Publ. RIMS Kyoto Univ. **53** (2017), 419–429
DOI 10.4171/PRIMS/53-3-4

Paradoxical Partition of Unity for Hypergroups

by

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Abstract

The paradoxical partition of unity of a discrete hypergroup is defined. It is shown that a discrete hypergroup is amenable if and only if it admits no paradoxical partition of unity. We introduce the Tarski number for discrete hypergroups and present a constructive way to compute an upper bound for this number.

2010 Mathematics Subject Classification: Primary 43A62; Secondary 43A07.
Keywords: Hypergroup, paradoxical partition of unity, amenability.

§1. Introduction

Beginning in the last century, many studies have been carried out on the amenability of discrete and nondiscrete groups; see for example [G] , [Pa] and [Pi]. We would like to point out the Tarski alternative. This alternative specifies that an arbitrary group is either amenable or paradoxical [C]. There are also other theorems that characterize the amenability of groups. Among them is a theorem proved by Rosenblatt and Willis in 2001 [RW]. In that paper, the authors defined the configuration equations of groups and showed that a group G is amenable if and only if every system of configuration equations associated to G has normalized solutions.

Locally compact hypergroups as generalizations of locally compact topological groups were introduced in 1973 by Dunkl [D] and then studied by Jewett [J] and Spector [Sp]. This concept has been of interest to many authors ever since (see [A], [BH], [HK] and [V]). The amenability of hypergroups is an interesting area of research as well (see [Sk] and [W]). In this paper introducing paradoxical partitions of unity, we state and prove an analogy of the Tarski alternative for hypergroups.

§2. Preliminaries

We start this section with a few definitions and some known theorems.

Communicated by N. Ozawa. Received June 24, 2016. Revised September 28, 2016.

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Definition 2.1. Consider a nonempty subset C of $\mathbb{R}^n$. Then C is a cone in $\mathbb{R}^n$ if for any vector $y \in C$ and any $k > 0$ we have $ky \in C$. The cone C is called pointed if $C \cap (-C) = \{0\}$.

In what follows, C^* will denote the polar cone of an arbitrary cone C in $\mathbb{R}^n$; that is,

$$C^* = \{y^* \in \mathbb{R}^n;\ y^*y \geq 0 \text{ for all } y \in C\}.$$

Then the interior of C^* is given by

$$\operatorname{int} C^* = \{y \in C^* : 0 \neq x \in C \Rightarrow xy > 0\}.$$

The following theorem is the well-known Gordan theorem, described over cone domains.

Theorem 2.2. *Let M be a nonzero $m \times n$ matrix and let C be a cone in $\mathbb{R}^n$ such that it is closed, convex and pointed. Then one and only one of the following statements is consistent, where M' stands for the transpose of the matrix M:*

(1) $Mx = 0$ for some $x \in C$, $x \neq 0$;

(2) $M'y \in \operatorname{int}(-C^*)$ for some $y \in \mathbb{R}^m$.

Proof. See [SS, Lemma 2]. □

The concept of configuration is defined in [RW]. In that paper, the authors use this notion to give an equivalence condition for the amenability of groups. A conclusion of that paper is to construct a paradoxical decomposition for nonamenable discrete groups (see [Y]). The definition of configuration and the related topics for hypergroups was given in 2014 in [W]. We recall some definitions and basic theorem directly from [W] on an arbitrary locally compact hypergroup, but we shall not be involved with nondiscrete hypergroups throughout this paper.

Definition 2.3 ([BH]). A hypergroup is a locally compact space H with the following conditions:

(1) There exists an associative binary operation $*$ called convolution on $M(H)$ under which $M(H)$ is an algebra. Moreover, for every x, y in H, $\delta_x * \delta_y$ is a probability measure with compact support.
(2) The mapping $(x, y) \mapsto \delta_x * \delta_y$ is a continuous map from $H \times H$ into $M(H)$ equipped with the weak* topology.
(3) The mapping $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ is a continuous mapping from $H \times H$ into the compact subsets of H equipped with the Michael topology.
(4) There exists a unique element e in H such that $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all x in H.

