

Twisted Alexander Polynomials of Hyperbolic Links

by

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Abstract

In this paper we apply the twisted Alexander polynomial to study fibering and genus detecting problems for oriented links. In particular we generalize a conjecture of Dunfield, Friedl and Jackson on the torsion polynomial of hyperbolic knots to hyperbolic links, and confirm it for an infinite family of hyperbolic 2-bridge links. Moreover, we consider a similar problem for parabolic representations of 2-bridge link groups.

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§1. Introduction

The twisted Alexander polynomial, a generalization of the classical Alexander polynomial [2], is defined for a pair consisting of a group and its representation into a linear group. This invariant was first introduced by Lin [14] for knots in the 3-sphere S^3 and by Wada [28] for finitely presentable groups which include the link groups. In recent years, a theory of twisted Alexander polynomials has been rapidly developed and has contributed to solving various important problems in low-dimensional topology, especially in the theory of knots and links. However, it seems that this invariant still remains to be studied. As for recent developments on twisted Alexander polynomials and their applications, we refer to the survey papers [9, 19] and the references therein.

In [6], based on huge experimental calculations, Dunfield, Friedl and Jackson conjectured that for a hyperbolic knot K in S^3 , i.e., the complement $S^3 \setminus K$ has a

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complete hyperbolic structure of finite volume, the twisted Alexander polynomial associated to a lift of the holonomy representation detects the genus and fiberedness of K . At the present time, the conjecture has been confirmed for all hyperbolic knots with at most 15 crossings [6], a certain infinite family of hyperbolic 2-bridge knots (see [18, 20]). Moreover, Agol and Dunfield showed in [1] that the twisted Alexander polynomial detects the genus for a large class of hyperbolic knots, which includes many knots whose ordinary Alexander polynomial is trivial.

In this paper we apply the twisted Alexander polynomial to study the fibering and genus detecting problems for *oriented links*. In particular, we will generalize the conjecture of Dunfield, Friedl and Jackson to a hyperbolic link L in S^3 . In fact, we conjecture that the twisted Alexander polynomial associated to a lift of the holonomy representation, say $\rho_0 : \pi_1(S^3 \setminus L) \rightarrow \mathrm{SL}(2, \mathbb{C})$, detects the Thurston norm [26] and fiberedness of an oriented hyperbolic link, and show that the conjecture holds true for an infinite family of hyperbolic 2-bridge links. Actually, in Section 4 we will show the following theorem.

Theorem 1.1 (Theorem 4.12). *For the double twist link L as in Figure 2, the twisted Alexander polynomial $\Delta_{L, \rho_0}(t)$ associated to ρ_0 determines the Thurston norm. Moreover, L is fibered if and only if $\Delta_{L, \rho_0}(t)$ is monic.*

As is well known, these topological properties of a 2-bridge link are detected by the reduced Alexander polynomial (see [4, 21, 22, 23]). However, there seems to be no a priori reason that the same must be true for the twisted Alexander polynomial.

Since a lift of the holonomy representation of a hyperbolic link L is one of the *parabolic* representations (namely, it is a nonabelian $\mathrm{SL}(2, \mathbb{C})$ -representation and the traces of the images of all the meridians of L are 2), it is natural to consider the following problem: *For an oriented hyperbolic link L and its parabolic representation $\rho : \pi_1(S^3 \setminus L) \rightarrow \mathrm{SL}(2, \mathbb{C})$, does the twisted Alexander polynomial associated to ρ determine the Thurston norm and fiberedness of L ?* In Section 5, we give a partial answer to this question in the case of a 2-bridge link. More precisely, we show that not all parabolic representations detect the genus (in this case, the Thurston norm is equivalent to the genus) of a hyperbolic 2-bridge link.

This paper is organized as follows. In Section 2, we briefly review some basic materials for the $\mathrm{SL}(2, \mathbb{C})$ -character variety of a finitely generated group and the twisted Alexander polynomial of an oriented link associated to a two-dimensional linear representation. In Section 3 we review a conjecture of Dunfield, Friedl and Jackson for hyperbolic knots and state its generalization for oriented hyperbolic links. Section 4 is devoted to the calculation of the loci of the character variety, which characterize fiberedness and the genus of a wide family of 2-bridge links.

This result can be regarded as a generalization of [12] and [11], which discussed the same problem in the case of knots. In the final section, we give an answer to the question on parabolic representations mentioned above.

§2. Preliminaries

In this section we give several standard definitions, and put a convention on the links that we will handle throughout the paper.

§2.1. Oriented links

A μ -component *link* L is the union of μ ordered, oriented and pairwise disjoint circles L_i embedded in the 3-sphere S^3 . Two links L and L' are *equivalent* if and only if $\mu = \mu'$ and there exists an orientation-preserving homeomorphism f of S^3 to itself such that $f(L_i) = L'_i$ and $f|_{L_i}$ is also orientation preserving for any i . A *knot* is nothing but a 1-component link.

A *Seifert surface* of an oriented link $L \subset S^3$ is a connected, oriented, compact surface S embedded in S^3 whose boundary is L such that the orientation on L is just the induced orientation from S . The link L is called *fibred* if the exterior $E_L = S^3 \setminus \text{int}(N(L))$ has the structure of a surface bundle over the circle such that a Seifert surface of L represents a fiber. The *genus* $g = g(L)$ of L is the minimum of the genera of all its Seifert surfaces. We note that fibredness and the genus of a link depend on the choice of orientations of the link.

A link L in S^3 is called *hyperbolic* if the exterior E_L has a complete hyperbolic structure of finite volume.

Let N be a compact, connected, orientable 3-manifold and $\sigma \in H^1(N; \mathbb{Z})$. The *Thurston norm* of σ is defined as

$$\|\sigma\|_T = \min\{\chi_-(S) \mid S \subset N \text{ properly embedded surface dual to } \sigma\},$$

where for a given surface S with connected components $S_1 \cup \cdots \cup S_k$, we define $\chi_-(S) = \sum_{i=1}^k \max\{-\chi(S_i), 0\}$. Thurston showed in [26] that this defines a seminorm on $H^1(N; \mathbb{Z})$ and, moreover, on $H^1(N; \mathbb{R})$.

Convention 2.1. In this paper we always consider *oriented* links. In addition, we assume throughout this paper that links are *nonsplit*.

§2.2. Character varieties

Let G be a finitely generated group. The *variety of representations* $R(G)$ of G is the set of $\text{SL}(2, \mathbb{C})$ -representations, i.e., $R(G) = \text{Hom}(G, \text{SL}(2, \mathbb{C}))$. Since G is finitely generated, $R(G)$ can be embedded in a product $\text{SL}(2, \mathbb{C}) \times \cdots \times \text{SL}(2, \mathbb{C})$

by mapping each representation to the image of a generating set. In this manner, $R(G)$ is an affine algebraic set whose defining polynomials are induced by the relators of a presentation of G . It is known that this structure is independent of the choice of presentations of G up to isomorphism.

A representation $\rho : G \rightarrow \text{SL}(2, \mathbb{C})$ is said to be *abelian* if $\rho(G)$ is an abelian subgroup of $\text{SL}(2, \mathbb{C})$. A representation ρ is called *reducible* if there exists a proper invariant subspace in \mathbb{C}^2 under the action of $\rho(G)$. This is equivalent to saying that ρ can be conjugated to a representation whose image consists of upper triangular matrices. When ρ is not reducible, it is called *irreducible*.

Given a representation $\rho \in R(G)$, its *character* is the map $\chi_\rho : G \rightarrow \mathbb{C}$ defined by $\chi_\rho(\gamma) = \text{tr}(\rho(\gamma))$ for $\gamma \in G$. We denote the set of all characters by $X(G)$. For a given element $\gamma \in G$, we define the map $\tau_\gamma : X(G) \rightarrow \mathbb{C}$ by $\tau_\gamma(\chi) = \chi(\gamma)$. Then it is known that $X(G)$ is an affine algebraic set that embeds in \mathbb{C}^N with coordinates $(\tau_{\gamma_1}, \dots, \tau_{\gamma_N})$ for some $\gamma_1, \dots, \gamma_N \in G$. This affine algebraic set is called the *character variety* of G . We note that the set $\{\gamma_1, \dots, \gamma_N\}$ can be chosen to contain a generating set of G . The projection $R(G) \rightarrow X(G)$ given by $\rho \mapsto \chi_\rho$ is surjective.

