-itM}\}_{t\in\mathbf{R}}$ in $\mathcal{H}_E$. The problems have been studied in Mochizuki–Motai [11] in the Schrödinger case and Murai [13] in the Klein–Gordon case, when $b = 0$ and $\Omega = \mathbf{R}^n$ with $n \geq 3$. In each case, the proof is based on the Strichartz estimates. In this paper, a decaying condition is not required on the magnetic potential $b(x)$ itself. So, there are no effective Strichartz estimates on our operators L and M.

The purpose of this paper is to settle our approach in the L^2-framework of applying smoothing properties. Necessary resolvent estimates for L were obtained in Mochizuki [9, 10] when $n \geq 3$ and in Mochizuki–Nakazawa [12] when $n = 2$. As will be seen, it is possible to use them to obtain similar estimates for M in the energy space, and this becomes one of our main results. Note that corresponding estimates have not been established so far for the acoustic wave equation (the case $m = 0$). As studied in Mochizuki [7] (see also [8]), space-time weighted energy methods are effective in this case if $n \geq 3$. On the other hand, weighted energy methods give rise to a difficulty with the Klein–Gordon equation.

We treat the two equations (6) and (10) in a bundle as an integral equation

$$u(t) = e^{-it\Lambda} f - i \int_0^t e^{-i(t-\tau)\Lambda} V(\tau) u(\tau)\, d\tau \tag{11}$$

in a Hilbert space $\mathcal{H}$. Here $\mathcal{H} = L^2$, $\Lambda = L$ and $V(t) = c(x,t)$ in the case of (6), and $\mathcal{H} = \mathcal{H}_E$, $\Lambda = M$ and $V(t) = \begin{pmatrix} 0 & 0 \\ -ic(x,t) & -ib_0(x,t) \end{pmatrix}$ in the case of (10). Moreover, let X stand for the weighted spaces

$$X_0 = \Big\{ f(x);\ \|f\|_{X_0}^2 = \textstyle\int_\Omega (1 + [r])^{-2} |f(x)|^2\, dx < \infty \Big\} \tag{12}$$

in the Schrödinger case (6) and

$$\begin{aligned} X_E = \Big\{ & f(x) = \{f_1(x), f_2(x)\}; \\ & \|f\|_{X_E}^2 = \tfrac{1}{2} \textstyle\int_\Omega (1 + [r])^{-2} \{|\nabla_b f_1|^2 + m^2 |f_1|^2 + |f_2|^2\}\, dx < \infty \Big\} \end{aligned} \tag{13}$$

in the Klein–Gordon case (10).

For an interval $I \subset \mathbf{R}$ and a Banach space W, we denote by $L^2(I;W)$, the space of all W-valued functions $h(t)$ satisfying

$$\|h\|_{L^2(I;W)} = \left(\int_I \|h(t)\|_W^2 \, dt \right)^{1/2} < \infty.$$

Similarly, $C(I;W)$ denotes the space of all W-valued continuous functions of $t \in I$. Further, we denote by $\mathcal{B}(W)$ the space of bounded operators on W.

Now, the main results of this paper are summarized in the following theorems.

Theorem 1. *For $\zeta \in \mathbf{C}\backslash\mathbf{R}$ put $\mathcal{R}(\zeta) = (\Lambda - \zeta)^{-1}$. If ϵ_0 in (A1) is chosen small enough, then there exists $C_0 > 0$ such that*

$$\sup_{\zeta \in \mathbf{C}\backslash\mathbf{R}} \|\mathcal{R}(\zeta)f\|_X \leq C_0 \|f\|_{X'}$$

for each $f \in X'$, where X' is the dual space of X with respect to $\mathcal{H}$.

Theorem 2. *Assume (A1) and (A2) with small ϵ_0 and ϵ_1. Then for each $f \in \mathcal{H}$ there exists a unique solution $u(t) \in C(\mathbf{R};\mathcal{H})$ to the integral equation (11). Let $U(t,s)$, s, $t \in \mathbf{R}_\pm$ be the evolution operator that maps $u(s)$ to $u(t) = U(t,s)u(s)$. Then there exists $C_1 > 0$ such that*

$$\|U(\cdot,s)g\|^2_{L^2_t(\mathbf{R}_\pm;X)} \leq C_1 \|g\|^2_{\mathcal{H}} \tag{14}$$

for each $s \in \mathbf{R}_+ = (0,\infty)$ (or $\in \mathbf{R}_- = (-\infty,0)$) and $g \in \mathcal{H}$.

Theorem 3. *Under the same conditions as above, we have*

(i) $\{U(t,s)\}_{t,s\in\mathbf{R}}$ is a family of uniformly bounded operators in $\mathcal{H}$:

$$\sup_{t,s\in\mathbf{R}} \|U(t,s)\|_{\mathcal{B}(\mathcal{H})} = C_U < \infty;$$

(ii) for every $s \in \mathbf{R}_\pm$, there exists the strong limit

$$Z^\pm(s) = \operatorname*{s-lim}_{t\to\pm\infty} e^{-i(-t+s)\Lambda} U(t,s);$$

(iii) the operator $Z^\pm = Z^\pm(0)$ satisfies

$$\operatorname*{w-lim}_{s\to\pm\infty} Z^\pm U(0,s) e^{-is\Lambda} = I \quad (\textit{weak limit});$$

(iv) if ϵ_1 is chosen smaller so that it satisfies $\epsilon_V\sqrt{2C_0C_1} < 1$, where $\epsilon_V = \epsilon_1$ (Schrödinger) or $\epsilon_V = \max\{1, m^{-1}\}\epsilon_1$ (Klein–Gordon), then $Z^\pm : \mathcal{H} \longrightarrow \mathcal{H}$ is a bijection on $\mathcal{H}$. Thus, the scattering operator $S = Z^+(Z^-)^{-1}$ is well defined and also gives a bijection on $\mathcal{H}$.

There are several works that investigate time-dependent perturbations. See, e.g., Yafaev [17], Howland [2], Yajima [18], Kitada–Yajima [6] and Jensen [3] for the Schrödinger equations, and Cooper–Menzala–Strauss [1], Petkov [14] and Wirth [16] for wave equations. Except for [16], these works treat time-dependent real potentials. So, for each fixed t, the operator $-\Delta + c(x,t)$ becomes self-adjoint, and this fact plays an important role in the theory. Ref. [16] studies perturbations depending only on t. As for the case of non-self-adjoint perturbations, there is Kato's classical paper [4] (cf. also Kato–Yajima [5]), where time-independent potentials are treated. In the current paper, we partly extend results from [4] to time-dependent potentials in magnetic fields.

This paper is organized as follows. In the next section we discuss the resolvent estimates for the Schrödinger operator L. Two propositions are given with briefly summarized proofs. Proposition 1 attains Theorem 1 for L and both propositions are used in Section 3 to show Theorem 1 for the Klein–Gordon system. After summarizing smoothing properties for $e^{-it\Lambda}$, Theorem 2 is proven in Section 4 by use of successive approximation. Finally, in Section 5 we apply these results to show Theorem 3.

§2. Uniform resolvent estimates for the magnetic Laplacian

In this section we treat uniform resolvent estimates for the magnetic Laplacian $L = -\Delta_b$ in an exterior domain.

For $\kappa \in \mathbf{C}_+ = \{\kappa \in \mathbf{C};\ \operatorname{Im}\kappa > 0\}$ we put $R(\kappa^2) = (L-\kappa^2)^{-1}$. Assume (A1) with sufficiently small $\epsilon_0 > 0$. Then the following two propositions hold.