(5) There exists a homeomorphism $x \mapsto \check{x}$ of H called involution satisfying $\check{\check{x}} = x$ for all $x \in H$ and $(\delta_x * \delta_y)\check{} = \delta_{\check{y}} * \delta_{\check{x}}$ for all $x, y \in H$, where $\check{\mu}(A) = \mu(\check{A})$, for any Borel subset A.

(6) The element e belongs to $\operatorname{supp}(\delta_x * \delta_y)$ if and only if $y = \check{x}$.

Let f be a Borel function on H and $\mu \in M(H)$. The left translation $\mu * f$ is defined by $\mu * f(x) = (\check{\mu} * \delta_x)(f)$. We say that H is amenable if there exists a positive linear functional of norm 1 on $C_b(H)$ that is invariant under left translation. For each $f \in C_b(H)$ and $x, y \in H$, we have $\delta_x * f(y) = \delta_{\check{x}} * \delta_y(f)$. A Borel measure λ on H is called a (left) Haar measure if $\lambda(\delta_x * f) = \lambda(f)$ for all $f \in C_b(H)$ and $x \in H$.

Definition 2.4. Let H be a hypergroup with left Haar measure λ. Let $E = \{E_1, \dots, E_m\}$ be a finite measurable partition of H and choose an n-tuple of elements of H, $h = \{h_1, \dots, h_n\}$. A configuration is an $(n+1)$-tuple $C = (C_0, C_1, \dots, C_n)$ where each $C_j \in \{1, \dots, m\}$.

For a fixed configuration C, we define $\xi_0(C)$ to be the real-valued function on H given by

$$\xi_0(C)(x) := \prod_{j=0}^{n} \delta_{h_j} * \delta_x(E_{C_j})$$

using the convention that $h_0 = e$. An alternative expression for $\xi_0(C)$ is

$$\xi_0(C) = \prod_{j=0}^{n} \delta_{\check{h}_j} * \chi_{E_{C_j}}.$$

From this we see that $\xi_0(C)$ is the pointwise product of finitely many nonnegative measurable functions bounded by 1 and so is itself in $L_\infty(H)^+$ and has norm bounded by 1.

Definition 2.5. Fix $\mathcal{E}$ and $\mathfrak{h}$ as before. Let $\{z_C : C \in \operatorname{Con}(\mathfrak{h}, \mathcal{E})\}$ be variables corresponding to the m^{n+1} configurations. Consider the $m \times n$ configuration equations

$$\sum_{\substack{C \in \operatorname{Con}(\mathfrak{h},\mathcal{E}),\\ C_0 = i}} z_C = \sum_{\substack{C \in \operatorname{Con}(\mathfrak{h},\mathcal{E}),\\ C_j = i}} z_C$$

for each $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. We say that a solution to these configuration equations is *positive* if, for each $C \in \operatorname{Con}(\mathfrak{h}, \mathcal{E})$, we have $z_C \geq 0$; *normalized* if $\sum_{C \in \operatorname{Con}(\mathfrak{h},\mathcal{E})} z_C = 1$; and *inequality preserving* if for every choice of m^{n+1} real numbers $\{\alpha_C : C \in \operatorname{Con}(\mathfrak{h}, \mathcal{E})\}$,

$$0 \leq \sum_{C \in \operatorname{Con}(\mathfrak{h},\mathcal{E})} \alpha_C \xi_0(C) \text{ a.e. } \Rightarrow\ 0 \leq \sum_{C \in \operatorname{Con}(\mathfrak{h},\mathcal{E})} \alpha_C\, z_C.$$

It is clear that if a system of configuration equations admits a nonnegative nonzero inequality-preserving solution, then it admits a positive normalized inequality-preserving solution as well.