For a *link group* $G(L) = \pi_1(E_L)$, namely the fundamental group of the exterior E_L of L in S^3 , we write $R(L) = R(G(L))$ and $X(L) = X(G(L))$ for simplicity.

§2.3. Twisted Alexander polynomials

Let $L = L_1 \sqcup \dots \sqcup L_\mu$ be a μ -component oriented link in S^3 . We choose and fix a Wirtinger presentation of $G(L)$. That is, given a regular projection of the link L , we assign to each overpass a generator x_i , and to each crossing a relator $x_i x_j x_k^{-1} x_j^{-1}$, as in Figure 1 (the orientation of the under crossing arc is irrelevant). Thus we obtain a presentation of $G(L)$ with q generators and q relators, $\langle x_1, \dots, x_q \mid r_1, \dots, r_q \rangle$. After some reordering of the indices, the relators r_1, \dots, r_q satisfy $\prod_{i=1}^q r_i^{\pm 1} = 1$. This means that any one of the relators is a consequence of the other $q - 1$ relators. We remove one of the relators and call the resulting presentation

$$G(L) = \langle x_1, \dots, x_q \mid r_1, \dots, r_{q-1} \rangle$$

a *Wirtinger presentation* of $G(L)$.

The abelianization homomorphism

$$\alpha_L : G(L) \rightarrow H_1(E_L; \mathbb{Z}) \cong \mathbb{Z}^{\oplus \mu} = \langle t_1 \rangle \oplus \dots \oplus \langle t_\mu \rangle$$

is given by assigning to each generator x_i the meridian element $t_k \in H_1(E_L; \mathbb{Z})$ of the corresponding component L_k of L . Here we denote the sum in \mathbb{Z} multiplicatively. Moreover, we consider the surjective homomorphism $p : H_1(E_L; \mathbb{Z}) \rightarrow$

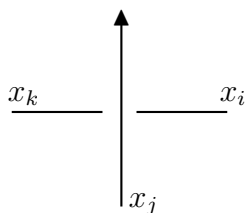


Figure 1. A relator $x_i x_j x_k^{-1} x_j^{-1}$.

$\mathbb{Z} = \langle t \rangle$ defined by $t_i \mapsto t$ and for simplicity denote the composition map $p \circ \alpha_L : G(L) \rightarrow \mathbb{Z}$ by α .

In this paper we consider a representation of $G(L)$ into the two-dimensional special linear group $SL(2, \mathbb{C})$, say $\rho : G(L) \rightarrow SL(2, \mathbb{C})$. The maps ρ and α naturally induce two ring homomorphisms $\tilde{\rho} : \mathbb{Z}[G(L)] \rightarrow M(2, \mathbb{C})$ and $\tilde{\alpha} : \mathbb{Z}[G(L)] \rightarrow \mathbb{Z}[t^{\pm 1}]$, where $\mathbb{Z}[G(L)]$ is the group ring of $G(L)$ and $M(2, \mathbb{C})$ is the matrix algebra of degree 2 over \mathbb{C} . Then $\tilde{\rho} \otimes \tilde{\alpha}$ defines a ring homomorphism $\mathbb{Z}[G(L)] \rightarrow M(2, \mathbb{C}[t^{\pm 1}])$. Let F_q denote the free group on generators x_1, \dots, x_q and

$$\Phi : \mathbb{Z}[F_q] \rightarrow M(2, \mathbb{C}[t^{\pm 1}])$$

the composition of the surjection $\tilde{\phi} : \mathbb{Z}[F_q] \rightarrow \mathbb{Z}[G(L)]$ induced by the presentation of $G(L)$ and the map $\tilde{\rho} \otimes \tilde{\alpha} : \mathbb{Z}[G(L)] \rightarrow M(2, \mathbb{C}[t^{\pm 1}])$.

Let us consider the $(q - 1) \times q$ matrix M whose (i, j) -entry is the 2×2 matrix

$$\Phi \left(\frac{\partial r_i}{\partial x_j} \right) \in M(2, \mathbb{C}[t^{\pm 1}]),$$

where $\frac{\partial}{\partial x}$ denotes the free differential. For $1 \leq j \leq q$, let us denote by M_j the $(q - 1) \times (q - 1)$ matrix obtained from M by removing the j th column. We regard M_j as a $2(q - 1) \times 2(q - 1)$ matrix with coefficients in $\mathbb{C}[t^{\pm 1}]$. Then Wada's *twisted Alexander polynomial* [28] of a link L associated with a representation $\rho : G(L) \rightarrow SL(2, \mathbb{C})$ is defined to be the rational function

$$\Delta_{L,\rho}(t) = \frac{\det M_j}{\det \Phi(1 - x_j)}$$

and is well defined up to multiplication by t^{2k} ($k \in \mathbb{Z}$).

Remark 2.2. By definition, $\Delta_{L,\rho}(t)$ is a rational function in the variable t , but it will be a Laurent polynomial if L is a link with two or more components [28, Proposition 9], or L is a knot K and ρ is nonabelian [13, Theorem 3.1].

We note that if $\rho, \rho' \in R(L)$ are conjugate representations, then $\Delta_{L,\rho}(t) = \Delta_{L,\rho'}(t)$ holds (see [28, Section 3]). If $\rho, \rho' \in R(L)$ are irreducible representations with $\chi_\rho = \chi_{\rho'}$, then ρ is conjugate to ρ' (see [5, Proposition 1.5.2]), and hence $\Delta_{L,\rho}(t) = \Delta_{L,\rho'}(t)$. Moreover, if $\rho, \rho' \in R(L)$ are reducible representations with $\chi_\rho = \chi_{\rho'}$, then $\Delta_{L,\rho}(t)$ and $\Delta_{L,\rho'}(t)$ are determined by diagonal entries of images of ρ and ρ' and hence they are equivalent. Therefore, we can define the twisted Alexander polynomial associated with $\chi \in X(L)$ to be $\Delta_{L,\rho}(t)$, where $\chi = \chi_\rho$, and we denote it by $\Delta_{L,\chi}(t)$ if we emphasize that it is a function on $X(L)$.

It is known that we can write the twisted Alexander polynomial $\Delta_{L,\chi}(t)$ without any ambiguity as

$$\Delta_{L,\chi}(t) = \sum_{j=0}^{2l} \psi_j(\chi) t^j$$

with \mathbb{C} -valued functions ψ_j on $X(L)$ such that $\psi_k = \psi_{2l-k}$ ($0 \leq k \leq l$) (see [8, Theorem 1.5]), where $l = \|\alpha\|_T$ is the Thurston norm of α (see [7, Theorem 1.1]).

For a subvariety X_0 of $X(L)$, we say that ψ_n is the *coefficient of the highest-degree term of $\Delta_{L,\chi}(t)$ on X_0* if $\psi_m \equiv 0$ for $m > n$ and $\psi_n \not\equiv 0$ on X_0 . Moreover, we say $\Delta_{L,\chi}(t)$ (respectively, $\Delta_{L,\rho}(t)$) is *monic* if the coefficient of the highest-degree term of $\Delta_{L,\chi}(t)$ (respectively, $\Delta_{L,\rho}(t)$) is 1. This makes sense because the twisted Alexander polynomial is well defined up to multiplication by t^{2k} ($k \in \mathbb{Z}$). It is known that the twisted Alexander polynomial of a fibered link is monic for every nonabelian representation [7, Theorem 1.1] (see [3, 10] for the case of fibered knots).

§3. Hyperbolic torsion polynomials

§3.1. A conjecture of Dunfield, Friedl and Jackson

Let N be a hyperbolic 3-manifold of finite volume. Then there is a faithful representation $\bar{\rho}_0 : \pi_1(N) \rightarrow \mathrm{PSL}(2, \mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$, where \mathbb{H}^3 denotes the upper half-space model of the hyperbolic 3-space, with discrete image such that $\mathbb{H}^3 / \mathrm{Im} \bar{\rho}_0 \cong N$. The representation $\bar{\rho}_0$ is called the *holonomy representation* and is unique up to conjugation. It is known that a peripheral torus subgroup of $\bar{\rho}_0(\pi_1(N)) \subset \mathrm{PSL}(2, \mathbb{C})$ is conjugate to a group of cosets of matrices of the form $\begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$, where $\nu \in \mathbb{C}$. In particular, the traces (defined up to sign) of the elements of such a subgroup are ± 2 . Using a result of Thurston, $\bar{\rho}_0$ may be lifted to a representation in $\mathrm{SL}(2, \mathbb{C})$ that is also faithful and has discrete image.

