



Mathematical Physics — *Physical significance of the curvature varifold-based description of crack nucleation*, by PAOLO MARIA MARIANO, communicated on 12 February 2010.

ABSTRACT. — The nucleation and/or growth of cracks in elastic-brittle solids has been recently described in [14] in terms of a special class of measures and with a variational technique requiring the minimization of a certain energy over classes of bodies. Here, the physical foundations of the theory and the basic ideas leading to it are described and commented further on. A view on certain possible developments and shifts toward different settings is also given. This article has expository character.

KEY WORDS: Fracture mechanics, elastic-brittle materials, currents, curvature varifolds, classes of bodies, extended weak diffeomorphisms.

MATHEMATICS SUBJECT CLASSIFICATION: 74R10, 49Q15, 49Q20.

When the adjective ‘variational’ is attributed to models of some physical phenomena, impulsively one may think that the physical situation under scrutiny is conservative. Variational means, in fact, that one is managing some functional, essentially an energy, and is asking that it attains its minimum on some function space. The situation is well known and is typical, for example, of the determination of ground states of elastic bodies under prescribed boundary conditions. The energy is defined on the reference place \mathcal{B} occupied by the body in the ambient space, a place fixed once and for all. Its minimum is required to be attained over a class of one-to-one orientation preserving maps.

Variational descriptions of the nucleation and the evolution of defects like cracks however exist. At least in principle, the question of their physical appropriateness can be posed, due to the dissipative character of the phenomena they are referred to. Of course, one can say that the quest of something to be minimized has rather instrumental character because in nature some economic principle always appears somewhere and the calculus of variations is at a stage of development that it can be desirable to be under conditions of using it. This point of view can be an answer. However, the answer can be even deeper.

When one think of the nucleation of defects, in fact, one manages a *mutant body*. Mutations occur, in fact, in the gross shape of the body at the continuum level, and they can be naturally pictured as mutations of the reference place \mathcal{B} which is now *not* fixed once and for all as for example in the standard formulation of elasticity. In principle, one can imagine of having at disposal a class of possible bodies occupying \mathcal{B} , every member of the class differing from the others by the defect pattern. Once boundary conditions are prescribed, a variational

principle can select the resulting body within a class. This way, a variational approach to the defect nucleation—the latter imposed by the interplay between boundary condition and nature of the material—is physically significant when it involves a minimization process of the free energy (for example) over an entire class of possible bodies. This point of view is the one adopted here and is also the answer to the question of physical appropriateness of some variational approaches. The energy dissipated in the nucleation (or growth) process is individuated in the gap arising from the body in the original shape—the one existing before the assignment of boundary conditions—and the actual shape (i.e. the body plus the defect pattern) obtained by the minimization procedure. Of course the explicit evaluation of this gap can be arduous.

This point of view interprets the physical aspects of two existing variational models of fracture mechanics. Both models are recalled below. The attention is focused on the second one in which cracks are described through appropriate measures, the so-called curvature varifolds. The physical significance of the approach is discussed here. It is shown also that the point of view can be adopted in other circumstances dealing with the nucleation of defects stratified over manifolds with different dimensions.

1

Under the suggestion of the pioneering Griffith's approach to fracture in brittle-elastic solids, a variational model in fracture mechanics has been proposed in [10]. It is based on a requirement of minimality of the overall energy \mathfrak{E} which accounts for the macroscopic deformation and the possible presence of cracks, and is defined by

$$\mathfrak{E}(\mathcal{C}, u) := \int_{\mathcal{B}} e(x, Du(x)) dx + \int_{\mathcal{C}} \phi d\mathcal{H}^2,$$

where \mathcal{B} is the regular¹ region occupied in the three-dimensional ambient space by the body, $Du(x)$ the spatial derivative evaluated at $x \in \mathcal{B}$ of the differentiable transplacement map $x \mapsto u(x) \in \mathbb{R}^3$ —the map defining the actual (deformed) place $u(\mathcal{B})$ — \mathcal{C} the representation in \mathcal{B} of a surface-like crack occurring in $u(\mathcal{B})$, e the elastic energy, ϕ a constant surface energy, $d\mathcal{H}^2$ the two-dimensional Hausdorff measure. The map $x \mapsto u$ is also assumed, as usual, to be one-to-one outside \mathcal{C} , orientation preserving (i.e. $\det Du(x) > 0$), and such that the global invertibility condition

$$\int_{\mathcal{B}} \tilde{f}(x, u(x)) \det Du(x) dx \leq \int_{\mathbb{R}^3} \sup_{x \in \mathcal{B}} \tilde{f}(x, z) dz,$$

holds for all $\tilde{f} \in C_0^\infty(\mathcal{B} \times \mathbb{R}^3)$, a condition allowing frictionless self-contact of the boundary while still preventing self-penetration (for details on this last condition

¹The type of regularity is specified later.

see [17]). In other words, $x \mapsto u$ is an orientation preserving homeomorphism outside the subset of \mathcal{C} containing its jump set.

The requirement is then that *at each instant $t \in [0, \bar{t}]$ of a cracking process the pair $(\mathcal{C}, u)(t)$ realize a minimum of the global energy \mathfrak{E} with \mathcal{C} an admissible crack.* Admissibility is intended in the sense that \mathcal{C} is a rectifiable set (specifically the image of a countable number of Lipschitz maps) with zero volume measure. The interval of time is then discretized and minimality is required at time steps. Minimizers are pairs (\mathcal{C}, u) : the minimum problem has two variables.

Notwithstanding simplicity and elegance of this model, the evaluation of minima of the energy at each time step involves a number of analytical problems. The essential difficulty arises in controlling in three dimensions minimizing sequences of surfaces leading to the possible actual crack, or better to its picture \mathcal{C} in the reference place. However, when the entire crack is open, \mathcal{C} coincides with the jump set of the transplacement field $x \mapsto u(x)$. The convenient simplification of identifying cracks with the jump sets of displacement fields has been then adopted. Bounded variation (BV) or special bounded variation (SBV) functions have been then involved as candidates to be minimizers of the energy considered as a functional of the sole u . The energy is then considered as the one of a simple Cauchy's body² and minimizers are sought in a space of maps including candidates to be reasonable descriptors of the elastic-brittle behavior. Remind that the choice of function spaces where one researches minimizers has constitutive nature. A difficulty has been however encountered: theorems allowing the selection of fields with discontinuity sets describing reasonable (physically significant) crack patterns seem to be not available at least in the current literature (see [4] for a pertinent review of the results along this line; also [7], [11], [6]). Moreover, this kind of approach seems to be not able to account for partially opened cracks. In fact, in this case the transplacement field is continuous across the closed part of the crack although the material bonds are broken in the actual place. The description of partially closed cracks can have even stringent interest in the time-discretized procedure sketched above. In fact, when such a procedure is applied by taking into account a loading program described by time-dependent boundary conditions, a program implying even the growth of pre-existing cracks besides the nucleation of a new fracture at the n -th instant, it can happen that the cracks may close even partially, and then they re-open at subsequent time steps. Moreover, the discovery of physically appropriate crack patterns is another key point.