Proposition 1. *There exists $C_2 > 0$ independent of κ such that*

$$\int_\Omega \frac{\operatorname{Im}\kappa r + 1}{[r]^2} \left|R(\kappa^2)f(x)\right|^2 dx \le C_2 \int_\Omega [r]^2 |f(x)|^2\, dx.$$

Proposition 2. *There exists $C_3 > 0$ independent of κ such that*

$$\int_\Omega \mu \left\{ \left|\nabla_b R(\kappa^2) f\right|^2 + \left|\kappa R(\kappa^2) f\right|^2 \right\} dx \le C_3 \int_\Omega \max\left\{\mu^{-1}, [r]^2\right\} |f|^2\, dx,$$

where $\mu = \mu(r)$ is a positive, smooth, decreasing function of $r > 0$ such that

$$\mu(r) = o(r^{-1}), \quad \mu(r) \ge (1+[r])^{-2} \quad \textit{and} \quad \|\mu\|_{L^1} = \int_0^\infty \mu(r)\, dr < \infty.$$

Remark 1. As will be seen below in the proofs of the propositions, ϵ_0 is concretely given as

$$0 \le \epsilon_0 < \begin{cases} \frac{1}{4\sqrt{21}} & (n=2), \\ \frac{1}{4\sqrt{2}} & (n=3), \\ \frac{\sqrt{2(n-1)(n-3)}}{4} & (n \ge 4). \end{cases}$$

The constants C_2 and C_3 depend on ϵ_0 and $\|\mu\|_{L^1}$.

Proposition 1 attains Theorem 1 for $\Lambda = L$. Moreover, in the next section these two propositions will play a fundamental role in proving Theorem 1 for $\Lambda = M$. In this section we briefly summarize the proof, which is separately given in [9, Theorem 1], [10, Theorem 4] when $n \ge 3$ and in [12, Theorem 1] when $n = 2$.

Lemma 1. *Let $u = R(\kappa^2)f$. Then we have*

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left\{ \left(\mu \operatorname{Im}\kappa \frac{n-1}{r} - \mu' \frac{n-1}{2r} \right)|u|^2 + \mu(|\nabla_b u|^2 + |\kappa u|^2) \right\} dx \\ &\qquad + \operatorname{Im}\kappa \int_{r_0}^\infty \mu(t)\,dt \int_{\Omega_t} \left\{ |\nabla_b u|^2 + |u|^2 \right\} dx \\ &\qquad = \frac{1}{2}\int_\Omega \mu \left\{ |\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2 \right\} dx - \operatorname{Re}\int_{r_0}^\infty \mu(t)\,dt \int_{\Omega_t} f\overline{i\kappa u}\,dx, \end{aligned}$$

where $\Omega_t = \{x \in \Omega;\ |x| < t\}$ and $\theta = \nabla_b u + \tilde{x}\left(\frac{n-1}{2r} - i\kappa\right)u$.

Lemma 2. *Let $\varphi = \varphi(r)$ be a positive increasing function of $r > 0$ satisfying*

$$\frac{\varphi'(r)}{\varphi(r)} \le \frac{1}{r}. \tag{15}$$

Then we have for $u = R(\kappa^2)f$,

$$\begin{aligned} &-\int_{\partial\Omega} \varphi \left\{ (\nu \cdot \nabla u)(\tilde{x} \cdot \overline{\nabla u}) - \frac{1}{2}(\nu \cdot \tilde{x})|\nabla u|^2 \right\} dS + \int_\Omega \varphi \Bigg[-\left(\frac{1}{r} - \frac{\varphi'}{\varphi} \right)|\tilde{x} \cdot \theta|^2 \\ &\qquad + \left(\operatorname{Im}\kappa + \frac{1}{r} - \frac{\varphi'}{2\varphi} \right)\left\{ |\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2 \right\} \\ &\qquad + \operatorname{Re}\{-(\nabla \times ib)u \cdot \overline{(\tilde{x} \times \theta)}\} \Bigg] dx \\ &\qquad = \operatorname{Re}\int_\Omega \varphi f \overline{\tilde{x} \cdot \theta}\,dx, \end{aligned}$$

where $\nu = \nu(x)$ is the outer unit normal to the boundary $\partial\Omega$ and $\tilde{x} \times \theta = \{\tilde{x}_j\theta_k - \tilde{x}_k\theta_j\}_{j<k}$.

We omit the proof of these identities, but note the following: $u = R(\kappa^2)f$ satisfies the equation

$$-\Delta_b u - \kappa^2 u = f(x) \quad \text{in } \Omega \text{ with } u|_{\partial\Omega} = 0. \tag{16}$$

We multiply by $-\overline{i\kappa u}$ on both sides of (16) and integrate the real part over Ω_t. Then Lemma 1 results from this equation multiplied by $\mu(t)$ and integrated over (r_0, ∞). Next, we rewrite (16) as an equation in θ:

$$-\nabla_b \theta + \left(\frac{n-1}{2r} - i\kappa\right)\tilde{x} \cdot \theta + \frac{(n-1)(n-3)}{4r^2} u = f. \tag{17}$$

Multiply by $\varphi \tilde{x} \cdot \overline{\theta}$ and then integration by parts of the real part over Ω leads us to Lemma 2.

The third identity is the following.

Lemma 3. *Let $\eta = \eta(r)$ and $\xi = \xi(r)$ be smooth, positive functions of $r > 0$ and let t be chosen large. Then the following identity holds for each $u \in H^1_{b0}$:*

$$\begin{aligned}
&\int_{\Omega_t} \xi \left\{ |\tilde{x} \cdot \theta|^2 + \frac{(n-1)(n-3)}{4r^2} |u|^2 \right\} dx \\
&\quad = \int_{\Omega_t} \xi |\tilde{x} \cdot \nabla_b u - i\kappa u - \eta u|^2 \, dx \\
&\qquad + \int_{S_t} \xi \left(\frac{n-1}{2r} + \eta\right) |u|^2 \, dS - \int_{\Omega_t} \xi' \left(\frac{n-1}{2r} + \eta\right) |u|^2 \, dx \\
&\qquad + \int_{\Omega_t} \xi \left\{ 2 \operatorname{Im} \kappa \left(\frac{n-1}{2r} + \eta\right) |u|^2 - \left(\frac{n-1}{r}\eta + \eta' + \eta^2\right) |u|^2 \right\} dx,
\end{aligned}$$

where $S_t = \{x \in \Omega;\ |x| = t\}$.

Proof. Note the identity

$$\begin{aligned}
|\tilde{x} \cdot \theta|^2 &= \left| \tilde{x} \cdot \nabla_b u + \frac{n-1}{2r} u - i\kappa u - \eta u + \eta u \right|^2 \\
&= |\tilde{x} \cdot \nabla_b u - i\kappa u - \eta u|^2 + \nabla \cdot \left\{ \tilde{x} \left(\frac{n-1}{2r} + \eta\right) |u|^2 \right\} \\
&\quad + 2 \operatorname{Im} \kappa \left(\frac{n-1}{2r} + \eta\right) |u|^2 - \frac{(n-1)(n-3)}{4r^2} |u|^2 - \left(\frac{1}{r}\eta + \eta' + \eta^2\right) |u|^2.
\end{aligned}$$

Multiply by $\xi(r)$ on both sides and integrate over Ω_t. Then since $u|_{\partial\Omega} = 0$, the desired identity follows. $\square$

Now we have $\nabla u = (\nu \cdot \nabla u)\nu$ on $\partial\Omega$ in Lemma 2. The star-shapedness of $\partial\Omega$ then shows that

$$\int_{\partial\Omega} \varphi\Big\{ -(\nu\cdot\nabla u)(\tilde{x}\cdot\overline{\nabla u}) + \frac{1}{2}(\nu\cdot\tilde{x})|\nabla u|^2\Big\}\,dS = -\frac{1}{2}\int_{\partial\Omega}\varphi(\nu\cdot\tilde{x})|\nu\cdot\nabla u|^2\,dS \geq 0.$$

Moreover, we have

$$\Big(\frac{1}{r}-\frac{\varphi'}{\varphi}\Big)\{|\theta|^2-|\tilde{x}\cdot\theta|^2\}\geq 0,$$
$$|-(\nabla\times ib)u\,\overline{\tilde{x}\times\theta}|\leq|\nabla\times b|\,|u|\,|\theta|.$$