In [6], Dunfield, Friedl and Jackson studied the twisted Alexander polynomial $\Delta_{K,\rho_0}(t)$ for $N = E_K$, the exterior of a hyperbolic knot K in S^3 and $\rho_0 : G(K) \rightarrow$

$SL(2, \mathbb{C})$, a lift of $\bar{\rho}_0$ such that $\text{tr } \rho_0(\mu_K) = 2$ for a meridian μ_K . They call $\Delta_{K, \rho_0}(t)$ the *hyperbolic torsion polynomial* and denote it by $\mathcal{T}_K(t)$. Moreover, based on huge numerical calculations, they conjectured the following.

Conjecture 3.1 (Dunfield–Friedl–Jackson [6, Conjecture 1.4]). *For a hyperbolic knot K in S^3 , the hyperbolic torsion polynomial $\mathcal{T}_K(t)$ determines the Thurston norm $x(K)$ or equivalently the genus of K . Moreover, the knot K is fibered if and only if $\mathcal{T}_K(t)$ is monic.*

Remark 3.2. Let $\alpha_K \in H^1(E_K; \mathbb{Z}) = \text{Hom}(G(K), \mathbb{Z})$ be the abelianization, then we can see $\|\alpha_K\|_T = 2g(K) - 1$ and denote it by $x(K)$.

Conjecture 3.1 has been verified for all hyperbolic knots with at most 15 crossings [6], hyperbolic twist knots [18] and a certain wide class of hyperbolic 2-bridge knots [20]. Recently Agol and Dunfield showed the former part of the conjecture for a large class of hyperbolic knots in S^3 that includes all special arborescent knots and many knots whose ordinary Alexander polynomial is trivial (see [1] for details).

§3.2. A generalization

For a μ -component hyperbolic link L there are 2^μ possible lifts of the holonomy representation $\bar{\rho}_0 : G(L) \rightarrow \text{PSL}(2, \mathbb{C})$ to an $SL(2, \mathbb{C})$ -representation. It is known that there is a canonical one-to-one correspondence between the set of lifts of the holonomy representation and the set of spin structures of the exterior of a link (see [17]). Among them we focus on the lift $\rho_0 : G(L) \rightarrow SL(2, \mathbb{C})$ such that the images of the meridians of L by ρ_0 are matrices in $SL(2, \mathbb{C})$ with trace 2. Similar to the case of knots, in this paper, we call this kind of nonabelian $SL(2, \mathbb{C})$ -representation (or character) of the link group parabolic. That is, a nonabelian $SL(2, \mathbb{C})$ -representation $\rho : G(L) \rightarrow SL(2, \mathbb{C})$ (or character χ_ρ) is called *parabolic* if the images of the meridians of L by ρ are matrices with trace 2. Then we propose the following conjecture.

Conjecture 3.3. *For a μ -component oriented hyperbolic link L in S^3 , the twisted Alexander polynomial $\Delta_{L, \rho_0}(t)$ determines the Thurston norm, namely, we have*

$$\text{deg } \Delta_{L, \rho_0}(t) = 2\|\alpha\|_T$$

for $\alpha \in H^1(E_L; \mathbb{Z})$ given by sending each meridian to 1. Moreover, the link L is fibered if and only if $\Delta_{L, \rho_0}(t)$ is monic.

Remark 3.4. For an alternating link L , it is known that the Thurston norm $\|\alpha\|_T$ coincides with the Alexander norm $\|\alpha\|_A$ for $\alpha \in H^1(E_L; \mathbb{Z})$ as above (see [16])

for details). The Alexander norm is determined by the (multivariable) Alexander polynomial of L , so that we have $\|\alpha\|_A = \deg \Delta_L(t) - 1$, where $\Delta_L(t)$ denotes the reduced Alexander polynomial. Hence the equality in Conjecture 3.3 will be $\deg \Delta_{L, \rho_0}(t) = 4g(L) + 2(\mu - 2)$ for a μ -component alternating hyperbolic link L .

In the next section, we show that Conjecture 3.3 holds for a wide family of 2-bridge links that are called the double twist links.

§4. Fiberings and genus detecting problems

In this section we discuss the fibering and genus detecting problems for the double twist links. In particular, for these links, we specify a finite number of loci in the character variety that characterize fiberedness and the genus of the links. This result can be regarded as a generalization of [12, Theorems 4.3 and 4.4]. As a consequence, we will show in Section 4.7 that Conjecture 3.3 holds true for the double twist links.

We start by reviewing Chebyshev polynomials and their properties.

§4.1. Chebyshev polynomials (1)

Let $S_k(q)$ be the Chebyshev polynomials of the second kind defined by $S_0(q) = 1$, $S_1(q) = q$ and $S_{k+1}(q) = qS_k(q) - S_{k-1}(q)$ for all integers k . Similarly, let $T_k(q)$ be the Chebyshev polynomials of the first kind defined by $T_0(q) = 2$, $T_1(q) = q$ and $T_{k+1}(q) = qT_k(q) - T_{k-1}(q)$ for all integers k . Note that $T_k(q) = S_k(q) - S_{k-2}(q)$.

The following three lemmas are elementary and hence their proofs are omitted.

Lemma 4.1. *Write $q = v + v^{-1}$. Then*

$$T_k(q) = v^k + v^{-k}.$$

We have $S_k(2) = k + 1$ and $S_k(-2) = (-1)^k(k + 1)$. If $v \neq \pm 1$ then

$$S_k(q) = \frac{v^{k+1} - v^{-(k+1)}}{v - v^{-1}}.$$

In particular, if $q = 2 \cos \beta$, where $\frac{\beta}{\pi} \in \mathbb{R} \setminus \mathbb{Z}$, then $S_k(q) = \frac{\sin(k+1)\beta}{\sin \beta}$.

Lemma 4.2. *We have*

$$S_k^2(q) + S_{k-1}^2(q) - qS_k(q)S_{k-1}(q) = 1.$$

Lemma 4.3. *For any positive integer k we have*

$$S_k(q) = \prod_{j=1}^k \left(q - 2 \cos \frac{j\pi}{k+1} \right).$$

In particular, all the roots of $S_k(q)$ are real numbers strictly between -2 and 2 .

Lemma 4.4. (1) *If $k \geq 1$ is even, then*

$$T_k(q) - 2 = (q - 2)(q + 2) \prod_{j=1}^{(k/2)-1} \left(q - 2 \cos \frac{2j\pi}{k} \right)^2.$$

(2) *If $k \geq 1$ is odd, then*

$$T_k(q) - 2 = (q - 2) \prod_{j=1}^{(k-1)/2} \left(q - 2 \cos \frac{2j\pi}{k} \right)^2.$$

Proof. Write $q = v + v^{-1}$. By Lemma 4.1, we have $T_k(q) = v^k + v^{-k}$. Then

$$\begin{aligned} T_k(q) - 2 &= v^{-k}(v^k - 1)^2 \\ &= v^{-k}(v - 1)^2 \prod_{1 \leq j \leq k-1} \left(v - e^{2\pi i j/k} \right)^2. \end{aligned}$$

Grouping the terms involving j and $k - j$ we have

$$\begin{aligned} \left(v - e^{2\pi i j/k} \right) \left(v - e^{2\pi i (k-j)/k} \right) &= \left(v - e^{2\pi i j/k} \right) \left(v - e^{-2\pi i j/k} \right) \\ &= v^2 + 1 - 2v \cos \frac{2j\pi}{k}. \end{aligned}$$

Suppose k is even. Since $e^{2\pi i j/k} = -1$ if $j = \frac{k}{2}$, we have

$$\begin{aligned} T_k(q) - 2 &= v^{-k}(v - 1)^2(v + 1)^2 \prod_{1 \leq j \leq (k/2)-1} \left(v^2 + 1 - 2v \cos \frac{2j\pi}{k} \right)^2 \\ &= (q - 2)(q + 2) \prod_{j=1}^{(k/2)-1} \left(q - 2 \cos \frac{2j\pi}{k} \right)^2. \end{aligned}$$

The case of odd k is similar. □

Lemma 4.5. (1) *If $k \geq 1$ is odd, then*

$$T_k(q) - q = (q - 2)(q + 2) \prod_{j=1}^{(k-3)/2} \left(q - 2 \cos \frac{2j\pi}{k-1} \right) \prod_{j=1}^{(k-1)/2} \left(q - 2 \cos \frac{2j\pi}{k+1} \right).$$

(2) If $k \geq 1$ is even, then

$$T_k(q) - q = (q - 2) \prod_{j=1}^{(k-2)/2} \left(q - 2 \cos \frac{2j\pi}{k-1} \right) \prod_{j=1}^{k/2} \left(q - 2 \cos \frac{2j\pi}{k+1} \right).$$

Proof. The proof of Lemma 4.5 is similar to that of Lemma 4.4. □

The following lemma is known; see, e.g., [27, Proposition 2.4].

