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Resonances of the Square Root of the Pauli Operator

by

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Abstract

We investigate the spectral properties of two relativistic Hamiltonians: one is the square root of a Pauli operator with an electric potential growing polynomially at infinity, and the other differs from it only in the sign of the potential. Moreover, we show that resonances (eigenvalues) of each of them converge to resonances (eigenvalues) of the corresponding Pauli operators with the same potential in the nonrelativistic limit.

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§1. Introduction

In this paper we consider the following two Hamiltonians $L_\pm(c)$ acting on the Hilbert space $L^2(\boldsymbol{R}^3)^2$:

$$L_\pm(c) = \pm\sqrt{(c\sigma\cdot D^b)^2 + m^2c^4} - mc^2 + V(x). \tag{1.1}$$

Here $c > 0$ is the speed of light, $m > 0$ is the rest mass of a relativistic particle, $D^b := D - b(x) = -i\nabla - b(x)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where σ_1, σ_2, σ_3 are Pauli matrices, i.e., (2×2)-Hermitian matrices satisfying

$$\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}I_2, \tag{1.2}$$

$$\sigma_1\sigma_2 = i\sigma_3, \qquad \sigma_2\sigma_3 = i\sigma_1, \qquad \sigma_3\sigma_1 = i\sigma_2, \tag{1.3}$$

for $j, k = 1, 2, 3$, where I_n is the $n\times n$ unit matrix. For example, the following σ_1, σ_2, σ_3 satisfy (1.2) and (1.3):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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The square root $\sqrt{(c\sigma\cdot D^b)^2+m^2c^4}$ is well defined because the Pauli operator $(c\sigma\cdot D^b)^2+m^2c^4$ is a positive self-adjoint operator under our assumption below. Since $(\sigma\cdot D)^2=-\Delta I_2$ follows from (1.2) and (1.3), $L_\pm(c)$ are relativistic Schrödinger operators when $b=0$. The potential $V(x)$ is an Hermitian (2×2)-matrix-valued function with a scalar principal part $v(x)I_2$ diverging at infinity, and the magnetic potential $b:\boldsymbol{R}^3\to\boldsymbol{R}^3$ is bounded. Moreover, we assume that they are all dilation analytic as in [1].

Let us state our assumptions more precisely. Conditions (1.4) and (1.5) below are used in the studies of $L_+(c)$ and $L_-(c)$, respectively.

Assumption 1.1. The positive constants M, M_1, M_2 with $M_1\le M_2$ and a_0 are supposed to satisfy one of the following conditions:

$$a_0<\frac{\pi}{M_2+2}\quad\text{and}\quad a_0(M_2-M_1)<\pi \tag{1.4}$$

or

$$a_0<\min\left\{\frac{2\pi}{M_2+2},\pi/2\right\}\quad\text{and}\quad a_0(M_2-M_1)<\pi. \tag{1.5}$$

Let us define stripes $\Omega:=\{\theta\in\boldsymbol{C};|\operatorname{Im}\theta|<a_0\}$ and $\Omega_\pm:=\{\theta\in\boldsymbol{C};0<\pm\operatorname{Im}\theta<a_0\}$.

The potential $V(x)$ is decomposed as $V(x)=v(x)I_2+W(x)$, where $v(x)$ and $W(x)$ are continuous real-valued and Hermitian (2×2)-matrix-valued functions, respectively, and $b\in C^\infty(\boldsymbol{R}^3,\boldsymbol{R}^3)$. Moreover, they satisfy the following conditions:

(v) For each $\theta\in\Omega$, there exist a continuous function $v_\theta(x)$ such that $v_t(x)=v(e^tx)$ for $t\in\boldsymbol{R}$ and $(v_\theta f,f)$ is analytic in $\theta\in\Omega$ for each $f\in\mathcal{S}$, the Schwartz space. Moreover, there exist positive constants R_0, K_0 and K_α for multiindices α such that $v_{i\tau}\in C^\infty(|x|>R_0)$ and

$$|\partial_x^\alpha v_{i\tau}(x)|\le K_\alpha|x|^{M-|\alpha|},\quad |\alpha|\ge 0, \tag{1.6}$$

$$K_0^{-1}|x|^M\le|v_{i\tau}(x)|\le K_0|x|^M, \tag{1.7}$$

$$\tau M_1\le\arg(v_{i\tau}(x))\le\tau M_2 \tag{1.8}$$

are satisfied uniformly in $|x|\ge R_0$ and $0\le\tau<a_0$, and

$$\sup_{|x|\le R_0,|\tau|<a_0}|v_{i\tau}(x)|<\infty. \tag{1.9}$$

(W) For each $\theta\in\Omega$, there exist a (2×2)-matrix-valued continuous function $W_\theta(x)$ such that $W_t(x)=W(e^tx)$ for $t\in\boldsymbol{R}$ and $(W_\theta f,f)$ is analytic in $\theta\in\Omega$ for each

$f \in \mathcal{S}^2$ and that

$$\sup_{x \in \boldsymbol{R}^3, \tau \in [0,a_0)} |W_{i\tau}(x)\langle x\rangle^{-M}|_{M_2} < \infty,$$
$$\lim_{|x| \to \infty} |W_{i\tau}(x)\langle x\rangle^{-M}|_{M_2} = 0 \tag{1.10}$$

uniformly in $\tau \in [0, a_0)$, where $\langle x\rangle = \sqrt{|x|^2+1}$ and $|\cdot|_{M_2}$ denotes the usual matrix norm.

(b) For each $\theta \in \Omega$, there exists a C^∞-function $b_\theta = {}^t(b_{1,\theta}, b_{2,\theta}, b_{3,\theta}) : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ such that $b_t(x) = b(e^t x)$ for $t \in \boldsymbol{R}$ and $(b_{j,\theta} f, f)$, $j = 1, 2, 3$ is analytic in $\theta \in \Omega$ for each $f \in \mathcal{S}$ and that

$$|\partial_x^\alpha b_{i\tau}(x)| \le K_\alpha \langle x\rangle^{-|\alpha|} \tag{1.11}$$

is satisfied uniformly in $\tau \in [0, a_0)$ for $|\alpha| \ge 0$ with some $K_\alpha > 0$.

In the sequel we write $V_\theta(x) := v_\theta(x) I_2 + W_\theta(x)$ and assume $c \ge 1$.

Remark 1.2. For sufficiently small a_0, condition (1.8) is satisfied for some positive M_1 and M_2 if there exists $K > 0$ such that

$$x \cdot \nabla v(x) \ge K|x|^M$$

for $|x| \ge R_0$ (see, e.g., [16, Lemma 2.2]).

Remark 1.3. By the dilation analyticity above we see that

$$v_{\theta+t}(x) = v_\theta(e^t x)$$

etc., for all $t \in \boldsymbol{R}$ and $\theta \in \Omega$, and that $v_{-i\tau}(x) = \overline{v_{i\tau}(x)}$ etc., for all $\tau \in (-a_0, a_0)$. Thus, estimates on v_θ, W_θ and b_θ for $\theta \in \Omega$ follow from those for $\theta = i\tau$, $\tau \in [0, a_0)$.

Remark 1.4. Let us denote by $\|\cdot\|_\infty$ the L^∞-norm on $\boldsymbol{R}^3$. Then, **(v)** implies

$$\sup_{|\tau| < a_0} \|v_{i\tau}(\cdot)\langle\cdot\rangle^{-M}\|_\infty < \infty \tag{1.12}$$

and $\Omega \ni \theta \to v_\theta(\cdot)\langle\cdot\rangle^{-M}$ is analytic in the operator norm. Further, by virtue of the Cauchy integral formula, the above estimate is valid even when $v_{i\tau}$ is replaced by $\frac{d^k}{d\tau^k} v_{i\tau}$ for each positive integer k.

Remark 1.5. A typical example of $v(x)$ is $v(x) = q(x/|x|)|x|^M$ with a C^∞-function q on S^2, the unit sphere in $\boldsymbol{R}^3$. In this case, $M_1 = M_2 = M$ and a_0 is arbitrarily chosen if it satisfies (1.4) or (1.5).

Here we make some comments on the notation used in this paper. We denote by $\|\cdot\|$ (resp. $(\cdot,\cdot)$) the norm (resp. the scalar product) of the Hilbert space $(L^2)^2 = L^2(\boldsymbol{R}^3)^2$ and also use this notation for other Hilbert spaces if they do not cause confusion. Moreover, the notation $\|\cdot\|$ is also used for operator norms. We denote by $\mathcal{S}$ the Schwartz space on $\boldsymbol{R}^3$ and by $H^s = H^s(\boldsymbol{R}^3)$ the Sobolev space of order s. We also define $D_M := L^2_M \cap H^1$, where $L^2_M = L^2_M(\boldsymbol{R}^3) := L^2(\boldsymbol{R}^3; \langle x\rangle^{2M} dx)$ is a weighted L^2-space. We sometimes write p for pI_n, omitting identity matrices I_n, when p is a scalar function or an operator acting on $L^2(\boldsymbol{R}^3)$. We denote the numerical range of an operator T by $\mathrm{Num}(T)$:

$$\mathrm{Num}(T) := \{(Tu,u); u \in D(T), \|u\| = 1\},$$

where $D(T)$ denotes the domain of T. We denote the spectrum of T by $\sigma(T)$, the discrete spectrum $\sigma_{\mathrm{d}}(T)$, the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(T)$, the singular continuous spectrum $\sigma_{\mathrm{sc}}(T)$ and the resolvent set $\rho(T)$. The letter K will denote various constants that may change from line to line. The square root $\sqrt{z}$ is defined to have the branch on the negative real line.

Let m, $s \in \boldsymbol{R}$ and denote by $S^{m,s}$ the space of functions $p(x,\xi) \in C^\infty(\boldsymbol{R}^3 \times \boldsymbol{R}^3)$ satisfying

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \le K_{\alpha\beta} \langle x\rangle^{m-|\beta|} \langle \xi\rangle^{s-|\alpha|} \quad \text{on } \boldsymbol{R}^3 \times \boldsymbol{R}^3,$$

for all α and β. We denote by $\Sigma^{m,s}$ the set of pseudodifferential operators $p(x,D)$, $p \in S^{m,s}$ defined by

$$p(x,D)u(x) = \frac{1}{(2\pi)^3} \iint e^{i(x-y)\cdot\xi} p(x,\xi) u(y)\, d\xi\, dy.$$

The Pauli operator $(\sigma\cdot D^b)^2$ is a nonnegative self-adjoint operator with domain $(H^2)^2$ and is written

$$(\sigma\cdot D^b)^2 = (D - b(x))^2 I_2 - \sigma\cdot(\nabla\times b(x)), \tag{1.13}$$

using (1.2) and (1.3). Since $(\sigma\cdot D^b)^2 + mc^2$ is a positive self-adjoint operator, the square root $\sqrt{(c\sigma\cdot D^b)^2 + m^2c^4}$ is well defined.

The following theorem is proved in the same way as in [11] (see also [12]) by using the commutator theorem (see, e.g., [20, Theorem X.36]).

Theorem 1.6. *The Hamiltonians $L_\pm(c)$ defined on $\mathcal{S}^2$ are essentially self-adjoint.*

We also denote the unique self-adjoint extension with the same notation, $L_\pm(c)$.

Due to the dilation analyticity of potentials we can define the following dilated operators for θ:

$$L_{\pm}(c,\theta) := \pm\sqrt{(c\sigma\cdot D_\theta^b)^2 + m^2c^4} - mc^2 + V_\theta(x),$$

where $D_\theta^b = e^{-\theta}D - b_\theta(x)$.

If $b = 0$, then $\sqrt{(c\sigma\cdot D_\theta^b)^2 + m^2c^4} = \sqrt{-c^2e^{-2\theta}\Delta + m^2c^4}I_2$ is well defined for $|\operatorname{Im}\theta| < \pi/2$ as a closed operator with domain $(H^1)^2$. However, if $b \neq 0$ and $\operatorname{Im}\theta \neq 0$, there is a possibility that the spectrum of $(c\sigma\cdot D_\theta^b)^2 + m^2c^4$ contains the zero so that its square root is not well defined in the usual way. But, for each $c \geq 1$ we can take a constant $\tau_0(c) \in (0, a_0]$ satisfying the following condition: there exist two small constants $\delta > 0$ and $\delta' > 0$ independent of $c \geq 1$ such that

$$C_\delta \subset \rho((c\sigma\cdot D_{i\tau}^b)^2 + m^2c^4) \quad \text{and} \quad C_\delta \cap \operatorname{Num}((c\sigma\cdot D_{i\tau}^b)^2 + m^2c^4 - \delta') = \phi$$

are valid for all $\tau \in [0, \tau_0(c))$, where

$$C_\delta := \{z; \pi - \delta \leq \arg z \leq \pi + \delta\},$$

a sector containing the half-line $(-\infty, 0]$. Then, by the Dunford integral, the square root of $(c\sigma\cdot D_\theta^b)^2 + m^2c^4$ is well defined (see (3.4)). Indeed, we show that there exists $\tau_0 > 0$ such that $\tau_0 \leq \tau_0(c)$ for all $c \geq 1$ (Lemma 3.1) and that we can take $\tau_0(c) = a_0$ for a sufficiently large number c (Lemma 3.2). We also note that $\tau_0(c) = a_0$ for all $c \geq 1$ if $b = 0$. We set

$$\Omega(c) := \{\theta \in \Omega; |\operatorname{Im}\theta| < \tau_0(c)\},$$
$$\Omega_+(c) := \{\theta \in \Omega; 0 < \operatorname{Im}\theta < \tau_0(c)\}.$$

Then, from (3.4) it is easily seen that $\sqrt{(c\sigma\cdot D_\theta^b)^2 + m^2c^4}$ is an analytic family of type (A) in $\theta \in \Omega(c)$ (see, e.g., [17, VII], [21, XII]).