Thus the following inequality is obtained from Lemma 2:

$$\begin{aligned}(18)\qquad &\int_\Omega\Big\{\Big(\operatorname{Im}\kappa\varphi+\frac{\varphi'}{2}\Big)|\theta|^2+\Big(\operatorname{Im}\kappa\varphi+\frac{\varphi}{r}-\frac{\varphi'}{2}\Big)\frac{(n-1)(n-3)}{4r^2}|u|^2\Big\}\,dx\\ &\quad\leq\int_\Omega\varphi\{|f|+|\nabla\times b|\,|u|\}|\theta|\,dx.\end{aligned}$$

We choose here $\varphi = r$ and use the Schwarz inequality. Then

$$\begin{aligned}&\int_\Omega\Big(\operatorname{Im}\kappa r+\frac{1}{2}\Big)\Big\{|\theta|^2+\frac{(n-1)(n-3)}{4r^2}|u|^2\Big\}\,dx\\ &\quad\leq\int_\Omega\Big\{\frac{[r]^2}{4\epsilon}|f|^2+\frac{\epsilon_0^2}{4\epsilon[r]^2}|u|^2\Big\}\,dx+\int 2\epsilon\frac{r^2}{[r]^2}|\theta|^2\,dx\end{aligned}$$

and we conclude that for any $\epsilon > 0$,

$$\begin{aligned}(19)\qquad &\int_\Omega\Big(\operatorname{Im}\kappa r+\frac{1}{2}-2\epsilon\frac{r^2}{[r]^2}\Big)\Big\{|\theta|^2+\frac{(n-1)(n-3)}{4r^2}|u|^2\Big\}\,dx\\ &\quad\leq\frac{1}{4\epsilon}\int_\Omega[r]^2|f|^2\,dx+\Big(\frac{\epsilon_0^2}{\epsilon}-2\epsilon(n-1)(n-3)\Big)\int_\Omega\frac{1}{4[r]^2}|u|^2\,dx.\end{aligned}$$

Lemma 4. *(i) If $n \geq 3$, then for any $u \in H^1_{b,0}(\Omega)$,*

$$\int_\Omega\frac{\operatorname{Im}\kappa r+1}{4r^2}|u|^2\,dx\leq\int_\Omega|\tilde{x}\cdot\theta|^2\,dx.$$

(ii) If $n = 2$, then for any $u \in H^1_{b,0}(\Omega)$ and $\epsilon > 0$,

$$\begin{aligned}&\int_\Omega\Big(\operatorname{Im}\kappa r+\frac{1}{2}-18\epsilon-8\epsilon^2\Big)\frac{1}{4[r]^2}|u|^2\,dx\\ &\quad\leq\int_\Omega\Big(\operatorname{Im}\kappa r+\frac{1}{2}-2\epsilon\frac{r^2}{[r]^2}\Big)\Big\{|\tilde{x}\cdot\theta|^2-\frac{1}{4r^2}|u|^2\Big\}\,dx.\end{aligned}$$

Proof.

(i) We choose $\xi \equiv 1$ and $\eta = -\frac{n-2}{2r}$ in Lemma 3. Then since

$$\frac{n-1}{2r} + \eta = \frac{1}{2r}, \qquad \frac{n-1}{r}\eta + \eta' + \eta^2 = -\frac{(n-2)^2}{4r^2}, \tag{20}$$

letting $t \to \infty$, we have the assertion.

(ii) We choose

$$\xi = \operatorname{Im}\kappa r + \frac{1}{2} - 2\epsilon\frac{r^2}{[r]^2} \quad \text{and} \quad \eta = \frac{1}{2[r]}$$

in Lemma 3. Then by assumption, $\xi(r) > 0$ and also

$$\liminf_{t\to\infty} \int_{S_t} \xi\left(\frac{1}{2r} + \eta\right)|u|^2\, dS = 0.$$

Moreover, since

$$\frac{1}{r}\eta + \eta' + \eta^2 = \frac{-1}{4[r]^2} = \frac{-1}{4r^2(1+\log r/r_0)^2}, \tag{21}$$

it follows that

$$\begin{aligned}
&\int_\Omega \left(\operatorname{Im}\kappa r + \frac{1}{2} - 2\epsilon\frac{r^2}{[r]^2}\right)\left\{|\tilde{x}\cdot\theta|^2 - \frac{1}{4r^2}|u|^2\right\} dx \\
&\quad \geq -\int_\Omega \left(\operatorname{Im}\kappa - 2\epsilon\left(\frac{r^2}{[r]^2}\right)'\right)\left(\frac{1}{2r} + \eta\right)|u|^2\, dx \\
&\qquad + \int_\Omega \left(\operatorname{Im}\kappa r + \frac{1}{2} - 2\epsilon\frac{r^2}{[r]^2}\right)\left\{2\operatorname{Im}\kappa\left(\frac{1}{2r} + \eta\right)|u|^2 + \frac{1}{4[r]^2}|u|^2\right\} dx.
\end{aligned}$$

Thus the inequalities

$$2(\operatorname{Im}\kappa)^2 r - 4\epsilon\operatorname{Im}\kappa\frac{r^2}{[r]^2} \geq -\frac{2\epsilon^2}{r}\frac{r^4}{[r]^4}$$

and

$$\left\{-\frac{2\epsilon^2}{r}\frac{r^4}{[r]^4} + 2\epsilon\left(\frac{r^2}{[r]^2}\right)'\right\}\left(\frac{1}{2r} + \eta\right) \geq -\frac{8(\epsilon^2 + 2\epsilon)}{4[r]^2}$$

lead us to the desired conclusion.

□

Proof of Proposition 1.

The case $n \geq 3$. We choose $\epsilon < \frac{1}{4}$ in (19) and apply Lemma 4(i). Then

$$\begin{aligned}
&\int_\Omega \left(\operatorname{Im}\kappa r + \frac{1}{2} - 2\epsilon\right)\frac{1}{4r^2}|u|^2\, dx + \int_\Omega \frac{(n-1)(n-3)}{8r^2}|u|^2\, dx \\
&\quad \leq \frac{1}{4\epsilon}\int_\Omega r^2|f|^2\, dx + \frac{\epsilon_0^2}{\epsilon}\int_\Omega \frac{1}{4r^2}|u|^2\, dx,
\end{aligned}$$

and hence

$$\int_\Omega \left\{ \operatorname{Im}\kappa r + \frac{(n-2)^2\epsilon - 4\epsilon^2 - 2\epsilon_0^2}{4\epsilon} \right\} \frac{1}{r^2}|u|^2\,dx \le \frac{1}{4\epsilon}\int_\Omega r^2|f|^2\,dx.$$

Since

$$\sup_{\epsilon<1/4}\{(n-2)^2\epsilon - 4\epsilon^2\} = \begin{cases} \frac{(n-1)(n-3)}{4} & (n \ge 4), \\ \frac{1}{16} & (n=3), \end{cases}$$

the desired inequality holds if $\epsilon_0^2 < \frac{(n-1)(n-3)}{8}$ (when $n \ge 4$) and $\epsilon_0^2 < \frac{1}{32}$ (when $n = 3$).