Remark 2.6. It is shown in [W] that

$$\sum_{\substack{C\in \mathrm{Con}(\mathfrak{h},\mathcal{E}),\\ C_0=i}} \xi_0(C)=\chi_{E_i} \quad \text{and} \quad \sum_{\substack{C\in \mathrm{Con}(\mathfrak{h},\mathcal{E}),\\ C_j=i}} \xi_0(C)=\delta_{\check{h}_j}*\chi_{E_i}.$$

Theorem 2.7 ([W]). *Let H be a hypergroup with left Haar measure λ. Then H is amenable if and only if for all choices of $m, n, \mathfrak{h}$ and $\mathcal{E}$ the $m\times n$ configuration equations have a positive normalized inequality-preserving solution.*

Definition 2.8. A group G is paradoxical if it admits a paradoxical decomposition, that is, if there exist disjoint subsets $P_1, P_2, \ldots, P_m, Q_1, Q_2, \ldots, Q_n$ of G and elements $g_1, g_2, \ldots, g_m, h_1, h_2, \ldots, h_n$ of G such that

$$G=\bigcup_{i=1}^{m} g_iP_i=\bigcup_{j=1}^{n} h_jQ_j.$$

The minimal value of $m+n$ for possible paradoxical decompositions of G is called the Tarski number of G. For the case where we replace the group G with an arbitrary hypergroup H, this definition does not exactly work. Note that in the group case, one can interpret the paradoxical decomposition as

$$\begin{aligned}
1&=\sum_{i=1}^{m}\chi_{P_i}+\sum_{j=1}^{m}\chi_{Q_j},\\
1&=\sum_{i=1}^{m}\delta_{g_i^{-1}}*\chi_{P_i},\\
1&=\sum_{j=1}^{n}\delta_{h_j^{-1}}*\chi_{Q_j},
\end{aligned}$$

where χ_E denotes the characteristic function of the set E and $\delta_{x^{-1}}*\chi_E=\chi_{xE}$ is the left translation of function χ_E by the element $x\in G$. But in the hypergroup case, $\delta_{\check{x}}*\chi_E$ is not equal to χ_{xE}. Indeed, $\delta_{\check{x}}*\chi_E$ has values in $[0,1]$ instead of $\{0,1\}$. So a new definition seems to be needed. The terminology used in this paper focuses more on the sentence $1=\sum_{i=1}^{m}\chi_{P_i}+\sum_{j=1}^{m}\chi_{Q_j}$ and then we call it a paradoxical partition of unity.

§3. Paradoxical partition of unity

As described in the introduction, paradoxical decompositions for hypergroups cannot be easily defined. We define the concept of a paradoxical partition of unity, which is, to some extent, a suitable substitute.

Definition 3.1. Suppose that H is a discrete hypergroup. Let

$$\mathcal{F} = \{f_1, \dots, f_n, g_0\}$$

be a finite family of nonnegative real-valued bounded functions on H and $\mathfrak{h} = \{h_1, \dots, h_n, h'_1, \dots, h'_n\}$ be a subset of H. We say that $(\mathcal{F}, \mathfrak{h})$ is a paradoxical partition of unity of H if

(1) $\sum_{i=1}^n f_i = 1$;

(2) $\sum_{i=1}^n \delta_{h_i} * f_i = g_0 + \sum_{i=1}^n \delta_{h'_i} * f_i$;

(3) $M(g_0) > 0$ for every nonzero bounded positive linear functional M on $C(H)$.

Definition 3.2. The number n in the definition of a paradoxical partition of unity of a hypergroup is called the h-Tarski number of that decomposition. The least such number is called the h-Tarski number of the hypergroup H and is denoted by $\theta(H)$.

By the first two conditions of Definition 3.1 it is clear that the h-Tarski number of hypergroups is at least 2.