Now let us state our results on the spectral properties of $L_\pm(c)$ for fixed $c \geq 1$. Let $\{\mathcal{U}(t)\}_{t\in\boldsymbol{R}}$ be the dilation group on $\boldsymbol{R}^3$,

$$(\mathcal{U}(t)f)(x) = e^{(3t/2)}f(e^tx),$$

and write $\mathcal{U}_2(t) = \mathcal{U}(t)I_2$. We will see that $\{L_+(c,\theta)\}_{\theta\in\Omega(c)}$ is an analytic family of type (A) with compact resolvent, and satisfies

$$\mathcal{U}_2(t)L_+(c,\theta)\mathcal{U}_2(t)^{-1} = L_+(c,\theta+t), \quad t \in \boldsymbol{R},\ \theta \in \Omega(c) \tag{1.14}$$

(see Proposition 3.4). Thus $L_+(c,\theta)$ has a purely discrete spectrum and, since $L_+(c) = L_+(c,0)$, the standard argument on the dilation analyticity gives the following theorem.

Theorem 1.7. *Suppose Assumption 1.1 with (1.4) and fix* $c \geq 1$*:*

(a) The discrete spectrum $\sigma_{\mathrm{d}}(L_+(c,\theta))$ *is independent of* $\theta \in \Omega(c)$*, denoted by* $\Sigma_+(c)$*, and coincides with* $\sigma_{\mathrm{p}}(L_+(c))$*. Moreover, the multiplicity of each eigenvalue is independent of* $\theta \in \Omega(c)$*.*

(b) $L_+(c)$ *has a purely discrete spectrum.*

We shall see that $\{L_-(c,\theta)\}_{\theta\in\Omega_+(c)}$ is an analytic family of type (A) *only* in $\Omega_+(c)$ (not in $\Omega(c)$) and that each $L_-(c,\theta)$ also has compact resolvent for each $\theta \in \Omega_+(c)$ and satisfies

$$\mathcal{U}_2(t)L_-(c,\theta)\mathcal{U}_2(t)^{-1} = L_-(c,\theta+t), \quad t \in \boldsymbol{R},\ \theta \in \Omega_+(c). \tag{1.15}$$

Thus, the spectrum of $L_-(c,\theta)$ (also consisting of a discrete spectrum only) is independent of $\theta \in \Omega_+(c)$. On the other hand, it will be proved that the resolvent $(L_-(c)-z)^{-1}$, $\operatorname{Im} z < 0$ is the strong limit of $(L_-(c,\theta)-z)^{-1}$ as $\Omega_+(c) \ni \theta \to 0$ (Proposition 3.4). Namely, $L_-(c)$ is obtained as the limit of $L_-(c,\theta)$. However, the spectral properties of $L_-(c)$ and $L_-(c,\theta)$ are quite different. Indeed, we have the following theorem.

Theorem 1.8. *Suppose Assumption 1.1 with (1.5) and fix* $c \geq 1$*:*

(a) The discrete spectrum $\sigma_{\mathrm{d}}(L_-(c,\theta))$ *is independent of* $\theta \in \Omega_+(c)$*, denoted by* $\Sigma_-(c)$*, and satisfies*

$$\Sigma_-(c) \subset \overline{\boldsymbol{C}_+}, \qquad \Sigma_-(c) \cap \boldsymbol{R} = \sigma_{\mathrm{p}}(L_-(c)),$$

where $\overline{\boldsymbol{C}}_+ := \{z \in \boldsymbol{C}; \operatorname{Im} z \geq 0\}$ *is the closed upper half-plane. Moreover, the multiplicity of each eigenvalue is independent of* $\theta \in \Omega_+$*.*

(b) $L_-(c)$ *has at most finitely many eigenvalues, and the multiplicity of each of them is finite.*

(c) $\sigma(L_-(c)) = \boldsymbol{R}$ *and* $\sigma_{\mathrm{sc}}(L_-(c)) = \phi$*. In particular,* $\sigma(L_-(c)) \setminus \sigma_{\mathrm{p}}(L_-(c)) \subset \sigma_{\mathrm{ac}}(L_-(c))$*.*

Remark 1.9. We emphasize that, even though the spectra of $L_+(c)$ and $L_-(c)$ are quite different, they can be treated in the same framework (Theorem 2.3). Indeed, they are regarded as *boundary values* of analytic families $\{L_+(c,\theta)\}_{\theta\in\Omega_-(c)}$ and $\{L_-(c,\theta)\}_{\theta\in\Omega_-(c)}$, respectively (see Section 2 for the definition of a boundary value of an analytic family). In [16] we proved that self-adjoint operators T defined as boundary values of a certain analytic family are classified into two categories: type (I) where $\sigma(T) = \sigma_{\mathrm{d}}(T)$, and type (II) where $\sigma(T) = (-\infty,+\infty)$, $\sigma_{\mathrm{sc}}(T) = \emptyset$ (Theorem 2.3). In Section 3, we shall show that $L_+(c)$ is of type (I) and $L_-(c)$ is of type (II).

Under some conditions, there is no embedded eigenvalue of $L_-(c)$ in $\sigma(L_-(c)) = \boldsymbol{R}$.

Corollary 1.10. *In addition to Assumption 1.1 with (1.5), suppose $b = 0$ and there exists $\theta_1 \in \Omega_+$ such that $\operatorname{Im} V_{\theta_1}(x) := (2i)^{-1}(V_{\theta_1}(x) - V_{\theta_1}^*(x))$ is a nonnegative Hermitian matrix for each $x \in \boldsymbol{R}^3$. Then, $\sigma_{\mathrm{p}}(L_-(c)) = \phi$. Moreover, there exists $\lambda_0 > 0$ independent of $c \geq 1$ such that $\Sigma_-(c) \subset \{\operatorname{Im} z \geq \lambda_0\}$.*

Remark 1.11. (i) Each element of $\Sigma_-(c)$ is called a *resonance* of $L_-(c)$.

(ii) As far as the author knows there are only a few studies on the relativistic Schrödinger operator $\sqrt{-\Delta + 1} - v(x)$ with a potential $v(x)$ satisfying $v(x) \to \infty$ as $|x| \to \infty$ (e.g., [11, 12, 16]).

Now we investigate a resonance-free region for each $c \geq 1$. Let us denote by $\mathcal{N}(\mathcal{C}, d)$, $d \geq 0$ the d-neighborhood of a set $\mathcal{C}$:

$$\mathcal{N}(\mathcal{C}, d) = \{z; \operatorname{dist}(z, \mathcal{C}) \leq d\},$$

and let us define a sector

$$C(w, \tau_1, \tau_2) := \{z \in \boldsymbol{C}; \tau_1 \leq \arg(z - w) \leq \tau_2\}$$

for $w \in \boldsymbol{C}$ and $-\pi \leq \tau_1 \leq \tau_2 \leq \pi$.

Theorem 1.12. *Suppose Assumption 1.1 with (1.5) and fix $c \geq 1$. Then, for any $\varepsilon > 0$ there exists $d \geq 0$ such that*

$$\Sigma_-(c) \subset \mathcal{N}(C(-2mc^2, \Theta_1(c) - \varepsilon, \Theta_2(c) + \varepsilon), d) \cap \overline{\boldsymbol{C}_+}, \tag{1.16}$$

where

$$\Theta_1(c) := \min\left\{\frac{M_1}{M_1 + 2}\pi, M_1\tau_0(c)\right\},$$
$$\Theta_2(c) := \max\left\{\frac{M_2}{M_2 + 1}\pi, \pi - \tau_0(c)\right\}.$$

Moreover, (1.16) is valid for $\varepsilon = 0$ if $b = 0$ and $W = 0$.

Remark 1.13. Let $b = 0$, $W = 0$ and $v(x) = \kappa_0|x|^M$ for some $\kappa_0 > 0$ and $M \geq 2$. Then, we can see that $M_1 = M_2 = M$ and $\varepsilon = d = 0$ so that the resonances of $L_-(c) + 2mc^2$ are contained in the sector (5.2). In the final section we show that the result of this theorem is optimal in this case.

We next consider two Pauli operators $P_\pm$ acting on $(L^2)^2$,

$$P_\pm = \pm\frac{1}{2m}(\sigma \cdot D^b)^2 + v(x)I_2 + W(x), \tag{1.17}$$

and their dilated operators

$$P_{\pm}(\theta) = \pm\frac{1}{2m}(\sigma \cdot D_\theta^b)^2 + v_\theta(x)I_2 + W_\theta(x) \tag{1.18}$$

for $\theta \in \Omega$. Here we assume (1.4) and $\theta \in \Omega$ for $P_+(\theta)$, and (1.5) and $\theta \in \Omega_+$ for $P_-(\theta)$, respectively. We will show that both $P_+(\theta)$ and $P_-(\theta)$ are closed operators with domain $(H^2)^2 \cap (L_M^2)^2$ and that P_+ is self-adjoint with the same domain and that $\mathcal{S}^2$ is its core. But, the essential self-adjointness of P_- defined on $\mathcal{S}^2$ is guaranteed only for $M \le 2$ (e.g., [13]). Indeed, the Schrödinger operator $-\Delta - |x|^s$, $s > 0$, defined on $\mathcal{S}^2$, is essentially self-adjoint if and only if $0 < s \le 2$ (see, e.g., [7], [20]).

Proposition 1.14. *Suppose Assumption 1.1 with (1.4):*

(a) P_+ ($= P_+(0)$) defined on $\mathcal{S}^2$ is essentially self-adjoint and its self-adjoint extension, also denoted by P_+, is bounded from below and has compact resolvent. In particular, it has a purely discrete spectrum with its eigenvalues accumulating at infinity.

(b) For $\theta \in \Omega$, $P_+(\theta)$ defined on $\mathcal{S}^2$ has a unique closed extension, also denoted by $P_+(\theta)$, with domain $(H^2)^2 \cap (L_M^2)^2$, and its spectrum $\sigma_{\mathrm{d}}(P_+(\theta))$ consists only of eigenvalues with finite multiplicity. Moreover, $\sigma_{\mathrm{d}}(P_+(\theta))$ is independent of θ, i.e., $\sigma(P_+(\theta)) = \sigma_{\mathrm{d}}(P_+(\theta)) = \sigma_{\mathrm{d}}(P_+) = \sigma(P_+)$, and the multiplicity of each eigenvalue is independent of $\theta \in \Omega$.

Proposition 1.15. *Suppose Assumption 1.1 with (1.5) . Let $\theta \in \Omega_+$.*

(a) $P_-(\theta)$ defined on $\mathcal{S}^2$ has a unique closed extension, also denoted by $P_-(\theta)$, with domain $(H^2)^2 \cap (L_M^2)^2$. Also, $P_-(\theta)$ has compact resolvent, and so its spectrum $\sigma(P_-(\theta)) = \sigma_{\mathrm{d}}(P_-(\theta))$ consists only of eigenvalues with finite multiplicity. Moreover, $\sigma_{\mathrm{d}}(P_-(\theta))$ is independent of θ, and satisfies $\sigma_{\mathrm{d}}(P_-(\theta)) \subset \overline{\boldsymbol{C}_+}$, and the multiplicity of each eigenvalue is independent of $\theta \in \Omega_+$.

(b) If $M \le 2$, then P_- defined on $\mathcal{S}^2$ is essentially self-adjoint and its self-adjoint extension, also denoted by P_-, has at most finitely many eigenvalues, and the multiplicity of each of them is finite. Moreover,

$$\sigma_{\mathrm{d}}(P_-(\theta)) \cap \boldsymbol{R} = \sigma_{\mathrm{p}}(P_-)$$

for $\theta \in \Omega_+$, and $\sigma(P_-) = \boldsymbol{R}$, $\sigma_{\mathrm{sc}}(P_-) = \phi$. In particular, $\sigma(P_-) \setminus \sigma_{\mathrm{p}}(P_-) \subset \sigma_{\mathrm{ac}}(P_-)$.

We call an eigenvalue of $P_-(\theta)$ a *resonance* of P_-, and the *multiplicity* of the resonance is defined to be the algebraic multiplicity of the eigenvalue.

Remark 1.16. Let us consider a typical case: $v(x) = |x|^M$, $b = 0$, $W = 0$ and $m = 1/2$, i.e., $P_\mp = \pm\Delta + |x|^M$. Denote by $\{\nu_j\}_{j=1}^\infty$ the eigenvalues of P_+ and $\beta_0 := \pi(2+M)^{-1}$. Then $\{-e^{-2i\beta_0}\nu_j\}_{j=1}^\infty$ is the set of resonances of P_- (see [16, p. 1335]). In particular, the resonances of $P_- = \Delta + |x|^2$ lie on the positive imaginary axis when $M = 2$.