Lemma 4.6. *Suppose $Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \in \text{SL}(2, \mathbb{C})$. Let $q = \text{tr } Q$. Then for any positive integer k we have*

$$Q^k = \begin{pmatrix} S_k(q) - Q_{22}S_{k-1}(q) & Q_{12}S_{k-1}(q) \\ Q_{21}S_{k-1}(q) & S_k(q) - Q_{11}S_{k-1}(q) \end{pmatrix}$$

and

$$\det(I + Q + \dots + Q^{k-1}) = \frac{T_k(q) - 2}{q - 2}.$$

§4.2. Double twist links

For integers m and n , consider the double twist link $L = J(2m + 1, 2n + 1)$, which is the 2-bridge link corresponding to the rational number $\frac{2n+1}{4mn+2(m+n)}$ (see Figure 2). By [24, Lemma 3.2] (and [15, Section 2.3] also), the link group of L has a presentation

$$G(L) = \langle a, b \mid ((ab^{-1})^m ab(a^{-1}b)^m)^n (ab^{-1})^m = (b^{-1}a)^m ((ba^{-1})^m ba(b^{-1}a)^m)^n \rangle,$$

where a and b are meridians of L depicted in Figure 2.

By setting $w = (b^{-1}a)^m ((ba^{-1})^m ba(b^{-1}a)^m)^n$, we can rewrite the link group as

$$G(L) = \langle a, b \mid awa^{-1} = w \rangle.$$

Let $\rho : G(L) \rightarrow \text{SL}(2, \mathbb{C})$ be a nonabelian representation and $r = awa^{-1}w^{-1}$. We have

$$\frac{\partial r}{\partial b} = a \left(\frac{\partial w}{\partial b} - wa^{-1}w^{-1} \frac{\partial w}{\partial b} \right) = a(1 - wa^{-1}w^{-1}) \frac{\partial w}{\partial b}.$$

Hence $\Delta_{L,\rho}(t) = \det \Phi \left(\frac{\partial r}{\partial b} \right) / \det \Phi(1 - a) = \det \Phi \left(\frac{\partial w}{\partial b} \right)$.

We consider only the case $m, n \geq 1$. The other cases are similar. Let $u = (ba^{-1})^m ba(b^{-1}a)^m$. Then $w = (b^{-1}a)^m u^n$. We have

$$\begin{aligned} \frac{\partial w}{\partial b} &= (1 + (b^{-1}a) + \dots + (b^{-1}a)^{m-1}) (-b^{-1}) \\ &\quad + (b^{-1}a)^m (1 + u + \dots + u^{n-1}) \frac{\partial u}{\partial b}, \end{aligned}$$

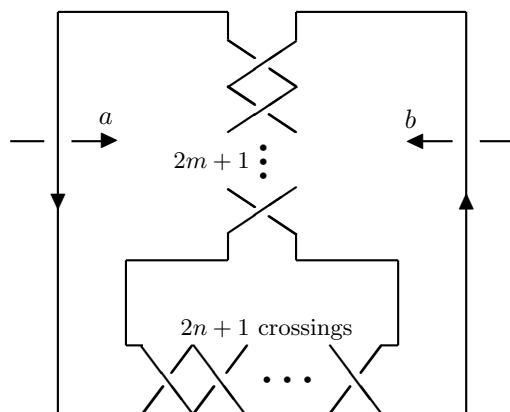


Figure 2. The double twist link $J(2m + 1, 2n + 1)$ with $m, n \geq 1$.

where

$$\begin{aligned} \frac{\partial u}{\partial b} &= \frac{\partial(b(a^{-1}b)^m(ab^{-1})^m a)}{\partial b} \\ &= 1 + b(1 + (a^{-1}b) + \dots + (a^{-1}b)^{m-1})a^{-1} \\ &\quad - b(a^{-1}b)^m(1 + (ab^{-1}) + \dots + (ab^{-1})^{m-1})ab^{-1}. \end{aligned}$$

For $h \in G(L)$ we denote $\rho(h)$ by H . With the orientation of L as in Figure 2, the genus is given by $g(L) = n$ and L is fibered if and only if $m = 1$. (These facts can be proved by computing the reduced Alexander polynomial of L and then applying [23, Theorem 1.1].) Moreover, $\Phi(a) = tA$ and $\Phi(b) = tB$. Then $\Phi(u) = t^2U$. The highest-degree term (in t) of $\Delta_{L,\rho}(t)$ is

$$\begin{aligned} &\det \Phi \left(-(b^{-1}a)^m u^{n-1} b(a^{-1}b)^m (1 + ab^{-1} + \dots + (ab^{-1})^{m-1}) ab^{-1} \right) \\ &= (t^{2n-1})^2 \det \Phi(1 + ab^{-1} + \dots + (ab^{-1})^{m-1}) \\ &= t^{4n-2} \frac{T_m(z) - 2}{z - 2}, \end{aligned}$$

where $z = \text{tr } AB^{-1}$. Here we apply Lemma 4.6 with $Q = AB^{-1}$. Similarly, the lowest-degree term of $\Delta_{L,\rho}(t)$ is $t^{-2} \left(\frac{T_m(z)-2}{z-2} \right)$.

By [24, Theorem 1.1], the nonabelian character variety of $L = J(2m+1, 2n+1)$ is the zero set in \mathbb{C}^3 of the polynomial

$$R(x, y, z) = S_{m-1}(z)S_n(v) - S_m(z)S_{n-1}(v),$$

where $x = \text{tr } A$, $y = \text{tr } B$ and

$$v = \text{tr } U = (xS_m(z) - yS_{m-1}(z))(yS_m(z) - xS_{m-1}(z)) - z(S_m^2(z) + S_{m-1}^2(z)) + 4S_m(z)S_{m-1}(z).$$

§4.3. Genus

By Lemma 4.4, we have $\frac{T_m(z)-2}{z-2} = 0$ if and only if $z = 2 \cos \frac{2j\pi}{m}$ for some $1 \leq j \leq \frac{m}{2}$.

Suppose $z = 2 \cos \frac{2j\pi}{m}$ for some $1 \leq j < \frac{m}{2}$. Then, by Lemma 4.1,

$$S_{m-1}(z) = \frac{\sin m \frac{2j\pi}{m}}{\sin \frac{2j\pi}{m}} = 0 \quad \text{and} \quad S_m(z) = \frac{\sin(m+1) \frac{2j\pi}{m}}{\sin \frac{2j\pi}{m}} = 1.$$

Hence $R(x, y, z) = -S_{n-1}(v)$, where $v = xy - z$. Then, by Lemma 4.3, $R(x, y, z) = 0$ if and only if $v = 2 \cos \frac{k\pi}{n}$ for some $1 \leq k \leq n - 1$.

Suppose $z = -2$ (in this case m must be even). Then $S_{m-1}(z) = (-1)^{m-1}m = -m$ and $S_m(z) = (-1)^m(m+1) = m+1$. Hence

$$R(x, y, z) = -(mS_n(v) + (m+1)S_{n-1}(v)),$$

where $v = xy + 2 + (m^2 + m)(x + y)^2$.