²In continuum mechanics a body is considered in primitive sense as a set with elements called the material elements and identified even vaguely with molecular or atomic aggregates. The selection of material elements, also called representative volume elements, is matter of modelling. Once even a rough idea of them is formulated—in a sense one is specifying what are the peculiar physical aspects of the material texture—the essential step is to furnish geometrical structure to the body which would be otherwise just an abstract set. The representation of the interactions is then straightforward, dual in the sense of power. The geometrical representation of the material elements is then matter of modelling and can be even minimalist: in fact, one can choose to assign to every material element only the place that it occupies in the ambient space. I use to call Cauchy's bodies those bodies for which the minimalist approach summarized above is sufficient to represent the essential peculiarities of their morphology, and the representation of inner actions is just in terms of standard stresses.

Such questions have been tackled from a different point of view in [14] (see also additional comments in [15], [13]) by using tools from geometric analysis. The skeletal features of the theory are listed in the ensuing items.

- (i) Distinction is made between cracks and jump set of the transplacement field, as in [10]. The latter set is however constrained to be contained within the crack pattern. Differently from all previous proposals, the crack pattern is described through measures over a fiber bundle with typical fiber the Grassmanian of ‘planes’ through \mathcal{B} , the so-called curvature varifolds.
- (ii) A generalized notion of curvature can be associated with curvature varifolds. It enters the constitutive structure of the surface energy along the crack margins. The resulting energy differs from Griffith’s proposal for the presence of the generalized curvature of the varifold and the energy along the tip in three dimensions. In this sense the model is an evolution of Griffith’s scheme.

The curvature-dependence of the surface energy has analytical advantages and permits the control of minimizing sequences. A theorem showing the existence of pairs of crack patterns and transplacement fields in appropriate measure and function spaces is then available (see [14]). It has some implications:

- (a) The crack pattern results a rectifiable set. Although it can be very irregular, it has the features that our intuition assigns to a fracture.
- (b) Balance equations can be derived in weak form from the first variation even for a *generic* rectifiable set. Notice that if a crack is assumed to be coincident with the discontinuity set of the transplacement field only, to obtain balance equations, stronger regularity assumptions on the geometry of the crack pattern are necessary (see detailed analyses in [4], [5], [7], [11], [6]).

These consequences imply that, beyond the analytical advantages, the physical meaning of items (i) and (ii) deserves additional analyses. They can be developed by going along the essential steps of the theory.

2

Consider the place \mathcal{B} of a body, selected as a reference, as an open, connected set in the three-dimensional ambient space, with surface-like boundary oriented by the normal at each point, to within a finite number of corners and edges. If a 2D crack is formed in the deformed configuration $u(\mathcal{B})$ — u the transplacement—and crosses a generic point $u(x)$ with x in \mathcal{B} , its ‘direction’ is locally described by tangent plane to the crack at $u(x)$, when the crack is smooth. When the crack margins have a corner at $u(x)$, a cone of planes has to be considered. Crack patterns can be however very irregular. One can accept a set as a representative of a crack pattern when it is just rectifiable as mentioned hitherto. So, an approximate notion of tangent plane is available in geometric measure theory (see [9]).

Crack patterns can have a fictitious representation in the reference place \mathcal{B} (let say through sets with zero volume measure)—the reference place is now mutant

because the macroscopic structure of the body is changing with the nucleation and growth of a crack pattern. Let Π indicate a two-dimensional plane or a straight line in three dimensional ambient space where \mathcal{B} is contained. The pair (x, Π) gives in \mathcal{B} local information on the geometry of the crack crossing possibly $u(x)$. Of course, up to when a real crack is not realized, any Π in the star at x can be a candidate to describe locally the direction of a possible crack pattern. The pair (x, Π) can be viewed as a typical point of a fiber bundle $\mathcal{G}_k(\mathcal{B})$, $k = 1, 2$, with natural projector $\pi : \mathcal{G}_k(\mathcal{B}) \rightarrow \mathcal{B}$ and typical fiber $\pi^{-1}(x) = \mathcal{G}_{k,3}$, the Grassmanian of $2D$ -planes or straight lines associated with \mathcal{B} . A k -varifold over \mathcal{B} is a non-negative Radon measure V over the bundle $\mathcal{G}_k(\mathcal{B})$ (see [1], [2], [3], [18], [19]). The measure V has a projection over \mathcal{B} obtained by using the *projector of measures*—indicated here by $\pi_{\#}$ —associated with the natural projection π defining the fiber bundle $\mathcal{G}_k(\mathcal{B})$. Such a projection, namely $\pi_{\#}V$, is Radon measure over \mathcal{B} and is also indicated for short by μ_V . It is called the weighed measure of the varifold and defines the so-called *mass* $\mathbf{M}(V)$ of the varifold itself through the relation $\mathbf{M}(V) := V(\mathcal{G}_k(\mathcal{B})) = \mu_V(\mathcal{B})$. For the purpose of describing crack patterns through measures, the essential point is the possibility of constructing varifolds over a subset \mathfrak{b} of \mathcal{B} . The subset \mathfrak{b} can have variegate nature. Here, for the physical purpose at hand, it is assumed that it is an admissible crack in the sense sketched above. Let \mathcal{H}^k be the k -dimensional Hausdorff measure in \mathbb{R}^3 , $k = 1, 2$. It is assumed that \mathfrak{b} is a \mathcal{H}^k -measurable, k -rectifiable subset of \mathcal{B} . For θ a function in $L^1(\mathfrak{b}, \mathcal{H}^k)$, the approximate tangent k -space³ (here $1D$ or $2D$) $T_x\mathfrak{b}$ to \mathfrak{b} at almost every x in \mathfrak{b} is defined for $\theta\mathcal{H}^k \llcorner \mathfrak{b}$ a.e. $x \in \mathcal{B}$. The symbol $\Pi(x)$ indicates the orthogonal projection onto $T_x\mathfrak{b}$. A varifold $V_{\mathfrak{b},\theta}$, restricted to \mathfrak{b} , can be then defined. It is called the *rectifiable varifold* associated with \mathfrak{b} , with density θ , and is such that

$$\int_{\mathcal{G}_k(\mathcal{B})} \varphi(x, \Pi) dV_{\mathfrak{b},\theta}(x, \Pi) = \int_{\mathfrak{b}} \theta(x)\varphi(x, \Pi) d\mathcal{H}^k,$$

for any $\varphi \in C^0(\mathcal{G}_k(\mathcal{B}))$. Rectifiable sets can be considered a sort of generalized surfaces (see [1]). A subclass of them admits a notion of generalized mean curvature vector (see [2], [3]). For members of such subclass (not all) a notion of second fundamental form can be defined (see [18]). Here the attention is on varifolds admitting density θ with integer values, the so-called *integer rectifiable varifolds*. They allow the definition of a special class of varifolds (see [19]) which is essential for the ensuing developments⁴.

