Publ. RIMS Kyoto Univ. **53** (2017), 457–516
DOI 10.4171/PRIMS/53-4-1

Moduli of Galois Representations

by

Yuichiro Taguchi

Abstract

We develop a theory of moduli of Galois representations. More generally, for an object in a rather general class $\mathfrak{A}$ of noncommutative topological rings, we construct a moduli space of its absolutely irreducible representations of a fixed degree as a (so we call) "f-$\mathfrak{A}$ scheme". Various problems on Galois representations can be reformulated in terms of such moduli schemes. As an application, we show that the "difference" between the strong and weak versions of the finiteness conjecture of Fontaine–Mazur is filled in by the finiteness conjecture of Khare–Moon.

2010 Mathematics Subject Classification: Primary 11G99; Secondary 14G99, 16G99.
Keywords: Moduli, Galois representation, f-profinite ring, Fontaine–Mazur conjecture, Khare–Moon conjecture.

§1. Introduction

In this paper, we develop a theory of moduli of Galois representations, which generalizes Mazur's deformation theory ([25]). In fact, we formulate the theory for absolutely irreducible representations of a rather general class of topological (noncommutative) rings (rather than groups). Thus it is a topological version of Procesi's theory ([35]). (For another approach, see Chenevier [7] and the remark at the end of this introduction.) The main differences between our theory and Mazur's are

– we do not fix a residual representation ρ_0 to start with, so that we can construct a moduli space that parametrizes all absolutely irreducible representations having various residual representations;
– we are interested in parametrizing the isomorphism classes of absolutely irreducible $\mathbb{Q}_p$-representations as well as $\mathbb{Z}_p$-representations;

Communicated by S. Mochizuki. Received February 22, 2017. Revised April 29, 2017; May 2, 2017.

Y. Taguchi: Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, Japan;
e-mail: taguchi@math.titech.ac.jp

– to parametrize absolutely irreducible p-adic representations having a fixed residual representation ρ_0 defined over a finite field k, we do not need an assumption such as $\mathrm{End}(\rho_0) \simeq k$ to ensure the universality of the moduli space, although this is only at the expense of localization of the coefficient rings (e.g., making the prime p invertible).

As applications, we have the following:

(1) We reformulate, in moduli-theoretic language, the finiteness conjectures of Khare and Moon ([18], [27]) on mod p Galois representations with bounded conductor.
(2) We prove a certain relation between some finiteness conjectures on Galois representations (the conjecture mentioned above in (1) and two versions of the finiteness conjecture of Fontaine–Mazur ([11])).

The last application (2) is the original motivation for this work. If all the p-adic representations involved in the finiteness conjecture of Fontaine–Mazur had residual representations ρ_0 with the property $\mathrm{End}(\rho_0) \simeq k$, then Mazur's deformation theory would be enough for our application in (2) (see [44, Sect. 4]), but of course this is far from reality. Thus we were led to a moduli theory over $\mathbb{Q}_p$ that can deal with $\mathbb{Q}_p$-representations having "small" residual representations ρ_0 (or large $\mathrm{End}(\rho_0)$).

Now we explain the contents in more detail. In Section 2, we introduce the notions of "pro-$\mathfrak{a}$" and "f-pro-$\mathfrak{a}$ ring" for a category $\mathfrak{a}$ of rings (Definitions 2.1 and 2.2). Two typical examples of pro-$\mathfrak{a}$ rings are profinite and proartinian rings. Typically, f-pro-$\mathfrak{a}$ rings are obtained from pro-$\mathfrak{a}$ rings by localization (or, adjoining **f**ractions). The terminology "f-pro-$\mathfrak{a}$ ring" is chosen after "f-adic ring" of Huber ([16]), although we do not require any finiteness conditions on the rings. The main result here is that, for any f-pro-$\mathfrak{a}$ ring R and any integer $n \geq 1$, there exist a commutative f-pro-$\mathfrak{a}$ ring $\mathrm{F}_n(R)$ and a morphism $\Phi_{R,n} : R \to \mathrm{M}_n(\mathrm{F}_n(R))$ which is universal for morphisms of R into matrix algebras of degree n. This ring $\mathrm{F}_n(R)$ will be the main ingredient for the construction of the moduli scheme $X_{R,n}$ which parametrizes all degree-n absolutely irreducible representations of R.

After recalling basic facts on Azumaya algebras in Section 3, we define in Section 4 the notion of an "absolutely irreducible representation" of an f-$\mathfrak{A}$ ring R and then construct the moduli scheme $X_{R,n}$ (Theorem 4.20). Here, we work in a fixed category $\mathfrak{A}$ of topological rings, assuming certain axioms. The scheme $X_{R,n}$ is in the category of schemes obtained by globalizing the spectra of commutative objects in the category ${}^{\mathrm{f}}\mathfrak{A}$ of f-$\mathfrak{A}$ rings; thus if ${}^{\mathrm{f}}\mathfrak{A}$ is the category of f-pro-$\mathfrak{a}$ rings, then $X_{R,n}$ is an "f-pro-$\mathfrak{a}$ scheme". Clearly the theory extends to the case of a sheaf of rings instead of the ring R, but we content ourselves with the usual ring case to fix ideas, which will be enough in most applications.

Section 5 is a variant of this moduli theory for "τ-algebras", which will be used in an application in Section 8. The τ-algebras R over a commutative ring E are analogous to Azumaya algebras in that they have E-linear maps $\tau : R \to E$ in place of the reduced trace maps. As the τ-structure "rigidifies" representations of R, the coefficient ring E already knows enough of the absolutely irreducible τ-representations of R (Proposition 5.6).

Problems on representations of a group may often be reduced to those of a ring by considering the group ring. This is especially the case if the Brauer group of the coefficient ring vanishes. This, in our context, is explained in Section 6.

The first version of this paper contained a section on a certain zeta function

$$Z_R(T) = Z_{R,n,\mathbb{F}_q}(T) := \exp\left(\sum_{\nu=1}^{\infty} \frac{N_\nu}{\nu} T^\nu\right),$$

which is defined as the generating function of the number N_ν of degree-n absolutely irreducible representations of a profinite ring R into $\mathrm{M}_n(\mathbb{F}_{q^\nu})$ over various finite extensions $\mathbb{F}_{q^\nu}$ of a fixed finite field $\mathbb{F}_q$ (similar zeta functions were studied in [14] for *discrete* rings R). Since the results obtained in this section are rather trivial modulo the moduli theory, we have decided to omit the zeta function section. For instance, it can be shown that, if the moduli scheme $X_{R,n}$ is of finite type as a profinite scheme over $\mathbb{F}_q$, then $Z_{R,n,\mathbb{F}_q}(T)$ has nontrivial radius of convergence in the complex plane. This is trivially true if R is topologically finitely generated. So the interesting case is where R is not known to be finitely generated and still $X_{R,n}$ turns out to be of finite type. We suspect that the completed group rings of a certain kind of Galois groups of an algebraic number field would provide such examples (cf. Question 4.26).

In Section 7, we formulate a finiteness conjecture on the moduli scheme of the representations of the group ring over a finite field of a certain Galois group of a global field, and show that it implies the finiteness conjecture, which we call ($\mathbb{F}$), of Khare and Moon ([18], [27]) on mod p Galois representations of a global field with bounded conductor.

Finally in Section 8, we prove (Theorem 8.1) the equivalence under ($\mathbb{F}$) of the two versions ([11, Conj. 2a, 2b]) of the finiteness conjecture of Fontaine–Mazur on geometric Galois representations. The conjecture states that there are only finitely many isomorphism classes of geometric Galois representations of a certain type, and the difference between the two versions is whether the representations considered are defined over $\overline{\mathbb{Q}}_p$ or a fixed finite extension of $\mathbb{Q}_p$. This difference seems rather substantial. Our result shows that Conjecture ($\mathbb{F}$) fills in this gap in some sense. In proving the theorem, we use both our moduli theory and Mazur's deformation theory. As the latter can be reconstructed by using our moduli theory, we could make this section more self-contained. However, we find it more convenient to refer to the existing literature.

Some of the results in this paper might be proved also by employing the theory of Chenevier [7] of pseudorepresentations; in particular, he has also constructed (at least when R is a profinite ring) a moduli space of absolutely irreducible representations (which may be residually reducible) using a different approach. The author hopes to work out more precise relations of our theory to Chenevier's in the future.

Convention. In this paper, a *ring* is an associative (but not necessarily commutative) ring with unity. If E is a commutative ring, an *E-algebra* R is a ring that is endowed with a ring homomorphism $\alpha : E \to A$ whose image is contained in the center of A. By abuse of notation, we write $X \in \mathfrak{A}$ if X is an object of a category $\mathfrak{A}$. If $\mathfrak{A}$ is a category of rings, then $\mathfrak{C}$ will denote the full subcategory of $\mathfrak{A}$ consisting of commutative objects. Furthermore, if $E \in \mathfrak{C}$, then $\mathfrak{A}_E$ and $\mathfrak{C}_E$ denote the category of E-algebras whose underlying rings are in $\mathfrak{A}$ and $\mathfrak{C}$ respectively. When a category of topological rings is defined only by specifying what its objects are, we understand that the morphisms in the category are all the continuous ring homomorphisms. We denote by $\mathrm{Hom}_{\mathfrak{A}}(A, B)$ the set of morphisms from A to B in the category $\mathfrak{A}$. For a commutative ring E and an E-module M, we denote by $\mathrm{End}_E(M)$ the ring of E-module endomorphisms of M. For any ring R, we denote by $\mathrm{M}_n(R)$ the ring of $n \times n$ matrices with entries in R.

§2. f-pro-$\mathfrak{a}$ rings

Throughout this section, fix a category $\mathfrak{a}$ of rings, and let $\mathfrak{c}$ denote its full subcategory consisting of commutative objects.

Definition 2.1. A *pro-$\mathfrak{a}$ ring* R is a topological ring that is canonically isomorphic to the topological ring $\varprojlim_\lambda R/I_\lambda$, where $(I_\lambda)_{\lambda\in\Lambda}$ is a family, indexed with a right-directed set Λ, of open two-sided ideals of R such that the quotient rings R/I_λ are in $\mathfrak{a}$. A *morphism* of pro-$\mathfrak{a}$ rings is a continuous homomorphism of rings. We denote by $\widehat{\mathfrak{a}}$ the category of pro-$\mathfrak{a}$ rings, and by $\widehat{\mathfrak{c}}$ the full subcategory of $\widehat{\mathfrak{a}}$ consisting of commutative objects.

Next we shall define the notion of an f-pro-$\mathfrak{a}$ ring. The definition is given more generally as follows. Let $\mathfrak{A}$ be any category of topological rings (say, $\mathfrak{A} = \widehat{\mathfrak{a}}$ above).

Definition 2.2. An f-$\mathfrak{A}$ *ring* is a triple (R_{o}, R, f) consisting of an object R_{o} of $\mathfrak{A}$, a ring R and a ring homomorphism $f : R_{\mathrm{o}} \to R$. It is often denoted simply by (R_{o}, R) (or even R), with f (or (R_{o}, f)) understood. We call R_{o} (resp. R) the *first* (or *topological*) *factor* (resp. *second* (or *algebraic*) *factor*) of the f-$\mathfrak{A}$ ring (R_{o}, R, f). A *morphism* $(\phi_{\mathrm{o}}, \phi) : (R_{\mathrm{o}}, R, f) \to (R'_{\mathrm{o}}, R', f')$ (or simply $\phi : R \to R'$) is a commutative diagram

$$\begin{array}{ccc} R_{\mathrm{o}} & \xrightarrow{\phi_{\mathrm{o}}} & R'_{\mathrm{o}} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ R & \xrightarrow[\phi]{} & R' \end{array}$$

of rings in which ϕ_{o} is a morphism in $\mathfrak{A}$. We say an f-$\mathfrak{A}$ ring (R_{o}, R, f) is *commutative* if both R_{o} and R are commutative. We denote by ${}^{\mathrm{f}}\mathfrak{A}$ the category of f-$\mathfrak{A}$ rings, and by ${}^{\mathrm{f}}\mathfrak{C}$ its full subcategory consisting of commutative objects. In particular, we write ${}^{\mathrm{f}}\mathfrak{A} = {}^{\mathrm{f}}\widehat{\mathfrak{a}}$ and ${}^{\mathrm{f}}\mathfrak{C} = {}^{\mathrm{f}}\widehat{\mathfrak{c}}$ if $\mathfrak{A} = \widehat{\mathfrak{a}}$.

If $\mathfrak{a}$ is the category of artinian (resp. finite) rings, then we say also "proartinian" and "f-proartinian" (resp. "profinite" and "f-profinite") instead of pro-$\mathfrak{a}$ and f-pro-$\mathfrak{a}$.

The category $\mathfrak{A}$ is naturally identified with a full subcategory of ${}^{\mathrm{f}}\mathfrak{A}$ by $R_{\mathrm{o}} \mapsto (R_{\mathrm{o}}, R_{\mathrm{o}}, \mathrm{id})$.

Definition 2.3. An f-$\mathfrak{A}$ *subring* (or simply, *closed subring*) (S_{o}, S, g) of (R_{o}, R, f) is an object of ${}^{\mathrm{f}}\mathfrak{A}$ such that S_{o} is a closed subring of R_{o}, S is a subring of R and $g = f|_{S_{\mathrm{o}}}$.

An ideal I_{o} of an object R_{o} of $\mathfrak{A}$ is said to be *closed* if it is topologically closed in R_{o} and the quotient ring $R_{\mathrm{o}}/I_{\mathrm{o}}$ is again an object of $\mathfrak{A}$. An *ideal* of an f-$\mathfrak{A}$ ring (R_{o}, R, f) is a pair (I_{o}, I) consisting of an ideal I_{o} of R_{o} and an ideal I of R such that $f(I_{\mathrm{o}}) \subset I$. An ideal (I_{o}, I) is said to be *closed* if I_{o} is closed. If (I_{o}, I) is a closed ideal of (R_{o}, R, f), then the *quotient* $(R_{\mathrm{o}}/I_{\mathrm{o}}, R/I, \overline{f})$ is again an f-$\mathfrak{A}$ ring, where $\overline{f}$ is the ring homomorphism $R_{\mathrm{o}}/I_{\mathrm{o}} \to R/I$ induced by f.

A morphism $(\phi_{\mathrm{o}}, \phi) : (R_{\mathrm{o}}, R, f) \to (R'_{\mathrm{o}}, R', f')$ of f-$\mathfrak{A}$ rings is said to be *injective* (resp. *surjective*) if ϕ is injective (resp. surjective).

For a morphism $(\phi_{\mathrm{o}}, \phi) : (R_{\mathrm{o}}, R, f) \to (R'_{\mathrm{o}}, R', f')$ of f-$\mathfrak{A}$ rings, define

$$\mathrm{Ker}(\phi_{\mathrm{o}}, \phi) := (\mathrm{Ker}(\phi_{\mathrm{o}}), \mathrm{Ker}(\phi)).$$

This is an ideal of (R_{o}, R, f). If $\mathrm{Im}(\phi_{\mathrm{o}})$ exists in $\mathfrak{A}$ (in particular, if $\mathrm{Ker}(\phi_{\mathrm{o}})$ is a closed ideal of R_{o}), then we define

$$\mathrm{Im}(\phi_{\mathrm{o}}, \phi) := (\mathrm{Im}(\phi_{\mathrm{o}}), \mathrm{Im}(\phi), f'|_{\mathrm{Im}(\phi_{\mathrm{o}})}).$$

This is an f-$\mathfrak{A}$ subring of (R'_o, R', f') if $\mathrm{Im}(\phi_o)$ is closed in R'_o.

Let (G_o, G) be a pair consisting of a subset G_o of R_o and a subset G of R. We say the f-$\mathfrak{A}$ ring (R_o, R, f) is *generated* by (G_o, G) if R_o is topologically generated by G_o (i.e., R_o is the smallest closed subring of R_o containing G_o) and R is generated as an R_o-algebra (via f) by G. We say that (R_o, R, f) is *finitely generated* if there exist finite subsets G_o and G as above such that (G_o, G) generates (R_o, R, f).

Let $E = (E_o, E)$ be a commutative f-$\mathfrak{A}$ ring.

Definition 2.4. An *f-$\mathfrak{A}$ E-algebra* is an f-$\mathfrak{A}$ ring (A_o, A, f) that is endowed with a morphism $\alpha : (E_o, E) \to (A_o, A)$ in ${}^{\mathrm{f}}\mathfrak{A}$ whose image is contained in the center of (A_o, A) (the center of (A_o, A, f) is $(\mathrm{Center}(A_o), \mathrm{Center}(A), f|_{\mathrm{Center}(A_o)})$, by definition). A *morphism* $\phi : A \to B$ of f-$\mathfrak{A}$ E-algebras is a morphism of f-$\mathfrak{A}$ rings that is compatible with the structure morphisms $\alpha : E \to A$ and $\beta : E \to B$; i.e., $\phi \circ \alpha = \beta$. We denote by ${}^{\mathrm{f}}\mathfrak{A}_E$ and ${}^{\mathrm{f}}\mathfrak{C}_E$, respectively, the category of f-$\mathfrak{A}$ E-algebras and its full subcategory consisting of commutative objects. In particular, ${}^{\mathrm{f}}\widehat{\mathfrak{a}}_E$ and ${}^{\mathrm{f}}\widehat{\mathfrak{c}}_E$ are respectively the category of f-pro-$\mathfrak{a}$ E-algebras and its full subcategory of commutative f-pro-$\mathfrak{a}$ E-algebras.

For example, if $\mathfrak{a}$ is the category of finite rings, then we have ${}^{\mathrm{f}}\widehat{\mathfrak{a}}_{\widehat{\mathbb{Z}}} = {}^{\mathrm{f}}\widehat{\mathfrak{a}}$ and ${}^{\mathrm{f}}\widehat{\mathfrak{c}}_{\widehat{\mathbb{Z}}} = {}^{\mathrm{f}}\widehat{\mathfrak{c}}$, where $\widehat{\mathbb{Z}}$ is the profinite completion of the integer ring $\mathbb{Z}$.

Remark 2.5. In a former version of this paper, we tried to define an f-$\mathfrak{A}$ ring as a kind of topological ring. As K. Kato and S. Yasuda pointed out, however, it seems difficult to define a natural topology on localizations of a pro-$\mathfrak{a}$ ring. For example, if R_o is the formal power series ring $\mathbb{Z}_p[\![x]\!]$ over the p-adic integer ring $\mathbb{Z}_p$ with the profinite topology, then in $R = R_o[1/p]$, the image pR_o of R_o by multiplication by p, which is an automorphism of R, is not an open subset of R_o. Thus we decided to abandon the topology of R but keep R_o in the data of an f-$\mathfrak{A}$ ring to "remember" the topology of R_o.

Example 2.6. If R_o is a pro-$\mathfrak{a}$ ring and R is the localization $S^{-1}R_o$ of R_o with respect to a multiplicative subset S of the center of R_o, then (R_o, R, f) is an f-pro-$\mathfrak{a}$ ring, where $f : R_o \to R$ is the localization map. The same applies if R_o is a pro-$\mathfrak{a}$ E_o-algebra and S is a multiplicative subset of E_o. For example, if G is a profinite group, then the completed group ring $\widehat{\mathbb{Z}}[\![G]\!] = \varprojlim_{n,H} (\mathbb{Z}/n\mathbb{Z})[G/H]$, where the projective limit is over all nonzero integers n and open normal subgroups H of G, is a profinite ring. For any nonzero integer N, the triple $(\widehat{\mathbb{Z}}[\![G]\!], \widehat{\mathbb{Z}}[\![G]\!][1/N], f)$ is an f-profinite ring, where $f : \widehat{\mathbb{Z}}[\![G]\!] \to \widehat{\mathbb{Z}}[\![G]\!][1/N]$ is the localization map.

Example 2.7. Let (R_o, R, f) be an f-$\mathfrak{A}$ ring. If $g : R \to R'$ is a ring homomorphism, then $(R_o, R', g \circ f)$ is also an f-$\mathfrak{A}$ ring. We denote by R^{op} the opposite algebra to R (i.e., the ring whose underlying module is R itself and the multiplication $x \cdot y$ in R^{op} is defined to be yx in R). If the category $\mathfrak{A}$ is closed under opposition, then $(R_{\mathrm{o}}^{\mathrm{op}}, R^{\mathrm{op}}, f^{\mathrm{op}})$ is also an f-$\mathfrak{A}$ ring. If $\mathfrak{A}$ is closed under taking the matrix algebra $\mathrm{M}_n(\text{-})$, then $(\mathrm{M}_n(R_{\mathrm{o}}), \mathrm{M}_n(R), \mathrm{M}_n(f))$ is also an f-$\mathfrak{A}$ ring. Thus, for example, $(R_{\mathrm{o}}, \mathrm{M}_n(R), \iota \circ f)$, $(R_{\mathrm{o}}^{\mathrm{op}}, \mathrm{M}_n(R)^{\mathrm{op}}, \iota \circ f^{\mathrm{op}})$, $(\mathrm{M}_n(R_{\mathrm{o}}), \mathrm{M}_n(R), \mathrm{M}_n(f))$, etc. are f-$\mathfrak{A}$ rings, where $\iota : R \to \mathrm{M}_n(R)$ is the map identifying R with the scalar matrices.

In the rest of this section, we assume the following axiom on the category $\mathfrak{a}$:

(a1) The category $\mathfrak{a}$ is closed under taking subrings, quotients and finite direct-products.

For example, if Λ is a fixed commutative ring, the category of Λ-algebras that are of finite length as Λ-modules has this property, while the category of all artinian rings does not.

Example 2.8. For any ring R, the family $(R_\lambda)_\lambda$ of all quotients R_λ of R that are in $\mathfrak{a}$ forms a projective system. Indeed, if $R_1 = R/I_1$ and $R_2 = R/I_2$ are two such quotients, then the quotient $R_3 := R/(I_1 \cap I_2)$ may be identified with a subring of the direct-product $R_1 \times R_2$ and hence it is in $\mathfrak{a}$ by Axiom (a1) above. Let $\widehat{R} = \varprojlim_\lambda R_\lambda$ be the projective limit of this projective system. Then $\widehat{R}$ is pro-$\mathfrak{a}$, and is called the *pro-$\mathfrak{a}$ completion* of R.

In the rest of this section, we fix a commutative f-pro-$\mathfrak{a}$ ring Z. Our purpose in this section is to prove the following lemma.

Lemma 2.9. *For any $R \in {}^{\mathrm{f}}\widehat{\mathfrak{a}}_Z$, $F \in {}^{\mathrm{f}}\widehat{\mathfrak{c}}_Z$ and an integer $n \geq 1$, there exist a unique object $\mathrm{F}_n(R) \in {}^{\mathrm{f}}\widehat{\mathfrak{c}}_F$ and a universal morphism $\Phi : R \to \mathrm{M}_n(\mathrm{F}_n(R))$ in ${}^{\mathrm{f}}\widehat{\mathfrak{a}}_Z$, the universality meaning that for any $O \in {}^{\mathrm{f}}\widehat{\mathfrak{c}}_F$ and any morphism $\phi : R \to \mathrm{M}_n(O)$ in ${}^{\mathrm{f}}\widehat{\mathfrak{a}}_Z$, there exists a unique morphism $f : \mathrm{F}_n(R) \to O$ in ${}^{\mathrm{f}}\widehat{\mathfrak{c}}_F$ such that $\phi = f_*\Phi := \mathrm{M}_n(f) \circ \Phi$. In other words, we have a canonical bijection*

$$\mathrm{Hom}_{{}^{\mathrm{f}}\widehat{\mathfrak{c}}_F}(\mathrm{F}_n(R), O) \simeq \mathrm{Hom}_{{}^{\mathrm{f}}\widehat{\mathfrak{a}}_Z}(R, \mathrm{M}_n(O)),$$
$$f \mapsto f_*\Phi,$$

functorially in $R \in {}^{\mathrm{f}}\widehat{\mathfrak{a}}_F$ and $O \in {}^{\mathrm{f}}\widehat{\mathfrak{c}}_F$. Thus the functor $\mathrm{F}_n : {}^{\mathrm{f}}\widehat{\mathfrak{a}}_Z \to {}^{\mathrm{f}}\widehat{\mathfrak{c}}_F$ is a left adjoint of the functor $\mathrm{M}_n : {}^{\mathrm{f}}\widehat{\mathfrak{c}}_F \to {}^{\mathrm{f}}\widehat{\mathfrak{a}}_Z$. If R is pro-$\mathfrak{a}$*, then so is $\mathrm{F}_n(R)$.*

Proof. First assume that Z is pro-$\mathfrak{a}$ and R is in $\mathfrak{a}$. Then we can write

$$R \simeq Z\langle X_\mu \rangle_{\mu \in M}/I,$$

where $Z\langle X_\mu\rangle_{\mu\in M}$ (or $Z\langle X_\mu\rangle$ for short) denotes the noncommutative polynomial ring over Z in variables X_μ indexed by $\mu \in M$, and I is a two-sided ideal of $Z\langle X_\mu\rangle$. For each μ, let $\underline{X}_\mu = (x_{\mu ij})_{1\le i,j\le n}$ be an $(n\times n)$-matrix with variable components $x_{\mu ij}$, and let $F[x_{\mu ij}]_{\mu\in M,1\le i,j\le n}$ (or $F[x_{\mu ij}]$ for short) be the commutative polynomial ring over F in the variables $x_{\mu ij}$. Then there is a Z-algebra homomorphism

$$\varphi : Z\langle X_\mu\rangle \to \mathrm{M}_n(F[x_{\mu ij}]) \tag{2.1}$$

that extends the structure morphism $\iota : Z \to F$ and that maps the variable X_μ to the matrix $\underline{X}_\mu$. For each $f \in I$, let $f_{ij} \in F[x_{\mu ij}]$ be the (i,j)-component of the matrix $\varphi(f) \in \mathrm{M}_n(F[x_{\mu ij}])$. Let $\overline{I}$ be the ideal of $F[x_{\mu ij}]$ generated by the f_{ij} for all $f \in I$ and $1 \le i,j \le n$. Then the map φ descends to a Z-algebra homomorphism

$$\overline{\varphi} : R \to \mathrm{M}_n(F[x_{\mu ij}]/\overline{I}).$$

Let $\mathrm{F}_n(R)$ be the pro-$\mathfrak{a}$ completion of the ring $F[x_{\mu ij}]/\overline{I}$. Composing the above $\overline{\varphi}$ with the natural map $\mathrm{M}_n(F[x_{\mu ij}]/\overline{I}) \to \mathrm{M}_n(\mathrm{F}_n(R))$, we obtain a morphism

$$\Phi : R \to \mathrm{M}_n(\mathrm{F}_n(R))$$

of pro-$\mathfrak{a}$ Z-algebras. By construction, we have

$$\begin{gathered}\mathrm{Hom}_{\widehat{\mathfrak{c}}_F}(\mathrm{F}_n(R), O) \simeq \mathrm{Hom}_{\widehat{\mathfrak{a}}_Z}(R, \mathrm{M}_n(O)),\\ f \mapsto f_*\Phi,\end{gathered}$$

functorially in $O \in \mathfrak{c}_F$.

Next suppose that both Z and R are pro-$\mathfrak{a}$. If R is a projective limit of Z-algebras R_λ in $\mathfrak{a}$, then apply the above construction to each R_λ to obtain pro-$\mathfrak{a}$ F-algebras $\mathrm{F}_n(R_\lambda)$ and morphisms $\Phi_\lambda : R_\lambda \to \mathrm{M}_n(\mathrm{F}_n(R_\lambda))$ in $\widehat{\mathfrak{a}}_Z$. These maps Φ_λ form a projective system

$$\begin{array}{ccc} R_\lambda & \xleftarrow{\pi_{\lambda'\lambda}} & R_{\lambda'} \\ {\scriptstyle\Phi_\lambda}\downarrow & & \downarrow{\scriptstyle\Phi_{\lambda'}} \\ \mathrm{M}_n(\mathrm{F}_n(R_\lambda)) & \xleftarrow[\varpi_{\lambda'\lambda}]{} & \mathrm{M}_n(\mathrm{F}_n(R_{\lambda'})). \end{array}$$

Indeed, if $\lambda < \lambda'$ and $\pi_{\lambda'\lambda} : R_{\lambda'} \to R_\lambda$ is the transition map, then

$$\Phi_\lambda \circ \pi_{\lambda'\lambda} : R_{\lambda'} \to \mathrm{M}_n(\mathrm{F}_n(R_\lambda))$$

is a Z-algebra homomorphism, and hence, by the universality of $\Phi_{\lambda'}$, it factors through $\Phi_{\lambda'}$, yielding a unique morphism $f_{\lambda'\lambda} : \mathrm{F}_n(R_{\lambda'}) \to \mathrm{F}_n(R_\lambda)$ that makes the above diagram commutative with $\varpi_{\lambda'\lambda} = (f_{\lambda'\lambda})_*\Phi_{\lambda'}$. Now put

$$\mathrm{F}_n(R) := \varprojlim_{\lambda\in\Lambda} \mathrm{F}_n(R_\lambda).$$

Noting that $\varprojlim_{\lambda\in\Lambda} \mathrm{M}_n(\mathrm{F}_n(R_\lambda)) = \mathrm{M}_n(\varprojlim_{\lambda\in\Lambda} \mathrm{F}_n(R_\lambda))$, we then obtain a morphism

$$\Phi := \varprojlim_{\lambda\in\Lambda} \Phi_\lambda : R \to \mathrm{M}_n(\mathrm{F}_n(R))$$

in $\widehat{\mathfrak{a}}_Z$ which, by construction, is continuous and has the required universal property. This proves the last assertion of the lemma.