We have shown the following. Let

$$Y_{j,k} = \{x, y, z \in \mathbb{C} \mid z = 2 \cos \frac{2j\pi}{m}, xy - z = 2 \cos \frac{k\pi}{n}\}.$$

Let $v_1 = xy + 2 + (m^2 + m)(x + y)^2$ and

$$Y = \{x, y, z \in \mathbb{C} \mid z = -2, mS_n(v_1) + (m+1)S_{n-1}(v_1) = 0\}.$$

Proposition 4.7. *For the double twist link $L = J(2m + 1, 2n + 1)$ with $m, n \geq 1$ and orientation as in Figure 2,*

(1) *if m is odd, then $\text{deg } \Delta_{L,\chi}(t) = 4n = 4g(L)$ on*

$$\{R(x, y, z) = 0\} \setminus \bigcup_{\substack{1 \leq j < m/2 \\ 1 \leq k \leq n-1}} Y_{j,k};$$

(2) *if m is even, then $\text{deg } \Delta_{L,\chi}(t) = 4n = 4g(L)$ on*

$$\{R(x, y, z) = 0\} \setminus Y \setminus \bigcup_{\substack{1 \leq j < m/2 \\ 1 \leq k \leq n-1}} Y_{j,k}.$$

§4.4. Fiberedness

Here we consider nonfibered links L . Then we have $m > 1$.

Since $T_k(q) - q = T_k(q) - 2 - (q - 2)$, by Lemma 4.5 we have $\frac{T_m(z)-2}{z-2} = 1$ if and only if $z = 2 \cos \frac{2j\pi}{m-1}$ for some $1 \leq j \leq \frac{m-1}{2}$, or $z = 2 \cos \frac{2k\pi}{m+1}$ for some $1 \leq k \leq \frac{m+1}{2}$.

Suppose $z = 2 \cos \frac{2j\pi}{m-1}$ for some $1 \leq j < \frac{m-1}{2}$. Then, by Lemma 4.1,

$$S_{m-1}(z) = \frac{\sin m \frac{2j\pi}{m-1}}{\sin \frac{2j\pi}{m-1}} = 1 \quad \text{and} \quad S_m(z) = \frac{\sin(m+1) \frac{2j\pi}{m-1}}{\sin \frac{2j\pi}{m-1}} = z.$$

Hence $R(x, y, z) = S_n(v) - zS_{n-1}(v)$, where $v = xy - (x^2 + y^2 - 3)z + xyz^2 - z^3$.

Suppose $z = 2 \cos \frac{2j\pi}{m+1}$ for some $1 \leq j < \frac{m+1}{2}$. Then, by Lemma 4.1,

$$S_{m-1}(z) = \frac{\sin m \frac{2j\pi}{m+1}}{\sin \frac{2j\pi}{m+1}} = -1 \quad \text{and} \quad S_m(z) = \frac{\sin(m+1) \frac{2j\pi}{m+1}}{\sin \frac{2j\pi}{m+1}} = 0.$$

Hence $R(x, y, z) = -S_n(v)$, where $v = xy - z$. Then, by Lemma 4.3, $R(x, y, z) = 0$ if and only if $v = 2 \cos \frac{k\pi}{n+1}$ for some $1 \leq k \leq n$.

Suppose $z = -2$ (in this case m must be odd). Then $S_{m-1}(z) = (-1)^{m-1}m = m$ and $S_m(z) = (-1)^m(m+1) = -(m+1)$. Hence

$$R(x, y, z) = mS_m(v) + (m+1)S_{n-1}(v),$$

where $v = xy + 2 + (m^2 + m)(x + y)^2$.

We have shown the following. Let

$$Z_{j,k} = \left\{ x, y, z \in \mathbb{C} \mid z = 2 \cos \frac{2j\pi}{m+1}, xy - z = 2 \cos \frac{k\pi}{n+1} \right\}.$$

Let $v_2 = xy - (x^2 + y^2 - 3)z + xyz^2 - z^3$ and let

$$Z_l = \left\{ x, y, z \in \mathbb{C} \mid z = 2 \cos \frac{2l\pi}{m-1}, S_n(v_2) - zS_{n-1}(v_2) = 0 \right\}.$$

Proposition 4.8. *For the (nonfibered) double twist link $L = J(2m + 1, 2n + 1)$ with $m > 1, n \geq 1$ and orientation as in Figure 2,*

(1) *if m is even, then $\Delta_{L,\chi}(t)$ is nonmonic on*

$$\{R(x, y, z) = 0\} \setminus \bigcup_{\substack{1 \leq j < (m+1)/2 \\ 1 \leq k \leq n}} Z_{j,k} \setminus \bigcup_{1 \leq l < (m-1)/2} Z_l;$$

(2) *if m is odd, then $\Delta_{L,\chi}(t)$ is nonmonic on*

$$\{R(x, y, z) = 0\} \setminus Y \setminus \bigcup_{\substack{1 \leq j < (m+1)/2 \\ 1 \leq k \leq n}} Z_{j,k} \setminus \bigcup_{1 \leq l < (m-1)/2} Z_l.$$

We now consider another orientation of L that is obtained from the one in Figure 2 by changing the orientation of the component corresponding to the meridian b . With this orientation we have $g(L) = m$ and L is fibered if and only if $n = 1$.

Moreover, $\Phi(a) = tA$ and $\Phi(b) = t^{-1}B$. Then $\Phi(u) = U$. The highest-degree term (in t) of $\Delta_{L,\rho}(t)$ is

$$\begin{aligned} \det \Phi \left((b^{-1}a)^m (1 + u + \dots + u^{n-1}) \right) &= (t^{2m})^2 \det \Phi(1 + u + \dots + u^{n-1}) \\ &= t^{4m} \frac{T_n(v) - 2}{v - 2}. \end{aligned}$$

Similarly, the lowest-degree term of $\Delta_{L,\rho}(t)$ is $t^0 \frac{T_n(v) - 2}{v - 2}$.

§4.5. Genus

Recall that

$$\begin{aligned} v &= (xS_m(z) - yS_{m-1}(z))(yS_m(z) - xS_{m-1}(z)) \\ &\quad - z(S_m^2(z) + S_{m-1}^2(z)) + 4S_m(z)S_{m-1}(z). \end{aligned}$$

A similar argument as in Section 4.3 shows the following. Let

$$Y'_{j,k} = \left\{ x, y, z \in \mathbb{C} \mid xy - z = 2 \cos \frac{2j\pi}{n}, z = 2 \cos \frac{k\pi}{m} \right\}$$

and

$$Y' = \left\{ x, y, z \in \mathbb{C} \mid v = -2, nS_m(z) + (n + 1)S_{m-1}(z) = 0 \right\}.$$

Proposition 4.9. *For the double twist link $L = J(2m + 1, 2n + 1)$ with $m, n \geq 1$ and orientation obtained from that in Figure 2 by changing the orientation of the component corresponding to the meridian b ,*

(1) *if n is odd, then $\deg \Delta_{L,\chi}(t) = 4m = 4g(L)$ on*

$$\{R(x, y, z) = 0\} \setminus \bigcup_{\substack{1 \leq j < n/2, \\ 1 \leq k \leq m-1}} Y'_{j,k};$$

(2) *if n is even, then $\deg \Delta_{L,\chi}(t) = 4m = 4g(L)$ on*

$$\{R(x, y, z) = 0\} \setminus Y' \setminus \bigcup_{\substack{1 \leq j < n/2, \\ 1 \leq k \leq m-1}} Y'_{j,k}.$$

§4.6. Fiberedness

As for the fiberedness of $J(2m + 1, 2n + 1)$ with the same orientation as in Proposition 4.9, we have the following. Let

$$Z'_{j,k} = \left\{ x, y, z \in \mathbb{C} \mid xy - z = 2 \cos \frac{2j\pi}{n+1}, z = 2 \cos \frac{k\pi}{m+1} \right\}$$

and

$$Z'_l = \left\{ x, y, z \in \mathbb{C} \mid v = 2 \cos \frac{2l\pi}{n-1}, S_m(z) - vS_{m-1}(z) = 0 \right\}.$$

Proposition 4.10. *For the nonfibered double twist link $L = J(2m + 1, 2n + 1)$ with $m \geq 1, n > 1$ and orientation obtained from that in Figure 2 by changing the orientation of the component corresponding to the meridian b ,*

(1) *if n is even, then $\Delta_{L,\chi}(t)$ is nonmonic on*

$$\{R(x, y, z) = 0\} \setminus \bigcup_{\substack{1 \leq j < (n+1)/2 \\ 1 \leq k \leq m}} Z'_{j,k} \setminus \bigcup_{1 \leq l < (n-1)/2} Z'_l;$$

(2) *if m is odd, then $\Delta_{L,\chi}(t)$ is nonmonic on*

$$\{R(x, y, z) = 0\} \setminus Y' \setminus \bigcup_{\substack{1 \leq j < (n+1)/2 \\ 1 \leq k \leq m}} Z'_{j,k} \setminus \bigcup_{1 \leq l < (n-1)/2} Z'_l.$$

The remaining two orientations of L are similar. They are just mirror images of the previous two. In general, there are 2^μ ways to orient a μ -component link. Moreover, up to taking mirror images, there are only $2^{\mu-1}$ orientations to consider.