The case $n = 2$. We combine (19) and Lemma 4(ii) to obtain

$$\begin{aligned} &\int_\Omega \left(\operatorname{Im}\kappa r + \frac{1}{2} - 18\epsilon - 8\epsilon^2 \right) \frac{1}{4[r]^2}|u|^2\,dx \\ &\qquad \le \frac{1}{4\epsilon}\int_\Omega [r]^2|f|^2\,dx + \left(\frac{\epsilon_0^2}{\epsilon} + 2\epsilon\right)\int_\Omega \frac{1}{4[r]^2}|u|^2\,dx \end{aligned}$$

for any $\epsilon < \frac{1}{4}$, which implies

$$\int_\Omega \left\{ \operatorname{Im}\kappa r + \frac{\epsilon - 40\epsilon^2 - 16\epsilon^3 - 2\epsilon_0^2}{2\epsilon} \right\} \frac{1}{4[r]^2}|u|^2\,dx \le \frac{1}{4\epsilon}\int_\Omega [r]^2|f|^2\,dx.$$

The desired inequality then holds if ϵ_0^2 is less than

$$\frac{1}{2}\sup_{\epsilon<1/4}\{\epsilon - 40\epsilon^2 - 16\epsilon^3\}.$$

As is easily seen, this number is between $\frac{1}{336}$ and $\frac{1}{320}$. □

To proceed to the proof of Proposition 2 we return to (18). The Schwarz inequality then implies

$$\begin{aligned} &\int_\Omega \Bigg\{ \left(\operatorname{Im}\kappa\varphi + \frac{\varphi'}{2} - 2\epsilon\varphi' \right)|\theta|^2 \\ &\qquad + \left(\operatorname{Im}\kappa\varphi + \frac{\varphi}{r} - \frac{\varphi'}{2} \right)\frac{(n-1)(n-3)}{4r^2}|u|^2 \Bigg\}\,dx \\ &\qquad \le \int_\Omega \frac{\varphi^2}{4\epsilon\varphi'}\{|f|^2 + |\nabla\times b|^2|u|^2\}\,dx \end{aligned} \tag{22}$$

for any $\epsilon > 0$. In the following we fix $\epsilon < \frac{1}{8}$.

Lemma 5. *Assume* (A1). *Then*

$$\int_\Omega \mu\left\{|\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2\right\}dx \le C\int_\Omega \max\{[r]^2, \mu^{-1}\}|f|^2\,dx$$

holds for some $C = C(\epsilon_0, \|\mu\|_1) > 0$.

Proof. In the case $n \geq 3$ we choose $\varphi(r) = \int_0^r \mu(\tau)\,d\tau$ in (22). Since $r\mu \leq \varphi \leq \|\mu\|_{L^1}$, by use of (A1) we have

$$\begin{aligned}\int_\Omega \left\{ \frac{1-4\epsilon}{2}\mu|\theta|^2 + \frac{1}{2}\mu\frac{(n-1)(n-3)}{4r^2}|u|^2 \right\} dx \\ \leq \|\mu\|_{L^1}^2 \left\{ \int_\Omega \frac{\mu^{-1}}{4\epsilon}|f|^2\,dx + \int_\Omega \frac{\epsilon_0^2[r]^{-4}\mu^{-1}}{4\epsilon}|u|^2\,dx \right\}.\end{aligned}$$

Hence, the use of Proposition 1 leads to the assertion.

In the case $n = 2$ we choose $\varphi = \frac{r}{(4+\log r/r_0)^2}$ in (22). Then since

$$\begin{aligned}\varphi' &= \frac{1}{(4+\log r/r_0)^2} - \frac{2}{(4+\log r/r_0)^3} \geq \frac{1}{2(4+\log r/r_0)^2}, \\ \frac{\varphi}{r} - \frac{\varphi'}{2} &\leq \frac{3}{4(4+\log r/r_0)^2}, \qquad \frac{\varphi^2}{\varphi'} \leq \frac{2r^2}{(4+\log r/r_0)^2},\end{aligned}$$

it follows that

$$\begin{aligned}\int_\Omega \frac{1-4\epsilon}{4(4+\log r/r_0)^2}|\theta|^2\,dx - \int \left\{ \operatorname{Im}\kappa r + \frac{1}{4} \right\} \frac{3}{4r^2(4+\log r/r_0)^2}|u|^2\,dx \\ \leq \int_\Omega \frac{r^2}{2\epsilon(4+\log r/r_0)^2}|f|^2\,dx + \int_\Omega \frac{\epsilon_0^2 r^2}{2\epsilon[r]^4(4+\log r/r_0)^2}|u|^2\,dx.\end{aligned}$$

Hence we have

$$\int_\Omega \frac{1-4\epsilon}{4(4+\log r/r_0)^2}|\theta|^2\,dx \leq \frac{1}{32\epsilon}\int_\Omega [r]^2|f|^2\,dx + \int_\Omega \left\{ \operatorname{Im}\kappa r + \frac{3}{4} + \frac{\epsilon_0^2}{8\epsilon} \right\} \frac{|u|^2}{4[r]^2}\,dx.$$

The use of Proposition 1 leads to the assertion if we note that

$$\mu \leq \frac{o(1)}{(4+\log r/r_0)^2} \quad \text{and} \quad \frac{(n-1)(n-3)}{4r^2} \leq 0$$

in this case. $\square$

Proof of Proposition 2. We start from the identity of Lemma 1. Since $\operatorname{Im}\kappa > 0$, it follows that

$$\begin{aligned}(23)\qquad \frac{1}{2}\int_\Omega \left\{ \left(\mu \operatorname{Im}\kappa \frac{n-1}{r} - \mu' \frac{n-1}{2r} \right)|u|^2 + \mu(|\nabla_b u|^2 + |\kappa u|^2) \right\} dx \\ \leq \frac{1}{2}\int_\Omega \mu \left\{ |\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2 \right\} dx \\ + \int_{r_0}^\infty \mu(t)\,dt \int_{\Omega_t} |f(x)|\,|i\kappa u|\,dx.\end{aligned}$$

Here $\mu' \leq 0$ by assumption and we have from the Schwarz inequality,

$$\|\mu\|_{L^1} \int_\Omega |f|\,|i\kappa u|\,dx \leq \|\mu\|_{L^1}^2 \int_\Omega \mu^{-1}|f|^2\,dx + \frac{1}{4}\int_\Omega \mu|\kappa u|^2\,dx.$$

Applying these inequalities and Lemma 5 in (23), we conclude with the desired inequality. $\square$

§3. Proof of Theorem 1 for the Klein–Gordon equation

Proposition 1 attains Theorem 1 for L. To show the theorem for the Klein–Gordon operator M, we need some more estimates concerning the resolvent $R_m(\kappa^2)$ of the operator $L+m^2$. To this aim we put $\xi(r)=(1+[r]^2)^{-1/2}$ and choose $\mu(r)$ in Proposition 2 to satisfy $\xi(r)\le\sqrt{\mu(r)}$.

Lemma 6. *There exists $C>0$ such that*

$$(1+|\kappa|)\|\xi R_m(\kappa^2)f\|+\|\xi\nabla_b(R_m(\kappa^2)f)\|\le C\|\xi^{-1}f\|, \tag{24}$$

$$\|\xi\Delta_b(R_m(\kappa^2)f)\|\le C\{\|\xi^{-1}\nabla_b f\|+\|\xi^{-1}f\|\}, \tag{25}$$

$$|\kappa|\,\|\xi\nabla_b(R_m(\kappa^2)f)\|\le C\{\|\xi^{-1}\nabla_b f\|+\|\xi^{-1}f\|\}, \tag{26}$$

for each $\kappa\in\mathbf{C}_+$ and $f\in X_0'$ also satisfying $\nabla_b f\in X_0'$.

Proof. Note that

$$|\kappa|^2\|\xi R_m(k^2)f\|^2\le m^2\|\xi R_m(\kappa^2)f\|^2+|-m^2+\kappa^2|\,\|\xi R_m(\kappa^2)f\|^2.$$

Then (24) follows directly from Propositions 1 and 2.