Finally, suppose $Z_\mathrm{o} \to Z$ and $R_\mathrm{o} \to R$ are f-pro-$\mathfrak{a}$. By the above arguments, there is an $\mathrm{F}_n(R_\mathrm{o}) \in \widehat{\mathfrak{c}}_{F_\mathrm{o}}$ and a universal morphism $\Phi_\mathrm{o} : R_\mathrm{o} \to \mathrm{M}_n(\mathrm{F}_n(R_\mathrm{o}))$ in $\widehat{\mathfrak{a}}_{Z_\mathrm{o}}$. Extend it Z-linearly to a morphism

$$\Phi_{\mathrm{o},Z} : Z \otimes_{Z_\mathrm{o}} R_\mathrm{o} \to \mathrm{M}_n(F \otimes_{F_\mathrm{o}} \mathrm{F}_n(R_\mathrm{o})) \tag{2.2}$$

in ${}^{\mathrm{f}}\widehat{\mathfrak{a}}_Z$. Suppose $\{\eta_\nu\}_{\nu\in N}$ is a subset of R that generates R over $Z \otimes_{Z_\mathrm{o}} R_\mathrm{o}$. Then there is a surjection $(Z \otimes_{Z_\mathrm{o}} R_\mathrm{o})\langle Y_\nu\rangle_{\nu\in N} \to R$ that maps Y_ν to η_ν, so that

$$R \simeq (Z \otimes_{Z_\mathrm{o}} R_\mathrm{o})\langle Y_\nu\rangle_{\nu\in N}/J$$

for some two-sided ideal J of $(Z \otimes_{Z_\mathrm{o}} R_\mathrm{o})\langle Y_\nu\rangle$. For each ν, let $\underline{Y}_\nu = (y_{\nu ij})_{1\le i,j\le n}$ be an $(n \times n)$-matrix with variable components $y_{\nu ij}$. Then the morphism $\Phi_{\mathrm{o},Z}$ of (2.2) extends to a morphism

$$\varphi : (Z \otimes_{Z_\mathrm{o}} R_\mathrm{o})\langle Y_\nu\rangle \to \mathrm{M}_n((F \otimes_{F_\mathrm{o}} \mathrm{F}_n(R_\mathrm{o}))[y_{\nu ij}])$$

in ${}^{\mathrm{f}}\widehat{\mathfrak{a}}_Z$ by mapping Y_ν to $\underline{Y}_\nu$. Here, $(F \otimes_{F_\mathrm{o}} \mathrm{F}_n(R_\mathrm{o}))[y_{\nu ij}]$ is the commutative polynomial ring over $F \otimes_{F_\mathrm{o}} \mathrm{F}_n(R_\mathrm{o})$ in the variables $y_{\nu ij}$ for all $\nu \in N$ and $1 \le i,j \le n$. Now define $\mathrm{F}_n(R)$ to be the quotient of $(F \otimes_{F_\mathrm{o}} \mathrm{F}_n(R_\mathrm{o}))[y_{\nu ij}]$ by the ideal generated by all the (i,j)-components of $\varphi(g)$ for all $g \in J$. Then φ descends to a morphism

$$\Phi : R \to \mathrm{M}_n(\mathrm{F}_n(R))$$

in ${}^{\mathrm{f}}\widehat{\mathfrak{a}}_Z$, which extends Φ_o. By construction, the morphism $(\Phi_\mathrm{o}, \Phi) : (R_\mathrm{o}, R) \to (\mathrm{M}_n(\mathrm{F}_n(R_\mathrm{o})), \mathrm{M}_n(\mathrm{F}_n(R)))$ has the required properties. □

§3. Algebraic preliminaries on Azumaya algebras

In this section, we fix a category $\mathfrak{A}$ of rings and let $\mathfrak{C}$ be its full subcategory consisting of commutative objects (in applications, $\mathfrak{A}$ will be a category of topological rings, but in this section we are concerned only with algebraic structures). For each $E \in \mathfrak{C}$, we denote by $\mathfrak{A}_E$ the category of E-algebras in $\mathfrak{A}$, and by $\mathfrak{C}_E$ the full subcategory of $\mathfrak{A}_E$ consisting of commutative objects. Throughout this section, F is an object of $\mathfrak{C}$, and all locally free F-modules are assumed to be *of finite and constant rank*. Recall (e.g., [21, Chap. I, Sect. 5]) that an F-module M is said to be locally free of finite type if there is a Zariski covering $\prod_{i=1}^n F_{f_i}$ of F (i.e., a finite direct-product of rings of fractions $F_{f_i} = F[1/f_i]$, $f_i \in F$, which is faithfully flat over F) such that $F_{f_i} \otimes_F M$ is free of finite type over F_{f_i}, and that this is equivalent to saying that M is of finite presentation over F and, for each maximal ideal $\mathfrak{m}$ of M, the localization $M_\mathfrak{m}$ of M at $\mathfrak{m}$ is free over the local ring $F_\mathfrak{m}$.

In what follows, for a ring A, we denote by A^{op} the opposite algebra of A.

Definition 3.1. An *Azumaya algebra* A over F of *degree* n is an object of $\mathfrak{A}_F$ such that

(1) A is a locally free F-module of rank n^2;

(2) the map

$$\begin{aligned} \iota : A \otimes_F A^{\mathrm{op}} &\to \mathrm{End}_F(A), \\ a \otimes b &\mapsto (x \mapsto axb) \end{aligned}$$

is an isomorphism of rings.

Remark 3.2. It follows from (1) that the structure morphism $F \to A$ is injective. Henceforth we identify F with its image in A. It follows from (2) that the center of A coincides with F. Note that $A \otimes_F A^{\mathrm{op}}$ and $\mathrm{End}_F(A)$ are both locally free as F-modules.

Recall from basic facts on Azumaya algebras (see, e.g., [12], [21]) that, for any Azumaya algebra A over F, there exists a faithfully flat morphism $F \to F'$ of commutative rings such that A *splits* over F', i.e., the F'-algebra $F' \otimes_F A$ is isomorphic to the matrix algebra $\mathrm{M}_n(F')$. Then A may be identified with a subalgebra of $\mathrm{M}_n(F')$. We say that such a morphism $F \to F'$ is a *splitting* of A. In fact, a splitting $F \to F'$ can be taken to be finite étale (cf. [12, Thm. 5.1(iii)]).

Recall also that there is an F-linear map $\mathrm{Tr}_{A/F} : A \to F$, called the *reduced trace map* of A. It commutes with any base change, i.e., for any $f : F \to F'$, we have $\mathrm{Tr}_{A'/F'} = F' \otimes_F \mathrm{Tr}_{A/F}$, where $A' = F' \otimes_F A$, and a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{Tr}_{A/F}} & F \\ \downarrow & & \downarrow \\ A' & \xrightarrow[\mathrm{Tr}_{A'/F'}]{} & F', \end{array}$$

where the left vertical arrow is the natural map induced by f. If $F \to F'$ is a splitting of A, then the two vertical arrows in the above diagram are injective and $\mathrm{Tr}_{A/F}$ coincides with the restriction of the usual trace map $\mathrm{Tr}_{A'/F'} : A' \to F'$ of the matrix algebra $A' \simeq \mathrm{M}_n(F')$. Note that $\mathrm{Tr}_{A/F}$ is surjective. This is clear for $\mathrm{Tr}_{A'/F'}$, and the surjectivity for $\mathrm{Tr}_{A/F}$ follows from this by faithfully flat descent.

Next we consider conditions for a subalgebra of an Azumaya algebra to become an Azumaya algebra.

Definition 3.3. Let M and N be two F-modules. An F-bilinear map $\beta : M \times N \to F$ is said to be *nondegenerate* if, for every maximal ideal $\mathfrak{m}$ of F, the induced $\kappa(\mathfrak{m})$-bilinear map $\beta(\mathfrak{m}) : M(\mathfrak{m}) \times N(\mathfrak{m}) \to \kappa(\mathfrak{m})$ is nondegenerate in the usual sense, where $\kappa(\mathfrak{m})$ denotes the residue field of $\mathfrak{m}$ and $M(\mathfrak{m}) := \kappa(\mathfrak{m}) \otimes_F M$, and so on.

For example, if A is an Azumaya algebra over F, then the map

$$\begin{aligned} A \times A &\to F, \\ (a, b) &\mapsto \mathrm{Tr}_{A/F}(ab) \end{aligned}$$

is nondegenerate. This is easily checked if A is a matrix algebra, and the general case follows by faithfully flat descent.

Suppose that M and N are free F-modules of the same rank m, and that $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ are respectively F-bases of M and N. Let $\beta : M \times N \to F$ be an F-bilinear map. If we set

$$D := (\beta(x_i, y_j))_{1 \le i,j \le m} \quad \in \; \mathrm{M}_m(F),$$

then for $x = \sum_{i=1}^m a_i x_i \in M$ and $y = \sum_{i=1}^m b_i y_i \in N$, we have

$$\beta(x, y) = (a_1 \cdots a_m) D\, {}^{\mathrm{t}}(b_1 \cdots b_m),$$

where ${}^{\mathrm{t}}(b_1 \cdots b_m)$ denotes the transpose of $(b_1 \cdots b_m)$. It follows in particular that

(3.1) β is nondegenerate if and only if D is invertible.

In this discussion, we assumed that M and N are free on bases (x_i) and (y_i), but in some sense this is implied by the invertibility of the matrix D, as we shall see presently. Before that, we prove the following lemma.

Lemma 3.4. *Let M and N be two F-modules, and let $\beta : M \times N \to F$ be an F-bilinear map. Let $x_1, \ldots, x_m \in M$ and $y_1, \ldots, y_m \in N$, and put $D = (\beta(x_i, y_j))_{1 \le i,j \le m}$. For any $x \in M$, consider the equation*

(3.2) $$a_1 x_1 + \cdots + a_m x_m = x \quad \text{in } M,$$

in which a_i are unknowns in F. Suppose D is invertible. Then the solution to the equation (3.2) is unique if it exists and, when it exists, it is given by

$$(a_1, \dots, a_m) = (\beta(x, y_1), \dots, \beta(x, y_m))D^{-1}. \tag{3.3}$$

Proof. Equation (3.2) implies

$$a_1\beta(x_1, y_j) + \cdots + a_m\beta(x_m, y_j) = \beta(x, y_j)$$

for each j, i.e.,

$$(a_1, \dots, a_m)D = (\beta(x, y_1), \dots, \beta(x, y_m)). \tag{3.4}$$

Then the lemma follows. □

Lemma 3.5. *Let M and N be two locally free F-modules of rank m, and let $\beta : M \times N \to F$ be an F-bilinear map. Let $x_1, \dots, x_m \in M$ and $y_1, \dots, y_m \in N$, and put $d = \det(\beta(x_i, y_j))_{1 \le i,j \le m}$. Then the following conditions are equivalent:*

(1) $d \in F^\times$;

(2) the F-bilinear map β is nondegenerate, and M and N are free F-modules, of which (x_i) and (y_i) are respective F-bases.

Proof. Considering modulo each maximal ideal of F, we may assume that F is a field. Indeed, each of the conditions above ($d \in F^\times$, (x_i) is a basis, β is nondegenerate) holds true if and only if it holds modulo $\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of F. In view of (3.1), we need to prove only that (1) implies that M and N are free on (x_i) and (y_i) respectively. Since M and N are assumed locally free, it is enough to show that (x_i) and (y_i) are, respectively, linearly independent. If $d \in F^\times$, this follows from Lemma 3.4. □

We generalize this lemma as follows, so as to be convenient in our applications.

Lemma 3.6. *Let M and N be two locally free F-modules of rank m, and let $\beta : M \times N \to F$ be an F-bilinear map. Let F_0 be a subring of F, and M_0, N_0 be F_0-submodules of M, N respectively. Assume that $\beta(M_0, N_0) \subset F_0$. Let $x_1, \dots, x_m \in M_0$ and $y_1, \dots, y_m \in N_0$, and put $d = \det(\beta(x_i, y_j))_{1 \le i,j \le m}$.*

(i) The following conditions are equivalent:

(1) $d \in {F_0}^\times$;

(2) the F_0-bilinear map $\beta_0 : M_0 \times N_0 \to F_0$ induced by β is nondegenerate, and M_0, N_0 are free F_0-modules, of which (x_i), (y_i) are respective F_0-bases.

(ii) If the above conditions (1), (2) hold, then M, N are free F-modules, and (x_i), (y_i) are respectively their F-bases. One has $M \simeq F \otimes_{F_0} M_0$ and $N \simeq F \otimes_{F_0} N_0$.

Proof. (i) By (3.1), we need to prove only that (1) implies that M_0 and N_0 are free on (x_i) and (y_i) respectively. For a given $x \in M_0$, consider the equation

$$a_1 x_1 + \cdots + a_m x_m = x \quad \text{in } M_0,$$

where $a_1, \ldots, a_m$ are unknowns. By Lemma 3.5, this has a unique solution $(a_1, \ldots, a_m)$ in $F^{\oplus m}$, which is given by (3.3). Since $\beta(x, y_j) \in F_0$ and D is invertible in $\mathrm{M}_m(F_0)$, the solution is in fact in ${F_0}^{\oplus m}$. Hence (x_i) is an F_0-basis of M_0. The same is true for (y_i) and N_0.

Part (ii) follows from (i) upon noticing that ${F_0}^{\times} \subset F^{\times}$. □

This lemma implies the following proposition, except for part (ii–1) of it:

Proposition 3.7. *Let A be an Azumaya algebra over F of degree n. Let F_0 be a subring of F, and let A_0 be an F_0-subalgebra of A. Assume that $\mathrm{Tr}_{A/F}(A_0) \subset F_0$. For any n^2-tuple $\boldsymbol{a} = (a_1, \ldots, a_{n^2})$ of elements a_i of A_0, set*

$$d(\boldsymbol{a}) := \det(\mathrm{Tr}_{A/F}(a_i a_j))_{1 \le i,j \le n^2} \quad \in \ F_0.$$

(i) The following conditions are equivalent:

(i–1) $d(\boldsymbol{a}) \in {F_0}^{\times}$;

(i–2) the F_0-bilinear map

$$\begin{aligned} \beta_0 : A_0 \times A_0 &\to F_0, \\ (a, b) &\mapsto \mathrm{Tr}_{A/F}(ab) \end{aligned}$$

is nondegenerate, and A_0 is a free F_0-module, of which $(a_1, \ldots, a_{n^2})$ is an F_0-basis.

(ii) If the above conditions (i–1), (i–2) hold, then

(ii–1) A_0 is an Azumaya algebra over F_0 of degree n and, in particular, F_0 coincides with the center of A_0;

(ii–2) A is a free F-module, and $(a_1, \ldots, a_{n^2})$ is an F-basis of A; one has $A \simeq F \otimes_{F_0} A_0$.

To prove (ii–1), recall that if A is a locally free F-module of rank m, then the ring $\boldsymbol{E} := \mathrm{End}_F(A)$ is an Azumaya algebra over F of degree m (cf. [21, Chap. III, Thm. 5.1]). In particular, we have the reduced trace map $\mathrm{Tr}_{\boldsymbol{E}/F} : \boldsymbol{E} \to F$. If, moreover, A is an F-algebra, then we have a natural ring homomorphism $\iota : A \otimes_F A^{\mathrm{op}} \to \boldsymbol{E}$; $a \otimes b \mapsto (x \mapsto axb)$. If A is an Azumaya algebra over F, then we have $\mathrm{Tr}_{\boldsymbol{E}/F}(\iota(a \otimes b)) = \mathrm{Tr}_{A/F}(a)\mathrm{Tr}_{A/F}(b)$ for any $a, b \in A$. This can be seen, for example, by choosing a splitting $F \to F'$ for A and considering with the matrix algebra $\mathrm{M}_n(F')$ in place of A. The next lemma is extracted from the proof of [35, Theorem 2.2].

Lemma 3.8. *Let A be an F-algebra that is locally free of rank m as an F-module. Suppose there is given an F-linear map $\tau : A \to F$ that satisfies*

$$\mathrm{Tr}_{\boldsymbol{E}/F}(\iota(a \otimes b)) = \tau(a)\tau(b) \tag{3.5}$$

for any $a \otimes b \in A \otimes_F A^{\mathrm{op}}$. If there exists an m-tuple $(a_1, \ldots, a_m)$ of elements of A such that

$$\det(\tau(a_i a_j))_{1 \le i,j \le m} \quad \in \ F^{\times}, \tag{3.6}$$

then A is an Azumaya algebra over F of degree $n = \sqrt{m}$. It is free as an F-module, and $(a_1, \ldots, a_m)$ is an F-basis of it.

Remark 3.9. Conversely, if A/F is an Azumaya algebra that is free over F with a basis $(a_1, \ldots, a_m)$, then Proposition 3.7(i) implies that (3.6) holds with $\tau = \mathrm{Tr}_{A/F}$. Thus the conditions (3.5) and (3.6) characterize Azumaya algebras that are free over their centers.

Proof. We shall show that the ring homomorphism

$$\iota : A \otimes_F A^{\mathrm{op}} \to \mathrm{End}_F(A)$$

is an isomorphism. To prove this, by Lemma 3.10 below, it is enough to show that ι is surjective, for which in turn it is enough to show that the family $(\iota(a_i \otimes a_j))_{1 \le i,j \le m}$ is an F-basis of $\mathrm{End}_F(A)$. This follows from Proposition 3.7(ii–2) (with $\iota(a_i \otimes a_{i'})$ in place of a_i), since we have

$$\begin{aligned}
&\det(\mathrm{Tr}_{\boldsymbol{E}/F}(\iota(a_i \otimes a_{i'}) \circ \iota(a_j \otimes a_{j'})))_{1 \le i,i',j,j' \le m} \\
&\quad = \det(\mathrm{Tr}_{\boldsymbol{E}/F}(\iota(a_i a_j \otimes a_{i'} a_{j'})))_{1 \le i,i',j,j' \le m} \\
&\quad = \det(\tau(a_i a_j)\tau(a_{i'} a_{j'}))_{1 \le i,i',j,j' \le m} \qquad (\text{by } (3.5)) \\
&\quad = \det((\tau(a_i a_j))_{1 \le i,j \le m} \otimes (\tau(a_{i'} a_{j'}))_{1 \le i',j' \le m}) \\
&\quad = \det(\tau(a_i a_j))^{2m}_{1 \le i,j \le m},
\end{aligned}$$

which is in $F^{\times}$ by assumption (3.6). Here, in the fourth det, the $\otimes$ means the Kronecker product of the $m \times m$ matrix $(\tau(a_i a_j))$ with itself. □

Lemma 3.10. *Let M and N be locally free F-modules of rank m and n respectively, with $m \leq n$. If there is a surjective homomorphism $f : M \to N$ of F-modules, then it is in fact bijective.*

Proof. Let $K = \mathrm{Ker}(f)$. To show that $K = 0$, by faithfully flat descent, we may assume that M and N are free over F. Then there is a section $g : N \to M$ to f, so that we have $M = K \oplus \mathrm{Im}(g)$. This shows that K is a projective F-module. Since the rank of a projective F-module is additive with respect to short exact sequences, it follows that $K = 0$. □

Now we prove Proposition 3.7(ii–1).

Proof of Proposition 3.7(ii–1). By Proposition 3.7(ii–2) and the assumption that $d \in {F_0}^\times$, the F_0-module A_0 is free with a basis $(a_1, \dots, a_{n^2})$, and we have $A \simeq F \otimes_{F_0} A_0$. Hence we have the following commutative diagram:

$$\begin{array}{ccc} A_0 \otimes_{F_0} A_0^{\mathrm{op}} & \longrightarrow & \mathrm{End}_{F_0}(A_0) \\ \big\downarrow & & \big\downarrow {\scriptstyle F\text{-linear ext.}} \\ A \otimes_F A^{\mathrm{op}} & \xrightarrow[\iota]{\simeq} & \mathrm{End}_F(A), \end{array}$$

in which the two vertical arrows are injective and ι is bijective. We apply Lemma 3.8 with A_0/F_0 in place of A/F and $\tau = \mathrm{Tr}_{A/F}|_{A_0}$. For $a \otimes b \in A \otimes_F A^{\mathrm{op}}$, we have

$$\mathrm{Tr}_{\boldsymbol{E}/F}(\iota(a \otimes b)) = \mathrm{Tr}_{A/F}(a)\mathrm{Tr}_{A/F}(b),$$

since A is Azumaya. Hence condition (3.5) of Lemma 3.8 is satisfied for all $a \otimes b \in A_0 \otimes_{F_0} A_0^{\mathrm{op}}$. Condition (3.6) holds by the assumption of Proposition 3.7. It follows that A_0 is an Azumaya algebra over F_0 of degree n. □

Corollary 3.11. *Let A be an Azumaya algebra over F of degree n. Let F_0 be a subring of F, and let A_0 be a subring of A that is an F_0-algebra. Assume that $\mathrm{Tr}_{A/F}(A_0) \subset F_0$. For any n^2-tuple $(a_1, \dots, a_{n^2})$ of elements of A_0, put*

$$d = d(\boldsymbol{a}) := \det(\mathrm{Tr}_{A/F}(a_i a_j))_{1 \leq i,j \leq n^2} \quad \in \; F_0.$$

Then $A_0[d^{-1}]$ is an Azumaya algebra over $F_0[d^{-1}]$ of degree n. Moreover, it is a free $F_0[d^{-1}]$-module with an $F_0[d^{-1}]$-basis $(\overline{a}_1, \dots, \overline{a}_{n^2})$.

Here, $F_0[d^{-1}]$ denotes the ring $F_0[X]/(dX-1)$ obtained from F_0 by making d invertible, and $A_0[d^{-1}] := F_0[d^{-1}] \otimes_{F_0} A_0$. For an element x of F_0 or A_0, we denote by $\overline{x}$ its image in $F_0[d^{-1}]$ or $A_0[d^{-1}]$. Note that d may be a zero-divisor, in which case $F_0[d^{-1}]$ and $A_0[d^{-1}]$ are the zero-ring. When we make any statements on $F_0[d^{-1}]$ and $A_0[d^{-1}]$, we shall assume tacitly that they are not the zero-ring.

Proof. The $F_0[d^{-1}]$-algebra $A_0[d^{-1}]$ is a subring of the Azumaya algebra $A[d^{-1}]$ over $F[d^{-1}]$. Since $F_0 \subset F \cap A_0$, we have $F_0[d^{-1}] \subset F[d^{-1}] \cap A_0[d^{-1}]$. Let Tr denote the reduced trace map $A[d^{-1}] \to F[d^{-1}]$. Then we have $\mathrm{Tr}(A_0[d^{-1}]) \subset F_0[d^{-1}]$ and

$$\overline{d} = \det(\mathrm{Tr}(\overline{a}_i \overline{a}_j)) \quad \in F_0[d^{-1}]^{\times}.$$

Thus the corollary follows from Proposition 3.7. □

§4. Moduli of absolutely irreducible representations

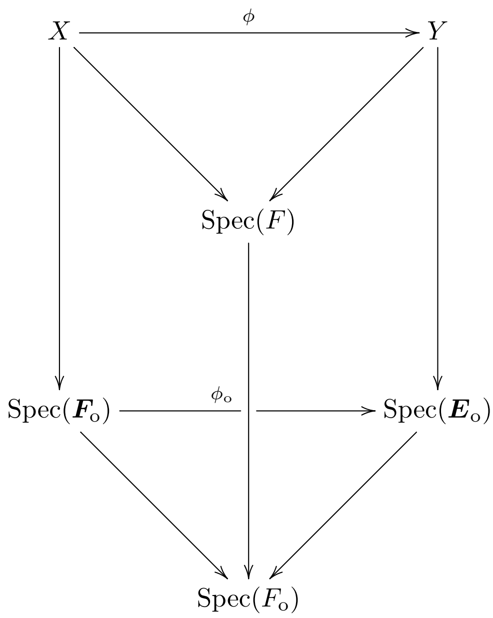
In this section, we construct the moduli scheme of absolutely irreducible representations of a ring in a certain category. Fix a category $\mathfrak{A}$ of topological rings, and let $\mathfrak{C}$ be the full subcategory of $\mathfrak{A}$ consisting of commutative objects.

§4.1. f-pro-$\mathfrak{a}$ schemes

In Section 2, we defined the category ${}^{\mathrm{f}}\mathfrak{A}$ of f-$\mathfrak{A}$ rings and its full subcategory ${}^{\mathrm{f}}\mathfrak{C}$ of commutative objects. To work globally, let ${}^{\mathrm{f}}\mathfrak{S}$ be the category of schemes obtained by globalizing the opposite category ${}^{\mathrm{f}}\mathfrak{C}^{\mathrm{op}}$ of ${}^{\mathrm{f}}\mathfrak{C}$. Precisely speaking, this means the following: Let $F_{\mathrm{o}} \to F$ be a commutative f-$\mathfrak{A}$ ring. An f-$\mathfrak{A}$ *scheme over* $F_{\mathrm{o}} \to F$ is a triple $(\boldsymbol{F}_{\mathrm{o}}, X, f)$ consisting of an object $\boldsymbol{F}_{\mathrm{o}} \in \mathfrak{C}_{F_{\mathrm{o}}}$, an F-scheme X and a morphism $f : \boldsymbol{F}_{\mathrm{o}} \to \mathcal{O}_X$ of sheaves of rings on X, where $\boldsymbol{F}_{\mathrm{o}}$ is regarded as a constant sheaf on X. We often simplify the notation to say X is an f-$\mathfrak{A}$ scheme over F relative to $\boldsymbol{F}_{\mathrm{o}}$, with f being understood. The morphism f is called the *topological structure morphism* of X. Equivalently, an f-$\mathfrak{A}$ scheme over $F_{\mathrm{o}} \to F$ relative to $\boldsymbol{F}_{\mathrm{o}}$ may be thought of just as a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Spec}(\boldsymbol{F}_{\mathrm{o}}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(F) & \longrightarrow & \mathrm{Spec}(F_{\mathrm{o}}) \end{array}$$

of schemes. Let X and Y be f-$\mathfrak{A}$ schemes over $F_{\mathrm{o}} \to F$ relative to $\boldsymbol{F}_{\mathrm{o}}$ and $\boldsymbol{E}_{\mathrm{o}}$, respectively. A morphism $\phi : X \to Y$ of f-$\mathfrak{A}$ schemes over F is a commutative diagram



of schemes, where ϕ_{o} is a morphism of F_{o}-schemes coming from a morphism $\boldsymbol{E}_{\mathrm{o}} \to \boldsymbol{F}_{\mathrm{o}}$ in $\mathfrak{A}$. Thus we apply the usual terminologies on (morphisms of) schemes also to (morphisms of) f-$\mathfrak{A}$ schemes by forgetting the topological structures.

Let ${}^{\mathrm{f}}\mathfrak{S}_F$ denote the category of f-$\mathfrak{A}$ schemes over $F_{\mathrm{o}} \to F$, and ${}^{\mathrm{f}}\mathfrak{S}_F(\boldsymbol{F}_{\mathrm{o}})$ its full subcategory of those relative to $\boldsymbol{F}_{\mathrm{o}}$. Without specifying the base ring F, we denote by ${}^{\mathrm{f}}\mathfrak{S}$ the category of all f-$\mathfrak{A}$ schemes. Note that the spectra of objects in $\mathfrak{C}$ are naturally regarded as objects of ${}^{\mathrm{f}}\mathfrak{S}$, and hence $\mathfrak{C}^{\mathrm{op}}$ is naturally regarded as a full subcategory of ${}^{\mathrm{f}}\mathfrak{S}$.

When $\mathfrak{A} = \widehat{\mathfrak{a}}$ is the category of pro-$\mathfrak{a}$ rings (cf. Section 2), then we call an object of ${}^{\mathrm{f}}\mathfrak{S}$ an f-*pro-*$\mathfrak{a}$ *scheme*. An affine object $\mathrm{Spec}(A)$ in ${}^{\mathrm{f}}\mathfrak{S}$ is said to be a *pro-*$\mathfrak{a}$ *scheme* if A is a commutative pro-$\mathfrak{a}$ ring, i.e., if it comes from $\widehat{\mathfrak{c}}^{\mathrm{op}}$.