A new ingredient has to be inserted. It is a third-rank tensor field $(x, \Pi) \mapsto \mathbf{A}(x, \Pi) \in \mathbb{R}^{3*} \otimes \mathbb{R}^3 \otimes \mathbb{R}^{3*}$ over $\mathcal{G}_k(\mathcal{B})$, with components A_j^{li} . It plays the role of a *generalized curvature*. A varifold V is called a *curvature k -varifold with boundary* if (i) V is an integer, rectifiable k -varifold $V_{\mathfrak{b},\theta}$ associated with the triple $(\mathfrak{b}, \theta, \mathcal{H}^k)$, (ii) there exists a function $\mathbf{A} \in L^1(\mathcal{G}_k(\mathcal{B}), \mathbb{R}^{3*} \otimes \mathbb{R}^3 \otimes \mathbb{R}^{3*})$, and a vector Radon measure ∂V such that, for every $\varphi \in C_c^\infty(\mathcal{G}_k(\mathcal{B}))$, one gets

³The choice $k = 1, 2$ allows one to treat in a unitary way surface and linear cracks.

⁴See [16], last chapter, for a nimble presentation of varifolds.

$$\int_{\mathcal{G}_k(\mathcal{B})} (\Pi_j^i D_{x_j} \varphi + A_j^{il} D_{\Pi_j^l} \varphi + A_j^{ij} \varphi) dV(x, \Pi) = - \int_{\mathcal{G}_k(\mathcal{B})} \varphi d\delta V^i(x, \Pi).$$

The vector measure δV is called the *varifold boundary measure* [19]. The subclass of varifolds with generalized curvature \mathbf{A} in $L^p(\mathcal{G}_k(\mathcal{B}))$ is indicated here by $CV_k^p(\mathcal{B})$. It is possible to show (see [18]) that if $V = V_{b, \theta} \in CV_k^p(\mathcal{B})$, with $p > k$, V is locally the graph of a multivalued function of class $C^{1, \alpha}$, $\alpha = 1 - \frac{k}{p}$, far from ∂V .

- Varifolds with boundary can be used as descriptors of crack patterns: (i) The set b has the minimal geometrical properties of an admissible crack, at least in the sense mentioned above. (ii) The density θ furnishes information on its possible faceted shape⁵. (iii) The local orientation of the crack pattern is accounted for through Π . (iv) Curvature is considered, although in the generalized (weak) form specified above. (v) The boundary of the crack—it includes the tip—is described by the boundary of the varifold.
- Consider a smooth crack crossing a body in \mathcal{B} and intersecting somewhere its boundary but maintaining the tip in the interior of \mathcal{B} . A two-dimensional ($k = 2$) varifold V_2 describes the crack, its boundary measure δV_2 is supported by the entire boundary of the crack itself. To represent separately the crack tip, that is the part of the boundary of the crack in the interior of \mathcal{B} , a specific one-dimensional varifold V_1 has to be inserted. It is supported by the tip alone. Its boundary is supported by possible corners along the tip and the points determining the intersection of the tip with the external boundary of the body $\partial \mathcal{B}$. The insertion of V_1 allows one to assign later energy to the tip of the crack. Different properties can be also assigned to the corners of the tip by using δV_1 . Of course, to capture the intuitive structure of the geometry under scrutiny, the varifolds V_2 and V_1 have to satisfy a certain link. A definition presented in [14] specifies the link: a family $\{V_k\}_{k=1}^{d-1}$ of k -varifolds with boundary in d -dimensional ambient space is said to be *stratified* when $\pi_{\#} |\delta V_k| \leq \mu_{V_{k-1}}$ for all k 's. In the special case treated here the condition of stratification reduces to $\pi_{\#} |\delta V_2| \leq \mu_{V_1}$.
- The choice $k = 1, 2$ for constructing $\mathcal{G}_k(\mathcal{B})$ and the associated varifolds allows one to consider not only two-dimensional cracks with the relative tips but also additional linear defects ($k = 1$) which can be cracks included in very thin tubes—material bonds are broken along a line for some reason—or even dislocations. In the latter case, through a one-dimensional varifold one can describe dislocations emanating from a crack tip in a three-dimensional body. In the former case one manages crack patterns stratified over various dimensions. Of course, the analogous description can be adopted in space dimension $d \geq 2$ and stratification of defects of various nature can be accounted for.

A new form of the energy for a body undergoing fractures, based on the description of cracks in terms of varifolds, has been proposed in [14] (see also comments

⁵ If in a neighborhood of x there is a smooth surface, $\theta = 1$, when there is a net fold, $\theta = 2$, and so on.

in [15], [13]). Such an energy is indicated below by $\mathcal{E}(u, \{V_k\}, \mathcal{B})$. It differs from the expression $\mathfrak{G}(\mathcal{C}, u)$ coming from the traditional Griffith’s proposal and is an extension of it. For a three-dimensional body it reads:

$$\begin{aligned} \mathcal{E}(u, \{V_k\}, \mathcal{B}) := & \int_{\mathcal{B}} e(x, u(x), Du(x)) dx + \sum_{k=1}^2 \alpha_k \int_{\mathcal{G}_k(\mathcal{B})} |\mathbf{A}_{(k)}|^{p_k} dV_k \\ & + \sum_{k=1}^2 \beta_k \mathbf{M}(V_k) + \gamma \mathbf{M}(\partial V_1), \end{aligned}$$

where $\alpha_k, \beta_k, \gamma$ and p_k are constitutive coefficients. In particular, $\alpha_k, \beta_k, \gamma$ are positive numbers, so the contribution of the generalized curvature of the varifolds is always present, even if it can be extremely small. The density $e(x, u, Du)$ is defined as the difference $e(x, u, Du) := \tilde{e}(x, Du) - w(u)$ between the *bulk elastic energy* $\tilde{e}(x, Du)$ and the *potential $w(u)$ of external body forces*. A number of comments on its physical meaning are necessary.