Remark 4.11. Similar results to Propositions 4.7–4.10 can be stated for all double twist links $J(2m + 1, 2n + 1)$, where m, n are integers. However, for simplicity, we choose to present only the case that $m, n \geq 1$.

§4.7. Main theorem

Note that $L = J(2m + 1, 2n + 1)$ is hyperbolic if and only if $m, n \notin \{-1, 0\}$. Since the double twist link L is alternating, the notions of the genus and the Thurston norm of L are equivalent (see Remark 3.4).

We are now ready to show our main theorem of this paper.

Theorem 4.12. *For the (hyperbolic) double twist link $L = J(2m + 1, 2n + 1)$ with $m, n \notin \{-1, 0\}$ and with any orientation, the twisted Alexander polynomial $\Delta_{L,\rho_0}(t)$ determines the genus $g(L)$ (or equivalently the Thurston norm). Moreover, L is fibered if and only if $\Delta_{L,\rho_0}(t)$ is monic.*

Note that the character of our lift $\rho_0 : G(L) \rightarrow \text{SL}(2, \mathbb{C})$ has the form $\chi_{\rho_0} = (2, 2, z_0)$ for some $z_0 \notin \mathbb{R}$ satisfying $R(2, 2, z_0) = 0$.

As mentioned in Remark 4.11, we consider only the case $m, n \geq 1$. The other cases are similar. To prove Theorem 4.12, by Propositions 4.7–4.10, it suffices to show that neither $Y, Y', Y_{j,k}, Y'_{j,k}, Z_l, Z'_l, Z_{j,k}$ nor $Z'_{j,k}$ contains characters of the form $(2, 2, z)$ with $z \notin \mathbb{R}$. Except for the cases of Y' and Z'_l , the others are clear.

We now show these two cases.

Lemma 4.13. *If $(x, y, z) \in Z'_l$ then $v = 2 \cos \frac{2l\pi}{n-1} \in \mathbb{R}$ and*

$$z = -v^3 + v^2xy - v(x^2 + y^2 - 3) + xy.$$

Hence, Z'_l does not contain characters of the form $(2, 2, z)$ with $z \notin \mathbb{R}$.

Proof. Since $(x, y, z) \in Z'_l$, we have $S_m(z) = vS_{m-1}(z)$. Combining this with the equality $S_m^2(z) + S_{m-1}^2(z) - zS_m(z)S_{m-1}(z) = 1$ in Lemma 4.2, we get

$$S_{m-1}^2(v) = \frac{1}{v^2 + 1 - zv}.$$

Hence

$$\begin{aligned} v &= (xS_m(z) - yS_{m-1}(z))(yS_m(z) - xS_{m-1}(z)) \\ &\quad - z(S_m^2(z) + S_{m-1}^2(z)) + 4S_m(z)S_{m-1}(z) \\ &= (4v - vx^2 + xy + v^2xy - vy^2 - z - v^2z)S_{m-1}^2(v) \\ &= \frac{4v - vx^2 + xy + v^2xy - vy^2 - z - v^2z}{v^2 - zv + 1}. \end{aligned}$$

By solving for z (in terms of v, x, y), we obtain the desired formula. □

Lemma 4.14. *If $(x, y, z) \in Y'$ then*

$$z = (n^2 + n)(x + y)^2 + xy + 2.$$

Hence, Y' does not contain characters of the form $(2, 2, z)$ with $z \notin \mathbb{R}$.

Proof. The proof of Lemma 4.14 is similar to that of Lemma 4.13. □

In view of Theorem 4.12 and its proof, we may propose the following problem.

Problem 4.15. *For an oriented hyperbolic 2-bridge link L and its parabolic representation $\rho : G(L) \rightarrow \text{SL}(2, \mathbb{C})$, does the twisted Alexander polynomial $\Delta_{L,\rho}(t)$ determine the genus $g(L)$ and fiberedness of L ?*

If Problem 4.15 has an affirmative answer, then Conjecture 3.3 holds true for all hyperbolic 2-bridge links. However, as we will see in the next section, a part of Problem 4.15 has a negative answer.

§5. Parabolic representations

Let us recall that a nonabelian representation $\rho : G(L) \rightarrow \text{SL}(2, \mathbb{C})$ is parabolic if the images of all the meridians of L by ρ are matrices with trace 2.

For positive integers m, n and p , consider the 2-bridge link $C(2m, 2n, -2p)$ in the Conway notation (see Figure 3), where positive numbers $2m, 2n$ correspond to right-handed twists and the negative number $-2p$ corresponds to left-handed twists. It is the rational link corresponding to the continued fraction

$$[2m, 2n, -2p] = 2m + \frac{1}{2n - \frac{1}{2p}}$$

and is the 2-bridge link $(2m(4np - 1) + 2p, 4np - 1)$ in Schubert notation. Note that $C(2m, 2n, -2p)$ is a hyperbolic link.

Theorem 5.1. *For the 2-bridge link $L = C(2m, 2n, -2p)$, with m, n, p being odd positive integers and $m \neq p$, then the twisted Alexander polynomial $\Delta_{L, \rho_0}(t)$ detects the genus of L . Moreover,*

- (1) if $\gcd(m, p) = 1$, then all parabolic representations detect $g(L)$;
- (2) if $\gcd(m, p) \geq 3$, then not all parabolic representations detect $g(L)$.

Remark 5.2. We do not know whether there is a parabolic representation of a 2-bridge knot K that does not detect the genus of K (see [20, Theorem 1.2]).

§5.1. Chebyshev polynomials (2)

In this subsection we prepare two lemmas for the Chebyshev polynomials.

Lemma 5.3. *For integers k, l we have*

$$(5.1) \quad S_k(q)S_{l-1}(q) - S_{k-1}(q)S_l(q) = S_{l-k-1}(q).$$

Proof. It suffices to show (5.1) for $q \neq \pm 2$. When $q \neq \pm 2$, we write $q = v + v^{-1}$ for some $v \neq \pm 1$. By Lemma 4.1, the LHS of (5.1) is equal to

$$\begin{aligned} & \frac{v^{k+1} - v^{-(k+1)}}{v - v^{-1}} \cdot \frac{v^l - v^{-l}}{v - v^{-1}} - \frac{v^k - v^{-k}}{v - v^{-1}} \cdot \frac{v^{l+1} - v^{-(l+1)}}{v - v^{-1}} \\ &= \frac{-(v^{k+1-l} + v^{l-(k+1)}) + (v^{k-(l+1)} + v^{l+1-k})}{(v - v^{-1})^2} \\ &= \frac{v^{l-k} - v^{k-l}}{v - v^{-1}}, \end{aligned}$$

which is also equal to the RHS of (5.1). □

Lemma 5.4. *For integers k, l we have*

$$\gcd(S_{k-1}(q), S_{l-1}(q)) = S_{\gcd(k,l)-1}(q).$$

Proof. The lemma follows from the fact that

$$\gcd(v^k - v^{-k}, v^l - v^{-l}) = v^{\gcd(k,l)} - v^{-\gcd(k,l)}$$

and Lemma 4.1. □

§5.2. Two bridge links $C(2m, 2n, -2p)$

In this subsection we prove Theorem 5.1. To this end we first give a presentation of the link group of $C(2m, 2n, -2p)$.