Now we state our results on the nonrelativistic limits of $L_\pm(c)$. We denote by $B_\varepsilon(\lambda)$ the disc with center λ and radius $\varepsilon > 0$:

$$B_\varepsilon(\lambda) := \{z \in \boldsymbol{C}; |z - \lambda| \le \varepsilon\}.$$

Theorem 1.17. *Suppose Assumption 1.1 with (1.4). Fix $L > 0$ and denote by $\{\lambda_j\}_{j=1}^{N_+}$ the set of all eigenvalues of P_+ in the interval $(-\infty, L)$, and denote by m_j the multiplicity of λ_j. Then, for any small $\varepsilon > 0$ there exists $c_0 > 0$ such that if $c \ge c_0$, there are m_j eigenvalues of $L_+(c)$ in $B_\varepsilon(\lambda_j)$ for each $j = 1, \dots, N_+$. Moreover, there is no eigenvalue in $(-\infty, L) \setminus \big(\cup_{j=1}^{N_+} B_\varepsilon(\lambda_j)\big)$.*

Theorem 1.18. *Suppose Assumption 1.1 with (1.5). Let $\mathcal{O}$ be a bounded open set in $\boldsymbol{C}$ and denote by $\{\mu_k\}_{k=1}^{N_-}$ the set of all resonances of P_- in it, and denote by n_k the (algebraic) multiplicity of μ_k. Then, for any small $\varepsilon > 0$ there exists $c_0 > 0$ such that if $c \ge c_0$, there are n_k resonances of $L_-(c) + 2mc^2$ in $B_\varepsilon(\mu_k) \cap \overline{\boldsymbol{C}}_+$ for each $k = 1, \dots, N_-$. Moreover, there is no resonance in $\mathcal{O} \setminus \big(\cup_{k=1}^{N_-} B_\varepsilon(\mu_k)\big)$.*

Remark 1.19. The resonances of $L_-(c)$ locate in its numerical range. It is known that the resolvent of a non-self-adjoint operator is hard to handle if the spectrum parameter belongs to the numerical range. Indeed, in the one-dimensional case, the norm

$$\left\| \left(-e^{-it}\frac{d^2}{dx^2} + e^{it}x^2 - re^{i\tau} \right)^{-1} \right\|$$

diverges as $r \to \infty$ if $|\tau| < |t|$, even though the spectrum of the $-e^{-it}d^2/dx^2 + e^{it}x^2$ is a subset of $\boldsymbol{R}$ ([6, Thm. 14.5.4]). Taking account of this fact, we cannot expect to obtain a useful uniform estimate of $\|(L_-(c,\theta) - z)^{-1}\|$ when z belongs to the numerical range even though z is not near the spectrum. Since our approach depends heavily on uniform resolvent estimates of $L_\pm(c,\theta)$, it is difficult for us to eliminate the boundedness condition on $\mathcal{O}$ in Theorem 1.18.

In [14] the spectral property and the nonrelativistic limit of resonances of the Dirac operator with the magnetic and electric potentials b and v, respectively, are studied by using the result obtained in the present work. Here we briefly explain a relation between $L_\pm(c)$ and the Dirac operator. See [14] (but [16] for the case $b = 0$) for the details.

Let us consider the Dirac operator with a magnetic potential b and an electric potential v:

$$H(c) = c\alpha \cdot D^b + mc^2\beta + vI_4$$

in $(L^2)^4$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, α_j and β are the Dirac matrices

$$\alpha_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

and denote $L_\pm(c,\theta)$ with $W = 0$ by $L_{\pm 1}(c,\theta)$. A resonance of $H(c)$ is defined by an eigenvalue of the dilated Dirac operator

$$H(c,\theta) = c\alpha \cdot D^b_\theta + mc^2\beta + v_\theta I_4$$

for $\theta \in \Omega_+(c)$ with assumption (1.4). There exists an invertible bounded operator $U(c,\theta)$ such that $U(c,\theta)(H(c,\theta) - mc^2)U(c,\theta)^{-1} = L_1 + Q_1$, where

$$L_1 := \begin{pmatrix} L_{+1}(c,\theta)I_2 & 0 \\ 0 & L_{-1}(c,\theta)I_2 \end{pmatrix},$$
$$Q_1 := U(c,\theta)v_\theta I_4 U(c,\theta)^{-1} - v_\theta I_4.$$

Roughly speaking, $Q_1\langle x\rangle^{-M+1}$ is bounded and its norm goes to zero as $c \to \infty$. Namely, L_1 is the principal term and Q_1 is its perturbation, and hence the analysis of $L_{\pm 1}(c,\theta)$ is necessary for investigating the resonances of $H(c)$. See [14] (but [16] for $b = 0$) to know how we use results on $L_\pm(c,\theta)$ to study the Dirac operator. When $\theta = 0$, this transformation by $U(c,\theta)$ is called the Foldy–Wouthuysen transformation.

This work is closely related to the nonrelativistic limit of Dirac operators, and there are many papers on the subject (see [23], [15], [16]). However, relativistic Schrödinger operators and, more generally, the square root of Pauli operators themselves, are interesting subject to study. Though there are many studies on relativistic Schrödinger operators (see, e.g., [3], [18], and see also [9] for semirelativistic Pauli–Fierz Hamiltonians), as far as the author knows there is no study on the spectral property and resonances of the square root of a Pauli operator (or a Schrödinger operator) with a potential diverging to $-\infty$ as $|x| \to \infty$, except for [11, 12, 16]. In [11, 12] the essential self-adjointness is investigated, and in [16] several results, as in the present work, are obtained for $b = 0$ and $W = 0$. The spectral properties of Schrödinger operators with a similar potential are studied by several authors (see, e.g., [5, 26]). But, they do not discuss the resonances. We impose the dilation analyticity on the potentials to discuss the resonances. The dilation analytic method, which dates back to [1], has been applied to many Schrödinger operators and Dirac operators (e.g., [2, 4, 8, 10, 16, 22, 25]). In particular, Amour, Brummelhuis and Nourrigat [2] used the method to study the nonrelativistic limit of resonances of Dirac operators with an electric potential diverging at infinity (see also [24]). Inspired by their work, the author and Yamada [16] studied the same problem by introducing two relativistic Schrödinger operators, and in this work the study of the relativistic Schrödinger operators plays a crucial role. However, Dirac operators have no magnetic potentials in [2, 16]. In [14] their results are extended to Dirac operators with a magnetic potential.

Finally, we compare the present results with those in [16], in which Dirac operators and relativistic Schrödingers are considered. In [16] the same results as Theorems 1.7, 1.8 and Propositions 1.14 and 1.15 have been proved under the assumption $b = 0$ and $W = 0$, and Theorems 1.17 and 1.18 in the case of $b = 0$ and $W = 0$ follow immediately from results obtained in [16]. However, there is no result in [16] on a resonance-free region, such as Theorem 1.12.

The plan of this paper is as follows. In Section 2 we prepare an abstract theorem (Theorem 2.3) without proof, since it has already been proved in [16]. In Section 3 we study $L_\pm(c,\theta)$ in detail to give the proofs of Theorems 1.7 and 1.8, Corollary 1.10 and Theorem 1.12. In Section 4 we investigate $P_\pm(\theta)$ in detail and show that $P_+(\theta)$ and $P_-(\theta)$ are the nonrelativistic limits of $L_+(c,\theta)$ and $L_-(c,\theta) + 2mc^2$, respectively. In Section 5 we show that Theorems 1.17 and 1.18 follow immediately from results in the previous section and that the result of Theorem 1.12 is optimal in some cases.

§2. Abstract results

In this section we state an abstract result on the self-adjoint operators defined as boundary values of some kind of operator-valued analytic functions. This result is used to determine the spectral properties of several operators appearing in this work. See [16] for the proof.

Let T be a self-adjoint operator and $\{T(\theta)\}_{\theta\in\boldsymbol{\Omega}_+}$ a family of closed operators in a Hilbert space $\mathcal{H}$, where $\boldsymbol{\Omega}_+ = \{\theta \in \boldsymbol{C}; 0 < \operatorname{Im}\theta < a\}$ for some $a > 0$. We assume the following:

(A1) $\{T(\theta)\}_{\theta\in\boldsymbol{\Omega}_+}$ is an analytic family in the sense of Kato (see [17, VII], [21, XII]).

(A2) Each $T(\theta)$ has compact resolvent.

(A3) There is a strongly continuous one-parameter unitary group $\{U(t)\}_{t\in\boldsymbol{R}}$ such that

$$U(t)T(\theta)U(t)^* = T(\theta + t) \tag{2.1}$$

for $t \in \boldsymbol{R}$ and $\theta \in \boldsymbol{\Omega}_+$.

By (A1) and (A2), each $T(\theta)$ has a purely discrete spectrum and the eigenvalues are analytic functions or branches of one or several analytic functions, and (A3) implies that the eigenvalues of $T(\theta)$ are invariant when θ is changed to $\theta+t$ if t is real. Thus, each eigenvalue is a constant function of $\theta \in \boldsymbol{\Omega}_+$ (see, e.g., [1, 21]). Therefore we obtain the following result.

Proposition 2.1. *Suppose (A1)∼(A3). Then there is a discrete set* Σ *in* $\boldsymbol{C}$ *such that* $\sigma(T(\theta)) = \sigma_{\mathrm{d}}(T(\theta)) = \Sigma$ *for all* $\theta \in \boldsymbol{\Omega}_+$.

Let $\boldsymbol{C}_\pm = \{z \in \boldsymbol{C}; \pm \operatorname{Im} z > 0\}$. A self-adjoint operator T is supposed to be related to the analytic family $\{T(\theta)\}_{\theta \in \boldsymbol{\Omega}_+}$ in the following sense:

(A4) There is a nonempty open set $\mathcal{O}_0 \subset \boldsymbol{C}_- \setminus \Sigma$ such that

$$w - \lim_{t \to +0} (T(it) - z)^{-1} = (T - z)^{-1} \quad \text{(weakly)}$$

for each $z \in \mathcal{O}_0$.

For each $s \in \boldsymbol{R}$ define a self-adjoint operator $T(s)$ by $T(s) := U(s) T U(s)^*$. Then $T(0) = T$ and

$$\begin{aligned} w - \lim_{t \to +0} (T(s+it) - z)^{-1} &= w - \lim_{t \to +0} U(s)(T(it) - z)^{-1} U(s)^* \\ &= U(s)(T - z)^{-1} U(s)^* = (T(s) - z)^{-1} \end{aligned}$$

by (A3). Thus the self-adjoint operators $T(s)$, $s \in \boldsymbol{R}$ are regarded as *boundary values* of the operator-valued function $T(\theta)$ defined on $\boldsymbol{\Omega}_+$. The following proposition shows that the eigenvalues of $T(\theta)$ are located in the closed upper half-plane $\overline{\boldsymbol{C}}_+$.

Proposition 2.2. *Suppose (A1)∼(A4). Then* $\Sigma \subset \overline{\boldsymbol{C}_+}$.

For $E \in \boldsymbol{R}$, let γ be a positively oriented small circle $|z - E| = \varepsilon$ enclosing E with $\{z \in \boldsymbol{C}; 0 < |z - E| \le \varepsilon\} \cap \Sigma = \phi$ and let

$$P_\theta(E) = -\frac{1}{2\pi i} \int_\gamma (T(\theta) - z)^{-1}\, dz.$$

Then this operator is the eigenprojection associated with $E \in \sigma_d(T(\theta)) = \Sigma$ if $E \in \Sigma$ and $P_\theta(E) = 0$ otherwise. Moreover, for each $E \in \Sigma$ the projection-valued function $P_\theta(E)$ is analytic in $\theta \in \boldsymbol{\Omega}_+$. In particular, the dimension of the range of $P_\theta(E)$ is independent of θ for each E. Let $\mathbf{P}_s(\cdot)$ be the spectral projection of $T(s)$ for $s \in \boldsymbol{R}$.

Theorem 2.3. *Suppose (A1)∼(A4).*

(a) $\sigma_{\mathrm{d}}(T(\theta)) \cap \boldsymbol{R} = \sigma_{\mathrm{p}}(T)$ *for all* $\theta \in \boldsymbol{\Omega}_+$. *Moreover, for each* $E \in \sigma_{\mathrm{p}}(T)$ *and* $s \in \boldsymbol{R}$, *we have*

$$\lim_{\boldsymbol{\Omega}_+ \ni \theta \to s} \|P_\theta(E) - \mathbf{P}_s(\{E\})\| = 0. \tag{2.2}$$

In particular, the eigenvalues of T *are discrete and each eigenvalue has finite multiplicity.*

(b) Either

(I) T *has a purely discrete spectrum, i.e.,* $\sigma(T) = \sigma_{\mathrm{d}}(T)$

or

(II) $\sigma(T) = \boldsymbol{R}$, $\sigma_{\mathrm{sc}}(T) = \phi$

holds. In particular, we have $\sigma(T) \setminus \sigma_{\mathrm{p}}(T) \subset \sigma_{\mathrm{ac}}(T)$ *in case (II).*

(c) If $\Sigma \cap (\boldsymbol{C} \setminus \boldsymbol{R}) \neq \phi$ *or* $\Sigma = \phi$, *then case (II) holds. Thus,* $\Sigma = \sigma_{\mathrm{p}}(T)$ *in case (I).*

(d) Suppose case (I) above holds and fix $z \notin \sigma_{\mathrm{d}}(T)$. *Then the resolvent* $(T(\theta)-z)^{-1}$ *has an analytic continuation of* θ *from* $\boldsymbol{\Omega}_+$ *to* $\boldsymbol{\Omega} := \{\theta \in \boldsymbol{C}; |\operatorname{Im}\theta| < a\}$.

§3. Relativistic Pauli operators

We define a dilated Pauli operator $S_b(c,\theta)$ by

$$\begin{aligned} S_b(c,\theta) &:= (c\sigma \cdot D_\theta^b)^2 + m^2c^4 \\ &= c^2\left((e^{-\theta}D - b_\theta(x))^2 I_2 - \sigma \cdot e^{-\theta}\nabla \times b_\theta(x)\right) + m^2c^4 \end{aligned} \tag{3.1}$$

for $\theta \in \Omega$, which is a closed operator with domain $(H^2)^2$ and a core $\mathcal{S}^2$. We also write

$$\begin{aligned} S_b(c,\theta) &= S_0(c,\theta) \\ &\quad + c^2\left((-2e^{-\theta}b_\theta \cdot D + ie^{-\theta}\nabla \cdot b_\theta + b_\theta^2)I_2 - \sigma \cdot e^{-\theta}\nabla \times b_\theta(x)\right), \end{aligned} \tag{3.2}$$

where $S_0(c,\theta) = (-c^2e^{-2\theta}\Delta + m^2c^4)I_2$.