- The addendum $\beta_2 \mathbf{M}(V_2)$ has the role of the last integral in $\mathfrak{G}(\mathcal{C}, u)$, that is the role of Griffith’s surface energy: β_2 has the same meaning of ϕ in $\mathfrak{G}(\mathcal{C}, u)$. Of course, the first addendum is the bulk elastic energy with density $e(x, Du)$ —the body is then simple but anisotropic—and coincides with the first integral in $\mathfrak{G}(\mathcal{C}, u)$. The other terms are not standard.
- The addendum $\beta_1 \mathbf{M}(V_1)$ counts energy along the tip. Such an energy is proportional to the length of the tip itself, namely to the mass $\mathbf{M}(V_1)$ of the one-dimensional varifold supported by the tip itself.
- The term $\gamma \mathbf{M}(\partial V_1)$ adds possible energy concentrated at the tip corners where material bonds can be entangled in principle in a way different from the other parts of the tip.
- The two addenda

$$\alpha_2 \int_{\mathcal{G}_2(\mathcal{B})} |\mathbf{A}_{(2)}|^{p_2} dV_2 + \alpha_1 \int_{\mathcal{G}_1(\mathcal{B})} |\mathbf{A}_{(1)}|^{p_1} dV_1$$

mark in a more pronounced manner the difference with respect to $\mathfrak{G}(\mathcal{C}, u)$. They have pure configurational nature: they do not involve directly the gradient of deformation $Du(x)$. The first one accounts for the (generalized) curvature of the varifold describing the surface of the crack, the second one includes the curvature of the varifold describing the tip. Influence of the curvature of the crack, above all in the proximity of the tip, has been recognized in [24], a work devoted however to other aspects of fracture processes, namely Grinfeld’s instability. The curvature here aims to account at macroscopic level of local microstructural effects at low scale, occurring in the cracking process. Curvature energy can be associated with bending effects in breaking material bonds when a crack is determined—consider for example a material in which the inner bonds are modeled through beam-like interactions. A point has to be stressed: Bending occurs in the current configuration $u(\mathcal{B})$ while, as a function

of x , \mathbf{A} is defined over the reference configuration \mathcal{B} . However, when bending of material bonds in the actual configuration breaks the bonds themselves, such a bending has a configurational effect because it contributes to the mutation of \mathcal{B} due to the nucleation and possible propagation of a crack. All configurational effects are measured in \mathcal{B} , as it is commonly accepted. Additionally, analogies with different approaches to crack analyses in complex bodies (see [21]) and bodies with strain-gradient effects (see [25], [20]) can be called upon to enforce the interpretation of the presence of \mathbf{A} as a configurational indicator of the effects due to latent microstructures. The terms including \mathbf{A} in the energy tell us essentially that, once boundary conditions are prescribed, one has to pay in energy for curving the crack. Consider a rectangular planar sheet of a homogeneous material half of it including a straight crack in the middle as in Figure 1. Apply boundary conditions in terms of transplacement (Dirichlet boundary conditions) in mode 1. If the boundary conditions are such that the crack can grow, it remains straight, unless some additional agency occurs paying energy in curving it.

- Consider a body in an initial configuration in which no crack is present. For example, \mathcal{B} is a three-dimensional ball. After some loading program, a certain configuration in which a crack occurs is reached. Clearly one could consider such a configuration as the initial one. The possibility stresses the point that ‘being cracked’ is only a relative concept. Configurations—or better states—have to be compared to affirm that a body is cracked. Moreover, in sequences of configurations (or states) in which one excludes the possibility of restoring cracks by gluing the matter across crack facies, an order relation has to be considered. Such an order is given by monotonicity in crack patterns: if in a certain configuration there is a crack with respect to an uncracked configuration in the sequence, a subsequent configuration can have a crack pattern that coincides with or includes the previous crack. In terms of varifolds such a point of view is expressed by affirming the existence a family of *comparison varifolds* $\{\tilde{V}_k\}$ such that the family of varifolds $\{V_k\}$ describing the actual crack pattern is constrained by $\mu_{\tilde{V}_k} \leq \mu_{V_k}$ for any k and $\tilde{V}_k \in CV_k^{pk}(\mathcal{B})$. The assignment of $\{\tilde{V}_k\}$ does not mean that one is considering in a given configuration a pre-

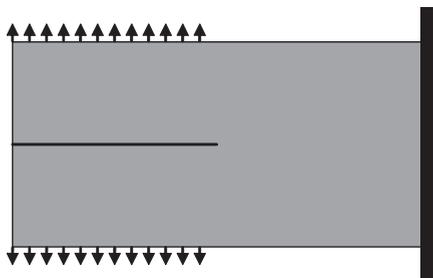


Figure 1. Clamped two-dimensional sheet with a horizontal crack in the middle, subjected to Dirichlet boundary conditions—the arrows represent applied transplacements, the black zone is the clamping device.

existing crack pattern always, because the comparison varifold family can be also empty.

- The assumption used so far that the ambient space is three-dimensional has been accepted only to be close to the standard physical intuition. There is no obstruction to consider higher dimensional spaces. Consider the dimension of the ambient space to be $d \geq 2$. The previous treatment can be extended to this case straight away. The only modification, including the expression of the energy is that the range of k must be considered to be $1 \leq k \leq d - 1$, at least in principle. The rest remains unchanged. This choice allows one to consider stratified families of defects at various scales with various physical meanings, depending on the circumstance. Of course, there could be cases in which k could not take all the natural values from 1 to $d - 1$. Even in three-dimensional ambient space one could have only $k = 1$, for example, being in the condition to consider only linear defects like nets of dislocations. Such a case would deserve perhaps a treatment a part. The case of four manifolds, $d = 4$, could be called upon when the description of crack in relativistic elastic bodies is considered, with all the necessary changes in the representation of the energy, adapted to the relativistic setting. In this case, however, care must be taken because in the relativistic setting a representation of continuous bodies based on the back-to-label representation seems to be preferable. The circumstance then would change the stage and would include a sort of ‘mixed’ representation of the energy.

In the setting described so far, a minimality requirement is prescribed for the energy:

Minimize $E(u, \{V_k\}, \mathcal{B})$ with V_k in $CV_k^{pk}(\mathcal{B})$, comparison varifolds $\{\tilde{V}_k\}$, u in an appropriate function space, with assigned boundary conditions.

Solution to this problem, if any, is a pair $(u, \{V_k\})$. Minimization over a class of varifolds has the meaning of minimization over a class of bodies: every possible crack pattern represented over \mathcal{B} (remind that nucleation and/or growth of cracks occur in the current place) defines a body. Different crack patterns indicate different bodies: \mathcal{B} , in fact, changes. The family of varifolds $\{V_k\}$ represents the crack pattern—so it selects a body—and the field $x \mapsto u(x)$ describes the deformation of such a body. The two ‘objects’ are correlated. In fact, in the actual configuration possible nucleation, growth and/or opening of a crack are consequences of the deformation, the varifolds over \mathcal{B} are representatives of what happens in the actual configuration. The choice of the function space hosting the generic u is then another key point of the treatment. It has to be linked to the varifolds supported on \mathcal{B} .