To show (25) we start from the equation

$$\xi\Delta_b(R_m(\kappa^2)g)=\Delta_b(\xi R_m(\kappa^2)g)-2\nabla_b\cdot\{(\nabla\xi)R_m(\kappa^2)g\}+(\Delta\xi)R_m(\kappa^2)g.$$

Put $\vec f=(\nabla\xi)R_m(\kappa^2)g$. Then since $\vec f|_{\partial\Omega}=\vec 0$, we have

$$|(\nabla_b\cdot\vec f,h)|=|(\vec f,-\nabla_b h)|\le\|\vec f\|\,\|h\|_{\dot H_b^1},$$

where $\dot H_b^1$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla_b h\|$. Let $\dot H_b^{-1}$ denote the dual space of $\dot H_b^1$. Then we have $\|\nabla_b\cdot\vec f\|_{\dot H_b^{-1}}\le\|\vec f\|$, and hence

$$\begin{aligned}&\|\xi\Delta_b(R_m(\kappa^2)g)\|_{\dot H_b^{-1}}\\&\quad\le\|\Delta_b(\xi R_m(\kappa^2)g)\|_{\dot H_b^{-1}}+2\|(\nabla\xi)R_m(\kappa^2)g\|+\|(\Delta\xi)R_m(\kappa^2)g\|_{\dot H_b^{-1}}\end{aligned}$$

Here, since

$$\Delta_b(\xi R_m(\kappa^2)g)=\nabla_b\cdot\{(\nabla\xi)R_m(\kappa^2)g+\xi\nabla_b(R_m(\kappa^2)g)\}$$

and $|\nabla\xi|\le C|\xi|$, it follows from (24) that

$$\|\Delta_b(\xi R_m(\kappa^2)g)\|_{\dot H_b^{-1}}\le\|(\nabla\xi)R_m(\kappa^2)g\|+\|\xi\nabla_b(R_m(\kappa^2)g)\|\le C\|\xi^{-1}g\|.$$

Moreover, noting that $|\Delta\xi| \le C[r]^{-1}\xi$, we can use the Hardy inequality (see [9, Lemma 3] when $n \ge 3$ and [12, Lemma 4] when $n = 2$)

$$\int_\Omega \frac{[n-2]}{4[r]^2}|f(x)|^2\,dx \le \int_\Omega |\nabla_b f(x)|^2\,dx \tag{27}$$

with $[n-2] = 1$ (when $n = 2$) and $[n-2] = n-2$ (when $n \ge 3$) to obtain

$$\|(\Delta\xi)R_m(\kappa^2)g\|_{\dot{H}_b^{-1}} \le C\|\xi R_m(\kappa^2)g\| \le C\|\xi^{-1}g\|.$$

These lead us to the inequality

$$\|\xi\Delta_b(R_m(\kappa^2)g)\|_{\dot{H}_b^{-1}} \le C\|\xi^{-1}g\|.$$

Equation (25) then follows from the equality

$$(\Delta_b(R_m(\kappa^2)f, g)) = (R_m(\kappa^2)\Delta_b f, g) = (\xi^{-1}f, \xi\Delta_b(R_m(\overline{\kappa}^2)g))$$

since we have

$$\|\xi^{-1}f\|_{\dot{H}_b^1} \le \|\xi^{-1}\nabla_b f\| + \|\nabla(\xi^{-1})f\|$$

and $|\nabla\xi^{-1}| \le C\xi^{-1}$.

Next note that

$$\kappa^2 R_m(\kappa^2)f = -f - (\Delta_b - m^2)R_m(\kappa^2)f.$$

Then the use of (24) and (25) shows

$$\|\xi(\Delta_b - m^2)R_m(\kappa^2)f\| \le C\{\|\xi^{-1}\nabla_b f\| + \|\xi^{-1}f\|\}.$$

Since $\|\xi f\| \le \|\xi^{-1}f\|$, this proves

$$|\kappa|^2\|\xi R_m(\kappa^2)f\| \le C\{\|\xi^{-1}\nabla_b f\| + \|\xi^{-1}f\|\}. \tag{28}$$

By use of (28), (24) and (25) we have

$$\begin{aligned}
&|\kappa|^2\|\xi\nabla_b(R_m(\kappa^2)f)\|^2 \\
&\quad = -|\kappa|^2(\{\xi\Delta_b(R_m(\kappa^2)f) + 2\nabla\xi\cdot\nabla_b(R_m(\kappa^2)f)\}, \xi R_m(\kappa^2)f) \\
&\quad \le \{\|\xi\Delta_b(R_m(\kappa^2)f)\| + 2\|\nabla\xi\cdot\nabla_b(R_m(\kappa^2)f)\|\}|\kappa|^2\|\xi R_m(\kappa^2)f\| \\
&\quad \le C\{\|\xi^{-1}\nabla_b f\| + \|\xi^{-1}f\|\}\{\|\xi^{-1}\nabla_b f\| + \|\xi^{-1}f\|\},
\end{aligned}$$

which proves (26). □

With this lemma we can prove the following proposition, which attains Theorem 1 for M.

Proposition 3. *Assume* (A1) *with small* ϵ_0. *For* $\kappa \in \mathbf{C}\backslash\mathbf{R}$ *put* $\mathcal{R}(\kappa) = (M-\kappa)^{-1}$. *Then there exists* $C_4 > 0$ *independent of* κ *and* $f \in X_E$ *such that*

$$\|\mathcal{R}(\kappa)f\|_{X_E} \le C_4\|f\|_{X'_E}, \tag{29}$$

where X_E *is the weighted energy space defined by* (13).

Proof. Note that

$$\begin{aligned}
&|(\mathcal{R}(\kappa)f, g)_{\mathcal{H}_E}| \\
&\quad = \tfrac{1}{2}\big[\,\big|\,\big(\nabla_b\left\{\kappa R_m(\kappa^2)f_1 + iR_m(\kappa^2)f_2\right\}, \nabla_b g_1\big) \\
&\qquad + \big((c+m^2)\left\{\kappa R_m(\kappa^2)f_1 + iR_m(\kappa^2)f_2\right\}, g_1\big) \\
&\qquad + \big(\left\{i(\Delta_b - m^2)R_m(\kappa^2)f_1 + \kappa R_m(\kappa^2)f_2\right\}, g_2\big)\,\big|\,\big] \\
&\quad \le \tfrac{1}{2}\big[\left\{|\kappa|\,\|\xi\nabla_b(R_m(\kappa^2)f_1)\| + \|\xi\nabla_b R_m(\kappa^2)f_2\|\right\}\|\xi^{-1}\nabla_b g_1\| \\
&\qquad + m^2\left\{|\kappa|\,\|\xi R_m(\kappa^2)f_1\| + \|\xi R_m(\kappa^2)f_2\|\right\}\|\xi^{-1}g_1\| \\
&\qquad + \left\{\|\xi\Delta_b(R_m(\kappa^2)f_1)\| + m^2\|\xi R_m(\kappa^2)f_1\| + |\kappa|\,\|\xi R_m(\kappa^2)f_2\|\right\} \\
&\qquad \times \|\xi^{-1}g_2\|\,\big].
\end{aligned} \tag{30}$$

Then applying the inequalities of Lemma 6 to each component on the right and noting that $m > 0$, we see that (29) holds. □

§4. Proof of Theorem 2

The resolvent estimates of Theorem 1 lead us to the smoothing properties summarized in the following proposition.

Proposition 4. *Assume* (A1) *with small* ϵ_0. *Then for each* $h(t) \in L^2(\mathbf{R}_\pm; X')$ *and* $f \in \mathcal{H}$, *we have*

$$\left\|\int_0^t e^{-i(t-\tau)\Lambda}h(\tau)\,d\tau\right\|^2_{L^2(\mathbf{R}_\pm;X)} \le C_0^2\|h\|^2_{L^2(\mathbf{R}_\pm;X')}, \tag{31}$$

$$\sup_{t\in\mathbf{R}_\pm}\left\|\int_0^t e^{i\tau\Lambda}h(\tau)\,d\tau\right\|^2_{\mathcal{H}} \le 2C_0\|h\|^2_{L^2(\mathbf{R}_\pm;X')}, \tag{32}$$

$$\|e^{-it\Lambda}f\|^2_{L^2(\mathbf{R}_\pm;X)} \le 2C_0\|f\|^2_{\mathcal{H}}, \tag{33}$$

where $\mathbf{R}_+ = (0,\infty)$ *and* $\mathbf{R}_- = (-\infty, 0)$.

Proof (Cf. Reed–Simon [15]). By the standard approximation procedure, we can assume $h(t) \in C_0^\infty(I; X')$ for some interval $I \subset \mathbf{R}_\pm$.