Theorem 3.3. *The hypergroup H is amenable if and only if H admits no paradoxical partition of unity.*

Proof. If $\mathcal{M}$ is a left-invariant mean on $C(H)$, and the above definition satisfied, then

$$\mathcal{M}\biggl(\sum_{i=1}^n \delta_{h_i} * f_i\biggr) = \mathcal{M}\biggl(\sum_{i=1}^n \delta_{h'_i} * f_i + g_0\biggr).$$

So

$$\mathcal{M}\biggl(\sum_{i=1}^n f_i\biggr) = \mathcal{M}\biggl(\sum_{i=1}^n f_i\biggr) + \mathcal{M}(g_0),$$

thus $M(g_0) = 0$, which is impossible.

Now let H be not amenable. Then by Theorem 2.7 there exists a system of configuration equations with no nonnegative nonzero inequality-preserving solution. Write this system as $AX = 0$, where the rows of A are the coefficient vectors of equations $\sum_{C_0=i} z_C = \sum_{C_j=i} z_C$, $1 \le i \le m$, $1 \le j \le n$ and X is the vector of variables. Let $\mathcal{P}$ be the set

$$\mathcal{P} = \{(z_C)_{C\in\mathrm{Con}(\mathfrak{h},\mathcal{E})};\ z_C \ge 0 \text{ and } (z_C) \text{ is inequality preserving}\}.$$

It is easily seen that $\mathcal{P}$ is a pointed closed convex cone in $\mathbb{R}^{m^{n+1}}$. By assumption, the homogenous system $AX = 0$ has no nonzero solution in $\mathcal{P}$. Now by Theorem 2.2 there exists a vector $y_1 \in \mathbb{R}^{mn}$ such that $A'y_1 \in \operatorname{int}(-\mathcal{P}^*)$ or equivalently, there exists (a nonzero vector) $y \in \mathbb{R}^{mn}$ such that for each nonzero $(z_C)_C \in \mathcal{P}$, we have $(y'A).(z_C)_C > 0$. Obviously for every $x \in H$, we have $0 \neq \big(\xi_0(C)(x)\big)_C \in \mathcal{P}$. Define

$$g := y' \left[A\big(\xi_0(C)\big)_C\right].$$

Then

$$g(x) > 0 \quad (x \in H).$$

Also, for each nonzero positive linear functional $\mathcal{M}$ on $C(H)$,

$$0 \neq \big(\mathcal{M}(\xi_0(C))\big)_C \in \mathcal{P}.$$

Therefore

$$\mathcal{M}(g) = \mathcal{M}\left(y'A\big(\xi_0(C)\big)_C\right) = y'\left[A\big(\mathcal{M}(\xi_0(C))\big)_C\right] > 0.$$

Let $y = (y_{ij})_{i,j=1}^{m,n}$. It is easily seen that each y_{ij} can be chosen as an integer. By the definition of A and Remark 2.6,

$$y'\left[A\big(\xi_0(C)(x)\big)_C\right] = \sum_{i,j} y_{ij}\left(\sum_{C_j=i}\xi_0(C) - \sum_{C_0=i}\xi_0(C)\right) = \sum_{i,j} y_{ij}(\delta_{\check{h}_j} * \chi_{E_i} - \chi_{E_i}).$$

Hence

$$\sum_{i,j} y_{ij}(\delta_{\check{h}_j} * \chi_{E_i}) = g + \sum_{i,j} y_{ij}\chi_{E_i}.$$

We use the following notation to achieve the paradoxical partition of unity

$$\begin{aligned}
f_{ij} &= \frac{|y_{ij}|}{\sum_{s,t=1}^{m,n}|y_{st}|}\chi_{E_i} \quad (1 \le i \le m,\ 1 \le j \le n),\\
f_i &= \left(\frac{\sum_{s,t=1}^{m,n}|y_{st}| - \sum_{j=1}^{n}|y_{ij}|}{\sum_{s,t=1}^{m,n}|y_{st}|}\right)\chi_{E_i} \quad (1 \le i \le m),\\
g_0 &= \frac{g}{\sum_{s,t=1}^{m,n}|y_{st}|},\\
x_{ij} &= \begin{cases} h_j, & y_{ij} > 0,\\ e, & y_{ij} < 0,\end{cases} \quad \text{and} \quad x'_{ij} = \begin{cases} e, & y_{ij} > 0,\\ h_j, & y_{ij} < 0.\end{cases}
\end{aligned}$$