In continuum mechanics the choice of function spaces as ambient for solutions to equilibrium or evolution problems has constitutive nature. The properties of the

members of a given space carry a physical meaning about. The characteristic physical features of the problem under scrutiny have then to address the functional choice. In the case under analysis, the idea is that the material is elastic-brittle. It means that the material is elastic up to a certain threshold after which a crack is created while outside the crack the material is still in elastic phase. The threshold can be expressed in terms of deformation, stress, energy, depending on circumstances. Moreover, as it will be clear later, the threshold can be also not expressed directly. It is, in a sense, included in the choice of the function space that one selects.

Let us focus the attention on pure elasticity first. If the material is purely elastic, it can deform at will, without any threshold and, in principle, without end. The deformation can be also perfectly recovered, after unloading. In this sense, phenomena like cavitation in solids are ascribed to elastic-brittle behavior rather than perfect elastic setting. If this view is accepted, the consequence would be that a perfectly (hyper)elastic body should be such that any compatible transplacement (or displacement, depending on the choice) field does not describe the nucleation of fractures or holes.

It is well known that under conditions of polyconvexity of the elastic energy density, minimizers of the elastic energy of a simple body can be found in the Sobolev space $W^{1,p}(\mathcal{B}, \mathbb{R}^3)$ —a space hosting maps with first distributional derivative having integrable p -power, i.e. the first derivative is in L^p . However, when $p < 3$ non negligible is the presence of transplacement maps with graphs admitting boundaries with projections into the interior of \mathcal{B} . Such boundaries describe the formation of ‘holes’ and/or open ‘fractures’ of various nature, so they are undesirable when a purely elastic material is under analysis, at least if the view on elasticity sketched above is accepted.

The difficulty can be overcome. In fact, there is a global way to check—eventually to control—the presence of boundaries in the graph of a map with projection into the domain of the map itself through linear functionals. Indicate them by G_u , with the indices u suggesting that the functional G is associated with the transplacement u . In the case treated here, such functionals are linear over smooth 3-forms with compact support in $\mathcal{B} \times \mathbb{R}^3$. The general technique, which is valid in any finite dimension d of the ambient space, is described in the monograph [17]. Here, I sketch only minimal ideas, furnishing the essential picture and physical interpretations in the case under scrutiny.

Preliminarily, remind that a r -vector over a linear space E is a rank- r skew-symmetric tensor, that is an element of the skew-symmetrization of $E \otimes \cdots \otimes E$. The space of r -vectors is indicated here by $\Lambda_r(E)$. It has a natural dual $\Lambda^r(E)$. Any map of the type $\omega : \mathcal{B} \rightarrow \Lambda^r(E)$ is called a r -form⁶. The space of all r -forms of the type just defined is indicated by $\mathcal{D}^r(E)$.

For the mechanics treated here interesting is the case of 3-vectors over $\mathbb{R}^3 \times \mathbb{R}^3$, the space hosting the graph of the deformation. Consider a deformation (trans-

⁶More specifically, one should consider E as a real vector bundle over \mathcal{B} of fiber dimension d , that is a family of d -dimensional vector spaces parametrized by points of \mathcal{B} .

placement) $x \mapsto u(x) \in \mathbb{R}^3$, $x \in \mathcal{B}$. Amid all possible 3-vectors over $\mathbb{R}^3 \times \mathbb{R}^3$, it is possible to define at every $x \in \mathcal{B}$ the 3-vector $M(Du)$ associated with the gradient of deformation Du . Detailed definition and various properties can be found in [17], here, for the expository purposes declared at the beginning, it is only necessary to know that at each x its components are the entries of $Du(x)$, $adjDu(x)$, $detDu(x)$. In $M(Du)$, then, all elements characterizing the deformation of lines, areas, and volume of the body in \mathcal{B} are included. In this sense, $M(Du)$ characterizes completely the deformation. Its dual counterpart—the value at the same x of some form in $\mathcal{D}^3(\mathbb{R}^3 \times \mathbb{R}^3)$ —is then a sort of generalized stress.

Given a transplacement $x \mapsto u(x) \in \mathbb{R}^3$, $x \in \mathcal{B}$, the 3-current integration G_u (current for short) over the graph of u is defined to be a linear functional over smooth 3-forms $\omega \in \mathcal{D}^3(\mathbb{R}^3 \times \mathbb{R}^3)$ with compact support in $\mathcal{B} \times \mathbb{R}^3$, namely

$$G_u(\omega) := \int_{\mathcal{B}} \langle \omega(x, u(x)), M(Du(x)) \rangle dx,$$

where the angle brackets indicate the natural action over $M(Du)$ of its dual counterpart⁷. Essentially, $G_u(\omega)$ plays the role of generalized internal power. The number $\mathbf{M}(G_u)$ indicates here the so-called *mass of the current* and is defined by

$$\mathbf{M}(G_u) := \int_{\mathcal{B}} |M(Du(x))| dx,$$

where $|M(Du(x))|$ is the modulus of $M(Du(x))$, evaluated in the standard way for tensors. The symbol $|G_u|$ indicates the *total variation* of the current and is defined as usual for functionals. A *boundary current* can be associated with G_u ; it is indicated by ∂G_u and defined by duality, that is⁸

$$\partial G_u(\omega) := G_u(d\omega), \quad \forall \omega \in \mathcal{D}^2(\mathcal{B} \times \mathbb{R}^3),$$

with $\mathcal{D}^2(\mathcal{B} \times \mathbb{R}^3)$ the space of 2-forms over $\mathbb{R}^3 \times \mathbb{R}^3$ with compact support⁹ in $\mathcal{B} \times \mathbb{R}^3$. The notion of boundary current has not only formal nature. It has an immediate physical interpretation: when the graph of u is free of boundaries inside the interior of \mathcal{B} , $\partial G_u(\omega) = 0$ for any $\omega \in \mathcal{D}^2(\mathcal{B} \times \mathbb{R}^3)$. Essentially, this zero boundary condition prevents the formation of cracks or holes inside the actual place $u(\mathcal{B})$ of the body. Such a condition has been used (see [17] and the other references of its authors mentioned therein) to define a class of transplacements—the so-called *weak diffeomorphisms*—which is ‘constitutively’ an appropriate choice for describing what one imagines to be a pure elastic deformation, as sketched above.

⁷ A bit more precisely, the current is defined by taking the rectifiable part of the graph of u , that is the part of the graph that can be seen as the graph of Lipschitz maps.

⁸ The so-called external differentiation over forms is indicated by d and acts as $d : \mathcal{D}^n(E) \rightarrow \mathcal{D}^{n+1}(E)$.

⁹ The definitions of currents and related boundaries can be also available in spaces with higher dimension (see [17] for the complete theory).