For $t \in \mathbf{R}_\pm$ we put $v(t) = \int_0^t e^{-i(t-\tau)\Lambda} h(\tau)\, d\tau$, where $h(t)$ is regarded as 0 outside I, and consider its Laplace transform

$$\tilde{v}(\zeta) = \pm \int_0^{\pm\infty} e^{i\zeta t} v(t)\, dt, \qquad \pm \operatorname{Im} \zeta > 0,$$

Then since $\tilde{v}(\zeta) = -i\mathcal{R}(\zeta)\tilde{h}(\zeta)$, it follows from the Plancherel theorem and Theorem 1 that

$$\begin{aligned}\left| \int_I e^{\mp 2\epsilon t} (v(t), g(t))_{\mathcal{H}}\, dt \right| &= \left| (2\pi)^{-1} \int_{-\infty}^{\infty} (\tilde{v}(\lambda \pm i\epsilon), \tilde{g}(\lambda \pm i\epsilon))_{\mathcal{H}}\, d\lambda \right| \\ &\le \int_{-\infty}^{\infty} \|\mathcal{R}(\lambda \pm i\epsilon)\tilde{h}(\lambda \pm i\epsilon)\|_X \|\tilde{g}(\lambda \pm i\epsilon)\|_{X'}\, d\lambda \\ &\le C_0 \int_I e^{\mp 2\epsilon t} \|h(t)\|_{X'} \|g(t)\|_{X'}\, dt\end{aligned}$$

for any $g(t) \in C_0^\infty(I; X')$. Letting $\epsilon \downarrow 0$, we obtain inequality (31).

Next, note that the Fubini theorem implies

$$\begin{aligned}\left\| \int_0^t e^{is\Lambda} h(s)\, ds \right\|_{\mathcal{H}}^2 &= \int_0^t \left(\int_0^\sigma e^{-i(\sigma - s)\Lambda} h(s)\, ds, h(\sigma) \right)_{\mathcal{H}} d\sigma \\ &\quad + \int_0^t \left(h(s), \int_0^s e^{-i(s-\sigma)\Lambda} h(\sigma)\, d\sigma \right)_{\mathcal{H}} ds,\end{aligned}$$

where $(\cdot,\cdot)_{\mathcal{H}}$ is extended to the duality between X and X'. This and (31) show that (32) holds.

Equation (33) is obvious from (32). □

Lemma 7. *Under* (A2) *we have*

$$|(V(t)u, v)_{\mathcal{H}}| \le \tilde{\eta}(t)\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}} + \epsilon_V \|u\|_X \|v\|_X,$$

where $\tilde{\eta}(t) = \eta(t)$, $\epsilon_V = \epsilon_1$ *for the Schrödinger case and* $\tilde{\eta}(t) = \max\{1, m^{-1}\}\eta(t)$, $\epsilon_V = \max\{1, m^{-1}\}\epsilon_1$ *for the Klein–Gordon case.*

Proof. The lemma is obvious for the Schrödinger case since we have

$$|(V(t)u, v)| \le \int_\Omega (\eta(t) + \epsilon_1(1 + [r]^2)^{-1})|u(x)|\,|v(x)|\, dx.$$

On the other hand, for the Klein–Gordon case we have

$$\begin{aligned}|(V(t)u, v)_{\mathcal{H}_E}| &= \frac{1}{2}\left| \int_\Omega \{c(x,t)u_1 + b_0(x,t)u_2\}\overline{v_2}\, dx \right| \\ &\le \frac{1}{2} \int_\Omega (\eta(t) + \epsilon_1(1 + [r]^2)^{-1})\{|u_1| + |u_2|\}|v_2|\, dx \\ &\le \max\{1, m^{-1}\}\{\eta(t)\|u\|_{\mathcal{H}_E}\|v\|_{\mathcal{H}_E} + \epsilon_1\|u\|_{X_E}\|v\|_{X_E}\}.\end{aligned}$$

Thus, the lemma also holds in this case. □

For $0 \le \pm s \le \pm T \le \infty$ let $I_{+,s} = [s, T]$ or $I_{-,s} = [T, s]$. We do not exclude $T = \pm\infty$ and write $I_{+,s} = \mathbf{R}_{+,s} = [s, \infty)$ or $I_{-,s} = \mathbf{R}_{-,s} = (-\infty, s]$.

With this notation let $Y(I_{\pm,s})$ be the space of functions $v(t) \in \mathrm{BC}(I_{\pm,s}; \mathcal{H}) \cap L^2(I_{\pm,s}; X)$ (where BC means the space of bounded continuous functions) such that

$$\|v\|_{Y(I_{\pm,s})} = \sup_{t \in I_{\pm,s}} \|v(t)\|_{\mathcal{H}} + \|v\|_{L^2(I_{\pm,s};X)} < \infty. \tag{34}$$

Lemma 8. *We put*

$$\Phi_{\pm,s} v(t) = \int_s^t e^{-i(t-\tau)\Lambda} V(\tau) v(\tau)\, d\tau, \quad v(t) \in Y(I_{\pm,s}).$$

Then $\Phi_{\pm,s} \in \mathcal{B}(Y(I_{\pm,s}))$ *and we have*

$$\sup_{t \in I_{\pm,s}} \|\Phi_{\pm,s} v(t)\|_{\mathcal{H}} \le \|\tilde{\eta}\|_{L^1(I_{\pm,s})} \sup_{t \in I_{\pm,s}} \|v(t)\|_{\mathcal{H}} + \epsilon_V \sqrt{2C_0} \|v\|_{L^2(I_{\pm,s};X)}, \tag{35}$$

$$\|\Phi_{\pm,s} v\|_{L^2(I_{\pm,s};X)} \le 2\sqrt{2C_0} \|\tilde{\eta}\|_{L^1(I_{\pm,s})} \sup_{t \in I_{\pm,s}} \|v(t)\|_{\mathcal{H}} + 3\epsilon_V C_0 \|v\|_{L^2(I_{\pm,s};X)}. \tag{36}$$

Proof. Let $g \in \mathcal{H}$. Then it follows from Lemma 7 that

$$\begin{aligned} |(\Phi_{\pm,s} v(t), g)_{\mathcal{H}}| &= \left| \int_s^t \left(V(\tau) v(\tau), e^{-i(\tau-t)\Lambda} g \right)_{\mathcal{H}} d\tau \right| \\ &\le \left| \int_s^t \tilde{\eta}(\tau) \|v(\tau)\|_{\mathcal{H}} \|g\|_{\mathcal{H}}\, d\tau \right| \\ &\quad + \epsilon_V \left| \int_s^t \|v(\tau)\|_X \|e^{-i(\tau-t)\Lambda} g\|_X\, d\tau \right|. \end{aligned} \tag{37}$$

So, by use of (33) and the unitarity of $e^{-it\Lambda}$ we obtain

$$|(\Phi_{\pm,s} v(t), g)_{\mathcal{H}}| \le \|\tilde{\eta}\|_{L^1(I_{\pm,s})} \sup_{\tau \in I_{\pm,s}} \|v(\tau)\|_{\mathcal{H}} \|g\|_{\mathcal{H}} + \epsilon_V \|v\|_{L^2(I_{\pm,s};X)} \sqrt{2C_0} \|g\|_{\mathcal{H}},$$

which implies (35).