We say that a scheme X in ${}^{\mathrm{f}}\mathfrak{S}_F$ is of *finite type over* F if it has a finite affine open covering $X = U_1 \cup \cdots \cup U_c$, where $U_i = \mathrm{Spec}(O_i)$ with $O_i \in {}^{\mathrm{f}}\mathfrak{C}_F$, such that each O_i is finitely generated as an object of ${}^{\mathrm{f}}\mathfrak{C}_F$.

Let S be an object of ${}^{\mathrm{f}}\mathfrak{S}_F(\boldsymbol{F}_{\mathrm{o}})$, where $\boldsymbol{F}_{\mathrm{o}} \in \mathfrak{C}_{F_{\mathrm{o}}}$. A *sheaf of* f-$\mathfrak{A}$ *rings* $(\boldsymbol{A}_{\mathrm{o}}, \mathcal{A}, f)$ on S is a triple consisting of an $\boldsymbol{F}_{\mathrm{o}}$-algebra $\boldsymbol{A}_{\mathrm{o}} \in \mathfrak{A}$, a Zariski-sheaf $\mathcal{A}$ on S, where S is regarded as the usual scheme, and a morphism $f : \boldsymbol{A}_{\mathrm{o}} \to \mathcal{A}$ of sheaves of rings on S, where $\boldsymbol{A}_{\mathrm{o}}$ is regarded as a constant sheaf on S. A *morphism* $(\phi_{\mathrm{o}}, \phi) : (\boldsymbol{A}_{\mathrm{o}}, \mathcal{A}, f) \to (\boldsymbol{A}'_{\mathrm{o}}, \mathcal{A}', f')$ of sheaves of f-$\mathfrak{A}$ rings on S is a pair consisting of a morphism $\phi_{\mathrm{o}} : \boldsymbol{A}_{\mathrm{o}} \to \boldsymbol{A}'_{\mathrm{o}}$ in $\mathfrak{A}$ and a morphism $\phi : \mathcal{A} \to \mathcal{A}'$ of sheaves of rings that are compatible with f and f' in the obvious sense. We denote by ${}^{\mathrm{f}}\mathfrak{A}_S$ the category of sheaves of f-$\mathfrak{A}$ rings on S, and by ${}^{\mathrm{f}}\mathfrak{C}_S$ the full subcategory of ${}^{\mathrm{f}}\mathfrak{A}_S$ consisting of commutative objects. Similarly, we define the category ${}^{\mathrm{f}}\mathfrak{A}_O$ of sheaves of f-$\mathfrak{A}$ $\mathcal{O}$-algebras on S if $\mathcal{O}\in{}^{\mathrm{f}}\mathfrak{C}_S$, and the category of sheaves of $\mathcal{A}$-modules on S if $\mathcal{A}\in{}^{\mathrm{f}}\mathfrak{A}_S$.

§4.2. Azumaya algebras and absolutely irreducible representations

In this context, we define an Azumaya algebra in ${}^{\mathrm{f}}\mathfrak{A}_S$ as follows (cf. [12, Sect. 5]):

Definition 4.1. (i) Let S be a (usual) scheme and $\mathcal{O}$ a sheaf of commutative $\mathcal{O}_S$-algebras on S. An *Azumaya algebra* $\mathcal{A}$ over $\mathcal{O}$ of degree n is a sheaf of $\mathcal{O}$-algebras on S such that

(1) Zariski locally on S, the $\mathcal{O}$-module $\mathcal{A}$ is free of rank n^2;

(2) the map

$$\begin{aligned} \iota : \mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}^{\mathrm{op}} &\to \mathrm{End}_{\mathcal{O}}(\mathcal{A}), \\ a \otimes b &\mapsto (x \mapsto axb) \end{aligned}$$

is an isomorphism.

As in Section 3, an Azumaya algebra $\mathcal{A}$ over $\mathcal{O}$ has the reduced trace map $\mathrm{Tr}_{\mathcal{A}/\mathcal{O}} : \mathcal{A} \to \mathcal{O}$, which is a morphism of sheaves of $\mathcal{O}$-modules.

(ii) Let S be an object of ${}^{\mathrm{f}}\mathfrak{S}_F(\boldsymbol{F}_{\mathrm{o}})$, and let $\mathcal{O} = (\boldsymbol{O}_{\mathrm{o}}, \mathcal{O})$ be an object of ${}^{\mathrm{f}}\mathfrak{C}_S$. An *Azumaya algebra* in ${}^{\mathrm{f}}\mathfrak{A}_S$ over $(\boldsymbol{O}_{\mathrm{o}}, \mathcal{O})$ of degree n is an object $(\boldsymbol{A}_{\mathrm{o}}, \mathcal{A}, f)$ in ${}^{\mathrm{f}}\mathfrak{A}_S$ such that

(1) $\mathcal{A}$ is an Azumaya algebra over $\mathcal{O}$ of degree n in the sense of (i) above;

(2) for each Zariski-open subset U of S, the natural map $\mathcal{O}(U) \otimes_{\boldsymbol{F}_{\mathrm{o}}} \boldsymbol{A}_{\mathrm{o}} \to \mathcal{A}(U)$ is surjective;

(3) for each Zariski-open subset U of S, $\mathrm{Tr}_{\mathcal{A}(U)/\mathcal{O}(U)}(f_U(\boldsymbol{A}_{\mathrm{o}}))$ is contained in the image of $\boldsymbol{F}_{\mathrm{o}}$ in $\mathcal{O}(U)$.

We say also that $\mathcal{A}$ is an Azumaya algebra over S if $\mathcal{O} = \mathcal{O}_S$.

When we talk about the reduced trace map $\mathrm{Tr}_{\mathcal{A}/\mathcal{O}}$ of an Azumaya algebra $(\boldsymbol{A}_{\mathrm{o}}, \mathcal{A}, f)$ in ${}^{\mathrm{f}}\mathfrak{A}_S$, we always refer to that of its algebraic factor $\mathcal{A}$.

Remark 4.2. If $\sigma : \mathcal{A}_1 \to \mathcal{A}_2$ is an isomorphism of Azumaya algebras over S, it is automatically compatible with the reduced trace maps, i.e., the diagram

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\mathrm{Tr}_{\mathcal{A}_1/S}} & \mathcal{O}_S \\ \downarrow{\scriptstyle\sigma} & & \| \\ \mathcal{A}_2 & \xrightarrow[\mathrm{Tr}_{\mathcal{A}_2/S}]{} & \mathcal{O}_S \end{array}$$

is commutative. This follows from the fact that the reduced trace map of an Azumaya algebra is invariant under any automorphism of the Azumaya algebra (Theorem of Auslander–Goldman; cf. [12, Thm. 5.10 and n$^{\text{o}}$. 5.13]).

Let us define the objects that we shall parametrize:

Definition 4.3. Let $S \in {}^{\mathrm{f}}\mathfrak{S}_F$, $R \in {}^{\mathrm{f}}\mathfrak{A}_S$ and $\mathcal{O} \in {}^{\mathrm{f}}\mathfrak{C}_S$. A *representation* of R over $\mathcal{O}$ of *degree* n is a morphism $\rho : R \to \mathcal{A}$ in ${}^{\mathrm{f}}\mathfrak{A}_S$, where $\mathcal{A}$ is an Azumaya algebra over $\mathcal{O}$ of degree n. A representation $\rho : R \to \mathcal{A}$ of R over $\mathcal{O}$ is said to be *absolutely irreducible* if ρ is essentially surjective, meaning that the image $\rho(R)$ generates (the second factor of) $\mathcal{A}$ as a sheaf of $\mathcal{O}$-modules (or equivalently, as a sheaf of $\mathcal{O}$-algebras).

See Proposition 6.3 for motivation and background for this definition.

Note that absolute irreducibility is a local notion (cf. [15, Chap. II, Caution 1.2]). Namely, a morphism $\rho : R \to \mathcal{A}$ in ${}^{\mathrm{f}}\mathfrak{A}_S$ is absolutely irreducible if and only if the induced morphism $\rho_s : R \to \mathcal{A}_s$ in ${}^{\mathrm{f}}\mathfrak{A}$ is absolutely irreducible over $\mathcal{O}_s$ for all $s \in S$. Here, $\mathcal{A}_s = (\boldsymbol{A}_{\mathrm{o}}, \mathcal{A}_s, f_s)$ denotes the stalk of $\mathcal{A}$ at s, and similarly for $\mathcal{O}_s$. It is also equivalent to saying that, for each affine open subset U of S, the restriction $\rho_U : R|_U \to \mathcal{A}|_U$ of ρ to U is absolutely irreducible over $\mathcal{O}_U$.

Two representations $\rho_i : R \to \mathcal{A}_i$ $(i = 1, 2)$ over $\mathcal{O}$ are said to be *isomorphic* if there exists an isomorphism $\sigma : \mathcal{A}_1 \to \mathcal{A}_2$ in ${}^{\mathrm{f}}\mathfrak{A}_S$ that makes the diagram

$$\begin{array}{ccc} R & \xrightarrow{\rho_1} & \mathcal{A}_1 \\ \| & & \downarrow{\scriptstyle \sigma} \\ R & \xrightarrow[\rho_2]{} & \mathcal{A}_2 \end{array}$$

commutative.

Remark 4.4. In this paper, we are most interested in the case where R is a constant sheaf; thus, to avoid unnecessary complication, *we henceforth assume that* $R \in {}^{\mathrm{f}}\mathfrak{A}$.

If $\rho : R \to \mathcal{A}$ is a representation, we denote by $\mathrm{Tr}_{\mathcal{A}/\mathcal{O}}\rho$ the composite map $\mathrm{Tr}_{\mathcal{A}/\mathcal{O}} \circ \rho : R \to \mathcal{O}$.

For each $S \in {}^{\mathrm{f}}\mathfrak{S}_F$, let $\underline{\mathrm{Rep}}_{R,n}(S)$ denote the set of isomorphism classes of representations of R over S of degree n, and let $\underline{\mathrm{Rep}}^{\mathrm{ai}}_{R,n}(S)$ denote its subset consisting of those of absolutely irreducible ones. They may be thought of as fiber spaces (just as sets) over the n-torsion subgroup ${}_n\mathrm{Br}(S)$ of the Azumaya–Brauer group $\mathrm{Br}(S)$ of S as the usual scheme. (Recall that $\mathrm{Br}(S)$ is the group of equivalence classes of Azumaya algebras $\mathcal{A}$ over S, where $\mathcal{A}$ and $\mathcal{A}'$ are said to be equivalent if there exist locally free $\mathcal{O}_S$-modules $\mathcal{E}$, $\mathcal{E}'$ such that $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{E} \simeq \mathcal{A}' \otimes_{\mathcal{O}_S} \mathcal{E}'$. In the following, we call it simply the *Brauer group* of S. See [12, §2] for the n-torsion of $\mathrm{Br}(S)$.) The correspondences $S \mapsto \underline{\mathrm{Rep}}_{R,n}(S)$ and $S \mapsto \underline{\mathrm{Rep}}^{\mathrm{ai}}_{R,n}(S)$ are functorial and contravariant. Indeed, if $g : T \to S$ is a morphism in ${}^{\mathrm{f}}\mathfrak{S}_F$ and $\rho : R \to \mathcal{A}$ is a representation of R over S, then we can define the pull-back $g^*\rho : R \to g^*\mathcal{A}$ of ρ by g as follows: Let V be an open subset of T. For each open subset U of S, we have the ring homomorphism $\rho_U : R \to \mathcal{A}(U)$. This induces

$$R \to \mathcal{O}_T(g^{-1}(U)) \otimes_{\mathcal{O}_S(U)} \mathcal{A}(U) \to (g^*\mathcal{A})(V)$$

if $g(V) \subset U$ (here, the composite map does not depend on U). Gluing the maps $R \to (g^*\mathcal{A})(V)$ for various V, we obtain a morphism $R \to g^*\mathcal{A}$ in $\mathfrak{A}_T$. By construction, it is absolutely irreducible if ρ is so. Thus we have

$$g^* : \underline{\mathrm{Rep}}_{R,n}(S) \to \underline{\mathrm{Rep}}_{R,n}(T) \qquad \text{and} \qquad g^* : \underline{\mathrm{Rep}}^{\mathrm{ai}}_{R,n}(S) \to \underline{\mathrm{Rep}}^{\mathrm{ai}}_{R,n}(T).$$

In particular, if a representation $\rho : R \to \mathcal{A}$ over $\mathcal{O}$ is defined over a subalgebra $\mathcal{O}_0$ of $\mathcal{O}$, if it comes from some $\rho_0 : R \to \mathcal{A}_0$ over $\mathcal{O}_0$ by scalar extension $\mathcal{O}_0 \to \mathcal{O}$ and if ρ_0 is absolutely irreducible, then ρ is also absolutely irreducible.

In the other direction, we have a morphism $g_*\rho : R \to g_*\mathcal{A}$ of sheaves of rings on S for each $\rho : R \to \mathcal{A}$ in $\underline{\mathrm{Rep}}_{R,n}(T)$ defined by

$$(g_*\rho)(U) : R \to \mathcal{A}(g^{-1}(U)),$$

where U is any open subset of S. It may not be a representation of R over $g_*\mathcal{O}_T$ in our sense, because $g_*\mathcal{A}$ may not be an Azumaya algebra over $g_*\mathcal{O}_T$.

Now we consider localizing the base scheme to make a given representation absolutely irreducible. If $\mathcal{O}$ is a local ring, then by Proposition 3.7, a representation $\rho : R \to \mathcal{A}$ over $\mathcal{O}$ of degree n is absolutely irreducible if and only if there exists an n^2-tuple $(r_1, \ldots, r_{n^2})$ of elements of R such that $d = \det(\mathrm{Tr}\,\rho(r_i r_j)) \in \mathcal{O}^\times$. Since absolute irreducibility is a local notion, we have the following proposition.

Proposition 4.5. *Let $R \in {}^{\mathrm{f}}\mathfrak{A}$ and $\mathcal{A} \in {}^{\mathrm{f}}\mathfrak{A}_S$. A representation $\rho : R \to \mathcal{A}$ over $\mathcal{O}$ of degree n is absolutely irreducible if and only if, for each $s \in S$ and maximal ideal $\mathfrak{m}$ of $\mathcal{O}_s$, there exists an n^2-tuple $(r_1, \ldots, r_{n^2})$ of elements of R such that* $\det(\mathrm{Tr}\,\rho(r_i r_j)) \in \mathcal{O}^\times_{s,\mathfrak{m}}$.

Here, the $r, \ldots, r_{n^2}$ are elements of the second factor R of $R = (R_{\mathrm{o}}, R, f) \in {}^{\mathrm{f}}\mathfrak{A}$.

This implies the following corollary:

Corollary 4.6. *Let $\rho_i : R \to \mathcal{A}_i$ $(i = 1, 2)$ be two representations of R over $\mathcal{O}$. If one of them is absolutely irreducible and if* $\operatorname{Tr}\rho_1 = \operatorname{Tr}\rho_2$ *as a morphism of sheaves on S, then the other is also absolutely irreducible.*

We have another a corollary:

Corollary 4.7. *If $\rho : R \to \mathcal{A}$ is an absolutely irreducible representation over $\mathcal{O}$, then* $\operatorname{Im}(\operatorname{Tr}\rho)$ *generates $\mathcal{O}$ as an $\mathcal{O}$-module.*

Proof. It is enough to show the statement locally, so assume that $\mathcal{O}$ is a local ring. Then by Proposition 4.5, there is an n^2-tuple $(r_1, \dots, r_{n^2})$ of elements of R such that $\det(\operatorname{Tr}\rho(r_i r_j)) \in \mathcal{O}^\times$. Since $\mathcal{O}$ is local, there is some (i, j) for which $\operatorname{Tr}\rho(r_i r_j) \in \mathcal{O}^\times$. □

§4.3. Azumaya and absolute irreducibility loci

Next we define the Azumaya and absolute irreducibility loci.

Definition 4.8. Suppose $S \in {}^{\mathrm{f}}\mathfrak{S}_F(\boldsymbol{F}_{\mathrm{o}})$. For an $\mathcal{O}_S$-algebra $\mathcal{A}_0$, we define its *Azumaya locus* $S^{\mathrm{Az}}(\mathcal{A}_0)$ to be the locus in S over which $\mathcal{A}_0$ is Azumaya; i.e.,

$$S^{\mathrm{Az}}(\mathcal{A}_0) := \{s \in S | \text{ the stalk } \mathcal{A}_{0,s} \text{ is an Azumaya algebra over } \mathcal{O}_{S,s}\}.$$

Remark 4.9. More precisely, one may consider the nth Azumaya locus for each $n \geq 1$, the locus over which $\mathcal{A}_0$ is Azumaya of degree n. In what follows, however, our $\mathcal{A}_0$ will mostly be a subring of a fixed Azumaya algebra, so that the rank of $\mathcal{A}_0$ is essentially fixed.

Example 4.10. If $S = \operatorname{Spec}(\mathbb{Z}_p)$, $\mathcal{O} = \mathbb{Q}_p$, $\mathcal{A}$ a division algebra over $\mathbb{Q}_p$ and $\mathcal{A}_0$ is an order of $\mathcal{A}$, then by Witt's theorem, $S^{\mathrm{Az}}(\mathcal{A}_0)$ is the set consisting of the generic point of S. Indeed, $\mathcal{A}_0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is Azumaya by definition. If $\mathcal{A}_0$ is Azumaya over the whole $\mathbb{Z}_p$, then so is $\mathcal{A}_0 \otimes_{\mathbb{Z}_p} \mathbb{F}_p$, which splits by a theorem of Wedderburn ([46, Chap. 1, Sect. 1, Thm. 1]). By a theorem of Witt ([48]), $\mathcal{A}_0$ itself splits over $\mathbb{Z}_p$.

We claim that $S^{\mathrm{Az}}(\mathcal{A}_0)$ is in fact an open subscheme of S; then by defining the topological structure morphism $\boldsymbol{F}_{\mathrm{o}} \to \mathcal{O}_{S^{\mathrm{Az}}(\mathcal{A}_0)}$ to be the composite $\boldsymbol{F}_{\mathrm{o}} \to \mathcal{O}_S \to \mathcal{O}_S|_{S^{\mathrm{Az}}(\mathcal{A}_0)}$, we may regard $S^{\mathrm{Az}}(\mathcal{A}_0)$ as an object of ${}^{\mathrm{f}}\mathfrak{S}_F(\boldsymbol{F}_{\mathrm{o}})$. To show this, we may assume that S is integral and affine. Let s be a point of $S^{\mathrm{Az}}(\mathcal{A}_0)$. Then the stalk $\mathcal{A}_{0,s}$ is an Azumaya algebra that is free over the local ring $\mathcal{O}_{S,s}$. By Proposition 3.7, there exists an $\mathcal{O}_{S,s}$-basis $(a'_1, \dots, a'_{n^2})$ of $\mathcal{A}_{0,s}$ such that $\det(\operatorname{Tr}_{\mathcal{A}_{0,s}/\mathcal{O}_{S,s}}(a'_i a'_j)) \in \mathcal{O}_{S,s}^\times$. Here, each a'_i is of the form $a'_i = (1/f_i) \otimes a_i$ for some section a_i of $\mathcal{A}_0$ and

f_i of $\mathcal{O}_S$ with $f_i \in \mathcal{O}_{S,s}^\times$. Then we have

$$\det(\mathrm{Tr}_{\mathcal{A}_{0,s}/\mathcal{O}_{S,s}}(a_i' a_j')) = \frac{1}{(f_1 \cdots f_{n^2})^2} \det(\mathrm{Tr}_{\mathcal{A}_{0,s}/\mathcal{O}_{S,s}}(a_i a_j)).$$

Hence $d := \det(\mathrm{Tr}_{\mathcal{A}_{0,s}/\mathcal{O}_{S,s}}(a_i a_j))$ is also in $\mathcal{O}_{S,s}^\times$, and $S[1/d]$ is an open subscheme of S that contains s. The sheaf $\mathcal{A}_0[1/d]$ on $S[1/d]$ may be regarded as a subalgebra of $\mathcal{A}_{0,s}$. By Corollary 3.11, it is an Azumaya algebra over $S[1/d]$, and hence $S[1/d] \subset S^{\mathrm{Az}}(\mathcal{A}_0)$.

Corollary 3.11 also implies the following:

Proposition 4.11. *Let $\mathcal{A}$ be an Azumaya algebra of degree n over $\mathcal{O} \in {}^{\mathrm{f}}\mathfrak{C}_S$, and $\mathcal{A}_0$ an $\mathcal{O}_S$-subalgebra of $\mathcal{A}$. Let $\gamma : \mathcal{O}_S \to \mathcal{O}$ be the structure morphism of $\mathcal{O}$, and assume that $\mathrm{Tr}_{\mathcal{A}/\mathcal{O}}(\mathcal{A}_0) \subset \mathrm{Im}(\gamma)$. Then $S^{\mathrm{Az}}(\mathcal{A}_0)$ has the following presentation:*

$$S^{\mathrm{Az}}(\mathcal{A}_0) = \bigcup_U \bigcup_d \mathrm{Spec}(\mathcal{O}_S(U)[d^{-1}]) \quad \textit{in } S, \tag{4.1}$$

where $\{U\}$ is an affine open covering of S and we let $d = \det(\mathrm{Tr}_{\mathcal{A}/\mathcal{O}}(a_i a_j))_{1 \le i,j \le n^2}$, for each U, with $(a_1, \ldots, a_{n^2})$ moving through all n^2-tuples of elements of $\mathcal{A}_0(U)$.

This again proves the openness of $S^{\mathrm{Az}}(\mathcal{A}_0)$ in this case. Note that the map $\mathrm{Tr}_{\mathcal{A}/\mathcal{O}}|_{\mathcal{A}_0} : \mathcal{A}_0 \to \mathcal{O}_S$ is surjective on the Azumaya locus $S^{\mathrm{Az}}(\mathcal{A}_0)$, since the trace map of an Azumaya algebra is surjective.

Remark 4.12. If $\mathcal{A}_0$ is generated by a set G of global sections as an $\mathcal{O}_S$-module, then it is enough for us, in (4.1), to let $(a_1, \ldots, a_{n^2})$ move only in G^{n^2}. Indeed, suppose $\mathcal{A}_0$ is generated by G as an $\mathcal{O}_S$-module and is locally free on U. Then by Nakayama's lemma, at each point $s \in U$, one can choose an $\mathcal{O}_{S,s}$-basis of $\mathcal{A}_{0,s}$ from G^{n^2}.

Next let $\rho : R \to \mathcal{A}$ be a representation over T, where $T \in {}^{\mathrm{f}}\mathfrak{S}_F(\boldsymbol{F}_{\mathrm{o}})$.

Definition 4.13. We define the *absolute irreducibility locus* $T^{\mathrm{ai}}(\rho)$ of ρ to be the locus in T over which ρ is absolutely irreducible; i.e.,

$$T^{\mathrm{ai}}(\rho) := \{t \in T |\ \text{the representation } \rho_t : R \to \mathcal{A}_t \text{ over } \mathcal{O}_{T,t} \\ \text{is absolutely irreducible}\}.$$

By Proposition 4.5, we have the following result:

Proposition 4.14. *With the above notation, we have the following presentation:*

$$T^{\mathrm{ai}}(\rho) = \bigcup_V \bigcup_d \mathrm{Spec}(\mathcal{O}_T(V)[d^{-1}]) \quad \textit{in } T, \tag{4.2}$$

where $\{V\}$ is an affine open covering of T and we let $d = \det(\mathrm{Tr}_{\mathcal{A}/T}\rho(r_i r_j))_{1\leq i,j\leq n^2}$, for each V, with $(r_1, \ldots, r_{n^2})$ moving through all n^2-tuples of elements of R.

In particular, $T^{\mathrm{ai}}(\rho)$ is in fact an open subscheme of T, which we regard as an object of ${}^{\mathrm{f}}\mathfrak{S}_F(\boldsymbol{F}_{\mathrm{o}})$ by defining the topological structure morphism $\boldsymbol{F}_{\mathrm{o}} \to \mathcal{O}_{T^{\mathrm{ai}}(\rho)}$ to be the composite $\boldsymbol{F}_{\mathrm{o}} \to \mathcal{O}_T \to \mathcal{O}_T|_{T^{\mathrm{ai}}(\rho)}$.

The next proposition generalizes the lemma of Carayol–Serre (cf. [6, Thm. 2] and [26, Sect. 6, Prop.]):

Proposition 4.15. *Let $g : T \to S$ be a morphism in ${}^{\mathrm{f}}\mathfrak{S}_F(\boldsymbol{F}_{\mathrm{o}})$, and put $\mathcal{O}_0 := \mathrm{Im}(g^{\#} : \mathcal{O}_S \to g_*\mathcal{O}_T)$ and $\mathcal{O} := g_*\mathcal{O}_T$. Let $\mathcal{A}$ be an Azumaya algebra over T, and let $\rho : R \to \mathcal{A}$ be a representation of R over T such that $g_* \mathrm{Tr}_{\mathcal{A}/T}\, \rho(R) \subset \mathcal{O}_0$. Let $\mathcal{A}_0$ be the $\mathcal{O}_0$-subalgebra of $g_*\mathcal{A}$ generated by the image of ρ. Then we have*

$$g^{-1}(S^{\mathrm{Az}}(\mathcal{A}_0)) = T^{\mathrm{ai}}(\rho).$$

Furthermore, the representation ρ descends to a representation $\rho_0 : R \to \mathcal{A}_0$ over S, whose restriction to $S^{\mathrm{Az}}(\mathcal{A}_0)$ is absolutely irreducible.

To understand this proposition, it would be helpful to note the following particular case: If $\mathcal{A}_0$ is an Azumaya algebra over $\mathcal{O}_0$, then the condition $g_* \mathrm{Tr}_{\mathcal{A}/T}\, \rho(R) \subset \mathcal{O}_0$ ensures that $g_*\rho : R \to \mathcal{A}_0$ is an absolutely irreducible representation over O_0 (in this case, $g_*\mathcal{A}$ is also Azumaya over $g_*\mathcal{O}_T$, and $g_*\rho : R \to g_*\mathcal{A}$ is an absolutely irreducible representation over $g_*\mathcal{O}_T$).

Proof. Let $U_d = \mathrm{Spec}(\mathcal{O}_S(U)[d^{-1}])$ be an affine piece of $S^{\mathrm{Az}} = S^{\mathrm{Az}}(\mathcal{A}_0)$ as in (4.1). By Remark 4.12, we may assume $d = \det(\mathrm{Tr}_{\mathcal{A}/T}\, \rho(r_i r_j))$ for some $(r_1, \ldots, r_{n^2}) \in R^{n^2}$. Then $\mathrm{Spec}(\mathcal{O}_T(g^{-1}(U))[d^{-1}])$ is an affine piece of $T^{\mathrm{ai}} = T^{\mathrm{ai}}(\rho)$ as in (4.2). Hence $g^{-1}(S^{\mathrm{Az}}) \subset T^{\mathrm{ai}}$. Conversely, let $V_d = \mathrm{Spec}(\mathcal{O}_T(V)[d^{-1}])$ be an affine piece of T^{ai} as in (4.2). Let $\{U\}$ be a family of affine open subsets of S that covers $g(V)$. Then $\{U_d\}$, where $U_d = \mathrm{Spec}(\mathcal{O}_S(U)[d^{-1}])$, covers $g(V_d)$ and, in view of (4.1), the union of U_d's is contained in S^{Az}. Hence $g(T^{\mathrm{ai}}) \subset S^{\mathrm{Az}}$. The last statement of the proposition is clear. □

Remark 4.16. (i) If, in particular, $S = T$ and g is the identity morphism, then the above proposition asserts that the Azumaya and absolute irreducibility loci coincide. Thus, given any representation $\rho : R \to \mathcal{A}$ over S, we can "cut out" a unique maximal absolutely irreducible representation from ρ by restricting to $S^{\mathrm{Az}}(\mathcal{A}_0)$.

(ii) We have trivially $S^{\mathrm{Az}}(\mathcal{A}_0) \supset g(T^{\mathrm{ai}}(\rho))$, but in general $S^{\mathrm{Az}}(\mathcal{A}_0)$ may not be contained in $g(T^{\mathrm{ai}}(\rho))$: a trivial example of such a case occurs when ρ_0 is an absolutely irreducible representation over S and $\rho = g^*\rho_0$, while T is a proper subscheme of S.