Hence, $c^{-2}(S_b(c,\theta) - S_0(c,\theta))$ is $-\Delta I_2$ bounded with bound 0 uniformly in $c \geq 1$, and so $S_b(c,0)$ is self-adjoint.

Lemma 3.1. *Suppose assumption* (**b**) *with* $0 < a_0 < \pi/2$. *Then,* $\{S_b(c,\theta)\}_{\theta\in\Omega}$ *is an analytic family of type (A) for each* $c \geq 1$. *Moreover, there exist* $K > 0$ *and small* $\tau_0 > 0$ *such that* $S_b(c,\theta)$ *is a strictly* m*-accretive operator and* $\operatorname{Num}(S_b(c,\theta))$ *is contained in the sector* $C(m^2c^4/2, -K|\operatorname{Im}\theta|, K|\operatorname{Im}\theta|)$ *for* $\theta \in \Omega$ *with* $|\operatorname{Im}\theta| < \tau_0$ *and* $c \geq 1$. *In particular, the spectrum of* $S_b(c,\theta)$ *is contained in this sector.*

Proof. It follows from assumption (**b**) and (3.2) that $S_b(c,\theta)$, $\theta \in \Omega$ is an analytic family of type (A). For the next statement it suffices to prove the lemma for $\operatorname{Re}\theta = 0$, because

$$\mathcal{U}_2(\operatorname{Re}\theta) S_b(c, i \operatorname{Im}\theta) \mathcal{U}_2(\operatorname{Re}\theta)^{-1} = S_b(c,\theta). \tag{3.3}$$

Let $\theta = i\tau$, τ being small. Then, by (3.1) we can write $S_b(c,i\tau) = S_b(c,0) + r(i\tau)$, where $r(i\tau)$ satisfies $\|S_b(c,0)^{-1/2} r(i\tau) S_b(c,0)^{-1/2}\| \le K_0|\tau|$ uniformly in $c \ge 1$ for some $K_0 > 0$, and so

$$|(r(i\tau)f,f)| \le K_0|\tau|(S_b(c,0)f,f)$$

for any $f \in (H^2)^2$ with $\|f\|_{(L^2)^2} = 1$. Let $w = (S_b(c,i\tau)f,f)$ and $z = (S_b(c,0)f,f)$. Then, $|w - z| \le K_0|\tau||z|$. Therefore, since $z \ge m^2c^4$ we have

$$w \in \mathcal{N}\left(C(m^2c^4, -\sin^{-1}(K_0|\tau|), \sin^{-1}(K_0|\tau|)), m^2c^4K_0|\tau|\right)$$

and so $\operatorname{Num}(S_b(c,i\tau)) \subset C((1/2)m^2c^4, -K_1|\tau|, K_1|\tau|)$ for some $K_1 > 0$ independent of τ if $|\tau|$ is small enough. Thus, $S_b(c,i\tau)$ is a strictly m-accretive operator with the spectrum contained in the sector (e.g., in [19, Thm. VIII.17]). This completes the proof. □

Lemma 3.2. *Suppose assumption* (**b**) *with* $0 < a_0 < \pi/2$. *Then, for any* $\varepsilon > 0$ *there exists* $d \ge 0$ *independent of* $\theta \in \Omega$ *and* $c \ge 1$ *such that* $\operatorname{Num}(S_b(c,\theta))$ *is contained in the set* $\mathcal{N}(C(m^2c^4, -2\tau - \varepsilon, -2\tau + \varepsilon), c^2d)$, *where* $\tau = \operatorname{Im}\theta$. *In particular, there is* $c_0 > 0$ *such that* $e^{i\tau}S_b(c,\theta)$ *is a strictly* m*-accretive operator for* $\theta \in \Omega$ *and* $c \ge c_0$.

Proof. We prove the lemma for $\operatorname{Re}\theta = 0$ as in the previous lemma. Write $\widehat{S_b} := e^{i\tau}c^{-2}(S_b(c,i\tau) - m^2c^4)$ and $\widehat{S}_0 := e^{i\tau}c^{-2}(S_0(c,i\tau) - m^2c^4) = -e^{-i\tau}\Delta$ for simplicity, and set $w = (\widehat{S}_b f,f)$ and $z = (\widehat{S}_0 f,f)$ with $\|f\|_{(L^2)^2} = 1$. Then, the lemma follows from $\operatorname{Num}(\widehat{S}_0) = \{e^{-i\tau}t; t \ge 0\}$ and the fact, which follows immediately from (3.2), that for any $\varepsilon' > 0$ there is $K > 0$ such that $|w - z| \le \varepsilon'|z| + K$. Indeed, we obtain the lemma by putting $\varepsilon' = \sin\varepsilon$. □

We know that $S^{1/2} = S_b(c,\theta)^{1/2}$, for $\theta \in \Omega(c)$, is represented as

$$S^{1/2}f = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(S+\lambda)^{-1}Sf\,d\lambda, \quad f \in D(S). \tag{3.4}$$

Since it is not easy to analyze $S_b(c,\theta)^{1/2}$ directly, the following lemma is useful in the sequel because $S_0(c,\theta)^{1/2}$, i.e., $b = 0$, is the pseudodifferential operator with symbol $\sqrt{-c^2e^{-2\theta}|\xi|^2 + m^2c^4}$.

Lemma 3.3. *Let $\theta \in \Omega(c)$ and $c \geq 1$. Then, the operator*

$$E_\theta(c) := S_b(c,\theta)^{1/2} - S_0(c,\theta)^{1/2} - \frac{1}{2}(S_b(c,\theta) - S_0(c,\theta))(S_0(c,\theta))^{-1/2} \tag{3.5}$$

defined on $\mathcal{S}^2$ can be extended to a bounded operator on $(L^2)^2$ with $\|E_\theta(c)\| \leq K$ uniformly in θ and c. In particular, $D(S_b(c,\theta)^{1/2}) = D(S_0(c,\theta)^{1/2}) = (H^1)^2$ and the difference $S_b(c,\theta)^{1/2} - S_0(c,\theta)^{1/2}$ is bounded.

Proof. We may assume $\operatorname{Re}\theta = 0$ and $\tau = \operatorname{Im}\theta \in (-\tau_0(c), \tau_0(c))$. To begin with let us set

$$\widetilde{S} = c^{-2}S_b(c,\theta) = (\sigma \cdot D_\theta^b)^2 + m^2c^2,$$
$$\widetilde{S}_0 = c^{-2}S_0(c,\theta) = (\sigma \cdot D_\theta^0)^2 + m^2c^2,$$

$V_b = \widetilde{S} - \widetilde{S}_0$ and $r = m^2c^2/2$ for simplicity. Then, by (3.2), $V_b(-\Delta+1)^{-1/2}$ is bounded. We also have $\|(T+\lambda)^{-1}\| \leq K(\lambda+r)^{-1}$, $\lambda > 0$ for $T = \widetilde{S}$ and $\widetilde{S}_0$ (see the definition of $\tau_0(c)$ and Lemma 3.2). By the resolvent equation we have

$$\begin{aligned}(\widetilde{S}+\lambda)^{-1} - (\widetilde{S}_0+\lambda)^{-1} &= -(\widetilde{S}+\lambda)^{-1}V_b(\widetilde{S}_0+\lambda)^{-1} \\ &= -(\widetilde{S}_0+\lambda)^{-1}V_b(\widetilde{S}_0+\lambda)^{-1} \\ &\quad + (\widetilde{S}+\lambda)^{-1}V_b(\widetilde{S}_0+\lambda)^{-1}V_b(\widetilde{S}_0+\lambda)^{-1}\end{aligned}$$

for $\lambda > 0$. Since $\|(-\Delta+1)^{1/2}(\widetilde{S}_0+\lambda)^{-1}\| \leq K(r+\lambda)^{-1/2}$, the norm of the second term is bounded by $K(r+\lambda)^{-2}$. The first term is written

$$-V_b(\widetilde{S}_0+\lambda)^{-2} - (\widetilde{S}_0+\lambda)^{-1}[V_b, \widetilde{S}_0](\widetilde{S}_0+\lambda)^{-2}.$$

Since $\|[V_b, \widetilde{S}_0](\widetilde{S}_0+\lambda)^{-1}\|$ is uniformly bounded in $\lambda > 0$, the norm of the second term is bounded by $K(r+\lambda)^{-2}$. Thus, on $\mathcal{S}^2$ we can write

$$\begin{aligned}\widetilde{S}^{1/2} - \widetilde{S}_0^{1/2} &= -\frac{1}{\pi}\int_0^\infty \lambda^{1/2}[(\widetilde{S}+\lambda)^{-1} - (\widetilde{S}_0+\lambda)^{-1}]\,d\lambda \\ &= \frac{1}{\pi}\int_0^\infty \lambda^{1/2}V_b(\widetilde{S}_0+\lambda)^{-2}\,d\lambda - \frac{1}{\pi}\int_0^\infty \lambda^{1/2}\widetilde{R}(\lambda)\,d\lambda \\ &= I + II,\end{aligned}$$

where $R(\lambda)$ satisfies $\|\widetilde{R}(\lambda)\| \leq K(r+\lambda)^{-2}$. Hence, II can be extended to a bounded operator with $\|II\| \leq Kr^{-1/2}$. Since $\lim_{\lambda\to\infty} \lambda^{1/2}V_b(\widetilde{S}_0+\lambda)^{-1}f = 0$ for each $f \in \mathcal{S}^2$, we have

$$If = \frac{1}{2\pi}\int_0^\infty \lambda^{-1/2}V_b(\widetilde{S}_0+\lambda)^{-1}f\,d\lambda = \frac{1}{2}V_b\widetilde{S}_0^{-1/2}f,$$

by partial integration. Hence, I can be extended to a bounded operator, whose norm is uniformly bounded in $c \geq 1$. Since $E_{i\tau}(c) = c(\widetilde{S}^{1/2} - \widetilde{S}_0^{1/2}) - cI$, by keeping track of τ dependence we have obtained the desired result. □

Now, denoting $T_b(c,\theta) := S_b(c,\theta)^{1/2}$ we consider the two operators

$$L_\pm(c,\theta) := \pm T_b(c,\theta) - mc^2 + v_\theta(x) I_2 + W_\theta \tag{3.6}$$

defined on $\mathcal{S}^2$. Note that if t is a real number, they are written

$$L_\pm(c,t) = \mathcal{U}_2(t) L_\pm(c) \mathcal{U}_2(t)^{-1} \tag{3.7}$$

on $\mathcal{S}^2$.

The following proposition is the main result in this section.

Proposition 3.4. *In the following, we suppose Assumption 1.1 with (1.4) for (a) and (b), and (1.5) for (c)∼(f).*

(a) For each $\theta \in \Omega(c)$, $L_+(c,\theta)$ defined on $\mathcal{S}^2$ is closable, and its closure (also denoted by $L_+(c,\theta)$) has domain D_M. Moreover, its resolvent set is nonempty and, in particular, $L_+(c,\theta)$ has compact resolvent.

(b) For each $c \geq 1$ the family of closed operators $\{L_+(c,\theta)\}_{\theta\in\Omega(c)}$ is an analytic family of type (A) with property (1.14).

(c) For each $\theta \in \Omega_+(c)$, $L_-(c,\theta)$ defined on $\mathcal{S}^2$ is closable and its closure (also denoted by $L_-(c,\theta)$) has domain D_M. Moreover, its resolvent set is nonempty and, in particular, $L_-(c,\theta)$ has compact resolvent.

(d) The family of closed operators $\{L_-(c,\theta)\}_{\theta\in\Omega_+(c)}$ is an analytic family of type (A) with property (1.15).

(e) There is a constant $r_0 > 0$ independent of c and $\theta \in \Omega_+(c)$ such that $\{z \in \boldsymbol{C}; \operatorname{Im} z < -r_0\} \subset \rho(L_-(c,\theta))$.

(f) Let $c \geq 1$ and $\operatorname{Im} z < -r_0$. Then the resolvent $(L_-(c,\theta) - z)^{-1}$ converges to $(L_-(c) - z)^{-1}$ strongly as $\theta \to 0$:

$$s - \lim_{\Omega_+(c)\ni\theta\to 0} (L_-(c,\theta) - z)^{-1} = (L_-(c) - z)^{-1}. \tag{3.8}$$

To prove this proposition we prepare several lemmas.

Since

$$L_\pm(c,\theta) = \mathcal{U}_2(\operatorname{Re}\theta) L_\pm(c, i\operatorname{Im}\theta) \mathcal{U}_2(\operatorname{Re}\theta)^{-1}, \tag{3.9}$$

we have only to study the case $\operatorname{Re}\theta = 0$. To begin with we fix $a_2 \in (0, \pi/2)$ and set

$$p(c,\tau,\xi) := \sqrt{e^{-2\tau i} c^2 |\xi|^2 + m^2 c^4} - mc^2$$

for $|\tau| < \pi/2$, i.e., $p(c,\tau,D) = T_0(c,i\tau) - mc^2$.

Lemma 3.5. *(a) For $k = 1, 2, 3$,*

$$|\partial_{\xi_k} p(c,\tau,\xi)| \leq K|\xi_k|, \tag{3.10}$$

$$|\partial_{\xi_k} \operatorname{Im} p(c,\tau,\xi)| \leq K\tau|\xi_k|. \tag{3.11}$$

(b) For $|\alpha| \geq 2$,

$$|\partial_\xi^\alpha p(c,\tau,\xi)| \leq K_\alpha c(|\xi|+c)^{1-|\alpha|} \leq K_\alpha(|\xi|+c)^{2-|\alpha|}, \tag{3.12}$$

$$|\partial_\xi^\alpha \operatorname{Im} p(c,\tau,\xi)| \leq K_\alpha|\tau| c(|\xi|+c)^{1-|\alpha|} \leq K_\alpha|\tau|(|\xi|+c)^{2-|\alpha|}. \tag{3.13}$$

Here constants $K > 0$ and $K_\alpha > 0$ are uniformly in $c \geq 1$ and $\tau \in [-a_2, a_2]$.