Here the situation is a bit more complicated. The deformation has to be a weak diffeomorphism outside a subset of the support of the varifolds describing the crack pattern, if the minimizing procedure provides a non-empty family of minimizing varifolds. Inside that subset, the transplacement admits jumps, so it is not purely a weak diffeomorphism.

Extended weak diffeomorphisms are then necessary. They must have the physical properties just indicated. Such properties can be summarized in a formal definition.

DEFINITION 1. *Assigned a stratified curvature varifold $V = \{V_k\}_{k=1}^{n-1}$ with boundary, i.e., $V_k \in CV^{pk}$, a map $x \mapsto u$ is said to be an extended weak diffeomorphism (in short $u \in dif^{1,1}(\mathcal{B}, V, \mathbb{R}^3)$), when*

- (i) $u \in L^1(\mathcal{B})$ and is a.e. approximately differentiable,
- (ii) $|M(Du)| \in L^1(\mathcal{B})$,
- (iii) $\det Du(x) > 0$ for almost every $x \in \mathcal{B}$,
- (iv) for any $f \in C_c^\infty(\mathcal{B} \times \mathbb{R}^3)$

$$\int_{\mathcal{B}} f(x, u(x)) \det Du(x) dx \leq \int_{\mathbb{R}^3} \sup_{x \in \mathcal{B}} f(x, w) dw,$$

- (v) $\pi_{\#}|\partial G_u| \leq \sum_{j=1}^2 \mu_{V_j} + \pi_{\#}|\partial V_1|$ as measures on \mathcal{B} .

The definition has natural extension in \mathbb{R}^d : the summation in the item (v) should be extended up to $d - 1$. Of course, the definition of currents and related boundaries holds in dimension d : in that case $M(Du)$ is tested over d -forms. Here and in the whole paper, the restriction $d = 3$ is essentially motivated by the physics under scrutiny (see [14] for the abstract theory). The first item indicates the possibility of evaluating the gradient of deformation. The second item is another regularity condition. It implies that one can in principle measure the average of the gradient of deformation, the volume change, the overall deformation of surfaces. The third item is the standard condition that a transplacement be an orientation preserving map. Item (iv) is the condition mentioned at the beginning of Section 1. It permits to move \mathcal{B} along u into a region $u(\mathcal{B})$ in such a way that self-contact between parts of the boundary $\partial\mathcal{B}$ be allowed while self-penetration excluded. Notice that in item (v) the action of the projector $\pi_{\#}$ on the total variation of the boundary current is motivated by the fact that the latter behaves substantially as a measure. Essential properties for the space of extended weak diffeomorphisms are shown in [14] (see there the relevant theorems and proofs).

Standard weak diffeomorphisms are $W^{1,1}(\mathcal{B}, \mathbb{R}^3)$ maps that satisfy the items (ii), (iii), (iv) in previous definition while item (v) which is substituted by the zero boundary condition $\partial G_u(\omega) = 0$ for any $\omega \in \mathcal{D}^2(\mathcal{B} \times \mathbb{R}^3)$. From a kinematic point of view, in going from $dif^{1,1}(\mathcal{B}, \mathbb{R}^3)$ to $dif^{1,1}(\mathcal{B}, V, \mathbb{R}^3)$, one transits from pure elastic setting to elastic-brittle behavior.

In this sense, the requirement of minimality of the energy includes the possibility of finding minimizers—if any—in terms of weak diffeomorphisms and null

varifolds, and in terms of extended weak diffeomorphisms and non-null varifolds. The transition from a situation to another is morally the threshold from the elastic to the elastic-brittle behavior. In this sense, also, there is no need in principle of adding another condition defining the threshold itself. There could be also minimizers for which the transplacement field is simply a weak diffeomorphism but the varifolds are not null. This situation describes presence of closed cracks only: in other words, material bonds are broken but the crack remains closed and the transplacement field does not jump across the crack facies. Such a situation can occur in a step-by-step minimization program obtained by updating in time steps the boundary conditions and requiring minimality of the energy at each step. At the step $n - 1$ a crack pattern can occur, at the step n the deformation closes the cracks, at the step $n + 1$ there is a purely elastic continuation, then, at further steps, new crack patterns accrue.

Of course, proving existence of minimizers is a crucial step for attributing sense to the previous reasonings. Existence depends on the characteristic properties of the energy and the boundary conditions. Once the existence of minimizers is established, the characterization of them along the physical suggestions collected above is matter of regularity theorems. The question is open. Actually, any regularity theorem is available. Different is the case of the existence problem.

4

The existence result for the minimum problem stated above for the energy $\mathcal{E}(u, \{V_k\}, \mathcal{B})$ of an elastic brittle solids has been proven in [14]. Boundary conditions of Dirichlet type can be presumed. They are given by prescribing the transplacement field along the boundary $\partial\mathcal{B}$ of \mathcal{B} .

The discussion of the existence is developed by taking first a subspace of $dif^{1,1}(\mathcal{B}, V, \mathbb{R}^3)$, precisely the space $dif^{p,1}(\mathcal{B}, V, \mathbb{R}^3)$ defined by

$$dif^{p,1}(\mathcal{B}, V, \mathbb{R}^3) := \{u \in dif^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^3) \mid |M(Du)| \in L^p(\mathcal{B})\},$$

for some $p > 1$. Essentially, the choice of $dif^{p,1}(\mathcal{B}, V, \mathbb{R}^3)$ with $p > 1$ is a request of additional regularity which is sometimes necessary for physical needs. Combination with the space of varifolds allows one to recognize a natural ambient in which the existence of minimizers of the energy $\mathcal{E}(u, \{V_k\}, \mathcal{B})$ can be investigated. Such a space is indicated by $\mathcal{A}_{q,p,K,\{\tilde{V}_k\}}(\mathcal{B})$ and defined by

$$\begin{aligned} \mathcal{A}_{q,p,K,\{\tilde{V}_k\}}(\mathcal{B}) := & \{(u, \{V_k\}) \mid V_k \in CV_k^{pk}(\mathcal{B}), u \in dif^{q,1}(\mathcal{B}, V_k, \mathbb{R}^3), \\ & \{V_k\} \text{ is stratified, } \|u\|_{L^\infty(\mathcal{B})} \leq K, \mu_{\tilde{V}_k} \leq \mu_{V_k}, \forall k = 1, 2\}, \end{aligned}$$

where \tilde{V}_1 and \tilde{V}_2 are comparison varifolds describing possible initial cracks. In particular, the subspace

$$\mathcal{A}_{q,p,K,\{\tilde{V}_k\}}^{u_0}(\mathcal{B}) := \{(u, \{V_k\}) \in \mathcal{A}_{q,p,K,\{\tilde{V}_k\}}(\mathcal{B}) \mid u(x) = u_0(x), x \in \partial\mathcal{B}_u\},$$

with $\partial\mathcal{B}_u$ the part of the boundary of the body where the transplacement field is prescribed, takes into account the boundary conditions of Dirichlet type mentioned above.