With the same notation and assumptions as Proposition 4.15, suppose further that we are given a commutative diagram

$$\begin{array}{ccc} T & \xleftarrow{\;t\;} & Q \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ S & \xleftarrow[\;s\;]{} & P \end{array}$$

in ${}^{\mathrm{f}}\mathfrak{S}_F$, where $\pi : Q \to P$ is surjective. Then the above proposition says that the representation $t^*\rho : R \to t^*\mathcal{A}$ over Q is absolutely irreducible if and only if the ring $s^*\mathcal{A}_0$ is an Azumaya algebra over P. Moreover, if this is the case, Lemma 4.24 below implies that the morphism $(s\circ\pi)^*\mathcal{A}_0 \to t^*\mathcal{A}$ induced by ρ is an isomorphism. If the schemes are affine (as will be used later), we have the following result:

Corollary 4.17. *Let $\boldsymbol{F}$ be an object of ${}^{\mathrm{f}}\mathfrak{C}_F$, $\boldsymbol{A}$ an Azumaya algebra over $\boldsymbol{F}$ and $\rho : R \to \boldsymbol{A}$ a representation of R over $\boldsymbol{F}$. Let $\boldsymbol{F}_0$ be a closed subring of $\boldsymbol{F}$ that contains $\mathrm{Tr}_{\boldsymbol{A}/\boldsymbol{F}}\,\rho(R)$, and let $\boldsymbol{A}_0$ be the closed $\boldsymbol{F}_0$-subalgebra of $\boldsymbol{A}$ generated by $\rho(R)$. Then for any commutative diagram*

$$\begin{array}{ccc} \boldsymbol{F} & \xrightarrow{\;f\;} & O \\ \big\uparrow & & \big\uparrow{\scriptstyle \iota} \\ \boldsymbol{F}_0 & \xrightarrow[\;f_0\;]{} & O_0 \end{array}$$

in ${}^{\mathrm{f}}\mathfrak{C}_F$, where $\iota : O_0 \to O$ is assumed faithfully flat, the following conditions are equivalent:

(1) The representation $f_*\rho : R \to O \otimes_{\boldsymbol{F}} \boldsymbol{A}$ is absolutely irreducible.

(2) The O_0-algebra $O_0 \otimes_{\boldsymbol{F}_0} \boldsymbol{A}_0$ is Azumaya, where the tensor product is via f_0.

Moreover, if this is the case, the map $O \otimes_{\boldsymbol{F}_0} \boldsymbol{A}_0 \to O \otimes_{\boldsymbol{F}} \boldsymbol{A}$ induced by ρ is an isomorphism.

As a special case (where $O = \boldsymbol{F}$ and $O_0 = \boldsymbol{F}_0$), we have a corollary:

Corollary 4.18. *Let $\boldsymbol{F}$ and $\boldsymbol{A}$ be as above. Let $\boldsymbol{F}_0$ be a closed subring of $\boldsymbol{F}$, and $\boldsymbol{A}_0$ a closed $\boldsymbol{F}_0$-subalgebra of $\boldsymbol{A}$. Assume that $\boldsymbol{F}_0 \supset \mathrm{Tr}_{\boldsymbol{A}/\boldsymbol{F}}(\boldsymbol{A}_0)$ and that $\boldsymbol{F}$ is faithfully flat over $\boldsymbol{F}_0$. Then $\boldsymbol{F}\boldsymbol{A}_0 = \boldsymbol{A}$ if and only if $\boldsymbol{A}_0$ is Azumaya.*

§4.4. Construction of the moduli scheme

Fix a morphism $Z \to F$ in ${}^{\mathrm{f}}\mathfrak{C}$ (in most applications, Z will be the initial object of ${}^{\mathrm{f}}\mathfrak{A}$ and F will be the "coefficient ring" of the representations under consideration). Let $R = (R_{\mathrm{o}}, R)$ be an object of ${}^{\mathrm{f}}\mathfrak{A}_Z$. Fix an integer $n \geq 1$. We assume that there exists a morphism that is universal for absolutely irreducible representations of R into matrix algebras of degree n in ${}^{\mathrm{f}}\mathfrak{A}_F$. Precisely speaking, we assume the following:

($\mathrm{V}^{\mathrm{ai}}_{n,F}$) There exist an object $\boldsymbol{F} = (\boldsymbol{F}_{\mathrm{o}}, \boldsymbol{F})$ of ${}^{\mathrm{f}}\mathfrak{C}_F$ and a morphism $\Phi : R \to \mathrm{M}_n(\boldsymbol{F})$ in ${}^{\mathrm{f}}\mathfrak{A}_F$ such that, for any $O \in {}^{\mathrm{f}}\mathfrak{C}_F$ and any absolutely irreducible representation $\phi : R \to \mathrm{M}_n(O)$ in ${}^{\mathrm{f}}\mathfrak{A}_Z$, there exists a unique morphism $f : \boldsymbol{F} \to O$ in ${}^{\mathrm{f}}\mathfrak{C}_F$ such that $\phi = f_*\Phi := \mathrm{M}_n(f) \circ \Phi$, i.e., the following diagram is commutative:

$$\begin{array}{ccc} & & \mathrm{M}_n(\boldsymbol{F}) \\ & \overset{\Phi}{\nearrow} & \downarrow{\scriptstyle \mathrm{M}_n(f)} \\ R & \xrightarrow[\phi]{} & \mathrm{M}_n(O). \end{array}$$

In Lemma 2.9, we saw that, if $\mathfrak{A}$ is the category of pro-$\mathfrak{a}$ rings with $\mathfrak{a}$ satisfying Axiom (a1), then ($\mathrm{V}^{\mathrm{ai}}_{n,F}$) holds for any $n \geq 1$ and R in ${}^{\mathrm{f}}\mathfrak{A}_Z$.

In the following, we set

$$\boldsymbol{A} = (\boldsymbol{A}_{\mathrm{o}}, \boldsymbol{A}) := (\mathrm{M}_n(\boldsymbol{F}_{\mathrm{o}}), \mathrm{M}_n(\boldsymbol{F}))$$

for brevity of notation. We write simply $\mathrm{Tr}_{\boldsymbol{A}/\boldsymbol{F}}$ for the pair of the trace maps $\mathrm{Tr}_{\boldsymbol{A}_{\mathrm{o}}/\boldsymbol{F}_{\mathrm{o}}} : \boldsymbol{A}_{\mathrm{o}} \to \boldsymbol{F}_{\mathrm{o}}$ and $\mathrm{Tr}_{\boldsymbol{A}/\boldsymbol{F}} : \boldsymbol{A} \to \boldsymbol{F}$. We shall prove that the contravariant functor

$$\underline{\mathrm{Rep}}^{\mathrm{ai}}_{R,n} : {}^{\mathrm{f}}\mathfrak{S}_F \to (\mathrm{Sets}),$$
$$S \mapsto \underline{\mathrm{Rep}}^{\mathrm{ai}}_{R,n}(S)$$

is representable by an object of ${}^{\mathrm{f}}\mathfrak{S}_F$. Define two rings $\boldsymbol{F}^{\mathrm{tr}} = \boldsymbol{F}^{\mathrm{tr}}_{R,\Phi} \in {}^{\mathrm{f}}\mathfrak{C}_F$ and $\boldsymbol{A}^{\mathrm{tr}} = \boldsymbol{A}^{\mathrm{tr}}_{R,\Phi} \in {}^{\mathrm{f}}\mathfrak{A}_F$:

- $\boldsymbol{F}^{\mathrm{tr}}$ is the closed F-subalgebra of $\boldsymbol{F}$ generated by $\mathrm{Tr}_{\boldsymbol{A}/\boldsymbol{F}}(\Phi(R))$.
- $\boldsymbol{A}^{\mathrm{tr}}$ is the closed $\boldsymbol{F}^{\mathrm{tr}}$-subalgebra of $\boldsymbol{A}$ generated by the image $\Phi(R)$ of R.

We write Φ again to denote the obvious morphism $R \to \boldsymbol{A}^{\mathrm{tr}}$. The reduced trace map $\mathrm{Tr}_{\boldsymbol{A}/\boldsymbol{F}} : \boldsymbol{A} \to \boldsymbol{F}$ induces an $\boldsymbol{F}^{\mathrm{tr}}$-linear map $\tau : \boldsymbol{A}^{\mathrm{tr}} \to \boldsymbol{F}^{\mathrm{tr}}$; we have the commutative diagram

$$\begin{array}{ccc} \boldsymbol{A}^{\mathrm{tr}} & \longrightarrow & \boldsymbol{A} \\ \tau\downarrow & & \downarrow \mathrm{Tr}_{\boldsymbol{A}/\boldsymbol{F}} \\ \boldsymbol{F}^{\mathrm{tr}} & \longrightarrow & \boldsymbol{F} \end{array} \tag{4.3}$$

in ${}^{\mathrm{f}}\mathfrak{A}_F$, where the horizontal arrows are the inclusion maps. Recall that τ is surjective on the Azumaya locus of $\boldsymbol{A}^{\mathrm{tr}}$ in $\mathrm{Spec}(\boldsymbol{F}^{\mathrm{tr}})$ (cf. after Definition 4.8).

Definition 4.19. Define $X_{R,n,F}$, as an object of ${}^{\mathrm{f}}\mathfrak{S}_F$, to be the Azumaya locus in $\mathrm{Spec}(\boldsymbol{F}^{\mathrm{tr}})$ of $\boldsymbol{A}^{\mathrm{tr}}$. Let $\mathcal{A}_{R,n,F}$ be the restriction to $X_{R,n,F}$ of the sheafification of $\boldsymbol{A}^{\mathrm{tr}}$.

It will follow from Theorem 4.20 below that there are canonical isomorphisms $X_{R,n,F'} = X_{R,n,F} \otimes_F F'$ and $\mathcal{A}_{R,n,F'} = \mathcal{A}_{R,n,F} \otimes_F F'$ if F' is finite over F. If the coefficient ring F is understood, we may often drop the "F" from the notation to write $X_{R,n,F} = X_{R,n}$ and $\mathcal{A}_{R,n,F} = \mathcal{A}_{R,n}$.

As in (4.1) (cf. Remark 4.12), $X_{R,n}$ is presented as

$$X_{R,n} := \bigcup_{(r_1,\dots,r_{n^2})} \mathrm{Spec}\left(\boldsymbol{F}^{\mathrm{tr}}[d^{-1}]\right) \tag{4.4}$$

$$= \mathrm{Spec}(\boldsymbol{F}^{\mathrm{tr}}) \setminus \mathrm{Spec}(\boldsymbol{F}^{\mathrm{tr}}/\mathfrak{d}), \tag{4.5}$$

where $d = \det(\tau(\Phi(r_i r_j)))_{1\le i,j\le n^2}$ and $\mathfrak{d}$ is the ideal of $\boldsymbol{F}^{\mathrm{tr}}$ generated by these elements d, with $(r_1,\dots,r_{n^2})$ moving through all the n^2-tuples of elements of R.

By Proposition 4.15, we have an absolutely irreducible representation

$$\rho_{R,n} : R \to \mathcal{A}_{R,n} \tag{4.6}$$

over $X_{R,n}$ of degree n. Let $S \in {}^{\mathrm{f}}\mathfrak{S}_F$, and let $g \in X_{R,n}(S) := \mathrm{Hom}_{{}^{\mathrm{f}}\mathfrak{S}_F}(S, X_{R,n})$. Pulling back the representation $\rho_{R,n} : R \to \mathcal{A}_{R,n}$ over $X_{R,n}$ by g, we obtain an absolutely irreducible representation $g^*\rho_{R,n} : R \to g^*\mathcal{A}_{R,n}$ over S that makes the following diagram commutative:

$$\begin{array}{ccccc} R & \xrightarrow{\rho_{R,n}} & \mathcal{A}_{R,n} & \xrightarrow{\mathrm{Tr}_{\mathcal{A}_{R,n}/X_{R,n}}} & \mathcal{O}_{X_{R,n}} \\ \| & & \downarrow g^{\natural} & & \downarrow g^{\#} \\ R & \xrightarrow[g_*g^*\rho_{R,n}]{} & g_*g^*\mathcal{A}_{R,n} & \xrightarrow[g_*\,\mathrm{Tr}_{g^*\mathcal{A}_{R,n}/S}]{} & g_*\mathcal{O}_S, \end{array} \tag{4.7}$$

where $g^{\natural} : \mathcal{A}_{R,n} \to g_*g^*\mathcal{A}_{R,n} = g_*\mathcal{O}_S \otimes_{\mathcal{O}_{X_{R,n}}} \mathcal{A}_{R,n}$ is the natural morphism. Here, the objects in the diagram are regarded as sheaves of rings on $X_{R,n}$. Let $\boldsymbol{r}(g)$

denote the isomorphism class of the representation $g^*\rho_{R,n}$. Thus we have a map

$$\boldsymbol{r} : X_{R,n}(S) \to \underline{\mathrm{Rep}}^{\mathrm{ai}}_{R,n}(S)$$

of sets. By construction, it is functorial in $S \in {}^{\mathrm{f}}\mathfrak{S}_F$. Our main result of this section is that this map $\boldsymbol{r}$ is bijective. In other words, we have the following theorem:

Theorem 4.20. *The scheme $X_{R,n}$ represents the functor $\underline{\mathrm{Rep}}^{\mathrm{ai}}_{R,n}$ in ${}^{\mathrm{f}}\mathfrak{S}_F$.*

Proof. To construct the inverse $\boldsymbol{x} : \underline{\mathrm{Rep}}^{\mathrm{ai}}_{R,n}(S) \to X_{R,n}(S)$ to $\boldsymbol{r}$, we use the following lemma:

Lemma 4.21. *For any absolutely irreducible representation $\rho : R \to \mathcal{A}$ over S of degree n, there exist unique morphisms $g : S \to X_{R,n}$ and $a_S : \mathcal{A}_{R,n} \to g_*\mathcal{A}$ that make the following diagram commutative:*

$$\begin{array}{ccccc} R & \xrightarrow{\rho_{R,n}} & \mathcal{A}_{R,n} & \xrightarrow{\mathrm{Tr}_{\mathcal{A}_{R,n}/X_{R,n}}} & \mathcal{O}_{X_{R,n}} \\ \| & & \downarrow{\scriptstyle a_S} & & \downarrow{\scriptstyle g^\#} \\ R & \xrightarrow[g_*\rho]{} & g_*\mathcal{A} & \xrightarrow[g_*\,\mathrm{Tr}_{\mathcal{A}/S}]{} & g_*\mathcal{O}_S. \end{array} \tag{4.8}$$

The composite morphism $g^*a_S : g^*\mathcal{A}_{R,n} \to g^*g_*\mathcal{A} \to \mathcal{A}$ is an isomorphism and fits into the adjoint diagram

$$\begin{array}{ccccc} R & \xrightarrow{g^*\rho_{R,n}} & g^*\mathcal{A}_{R,n} & \xrightarrow{g^*\mathrm{Tr}_{\mathcal{A}_{R,n}/X_{R,n}}} & g^*\mathcal{O}_{X_{R,n}} \\ \| & & \downarrow{\scriptstyle g^*a_S} & & \| \\ R & \xrightarrow[\rho]{} & \mathcal{A} & \xrightarrow[\mathrm{Tr}_{\mathcal{A}/S}]{} & \mathcal{O}_S. \end{array} \tag{4.9}$$

In particular, we have $\rho \simeq g^*\rho_{R,n}$. The morphism g is also characterized as the unique morphism that makes the following diagram commutative:

$$\begin{array}{ccc} R & \xrightarrow{\mathrm{Tr}_{\mathcal{A}_{R,n}/X_{R,n}}\rho_{R,n}} & \mathcal{O}_{X_{R,n}} \\ \| & & \downarrow{\scriptstyle g^\#} \\ R & \xrightarrow[g_*\,\mathrm{Tr}_{\mathcal{A}/S}\,\rho]{} & g_*\mathcal{O}_S. \end{array} \tag{4.10}$$

Admitting this lemma, we first complete the proof of Theorem 4.20. For each $\rho \in \underline{\mathrm{Rep}}^{\mathrm{ai}}_{R,n}(S)$, define $\boldsymbol{x}(\rho)$ to be the morphism g in the above lemma. Then the lemma shows that $\boldsymbol{r}(\boldsymbol{x}(\rho)) = \rho$.

Conversely, given a morphism $g : S \to X_{R,n}$, we obtain $\boldsymbol{r}(g) = g^*\rho_{R,n} : R \to g^*\mathcal{A}_{R,n}$ as in diagram (4.7). For this $\rho = \boldsymbol{r}(g)$ and $\mathcal{A} = g^*\mathcal{A}_{R,n}$, we obtain a morphism $g' = \boldsymbol{x}(\rho) : S \to X_{R,n}$ as in diagram (4.8) with g' and $g^*\mathcal{A}_{R,n}$ in place of g and $\mathcal{A}$ respectively. Comparing diagrams (4.7) for g and (4.8) for $\rho = \boldsymbol{r}(g)$, and using the uniqueness of g in (4.10), we see that

$$\boldsymbol{x}(\boldsymbol{r}(g)) = g.$$

This shows that $\boldsymbol{r}$ and $\boldsymbol{x}$ are inverse to each other, thereby completing the proof of the theorem. □

Proof of Lemma 4.21. First we consider locally on S. Let $O \in {}^{\mathrm{f}}\mathfrak{C}_F$, and let $\rho : R \to A$ be an absolutely irreducible representation of R over O of degree n. If $A \hookrightarrow A' := O' \otimes_O A$ is a splitting of the Azumaya algebra A, then we have a commutative diagram

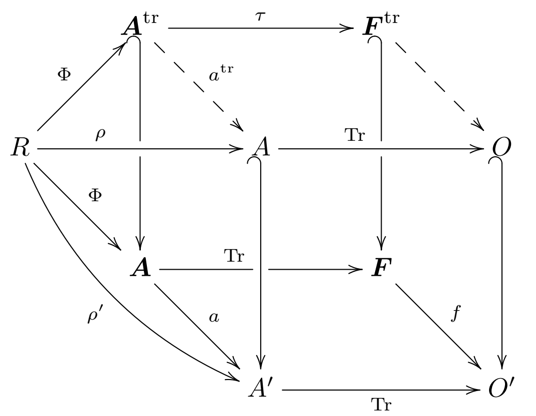
$$\begin{CD} A @>>> A' \\ @V{\mathrm{Tr}_{A/O}}VV @VV{\mathrm{Tr}_{A'/O'}}V \\ O @>>> O' \end{CD} \tag{4.11}$$

of O-modules. By our assumption $(\mathrm{V}^{\mathrm{ai}}_{n,F})$, the scalar extension $\rho' = O' \otimes_O \rho$ fits into the commutative diagram

$$\begin{CD} R @>{\Phi}>> \boldsymbol{A} @>{\mathrm{Tr}_{\boldsymbol{A}/\boldsymbol{F}}}>> \boldsymbol{F} \\ @| @VV{a}V @VV{f}V \\ R @>>{\rho'}> A' @>>{\mathrm{Tr}_{A'/O'}}> O', \end{CD} \tag{4.12}$$

where $f : \boldsymbol{F} \to O'$ is a morphism in ${}^{\mathrm{f}}\mathfrak{C}_F$ and $a := \mathrm{M}_n(f)$. Patching (4.3), (4.11) and (4.12) together, we have the diagram

(4.13)



in which all closed paths are commutative. By the definition of $\boldsymbol{F}^{\mathrm{tr}}$ (resp. $\boldsymbol{A}^{\mathrm{tr}}$), its image in O' (resp. A') is generated as an object of ${}^{\mathrm{f}}\mathfrak{C}_F$ (resp. ${}^{\mathrm{f}}\mathfrak{A}_{\boldsymbol{F}^{\mathrm{tr}}}$) by the image of R, and hence it is contained in the image of O (resp. A). Thus we obtain a morphism $f^{\mathrm{tr}} : \boldsymbol{F}^{\mathrm{tr}} \to O$ in ${}^{\mathrm{f}}\mathfrak{C}_F$ (resp. $a^{\mathrm{tr}} : \boldsymbol{A}^{\mathrm{tr}} \to A$ in ${}^{\mathrm{f}}\mathfrak{A}_F$) that is the restriction of f to $\boldsymbol{F}^{\mathrm{tr}}$ (resp. of $f_*\Phi$ to $\boldsymbol{A}^{\mathrm{tr}}$), thereby completing the broken lines in diagram (4.13); thus we have

$$\begin{array}{ccccc} R & \xrightarrow{\Phi} & \boldsymbol{A}^{\mathrm{tr}} & \xrightarrow{\tau} & \boldsymbol{F}^{\mathrm{tr}} \\ \| & & \downarrow{\scriptstyle a^{\mathrm{tr}}} & & \downarrow{\scriptstyle f^{\mathrm{tr}}} \\ R & \xrightarrow[\rho]{} & A & \xrightarrow[\mathrm{Tr}_{A/O}]{} & O. \end{array}$$

Note that f^{tr} is the unique morphism which makes the following diagram commutative:

$$\begin{array}{ccc} R & \xrightarrow{\tau\circ\Phi} & \boldsymbol{F}^{\mathrm{tr}} \\ \| & & \downarrow{\scriptstyle f^{\mathrm{tr}}} \\ R & \xrightarrow[\mathrm{Tr}_{A/O}\,\rho]{} & O. \end{array} \tag{4.14}$$

The morphism $a^{\mathrm{tr}}_O : O \otimes_{\boldsymbol{F}^{\mathrm{tr}}} \boldsymbol{A}^{\mathrm{tr}} \to A$ induced by a^{tr} is an isomorphism (Here, the tensor product $O \otimes_{\boldsymbol{F}^{\mathrm{tr}}} \cdot$ is via f^{tr}). Indeed, by Corollary 4.17 and the absolute irreducibility of ρ', the map $O' \otimes_{\boldsymbol{F}^{\mathrm{tr}}} \boldsymbol{A}^{\mathrm{tr}} \to A' \simeq O' \otimes_{\boldsymbol{F}} \boldsymbol{A}$ induced by a is bijective. Since the extension $O \to O'$ is faithfully flat, the map a^{tr}_O is also bijective.

Returning to the global situation, let $\rho : R \to \mathcal{A}$ be an absolutely irreducible representation of R over $S \in {}^{\mathrm{f}}\mathfrak{S}_F$ of degree n. The above arguments can be applied to each affine piece $\mathrm{Spec}(O) \subset S$. By the uniqueness of f in the condition $(\mathrm{V}^{\mathrm{ai}}_{n,F})$, various f^{tr}'s and a^{tr}'s patch together to give a unique morphism $g : S \to X_{R,n}$ (or $g^{\#} : \mathcal{O}_{X_{R,n}} \to g_*\mathcal{O}_S$) and a unique morphism $a_S : \mathcal{A}_{R,n} \to g_*\mathcal{A}$ that make diagram (4.8) commutative. Applying g^* to diagram (4.8) and composing with the natural morphism $g^*g_*\mathcal{A} \to \mathcal{A}$ etc., we obtain diagram (4.9), which is "adjoint" to (4.8). The morphism $g^*a_S : g^*\mathcal{A}_{R,n} \to \mathcal{A}$ is an isomorphism, being the globalization of the isomorphisms $a^{\mathrm{tr}}_O : O \otimes_{\boldsymbol{F}^{\mathrm{tr}}} \boldsymbol{A}^{\mathrm{tr}} \to A$. (The isomorphy of g^*a_S can be shown also as follows: It is surjective since ρ is absolutely irreducible. Since the $\mathcal{O}_S$-module $g^*\mathcal{A}_{R,n}$ and $\mathcal{A}$ are both locally free of rank n^2, Lemma 3.10 implies that g^*a_S is an isomorphism.) The uniqueness of g as in diagram (4.10) follows from the uniqueness of each affine piece f^{tr} as in diagram (4.14). □

The uniqueness of the morphism g in diagram (4.10) shows that, for $\rho \in \underline{\mathrm{Rep}}^{\mathrm{ai}}_{R,n}(S)$, the point $\boldsymbol{x}(\rho)$ is determined only by $\mathrm{Tr}_{\mathcal{A}/S}\,\rho$. Thus it follows that isomorphism classes of absolutely irreducible representations are determined by their traces. More precisely, we have the following result:

Corollary 4.22. *Let $\rho_i : R \to \mathcal{A}_i$ $(i = 1, 2)$ be two representations of R over S and assume that at least one of them is absolutely irreducible. Then we have $\rho_1 \simeq \rho_2$ if and only if $\operatorname{Tr}\rho_1 = \operatorname{Tr}\rho_2$ as a morphism $R \to \mathcal{O}_S$ of sheaves on S.*

Here, $\operatorname{Tr}\rho_i$ is the abbreviation of $\operatorname{Tr}_{\mathcal{A}_i/S}\rho_i$. Note that, by Corollary 4.6, we need to assume only that one of the ρ_i's is absolutely irreducible. See also [6, Thm. 1] and [26, Prop. in §5] for similar statements.

It may be useful to look at the following "trivial" example of the moduli scheme $X_{R,n}$:

Example 4.23. Let R be an Azumaya algebra in ${}^{\mathrm{f}}\mathfrak{A}_F$ of degree n over E. Then we have $X_{R,n} = \operatorname{Spec}(E)$ and the identity map $\rho : R \to R$ is the universal absolutely irreducible representation (cf. [35, Sect. 2.1]). This follows from Lemma 4.24 below. In the special case $R = \mathrm{M}_n(E)$, we can say more: The identity map $\rho : R \to \mathrm{M}_n(E)$ has universality as in $(\mathrm{V}^{\mathrm{ai}}_{n,F})$, and $\boldsymbol{F} = \boldsymbol{F}^{\mathrm{tr}} = E$, $\boldsymbol{A}^{\mathrm{tr}} = \mathrm{M}_n(E)$.

The following lemma, which was used in the above example, is quoted from [35, Prop. 1.7].

Lemma 4.24. *Let A and A' be Azumaya algebras over O and O' respectively, of the same degree n. Then any ring homomorphism $\phi : A \to A'$ maps O to O' and, via $\phi|_O : O \to O'$, it induces an isomorphism $O' \otimes_O A \to A'$. In particular, any representation of A into A' is absolutely irreducible. The map $\phi|_O : O \to O'$ depends only on the isomorphism class of f as a representation of A.*

This lemma implies in particular that there is a natural injective map

$$\operatorname{Hom}_{\mathfrak{A}}(A, A')/\operatorname{Aut}_{O'\text{-alg}}(A') \to \operatorname{Hom}_{\mathfrak{C}}(O, O')$$

of sets. It is also surjective if A and A' are matrix algebras, since then any $f : O \to O'$ induces a morphism $\mathrm{M}_n(f) : A \to A'$.

§4.5. Finiteness of the moduli

Next we turn to the problem of finite-dimensionality of the moduli scheme $X_{R,n}$. Recall from Sect. 4.1 that a scheme X in ${}^{\mathrm{f}}\mathfrak{S}_F$ is said to be of *finite type over* F if it has a finite affine open covering $X = U_1 \cup \cdots \cup U_c$, where $U_i = \operatorname{Spec}(O_i)$ with $O_i \in {}^{\mathrm{f}}\mathfrak{C}_F$, such that each O_i is finitely generated as an object of ${}^{\mathrm{f}}\mathfrak{C}_F$. The following proposition is an f-$\mathfrak{A}$ version of [35, Prop. 2.3], and is proved in a similar way, but we give a proof of it for the sake of completeness.

Proposition 4.25. *Let $\Phi : R \to \mathrm{M}_n(\boldsymbol{F})$ be a versal morphism as in* $(\mathrm{V}^{\mathrm{ai}}_{n,F})$. If $\Phi(R)$ is finitely generated as an object of ${}^{\mathrm{f}}\mathfrak{A}_F$, then the moduli scheme $X_{R,n}$ is of finite type over F.

Proof. We shall show that, in the presentation (4.4), we have that

(1) each $\boldsymbol{F}^{\mathrm{tr}}[1/d]$ is finitely generated, where $d = \det(\tau(\Phi(r_i r_j)))_{1\le i,j\le n^2}$ with $(r_1, \dots, r_{n^2})$ an n^2-tuple of elements of R; and

(2) the scheme $X_{R,n}$ is covered by a finite number of affine schemes $\mathrm{Spec}(\boldsymbol{F}^{\mathrm{tr}}[1/d])$ (and hence the ideal $\mathfrak{d}$ of $\boldsymbol{F}^{\mathrm{tr}}$ is finitely generated).