Proof. We write $p(c,\tau,\xi) + mc^2 = mc^2 g(\xi/(mc))$, where $g(\xi) := \sqrt{e^{-2\tau i}|\xi|^2 + 1}$. Since $0 < a_2 < \pi/2$, we see that $|g(\xi)| \geq K_1(1+|\xi|)$ for some $K_1 > 0$ uniformly in $|\tau| \leq a_2$. Thus, (3.10) and (3.12) follow from the estimates

$$|\partial_{\xi_k} g(\xi)| \leq K|\xi_k|(1+|\xi|)^{-1}, \quad k = 1, 2, 3,$$
$$|\partial_\xi^\alpha g(\xi)| \leq K_\alpha(1+|\xi|)^{1-|\alpha|}, \quad |\alpha| \geq 2.$$

Next we put $G(\xi) = |\xi|^2/(g(\xi) + \overline{g(\xi)})$. Then, $g(\xi) - \overline{g(\xi)} = -2i \sin 2\tau G(\xi)$, and

$$|\partial_{\xi_k} G(\xi)| \leq K|\xi_k|(1+|\xi|)^{-1}, \quad k = 1, 2, 3,$$
$$|\partial_\xi^\alpha G(\xi)| \leq K_\alpha(1+|\xi|)^{1-|\alpha|}, \quad |\alpha| \geq 2.$$

Thus, (3.11) and (3.13) follow immediately. □

Lemma 3.6. *There exists $K_0 > 0$ such that*

$$K_0 s(c,\xi) \leq |p(c,\tau,\xi)| \leq K_0^{-1} s(c,\xi) \tag{3.14}$$

and

$$-2\tau \leq \arg p(c,\tau,\xi) \leq -\tau, \quad \tau \in [0, a_2], \tag{3.15}$$

$$-\tau \leq \arg p(c,\tau,\xi) \leq -2\tau, \quad \tau \in [-a_2, 0] \tag{3.16}$$

for $c \geq 1$ and $\xi \in \boldsymbol{R}^3$, where

$$s(c,\xi) := \frac{c|\xi|^2}{\sqrt{|\xi|^2 + c^2}}. \tag{3.17}$$

Proof. We give the proof for $\tau \in [0, a_2]$ only. Let us write

$$p(c,\tau,\xi) = mc^2 e^{-2\tau i} q\left(\frac{|\xi|^2}{m^2c^2}\right), \tag{3.18}$$

where $q(s) := e^{2\tau i}(\sqrt{e^{-2\tau i}s+1}-1)$ for $s \geq 0$. We note that $q'(s) = 2^{-1}(e^{-2\tau i}s+1)^{-1/2}$ and $q(0) = 0$, $q'(0) = 1/2$. Hence, since $0 \leq \arg(e^{-2\tau i}s+1)^{-1/2} \leq \tau$ for $s > 0$ because of $-2\tau \leq \arg(e^{-2\tau i}s+1) \leq 0$, we get

$$0 \leq \arg q'(s) \leq \tau \tag{3.19}$$

and so $0 \leq \arg q(s) \leq \tau$ for $s \geq 0$. Thus, (3.15) follows. Furthermore, we have $d|q(s)|^2/ds = 2\operatorname{Re} q(s)\overline{q'(s)} > 0$ for $s > 0$, which implies that $|q(s)|$ is strictly increasing. Writing $q(s) = s(\sqrt{e^{-2\tau i}s+1}+1)^{-1}$, we see that there exists $K_1 > 0$ independent of $\tau \in [0, a_2]$ such that $|q(s)| \leq K_1 s(s^{1/2}+1)^{-1}$ for all $s > 0$ and that there exists $s_1 > 0$ and $K_2 > 0$ independent of $\tau \in [0, a_2]$ such that $|q(s)| \geq K_1 s^{1/2}$ for $s \geq s_1$. On the other hand, since $q(s) = s/2 + O(s^2)$ as $s \to +0$, we see that there exists $s_2 > 0$ independent of τ such that $|q(s)| \geq s/4$ for $s \in [0, s_2]$. Consequently, since $|q(s)|$ is strictly increasing, we see that there exists $K_2 > 0$ such that $|q(s)| \geq K_2 s(1+s^{1/2})^{-1}$ uniformly in $\tau \in [0, a_2]$ and $s \geq 0$. Thus, by (3.18) we have the desired result. □

If $\tau \in [0, \pi/2)$, we see $-\tau \leq \arg(\sqrt{c^2 e^{-2\tau i}t + m^2c^4}) \leq 0$ for any $t \geq 0$. Thus, setting

$$N_0(\tau, c) := \{z; \operatorname{Im} z \geq \tan(-\tau)(\operatorname{Re} z + mc^2)\},$$

we have a corollary:

Corollary 3.7. *We have*

$$\operatorname{Num}(T_0(c, i\tau) - mc^2) \subset C(0, -2\tau, -\tau) \cap N_0(\tau, c)$$

for $\tau \in [0, a_2]$. *Similarly,*

$$\operatorname{Num}(T_0(c, i\tau) - mc^2) \subset C(0, -\tau, -2\tau) \cap N_0(-\tau, c)$$

for $\tau \in [-a_2, 0]$.

By Lemma 3.3 we can write

$$T_b(c, i\tau) - mc^2 = T_0(c, i\tau) - mc^2 + E_1(c, \tau) + E_2(c, \tau), \tag{3.20}$$

where $E_1(c, \tau) := -c^2 b_{i\tau}(x) \cdot e^{-i\tau} DS_0(c, i\tau)^{-1/2}$, and $\|E_2(c)\|$ is uniformly bounded in $c \geq 1$ and $|\tau| < \tau_0(c)$.

Lemma 3.8. *For any* $\varepsilon > 0$ *there exists* $K_\varepsilon > 0$ *such that*

$$|(E_1(c, \tau)f, f)| \leq \varepsilon|((T_0(c, i\tau) - mc^2)f, f)| + K_\varepsilon \|f\|^2 \tag{3.21}$$

holds for $f \in \mathcal{S}^2$, $c \geq 1$ *and* $|\tau| < \tau_0(c)$.

Proof. First we show that for any $\varepsilon > 0$ there exists $K_\varepsilon > 0$ independent of $c \geq 1$ and $\tau \in (-\tau_0(c), \tau_0(c))$ such that

$$|(E_1(c,\tau)f,f)| \leq \varepsilon|(S(c,D)f,f)| + K_\varepsilon\|f\|^2, \quad f \in \mathcal{S}^2. \tag{3.22}$$

Since $0 < a_0 < \pi/2$, we see that there exists $\delta_0 > 0$ such that

$$|e^{-2\tau i}\xi^2 + m^2c^2|^2 \geq \delta_0(\xi^2 + m^2c^2)^2, \quad \xi \in \boldsymbol{R}^3,$$

for $|\tau| \leq \tau_0(c)$ and $c \geq 1$. Thus, for any $\varepsilon > 0$ we have

$$\begin{aligned}|(E_1(c,\tau)f,f)| &= c|(D(-e^{-2\tau i}\Delta + m^2c^2)^{-1/4}f, (-e^{2\tau i}\Delta + m^2c^2)^{-1/4}b_{-i\tau}f)| \\ &\leq c\big\{\varepsilon\|D(-e^{-2\tau i}\Delta + m^2c^2)^{-1/4}f\|^2 \\ &\quad + (4\varepsilon)^{-1}\|(-e^{2\tau i}\Delta + m^2c^2)^{-1/4}b_{-i\tau}f\|^2\big\} \\ &\leq c\delta_1\varepsilon(D^2(-\Delta + m^2c^2)^{-1/2}f,f) + K'_\varepsilon\|f\|^2\end{aligned}$$

for some $\delta_1 > 0$ and $K'_\varepsilon > 0$, which implies (3.22). Write

$$\operatorname{Re}\big(e^{2\tau i}((T_0(c,i\tau) - mc^2)f,f)\big) = \int |p(c,\tau,\xi)|\cos\theta|\widehat{f}(\xi)|^2\,d\xi,$$

where $\theta = \arg(p(c,\tau,\xi)) + 2\tau$ and $\widehat{f}$ is the Fourier transform of f. Then, since $\cos\theta \geq \cos a_0 > 0$ by (3.15), it follows from (3.14) that $|((T_0(c,i\tau) - mc^2)f,f)| \geq K(s(c,D)f,f)$ for some $K > 0$ independent of $c \geq 1$ and $\tau \in (-\tau_0(c), \tau_0(c))$. Hence, combining this with (3.22) we complete the proof. □

Lemma 3.9. *For any small $\varepsilon > 0$ there exists $d_\varepsilon > 0$ such that*

$$\begin{aligned}\operatorname{Num}(T_b(c,i\tau) - mc^2) &\subset \mathcal{N}(C(0,-2\tau-\varepsilon,-\tau+\varepsilon),d_\varepsilon), \quad \tau \in [0,\tau_0(c)), \\ \operatorname{Num}(T_b(c,i\tau) - mc^2) &\subset \mathcal{N}(C(0,-\tau-\varepsilon,-2\tau+\varepsilon),d_\varepsilon), \quad \tau \in (-\tau_0(c),0]\end{aligned}$$

for $c \geq 1$.

Proof. We consider only the case $\tau \geq 0$. Set $z = ((T_b(c,i\tau) - mc^2)f,f)$ and $w = ((T_0(c,i\tau) - mc^2)f,f)$ for $f \in \mathcal{S}^2$ with $\|f\| = 1$. Then, it follows from (3.20) and Lemma 3.8 that for any small $\varepsilon > 0$ there exists $d_\varepsilon > 0$ such that $|z - w| \leq (\sin\varepsilon)|w| + d_\varepsilon$, and $w \in C(0,-2\tau,-\tau)$ by Corollary 3.7. Hence, the desired result follows immediately. □

Lemma 3.10. *Let $V_{i\tau} = v_{i\tau} + W_{i\tau}$ be the multiplication operator with domain $\mathcal{S}^2$. Then, for any $\varepsilon > 0$ there exists $d_\varepsilon > 0$ such that*

$$\begin{aligned}\operatorname{Num}(V_{i\tau}) &\subset \mathcal{N}(C(0,M_1\tau-\varepsilon,M_2\tau+\varepsilon),d_\varepsilon), \quad \tau \in [0,a_0), \\ \operatorname{Num}(V_{i\tau}) &\subset \mathcal{N}(C(0,M_2\tau-\varepsilon,M_1\tau+\varepsilon),d_\varepsilon), \quad \tau \in (-a_0,0].\end{aligned}$$

If $W = 0$, then $\varepsilon = 0$ is allowed in the above.

Proof. We give the proof only for $\tau \geq 0$. Let $f \in \mathcal{S}^2$ with $\|f\| = 1$ and set $w = (V_{i\tau}f, f)$ and $z = (v_{i\tau}\chi f, f)$, where χ is the characteristic function of the disk $|x| \geq R_0$. Obviously, $z \in C(0, M_1\tau, M_2\tau)$. It follows from (**v**) that

$$\left|\arg\left(v_{i\tau}(x)e^{-i(M_2+M_1)\tau/2}\right)\right| \leq (M_2 - M_1)\tau/2 < \pi/2$$

for $|x| \geq R_0$ and

$$\begin{aligned}|(v_{i\tau}\chi f, f)| &\geq \mathrm{Re}(e^{-i(M_2+M_1)\tau/2}v_{i\tau}\chi f, f)\\ &\geq K_0(|x|^M\chi f, f) \geq K_0(|x|^M f, f) - K_1,\end{aligned}$$

for some $K_0 > 0$ and $K_1 > 0$. Moreover, it follows from the condition (**W**) that for any $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that $|W_{i\tau}(x)| \leq \varepsilon|x|^M + K_\varepsilon$. Hence, by virtue of (1.9) we see that for any $\varepsilon > 0$ there exists $K'_\varepsilon > 0$ such that $|w - z| \leq \varepsilon|z| + K'_\varepsilon$. Using the same strategy as in the proof of Lemma 3.9, we have the desired result. □

For $\tau \in (0, a_0)$ and $\varepsilon \geq 0$ we define a sector

$$C_-(c,\tau,\varepsilon) := C(0, \theta_3(\tau) - \varepsilon, \theta_4(\tau) + \varepsilon), \tag{3.23}$$

where $\theta_3(\tau) := \min\{M_1\tau, \pi - 2\tau\}$ and $\theta_4(\tau) := \max\{\pi - \tau, M_2\tau\}$. It is easily seen that $C_-(c,\tau,0)$ is the smallest sector containing both $C(0, \pi - 2\tau, \pi - \tau)$ and $C(0, M_1\tau, M_2\tau)$. If we assume (1.5) and fix small $t_0 > 0$, then we can verify

$$0 \leq \theta_4(\tau) - \theta_3(\tau) \leq \theta_0 < \pi \tag{3.24}$$

for all $\tau \in (t_0, a_0)$ with some constant θ_0.

Combining Lemmas 3.9 and 3.10, we have the following result.

Lemma 3.11. *Let $L_\pm(c, i\tau)$ be defined on $\mathcal{S}^2$:*

(a) Suppose Assumption 1.1 with (1.4) and let $\varepsilon > 0$ be small. Then there exists $d_\varepsilon^{(+)} \geq 0$ such that

$$\mathrm{Num}(L_+(c,i\tau)) \subset \mathcal{N}(C(0, -2\tau - \varepsilon, M_2\tau + \varepsilon), d_\varepsilon^{(+)}), \quad \tau \in [0, \tau_0(c)), \tag{3.25}$$

$$\mathrm{Num}(L_+(c,i\tau)) \subset \mathcal{N}(C(0, M_2\tau - \varepsilon, -2\tau + \varepsilon), d_\varepsilon^{(+)}), \quad \tau \in (-\tau_0(c), 0], \tag{3.26}$$

for $c \geq 1$.