In all these definitions, another regularity requirement is prescribed. In fact, the condition $\|u\|_{L^\infty(\mathcal{B})} \leq K$ imposes that the essential supremum of u is almost everywhere—with respect to the Lebesgue measure—bounded. In fact, a priori it is not possible to exclude that, if one is able to prove under some conditions the minimality of the energy $\mathcal{E}(u, \{V_k\}, \mathcal{B})$ over some space of extended weak diffeomorphisms and varifolds, the minimizing varifold does not describe a fragmentation of the body—let say a crack cutting a piece of matter from the rest. In this case, a transplacement field could be such that the cut piece, now free from boundary conditions, can be translated rigidly to infinity. By imposing that $\|u\|_{L^\infty(\mathcal{B})} \leq K$, with K a real number, then, one wants to avoid the situation just sketched. So, in this sense the assignment of K has not properly constitutive nature. It is not related to some property of the material, rather it is a parameter selecting admissible deformation processes, admissibility considered with reference to the possible unconstrained extraction of pieces of matter from the body.

For d the dimension of the ambient space and k ranging from 1 to $d - 1$, analogous definitions of $\mathcal{A}^{u_0}_{q,p,K,\{\tilde{V}_k\}}(\mathcal{B})$ hold and the theory can be generalized (see relevant results in [14]).

Another crucial point in the path leading to the proof of existence theorem of minimizers of the energy is the discussion of the structural properties of the energy. They have constitutive nature, of course.

In non-linear elasticity of simple Cauchy's bodies, common assumptions about the structure of the energy $e(x, u(x), Du(x))$ are well known. By indicating by $M_{3 \times 3}^+$ the space of 3×3 matrices with positive determinant, the energy density e is considered as a map

$$e : \mathcal{B} \times \mathbb{R}^3 \times M_{3 \times 3}^+ \rightarrow [0, +\infty]$$

with values $e(x, u(x), Du(x))$. Remind that positiveness of the determinant of $Du(x)$ is the condition assuring that the transplacement be orientation preserving. The properties H1–H4 below are then assumed to hold.

H1 $e : \mathcal{B} \times \mathbb{R}^3 \times M_{3 \times 3}^+ \rightarrow [0, +\infty]$ is *continuous* in (x, u) .

H2 The map $Du(x) \mapsto e(x, u(x), Du(x))$ is *polyconvex*: that is there exists a Borel function Pe acting as

$$Pe : \mathcal{B} \times \mathbb{R}^3 \times \Lambda_3(\mathbb{R}^3 \times \mathbb{R}^3) \rightarrow \bar{\mathbb{R}}^+,$$

with values $Pe(x, u(x), \xi(x))$, which is continuous in (x, u) for every $\xi \in \Lambda_3(\mathbb{R}^3 \times \mathbb{R}^3)$, convex and lower semicontinuous in ξ for every (x, u) , and such that¹⁰ $Pe(x, u, M(Du)) = e(x, u, Du)$ for any list of entries $(x, u, Du) \in \mathcal{B} \times \mathbb{R}^3 \times M_{3 \times 3}^+$ with $\det Du > 0$.

¹⁰The dependence of u and Du on x is now suppressed for the sake of brevity.

H3 The energy density e satisfies the growth condition

$$e(x, u, Du) \geq C_1 |M(Du)|^r.$$

H4 For every $x \in \mathcal{B}$ and $Du \in M_{3 \times 3}^+$, if for some $u \in \mathbb{R}^3$ the inequality $e(x, u, Du) < +\infty$ is satisfied, then $\det Du > 0$.

The physical nature of these assumptions is discussed in various treatises (see [22], [23]). The standard presence in the polyconvex energy of the determinant of the gradient of deformation and the relevant adjugate is summarized here in the functional dependence on $M(Du)$. The choice is not only formal. It furnishes a rapid path toward the extension of the treatment to d -dimensional cases (see [17]).

However, the essential point is to underline that, in the setting explored here, H1–H4 do not need to be supplemented by additional structural assumptions on the energy to assure the existence of minimizers of $\mathcal{E}(u, \{V_k\}, \mathcal{B})$. The relevant theorem reads as follows:

THEOREM 2 ([14]). *Assume $K > 0$, $q, p_k > 1$, and $\tilde{V}_k \in CV_k^{p_k}(\mathcal{B})$ for any k . If there exists $(u_0, \{V_k^0\}) \in \mathcal{A}_{q,p,K,\{\tilde{V}_k\}}^{u_0}(\mathcal{B})$ such that $\mathcal{E}(u_0, \{V_k^0\}, \mathcal{B}) < +\infty$, then $\mathcal{E}(u, \{V_k\}, \mathcal{B})$ attains in that space the minimum value.*

Proof is presented in [14]. Here just comments have to be added.

- Since no additional structural hypotheses besides H1–H4 of standard non-linear elasticity need to be added, the information about the possible nucleation of a crack or growth of an existing one is furnished by the presence of the terms ruled by varifolds in the energy and the constitutive choice of $\mathcal{A}_{q,p,K,\{\tilde{V}_k\}}^{u_0}(\mathcal{B})$ as functional setting. It is just the latter choice that avoids the introduction of an external criterion for the nucleation of a crack or the growth of an existing one. In fact, the energy is minimized over a class of possible bodies.
- More in general than other descriptions, it is possible to determine the weak form of balance equations for crack patterns which are just rectifiable sets. The result is not discussed here for the sake of conciseness. It is presented in [14] and opens the way to computational opportunities not explored yet. The balance equations derived naturally in [14] for very general crack geometries—as mentioned above the crack pattern has to be just a rectifiable set—are the ones obtained by horizontal variations, that are variations of the reference place¹¹. Thus they have configurational nature: they involve in fact the Hamilton-Eshelby tensor and non-standard terms deriving from the variations

¹¹See [8] for clear explanations on the connection between horizontal variations describing the potential movement of defects and the balance of configurational forces. Take into account also that there is basic difference between the balance of configurational actions associated with macroscopic mutations of the reference place and the balances of standard actions generated by the deformation. In the conservative case and with reference to smooth fields, the former balances are essentially the pull-back in the reference place of the latter balances. In general it is not so. The difference has been evidenced first in [12] vol. 1, pages 152–153.

of the terms including the varifolds, the ones directly related with the geometry of the crack pattern.