Write $\boldsymbol{A} = \mathrm{M}_n(\boldsymbol{F})$ and $\mathrm{Tr} = \mathrm{Tr}_{\boldsymbol{A}/\boldsymbol{F}}$ for simplicity. Recall that $\boldsymbol{F}^{\mathrm{tr}}$ is the closed F-subalgebra of $\boldsymbol{F}$ generated by $\mathrm{Tr}(\Phi(R))$ and $\boldsymbol{A}^{\mathrm{tr}}$ is the closed $\boldsymbol{F}^{\mathrm{tr}}$-subalgebra of $\boldsymbol{A}$ generated by $\Phi(R)$. Note that $\Phi(R) \in {}^{\mathrm{f}}\mathfrak{A}_F$ and $\boldsymbol{F}^{\mathrm{tr}} \in {}^{\mathrm{f}}\mathfrak{C}_F$ are in fact triples $(\Phi(R)_{\mathrm{o}}, \Phi(R), f_{\Phi(R)})$ and $(\boldsymbol{F}^{\mathrm{tr}}_{\mathrm{o}}, \boldsymbol{F}^{\mathrm{tr}}, f_{\boldsymbol{F}^{\mathrm{tr}}})$, respectively. Suppose $\Phi(R)$ is generated by $s_1, \dots, s_N$. This means (Definition 2.3) that some of them, say, $s_1, \dots, s_M$, generate $\Phi(R)_{\mathrm{o}}$ and the rest $s_{M+1}, \dots, s_N$ generate $\Phi(R)$ as a $\Phi(R)_{\mathrm{o}}$-algebra. For (1), we have to show both that $\boldsymbol{F}^{\mathrm{tr}}_{\mathrm{o}}[1/d]$ is topologically finitely generated and that $\boldsymbol{F}^{\mathrm{tr}}[1/d]$ is finitely generated over $\boldsymbol{F}^{\mathrm{tr}}_{\mathrm{o}}[1/d]$. Since the proofs are similar, we shall prove only the former part (so we assume $M = N$ in (1)). For (2), we need to show only the existence of a finite covering of $X_{R,n}$ as the usual scheme.

(1) For ease of notation, we write (here only) R, $\boldsymbol{F}$, $\boldsymbol{A}$, … for R_{o}, $\boldsymbol{F}_{\mathrm{o}}$, $\boldsymbol{A}_{\mathrm{o}}$, …, respectively (so these objects are in $\mathfrak{A}$). Let $d = \det(\mathrm{Tr}(\Phi(r_i r_j)))$ with $(r_1, \dots, r_{n^2}) \in R^{n^2}$ and put $a_i = \Phi(r_i)$. Then $(a_1, \dots, a_{n^2})$ is an $\boldsymbol{F}^{\mathrm{tr}}[1/d]$-basis of $\boldsymbol{A}^{\mathrm{tr}}[1/d]$ by Corollary 3.11. For each (i, j), write

$$a_i a_j = \sum_k f_{ijk} a_k \quad \text{with } f_{ijk} \in \boldsymbol{F}^{\mathrm{tr}}[1/d].$$

For each i, write

$$s_i = \sum_j f_{ij} a_j \quad \text{with } f_{ij} \in \boldsymbol{F}^{\mathrm{tr}}[1/d].$$

We claim that $\boldsymbol{F}^{\mathrm{tr}}[1/d]$ is topologically generated by

$$1/d,\ \mathrm{Tr}(a_i),\ f_{ij},\ f_{ijk} \quad \text{for all } i, j, k.$$

Let $\boldsymbol{F}_0$ be the F-subalgebra of $\boldsymbol{F}^{\mathrm{tr}}[1/d]$ algebraically generated by these elements. Let $\boldsymbol{A}_0$ be the $\boldsymbol{F}_0$-subalgebra $\sum_{i=1}^{n^2} \boldsymbol{F}_0 a_i$ of $\boldsymbol{A}^{\mathrm{tr}}[1/d]$. We have $\mathrm{Tr}(\boldsymbol{A}_0) \subset \boldsymbol{F}_0$. Let $\boldsymbol{S}_0$ be the F-subalgebra of $F[\Phi(R)]$ consisting of elements of the form

$$\sum_I c_I s^{m_I}, \quad c_I \in F,$$

where $I = (i_1, \ldots, i_l)$ are multiindices and $s^{m_I} = s_{i_1}^{m_{i_1}} \cdots s_{i_l}^{m_{i_l}}$ are monomials of exponents $m_I = (m_{i_1}, \ldots, m_{i_l})$. Since $\boldsymbol{S}_0$ is dense in $F[\Phi(R)]$, so is $F[\mathrm{Tr}(\boldsymbol{S}_0)]$ in $\boldsymbol{F}^{\mathrm{tr}}$. We claim that $\boldsymbol{S}_0 \subset \boldsymbol{A}_0$ (this will imply that $\mathrm{Tr}(\boldsymbol{S}_0) \subset \boldsymbol{F}_0$, and hence that $\boldsymbol{F}_0$ is dense in $\boldsymbol{F}^{\mathrm{tr}}[1/d]$). We shall show that each s^{m_I} lies in $\boldsymbol{A}_0$ by induction on the degree $|m_I| = m_{i_1} + \cdots + m_{i_l}$ of the monomial. This is true if $|m_I| = 1$. If $s^{m_I} = \sum_k f_k a_k$ with $f_k \in \boldsymbol{F}_0$, then

$$s_i s^{m_I} = \sum_j f_{ij} a_j \cdot \sum_k f_k a_k = \sum_{j,k} f_{ij} f_k a_j a_k = \sum_{i,j,h} f_{ij} f_k f_{jkh} a_h \quad \in \boldsymbol{A}_0.$$

Thus the induction proceeds.

(2) Choose an element r_i of R such that $\Phi(r_i) = s_i$ for each $i = 1, \ldots, N$. Let G be the subset of R consisting of monomials g_k (say) in $r_1, \ldots, r_N$ of total degree $\leq n^2$. Note that G is finite. We claim that, for any absolutely irreducible representation $\rho : R \to A$ over a local ring O in $\mathfrak{C}$, there exists a subset $\{g_{k_1}, \ldots, g_{k_{n^2}}\}$ of G whose image by ρ is an F-basis of A. (This will imply that the point of $X_{R,n,F}$ corresponding to ρ is contained in $\mathrm{Spec}(\boldsymbol{F}^{\mathrm{tr}}[1/d])$ with $d = \det(\mathrm{Tr}(\Phi(g_{k_i} g_{k_j})))$.) To prove the claim, we may assume, by Nakayama's lemma, that O is a field. Put $\bar{r}_i = \rho(r_i)$, and let V_j be the O-subspace of A generated by all monomials in $\bar{r}_i$'s of total degree at most j. Then we have an increasing sequence $V_0 \subset V_1 \subset \cdots \subset A$, in which $V_i = A$ for some i because $\rho : R \to A$ is absolutely irreducible (which factors through $\Phi : R \to \boldsymbol{A}^{\mathrm{tr}}$), $\Phi(R)$ is generated by $s_1, \ldots, s_N$ and A is finite-dimensional. On the other hand, once we have $V_{i+i} = V_i$ for some i, then V_i does not increase anymore. Thus V_i reaches A in at most $\dim_O A = n^2$ steps. □

The author has no idea whether there exists an example of R such that $\Phi(R)$ is not finitely generated and yet $X_{R,n}$ is of finite type. It is desirable to find a criterion for R that tells us precisely when $X_{R,n}$ is of finite type.

Let us conclude this section by asking a question:

Question 4.26. *Let G be a profinite group and F a commutative profinite ring. If G is topologically generated by a finite number of conjugacy classes, then is the f-profinite scheme $X_{F[\![G]\!],n,F}$ of finite type over F?*

Here, $F[\![G]\!]$ denotes the completed group ring of G over F.

For example, let K be an algebraic number field of finite degree over $\mathbb{Q}$. Let S be a finite set of places of K, and let $G_{K,S}$ be the Galois group over K of the maximal Galois extension of K that is unramified outside S. Then, though it is not known whether the profinite group $G_{K,S}$ is topologically finitely generated, it is known that $G_{K,S}$ is topologically generated by a finite number of conjugacy classes ([17], [32, Thm. 10.2.5 and Cor. 10.9.11]). Hence if the above question is answered in the affirmative in the case $F = \mathbb{F}_q$, then we can draw interesting information on the growth of the number of mod p Galois representations of $G_{K,S}$ as the coefficient field $\mathbb{F}_{q^\nu}$ grows.

§5. Variant: Representations of τ-algebras

In this section, we give a variant of the theory in Sections 2–4, replacing the ring R by a τ-algebra:

Definition 5.1. Let E be a commutative ring. A *τ-algebra* over E (or, *E-τ-algebra*) is a pair (R, τ) consisting of an E-algebra R and an E-linear map $\tau : R \to E$. The map τ is called the *trace map* of R, and is sometimes denoted by τ_R below. If we work in a category of topological rings, the trace map is assumed to be continuous.

Let $\alpha : E \to F$ be a homomorphism of commutative rings, and let R, S be τ-algebras over E, F, respectively. A *homomorphism* of τ-algebras (or simply, *τ-homomorphism*) $\phi : R \to S$ (with respect to α) is a ring homomorphism ϕ that makes the following diagram commutative:

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & F \\ {\scriptstyle\iota_R}\downarrow & & \downarrow{\scriptstyle\iota_S} \\ R & \xrightarrow{\phi} & S \\ {\scriptstyle\tau_R}\downarrow & & \downarrow{\scriptstyle\tau_S} \\ E & \xrightarrow[\alpha]{} & F, \end{array}$$

where ι_R, ι_S are respectively the structure morphisms of R, S.

Example 5.2. If A is an Azumaya algebra over E, then any E-subalgebra R of A can be regarded as an E-τ-algebra with trace map $\tau = \mathrm{Tr}_{A/E}|_R$. In what follows, subalgebras of an Azumaya algebra are regarded as τ-algebras in this way, unless otherwise stated.

Let A and B be Azumaya algebras of the same degree over E and F respectively. A ring homomorphism $\phi : A \to B$ induces a ring homomorphism $\alpha : E \to F$ on the center (Lemma 4.24). Then ϕ is a τ-homomorphism with respect to this α.

This is a motivating example for the notion of a τ-algebra. Indeed, a subalgebra R of an Azumaya algebra A is expected to share typical properties of Azumaya algebras, especially when R is "close" to A. This, however, may not be true in general, and the τ-structure helps to remedy the defect.

Remark 5.3. In the definition of a τ-algebra over E, we require only the E-linearity of τ, though in practice τ would share other properties with the reduced trace map of an Azumaya algebra. In [34], Procesi introduced and studied the notion of an *algebra with trace*. His purpose there was to characterize rings that can be embedded as a subring into the $n \times n$ matrix algebra over a commutative ring. Thus he required the trace map τ to be, most notably, central; $\tau(ab) = \tau(ba)$. Our purpose here, however, is to *rigidify* representations $\phi : R \to A$ (cf. Definition 5.5), and for this purpose, we need only to specify an E-linear map $\tau : R \to E$.

As a related topic, one should also note the theory of pseudo-representations ([47], [45], [33], [37], [2], [7]).

The purpose of this section is to show that a τ-version of Theorem 4.20 is true, i.e., that there exists a moduli scheme that parametrizes the absolutely irreducible representations of a τ-algebra in a certain category as in Section 4. To ensure that the theorem is applicable to f-pro-$\mathfrak{a}$ rings, we first give a τ-version of Lemma 2.9. Let $\mathfrak{A}$ be a category of topological rings and $\mathfrak{C}$ its full subcategory of commutative objects. In Section 2, we defined the category ${}^{\mathrm{f}}\mathfrak{A}_E$ of f-$\mathfrak{A}$ E-algebras and its full subcategory ${}^{\mathrm{f}}\mathfrak{C}_E$ of commutative objects, where $E = (E_{\mathrm{o}}, E)$ is a commutative f-$\mathfrak{A}$ ring. By an f-$\mathfrak{A}$ *E-τ-algebra*, we mean an object (R_{o}, R, f) of ${}^{\mathrm{f}}\mathfrak{A}_E$ such that $R_{\mathrm{o}} = (R_{\mathrm{o}}, \tau_{\mathrm{o}})$ is a E_{o}-τ-algebra in $\mathfrak{A}_{E_{\mathrm{o}}}$, $R = (R, \tau)$ is any E-τ-algebra, and f is a τ-homomorphism:

$$\begin{array}{ccc}
E_{\mathrm{o}} & \xrightarrow{f_E} & E \\
{\scriptstyle \iota_{R_{\mathrm{o}}}}\downarrow & & \downarrow{\scriptstyle \iota_R} \\
R_{\mathrm{o}} & \xrightarrow{f} & R \\
{\scriptstyle \tau_{R_{\mathrm{o}}}}\downarrow & & \downarrow{\scriptstyle \tau_R} \\
E_{\mathrm{o}} & \xrightarrow[f_E]{} & E.
\end{array}$$

A *morphism* of f-$\mathfrak{A}$ E-τ-algebras is a morphism in ${}^{\mathrm{f}}\mathfrak{A}_E$ that is at the same time a τ-homomorphism. We denote by ${}^{\mathrm{f}}\mathfrak{A}_E^{\tau}$ the category of f-$\mathfrak{A}$ E-τ-algebras. This in particular applies to the case where $\mathfrak{A} = \widehat{\mathfrak{a}}$ is the category of pro-$\mathfrak{a}$ rings with $\mathfrak{a}$ a fixed category of rings, so that ${}^{\mathrm{f}}\widehat{\mathfrak{a}}_E^{\tau}$ denotes the category of f-pro-$\mathfrak{a}$ E-τ-algebras.

Now fix a category $\mathfrak{a}$ of rings and let $\mathfrak{c}$ be its full subcategory consisting of commutative objects. Fix an object $Z \in {}^{\mathrm{f}}\widehat{\mathfrak{c}}$.

Lemma 5.4. *Assume that the category $\mathfrak{a}$ satisfies Axiom (a1) in Section 2. Let $E \in {}^{\mathrm{f}}\widehat{\mathfrak{c}}_Z$ and $R \in {}^{\mathrm{f}}\widehat{\mathfrak{a}}_E^{\tau}$. For any $F \in {}^{\mathrm{f}}\widehat{\mathfrak{c}}_Z$ and an integer $n \geq 1$, there exist a unique (up to canonical isomorphism) object $\mathrm{F}_n(R) \in {}^{\mathrm{f}}\widehat{\mathfrak{c}}_F$ and a morphism $\Phi : R \to \mathrm{M}_n(\mathrm{F}_n(R))$ of τ-algebras that has the following universality: for any*

$O \in {}^{\mathrm{f}}\widehat{\mathfrak{c}}_F$, any morphism $\alpha : E \to O$ in ${}^{\mathrm{f}}\widehat{\mathfrak{c}}_Z$ and any morphism $\phi : R \to \mathrm{M}_n(O)$ of τ-algebras with respect to α, there exists a unique morphism $f : \mathrm{F}_n(R) \to O$ in ${}^{\mathrm{f}}\widehat{\mathfrak{c}}_F$ such that $\phi = \mathrm{M}_n(f) \circ \Phi$.

In Section 8, this lemma will be applied with Z the ring $W(k)$ of Witt vectors over a finite field k, E a certain versal Galois deformation ring, R the E-subalgebra of $\mathrm{M}_n(E)$ generated by a certain Galois group (that is the image of the versal deformation) and F the fraction field of $W(k)$.

Proof. We give a proof when Z, E, F are in $\mathfrak{c}$ and R is in $\mathfrak{a}_E^\tau$, since the passage to ${}^{\mathrm{f}}\widehat{\mathfrak{a}}_E^\tau$ is the same as in the proof of Lemma 2.9 (but with the trace taken into account). Suppose R is presented as the quotient $E\langle X_\mu\rangle_{\mu\in M}/I$ of a non-commutative polynomial ring $E\langle X_\mu\rangle_{\mu\in M}$ by a two-sided ideal I. For each μ, let $\underline{X}_\mu = (x_{\mu ij})_{1\le i,j\le n}$ be an $(n \times n)$-matrix with variable components $x_{\mu ij}$, and let $E'[x_{\mu ij}] = E'[x_{\mu ij}]_{\mu\in M, 1\le i,j\le n}$ be the commutative polynomial ring over $E' := E \otimes_Z F$ in these variables. As in the proof of Lemma 2.9, there is an E-algebra homomorphism $\varphi : E\langle X_\mu\rangle \to \mathrm{M}_n(E'[x_{\mu ij}])$ that maps X_μ to $\underline{X}_\mu$. For each $f \in I$, let f_{ij} be the (i,j)-component of $\varphi(f) \in \mathrm{M}_n(E'[x_{\mu ij}])$. Let $\underline{I}$ be the ideal of $E'[x_{\mu ij}]$ generated by the f_{ij} for all $f \in I$ and $1 \le i,j \le n$. Then the map φ descends to a morphism $\underline{\varphi} : R \to \mathrm{M}_n(E'[x_{\mu ij}]/\underline{I})$. Let $\tau'(X_\mu) \in E'$ be the image of X_μ (mod I) by the composite map $\tau' : R \xrightarrow{\tau} E \to E'$, and let $\overline{I}$ be the ideal of $E'[x_{\mu ij}]$ generated by $\underline{I}$ and the elements $\mathrm{Tr}(\underline{X}_\mu) - \tau'(X_\mu)$ for all $\mu \in M$. Then the composite map $\overline{\varphi} : R \to \mathrm{M}_n(E'[x_{\mu ij}]/\overline{I})$ of $\underline{\varphi}$ and the natural projection $\mathrm{M}_n(E'[x_{\mu ij}]/\underline{I}) \to \mathrm{M}_n(E'[x_{\mu ij}]/\overline{I})$ is a τ-homomorphism. Let $\mathrm{F}_n(R)$ be the pro-$\mathfrak{a}$ completion of $E'[x_{\mu ij}]/\overline{I}$. Composing $\overline{\varphi}$ with the map induced by $E'[x_{\mu ij}]/\overline{I} \to \mathrm{F}_n(R)$, we obtain the desired morphism $\Phi : R \to \mathrm{M}_n(\mathrm{F}_n(R))$. □

Next we return to a general category $\mathfrak{A}$ of topological rings. Fix $Z \in {}^{\mathrm{f}}\mathfrak{C}$, and let ${}^{\mathrm{f}}\mathfrak{S}_Z$ be the category of f-$\mathfrak{A}$ schemes over Z (Section 4). If $S \in \mathfrak{S}_Z$, let ${}^{\mathrm{f}}\mathfrak{A}_S^\tau$ (resp. ${}^{\mathrm{f}}\mathfrak{C}_S$) be the category of sheaves of f-$\mathfrak{A}$ $\mathcal{O}_S$-τ-algebras (resp. commutative f-$\mathfrak{A}$ $\mathcal{O}_S$-algebras) on S.

Let $E \in {}^{\mathrm{f}}\mathfrak{C}_Z$, $R \in {}^{\mathrm{f}}\mathfrak{A}_E^\tau$, $S \in {}^{\mathrm{f}}\mathfrak{S}_Z$ and $\mathcal{O} \in {}^{\mathrm{f}}\mathfrak{C}_S$. Suppose we are given a morphism $\alpha : E \to \mathcal{O}$ in $\mathfrak{C}_Z$.

Definition 5.5. A *τ-representation* $\rho : R \to \mathcal{A}$ of R over $\mathcal{O}$ is a τ-homomorphism with respect to α, $\tau : R \to E$ and $\mathrm{Tr}_{\mathcal{A}/\mathcal{O}}$, where $\mathcal{A}$ is an Azumaya algebra over $\mathcal{O}$. Two τ-representations $\rho_i : R \to \mathcal{A}_i$ $(i = 1, 2)$ are said to be *isomorphic* if there is an isomorphism $f : \mathcal{A}_1 \to \mathcal{A}_2$ (which is automatically a τ-homomorphism) such that $\rho_2 = f \circ \rho_1$. We say that a τ-representation $\rho : R \to \mathcal{A}$ is *absolutely irreducible* if it is so as the usual representation.

Note that the structure morphism $\alpha : E \to \mathcal{O}$ is uniquely determined if the usual absolutely irreducible representation $\rho : R \to \mathcal{A}$ is given, because then the center E of R is mapped into the center $\mathcal{O}$ of $\mathcal{A}$. Thus we may (and do) speak of absolutely irreducible τ-representations *without a priori specifying the structure morphism* α.

The next proposition follows immediately from Corollary 4.22.

Proposition 5.6. *Fix a morphism $\alpha : E \to \mathcal{O}$ in ${}^{\mathrm{f}}\mathfrak{C}_Z$. For given R and $\mathcal{A}$ as above, there exists at most one isomorphism class of absolutely irreducible τ-representation $\rho : R \to \mathcal{A}$ with respect to α. More precisely, if $\rho_1,\ \rho_2 : R \to \mathcal{A}$ are two τ-homomorphisms one of which is absolutely irreducible, then the other is also absolutely irreducible and they are isomorphic as representations of R over $\mathcal{O}$.*

For $S \in {}^{\mathrm{f}}\mathfrak{S}_Z$, let $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{R,n}(S)$ denote the set of isomorphism classes of absolutely irreducible τ-representations of R of degree n over S (here, "over S" means "over $\mathcal{O}_S$"). The above proposition may be rephrased as follows:

Proposition 5.7. *The natural map $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{R,n}(S) \to \mathrm{Hom}_{\mathfrak{C}_Z}(E, \mathcal{O}_S)$ is injective.*

If R is an E-subalgebra of $\mathrm{M}_n(E)$, then this map is bijective, its inverse being given by $f \mapsto \mathrm{M}_n(f)$ (cf. Example 4.23). In general, however, it may not be surjective. Note in particular that the set $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{R,n}(S)$ depends on n but $\mathrm{Hom}_{\mathfrak{C}_Z}(E, \mathcal{O}_S)$ does not. For instance, if $R = \mathrm{M}_m(E)$ with the standard trace map, then $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{R,n}(S) = \emptyset$ unless $m = n$, while $\mathrm{Hom}_{\mathfrak{C}_Z}(E, E) \neq \emptyset$.

As in Section 4, the correspondence

$$\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{R,n} : {}^{\mathrm{f}}\mathfrak{S}_Z \to (\mathrm{Sets}),$$
$$S \mapsto \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{R,n}(S)$$

is a contravariant functor. By Proposition 5.6, $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{R,n}(S)$ may be identified with a subset of the Brauer group ${}_n\mathrm{Br}(S)$, and hence there is an injective natural transformation $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{R,n} \to {}_n\mathrm{Br}$ of functors.

In the rest of this section, we assume the following condition on $R \in {}^{\mathrm{f}}\mathfrak{A}^{\tau}_E$, which says that it has a universal τ-homomorphism of degree n with coefficients in $F \in {}^{\mathrm{f}}\mathfrak{C}_Z$:

($\mathrm{V}^{\mathrm{ai},\tau}_{n,F}$) There exist an object $\boldsymbol{F}$ of ${}^{\mathrm{f}}\mathfrak{C}_F$ and a morphism $\Phi : R \to \mathrm{M}_n(\boldsymbol{F})$ in ${}^{\mathrm{f}}\mathfrak{A}^{\tau}_Z$ such that, for any $O \in {}^{\mathrm{f}}\mathfrak{C}_F$ and any absolutely irreducible τ-representation $\rho : R \to \mathrm{M}_n(O)$, there exists a unique morphism $f : \boldsymbol{F} \to O$ in ${}^{\mathrm{f}}\mathfrak{C}_F$ such that $\rho = f_*\Phi := \mathrm{M}_n(f) \circ \Phi$, i.e., the following diagram is commutative:

$$\begin{array}{ccc} & & \mathrm{M}_n(\boldsymbol{F}) \\ & \overset{\Phi}{\nearrow} & \downarrow{\scriptstyle \mathrm{M}_n(f)} \\ R & \xrightarrow[\phi]{} & \mathrm{M}_n(O). \end{array} \tag{5.1}$$

Set $\boldsymbol{A} := \mathrm{M}_n(\boldsymbol{F})$. As in Section 4, we define

- $\boldsymbol{F}^{\mathrm{tr}}$: the closed E-subalgebra of $\boldsymbol{F}$ generated by $\mathrm{Tr}_{\boldsymbol{A}/\boldsymbol{F}}(\Phi(R))$;
- $\boldsymbol{A}^{\mathrm{tr}}$: the closed $\boldsymbol{F}^{\mathrm{tr}}$-subalgebra of $\boldsymbol{A}$ generated by $\Phi(R)$.

Let $\tau : \boldsymbol{A}^{\mathrm{tr}} \to \boldsymbol{F}^{\mathrm{tr}}$ be the restriction of $\mathrm{Tr}_{\boldsymbol{A}/\boldsymbol{F}}$ to $\boldsymbol{A}^{\mathrm{tr}}$. Then we have a morphism

$$\Phi : R \to \boldsymbol{A}^{\mathrm{tr}}$$

in ${}^{\mathrm{f}}\mathfrak{A}^{\tau}_{E}$. Let $X^{\tau}_{R,n}$ be the locus of $\mathrm{Spec}(\boldsymbol{F}^{\mathrm{tr}})$ over which $\boldsymbol{A}^{\mathrm{tr}}$ is Azumaya; we consider it as an object of ${}^{\mathrm{f}}\mathfrak{S}_F$. By Proposition 4.15, it is presented as

$$X^{\tau}_{R,n} = \bigcup_{(d)} \mathrm{Spec}(\boldsymbol{F}^{\mathrm{tr}}[d^{-1}]),$$

where the union is over the set of principal ideals (d) of $\boldsymbol{F}^{\mathrm{tr}}$ of the form $d = \det(\tau(\Phi(r_i r_j)))_{1\le i,j\le n^2}$ with $(r_1, \ldots, r_{n^2})$ moving through all n^2-tuples of R. It also follows from Proposition 4.15 that the restriction $\mathcal{A}^{\tau}_{R,n}$ to $X^{\tau}_{R,n}$ of the sheafification of $\boldsymbol{A}^{\mathrm{tr}}$ is an Azumaya algebra, and that we have an absolutely irreducible τ-representation $\rho^{\tau}_{R,n} : R \to \mathcal{A}^{\tau}_{R,n}$ over $X^{\tau}_{R,n}$.

A morphism $g : S \to X^{\tau}_{R,n}$ in ${}^{\mathrm{f}}\mathfrak{S}_F$ gives rise to an absolutely irreducible τ-representation $g^*\rho^{\tau}_{R,n} : R \to g^*\mathcal{A}^{\tau}_{R,n}$, whose isomorphism class we denote by $\boldsymbol{r}(g)$. By essentially the same arguments as in the proof of Theorem 4.20, we can prove that the map $\boldsymbol{r} : X^{\tau}_{R,n}(S) \to \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{R,n}(S)$ is bijective. Thus we have the following theorem:

Theorem 5.8. *The scheme $X^{\tau}_{R,n}$ represents the functor $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{R,n}$.*

In the next proposition, which will be used conveniently in Section 8, we assume for simplicity that $F = Z$.

Proposition 5.9. *Let A be an Azumaya algebra over $E \in {}^{\mathrm{f}}\mathfrak{C}_Z$. Let R be an E-τ-subalgebra of A, and let $\iota : R \hookrightarrow A$ denote the inclusion map. Then we have the following canonical isomorphisms:*

$$X^{\tau}_{R,n} = S^{\mathrm{Az}}(R) = T^{\mathrm{ai}}(\iota).$$

In particular, $X^{\tau}_{R,n}$ can be identified with an open subscheme of $\mathrm{Spec}(E)$. If further A is the matrix algebra $\mathrm{M}_n(E)$, then the inclusion map $\iota : R \hookrightarrow \mathrm{M}_n(E)$ is a versal *morphism as in* $(\mathrm{V}^{\mathrm{ai},\tau}_{n,F})$*, and hence* $X^\tau_{R,n}$ *is an open subscheme of* $\mathrm{Spec}(E^{\mathrm{tr}})$*, where* E^{tr} *is the* Z*-subalgebra generated by* $\tau(R)$*.*

Note that the two rings E^{tr} and E coincide "outside n" in the sense that $E^{\mathrm{tr}}[1/n] = E[1/n]$, because R contains E and $\tau(e) = ne$ for $e \in E$. In general, however, we cannot expect $E^{\mathrm{tr}} = E$. For instance, suppose $Z = \mathbb{Z}$ and $R = E[D_8]$, where D_8 is the dihedral group of order 8. Let $\iota : R \hookrightarrow \mathrm{M}_2(E)$ be the E-algebra homomorphism extending a two-dimensional irreducible representation $\rho : D_8 \to \mathrm{GL}_2(\mathbb{Z})$, and consider the τ-structure on R induced by $\mathrm{Tr} : \mathrm{M}_2(E) \to E$. Then, since $\mathrm{Tr}\,\rho$ has values $0, \pm 2$, one has $E^{\mathrm{tr}} = \mathbb{Z}[\tau(R)] = \mathbb{Z}[2E]$, which is an "order of E of conductor 2" and may not be equal to E in general.

Proof. The second equality follows from Proposition 4.15 applied with $S = T = \mathrm{Spec}(E)$ (note that then $S^{\mathrm{Az}}(R)$ and $T^{\mathrm{ai}}(\iota)$ are open subschemes of $\mathrm{Spec}(E)$).