(b) Suppose Assumption 1.1 with (1.5) and let $\varepsilon > 0$ be small. Then there exists $d_\varepsilon^{(-)} \geq 0$ such that

$$\mathrm{Num}(L_-(c,i\tau) + 2mc^2) \subset \mathcal{N}(C_-(c,\tau,\varepsilon), d_\varepsilon^{(-)}), \quad \tau \in (0, \tau_0(c)), \tag{3.27}$$

for $c \geq 1$.

If $b=0$ and $W=0$, then $\varepsilon=0$ is allowed in (3.25), (3.26) and (3.27).

We write $L^0_\pm(c,\theta)$ for $L_\pm(c,\theta)$ with $b=0$ and $W=0$:

$$L^0_\pm(c,\theta)=\pm\left(\sqrt{-c^2e^{-2\theta}\Delta+m^2c^4}-mc^2\right)I_2+v_\theta(x)I_2.$$

Lemma 3.12. *Suppose Assumption 1.1 with (1.4) and fix a small $\delta>0$. Then there exist positive constants $\tilde K_1$, $\tilde K_2$ such that*

$$\begin{aligned}\|(L^0_+(c,i\tau)-z)f\|^2+\tilde K_1\|f\|^2\\ \geq \tilde K_2(\|s(c,D)f\|^2+\|\langle x\rangle^M f\|^2+|z|^2\|f\|^2)\end{aligned} \tag{3.28}$$

is valid for all $z\in C(0,-2\tau-\delta,M_2\tau+\delta)^{\mathrm c}$, $c\geq 1$, $\tau\in[0,a_0)$ and $f\in\mathcal S^2$. The same estimate holds for $\tau\in(-a_0,0]$ if $z\in C(0,M_2\tau-\delta,-2\tau+\delta)^{\mathrm c}$.

Proof. We prove the lemma only for $\tau\geq 0$. Take $\chi\in C^\infty(\boldsymbol R^3)$ with $0\leq\chi(x)\leq 1$, $\chi(x)=0$ for $|x|\leq R_0$ and $\chi(x)=1$ for $|x|\geq 2R_0$, and set $v^{(\infty)}_{i\tau}(x)=\chi(x)v_{i\tau}(x)$. Then, $v^{(\infty)}_{i\tau}(x)\in C^\infty$ and $\mathrm{Num}(v^{(\infty)}_{i\tau})\subset C(0,M_1\tau,M_2\tau)$. Thus,

$$L:=e^{-i(M_2-2)\tau/2}(T_0(c,i\tau)-mc^2+v^{(\infty)}_{i\tau})+\gamma$$

satisfies $\mathrm{Num}(L)\subset C(0,-(M_2+2)\tau/2,(M_2+2)\tau/2)$, where $\gamma>0$ is a fixed constant, since both $\mathrm{Num}(e^{-i(M_2-2)\tau/2}(T_0(c,i\tau)-mc^2))$ and $\mathrm{Num}(e^{-i(M_2-2)\tau/2}v^{(\infty)}_{i\tau})$ are contained in this sector. Since the operator norm $\|v_{i\tau}-v^{(\infty)}_{i\tau}\|$ is bounded uniformly in τ, the proof of (3.28) is reduced to that of a similar inequality with $L^0_+(c,i\tau)$ replaced by L and $C(0,-2\tau-\delta,M_2\tau+\delta)$ replaced by $C(0,-(M_2+2)\tau/2-\delta,(M_2+2)\tau/2+\delta)$, respectively. Let $z=x_1+ix_2\notin C(0,-(M_2+2)\tau/2-\delta,(M_2+2)\tau/2+\delta)$ and $f\in\mathcal S^2$ with $\|f\|=1$. Then, by the equality

$$\begin{aligned}&(L-z)^*(L-z)\\&\quad=(\operatorname{Re}L)^2+(\operatorname{Im}L)^2-2(x_1\operatorname{Re}L+x_2\operatorname{Im}L)+|z|^2+i[\operatorname{Re}L,\operatorname{Im}L]\end{aligned}$$

and by the inequality

$$\begin{aligned}(2(x_1\operatorname{Re}L+x_2\operatorname{Im}L)f,f)&=2\{(x_1(\operatorname{Re}Lf,f)+x_2(\operatorname{Im}Lf,f)\}\\&\leq 2(\cos\delta)|z|((\operatorname{Re}Lf,f)^2+(\operatorname{Im}Lf,f)^2)^{1/2}\\&\leq(\cos\delta)(|z|^2+\|\operatorname{Re}Lf\|^2+\|\operatorname{Im}Lf\|^2)\\&\leq(\cos\delta)((|z|^2+(\operatorname{Re}L)^2+(\operatorname{Im}L)^2)f,f),\end{aligned}$$

we arrive at

$$(L-z)^*(L-z)\geq(1-\cos\delta)((\operatorname{Re}L)^2+|z|^2)+i[\operatorname{Re}L,\operatorname{Im}L]$$

as the form sense on $\mathcal{S}^2$. Write

$$P := \mathrm{Re}(e^{-i(M_2-2)\tau/2}(T_0(c,i\tau) - mc^2)),$$
$$Q := \mathrm{Re}(e^{-i(M_2-2)\tau/2} v_{i\tau}^{(\infty)} + \gamma),$$

i.e., $\mathrm{Re}\, L = P + Q$. Then, since $P \geq K_1 s(c,D)$ and $Q \geq K_1 \langle x\rangle^M$ for some $K_1 > 0$ by Lemma 3.6 and $(M_2+2)a_0 < \pi$, we have

$$(\mathrm{Re}\, L)^2 = P^2 + Q^2 + 2Q^{1/2}PQ^{1/2} + R \geq P^2 + Q^2 + R$$
$$\geq K_1^2 s(c,D)^2 + K_1^2 \langle x\rangle^{2M} + R, \tag{3.29}$$

where $R := [P, Q^{1/2}]Q^{1/2} + Q^{1/2}[Q^{1/2}, P] \in \Sigma^{1,M-1}$ and

$$K_2 := \sup_{\tau\in[0,a_0), c\geq 1} \|\langle D\rangle^{-1} R\, \langle x\rangle^{-(M-1)}\| < \infty$$

by Lemma 3.5. Thus,

$$|(Rf,f)| \leq K_3 \|\langle x\rangle^{M-1} f\|\, \|\langle D\rangle f\|$$
$$\leq (K_3/2)(\varepsilon_1(\langle D\rangle^2 f, f) + \varepsilon_1^{-1}(\langle x\rangle^{2M-2} f, f))$$

for any $\varepsilon_1 > 0$. Here we note that for any $\varepsilon > 0$ there exists $K'_\varepsilon > 0$ such that $\langle x\rangle^{2M-2} \leq \varepsilon\langle x\rangle^{2M} + K'_\varepsilon$ and that $s(c,\xi)^2 + (1/4) \geq |\xi|^2/4$ because $s(c,\xi)^2 \geq |\xi|^2/4$ for $|\xi| \geq 1$ and $c \geq 1$. Hence, we see that for any small $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that

$$R \geq -\varepsilon(s(c,D)^2 + \langle x\rangle^{2M}) - K_\varepsilon. \tag{3.30}$$

Similarly, we have

$$\sup_{\tau\in[0,a_0), c\geq 1} \|\langle D\rangle^{-1}[\mathrm{Re}\, L, \mathrm{Im}\, L]\langle x\rangle^{-(M-1)}\| < \infty,$$

and so we see that for any $\varepsilon > 0$ there exists $K'_\varepsilon > 0$ such that

$$i[\mathrm{Re}\, L, \mathrm{Im}\, L] \geq -\varepsilon(s(c,D)^2 + \langle x\rangle^{2M}) - K'_\varepsilon. \tag{3.31}$$

Consequently, we have proved the lemma. □

Lemma 3.13. *Suppose Assumption 1.1 with (1.5) and fix small $\delta > 0$ and $t_0 > 0$. Then there are positive constants K_1, K_2 such that*

$$\|(L^0_-(c,i\tau) + 2mc^2 - z)f\|^2 + K_1\|f\|^2$$
$$\geq K_2(\|s(c,D)f\|^2 + \|\langle x\rangle^M f\|^2 + |z|^2\|f\|^2) \tag{3.32}$$

is valid for all $z \in C_-(c,\tau,\delta)^{\mathrm{c}}$, $c \geq 1$, $\tau \in [t_0, a_0)$ and $f \in \mathcal{S}^2$.

Proof. The proof is carried out in a similar way to that of the previous lemma. Let

$$\widetilde{L} := e^{-i(\theta_3(\tau)+\theta_4(\tau))/2}(-T_0(c,i\tau) + mc^2 + v_{i\tau}^{(\infty)}) + \gamma$$

with a fixed constant $\gamma > 0$. Then,

$$\mathrm{Num}(\widetilde{L}) \subset C(0, -(\theta_4(\tau) - \theta_3(\tau))/2, (\theta_4(\tau) - \theta_3(\tau))/2)$$

since $\mathrm{Num}(-T_0(c,i\tau) + mc^2) \subset C(0, \pi - 2\tau, \pi - \tau)$, and it suffices to prove (3.32) with $L_-^0(c,i\tau)$ replaced by $\widetilde{L}$ and $C_-(c,\tau,\delta)$ by $C(0, -(\theta_4(\tau)-\theta_3(\tau))/2-\delta, (\theta_4(\tau)-\theta_3(\tau))/2+\delta)$. For $z = x_1+ix_2 \notin C(0, -(\theta_4(\tau)-\theta_3(\tau))/2-\delta, (\theta_4(\tau)-\theta_3(\tau))/2+\delta)$, we have in the same way as in the proof of Lemma 3.12,

$$(\widetilde{L} - z)^*(\widetilde{L} - z) \geq (1 - \cos\delta)((\mathrm{Re}\,\widetilde{L})^2 + |z|^2) + i[\mathrm{Re}\,\widetilde{L}, \mathrm{Im}\,\widetilde{L}].$$

Write

$$\begin{aligned}\widetilde{P} &:= \mathrm{Re}(e^{-i(\theta_3(\tau)+\theta_4(\tau))/2}(-T_0(c,i\tau) + mc^2)),\\ \widetilde{Q} &:= \mathrm{Re}(e^{-i(\theta_3(\tau)+\theta_4(\tau))/2}v_{i\tau}^{(\infty)} + \gamma).\end{aligned}$$

Since $\widetilde{P} \geq \widetilde{K}_0 s(c,D) > 0$ and $\widetilde{Q} \geq \widetilde{K}_0\langle x\rangle^M > 0$ for some $\widetilde{K}_0 > 0$ due to (3.24), we get

$$(\mathrm{Re}\,\widetilde{L})^2 \geq \widetilde{P}^2 + \widetilde{Q}^2 + \widetilde{R} \geq K_1(s(c,D)^2 + \langle x\rangle^{2M}) + \widetilde{R}$$

for some $K_1 > 0$, where $\widetilde{R} := [\widetilde{P}, \widetilde{Q}^{1/2}]\widetilde{Q}^{1/2} + \widetilde{Q}^{1/2}[\widetilde{Q}^{1/2}, \widetilde{P}]$. The rest of the proof is done in the same way as that of the previous lemma. □

We next prove the same results for $L_\pm(c,i\tau)$ by using (3.20) and (**W**). Write

$$\begin{aligned}\|E_1(c,i\tau)f\|^2 &\leq K\left(\frac{c^2|\xi|^2}{|\xi|^2+c^2}\widehat{f}, \widehat{f}\right)\\ &\leq K'\left\{\left(\frac{1}{\varepsilon}\frac{c^2}{|\xi|^2+c^2}\widehat{f}, \widehat{f}\right) + \left(\varepsilon\frac{c^2|\xi|^4}{|\xi|^2+c^2}\widehat{f}, \widehat{f}\right)\right\}\\ &\leq K'(\varepsilon^{-1}\|f\|^2 + \varepsilon\|s(c,D)f\|^2)\end{aligned}$$

for any $\varepsilon \in (0,1)$ and $f \in \mathcal{S}^2$. Moreover, for any $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that

$$\|W_{i\tau}f\| \leq \varepsilon\|\langle x\rangle^M f\| + K_\varepsilon\|f\|$$

for any $f \in \mathcal{S}^2$, $\varepsilon \in (0,1)$, $\tau \in [0, \tau_0(c))$. Therefore, we obtain the following.

Lemma 3.14. *Suppose Assumption 1.1 with (1.4) and fix a small $\delta > 0$. Then there exist positive constants $\tilde{K}_1$, $\tilde{K}_2$ such that*

$$\begin{aligned}\|(L_+(c,i\tau) - z)f\|^2 &+ \tilde{K}_1\|f\|^2\\ &\geq \tilde{K}_2(\|s(c,D)f\|^2 + \|\langle x\rangle^M f\|^2 + |z|^2\|f\|^2)\end{aligned} \tag{3.33}$$

is valid for all $z \in C(0, -2\tau - \delta, M_2\tau + \delta)^c$, $c \geq 1$, τ *with* $\tau \in [0, \tau_0(c))$ *and* $f \in \mathcal{S}^2$. *The same estimate holds for* $\tau \in (-\tau_0(c), 0]$ *if* $z \in C(0, M_2\tau - \delta, -2\tau + \delta)^c$.