Note that for every $1 \le i \le m$, $\sum_{j=1}^{n} f_{ij} + f_i = \chi_{E_i}$. Finally we have

$$\sum_{i,j=1}^{m,n} f_{ij} + \sum_{i}^{m} f_i = 1,$$

$$\sum_{i,j=1}^{m,n} \delta_{x_{ij}} * f_{ij} + \sum_{i=1}^{m} \delta_e * f_i = g_0 + \sum_{i,j=1}^{m,n} \delta_{x'_{ij}} * f_{ij} + \sum_{i=1}^{m} \delta_e * f_i,$$

and clearly $\mathcal{M}(g_0) > 0$ for every nonzero positive linear functional $\mathcal{M}$ on $C(H)$. □

Corollary 3.4. *Let m and n be as in the proof of Theorem 3.3. Then $\theta(H) \le m(n+1)$.*

Example 3.5. Let $K = H \vee \mathbb{F}_2$ be the hypergroup join of H and $\mathbb{F}_2$, where $\mathbb{F}_2$ is the free group on two generators a and b and H is an arbitrary finite hypergroup (see [BH, p. 59]). Suppose that $\mathfrak{g} = (a, b)$ and $\mathcal{E} = \{E_1, E_2, E_3, E_4\}$, where

$$\begin{aligned}
E_1 &= \{x,\ x \text{ is a reduced word starting with } a\},\\
E_2 &= \{x,\ x \text{ is a reduced word starting with } b\},\\
E_3 &= \{x,\ x \text{ is a reduced word starting with } a^{-1} \text{ or } b^{-1}\},\\
E_4 &= H.
\end{aligned}$$

Then $\mathrm{Con}(\mathfrak{h}, \mathcal{E})(\mathfrak{g}, \mathcal{E})$ consists of 64 configurations, say, $C_1, C_2, \ldots, C_{64}$. Without loss of generality, let $C_1 = (1,1,2)$, $C_2 = (2,1,2)$, $C_3 = (3,3,2)$, $C_4 = (3,4,2)$, $C_5 = (3,1,1)$, $C_6 = (3,1,3)$, $C_7 = (3,1,4)$, $C_8 = (4,1,2)$ and $C_9 = (3,2,2)$. By the construction of this hypergroup join, it is easy to see that

$$\begin{aligned}
&\xi_0(C_1) = \chi_{E_1}, \quad \xi_0(C_2) = \chi_{E_2},\\
&\xi_0(C_3) = \chi_{a^{-1}E_3},\\
&\xi_0(C_4) = \chi_{\{a^{-1}\}},\\
&\xi_0(C_5) = \chi_{\{x,\ x \text{ is a reduced word starting with } b^{-1}a\}},\\
&\xi_0(C_6) = \chi_{b^{-1}E_3,},\\
&\xi_0(C_7) = \chi_{\{b^{-1}\}},\\
&\xi_0(C_8) = \chi_H,\\
&\xi_0(C_9) = \chi_{\{x,\ x \text{ is a reduced word starting with } a^{-1}b\}},
\end{aligned}$$

and for the other configurations, $\xi_0(C) = 0$. Set $\mathcal{D} := \{C_1, \ldots, C_9\}$. Let M be the coefficient matrix of the system of configuration equations corresponding to $(\mathfrak{g}, \mathcal{E})$.