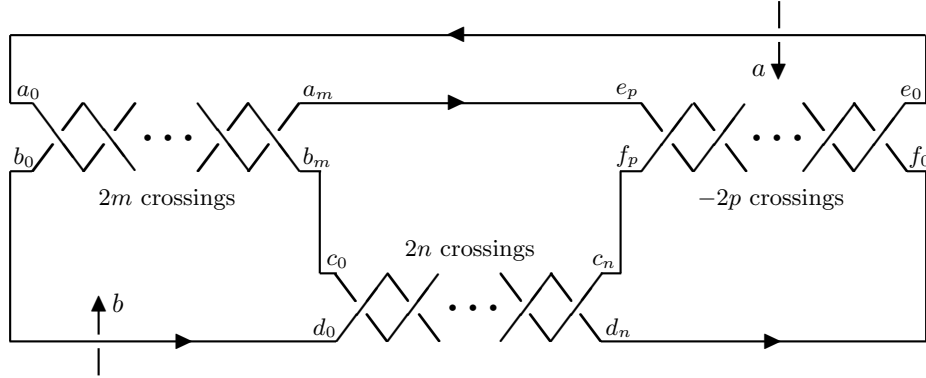


Figure 3. The 2-bridge link $L = C(2m, 2n, -2p)$ with $m, n, p \geq 1$ and the generators of $G(L)$.

Proposition 5.5. *The link group of $L = C(2m, 2n, -2p)$ has a presentation*

$$G(L) = \langle a, b \mid aw = wa \rangle,$$

where

$$w = (b^{-1}a)^m \left[a^{-1} \left((a^{-1}b)^m (ab^{-1})^m \right)^{-n} b \left((a^{-1}b)^m (ab^{-1})^m \right)^n \right]^p$$

and a and b are meridians of L depicted in Figure 3.

Proof. By applying the Wirtinger algorithm to the leftmost twist region in Figure 3 and by induction we have

$$a_m = (a_0^{-1}b_0)^m a_0 (b_0^{-1}a_0)^m, \quad b_m = (a_0^{-1}b_0)^m b_0 (b_0^{-1}a_0)^m.$$

Similarly, the middle and rightmost twist regions give

$$\begin{aligned} c_n &= (d_0 c_0^{-1})^n c_0 (c_0 d_0^{-1})^n, & d_n &= (d_0 c_0^{-1})^n d_0 (c_0 d_0^{-1})^n, \\ e_p &= (e_0^{-1} f_0)^p e_0 (f_0^{-1} e_0)^p, & f_p &= (e_0^{-1} f_0)^p f_0 (f_0^{-1} e_0)^p. \end{aligned}$$

We have $G(L) = \langle a, b \mid a_m = e_p \rangle$. Since $e_0 = a$ and $f_0 = d_n$, the relation $a_m = e_p$ is equivalent to $aw = wa$, where

$$w = (b^{-1}a)^m (a^{-1}d_n)^p.$$

Finally, since $d_n = (bb_m^{-1})^n b (b_m b^{-1})^n$ and $b_m = (a^{-1}b)^m b (b^{-1}a)^m$ we obtain the desired presentation of $G(L)$. \square

Let $\rho : G(L) \rightarrow \text{SL}(2, \mathbb{C})$ be a representation and $r = awa^{-1}w^{-1}$. We have

$$\frac{\partial r}{\partial b} = a \left(\frac{\partial w}{\partial b} - wa^{-1}w^{-1} \frac{\partial w}{\partial b} \right) = a(1 - wa^{-1}w^{-1}) \frac{\partial w}{\partial b}.$$

Hence $\Delta_{L,\rho}(t) = \det \Phi \left(\frac{\partial r}{\partial b} \right) / \det \Phi(1 - a) = \det \Phi \left(\frac{\partial w}{\partial b} \right)$.

For $k \geq 0$ and $h \in G(L)$, let $\delta_k(h) = 1 + h + \dots + h^k$. Let $v = (a^{-1}b)^m(ab^{-1})^m$ and $u = a^{-1}v^{-n}bv^n$. Then $w = (b^{-1}a)^m u^p$. We have

$$\begin{aligned} \frac{\partial w}{\partial b} &= \delta_{m-1}(b^{-1}a)(-b^{-1}) + (b^{-1}a)^m \delta_{p-1}(u) \frac{\partial u}{\partial b}, \\ \frac{\partial u}{\partial b} &= \frac{\partial(a^{-1}v^{-n}bv^n)}{\partial b} \\ &= a^{-1} \delta_{n-1}(v^{-1}) \frac{\partial v^{-1}}{\partial b} + a^{-1}v^{-n} \left(1 + b \delta_{n-1}(v) \frac{\partial v}{\partial b} \right) \\ &= a^{-1}v^{-n} \left(1 + (b-1) \delta_{n-1}(v) \frac{\partial v}{\partial b} \right), \\ \frac{\partial v}{\partial b} &= \frac{\partial(a^{-1}b)^m(ab^{-1})^m}{\partial b} \\ &= \delta_{m-1}(a^{-1}b)a^{-1} + (a^{-1}b)^m \delta_{m-1}(ab^{-1})(-ab^{-1}). \end{aligned}$$

For $h \in G(L)$, we denote $\rho(h)$ by the capital letter H for simplicity. With the orientation of $L = C(2m, 2n, -2p)$ as in Figure 3, the genus is given by $g(L) = 1$. (This fact can be proved by applying Seifert’s algorithm to the reduced alternating diagram of L corresponding to the continued fraction $[2m, 2n - 1, 1, 2p - 1]$.) Moreover, $\Phi(a) = tA$ and $\Phi(b) = tB$. Then $\Phi(v) = V$ and $\Phi(u) = U$.

The highest-degree term (in t) of $\Delta_{L,\rho}(t)$ is

$$\begin{aligned} &t^0 \det \rho \left((b^{-1}a)^m \delta_{p-1}(u) \times a^{-1}v^{-n}b \delta_{n-1}(v) \times (a^{-1}b)^m \delta_{m-1}(ab^{-1})(-ab^{-1}) \right) \\ &= t^0 \frac{T_p(\bar{u}) - 2}{\bar{u} - 2} \cdot \frac{T_n(\bar{v}) - 2}{\bar{v} - 2} \cdot \frac{T_m(z) - 2}{z - 2} \end{aligned}$$

by Lemma 4.6. Similarly, the lowest one is $t^{-4} \frac{T_p(\bar{u})-2}{\bar{u}-2} \cdot \frac{T_n(\bar{v})-2}{\bar{v}-2} \cdot \frac{T_m(z)-2}{z-2}$.

Consider a parabolic representation $\rho : G(L) \rightarrow \text{SL}(2, \mathbb{C})$ given by

$$A = \rho(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \rho(b) = \begin{pmatrix} 1 & 0 \\ 2-z & 1 \end{pmatrix},$$

where z satisfies the matrix equation $AW = WA$. Here $W = \rho(w)$. Note that $z = \text{tr } AB^{-1}$ holds.

Let $W_{ij} \in \mathbb{C}[z]$ denote the (i, j) -entry of W . Riley showed in [25] that $W_{21} = (2 - z)W'_{21}$ for some $W'_{21} \in \mathbb{C}[z]$ and that the matrix equation $AW = WA$ is equivalent to a single equation $W'_{21} = 0$. We call W'_{21} the *Riley polynomial* of L .

We now compute W'_{21} for $L = C(2m, 2n, -2p)$. Since

$$A^{-1}B = \begin{pmatrix} z-1 & -1 \\ 2-z & 1 \end{pmatrix} \quad \text{and} \quad AB^{-1} = \begin{pmatrix} z-1 & 1 \\ z-2 & 1 \end{pmatrix},$$

by Lemma 4.6 we have

$$\begin{aligned} V &= (A^{-1}B)^m (AB^{-1})^m = \begin{pmatrix} S_m(z) - S_{m-1}(z) & -S_{m-1}(z) \\ (2-z)S_{m-1}(z) & S_m(z) - (z-1)S_{m-1}(z) \end{pmatrix} \\ &\quad \times \begin{pmatrix} S_m(z) - S_{m-1}(z) & S_{m-1}(z) \\ (z-2)S_{m-1}(z) & S_m(z) - (z-1)S_{m-1}(z) \end{pmatrix} \\ &= \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} V_{11} &= S_m^2(z) - 2S_m(z)S_{m-1}(z) + (3-z)S_{m-1}^2(z), \\ V_{12} &= (z-2)S_{m-1}^2(z), \\ V_{21} &= -(z-2)^2S_{m-1}^2(z), \\ V_{22} &= S_m^2(z) + (2-2z)S_m(z)S_{m-1}(z) + (3-3z+z^2)S_{m-1}^2(z). \end{aligned}$$