Next, let $g(t) \in L^2(I_{\pm,s}; X')$. Then it similarly follows that

$$\begin{aligned} \left| \int_s^T (\Phi_{\pm,s} v(t), g(t))_{\mathcal{H}}\, dt \right| &= \left| \int_s^T \int_s^t \left(V(\tau) v(\tau), e^{-i(\tau-t)\Lambda} g(t) \right)_{\mathcal{H}} d\tau\, dt \right| \\ &\le \|\tilde{\eta}\|_{L^1(I_{\pm,s})} \sup_{\tau \in I_{\pm,s}} \left(\|v(\tau)\|_{\mathcal{H}} \left\| \int_\tau^T e^{it\Lambda} g(t)\, dt \right\|_{\mathcal{H}} \right) \\ &\quad + \epsilon_V \|v\|_{L^2(I_{\pm,s};X)} \left\| \int_\tau^T e^{-i(\tau-t)\Lambda} g(t)\, dt \right\|_{L^2(I_{\pm,s};X)}, \end{aligned}$$

where

$$\left\| \int_\tau^T e^{-i(\tau-t)\Lambda} g(t)\,dt \right\|_{L^2(I_{\pm,s};X)} \le \left\| \int_0^\tau e^{-i(\tau-t)\Lambda} g(t)\,dt \right\|_{L^2(I_{\pm,s};X)} + \left\| e^{-i\tau\Lambda} \int_0^T e^{it\Lambda} g(t)\,dt \right\|_{L^2(I_{\pm,s};X)}.$$

Thus, applying inequalities (31), (32) and (33), we obtain

$$\left| \int_s^T (\Phi_{\pm,s} v(t), g(t))_{\mathcal{H}}\,dt \right| \le \|\tilde{\eta}\|_{L^1(I_{\pm,s})} \sup_{\tau\in I_{\pm,s}} \|v(\tau)\|_{\mathcal{H}} 2\sqrt{2C_0}\|g\|_{L^2(I_{\pm,s};X')} + \epsilon_V \|v\|_{L^2(I_{\pm,s};X)} 3C_0 \|g\|_{L^2(I_{\pm,s};X')},$$

which implies (36). □

Now, since $\tilde{\eta}(t) \in L^1(\mathbf{R}_\pm)$, we can choose $0 < \delta \le 1$ and $\pm\sigma > 0$ to satisfy

$$(1 + 2\sqrt{2C_0})\|\tilde{\eta}\|_{L^1(I_{\pm,s})} < 1 \tag{38}$$

if $|I_{\pm,s}| = |T - s| \le \delta$ or $I_{\pm,s} = \mathbf{R}_{\pm,s}$ with $\pm s \ge \pm\sigma$. So, if ϵ_1 is chosen small enough to satisfy $\epsilon_V(2\sqrt{2C_0} + 3C_0) < 1$, then it follows from (34), (35) and (36) that

$$\begin{aligned} \|\Phi_{\pm,s} v\|_{Y(I_{\pm,s})} &\le \max\left\{ \left(1 + 2\sqrt{2C_0}\right)\|\tilde{\eta}\|_{L^1(I_{\pm,s})}, \epsilon_V\left(2\sqrt{2C_0} + 3C_0\right)\right\} \|v\|_{Y(I_{\pm,s})} \\ &< \|v\|_{Y(I_{\pm,s})}. \end{aligned} \tag{39}$$

Lemma 9. *For each fixed $I_{\pm,s}$ satisfying* (38)*, the integral equation*

$$u(t) = e^{-i(t-s)\Lambda} f - i \int_s^t e^{-i(t-\tau)\Lambda} V(\tau) u(\tau)\,d\tau \tag{40}$$

has a solution $u(t) \in Y(I_{\pm,s})$ and it satisfies

$$\|u\|_{Y(I_{\pm,s})} = \sup_{t\in I_{\pm,s}} \|u(t)\|_{\mathcal{H}} + \|u\|_{L^2(I_{\pm,s};X)} \le C_{\delta,\sigma}\|f\|_{\mathcal{H}} \tag{41}$$

for some $C_{\delta,\sigma} > 0$ independent of f.

Proof. We define $\{u_k(t)\}$ successively as follows:

$$u_0(t) = e^{-i(t-s)\Lambda} f, \qquad u_k(t) = u_0(t) - i\Phi_{\pm,s} u_{k-1}(t).$$

Note that the unitarity of $e^{-it\Lambda}$ and (33) show

$$\|u_0\|_{Y(I_{\pm,s})} = \|u_0(t)\|_{\mathcal{H}} + \|u_0\|_{L^2(I_{\pm,s};X)} \le (1 + \sqrt{2C_0})\|f\|_{\mathcal{H}}. \tag{42}$$

Thus, $u_0(t) \in Y(I_{\pm,s})$ and also each $u_k(t) \in Y(I_{\pm,s})$. Since

$$\|u_k - u_{k-1}\|_{Y(I_{\pm,s})} \leq \left(\|\Phi_{\pm,s}\|_{\mathcal{B}(Y(I_{\pm,s}))}\right)^k \|u_0\|_{Y(I_{\pm,s})}, \tag{43}$$

we see from (39) that

$$u_n(t) = u_0(t) + \sum_{k=1}^{n} \{u_k(t) - u_{k-1}(t)\}$$

converges in $Y_{I_{\pm,s}}$ as $n \to \infty$. The limit $u(t)$ obviously solves the integral equation (40). Inequality (41) with

$$C_{\delta,\sigma} = \frac{1 + \sqrt{2C_0}}{1 - \|\Phi_{\pm,s}\|_{\mathcal{B}(Y(I_{\pm,s}))}}$$

is a result of (42) and (43). □

Proof of Theorem 2. For δ and $\pm\sigma$ given in (38) we choose integer N to satisfy $N\delta \geq \pm\sigma$, and divide $\mathbf{R}_\pm$ into $N+1$ subintervals

$$I_{+,s_j} = [s_j, s_{j+1}] \quad \text{or} \quad I_{-,s_j} = [s_{j+1}, s_j] \quad (j = 0, 1, \ldots, N-1),$$
$$\text{and} \quad I_{\pm,s_N} = \mathbf{R}_{\pm,s_N},$$

where $s_j = \pm j\delta$ $(j = 0, 1, \ldots, N)$. Then by Lemma 9 the solution of (40) with $f = u(s_j)$ is constructed in each interval $I_{\pm,s_j}$, and by putting them together, a global solution of (11) is obtained. Moreover, the above argument and (41) imply that (14) holds with $C_1 = (N+1)C_{\delta,\sigma}^N$.

To show the uniqueness of solutions in $C(\mathbf{R}; \mathcal{H})$, note that the inequality

$$\|\Phi_{\pm,0}v(t)\|_{\mathcal{H}} \leq \pm \int_0^t \|V(\tau)v(\tau)\|_{\mathcal{H}}\, d\tau \leq \pm \int_0^t \{\tilde{\eta}(\tau) + \epsilon_V\}\|v(\tau)\|_{\mathcal{H}}\, d\tau$$

holds for each $t \in \mathbf{R}_\pm$. If $v(t)$ satisfies (11) with $f = 0$, then this inequality implies

$$\frac{d}{dt}\left[e^{\mp \int_0^t \{\tilde{\eta}(\tau)+\epsilon_V\}\, d\tau} \int_0^t \{\tilde{\eta}(\tau) + \epsilon_V\}\|v(\tau)\|_{\mathcal{H}}\, d\tau\right] \leq 0.$$

Integrating both sides, we conclude that $\|v(t)\|_{\mathcal{H}} = 0$ in $\mathbf{R}_\pm$. □

§5. Proof of Theorem 3

The proof of Theorem 3 will be based on Lemma 7 and the inequalities of Proposition 4 and Theorem 2.

We put $u(t,s) = U(t,s)f$, $u_0(t-s) = e^{-i(t-s)\Lambda} f_0$. Then we have from (11),

$$(u(t,s), u_0(t-s))_{\mathcal{H}} = (f, f_0)_{\mathcal{H}} - i \int_s^t (V(\tau)u(\tau,s), u_0(\tau-s))_{\mathcal{H}}\, d\tau.$$

On the right-hand side we apply the inequality of Lemma 7. It then follows from (33) and (14) that for any σ, $t \in \mathbf{R}_\pm$,

$$
\begin{aligned}
|(u(t,s), u_0(t-s))_{\mathcal{H}} - (u(\sigma,s), u_0(\sigma-s))_{\mathcal{H}}| \qquad\qquad \\
\le \left| \int_\sigma^t \tilde\eta(\tau) \|u(\tau,s)\|_{\mathcal{H}} \|u_0(\tau-s)\|_{\mathcal{H}}\, d\tau \right| \\
+ \epsilon_V \left| \int_\sigma^t \|u(\tau,s)\|_X^2\, d\tau \right|^{1/2} \left| \int_\sigma^t \|u_0(\tau-s))\|_X^2\, d\tau \right|^{1/2}.
\end{aligned} \tag{44}
$$

All the assertions of the theorem are verified from this inequality.