Then M is the blocked matrix

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & L \end{array}\right),$$

where A is the coefficient matrix of the system

$$\sum_{\substack{C\in\mathcal{D},\\ c_0=i}} Z_C = \sum_{\substack{C\in\mathcal{D},\\ c_j=i}} Z_C \quad (1 \le i \le m,\ 1 \le j \le n).$$

In other words, A is the coefficient matrix of

$$\begin{aligned}
&Z_{C_1} = Z_{C_1} + Z_{C_2} + Z_{C_5} + Z_{C_6} + Z_{C_7} + Z_{C_8},\\
&Z_{C_1} = Z_{C_5},\\
&Z_{C_2} = Z_{C_9},\\
&Z_{C_2} = Z_{C_1} + Z_{C_2} + Z_{C_3} + Z_{C_4} + Z_{C_8} + Z_{C_9},\\
&Z_{C_3} + Z_{C_4} + Z_{C_5} + Z_{C_6} + Z_{C_7} + Z_{C_9} = Z_{C_3},\\
&Z_{C_3} + Z_{C_4} + Z_{C_5} + Z_{C_6} + Z_{C_7} + Z_{C_9} = Z_{C_6},\\
&Z_{C_8} = Z_{C_4},\\
&Z_{C_8} = Z_{C_7}.
\end{aligned}$$

This new system has no nonzero nonnegative solution. In fact,

$$A = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0\\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & -1\\
0 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & -1\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0
\end{pmatrix}.$$

Putting $y' = (1, 0, 0, 1, \dots, 0)$, we have, for some $\alpha_{10}, \dots, \alpha_{64}$,

$$y'M = (1, 1, 1, 1, 1, 1, 1, 2, 1, \alpha_{10,} \dots, \alpha_{64}).$$

Let

$$g_0 := y'M(\xi_0(C))_C.$$

Then

$$g_0 = \left[\sum_{i=1}^{9} \xi_0(C_i)\right] + \xi_0(C_8) \ge \sum_{C\in\mathcal{D}} \xi_C = 1$$

and clearly for each mean $\mathcal{M}$ on $C(K)$, one has $\mathcal{M}(g_0) > 0$. On the other hand, if $f_1 = \chi_{E_1}$, $f_2 = \chi_{E_2}$ and $f_3 = \chi_{E_3 \cup E_4,}$, then

$$f_1 + f_2 + f_3 = 1$$

and

$$(\delta_{\check{a}} * f_1 + \delta_{\check{b}} * f_2 + \delta_e * f_3) - (\delta_e * f_1 + \delta_e * f_2 + \delta_e * f_3) = g_0$$

(see the proof of Theorem 3.3). Therefore

$$\{f_1, f_2, f_3, g_0\}$$

is a paradoxical partition of unity for K and K is nonamenable. Note that in this example, $\theta(K) \leq 3$.

The reader may compare the process of the construction of a paradoxical decomposition in [Y] and the construction of a paradoxical partition of unity in the above theorem. The second one is much easier! In the following theorem we give a relation between $\tau(G)$ and $\theta(G)$ for a group G.

Theorem 3.6. *Let G be a nonamenable group. Then $\theta(G)$ is at most the Tarski number of G.*

Proof. First method. Let the Tarski number of G be $n+m$ and

$$E_1, \dots, E_n, E_{n+1}, \dots, E_{n+m}, h_1^{-1}, \dots, h_n^{-1}, h_{n+1}^{-1}, \dots, h_{n+m}^{-1}$$

be the paradoxical decomposition of G and set $\mathfrak{h} = (h_1, \dots, h_{n+m})$ and $\mathcal{E} = \{E_1, \dots, E_{n+m}\}$. Since

$$G = \bigsqcup_{i=1}^{n} h_i^{-1} E_i = \bigsqcup_{j=n+1}^{n+m} h_j^{-1} E_j, \tag{3.1}$$

for each configuration $C = (C_0, C_1, \dots, C_{n+m}) \in \mathrm{Con}(\mathfrak{h}, \mathcal{E})$, there are unique $i \in \{1, \dots, n\}$ and $j \in \{n+1, \dots, n+m\}$ such that $C_i = i$ and $C_j = j$ (see [ARW, §2]). Thus

$$\sum_{C \in \mathrm{Con}(\mathfrak{h}, \mathcal{E})} z_C = \sum_{\substack{C \in \mathrm{Con}(\mathfrak{h}, \mathcal{E}), \\ C_i = i}} z_C \quad (1 \leq i \leq n)$$

and

$$\sum_{C \in \mathrm{Con}(\mathfrak{h}, \mathcal{E})} z_C = \sum_{\substack{C \in \mathrm{Con}(\mathfrak{h}, \mathcal{E}), \\ C_j = j}} z_C \quad (1 \leq j \leq m).$$