Then

$$V^n = \begin{pmatrix} S_n(\bar{v}) - V_{22}S_{n-1}(\bar{v}) & V_{12}S_{n-1}(\bar{v}) \\ V_{21}S_{n-1}(\bar{v}) & S_n(\bar{v}) - V_{11}S_{n-1}(\bar{v}) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where $\bar{v} = \text{tr } V$. Hence

$$\begin{aligned} U &= \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = A^{-1}V^{-n}BV^n \\ &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2-z & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ &= \begin{pmatrix} 1+(z-2)\alpha(\alpha+\beta) & -1+(z-2)\beta(\alpha+\beta) \\ -(z-2)\alpha^2 & 1-(z-2)\alpha\beta \end{pmatrix}. \end{aligned}$$

Since $W = (B^{-1}A)^m U^p$, we have

$$\begin{aligned} W &= \begin{pmatrix} S_m(z) - (z-1)S_{m-1}(z) & S_{m-1}(z) \\ (z-2)S_{m-1}(z) & S_m(z) - S_{m-1}(z) \end{pmatrix} \\ &\quad \times \begin{pmatrix} S_p(\bar{u}) - U_{22}S_{p-1}(\bar{u}) & U_{12}S_{p-1}(\bar{u}) \\ U_{21}S_{p-1}(\bar{u}) & S_p(\bar{u}) - U_{11}S_{p-1}(\bar{u}) \end{pmatrix}, \end{aligned}$$

where $\bar{u} = \text{tr } U$. This implies that

$$W_{21} = U_{21} (S_m(z) - S_{m-1}(z)) S_{p-1}(\bar{u}) + (z - 2)S_{m-1}(z) (S_p(\bar{u}) - U_{22}S_{p-1}(\bar{u})).$$

Since $U_{22} = 1 - (z - 2)\alpha\beta$ and $U_{21} = -(z - 2)\alpha^2$ we have

$$W_{21} = (2 - z)W'_{21},$$

where the Riley polynomial of L is given by

$$W'_{21} = \alpha^2(S_m(z) - S_{m-1}(z))S_{p-1}(\bar{u}) - S_{m-1}(z)(S_p(\bar{u}) + ((z - 2)\alpha\beta - 1)S_{p-1}(\bar{u})).$$

Since the holonomy representation $\rho_0 : G(L) \rightarrow \text{SL}(2, \mathbb{C})$ is one of the parabolic representations, it has the form

$$\rho_0(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho_0(b) = \begin{pmatrix} 1 & 0 \\ 2-z_0 & 1 \end{pmatrix}$$

for some $z_0 \notin \mathbb{R}$ satisfying $W'_{21}(z_0) = 0$.

To prove Theorem 5.1, we determine all complex numbers z satisfying both $W'_{21}(z) = 0$ and

$$\frac{T_p(\bar{u}) - 2}{\bar{u} - 2} \cdot \frac{T_n(\bar{v}) - 2}{\bar{v} - 2} \cdot \frac{T_m(z) - 2}{z - 2} = 0.$$

We consider the following three cases.

Case 1. Suppose $\frac{T_m(z)-2}{z-2} = 0$. Since m is odd, by Lemma 4.4 we have $z = 2 \cos \frac{2j\pi}{m}$ for some $1 \leq j \leq \frac{m-1}{2}$. Then, by Lemma 4.1,

$$S_{m-1}(z) = \frac{\sin m \frac{2j\pi}{m}}{\sin \frac{2j\pi}{m}} = 0 \quad \text{and} \quad S_m(z) = \frac{\sin(m+1) \frac{2j\pi}{m}}{\sin \frac{2j\pi}{m}} = 1.$$

It is easy to see that $V = I$ and

$$U = A^{-1}V^{-n}BV^n = A^{-1}B.$$

This implies that $W'_{21} = S_{p-1}(z)$.

By Lemma 5.4 we have $S_{\text{gcd}(m,p)-1}(z) = \text{gcd}(S_{m-1}(z), S_{p-1}(z))$.

If $\text{gcd}(m, p) = 1$, then $\text{gcd}(S_{m-1}(z), S_{p-1}(z)) = 1$. Since $S_{m-1}(z) = 0$, we have $W'_{21} = S_{p-1}(z) \neq 0$.

If $\text{gcd}(m, p) \geq 3$, we can choose $z' \in \mathbb{R}$ such that both $\frac{T_m(z')-2}{z'-2}$ and $S_{p-1}(z')$ are zero. Indeed, choose some $1 \leq j \leq \frac{1}{2} \text{gcd}(m, p)$ and take $z' = 2 \cos \frac{2j\pi}{\text{gcd}(m,p)}$. Then the parabolic representation ρ' corresponding to the root z' of W'_{21} does not detect the genus of L .

Case 2. Suppose $\frac{T_n(\bar{v})-2}{\bar{v}-2} = 0$. Since n is odd, we have $S_{n-1}(\bar{v}) = 0$ and $S_n(\bar{v}) = 1$. Then

$$V^n = \begin{pmatrix} S_n(\bar{v})-V_{22}S_{n-1}(\bar{v}) & V_{12}S_{n-1}(\bar{v}) \\ V_{21}S_{n-1}(\bar{v}) & S_n(\bar{v})-V_{11}S_{n-1}(\bar{v}) \end{pmatrix} = I$$

and $U = A^{-1}V^{-n}BV^n = A^{-1}B$. Hence, by Lemma 4.1 we have

$$W'_{21} = (S_m(z) - S_{m-1}(z))S_{p-1}(z) - S_{m-1}(z)(S_p(z) - S_{p-1}(z)) = S_{p-m-1}(z).$$

If $S_{p-m-1}(z) = 0$, then since $m \neq p$ we have $z \in \mathbb{R}$ (strictly between -2 and 2) by Lemma 4.3. By a direct calculation and Lemma 4.2 we have

$$\begin{aligned} \bar{v} = \text{tr } V &= 2\left(S_m^2(z) + S_{m-1}^2(z) - zS_m(z)S_{m-1}(z)\right) + (z-2)^2S_{m-1}^2(z) \\ &= 2 + (z-2)^2S_{m-1}^2(z), \end{aligned}$$

which is a real number greater than or equal to 2 . This contradicts $S_{n-1}(\bar{v}) = 0$.

Case 3. Suppose $\frac{T_p(\bar{u})-2}{\bar{u}-2} = 0$. Since p is odd, we have $S_{p-1}(\bar{u}) = 0$ and $S_p(\bar{u}) = 1$. Then $W'_{21} = -S_{m-1}(z)$.

If $S_{m-1}(z) = 0$, then $S_m^2(z) = 1$ by Lemma 4.2. It is easy to see that $V = I$ and

$$U = A^{-1}V^{-n}BV^n = A^{-1}B.$$

This implies that $\bar{u} = z$. Hence $S_{p-1}(z) = S_{p-1}(\bar{u}) = 0$. It cannot occur that both $S_{m-1}(z) = 0$ and $S_{p-1}(z) = 0$ if $\text{gcd}(m, p) = 1$.

From the above discussion we have proved Theorem 5.1 for the orientation in Figure 3.

We now consider another orientation of L that is obtained from the one in Figure 3 by changing the orientation of the component corresponding to the meridian a . With this orientation we have $g(L) = m + p - 1$. Moreover, $\Phi(a) = t^{-1}A$ and $\Phi(b) = tB$. Then $\Phi(v) = V$ and $\Phi(u) = t^2U$.

In this case the highest-degree term (in t) of $\Delta_{L,\rho}(t)$ is

$$\begin{aligned} &t^{4p-2} \det \left((b^{-1}a)^m u^{p-1} \times a^{-1}v^{-n} b \delta_{n-1}(v) \times \delta_{m-1}(a^{-1}b) a^{-1} \right) \\ &= t^{4p-2} \frac{T_{n-1}(\bar{v}) - 2}{\bar{v} - 2} \end{aligned}$$

and the lowest one is $t^{2-4m} \frac{T_{n-1}(\bar{v})-2}{\bar{v}-2}$. A similar argument to Case 2 above shows that all parabolic representations detect the genus of L .

This completes the proof of Theorem 5.1.

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