Proof of Theorem 3.

(i) We put $\sigma = s$ in (44). Then by (33) and (14),

$$
\begin{aligned}
|(u(t,s), u_0(t-s))_{\mathcal{H}} - (f, f_0)_{\mathcal{H}}| \le \left| \int_s^t \tilde\eta(\tau) \|u(\tau,s)\|_{\mathcal{H}} \|u_0(\tau-s)\|_{\mathcal{H}}\, d\tau \right| \\
+ \epsilon_V \sqrt{2C_0C_1} \|f\|_{\mathcal{H}} \|f_0\|_{\mathcal{H}}.
\end{aligned}
$$

Since $e^{-i(t-s)\Lambda}$ is unitary, it follows that

$$
\|u(t,s)\|_{\mathcal{H}} \le (1 + \epsilon_V \sqrt{2C_0C_1}) \|f\|_{\mathcal{H}} + \int_s^t \tilde\eta(\tau) \|u(\tau,s)\|_{\mathcal{H}}\, d\tau.
$$

The requirement $\eta(t) \in L^1(\mathbf{R})$ and the Gronwall inequality show the assertion with

$$
C_U = (1 + \epsilon_V \sqrt{2C_0C_1}) e^{\|\tilde\eta\|_{L^1}}.
$$

(ii) Noting (i), we have from (44), (33) and (14),

$$
\begin{aligned}
|(u(t,s), u_0(t-s))_{\mathcal{H}} - (u(\sigma,s), u_0(\sigma-s))_{\mathcal{H}}| \qquad\qquad \\
\le \left\{ C_U \|f\|_{\mathcal{H}} \left| \int_\sigma^t \tilde\eta(\tau)\, d\tau \right| + \epsilon_V \left| \int_\sigma^t \|u(\tau,s)\|_X^2\, d\tau \right|^{1/2} \sqrt{2C_0} \right\} \|f_0\|_{\mathcal{H}}.
\end{aligned}
$$

Here, for any fixed $s \in \mathbf{R}_\pm$, $\{\cdots\} \to 0$ as $\sigma, t \to \pm\infty$. Thus, $e^{-i(s-t)\Lambda} U(t,s)$ converges strongly in $\mathcal{H}$ as $t \to \pm\infty$.

(iii) Let $\sigma = s$ and $t \to \pm\infty$ in (44). Then noting (i) and (14), we have

$$
\begin{aligned}
|(Z^\pm(s)f, f_0)_{\mathcal{H}} - (f, f_0)_{\mathcal{H}}| \le \|f\|_{\mathcal{H}} \Big\{ C_U \left| \int_s^{\pm\infty} \tilde\eta(\tau)\, d\tau \right| \|f_0\|_{\mathcal{H}} \\
+ \epsilon_V \sqrt{C_1} \left| \int_s^{\pm\infty} \|u_0(\tau-s)\|_X^2\, d\tau \right|^{1/2} \Big\}.
\end{aligned} \tag{45}
$$

Choose here $f = e^{-is\Lambda}g$ and $f_0 = e^{-is\Lambda}g_0$. Then

$$|(\{e^{is\Lambda}Z^{\pm}(s)e^{-is\Lambda} - I\}g, g_0)_{\mathcal{H}}| \le \|e^{-is\Lambda}g\|_{\mathcal{H}}\left\{C_U\left|\int_s^{\pm\infty}\tilde{\eta}(\tau)\,d\tau\right|\|e^{-is\Lambda}g_0\|_{\mathcal{H}}+ \right.$$
$$\left. + \epsilon_V\sqrt{2C_0}\left|\int_s^{\pm\infty}\|e^{-i\tau\Lambda}g_0\|_X^2\,d\tau\right|^{1/2}\right\}.$$

With g and g_0 being arbitrary, this implies that as $s \to \pm\infty$,

$$Z^{\pm}U(0,s)e^{-is\Lambda} = e^{is\Lambda}Z^{\pm}(s)e^{-is\Lambda} \to I \quad \text{weakly in } \mathcal{H}.$$

(iv) Note that (45) and (33) imply

$$|(\{Z^{\pm}(s) - I\}f, f_0)_{\mathcal{H}}| \le \left\{\left|\int_s^{\pm\infty}\tilde{\eta}(\tau)\,d\tau\right|C_U + \epsilon_V\sqrt{2C_0C_1}\right\}\|f\|_{\mathcal{H}}\|f_0\|_{\mathcal{H}}.$$

Since $\epsilon_V\sqrt{2C_1C_0} < 1$, we can choose $\pm s > 0$ sufficiently large to satisfy

$$\left|\int_s^{\pm\infty}\tilde{\eta}(\tau)\,d\tau\right|C_U + \epsilon_V\sqrt{2C_0C_1} < 1.$$

Thus, $\|Z^{\pm}(s) - I\|_{\mathcal{B}(\mathcal{H})} < 1$ and $Z^{\pm}(s)$ gives a bijection on $\mathcal{H}$. The same property of $Z^{\pm}$ then easily follows.

□

§6. Final remarks

In the case $\Omega = \mathbf{R}^n$ ($n \ge 3$) and $b(x) = 0$, similar results have been obtained in [11] and [13], for complex potentials satisfying

$$c(x,t) \in L^{\nu}(\mathbf{R}; L^p) \cap BC(\mathbf{R}^{n+1})$$

with

$$0 \le \frac{1}{p} \le \frac{2}{n} \quad \text{and} \quad \frac{1}{\nu} = 1 - \frac{n}{2p}.$$

The smallness condition

$$\|c\|_{L^{\infty}(\mathbf{R}_{\pm}; L^{n/2})} \ll 1$$

is also required when $\nu = \infty$.

The arguments employed in this work are based on the Fourier transformation, and are not directly applicable to problems in an exterior domain. Moreover, note that the function

$$c(x,t) = c_0(1+r)^{-\alpha}(1+|t|)^{-\beta} \tag{46}$$

with α, $\beta \geq 0$ satisfies (A2) and also the above conditions if $\alpha/2+\beta > 1$. However, the function

$$c(x,t) = c_0 \sin t(1+r)^{-2} \quad \text{with small } |c_0| > 0$$

satisfies (A2) but slips out of the above conditions.

The potential (46) has been considered in Yafaev [17] when c is real and $\beta > 0$. For the Schrödinger equation (1) in $\mathbf{R}^n$ $(n \geq 3)$ with $b = 0$, his results include the following. The wave operator

$$W^{\pm} = \operatorname*{s-lim}_{t\to\pm\infty} U(0,t)e^{itL}$$

exists if $\alpha + \beta > 1$. It is in general incomplete, but becomes complete, i.e., the range of $W^{\pm}$ coincides with the whole space $L^2(\mathbf{R}^n)$, if the stronger condition $\frac{\alpha}{2} + \beta > 1$ is required.

In the case $m = 0$ and $n \geq 3$, weighted energy methods developed in [7] can be applied to our equation (2) in the magnetic fields without any essential modification. The requirements on the perturbation are

(A1′) $$|\nabla \times b(x)| \leq \epsilon \frac{n-2}{2r}\mu(r),$$

(A2′) $$\max\left\{|b_0(x,t)|, \frac{2r}{n-2}|c(x,t)|\right\} \leq \eta(t) + \epsilon\mu(r),$$

where $\epsilon > 0$ is a small constant and $\mu(r)$ is a positive, decreasing L^1-function of $r > 0$.

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