So by the definition of configuration equations, for $1 \leq i \leq n$ and $1 \leq j \leq m$ we have

$$\sum_{C \in \mathrm{Con}(\mathfrak{h},\mathcal{E})} z_C = \sum_{\substack{C \in \mathrm{Con}(\mathfrak{h},\mathcal{E}),\\ C_0 = i}} z_C = \sum_{\substack{C \in \mathrm{Con}(\mathfrak{h},\mathcal{E}),\\ C_0 = j}} z_C .$$

On the other hand, we know that

$$\sum_{j=1}^{n} \sum_{C_j = j} \chi_{x_0(C)} = \sum_{j=1}^{n} \sum_{C_0 = j} \delta_{\check{h}_j} * \chi_{x_0(C)}$$

and

$$\sum_{j=n+1}^{n+m} \sum_{C_j = j} \chi_{x_0(C)} = \sum_{j=n+1}^{n+m} \sum_{C_0 = j} \delta_{\check{h}_j} * \chi_{x_0(C)} .$$

Since $1 = \sum_{C \in \mathrm{Con}(\mathfrak{h},\mathcal{E})} \chi_{x_0(C)}$, there exist $\alpha_C > 0$, $C \in \mathrm{Con}(\mathfrak{h},\mathcal{E})$ such that

$$\sum_{j=1}^{n+m} \sum_{C_j = j} \chi_{x_0(C)} - \sum_{j=1}^{n+m} \sum_{C_0 = j} \delta_{\check{h}_j} * \chi_{x_0(C)} = \sum_{C \in \mathrm{Con}(\mathfrak{h},\mathcal{E})} \alpha_C \chi_{x_0(C)} . \tag{3.2}$$

Setting $g_0 = \sum_{C \in \mathrm{Con}(\mathfrak{h},\mathcal{E})} \alpha_C \chi_{x_0(C)}$ for each positive linear functional M on $C_b(G)$, we have $M(g_0) > 0$.

Equation (3.2) implies that

$$\sum_{j=1}^{n+m} \sum_{C_0 = j} \delta_{\check{h}_j} * \chi_{x_0(C)} - \sum_{j=1}^{n+m} \sum_{C_0 = j} \delta_e * \chi_{x_0(C)} = g_0 . \tag{3.3}$$

It is clear that $\sum_{j=1}^{n+m} \sum_{C_0 = j} \chi_{x_0(C)} = 1$. So (3.3) is a paradoxical partition of unity for G and we have $\theta(G) \leq m + n$. Therefore $\theta(G) \leq \tau(G)$.

Second method. By (3.1),

$$\sum \chi_{E_i} + \sum \chi_{F_j} = 1,$$
$$\sum \delta_{g_i} * \chi_{E_i} + \sum \delta_{h_j} * \chi_{F_j} = 2 = 1 + \sum \chi_{E_i} + \sum \chi_{F_j} .$$

Now it is enough to put $g_0 = 1$. Therefore $\theta(G) \leq \tau(G)$. □

Example 3.7. Let $\mathbb{F}_2$ be the free group on two generators a and b. Put $H = \{e\}$ in Example 3.5. It is seen that $\theta(\mathbb{F}_2) \leq 3 < 4 = \tau(\mathbb{F}_2)$ (see [C]).

Question 3.8. Let G be a group. Is there an exact relation between $\theta(G)$ and $\tau(G)$?

Acknowledgements

The author would like to thank the referee for making important suggestions about an earlier draft of this paper.

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