Lemma 3.15. *Suppose Assumption 1.1 with (1.5) and fix small* $\delta > 0$ *and* $t_0 > 0$. *Then there are positive constants* K_1, K_2 *such that*

$$\begin{aligned} &\|(L_-(c, i\tau) + 2mc^2 - z)f\|^2 + K_1\|f\|^2 \\ &\qquad \geq K_2(\|s(c, D)f\|^2 + \|\langle x\rangle^M f\|^2 + |z|^2\|f\|^2) \end{aligned} \tag{3.34}$$

is valid for all $z \in C_-(c, \tau, \delta)^c$, $c \geq 1$, $f \in \mathcal{S}^2$ *and* $\tau \in [t_0, \tau_0(c))$.

Proof of Theorem 1.6. By Lemma 3.3 we may assume $b = 0$. In this case the theorem has been proved in [11] when $W = 0$ (the dilation analyticity of v is not supposed in [11]) . Thus, we give only an outline of the proof. Since both $v(x)$ and $W(x)$ are smooth for $|x| > R_0$, we may assume both of them are smooth on $\boldsymbol{R}^3$. We first note that $L_{00} = \sqrt{-c^2\Delta + m^2c^4} + v(x)$ in $L^2(\boldsymbol{R}^3)$ is essential self-adjoint on $\mathcal{S}$ (see [11, Thm. 2.1]) due to assumption **(v)** and that the domain is D_M by Lemma 3.12. Thus, W is $L_{00}I_2$-bounded with bound 0, and so $L_+(c)$ is essential self-adjoint on $\mathcal{S}^2$. We next consider $L_-(c)$. The proof is carried out by using the commutator theorem (e.g., [20, Thm. X.37]) as in [11]. Let $A := L_-(c)$ with $b = 0$ defined on $\mathcal{S}^2$ and $N := A + 2\kappa\langle x\rangle^M I_2$ for a large $\kappa > 0$ such that $-v(x)I_2 + 2\kappa\langle x\rangle^M I_2 \geq \kappa\langle x\rangle^M I_2$. Then, by the same argument as above, N is self-adjoint with domain D_M. We have $N \geq 1$ for large $\kappa > 0$, and it is not difficult to prove that there exists $K > 0$ such that $\|Af\| \leq K\|Nf\|$ and

$$|(Af, Nf) - (Nf, Af)| \leq K\|N^{1/2}f\|^2$$

for all $f \in \mathcal{S}^2$. Then, [20, Thm. X.37] implies that A is essential self-adjoint on $\mathcal{S}^2$. □

Taking account of (3.9) and using the results obtained above, we can give the proof of Proposition 3.4.

Proof of Proposition 3.4.

(a) It follows from Lemma 3.14 that $L_+(c, \theta)$ defined on $\mathcal{S}^2$ is closable and the domain of the closure is D_M. Thus, since $\rho(L_+(c, \theta)) \neq \phi$ by Lemma 3.11, it has compact resolvent by Rellich's criterion.

(b) It is easy to see that $\{S_b(c, \theta)\}_{\theta\in\Omega}$ is an analytic family of type (A), and so is $\{T_b(c, \theta)\}_{\theta\in\Omega(c)}$ due to (3.4). Therefore, according to our assumptions, $\{L_+(c, \theta)\}_{\theta\in\Omega(c)}$ is also an analytic family of type (A).

(c), (d), (e) In the same way as above we obtain the desired results by Lemmas 3.11 and 3.15.

(f) By the resolvent equation we have for $\operatorname{Im} z < -r_0$ with large $r_0 > 0$,

$$(L_-(c,\theta) - z)^{-1} - (L_-(c) - z)^{-1} = -(L_-(c,\theta) - z)^{-1}(L_-(c,\theta) - L_-(c))(L_-(c) - z)^{-1}. \tag{3.35}$$

We have that $(L_-(c) - z)\mathcal{S}^2$ is dense in L^2 since $\mathcal{S}^2$ is core of $L_-(c)$, and $\|(L_-(c,\theta) - z)^{-1}\|$ is uniformly bounded in θ by Lemma 3.11. We also see that $L_-(c,\theta)f \to L_-(c)f$ strongly as $\Omega_+(c) \ni \theta \to 0$ for $f \in \mathcal{S}^2$. Hence, (3.8) follows from (3.35). This completes the proof.

□

Now we are in a position to prove Theorems 1.7 and 1.8 with the help of Theorem 2.3 as follows: $\mathcal{H} = L^2(\boldsymbol{R}^3)^2$, $\boldsymbol{\Omega}_+ = \Omega_+(c)$, $T = L_\pm(c)$, $T(\theta) = L_\pm(c,\theta)$, $U(t) = \mathcal{U}_2(t)$. Indeed, Proposition 3.4 shows that (A1)∼(A4) are satisfied in this case. It will be shown below that $L_+(c)$ is of type (I) and $L_-(c)$ is of type (II) in the sense of Theorem 2.3.

Proof of Theorem 1.7. Taking account of Proposition 3.4, we have the theorem as an immediate consequence of Theorem 2.3. □

Proof of Theorem 1.8. Since Proposition 3.4 shows that $L_-(c)$ is a boundary value of $\{L_-(c,\theta)\}_{\theta\in\Omega_+(c)}$, we have only to prove that $L_-(c)$ is of type (II). By Lemma 3.11, $\operatorname{Num}(L_-(c,i\tau)) \cap \boldsymbol{R}$ is bounded. But, if $L_-(c)$ is of type (I), the set contains all of the eigenvalues of $L_-(c)$, which is impossible. Thus, $L_-(c)$ is of type (II). □

Proof of Corollary 1.10. We may consider the case $\theta_1 = i\tau_1$ with $\tau_1 \in (0, \tau_0(c))$. Denote $T^0 := -T_0(c, i\tau_1) - mc^2$ for simplicity. Then, by Lemma 3.6 we have $\operatorname{Im} T^0 \geq K_1 s(c,D) \geq K_1 s(1,D) > 0$ for some $K_1 > 0$, and so $\operatorname{Im} T^0 + \operatorname{Im} V_{i\tau_1} > 0$. Thus, there is no eigenvalue of $L_-(c)$. Since $K_1 s(1,D) + \operatorname{Im} V_{i\tau_1}$ is positive and has compact resolvent, if we denote by $\lambda_0 > 0$ the lowest eigenvalue of it, then $\operatorname{Im} T^0 + \operatorname{Im} V_{i\tau_1} \geq \lambda_0$. Thus, we have proved the corollary. □

Proof of Theorem 1.12. First of all we note that the set of eigenvalues of $L_-(c,i\tau) + 2mc^2$ are contained in $\operatorname{Num}(L_-(c,i\tau) + 2mc^2)$ and that it is independent of $\tau \in (0, \tau_0(c))$. Thus, according to (3.27), for any $\varepsilon > 0$ there exists $d \geq 0$ such that it is contained in the set

$$\bigcap_{0 \leq \tau < \tau_0(c)} \mathcal{N}(C_-(c,\tau,\varepsilon), d) = \bigcap_{t_0 \leq \tau < \tau_0(c)} \mathcal{N}(C_-(c,\tau,\varepsilon), d)$$

for small $t_0 > 0$, where the equality follows from the fact that $C_-(c,\tau_2,\varepsilon)$ is contained in $C_-(c,\tau_1,\varepsilon)$ if $0 < \tau_1 < \tau_2$ are small. Here we remark that $M_1\tau$ and

$\pi-2\tau$ are increasing and decreasing, respectively, in τ, and so $\theta_3(\tau)$ has a maximum when $M_1\tau = \pi-2\tau$ if $0 < \tau < \tau_0(c)$. Thus

$$\sup_{t_0 \le \tau < \tau_0(c)} \theta_3(\tau) = \Theta_1(c),$$

and similarly

$$\inf_{t_0 \le \tau < \tau_0(c)} \theta_4(\tau) = \Theta_2(c).$$

Hence, we have the desired result. □

§4. Dilated Pauli operators

In this section we study the dilated Pauli operator $P_\pm(\theta)$ defined by (1.18) for $\theta \in \Omega$. First of all we consider $P_+(\theta)$.

Proposition 4.1. *(a) $P_+(\theta)$ defined on $\mathcal{S}^2$ is closable, and its closure (also denoted by $P_+(\theta)$) has domain $D(P_+(\theta)) = (H^2 \cap L^2_M)^2$.*

(b) $P_+(\theta)$ has compact resolvent. In particular, $P_+(\theta)$ has a purely discrete spectrum.

(c) $\{P_+(\theta)\}_{\theta\in\Omega}$ is an analytic family of type (A), and

$$\mathcal{U}_2(t)P_+(\theta)\mathcal{U}_2(t)^{-1} = P_+(\theta + t)$$

for all $\theta \in \Omega$ and $t \in \boldsymbol{R}$.

(d) The spectrum of $P_+(\theta)$ is independent of θ and, in particular, coincides with that of $P_+(0)$, denoted by P_+.

(e) P_+ is self-adjoint.

Remark 4.2. This proposition shows that the Pauli operator $P_+ = P_+(0)$ is a boundary value of $\{P_+(c,\theta)\}_{\theta\in\Omega_+}$ and of type (I) in Theorem 2.3.

Outline of the proof. Writing $P_0(\theta) := (2m)^{-1}(\sigma\cdot D^b_\theta)^2$ we have, on $\mathcal{S}^2$, the equality

$$(T_b(c,\theta) + mc^2)^2(T_b(c,\theta) - mc^2 - P_0(\theta)) = -2mc^2P_0(\theta)^2,$$

and hence

$$T_b(c,\theta) - mc^2 - P_0(\theta) = ((T_b(c,\theta) + mc^2)^{-1})^2(-2mc^2P_0(\theta)^2). \tag{4.1}$$

Thus, because the estimate

$$\|(T_b(c,\theta) + mc^2)^{-1}\| \le Kc^{-2} \tag{4.2}$$

follows from Lemma 3.9, we see that $L_+(c,\theta)f \to P_+(\theta)f$ as $c \to \infty$ for each $f \in \mathcal{S}^2$ and $\theta \in \Omega$. Thus, since $s(c,\xi) \to |\xi|^2$ as $c \to \infty$, we have by Lemma 3.14,

$$\|(P_+(\theta) - z)f\|^2 + \|f\|^2 \geq K_1(\|\Delta f\|^2 + \|\langle x\rangle^M f\|^2), \quad f \in \mathcal{S} \tag{4.3}$$

for some $K_1 > 0$. Using this estimate and arguments similar to those in the proof of Proposition 3.4, we can obtain (a) $\sim$ (d). In particular, $D(P_+(\theta)) = (H^2)^2 \cap (L^2_M)^2$ follows from (4.3). It is well known that $S_{00} := (-(2m)^{-1}\Delta + v(x))I_2$ on $\mathcal{S}^2$ is essentially self-adjoint in $(L^2)^2$, and it follows that $P_+(0) - S_{00}$ is S_{00}-bounded with relative bound zero, and so P_+ is essentially self-adjoint on $\mathcal{S}^2$. □

Proof of Proposition 1.14. Proposition 1.14 follows from the above proposition and Theorem 2.3 immediately. □

To study the nonrelativistic limit of $L_+(c,\theta)$, we first consider $T_b(c,\theta) - mc^2$. Lemmas 3.2 and 3.9 guarantee that we can take a bounded open set $\mathcal{B}_0$ contained in $\rho(P_0(\theta))$ and $\rho(T_b(c,\theta))$ for all $\theta \in \Omega$ and large $c \geq 1$.

Lemma 4.3. *Let $\mathcal{B}_0$ be as above. Then, there is a constant $K > 0$ such that*

$$\sup_{z\in\mathcal{B}_0} \|(T_b(c,\theta) - mc^2 - z)^{-1} - (P_0(\theta) - z)^{-1}\| \leq Kc^{-2} \tag{4.4}$$

for $\theta \in \Omega$ and large $c \geq 1$.

Proof. According to (4.1), we can write $(T_b(c,\theta) - mc^2 - z)^{-1} - (P_0(\theta) - z)^{-1} = F_1F_2F_3$. Here, $F_1 = (T_b(c,\theta)+mc^2)(T_b(c,\theta)-mc^2-z)^{-1}$, $F_2 = c^2(T_b(c,\theta)+mc^2)^{-1}$ and $F_3 = 2mP_0(\theta)^2(T_b(c,\theta)+mc^2)^{-2}(P_0(\theta)-z)^{-1}$. By the use of (4.2), we observe that $\|F_1\|$, $\|F_2\|$ and $c^4\|F_3\|$ are uniformly bounded in $\theta \in \Omega$, large $c \geq 1$ and $z \in \mathcal{B}_0$, which proves the lemma. □

Since the numerical range of $L_+(c,\theta)$ is contained in a sector independent of c (Lemma 3.11) and since $L_+(c,\theta)f$ converges to $P_+(\theta)f$ strongly for each $f \in \mathcal{S}^2$, the numerical range of $P_+(\theta)f$ is contained in the same sector. Thus, we can find a bounded open set $\mathcal{B}_1$ contained in $\rho(P_+(\theta))$ and $\rho(L_+(c,\theta))$ for all $\theta \in \Omega$ and large $c \geq 1$. Using the above lemma we can prove that $L_+(c,\theta)$ converges to $P_+(\theta)$ in the norm resolvent sense.

Lemma 4.4. *Let $\mathcal{B}_1$ be as above. Then there is a constant $K > 0$ such that*

$$\sup_{z\in\mathcal{B}_1} \|(L_+(c,\theta) - z)^{-1} - (P_+(\theta) - z)^{-1}\| \leq Kc^{-2}$$

for large $c \geq 1$.