- The existence result holds also at dimensions greater than 3, provided that the obvious variations in previous definitions (see [14], [15] for the abstract theory).
- An analogous result holds also for the generalized energy

$$\begin{aligned} \mathcal{E}(u, \{V_k\}, \mathcal{B}) &:= \int_{\mathcal{B}} e(x, u(x), Du(x)) dx + \sum_{k=1}^{d-1} \alpha_k \int_{\mathcal{G}_k(\mathcal{B})} \phi_k(|\mathbf{A}_{(k)}|) dV_k \\ &+ \sum_{k=1}^{d-1} \beta_k \mathbf{M}(V_k) + \gamma \mathbf{M}(\partial V_1), \end{aligned}$$

with $\phi_k(\cdot)$ a convex real-valued function satisfying the condition $\phi_k(t) \geq ct^{p_k}$. The interest of this remark is not only technical. It gives a greater degree of freedom in selecting further constitutive structures under the suggestions of possible experimental evidences and numerical tests.

- The existence result does not exclude the possibility that the minimization procedure foresees stratified varifolds supported on the boundary of \mathcal{B} . In this case one can say that a boundary crack appears. The meaning of such a boundary cracks is not exotic. In fact, on a part of the boundary where the transplacement is imposed a crack can occur so that the boundary condition is ‘broken’. Consider for example a beam jointed at one of its ends. Boundary conditions and properties of the material can be such that a crack occurs just at the interface between the joint and the beam. However, in principle a boundary crack can appear on a free part of the boundary. Such a situation can describe the fragmentation of a thin film at the boundary, which is, essentially, the abrasion of the boundary itself.

5

The technique discussed previously is a general tool for the description of phenomena in which energy can in principle be concentrated over submanifolds of a certain manifold, and this energy depends on the geometry of the submanifold itself. It can be used to analyze either specific situations of physical interest or to formulate and analyze abstract mathematical problems.

- Phenomena of physical interest that can be described by using the tools mentioned hitherto deal for example with the mechanics of linear defects like dislocations and discontinuity surfaces. For dislocations the choice of the varifolds play a crucial role. For interfaces, the functional setting has to be changed and a new special class of extended weak diffeomorphisms arises. The new choice requires proof of completeness of the new functional class and evaluation of the applicability of lower semicontinuity results.
- Another point of discussion is also the evaluation of the crack nucleation and growth in complex bodies. The adjective ‘complex’ distinguishes bodies charac-

terized by a prominent influence of changes in material texture (the *microstructure*) on the macroscopic behavior, an influence exerted through inner actions requiring a representation going beyond the common picture in terms of standard stresses. Quasicrystals, ferroelectrics, magnetoelastic materials, polymeric bodies of various nature, including elastomers, fullerene-based composites, porous bodies, bodies with continuous distributions of dislocations, multiphase materials are paradigmatic examples. Notwithstanding the variety of special models, a unitary picture of the mechanics of complex bodies exists. Within it, the representation of bodies goes beyond standard Cauchy's approach in the sense that every material element is viewed as a system rather than a black box individuated by a single point in the ambient space, which is Cauchy's view. Such a description is multifield and intrinsically multiscale. A *morphological descriptor* field $x \mapsto v(x)$, $x \in \mathcal{B}$, of the essential geometrical features of the material microstructure is then introduced. To construct the essential structures of the relevant mechanics, it is just necessary to presume that $v(x)$ is an element of a set \mathcal{M} which has just the structure of a differentiable manifold. It is assumed to be finite-dimensional for the sake of simplicity. \mathcal{M} is called the *manifold of substructural shapes*. The interest of mentioning here complex bodies is motivated by data showing that the microstructural changes may influence in non negligible way the force driving crack tips along evolution processes. Theoretical analysis of this phenomenon within the setting of the general model building framework of the mechanics of complex bodies (that is without specifying the type of microstructure) has been developed in [21]¹² from a point of view different from the one adopted here. The simplest extension of the theory discussed here to complex bodies is given by an energy of the type

$$\begin{aligned} \mathcal{E}(u, \{V_k\}, \mathcal{B}) := & \int_{\mathcal{B}} e(x, u(x), v(x), Du(x), Dv(x)) dx \\ & + \sum_{k=1}^2 \alpha_k \int_{\mathcal{G}_k(\mathcal{B})} |\mathbf{A}_{(k)}|^{p_k} dV_k + \sum_{k=1}^2 \beta_k \mathbf{M}(V_k) + \gamma \mathbf{M}(\partial V_1), \end{aligned}$$

with the natural modifications in generic dimension d . In analyzing the existence of minimizers, an essential point is the choice of the space hosting the morphological descriptor maps. Such a choice could require the embedding of \mathcal{M} into a linear space. Such embedding always exists because \mathcal{M} is finite dimensional, also it can be isometric when \mathcal{M} is Riemannian. In all cases, however, it is not unique so that it becomes an ingredient of the model, a sort of constitutive choice. Another point is the link of the jump set of the morphological descriptor field with the varifolds. Here the underlying physics is subtle. In principle one can accept that $x \mapsto v(x)$ may have jumps even outside the support of the varifolds and there it may be even continuous. Jumps outside the varifolds can be justified by the formation of domains of microstructures, like polarization or magnetization domains. The meaning of the possible continuity on the support

¹²There one can find appropriate references to works presenting and discussing experimental data.

of the varifolds is associated with the question whether in cracking a body one alters along the margins of the crack the microstructure, in a sense determining a new type of microstructure, or, else, the microstructure remains the same across the margins of the crack. In the philosophy of continuum mechanics, the remark above coincides with asking whether a crack just divides neighboring material elements or breaks the material elements met in front of the tip. The answer cannot be definitive and is matter of modelling. The situation becomes also more complicated when the structure of the surface energy involving the generalized curvature of the varifold is enriched by making more articulated assumptions. Relevant investigations are actually open.

- A point which may deserve to be noted is that the scheme discussed in previous sections has intrinsic similarity with the general framework of the mechanics of complex bodies which has been sketched rapidly in the last item. In fact, when curvature k -varifolds with boundary are chosen to represent cracks, in principle the region where they localize is not known—in other words one does not know where the support of the varifolds is placed in \mathcal{B} , that is where the crack is. In this setting, every point can be crossed in principle by a crack. The minimization procedure tells us that the crack is here or there, before nothing is known about its position. Instead of assigning to each material element a morphological descriptor ν selected in a finite dimensional differentiable manifold \mathcal{M} , one is then assigning to each material element a measure \cdot . In this sense, the scheme discussed here is driven by the ideas of the mechanics of complex bodies, and, in some sense, it goes beyond them a bit.

ACKNOWLEDGEMENTS. Discussions with Mariano Giaquinta, Giuseppe Modica, Gianfranco Capriz, Gilles Francfort, Luigi Ambrosio have pushed me in various manners to write this paper. I thank them for the impulse. This work has been developed within the activities of the research group in ‘Theoretical Mechanics’ of the ‘Centro di Ricerca Matematica Ennio De Giorgi’ of the Scuola Normale Superiore at Pisa. The support of the GNFM-INDAM is acknowledged.

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Received 13 November 2009,
and in revised form 14 December 2009.

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