Suppose we are given a point in $X^\tau_{R,n}(S)$, or, an absolutely irreducible τ-representation $\rho : R \to \mathcal{A}$ over S. It induces a morphism $\alpha : E \to \mathcal{O}_S$ in ${}^{\mathrm{f}}\mathfrak{C}_Z$ since $\rho(E)$ is contained in the center of $\mathcal{A}$, and then a τ-representation $\rho' : R \xrightarrow{\iota} A \to \mathcal{O}_S \otimes_E A$ over $\mathcal{O}_S$ with respect to α, where the tensor product is via α. Since $\mathrm{Tr}\,\rho' = \mathrm{Tr}\,\rho\ (= \alpha \circ \tau_R)$, we have $\rho' \simeq \rho$ by Proposition 5.6. Since ρ' is absolutely irreducible, the S-valued point α of $\mathrm{Spec}(E)$ is in $T^{\mathrm{ai}}(\iota)(S)$.

Conversely, let α be an S-valued point of $T^{\mathrm{ai}}(\iota)$; thus it is a morphism $\alpha : E \to \mathcal{O}_S$ such that the τ-representation $\rho : R \xrightarrow{\iota} A \to \mathcal{O}_S \otimes_E A$ induced by α is absolutely irreducible. Hence it gives an S-valued point of $X^\tau_{R,n}$. It is clear that these constructions $\rho \mapsto \alpha$ and $\alpha \mapsto \rho$ are inverse to each other.

Suppose $A = \mathrm{M}_n(E)$, and $\rho : R \to \mathrm{M}_n(O)$ is an absolutely irreducible τ-representation. Then it induces a map $\alpha : E \to O$ since $\rho(E)$ is contained in the center of $\mathrm{M}_n(O)$. Composing the map $\mathrm{M}_n(\alpha) : \mathrm{M}_n(E) \to \mathrm{M}_n(O)$ with $\Phi : R \to \mathrm{M}_n(E)$, we have another τ-representation $\rho' : R \to \mathrm{M}_n(O)$. By Proposition 5.6, this is also absolutely irreducible and isomorphic to ρ. The rest follows from the construction of $X^\tau_{R,n}$. □

§6. Relation with group representations

Let G be a profinite group. In this section, let ${}^{\mathrm{f}}\mathfrak{A} = {}^{\mathrm{f}}\widehat{\mathfrak{a}}$ and ${}^{\mathrm{f}}\mathfrak{C} = {}^{\mathrm{f}}\widehat{\mathfrak{c}}$ be respectively the category of f-profinite rings and its subcategory of commutative objects (cf. Section 2). Let ${}^{\mathrm{f}}\mathfrak{S}$ be the category of f-profinite schemes (cf. Section 4). Let $E = (E_{\mathrm{o}}, E, f)$ be an object of ${}^{\mathrm{f}}\mathfrak{C}$. For our purposes in this section, it is harmless and convenient to replace E_{o} by $f(E_{\mathrm{o}})$ and *assume* that E_{o} is a subring of E and f is the inclusion map. We define the *completed group ring* of G over E to be the object

$$E[\![G]\!] := (E_{\rm o}[\![G]\!], E \otimes_{E_{\rm o}} E_{\rm o}[\![G]\!]) \quad \text{in } {}^{\rm f}\mathfrak{A},$$

where $E_{\rm o}[\![G]\!]$ is the usual completed group ring

$$E_{\rm o}[\![G]\!] := \varprojlim_{I,H} (E_{\rm o}/I)[G/H],$$

with I and H running respectively through all the open ideals of $E_{\rm o}$ and open normal subgroups of G.

Remark 6.1. If G is finite, the completed group ring $E[\![G]\!]$ coincides with the f-version $(E_{\rm o}[G], E[G])$ of the usual group ring.

Definition 6.2. A *representation* of G over E of degree (or dimension) n is a group homomorphism $\rho : G \to \mathrm{GL}_n(E)$ that factors as $G \stackrel{\rho_{\rm o}}{\to} \mathrm{GL}_n(E_{\rm o}) \hookrightarrow \mathrm{GL}_n(E)$, where $\rho_{\rm o}$ is continuous with respect to the profinite topology of $\mathrm{GL}_n(E_{\rm o})$. It is said to be *absolutely irreducible* if $E_{\rm o}[\![\mathrm{Im}(\rho_{\rm o})]\!]$ generates $\mathrm{M}_n(E)$ as an E-module. Two representations ρ_1 and ρ_2 over E are said to be *isomorphic* (or *equivalent*) if there exists a $\sigma \in \mathrm{Aut}_{{}^{\rm f}\mathfrak{A}}(\mathrm{M}_n(E))$ such that $\sigma(\rho_1(g)) = \rho_2(g)$ for all $g \in G$. We denote by $\underline{\mathrm{Rep}}^{\rm ai}_{G,n}(E)$ the set of isomorphism classes of absolutely irreducible representations of G over E of degree n.

Be aware of the difference between $\underline{\mathrm{Rep}}^{\rm ai}_{G,n}(E)$ and $\underline{\mathrm{Rep}}^{\rm ai}_{F[\![G]\!],n}(E)$: the former, when G is a *group*, refers to representations of G into $\mathrm{GL}_n(E)$, whereas the latter refers to representations of the f-profinite ring $F[\![G]\!]$ into *any* Azumaya algebras of degree n over E.

Note that the image $\mathrm{Im}(\rho)$ of ρ may be identified with the closed subgroup $\mathrm{Im}(\rho_{\rm o})$ of the compact group $\mathrm{GL}_n(E_{\rm o})$.

Note also that, if E is noetherian, then the above definition of absolute irreducibility is equivalent to saying that the image of ρ generates $\mathrm{M}_n(E)$ as an E-module, because then the E-submodule of $\mathrm{M}_n(E)$ generated by $\mathrm{Im}(\rho)$ is of finite type and its coincidence with $\mathrm{M}_n(E)$ can be checked by reduction modulo each maximal ideal of E (and, over a field E ($\supset E_{\rm o}$), generation by $\mathrm{Im}(\rho)$ and generation by $E_{\rm o}[\![\mathrm{Im}(\rho_{\rm o})]\!]$ are equivalent; cf. Proposition 6.3 below). Note also that the automorphisms of $\mathrm{M}_n(E)$ are locally (i.e., Zariski-locally on $\mathrm{Spec}(E)$) inner by the theorem of Auslander–Goldman ([12, Thm. 5.10]).

Let ρ be a representation of G over $(E_{\rm o}, E)$ of degree n. Let $\alpha : F \to E$ be a morphism in ${}^{\rm f}\mathfrak{C}$. Then a representation $\rho : G \to \mathrm{GL}_n(E_{\rm o})$ induces a morphism $F_{\rm o}[\![\rho]\!] : F_{\rm o}[\![G]\!] \to \mathrm{M}_n(E_{\rm o})$ as the projective limit of $(F_{\rm o}/I')[\rho] : (F_{\rm o}/I')[G] \to \mathrm{M}_n(E_{\rm o}/IE_{\rm o})$, where I moves through the set of open ideals of $E_{\rm o}$ and $I' := \alpha^{-1}(I)$. Tensoring with F over $F_{\rm o}$ and composing with the natural map $F \otimes_{F_{\rm o}} \mathrm{M}_n(E_{\rm o}) \to$

$\mathrm{M}_n(E)$, we obtain a morphism $F[\![\rho]\!] : F[\![G]\!] \to \mathrm{M}_n(E)$ in ${}^{\mathrm{f}}\mathfrak{A}_E$. Then ρ is absolutely irreducible if and only if $F[\![\rho]\!]$ is absolutely irreducible in the sense of Section 4.

If E is a field, the above definition of absolute irreducibility is compatible with the usual one, since we have the following equivalence (cf. [5, Chap. 8, Sect. 13, n°. 4] or [22, Thm. 7.5]):

Proposition 6.3. *Let G and E be as above, and assume that (the second factor of) E is a field. Let $\rho : G \to \mathrm{GL}_n(E)$ be a finite-dimensional continuous E-linear representation of G. Then the following conditions are equivalent:*

(1) ρ is semisimple and $\mathrm{End}(\rho) = E$.

(2) The $\overline{E}$-linear extension $\overline{\rho} : G \to \mathrm{GL}_n(\overline{E})$ of ρ is irreducible, where $\overline{E}$ is an algebraic closure of E.

(2′) For any extension field E' of E, the E'-linear extension $\rho' : G \to \mathrm{GL}_n(E')$ of ρ is irreducible.

(3) For any subring F of E, the F-algebra homomorphism $F[\![\rho]\!] : F[\![G]\!] \to \mathrm{M}_n(E)$ induced by ρ is absolutely irreducible in the sense of Section 4.

Here, $\mathrm{End}(\rho) := \{M \in \mathrm{M}_n(E) |\ M\rho(g) = \rho(g)M \text{ for all } g \in G\}$, and the E in (1) is identified with the ring of scalar matrices in $\mathrm{M}_n(E)$.

The next proposition shows that the representation theory over E of a profinite group (at least for absolutely irreducible representations) reduces to that of the corresponding completed group ring if the n-torsion subgroup ${}_n\mathrm{Br}(E)$ of the Brauer group $\mathrm{Br}(E)$ vanishes (the Brauer group here is that of E as the second factor of $E = (E_\mathrm{o}, E)$):

Proposition 6.4. *Let $F \in {}^{\mathrm{f}}\mathfrak{C}$ and $E \in {}^{\mathrm{f}}\mathfrak{C}_F$. If ${}_n\mathrm{Br}(E) = 0$, then there is a canonical bijection*

$$X_{F[\![G]\!],n,F}(E) \simeq \underline{\mathrm{Rep}}^{\mathrm{ai}}_{G,n}(E).$$

Proof. An absolutely irreducible group representation $\rho : G \to \mathrm{GL}_n(E)$ gives rise to an absolutely irreducible representation $\phi : F[\![G]\!] \to \mathrm{M}_n(E)$ in ${}^{\mathrm{f}}\mathfrak{A}_F$. Suppose conversely that $\phi : F[\![G]\!] \to A$ is an absolutely irreducible representation of the ring $F[\![G]\!] \in {}^{\mathrm{f}}\mathfrak{A}_F$, where A is an Azumaya algebra over E of degree n. Since ${}_n\mathrm{Br}(E) = 0$, we have $A \simeq \mathrm{M}_n(E)$. Thus we have a group representation

$$\rho : G \hookrightarrow F[\![G]\!]^\times \overset{\phi}{\to} A^\times \simeq \mathrm{GL}_n(E),$$

which is also absolutely irreducible by construction.

Suppose there are two $\rho_1, \rho_2 \in \underline{\mathrm{Rep}}^{\mathrm{ai}}_{G,n}(E)$, and let ϕ_1, ϕ_2 be the corresponding ring representations of $F[\![G]\!]$. Then we have $\rho_1 \simeq \rho_2$ if and only if $\phi_1 \simeq \phi_2$ by our Definition 6.2 of equivalence of two group representations. Thus we obtain a bijective correspondence between $\underline{\mathrm{Rep}}^{\mathrm{ai}}_{F[\![G]\!],n}(E)$ and $\underline{\mathrm{Rep}}^{\mathrm{ai}}_{G,n}(E)$. □

§7. Finiteness conjecture of Khare and Moon

In this section, we give a reformulation of a finiteness conjecture of Khare and Moon using our moduli theory. Let K be a global field in the sense of the last section, and let G_K be the absolute Galois group of K. In the following, a $\mathbb{Q}$-divisor of K means a formal product $\prod_{\mathfrak{q}} \mathfrak{q}^{n_{\mathfrak{q}}}$ of prime divisors $\mathfrak{q}$ of K (= prime ideals of the integer ring $\mathcal{O}_K$ of K if K is an algebraic number field) with exponents $n_{\mathfrak{q}} \in \mathbb{Q}$ of which all but a finite number are zero. We say that a $\mathbb{Q}$-divisor $\prod_{\mathfrak{q}} \mathfrak{q}^{n_{\mathfrak{q}}}$ of K is *effective* if all the exponents $n_{\mathfrak{q}}$ are nonnegative. For two $\mathbb{Q}$-divisors $M = \prod_{\mathfrak{q}} \mathfrak{q}^{m_{\mathfrak{q}}}$ and $N = \prod_{\mathfrak{q}} \mathfrak{q}^{n_{\mathfrak{q}}}$, we write $M|N$ if $m_{\mathfrak{q}} \leq n_{\mathfrak{q}}$ for all $\mathfrak{q}$.

Fix a prime number p. For a continuous representation $\rho : G_K \to \mathrm{GL}_{\overline{\mathbb{F}}_p}(V) \simeq \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, where V is an n-dimensional $\overline{\mathbb{F}}_p$-vector space, we define its *Artin conductor* $N(\rho)$ *outside* p, as a $\mathbb{Q}$-divisor of K, as follows (cf. [40, §1.2]): If K is an algebraic number field, then define it as the product

$$N(\rho) = \prod_{\mathfrak{q} \nmid p} \mathfrak{q}^{n_{\mathfrak{q}}(\rho)}$$

over the primes $\mathfrak{q}$ of K not dividing p with exponents

$$n_{\mathfrak{q}}(\rho) := \sum_{i=0}^{\infty} \frac{1}{(G_{\mathfrak{q},0} : G_{\mathfrak{q},i})} \dim_{\overline{\mathbb{F}}_p}(V/V^{G_{\mathfrak{q},i}}), \qquad (7.1)$$

where $G_{\mathfrak{q},i}$ is the ith ramification subgroup of $\mathrm{Im}(\rho)$ at (an extension of) $\mathfrak{q}$. If K is an algebraic function field, then we define

$$N(\rho) = \prod_{\mathfrak{q}} \mathfrak{q}^{n_{\mathfrak{q}}(\rho)},$$

where the product is over all the prime divisors $\mathfrak{q}$ of K, while the exponent $n_{\mathfrak{q}}(\rho)$ is defined by the same formula (7.1) as above. It is known that $n_{\mathfrak{q}}(\rho)$ is in fact an integer if $\mathrm{char}(K) \neq p$ (though it may not be so if $\mathrm{char}(K) = p$), and that $n_{\mathfrak{q}}(\rho) = 0$ if and only if ρ is unramified at $\mathfrak{q}$.

Let N be an effective $\mathbb{Q}$-divisor of the global field K. Khare and Moon proposed independently the following statement (as a conjecture in [18] and as a problem in [27]; the function field case was formulated in [29]):

Conjecture ($\mathbb{F}$). *For any K, n, p and N as above, there exist only finitely many isomorphism classes of semisimple continuous representations $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ with $N(\rho)|N$.*

Not much is known about this conjecture. For some results in special cases when K is a number field, see [27], [28], [29], [30]. The odd and two-dimensional case with $K = \mathbb{Q}$ was proved by Khare and Wintenberger ([19]) as a result of their proof of Serre's modularity conjecture. See also [4], which reduces the function field case of the conjecture to a conjecture of de Jong ([8, Conj. 2.3]).

Conjecture ($\mathbb{F}$) can be formulated in different ways. First, it is equivalent to the finiteness of Galois extensions L/K whose Galois group can be embedded into $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$ with conductor bounded by N. Precisely speaking, it is equivalent to the following:

Conjecture ($\mathbb{F}'$). *For any K, n, p and N as above, there exist only finitely many Galois extensions L/K such that there exists a faithful representation $\rho : \mathrm{Gal}(L/K) \hookrightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ with $N(\rho)|N$.*

Indeed, suppose that ($\mathbb{F}$) is true, and let L/K and $\rho : G \hookrightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be as in ($\mathbb{F}'$). Let ρ^{ss} be the semisimplification of ρ. By ($\mathbb{F}$), the possibility of the subextension L^{ss}/K of L/K corresponding to $\mathrm{Ker}(\rho^{\mathrm{ss}})$ is finite. The extension L/L^{ss} is obtained as a succession of elementary p-extensions of length at most $\min\{e \in \mathbb{Z} |\ 2^e \geq n\}$ (cf. [27, Sect. 3]) that are unramified outside pN. Then by class field theory, the possibility of the extensions L/L^{ss} is also finite. Conversely, ($\mathbb{F}'$) implies ($\mathbb{F}$) since there exist only finitely many isomorphism classes of semisimple representations of a finite group into $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$.

The next reformulation uses a certain quotient of the Galois group G_K. For an effective $\mathbb{Q}$-divisor $N = \prod_{\mathfrak{q}} \mathfrak{q}^{n_{\mathfrak{q}}}$ of K, we denote by $G_K(N)$ the quotient of G_K by the normal subgroup generated by $G_{K_{\mathfrak{q}}}^{n_{\mathfrak{q}}}$ (and its conjugates) for all[1] $\mathfrak{q}$, where $G_{K_{\mathfrak{q}}}^{u}$ is the uth ramification subgroup (in the upper numbering filtration) of the absolute Galois group $G_{K_{\mathfrak{q}}}$ of the completion $K_{\mathfrak{q}}$ of K at $\mathfrak{q}$. Here, $G_{K_{\mathfrak{q}}}$ is identified with a subgroup of G_K by choosing an embedding $\overline{K} \hookrightarrow \overline{K}_{\mathfrak{q}}$. For an n-dimensional representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, we have the following relation ([29, Lem. 3.2]):

$$(7.2) \qquad N(\rho)|N \quad \Longrightarrow \quad \rho \text{ factors through } G_K(N) \quad \Longrightarrow \quad N(\rho)|N^n N_0,$$

where $N_0 := \prod_{\mathfrak{q}|N} \mathfrak{q}$. Hence the conjectures ($\mathbb{F}$) and ($\mathbb{F}'$) are equivalent to either of the following two statements:

Conjecture ($\mathbb{F}^*$). *For any K, n, p and N, there exist only finitely many isomorphism classes of semisimple continuous representations $\rho : G_K(N) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$.*

[1] Note that $n_{\mathfrak{q}} = 0$ for almost all $\mathfrak{q}$ and that we omit those $\mathfrak{q}|p$ if K is an algebraic number field.

Conjecture ($\mathbb{F}^{*\prime}$). *For any K, n, p and N, there exist only finitely many Galois extensions L/K whose Galois group is a quotient of $G_K(N)$ and can be embedded into $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$.*

Now recall the moduli scheme $X_{\mathbb{F}_p[\![G_K(N)]\!],n,\mathbb{F}_p}$ constructed in Section 4; it is an f-profinite scheme over $\mathbb{F}_p$. By Proposition 6.4, we have another proposition:

Proposition 7.1. *Conjecture* ($\mathbb{F}$) *is equivalent to the following:*

Conjecture ($\mathbb{X}$). *For any K, n, p and N, the f-profinite scheme $X_{\mathbb{F}_p[\![G_K(N)]\!],n,\mathbb{F}_p}$ has only finitely many $\overline{\mathbb{F}}_p$-rational points.*

Note that Conjecture ($\mathbb{X}$) does not assert that $X_{\mathbb{F}_p[\![G_K(N)]\!],n,\mathbb{F}_p}$ itself is finite over $\mathbb{F}_p$. It may well be something like $\mathrm{Spec}(\mathbb{F}_p(\!(t)\!))$.

§8. Relation between some finiteness conjectures on Galois representations

In this section, we prove a certain relation between versions of the finiteness conjecture of Fontaine and Mazur ([11, Conj. 2a,2b,2c]). First, let us recall the conjectures. Let K be an algebraic number field of finite degree over $\mathbb{Q}$, and $G_K = \mathrm{Gal}(\overline{K}/K)$ its absolute Galois group. Let E be a subfield of $\overline{\mathbb{Q}}_p$. A continuous representation $\rho : G_K \to \mathrm{GL}_d(E)$ is said to be *geometric* if it is

(1) unramified outside the union of the set of infinite places and a finite set S of finite places of K; and
(2) potentially semistable at all places v dividing p.

The condition "potentially semistable" is equivalent to being de Rham by Berger ([3]). Note that ρ is automatically potentially semistable at $v \nmid p$ by Grothendieck (cf. [42, Appendix, Proposition]; recall also that, for $v \nmid p$, ρ is said to be semistable at v if the action of inertia at v is unipotent).

In what follows, we always assume that S contains all places of K lying above p.

Let $\rho : G_{K,S} \to \mathrm{GL}_E(V)$ be a geometric representation. The *inertial level* $\mathcal{L}(\rho)$ of ρ is a family $(\mathcal{L}_v(\rho))_v$ of $\mathcal{L}_v(\rho)$ for all finite places v of K, where $\mathcal{L}_v(\rho)$ is the maximal open normal subgroup of an inertia group I_v at v such that ρ becomes semistable when restricted there. One has $\mathcal{L}_v(\rho) = I_v$ if $v \notin S$. An *inertial level* $\mathcal{L} = (\mathcal{L}_v)_v$ for (K, S) is a family of open normal subgroups $\mathcal{L}_v$ of I_v for all finite places v of K such that $\mathcal{L}_v = I_v$ if $v \notin S$. We say that a geometric representation ρ has inertial level *bounded by* $\mathcal{L}$ if $\mathcal{L}(\rho) \supset \mathcal{L}$, i.e., $\mathcal{L}_v(\rho) \supset \mathcal{L}_v$ for each $v \in S$. The *E-Hodge–Tate type* $h(\rho)$ of ρ is a family $(h_v(\rho))_{v|p}$ of $h_v(\rho)$'s for all primes v

dividing p, where $h_v(\rho)$ is the isomorphism class of the graded $(E\otimes_{\mathbb{Q}_p} K_v)$-module $\mathrm{Hom}_{\mathbb{Q}_p[G_{K_v}]}(V, \oplus_{r\in\mathbb{Z}}\mathbb{C}_p(r))$. An *E-Hodge–Tate type* for K at p is a family $(h_v)_{v|p}$, for all primes v dividing p, of isomorphism classes of graded free $(E\otimes_{\mathbb{Q}_p} K_v)$-modules h_v, all having the same rank (called the *degree* of h). The *Hodge–Tate weights* of h are the integers r such that the degree-r component of h_v is nonzero for some $v|p$.

We denote by $\mathrm{Geom}(K,S,h;E)$ (resp. $\mathrm{Geom}(K,S,\mathcal{L},h;E)$) the set of isomorphism classes of geometric semisimple[2] E-representations of $G_{K,S}$ with E-Hodge–Tate type h (resp. with inertial level bounded by $\mathcal{L}$ and E-Hodge–Tate type h). Then the three versions of the finiteness conjecture of Fontaine and Mazur ([11, Conj. 2a, 2b, 2c]) are the following statements, in which K, S, p, $\mathcal{L}$, h are as above and, in FM(b) and FM(c), the field E is a *finite* extension of $\mathbb{Q}_p$:

Conjecture. *We have*

FM(a) : *the set* $\mathrm{Geom}(K,S,\mathcal{L},h;\overline{\mathbb{Q}}_p)$ *is finite for any* K, S, p, $\mathcal{L}$, h;

FM(b) : *the set* $\mathrm{Geom}(K,S,\mathcal{L},h;E)$ *is finite for any* K, S, p, $\mathcal{L}$, h, E;

FM(c) : *the set* $\mathrm{Geom}(K,S,h;E)$ *is finite for any* K, S, p, h, E.

Very little is known on these conjectures, especially on FM(a), except when the nonexistence is proved (as in [1], [9], [10]). FM(a) has been proved in the "potentially abelian" case ([43]).

We have the following logical relations:

$$\mathrm{FM(a)} \Longrightarrow \mathrm{FM(b)} \Longleftrightarrow \mathrm{FM(c)}.$$

Here, the only nontrivial one is (b) $\Rightarrow$ (c), and this is proved by showing that, if E is of finite degree over $\mathbb{Q}_p$, then the inertial level of a geometric E-representation is automatically bounded (see [11, Sect. 4(a)]).

Now we are interested in the relation between Conjectures FM(a) and FM(b); there seems to be a substantial difference between them. Our main result in this section is that, in some sense, Conjecture ($\mathbb{F}$) of the previous section fills in this gap:

Theorem 8.1. *We have* $\mathrm{FM(b)} + (\mathbb{F}) \Longrightarrow \mathrm{FM(a)}$.

Actually, we shall prove this in a slightly different form (Theorem 8.4). To state our results properly, it is convenient to employ the language of deformation theory of Galois representations ([25], [26]). Let k be a finite field of characteristic

[2] In [11], these are defined to be the sets of *irreducible* geometric representations. For some technical reasons, we work with semisimple ones. Of course, there is no essential difference.

p, and let $\rho_0 : G_{K,S} \to \mathrm{GL}_d(k)$ be a continuous representation. Let $W = W(k)$ be the ring of Witt vectors over k, and let $\mathcal{C}_W$ be the category of noetherian complete local W-algebras with residue field k. Let $\mathfrak{D}$ be "deformation data", i.e., a full subcategory of the category $\mathfrak{Rep}^{\mathrm{fin}}_W(G_{K,S})$ of continuous $W[G_{K,S}]$-modules of finite length that is closed under taking subobjects, quotients and direct sums. Suppose that our ρ_0 is in $\mathfrak{D}$. Then by Ramakrishna ([36]; see also [26, Sect. 25]), there exists a versal deformation ring $R_{\mathfrak{D}}(\rho_0)$ of ρ_0 of type $\mathfrak{D}$ in $\mathcal{C}_W$ and a versal deformation $\rho_{\mathfrak{D}} : G_{K,S} \to \mathrm{GL}_d(R_{\mathfrak{D}}(\rho_0))$ of ρ_0 of type $\mathfrak{D}$. Moreover, if $\mathrm{End}(\rho_0) \simeq k$, then $\rho_{\mathfrak{D}}$ is universal. The versality (resp. universality) refers to the property that there exists a canonical surjection (resp. bijection)

$$\mathrm{Hom}_{W\text{-alg}}(R_{\mathfrak{D}}(\rho_0), A) \longrightarrow \{\text{type-}\mathfrak{D} \text{ deformations of } \rho_0 \text{ to } A\},$$

functorially in $A \in \mathcal{C}_W$. Note that $R_{\mathfrak{D}}(\rho_0)$ and $\rho_{\mathfrak{D}}$ are unique up to isomorphism (which is canonical if $\mathrm{End}(\rho_0) \simeq k$ but may not be so in general; cf. [38, (2.8)]).

Fontaine and Mazur formulated a deformation-theoretic version of their finiteness conjecture. To state it, for an inertial level $\mathcal{L}$ for (K, S) and two integers $a \leq b$, let us take, as our deformation data $\mathfrak{D}$, the full subcategory $\mathfrak{Rep}^{\mathrm{fin}}_W(G_{K,S})_{\mathrm{st},\mathcal{L},[a,b]}$ of $\mathfrak{Rep}^{\mathrm{fin}}_W(G_{K,S})$ consisting of objects T such that, for each $v \in S$,

(1) if $v \nmid p$, then $\mathcal{L}_v$ acts unipotently on T; and
(2) if $v \mid p$, there exists a semistable p-adic representation V of $\mathcal{L}_v$ with Hodge–Tate weights in $[a, b]$ (i.e., $(\mathbb{C}_v(-r) \otimes_{\mathbb{Q}_p} V)^{\mathcal{L}_v} = 0$ if $r \notin [a, b]$) such that T is isomorphic to a subquotient of V as a $\mathbb{Z}_p[\mathcal{L}_v]$-module.

In the rest of this section, $\mathfrak{D}$ is fixed to be this category $\mathfrak{Rep}^{\mathrm{fin}}_W(G_{K,S})_{\mathrm{st},\mathcal{L},[a,b]}$.

Now we have a deformation-theoretic version of the finiteness conjecture of Fontaine and Mazur ([11, Conj. 5]):

Conjecture FM(d)**.** *For any K, S, $\mathcal{L}$, $a \leq b$ and k, if $\rho_0 \in \mathfrak{D}$ is such that* $\mathrm{End}(\rho_0) \simeq k$*, then the universal deformation ring $R_{\mathfrak{D}}(\rho_0)$ is finite as a W-algebra.*

Since we have to deal with all residual representations ρ_0, which may not necessarily satisfy the condition $\mathrm{End}(\rho_0) \simeq k$, we shall formulate a conjecture using the moduli scheme as in Section 4 in addition to Mazur's deformation theory. Before doing so, we introduce a slightly wider (a priori) class of representations than geometric ones. For the moment, we relax the W-algebra structure to work in the category ${}^{\mathrm{f}}\mathfrak{C}_{\mathbb{Z}_p}$ of commutative f-profinite $\mathbb{Z}_p$-algebras (note that this is different from $\mathcal{C}_{\mathbb{Z}_p}$, which is the category of commutative complete noetherian local $\mathbb{Z}_p$-algebras) and with the deformation data $\mathfrak{D} = \mathfrak{Rep}^{\mathrm{fin}}_{\mathbb{Z}_p}(G_{K,S})_{\mathrm{st},\mathcal{L},[a,b]}$. In what follows, all profinite $\mathbb{Z}_p$-algebras R_{o} and their localizations R are regarded as the objects (R_o, R_o) and (R_o, R), respectively, of the category ${}^{\mathrm{f}}\mathfrak{A}_{\mathbb{Z}_p}$ of f-profinite $\mathbb{Z}_p$-algebras (or ${}^{\mathrm{f}}\mathfrak{C}_{\mathbb{Z}_p}$ of commutative f-profinite $\mathbb{Z}_p$-algebras). A finite extension F of $\mathbb{Q}_p$ is identified with the object $(\mathcal{O}_F, F, \mathcal{O}_F \hookrightarrow F)$ of ${}^{\mathrm{f}}\mathfrak{C}_{\mathbb{Z}_p}$, where $\mathcal{O}_F$ denotes the ring of integers of F.