Proof. Let $z \in \mathcal{B}_1$. Then the following resolvent equation holds:

$$(4.5)\qquad \begin{aligned}&(L_+(c,\theta)-z)^{-1}-(P_+(\theta)-z)^{-1}\\ &\quad=-(L_+(c,\theta)-z)^{-1}(L_+(c,\theta)-P_+(\theta))(P_+(\theta)-z)^{-1}\end{aligned}$$

on $(P_+(\theta)-z)\mathcal{S}^2$. Since $\mathcal{S}^2$ is a core of $P_+(\theta)$ it holds on the whole space $(L^2)^2$. Similarly, we have

$$(4.6)\qquad \begin{aligned}(P_+(\theta)-z)^{-1}&=(P_0(\theta)-z)^{-1}\\ &\quad-(P_0(\theta)-z)^{-1}(v_\theta I_2+W_\theta)(P_+(\theta)-z)^{-1},\end{aligned}$$

$$(4.7)\qquad \begin{aligned}(L_+(c,\theta)-z)^{-1}&=(T_b(c,\theta)-mc^2-z)^{-1}\\ &\quad-(L_+(c,\theta)-z)^{-1}(v_\theta I_2+W_\theta)(T_b(c,\theta)-mc^2-z)^{-1}.\end{aligned}$$

Now it follows from

$$(L_+(c,\theta)-z)^{-1}(v_\theta I_2+W_\theta)\subset((v_{\overline{\theta}}I_2+W_{\overline{\theta}})(L_+(c,\overline{\theta})-\overline{z})^{-1})^*$$

and $D(L_+(c,\theta))=D_M\subset D(v_\theta)$ that $(L_+(c,\theta)-z)^{-1}(v_\theta I_2+W_\theta)$ can be considered a bounded operator, and so it follows from Lemma 3.14 that $\|(L_+(c,\theta)-z)^{-1}(v_\theta I_2+W_\theta)\|$ is uniformly bounded for $\theta\in\Omega$ and large $c\geq 1$. Thus substituting (4.6) and (4.7) into the right-hand side of (4.5) and using the inequality

$$\|(P_0(\theta)-z)^{-1}(L_+(c,\theta)-P_+(\theta))(T_b(c,\theta)-z)^{-1}\|\leq Kc^{-2}$$

due to (4.4) , we arrive at the desired result. □

Proposition 4.5. *Let $\mathcal{B}$ be an arbitrary compact set in $\rho(P_+)$ and fix $\theta\in\Omega$. Then there are constants $c_0>0$ and $K>0$ such that $\mathcal{B}\subset\rho(L_+(c,\theta))$ for $c\geq c_0$ and*

$$\sup_{z\in\mathcal{B}}\|(L_+(c,\theta)-z)^{-1}-(P_+(\theta)-z)^{-1}\|\leq Kc^{-2}$$

for $c\geq c_0$.

Proof. Lemma 4.4 implies that $L_+(c,\theta)$ converges to $P_+(\theta)$ in the generalized sense and so the proposition follows immediately from [17, Thm. 2.25 and (3.10) in Chap. IV], since $\rho(P_+)=\rho(P_+(\theta))$. □

This result implies that for each eigenvalue λ (with multiplicity n) of P_+ there exist n eigenvalues (counting multiplicity) $\lambda_j(c)$, $j=1,\ldots,n$ of $L_+(c)$ near λ for large c and $\lambda_j(c)\to\lambda$ as $c\to\infty$.

We next consider $P_-(\theta)$.

Proposition 4.6. *Let $\theta \in \Omega_+$.*

(a) $P_-(\theta)$ defined on $\mathcal{S}$ is closable and its closure (also denoted by $P_-(\theta)$) has the domain $D(P_-(\theta)) = (H^2 \cap L^2_M)^2$.

(b) The resolvent set of $P_-(\theta)$ is not empty and its resolvent is compact. In particular, $P_-(\theta)$ has a purely discrete spectrum.

(c) $\{P_-(\theta)\}_{\theta \in \Omega_+}$ is an analytic family of type (A), and

$$\mathcal{U}(t)P_-(\theta)\mathcal{U}(t)^{-1} = P_-(\theta + t)$$

for all $\theta \in \Omega_+$ and $t \in \boldsymbol{R}$.

(d) The spectrum of $P_-(\theta)$ is independent of θ, denoted by $\widetilde{\Sigma}$.

Remark 4.7. We call an element of $\widetilde{\Sigma}$ a resonance of P_- even if P_- defined on $\mathcal{S}^2$ does not necessarily have a unique self-adjoint extension.

Outline of the proof. We see that $(L_-(c,\theta)+2mc^2)f \to P_-(\theta)f$ as $c \to \infty$ for each $f \in \mathcal{S}^2$ and $\theta \in \Omega_+$. Thus, since $s(c,\xi) \to |\xi|^2$ as $c \to \infty$, we have by Lemma 3.15,

$$\|(P_-(\theta) - z)f\|^2 + \|f\|^2 \geq K_1(\|\Delta f\|^2 + \|\langle x\rangle^M f\|^2), \quad f \in \mathcal{S}^2, \tag{4.8}$$

for some positive constants K_1. Using the estimate we can prove the proposition in the same way as Proposition 3.4. $\square$

Moreover, as in the proof of Proposition 4.5 we can prove the next result:

Proposition 4.8. *Let $\mathcal{B}$ be a compact set in $\boldsymbol{C} \setminus \widetilde{\Sigma}$ and fix $\theta \in \Omega_+$. Then there are constants $c_0 > 0$ and $K > 0$ such that $\mathcal{B} \subset \rho(L_-(c,\theta) + 2mc^2)$ for $c \geq c_0$ and*

$$\sup_{z \in \mathcal{B}} \|(L_-(c,\theta) + 2mc^2 - z)^{-1} - (P_-(\theta) - z)^{-1}\| \leq Kc^{-2}$$

for $c \geq c_0$.

By Propositions 4.6 and 4.8 we have the following corollary:

Corollary 4.9. *(a) $\widetilde{\Sigma} \subset \overline{\boldsymbol{C}_+}$.*

(b) If P_- defined on $\mathcal{S}^2$ is an essentially self-adjoint operator, then the self-adjoint extension (also denoted by $\widetilde{S}$) is of type (II) as a boundary value of the analytic family $\{P_-(\theta)\}_{\theta \in \Omega_+}$.

Proof. (a) If there exists an eigenvalue of $\widetilde{\Sigma}$ in $\boldsymbol{C}_-$, then Proposition 4.8 implies that there exist eigenvalues of $L_-(c,\theta) + 2mc^2$ near it for large $c > 0$. But, this contradicts the fact that the eigenvalues of $L_-(c,\theta)$ are all in $\overline{\boldsymbol{C}_+}$. Hence we have proved (a).

(b) Taking account of the fact that the numerical range of $P_-(\theta)$ is contained in the cone $\{w \in \boldsymbol{C}; A_1 \operatorname{Im}\theta < \arg(w-w_0) < \pi - A_1 \operatorname{Im}\theta\}$ for some $w_0 \in \boldsymbol{C}$ and $A_1 > 0$, we can prove, as in the proof of Proposition 3.4(f), that $(P_-(\theta)-z)^{-1}f$ converges to $(P_- - z)^{-1}f$ strongly as $\Omega_+ \ni \theta \to 0$ for all $f \in L^2(\boldsymbol{R}^3)$ and for all z with $(-\operatorname{Im} z) > 0$ sufficient large. Hence we can prove (b) as in the proof of Theorem 1.8. □

Remark 4.10. If $b = 0$ and $W = 0$, then $P_- = -(-(2m)^{-1}\Delta - v(x))I_2$. The essential self-adjointness and the absolute continuity of the spectrum of $-\Delta - v_0(x)$ with $v_0(x) \to \infty$ and $v_0(x) = O(|x|^M)$ as $|x| \to \infty$ under the condition $M \le 2$ have been studied by many papers (see, e.g., [26] and its references). In the case of P_- the essential self-adjointness can be proved in the same way as [20, Thm. X.38] (see also [13]).

Hence, we omit the proof of the following:

Proposition 4.11. *If $M \le 2$, then P_- defined on $\mathcal{S}^2$ is essentially self-adjoint.*

Proof of Proposition 1.15. Proposition 1.15 follows from Corollary 4.9, Proposition 4.11 and Theorem 2.3. □

§5. Nonrelativistic limits

Proofs of Theorems 1.17 and 1.18. We give the proof for Theorem 1.18 only. We already proved Theorem 1.17 after the proof of Proposition 4.5. Let $\varepsilon > 0$ be small. Then, it follows from Proposition 4.8 that there is no resonance of $L_-(c,\theta) + 2mc^2$ in $\mathcal{O} \setminus \left(\cup_{k=1}^{N_-} B_\varepsilon(\mu_k)\right)$ for large c.

Let

$$P_k(A) := \frac{-1}{2\pi i} \int_{|z-\mu_k|=\varepsilon} (A-z)^{-1}\, dz$$

be the eigenprojection for an operator A associated with the eigenvalues in the open disc $|z-\mu_k| < \varepsilon$. Then, by Proposition 4.8, we have

$$\lim_{c\to\infty} \|P_k(L_-(c,\theta) + 2mc^2) - P_k(P_-(\theta))\| = 0. \tag{5.1}$$

Thus, since $\dim P_k(P_-(\theta)) = \dim P_k(P_-)$, we have $\dim P_k(L_-(c,\theta) + 2mc^2) = n_k$ if c is large, and hence we have proved the theorem. □

We discuss the result of Theorem 1.12 for a simple case. To do so we prepare an elementary lemma.

Lemma 5.1. *Fix* $a_2 \in (0, \pi/2)$*. Then, for* $|\operatorname{Im}\theta| \le a_2$*,*

$$\left\| \sqrt{-c^2 e^{-2\theta}\Delta + m^2 c^4} - c e^{-\theta}\sqrt{-\Delta} \right\| \le \frac{mc^2}{\cos a_2}.$$

Proof. It suffices to prove

$$\sup_{\xi \in \boldsymbol{R}^3} \left| \sqrt{e^{-2i\tau} c^2 |\xi|^2 + m^2 c^4} - c e^{-i\tau}|\xi| \right| \le \frac{mc^2}{\cos a_2}$$

for $|\tau| \le a_2$. We write

$$\left| \sqrt{e^{-2i\tau} c^2 |\xi|^2 + m^2 c^4} - c e^{-i\tau}|\xi| \right| = \frac{m^2 c^4}{|e^{i\tau} p(c, \tau, \xi) + e^{i\tau} mc^2 + c|\xi|\,|}$$

and use Lemma 3.6 to have $\operatorname{Re}[e^{i\tau} p(c, \tau, \xi)] \ge 0$. Then, we have the desired result. □

Let $b = 0$, $W = 0$ and $v(x) = \kappa_0 |x|^M$ with $M \ge 2$ and $\kappa_0 > 0$. Then, $M_1 = M_2 = M$ and $2\pi(M+2)^{-1} \le \pi/2$. In this case we write $L_-(c, \theta) = L_{0-}(m, c, \theta)$, where

$$L_{0-}(m, c, \theta) := -\sqrt{-c^2 e^{-2\theta}\Delta + m^2 c^4} - mc^2 + \kappa_0 e^{M\theta}|x|^M.$$

Here we consider the mass $m > 0$ as a parameter. Since a_0 can be arbitrarily chosen if $0 < a_0 < 2\pi(M+2)^{-1}$ is satisfied, we choose a_0 sufficiently near $2\pi(M+2)^{-1}$, so that $\Theta_1(c) = M\pi/(M+2)$ and $\Theta_2(c) = M\pi/(M+1)$. We may set $\varepsilon = 0$ and $d = 0$ in Theorem 1.12. Thus, the resonances of $L_-(c) + 2mc^2$ are contained in the sector

$$C\Big(0, \frac{M\pi}{M+2}, \frac{M\pi}{M+1}\Big). \tag{5.2}$$

The corresponding P_- is $P_{0-} := (2m)^{-1}\Delta + \kappa_0|x|^M$ and its resonances are on the half-line $e^{iM(M+2)^{-1}\pi}[0, \infty)$ (see Remark 1.16). Since each resonance of $L_{0-}(m, c, 0) + 2mc^2$ converges to some resonance of P_{0-} as $c \to \infty$, there exist many resonances of $L_{0-}(m, c, 0) + 2mc^2$ near the half-line for large c. We next define

$$S_\pm(\theta) := \pm c e^{-\theta}\sqrt{-\Delta} + \kappa_0 e^{M\theta}|x|^M.$$

Then, according to the above lemma, $S_\pm(\theta)$ have similar properties as $L_\pm(c, \theta)$. Indeed, $S_+(\theta)$ is a closed operator with domain $D(S_+(\theta)) = D_M$ for $|\operatorname{Im}\theta| < \pi/(M+1)$ and $S_-(\theta)$ is also a closed operator with the same domain $D(S_-(\theta)) = D_M$ for $0 < \operatorname{Im}\theta < 2\pi/(M+1)$. Moreover, they are analytic families of type (A) in θ. Further, $S_+(0)$ is a positive self-adjoint operator with compact resolvent, and the eigenvalues are independent of θ. Here we observe that the following equality is valid:

$$S_-(\theta) = e^{iM\pi(M+1)^{-1}} S_+(\theta - i\pi/(M+1))$$

for $|\operatorname{Im}\theta| < 2\pi/(M+1)$. Thus, the eigenvalues of $S_-(\theta)$ are on the half-line $e^{iM\pi(M+1)^{-1}}[0,\infty)$. By the previous lemma we can also see that $L_{0-}(m,c,\theta)+2mc^2$ converges to $S_-(\theta) = -ce^{-\theta}\sqrt{-\Delta}+e^{M\theta}|x|^M$ in the norm-resolvent sense as $m \to 0$. Thus, there exist many resonances of $L_-(c)+2mc^2$ near the half-line for each c if m is small. Consequently, we know that the complement of the sector (5.2) is an optimal resonance-free region in this case.

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