Definition 8.2. Let $E = (E_o, E) \in {}^{\mathrm{f}}\mathfrak{C}_{\mathbb{Z}_p}$ and let V be a free E-module of finite rank. We say that a continuous representation $\rho : G_{K,S} \to \mathrm{GL}_E(V)$ is *piecewise geometric* with inertial level bounded by $\mathcal{L}$ and Hodge–Tate weights in $[a, b]$ (or, of *type* $\mathfrak{D}$, for short) if V admits a $G_{K,S}$-stable E_o-lattice T of which all quotients of finite length lie in $\mathfrak{D}$.

Here, an E_o-lattice of V means a free E_o-submodule T of V such that the natural E-module homomorphism $E \otimes_{E_o} T \to V$ is an isomorphism.

A typical example of such a representation arises from the type-$\mathfrak{D}$ versal deformation of some residual representation ρ_0 by way of a continuous ring homomorphism $\phi : R_{\mathfrak{D}}(\rho_0) \to E$.

For K, S, $\mathcal{L}$, $[a, b]$ and $E \in \mathfrak{C}_{\mathbb{Z}_p}$, let $\mathrm{Geom}'_n(K, S, \mathcal{L}, [a, b]; E)$ denote the set of isomorphism classes of n-dimensional semisimple piecewise geometric E-representations with inertial level bounded by $\mathcal{L}$ and Hodge–Tate weights in $[a, b]$. An algebraic extension E of $\mathbb{Q}_p$ is not f-profinite if it is of infinite degree over $\mathbb{Q}_p$. For such an E, we define the set $\mathrm{Geom}'_n(K, S, \mathcal{L}, [a, b]; E)$ to be the inductive limit of the sets $\mathrm{Geom}(K, S, \mathcal{L}, h; E')$ where E' moves through the set of finite extensions of $\mathbb{Q}_p$ contained in E. A geometric representation is trivially piecewise geometric, so that

$$\mathrm{Geom}(K, S, \mathcal{L}, h; E) \subset \mathrm{Geom}'_n(K, S, \mathcal{L}, [a, b]; E)$$

if n is the degree of the Hodge–Tate type h and $[a, b]$ contains all the Hodge–Tate weights of h. Fontaine and Mazur point out ([11, Rem. (a) after Conj. 5]) that it should not be very hard to prove the converse. Indeed, as a consequence of a theorem of Liu ([23, Thm. 1.0.2]), we have a proposition:

Proposition 8.3. *A piecewise geometric representation with bounded inertial level and Hodge–Tate weights is geometric.*

See also [20] for some related results.

On the contrary, a piecewise geometric representation with *unbounded* inertial level or Hodge–Tate weights may not be geometric; namely, an E-representation may not be geometric no matter how its subquotients of finite length come from geometric representations, if their inertial levels are unbounded or their Hodge–Tate weights do not stay in a fixed interval $[a, b]$.

By Proposition 8.3, the set $\mathrm{Geom}'_n(K,S,\mathcal{L},[a,b];E)$ is covered by the union of $\mathrm{Geom}(K,S,\mathcal{L},h;E)$'s for a finite number of Hodge–Tate types h. Thus Conjectures FM(a) and FM(b) are equivalent to the following:

Conjecture. *We have*

FM′(a) : *the set* $\mathrm{Geom}'_n(K,S,\mathcal{L},[a,b];\overline{\mathbb{Q}}_p)$ *is finite for any* $n,K,S,\mathcal{L},a\le b$;

FM′(b) : *the set* $\mathrm{Geom}'_n(K,S,\mathcal{L},[a,b];E)$ *is finite for any* $n,K,S,\mathcal{L},a\le b$ *and* E.

Here again, and also in Conjectures $\mathrm{FM(b)^{ai}}$ and $\mathrm{FM'(b)^{ai}}$ below, the field E is assumed to be *finite* over $\mathbb{Q}_p$.

We have the following logical relations:

$$\begin{array}{ccc} \mathrm{FM(a)} & \Leftrightarrow & \mathrm{FM'(a)} \\ \Downarrow & & \Downarrow \\ \mathrm{FM(b)} & \Leftrightarrow & \mathrm{FM'(b)} \\ \Updownarrow & & \\ \mathrm{FM(c)}. & & \end{array}$$

Instead of Theorem 8.1, we shall actually prove the following result:

Theorem 8.4. *We have* $\mathrm{FM'(b)}+(\mathbb{F})\Longrightarrow \mathrm{FM'(a)}$.

To prove this, we first note that there is an integral ideal N of K such that any residual representation ρ_0 that arises from ρ in $\mathrm{Geom}'_n(K,S,\mathcal{L},[a,b];\overline{\mathbb{Q}}_p)$ has conductor $N(\rho_0)$ bounded by N. Indeed, since $\mathcal{L}_v(\rho)$ contains the n_vth ramification subgroup $I_v^{n_v}$ (in the upper numbering) for some $n_v>0$, and ρ_0 restricted to $\mathcal{L}_v(\rho)$ for $v\nmid p$ is of the form

$$\rho_0|_{\mathcal{L}_v(\rho)}\sim\begin{pmatrix}1 & * & * \\ & \ddots & * \\ & & 1\end{pmatrix}$$

and is at most tamely ramified, it follows that ρ_0 factors through $G_K(N)$ for $N=\prod\mathfrak{q}_v^{n_v}$ in the notation of Section 7, where $\mathfrak{q}_v$ is the prime of K corresponding to v. By (7.2), the Artin conductor $N(\rho_0)$ is bounded uniformly in terms of $(\mathcal{L}_v(\rho))_{v\nmid p}$.

Now if we assume Conjecture $(\mathbb{F})$ (or equivalently, $(\mathbb{F}')$), there exists a finite extension field K' of K that neutralizes all such representations ρ_0 (i.e., $\rho_0|_{G_{K'}}=1$). Here we have the following lemma:

Lemma 8.5. *Let K'/K be a finite extension. If* $\rho\in\mathrm{Geom}'_n(K,S,\mathcal{L},[a,b];\overline{\mathbb{Q}}_p)$, *then its restriction* $\rho|_{G_{K'}}$ *to* K' *is in* $\mathrm{Geom}'_n(K',S',\mathcal{L}',[a,b];\overline{\mathbb{Q}}_p)$, *where* S' *and* $\mathcal{L}'$ *are appropriate data for* K'. *The restriction map* $\mathrm{Geom}'_n(K,S,\mathcal{L},[a,b];\overline{\mathbb{Q}}_p)\to\mathrm{Geom}'_n(K',S',\mathcal{L}',[a,b];\overline{\mathbb{Q}}_p)$ *has finite fibers.*

Proof. First, in view of the definition of piecewise geometricity, we have to check that, if a $\mathbb{Z}_p[G_{K,S}]$-module is in $\mathfrak{D} = \mathfrak{Rep}^{\mathrm{fin}}_{\mathbb{Z}_p}(G_{K,S})_{\mathrm{st},\mathcal{L},[a,b]}$, then its restriction to $G_{K',S'}$ is in $\mathfrak{D}' = \mathfrak{Rep}^{\mathrm{fin}}_{\mathbb{Z}_p}(G_{K',S'})_{\mathrm{st},\mathcal{L}',[a,b]}$ for a suitable choice of S' and $\mathcal{L}'$. This is clear. Next, to show the semisimplicity of the restriction $\rho|_{G_{K'}}$, let us first assume that K'/K is Galois. Then since $G_{K'}$ is normal in G_K, the semisimplicity follows from Clifford's theorem ([31, Chap. III, §3, Thm. 3.1]). The case of a general finite extension K' can be reduced to the Galois case by considering its Galois closure and using the next lemma. Last, the finiteness of the fibers follows from Lemma 8.7 below. □

Lemma 8.6. *Let G be a group and H a normal subgroup of G of finite index. Suppose $\rho : G \to \mathrm{GL}_n(E)$ is a linear representation of G over a field E of characteristic zero. If its restriction $\rho|_H$ to H is semisimple, then ρ itself is semisimple.*

Proof. This can be proved by the same arguments as the proof of the semisimplicity of a representation of a finite group over a field of characteristic zero. Let V be the representation space of ρ, and W a G-stable subspace of V. By assumption, we have a direct-sum decomposition $V = W \oplus W'$ as an $E[H]$-module. Let $\pi : V \to W$ be the $E[H]$-linear projection. Then one checks that the map $\tilde{\pi} : V \to W$ defined by

$$\tilde{\pi}(v) := \frac{1}{(G:H)} \sum_g (g\pi g^{-1})(v)$$

is an $E[G]$-linear projection, where the sum is over a complete system of representatives for G/H. Thus we obtain a decomposition $V = W \oplus \mathrm{Ker}(\tilde{\pi})$ as an $E[G]$-module. □

For any group G and a field E, let $\underline{\mathrm{Rep}}^{\mathrm{ss}}_G(E)$ denote the set of isomorphism classes of finite-dimensional semisimple E-linear representations of G.

Lemma 8.7. *Let G be a group, H a subgroup of G of finite index and E a field. Then the restriction map*

$$\underline{\mathrm{Rep}}^{\mathrm{ss}}_G(E) \to \underline{\mathrm{Rep}}^{\mathrm{ss}}_H(E)$$

has finite fibers.

Remark 8.8. This is false without the assumption of semisimplicity if E has characteristic > 0 (cf. [27, Rem. after Lem. 3.2]).

Proof. Let W be a semisimple $E[H]$-module of finite dimension over E. Consider semisimple $E[G]$-modules V whose restrictions to $E[H]$ are isomorphic to W. It is enough to show that there are only finitely many possibilities of isomorphism classes of $E[G]$-modules appearing as simple factors of such V's. Let V_1 be such a simple factor and W_1 its restriction to $E[H]$. Note that W_1 falls in one of the finitely many isomorphism classes of the $E[H]$-submodules of the semisimple $E[H]$-module W. By Frobenius' reciprocity law (e.g., [31, Chap. III, Thm. 1.19(ii)]), we have

$$\mathrm{Hom}_{E[H]}(W_1, W_1) \simeq \mathrm{Hom}_{E[G]}(V_1, \mathrm{Ind}_H^G(W_1)),$$

where $\mathrm{Ind}_H^G(W_1) := E[G] \otimes_{E[H]} W_1$ is the representation of G induced from W_1. Since the left-hand side is nonempty, so is the right-hand side, and hence V_1 is identified with a simple factor of $\mathrm{Ind}_H^G(W_1)$. Since $\mathrm{Ind}_H^G(W_1)$ has finite length as an $E[G]$-module, it has only finitely many composition factors by the Jordan–Hölder theorem. Hence the possibility for V_1 is finite. □

By Lemma 8.5, to prove Theorem 8.1, it is enough to prove that, under the assumption of FM′(b), there exist only finitely many elements of $\mathrm{Geom}'_n(K', S', \mathcal{L}', [a,b]; \overline{\mathbb{Q}}_p)$ that contain $\mathcal{O}_{\overline{\mathbb{Q}}_p}$-lattices with trivial residual representations. More generally, we shall prove the following Proposition 8.9. Let k be a finite field, and W the ring of Witt vectors over k. For a fixed residual representation $\rho_0 : G_K \to \mathrm{GL}_n(k)$ in $\mathfrak{D} = \mathfrak{Rep}^{\mathrm{fin}}_W(G_{K,S})_{\mathrm{st},\mathcal{L},[a,b]}$, let $\mathrm{Geom}'_n(K, S, \mathcal{L}, [a,b]; \overline{\mathbb{Q}}_p)(\rho_0)$ denote the set of elements of $\mathrm{Geom}'_n(K, S, \mathcal{L}, [a,b]; \overline{\mathbb{Q}}_p)$ that admits an $\mathcal{O}_{\overline{\mathbb{Q}}_p}$-lattice whose reduction is equivalent to $\rho_0 \otimes_k \overline{k}$. Note that a p-adic representation ρ may belong to $\mathrm{Geom}'_n(K, S, \mathcal{L}, [a,b]; \overline{\mathbb{Q}}_p)(\rho_i)$ for unisomorphic residual representations ρ_i (though their semisimplifications ρ_i^{ss} must be isomorphic (cf. [41, §15.2, Thm. 32], at least for the case of representations of finite groups; the same proof works for profinite groups as well)). Conjecture FM′(a) is equivalent to the combination of ($\mathbb{F}$) and the following:

Conjecture FM′(a)$^{\mathrm{fiber}}$**.** *The set* $\mathrm{Geom}'_n(K, S, \mathcal{L}, [a,b]; \overline{\mathbb{Q}}_p)(\rho_0)$ *is finite for any* $n, K, S, \mathcal{L}, a \leq b, k$ *and* ρ_0.

It remains to prove the proposition:

Proposition 8.9. *We have* FM′(b) $\Longleftrightarrow$ FM′(a)$^{\mathrm{fiber}}$.

Note that Conjecture FM′(b) is equivalent to its fiberwise version (whence the $\Leftarrow$ in the proposition) since, by Hermite–Minkowski, there exist only finitely many continuous representations $\rho_0 : G_{K,S} \to \mathrm{GL}_n(k)$ for a fixed finite field k.

The proof of $\Rightarrow$ in the proposition takes the rest of this section. To prove it, we shall first translate FM′(b) into an "absolutely irreducible version" (FM′(b)$^{\mathrm{ai}}$ below; see Proposition 8.10), and then into a "deformation-theoretic version" (FM′(d)$_{\mathbb{Q}_p}$ below; see Proposition 8.18). Once this has been done, then it will be easy to prove that $\mathrm{FM}'(\mathrm{d})_{\mathbb{Q}_p} \Rightarrow \mathrm{FM}'(\mathrm{a})^{\mathrm{fiber}}$ (see Proposition 8.17). Our plan is summarized as

$$(8.1)\qquad \begin{array}{ccc} \mathrm{FM}'(\mathrm{b}) & \overset{8.9}{\Longleftrightarrow} & \mathrm{FM}'(\mathrm{a})^{\mathrm{fiber}} \\ {\scriptstyle 8.10}\Big\Updownarrow & & \Big\Uparrow{\scriptstyle 8.17} \\ \mathrm{FM}'(\mathrm{b})^{\mathrm{ai}} & \underset{8.18}{\Longleftrightarrow} & \mathrm{FM}'(\mathrm{d})_{\mathbb{Q}_p}. \end{array}$$

Let $\mathrm{Geom}^{\mathrm{ai}}(K, S, \mathcal{L}, h; E)$ (resp. $\mathrm{Geom}'^{,\mathrm{ai}}_n(K, S, \mathcal{L}, [a, b]; E)$) denote the subset of $\mathrm{Geom}(K, S, \mathcal{L}, h; E)$ (resp. $\mathrm{Geom}'_n(K, S, \mathcal{L}, [a, b]; E)$) consisting of the isomorphism classes of absolutely irreducible representations. FM(b) and FM$'$(b) imply:

Conjecture. *We have*

$\mathrm{FM}(\mathrm{b})^{\mathrm{ai}}$: *the set* $\mathrm{Geom}^{\mathrm{ai}}(K, S, \mathcal{L}, h; E)$ *is finite for any* K, S, $\mathcal{L}$, h, $a \le b$ *and* E.

$\mathrm{FM}'(\mathrm{b})^{\mathrm{ai}}$: *the set* $\mathrm{Geom}'^{,\mathrm{ai}}_n(K, S, \mathcal{L}, [a, b]; E)$ *is finite for any* K, S, $\mathcal{L}$, n, $a \le b$ *and* E.

The converse is also true:

Proposition 8.10. *We have*

(1) $\mathrm{FM}(\mathrm{b}) \Longleftrightarrow \mathrm{FM}(\mathrm{b})^{\mathrm{ai}}$*;*

(2) $\mathrm{FM}'(\mathrm{b}) \Longleftrightarrow \mathrm{FM}'(\mathrm{b})^{\mathrm{ai}}$.

Proof. We prove only the latter equivalence, since the proof is the same for the other one. The implication $\Rightarrow$ is trivial. Suppose $\mathrm{Geom}'_n(K, S, \mathcal{L}, [a, b]; E)$ is infinite for some K, S, n, $\mathcal{L}$, $a \le b$ and E. Then the next lemma implies that there exists a finite extension E'/E such that $\mathrm{Geom}'^{,\mathrm{ai}}_m(K, S, \mathcal{L}, [a, b]; E')$ is also infinite for some $m \le n$. □

Lemma 8.11. *Let G be a group, E a finite extension of $\mathbb{Q}_p$ and n a positive integer. Then there exists a finite extension E'/E such that, for any semisimple representation V of G of dimension n, each irreducible factor of $E' \otimes_E V$ is absolutely irreducible.*

Proof. Let E' be the compositum (in $\overline{\mathbb{Q}}_p$) of all finite extensions of E of degree dividing n. Since there are only finitely many extensions of E of a given degree (e.g., by [39]), the extension E'/E is finite. We shall show that this extension has the required property. Let V be as in the lemma. We may assume it is irreducible. Then $E_V := \mathrm{End}_{E[G]}(V)$ is a finite extension field of E with $[E_V : E]$ dividing n.

By the definition of E', there are $[E_V : E]$ different E-embeddings of E_V into E'. Then we have

$$\mathrm{End}_{E'[G]}(E' \otimes_E V) \simeq E' \otimes_E E_V \simeq (E')^{[E_V:E]},$$

showing that each irreducible factor of $E' \otimes_E V$ has endomorphism ring isomorphic to E'. Note that semisimplicity is stable under extension of scalars by separable extensions ([5, Chap. 8, Sect. 13, n°. 4, Prop. 4]). By (the nontopological version of) Proposition 6.3, such a representation is absolutely irreducible. □

Now we return to the deformation data $\mathfrak{D} = \mathfrak{Rep}^{\mathrm{fin}}_W(G_{K,S})_{\mathrm{st},\mathcal{L},[a,b]}$, where W is the ring of Witt vectors over the finite field k (though parts of the following discussions are valid for more general deformation data). Fix a residual representation $\rho_0 : G_{K,S} \to \mathrm{GL}_n(k)$ in $\mathfrak{D}$. Let $R_{\mathfrak{D}}(\rho_0)$ be the type-$\mathfrak{D}$ versal deformation ring of ρ_0, and $\rho_{\mathfrak{D}} : G_{K,S} \to \mathrm{GL}_n(R_{\mathfrak{D}}(\rho_0))$ the versal deformation. Note that, by [23], both $R_{\mathfrak{D}}(\rho_0)$ and $\rho_{\mathfrak{D}}$ exist. Let $\mathcal{R}_{\mathfrak{D}}(\rho_0)$ be the closed $R_{\mathfrak{D}}(\rho_0)$-subalgebra of $\mathrm{M}_n(R_{\mathfrak{D}}(\rho_0))$ generated over $R_{\mathfrak{D}}(\rho_0)$ by the image of $\rho_{\mathfrak{D}}$. For simplicity of notation, we write $R = R_{\mathfrak{D}}(\rho_0)$ and $\mathcal{R} = \mathcal{R}_{\mathfrak{D}}(\rho_0)$ in what follows.[3] The representation theories of $\mathcal{R}$ and $\rho_{\mathfrak{D}}(G_{K,S})$ are closely related. To see the relation precisely, we consider the following objects: Let $T = T_{\mathfrak{D}}(\rho_0)$ be the closed W-subalgebra of R generated by $\mathrm{Tr}_{\mathrm{M}_n(R)/R}(\rho_{\mathfrak{D}}(G_{K,S}))$, and $\mathcal{T} = \mathcal{T}_{\mathfrak{D}}(\rho_0)$ the closed T-subalgebra of $\mathcal{R}$ generated by $\rho_{\mathfrak{D}}(G_{K,S})$. Being subalgebras of the matrix algebra, $\mathcal{R}$ and $\mathcal{T}$ have natural structures of τ-algebras in the sense of Section 5. Let ${}^{\mathrm{f}}\mathfrak{A}_W$ be the category of f-profinite W-algebras and ${}^{\mathrm{f}}\mathfrak{C}_W$ its subcategory consisting of commutative objects. Also, let ${}^{\mathrm{f}}\mathfrak{A}^{\tau}_W$ be the category of f-profinite W-τ-algebras. For each $E = (E_{\mathrm{o}}, E) \in {}^{\mathrm{f}}\mathfrak{C}_W$, define three sets $\mathrm{Geom}'_n(K, S, \mathcal{L}, [a, b]; E)(\rho_0)$, $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{R},\mathrm{M}_n}(E)$ and $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{T},\mathrm{M}_n}(E)$ as follows:

- $\mathrm{Geom}'^{,\mathrm{ai}}_n(K, S, \mathcal{L}, [a, b]; E)(\rho_0)$ denotes the set of isomorphism classes of absolutely irreducible representations $\rho : G_{K,S} \to \mathrm{GL}_n(E)$ that descends to a piecewise geometric representation of type $\mathfrak{D}$ defined over a closed noetherian local W-subalgebra E_1 of E that is a lifting of ρ_0;
- $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{R},\mathrm{M}_n}(E)$ (resp. $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{T},\mathrm{M}_n}(E)$) denotes the set of isomorphism classes of absolutely irreducible representations $\rho : \mathcal{R} \to \mathrm{M}_n(E)$ (resp. $\rho : \mathcal{T} \to \mathrm{M}_n(E)$), considered in ${}^{\mathrm{f}}\mathfrak{A}^{\tau}_W$.

[3] Caution: In Sections 2–5, the noncommutative ring to be represented was denoted by R. But to respect the common notation R for the (uni)versal deformation ring, we use $\mathcal{R}$ for the noncommutative ring to be represented and R for the deformation ring, which is the "coefficient ring" of $\mathcal{R}$ in the present context. So the R/E in Section 5 corresponds to $\mathcal{R}/R$ here.

In the rest of this section, let F be the fraction field of W. The above definitions extend naturally to the case where E is an infinite algebraic extension of F by taking the inductive limits with respect to finite extensions E'/F contained in E. Note that

$$\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{R},\mathrm{M}_n}(E) \subset \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{R},n}(E) \quad \text{and} \quad \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{T},\mathrm{M}_n}(E) \subset \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{T},n}(E).$$

We have the restriction map

$$\gamma : \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{R},\mathrm{M}_n}(E) \to \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{T},\mathrm{M}_n}(E).$$

In the rest of this section (except some places where otherwise stated), E will denote a subfield of $\overline{\mathbb{Q}}_p$ containing F. We shall define two maps

$$\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{R},\mathrm{M}_n}(E) \xrightarrow{\ \beta\ } \mathrm{Geom}'^{,\mathrm{ai}}_n(K,S,\mathcal{L},[a,b];E)(\rho_0) \xrightarrow{\ \alpha\ } \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{T},\mathrm{M}_n}(E)$$

in such a way that $\alpha \circ \beta = \gamma$.

Definition of α: Suppose $\rho : G_{K,S} \to \mathrm{GL}_n(E)$ is in $\mathrm{Geom}'^{,\mathrm{ai}}_n(K,S,\mathcal{L},[a,b];E)(\rho_0)$. Then it is induced by some $\phi : R \to E_1 \subset E$ by the versality of $\rho_{\mathfrak{D}}$, which induces also the representation $\mathrm{M}_n(\phi)|_{\mathcal{T}} : \mathcal{T} \to \mathrm{M}_n(E)$ in $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{T},\mathrm{M}_n}(E)$. Although the map ϕ may not be unique (since $\rho_{\mathfrak{D}}$ may not be *uni*versal), the restriction $\phi|_T : T \to E$ is uniquely determined by ρ, because T is topologically generated over W by $\mathrm{Tr}_{\mathrm{M}_n(R)/R}(\rho_{\mathfrak{D}}(g))$ for all $g \in G_{K,S}$, and each $\mathrm{Tr}_{\mathrm{M}_n(R)/R}(\rho_{\mathfrak{D}}(g))$ has to be mapped to $\mathrm{Tr}_{\mathrm{M}_n(E)/E}(\rho(g))$. It follows then that the isomorphism class of the representation $\mathrm{M}_n(\phi)|_{\mathcal{T}}$ is also uniquely determined by ρ, because $\mathcal{T}$ is topologically generated over T by $\rho_{\mathfrak{D}}(G_{K,S})$ and $\mathrm{GL}_n(\phi) \circ \rho_{\mathfrak{D}}$ is conjugate to ρ by an element of $\mathrm{GL}_n(E_1)$. Thus we obtain a map of sets

$$\alpha : \mathrm{Geom}'^{,\mathrm{ai}}_n(K,S,\mathcal{L},[a,b];E)(\rho_0) \to \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{T},\mathrm{M}_n}(E),$$
$$\rho \mapsto \mathrm{M}_n(\phi)|_{\mathcal{T}}.$$

This map α is injective. Indeed, suppose ρ_1 and ρ_2 give rise to isomorphic representations of $\mathcal{T}$. Then $\rho_i \sim \mathrm{GL}_n(\phi_i) \circ \rho_{\mathfrak{D}}$ for some $\phi_i : R \to E$ $(i = 1, 2)$ and $\mathrm{M}_n(\phi_1)|_{\mathcal{T}} \sim \mathrm{M}_n(\phi_2)|_{\mathcal{T}}$ as representations of $\mathcal{T}$, where $\sim$ means "conjugate by an element of $\mathrm{GL}_n(E)$", whence $\rho_1 \sim \rho_2$ as representations of $G_{K,S}$.

Remark 8.12. We cannot define a map $\mathrm{Geom}'^{,\mathrm{ai}}_n(K, S, \mathcal{L}, [a,b]; E)(\rho_0) \to \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{R},\mathrm{M}_n}(E)$ in the same way as above because the map ϕ may not be unique.

Definition of β: Suppose $\rho : \mathcal{R} \to \mathrm{M}_n(E)$ is a representation in $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{R},\mathrm{M}_n}(E)$. By restriction to $\rho_{\mathfrak{D}}(G_{K,S}) \subset \mathcal{R}^{\times}$, we obtain a representation $\rho' : G_{K,S} \to \mathrm{GL}_n(E)$, whose class is in fact in $\mathrm{Geom}_n^{\prime,\mathrm{ai}}(K, S, \mathcal{L}, [a, b]; E)(\rho_0)$. Indeed, ρ induces a morphism $\phi : R \to E$ since R is contained in the center of $\mathcal{R}$, and, by Proposition 5.7 (cf. also Proposition 5.9), ρ is isomorphic to $\mathrm{M}_n(\phi)|_{\mathcal{R}}$ (this is the point where we need the τ-structure), and hence ρ' is isomorphic to

$$\phi_* \rho_{\mathfrak{D}} = \mathrm{GL}_n(\phi) \circ \rho_{\mathfrak{D}} : G_{K,S} \to \mathrm{GL}_n(R) \to \mathrm{GL}_n(E_1) \hookrightarrow \mathrm{GL}_n(E),$$

where we set $E_1 := \mathrm{Im}(\phi : R \to E)$ (note that the first factor $E_{1,\mathrm{o}}$ of E_1 is in $\mathcal{C}_W$). Since $\rho_{\mathfrak{D}}$ is versal of type $\mathfrak{D}$, the representation $\phi_* \rho_{\mathfrak{D}}$ is in $\mathrm{Geom}_n^{\prime,\mathrm{ai}}(K, S, \mathcal{L}, [a, b]; E)(\rho_0)$. Thus we obtain a map of sets

$$\begin{aligned} \beta : \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{R},\mathrm{M}_n}(E) &\to \mathrm{Geom}_n^{\prime,\mathrm{ai}}(K, S, \mathcal{L}, [a, b]; E)(\rho_0), \\ \rho &\mapsto \rho'. \end{aligned}$$

This map β is surjective, because each ρ in $\mathrm{Geom}_n^{\prime,\mathrm{ai}}(K, S, \mathcal{L}, [a, b]; E)(\rho_0)$ arises from some $\phi : R \to E_1$ by the versality of $\rho_{\mathfrak{D}}$, and then the class of $\mathrm{M}_n(\phi)|_{\mathcal{R}} : \mathcal{R} \to \mathrm{M}_n(E_1) \hookrightarrow \mathrm{M}_n(E)$ in $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{R},\mathrm{M}_n}(E)$ is mapped by β to the class of ρ.

By construction, the composition $\alpha \circ \beta$ is the restriction map γ.

Thus we have proved the following result:

Proposition 8.13. *For each algebraic extension E of F, we have the following commutative diagram:*

$$\begin{array}{ccc} \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{R},\mathrm{M}_n}(E) & & \\ & \overset{\beta}{\searrow} & \\ \gamma \downarrow & & \mathrm{Geom}_n^{\prime,\mathrm{ai}}(K, S, \mathcal{L}, [a, b]; E)(\rho_0) \\ & \underset{\alpha}{\swarrow} & \\ \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{T},\mathrm{M}_n}(E), & & \end{array}$$

in which α is injective and β is surjective.

Set

$$\begin{aligned} \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathfrak{D},\rho_0,\mathrm{M}_n}(E) &:= \mathrm{Im}(\gamma : \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{R},\mathrm{M}_n}(E) \to \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{T},\mathrm{M}_n}(E)), \\ \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathfrak{D},\rho_0,n}(E) &:= \mathrm{Im}(\gamma' : \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{R},n}(E) \to \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{T},n}(E)), \end{aligned}$$

where γ' is also the restriction map (these are merely sets, not schemes). The latter set contains the former. By the above proposition, there is a bijection

$$\alpha : \mathrm{Geom}_n^{\prime,\mathrm{ai}}(K, S, \mathcal{L}, [a, b]; E)(\rho_0) \xrightarrow{\sim} \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathfrak{D},\rho_0,\mathrm{M}_n}(E). \tag{8.2}$$

The next proposition allows us to confine our attention to representations into matrix algebras.

Proposition 8.14. *The following statements are equivalent:*

(1) The set $\mathrm{Geom}'^{,\mathrm{ai}}_n(K, S, \mathcal{L}, [a, b]; E)(\rho_0)$ *is finite for any finite extension* E *of* F*.*
(2) The set $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathfrak{D},\rho_0,\mathrm{M}_n}(E)$ *is finite for any finite extension* E *of* F*.*
(3) The set $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathfrak{D},\rho_0,n}(E)$ *is finite for any finite extension* E *of* F*.*

Proof. The equivalence of (1) and (2) is trivial by (8.2). The implication (3) ⇒ (2) is also trivial. Suppose $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathfrak{D},\rho_0,n}(E)$ is infinite for some E. Recall that all Azumaya algebras of degree n over the field E split by any extension E'/E of degree n (e.g., combine [46, I-4, Prop. 5, IX-1, Th. 1 and IX-1, Prop. 3, Cor. 6]). The scalar extension $E \to E'$ then gives rise to a map

$$\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathfrak{D},\rho_0,n}(E) \to \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathfrak{D},\rho_0,\mathrm{M}_n}(E'),$$

which is injective since absolutely irreducible representations are determined by traces (Corollary 4.22). Thus (2) implies (3). □

Let $X^{\tau}_{\mathcal{R},n,W}$ be the moduli scheme of degree-n absolutely irreducible τ-representations of the ring $\mathcal{R} = \mathcal{R}_{\mathfrak{D}}(\rho_0)$ in ${}^{\mathrm{f}}\mathfrak{A}^{\tau}_W$ (cf. Theorem 5.8), so that we have $X^{\tau}_{\mathcal{R},n,W}(E) = \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{R},n}(E)$ for $E \in {}^{\mathrm{f}}\mathfrak{C}_W$. It is of finite type since $\mathcal{R}$ is finitely generated as an object of ${}^{\mathrm{f}}\mathfrak{A}_W$. Recall that it was constructed as an open subscheme of $\mathrm{Spec}(R^{\mathrm{tr}})$, where R^{tr} is the closed W-subalgebra of $R = R_{\mathfrak{D}}(\rho_0)$ generated by $\mathrm{Tr}_{\mathrm{M}_n(R)/R}(\mathcal{R})$ (note that, by Proposition 5.9, the inclusion map $\mathcal{R} \hookrightarrow \mathrm{M}_n(R)$ is a versal morphism as in $(\mathrm{V}^{\mathrm{ai},\tau}_{n,F})$ of Section 5). We have $R^{\mathrm{tr}}[1/n] = R[1/n]$ since $\mathcal{R}$ contains the scalar matrices $R1_n$. If ρ_0 is absolutely irreducible, then the universal deformation ring R serves well enough for our purpose:

Proposition 8.15. *If* ρ_0 *is absolutely irreducible, then we have* $\mathcal{R} = \mathrm{M}_n(R)$*, and the identity map* $\Phi : \mathcal{R} \to \mathrm{M}_n(R)$ *is the universal morphism. Hence we have* $R^{\mathrm{tr}} = R$*, and* $X^{\tau}_{\mathcal{R},n,W} = \mathrm{Spec}(R)$*. In particular, these are all profinite.*

Proof. If ρ_0 is absolutely irreducible, then the reduction of $\mathcal{R}$ modulo the maximal ideal of R is equal to $\mathrm{M}_n(k)$. By Nakayama's lemma, $\mathcal{R}$ coincides with $\mathrm{M}_n(R)$. The rest follows from Example 4.23 (note that, if $\mathcal{R}$ is a matrix algebra, there is no difference between the τ- and non-τ-versions). □

By the above proposition, Conjecture FM(d) may be rephrased (at least when ρ_0 is absolutely irreducible) as follows:

Conjecture $\mathrm{FM}'(\mathrm{d})^{\mathrm{ai}}$**.** *If ρ_0 is absolutely irreducible, then the profinite scheme $X^{\tau}_{\mathcal{R},n,W}$ is finite over W.*

Let $X^{\tau}_{\mathcal{T},n,W}$ be the moduli scheme of degree-n absolutely irreducible τ-representations of $\mathcal{T}$ in ${}^{\mathrm{f}}\mathfrak{A}^{\tau}_{W}$; we have $X^{\tau}_{\mathcal{T},n,W}(E) = \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{T},n}(E)$ for $E \in {}^{\mathrm{f}}\mathfrak{C}_W$. It is an open subscheme of $\mathrm{Spec}(T)$ (note that $T = T^{\mathrm{tr}}$, since T is the closed W-subalgebra of R generated by the traces of all elements of $\rho_{\mathfrak{D}}(G_{K,S}) \subset \mathcal{T}^{\times}$). In general, if ρ_0 is not absolutely irreducible, the bijectivity (8.2) suggests that one should look at the image of $X^{\tau}_{\mathcal{R},n,W} \to X^{\tau}_{\mathcal{T},n,W}$. Also, since we are interested in $\mathbb{Q}_p$-representations, we should look at

$$X^{\tau}_{\mathfrak{D},\rho_0,\mathbb{Q}_p} := \mathrm{Im}(X^{\tau}_{\mathcal{R},n,W,\mathbb{Q}_p} \to X^{\tau}_{\mathcal{T},n,W,\mathbb{Q}_p}),$$

where $X^{\tau}_{\mathcal{R},n,W,\mathbb{Q}_p} := X^{\tau}_{\mathcal{R},n,W}[p^{-1}]$ and $X^{\tau}_{\mathcal{T},n,W,\mathbb{Q}_p} := X^{\tau}_{\mathcal{T},n,W}[p^{-1}]$ are the f-profinite schemes obtained from $X^{\tau}_{\mathcal{R},n,W}$ and $X^{\tau}_{\mathcal{T},n,W}$ by making p invertible. These are all f-profinite schemes over $F = W[p^{-1}]$. Note that, for any algebraic extension E of F, we have $X^{\tau}_{\mathcal{R},n,W,\mathbb{Q}_p}(E) = X^{\tau}_{\mathcal{R},n,W}(E) \simeq \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{R},n}(E)$ and $X^{\tau}_{\mathcal{T},n,W,\mathbb{Q}_p}(E) = X^{\tau}_{\mathcal{T},n,W}(E) \simeq \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathcal{T},n}(E)$. We also have

$$X^{\tau}_{\mathfrak{D},\rho_0,\mathbb{Q}_p}(E) \supset \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathfrak{D},\rho_0,n}(E) \supset \underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathfrak{D},\rho_0,\mathrm{M}_n}(E) \simeq \mathrm{Geom}'^{,\mathrm{ai}}_{n}(K,S,\mathcal{L},[a,b];E)(\rho_0).$$

Remark 8.16. In general, the prime p may or may not be invertible in $X^{\tau}_{\mathcal{R},n,W}$ and $X^{\tau}_{\mathcal{T},n,W}$. That p is invertible in $X^{\tau}_{\mathcal{R},n,W}$ means that there exist no absolutely irreducible representations of $\mathcal{R}$ over an f-profinite W-algebra in which p is not invertible. If ρ_0 is absolutely irreducible, then p is not invertible in $X^{\tau}_{\mathcal{R},n,W}$ since $X^{\tau}_{\mathcal{R},n,W}(k)$ has a point corresponding to ρ_0. The converse may also be expected to be true. For example, the trivial representation $\rho_0 = 1$ of dimension n may lift to a representation over $\mathbb{F}_p[\![t]\!]$ that yields an absolutely irreducible representation $\rho : G_{K,S} \to \mathrm{GL}_n(\mathbb{F}_p(\!(t)\!))$. But such a representation does not seem to come from a representation of $\mathcal{R}$.

Now $\mathrm{FM}'(\mathrm{d})^{\mathrm{ai}}$ is generalized as follows:

Conjecture $\mathrm{FM}'(\mathrm{d})_{\mathbb{Q}_p}$**.** *For any residual representation $\rho_0 \in \mathfrak{D}$, the f-profinite scheme $X^{\tau}_{\mathfrak{D},\rho_0,\mathbb{Q}_p}$ is finite over F.*

This implies that the set $X^{\tau}_{\mathfrak{D},\rho_0,\mathbb{Q}_p}(\overline{\mathbb{Q}}_p)$ is, and a fortiori $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathfrak{D},\rho_0,n}(\overline{\mathbb{Q}}_p)$ is, finite. Hence we have the following proposition:

Proposition 8.17. *We have* $\mathrm{FM}'(\mathrm{d})_{\mathbb{Q}_p} \Longrightarrow \mathrm{FM}'(\mathrm{a})^{\mathrm{fiber}}$.

In fact, the converse $\Leftarrow$ is also true as plotted in diagram (8.1). To complete the plot, it remains for us to prove the following result:

Proposition 8.18. *We have* $\mathrm{FM}'(\mathrm{b})^{\mathrm{ai}} \Longleftrightarrow \mathrm{FM}'(\mathrm{d})_{\mathbb{Q}_\mathrm{p}}$.

Proof. The implication $\Leftarrow$ is easy: Assume that $\mathrm{FM}(\mathrm{d})_{\mathbb{Q}_\mathrm{p}}$ is true. Let E/F be a finite extension and k_E the residue field of E. Since there exist only finitely many continuous representations $\rho_0 : G_{K,S} \to \mathrm{GL}_n(k_E)$ by the Hermite–Minkowski theorem, it is enough to show that the set $\mathrm{Geom}_n^{\prime;\mathrm{ai}}(K,S,\mathcal{L},[a,b];E)(\rho_0)$ is finite for each ρ_0. But $\mathrm{FM}'(\mathrm{d})_{\mathbb{Q}_\mathrm{p}}$ implies the finiteness of $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathfrak{D},\rho_0,n}(\overline{\mathbb{Q}}_p)$, which contains $\mathrm{Geom}_n^{\prime;\mathrm{ai}}(K,S,\mathcal{L},[a,b];E)(\rho_0)$.

Next we prove the opposite implication $\Rightarrow$. Suppose that $X^\tau_{\mathfrak{D},\rho_0,\mathbb{Q}_p}$ is not finite over F. By Proposition 8.14, it is enough to show that the set $\underline{\mathrm{Rep}}^{\mathrm{ai},\tau}_{\mathfrak{D},\rho_0,n}(E) = \mathrm{Im}(\gamma : X^\tau_{\mathcal{R},n,W,\mathbb{Q}_p}(E) \to X^\tau_{\mathcal{T},n,W,\mathbb{Q}_p}(E))$ is infinite for some E. Recall that $X^\tau_{\mathcal{R},n,W,\mathbb{Q}_p}$ and $X^\tau_{\mathcal{T},n,W,\mathbb{Q}_p}$ are open subschemes of $\mathrm{Spec}(R[p^{-1}])$ and $\mathrm{Spec}(T[p^{-1}])$ respectively (note that $R^{\mathrm{tr}}[n^{-1}] = R[n^{-1}]$, so that we do not need "tr" here). By restricting our attention to an irreducible component of $\mathrm{Spec}(R)$ (or, factoring R modulo a minimal prime ideal), we are reduced to the case where R is integral. More precisely, we proceed as follows: By assumption, there is a minimal prime ideal $\mathfrak{p}$ of R such that, if $\overline{T}$ is the image of the composite map $T \hookrightarrow R \to \overline{R} := R/\mathfrak{p}$, then $\overline{T}[p^{-1}]$ is not finite over F. Note that $\overline{R}$ is a complete noetherian local domain in $\mathcal{C}_W$. Considering $X^\tau_{\mathcal{R},n,W,\mathbb{Q}_p} \otimes_{R[p^{-1}]} \overline{R}[p^{-1}] \to X^\tau_{\mathcal{T},n,W,\mathbb{Q}_p} \otimes_{T[p^{-1}]} \overline{T}[p^{-1}]$ instead of $X^\tau_{\mathcal{R},n,W,\mathbb{Q}_p} \to X^\tau_{\mathcal{T},n,W,\mathbb{Q}_p}$, we may and do assume that R is integral and T is a W-subalgebra of R such that $T[p^{-1}]$ is not finite over F.

Now choose nonzero elements $d \in R^{\mathrm{tr}}$ and $e \in T$ such that (the inclusion map $T \hookrightarrow R$ induces a morphism $T[e^{-1}] \hookrightarrow R[d^{-1}]$ and) the morphism $\mathrm{Spec}(R[d^{-1},p^{-1}]) \to \mathrm{Spec}(T[e^{-1},p^{-1}])$ is an "affine piece" of $\gamma : X^\tau_{\mathcal{R},n,W,\mathbb{Q}_p} \to X^\tau_{\mathcal{T},n,W,\mathbb{Q}_p}$. We are to show that there exist infinitely many morphisms $\varphi : T[e^{-1}, p^{-1}] \to E$ that extend to morphisms $\phi : R[d^{-1},p^{-1}] \to E$ and hence produce infinitely many points on $X^\tau_{\mathfrak{D},\rho_0,\mathbb{Q}_p}$. By the structure theorem of complete noetherian local domains (cf. e.g., [13, Chap. 0_{IV}, Thm. 19.8.8] or [24, Sect. 29]), R is finite over a subring R_0 that is isomorphic to the power series ring $W[\![X_1,\dots,X_m]\!]$ for some $m \geq 1$. Let $\delta = (d) \cap R_0$ be the ideal of R_0 lying below (d). Since $\delta \neq 0$, there exist infinitely many morphisms $\phi_0 : R_0 \to W$ in $\mathcal{C}_W$ such that $\phi_0(\delta) \neq 0$. Since there exist only finitely many finite extensions E/F of a given degree, Lemma 8.19 below allows us to find a finite extension E/F such that each of the morphisms $\phi_0 : R_0 \to W$ as above extends to a morphism $\phi : R \to \mathcal{O}_E$. Since $\phi(d) \neq 0$, it extends further to a morphism $\phi : R[d^{-1},p^{-1}] \to E$. Thus we obtain infinitely many points in $X^\tau_{\mathcal{R},n,W,\mathbb{Q}_p}(E)$.

Each such ϕ induces a morphism $\phi|_{T[e^{-1},p^{-1}]} : T[e^{-1},p^{-1}] \to E$, and hence a point of $X^\tau_{\mathcal{T},n,W,\mathbb{Q}_p}(E)$. We claim that there arise infinitely many different morphisms as $\phi|_{T[e^{-1},p^{-1}]}$ when $\phi : R[d^{-1},p^{-1}] \to E$ varies. This can be checked as follows: Let t be an element of T that is transcendental over W (such a t exists because the algebraic closure of F in the topologically finitely generated F-algebra $R[p^{-1}]$ is finite over F, while $T[p^{-1}]$ is assumed not finite over F). Then we will have infinitely many different values of $\phi(t)$. To be more precise, let $\sum a_i Y^i$ ($a_i \in R_0$) be the minimal polynomial of t over R_0. By Gauss' lemma, it is irreducible over the fraction field K_0 of R_0. Each a_i is a symmetric polynomial of the conjugates of t in the Galois closure over K_0 of the fraction field of R, and at least one of the a_i's, say a_j, is transcendental over W. When the variables $X_1, \dots, X_m$ vary in pW, the power series a_j assumes infinitely many different values, and hence so does our t. Thus there arise infinitely many different morphisms $\phi|_{T[e^{-1},p^{-1}]} : T[e^{-1}, p^{-1}] \to E$. □

Finally, we prove a lemma:

Lemma 8.19. *Let Λ be a commutative ring. Let R be an integral Λ-algebra and R_0 a Λ-subalgebra of R. Let A_0 be another integral Λ-algebra and E_0 the field of fractions of A_0. Assume R is generated as an R_0-module by at most N elements. Then for any Λ-algebra homomorphism $f_0 : R_0 \to A_0$, there exists a finite extension E/E_0 of degree $\leq N$ such that f_0 extends to a Λ-algebra homomorphism $f : R \to A$, where A is the integral closure of A_0 in E.*

Proof. Let $\mathfrak{p}_0 := \mathrm{Ker}(f_0)$. Since R is integral over R_0, there exists a prime ideal $\mathfrak{p}$ of R such that $\mathfrak{p} \cap R_0 = \mathfrak{p}_0$. Let $\kappa(\mathfrak{p}_0)$ and $\kappa(\mathfrak{p})$ be the residue fields of $\mathfrak{p}_0$ and $\mathfrak{p}$ respectively. Then $\kappa(\mathfrak{p}_0)$ is identified with a subfield of E_0 via f_0. By assumption, the extension $\kappa(\mathfrak{p})/\kappa(\mathfrak{p}_0)$ has degree $\leq N$. Let E be a compositum of E_0 and $\kappa(\mathfrak{p})$ in an algebraic closure of E_0. Then the extension E/E_0 has degree $\leq N$, and $f_0 : R_0 \to \kappa(\mathfrak{p}_0) \hookrightarrow E_0$ extends to a Λ-algebra homomorphism $f : R \to \kappa(\mathfrak{p}) \hookrightarrow E$. Since R is integral over R_0, the image of f is also integral over A_0. □

Acknowledgements

The author thanks Hyunsuk Moon for having drawn his attention to the paper [35] of Procesi several years ago, which was the starting point for this work. He thanks also Masanori Morishita for asking him many interesting questions related to the subject of this paper, and Xavier Caruso for informing him of Liu's paper [23], which improved the exposition of Section 8. The final stage of the writing of a preliminary version of this paper was done while the author was staying at POSTECH, March–August, 2007; he thanks YoungJu Choie and POSTECH for their hospitality. A thorough revision was made while the author was staying at the Korea Institute of Advanced Study in the academic year 2012/2013; he thanks Youn-Seo Choi and KIAS for their hospitality. The author has been partially supported by JSPS Grants-in-Aid for Scientific Research No. 19540036 and the KOFST Brain Pool Program. This work also received support from the JSPS Core-to-Core Program (coordinator: Makoto Matsumoto), Grants-in-Aid for Scientific Research No. 19204002 (representative: Makoto Matsumoto) and No. 17340008 (representative: Nobuo Tsuzuki).

In a preliminary version of this paper (2007), the notion of an f-pro-$\mathfrak{a}$ ring was defined as a certain topological ring. Then Kazuya Kato and Seidai Yasuda pointed out independently that the topology did not work properly in relevant cases. Later the author found the present formulation more convenient. Thus he thanks Kato and Yasuda very much.

Finally, the author thanks the referees for many useful comments on earlier versions of this paper, which helped the author to correct inaccuracies and improve the exposition.

References

[1] V. A. Abrashkin, Modular representations of the Galois group of a local field and a generalization of a conjecture of Shafarevich, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), 1135–1182; English transl.: Math. USSR-Izv. **35** (1990), 469–518. Zbl 0733.14008 MR 1039960

[2] J. Bellaiche and G. Chenevier, Families of Galois representations and Selmer groups, Astérisque **324** (2009). Zbl 1192.11035 MR 2656025

[3] L. Berger, Representations p-adiques et equations differentielles, Invent. Math. **148** (2002), 219–284. Zbl 1113.14016 MR 1906150

[4] G. Böckle and C. Khare, Finiteness results for mod ℓ Galois representations over function fields, Preprint (2006).

[5] N. Bourbaki, *Algèbre*, Hermann, Paris, 1958. Zbl 0102.27203 MR 0098114

[6] H. Carayol, Formes modulaires et représentations galoisiennes à valeurs dans un anneau local complet, in *p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991)*, Contemp. Math. 165, Amer. Math. Soc., Providence, RI, 1994, 213–237. Zbl 0812.11036 MR 1279611

[7] G. Chenevier, The p-adic analytic space of pseudocharacters of a profinite group, and pseudorepresentations over arbitrary rings, in *Automorphic forms and Galois representations, Vol. 1*, Proc. LMS Durham Symposium 2011, F. Diamond, P. L. Kassaei and M. Kim (eds.), London Math. Soc. Lecture Note Series 414, Cambridge University Press, Cambridge, 2014, 221–285. Zbl 1350.11063 MR 3444227

[8] A. J. de Jong, A conjecture on arithmetic fundamental groups, Israel J. Math. **121** (2001), 61–84. Zbl 1054.11032 MR 1818381

[9] J.-M. Fontaine, Il n'y a pas de variete abelienne sur $\mathbb{Z}$, Invent. Math. **81** (1985), 515–538. Zbl 0612.14043 MR 0807070

[10] J.-M. Fontaine, Schemas propres et lisses sur $\mathbb{Z}$, in *Proc. Indo-French Conference on Geometry (Bombay, 1989)*, Hindustan Book Agency, Delhi, 1993, 43–56. Zbl 0837.14014 MR 1274493

[11] J.-M. Fontaine and B. Mazur, Geometric Galois representations, in *Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993)*, International Press, Cambridge, MA, 1995, 41–78 (190–227 in the 2nd ed.). Zbl 0839.14011 MR 1363495

[12] A. Grothendieck, Le groupe de Brauer, I. Algebres d'Azumaya et interpretations diverses, Séminaire Bourbaki 1964/65, Exp. **290** (in the new edition: Vol. 9, Soc. Math. France, Paris, 1995, 199–219); also in *Dix Exposés sur la cohomologie des schémas*, North Holland, Amsterdam et Masson, Paris, 1968. Zbl 0193.21503 MR 1608798

[13] A. Grothendieck and J. Dieudonne, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I, I.H.E.S. Publ. Math. **20** (1964), 101–355. Zbl 0136.15901 MR 0173675

[14] S. Harada and H. Moon, On zeta functions of modular representations of a discrete group, J. Algebra **319** (2008), 4456–4471. Zbl 1183.20045 MR 2416730

[15] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics 52, Springer, 1977. Zbl 0367.14001 MR 0463157

[16] R. Huber, Continuous valuations, Math. Z. **212** (1993), 455–477. Zbl 0788.13010 MR 1207303

[17] Y. Ihara, How many primes decompose completely in an infinite unramified Galois extension of a global field?, J. Math. Soc. Japan **35** (1983), 81–106. Zbl 0518.12006 MR 0714470

[18] C. Khare, Conjectures on finiteness of mod p Galois representations, J. Ramanujan Math. Soc. **15** (2000), 23–42. Zbl 1017.11028 MR 1751924

[19] C. Khare and J.-P. Wintenberger, Serre's modularity conjecture, I, II, Invent. Math. **178** (2009), 485–586. Zbl 1304.11041 Zbl 1304.11042 MR 2551763 MR 2551764

[20] M. Kisin, Potentially semi-stable deformation rings, J. Amer. Math. Soc. **21** (2008), 513–546. Zbl 1205.11060 MR 2373358

[21] M.-A. Knus and M. Ojanguren, *Théorie de la descente et algèbres d'Azumaya*, Lecture Notes in Math. 389, Springer, 1974. Zbl 0284.13002 MR 0417149

[22] T. Y. Lam, *A first course in noncommutative rings*, 2nd ed., Graduate Texts in Math. 131, Springer, 2001. Zbl 0980.16001 MR 1838439

[23] T. Liu, Torsion p-adic Galois representations and a conjecture of Fontaine, Ann. Sci. École Norm. Sup. **40** (2007), 633–674. Zbl 1163.11043 MR 2191528

[24] H. Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics 8, Cambridge University Press, Cambridge, 1989. Zbl 0666.13002 MR 1011461

[25] B. Mazur, Deforming Galois representations, in *Galois groups over* $\mathbb{Q}$ *(Berkeley, CA, 1987)*, Y. Ihara, K. Ribet and J.-P. Serre (eds.), Math. Sci. Res. Inst. Publ. 16, Springer, New York, 1989, 385–437. Zbl 0714.11076 MR 1012172

[26] B. Mazur, An introduction to the deformation theory of Galois representations, in *Modular forms and Fermat's last theorem (Boston, MA, 1995)*, G. Cornell, J. H. Silverman and G. Stevens (eds.), Springer, New York, 1997, 243–311. Zbl 0901.11015 MR 1638481

[27] H. Moon, Finiteness results on certain mod p Galois representations, J. Number Theory **84** (2000), 156–165. Zbl 0967.11021 MR 1791814

[28] H. Moon, The number of monomial mod p Galois representations with bounded conductor, Tôhoku Math. J. **55** (2003), 89–98. Zbl 1047.11050 MR 1956082

[29] H. Moon and Y. Taguchi, Mod p Galois representations of solvable image, Proc. Amer. Math. Soc. **129** (2001), 2529–2534. Zbl 1033.11024 MR 1838373

[30] H. Moon and Y. Taguchi, Refinement of Tate's discriminant bound and non-existence theorems for mod p Galois representations, Doc. Math. Extra Vol. Kazuya Kato's Fiftieth Birthday (2003), 641–654. Zbl 1135.11319 MR 2046611

[31] H. Nagao and Y. Tsushima, *Representations of finite groups*, Academic Press, Boston, 1989. Zbl 0673.20002 MR 0998775

[32] J. Neukirch, A. Schmidt and K. Wingberg, *Cohomology of number fields*, Springer, 2000. Zbl 0948.11001 MR 1737196

[33] L. Nyssen, Pseudo-représentations, Math. Ann. **306** (1996), 257–283. Zbl 0863.16012 MR 1411348

[34] C. Procesi, A formal inverse to the Cayley-Hamilton theorem, J. Algebra **107** (1987), 63–74. Zbl 0618.16014 MR 0883869

[35] C. Procesi, Deformations of representations, in *Methods in ring theory (Levico Terme, 1997)*, Lecture Notes in Pure and Appl. Math. 198, Dekker, New York, 1998, 247–276. Zbl 0948.16012 MR 1767983

[36] R. Ramakrishna, On a variation of Mazur's deformation functor, Compos. Math. **87** (1993), 269–286. Zbl 0910.11023 MR 1227448

[37] R. Rouquier, Caractérisation des caractères et pseudo-caractères, J. Algebra **180** (1996), 571–586. Zbl 0857.16013 MR 1378546

[38] M. Schlessinger, Functors of Artin rings, Trans. A.M.S. **130** (1968), 208–222. Zbl 0167.49503 MR 0217093

[39] J.-P. Serre, Une "formule de masse" pour les extensions totalement ramifiees de degre donne d'un corps local, C. R. Acad. Sci. Paris **286** (1978), A1031–A1036. Zbl 0388.12005 MR 0500361

[40] J.-P. Serre, Sur les représentations modulaires de degré 2 de Gal($\overline{\mathbb{Q}}/\mathbb{Q}$), Duke Math. J. **54** (1987), 179–230. Zbl 0641.10026 MR 0885783

[41] J.-P. Serre, *Représentations linéaires des groupes finis*, 5th ed., Hermann, Paris, 1998. Zbl 0926.20003 MR 0543841

[42] J.-P. Serre and J. Tate, Good reduction of abelian varieties, Ann. of Math. **88** (1968), 492–517. Zbl 0172.46101 MR 0236190

[43] Y. Taguchi, On potentially abelian geometric representations, Ramanujan J. **7** (2003), 477–483. Zbl 1075.11043 MR 2040985

[44] Y. Taguchi, On the finiteness of various Galois representations, *Proc. JAMI Conference "Primes and knots"* (T. Kohno and M. Morishita, eds.), Contemporary Mathematics 416, Amer. Math. Soc., Providence, RI, 2006, 249–261. Zbl 1173.11061 MR 2276145

[45] R. Taylor, Galois representations associated to Siegel modular forms of low weight, Duke Math. J. **63** (1991), 281–332. Zbl 0810.11033 MR 1115109

[46] A. Weil, *Basic number theory*, 3rd ed., Springer, 1974. Zbl 0326.12001 MR 0427267

[47] A. Wiles, On ordinary λ-adic representations associated to modular forms, Invent. Math. **94** (1988), 529–573. Zbl 0664.10013 MR 0969243

[48] E. Witt, Schiefkörper über diskret bewerteten Körpern, J. reine angew. Math. **176** (1936), 153–156. Zbl 0016.05102 MR 1581528