

Convexity Properties of Intersections of Decreasing Sequences of q -Complete Domains in Complex Spaces

by

Mihnea COLȚOIU and Cezar JOIȚA

Abstract

We construct a decreasing sequence of 3-complete open subsets in \mathbb{C}^5 such that the interior of their intersection is not 3-complete. We also prove that, for every $q \geq 2$, there exists a normal Stein space X with only one isolated singularity and a decreasing sequence of open sets that are 2-complete, but the interior of their intersection is not q -complete with corners. In the concave case we show that, for every integer $n > 1$, there exists a connected complex manifold M of dimension n such that M is an increasing union of 1-concave open subsets and M is not weakly $(n-1)$ -concave.

2010 Mathematics Subject Classification: Primary 32F10; Secondary 32E10.

Keywords: Stein space, q -complete space, q -concave space.

§1. Introduction

Suppose that $\{D_\nu\}$ is a sequence of open subsets of \mathbb{C}^n and let $D := \text{Int}(\bigcap D_\nu)$. If each D_ν is a domain of holomorphy, then D is also a domain of holomorphy. More generally, if each D_ν is Hartogs q -convex (see Definition 6) then D has the same property. However Hartogs q -convexity is not a very useful notion since one does not get vanishing results for the cohomology groups of a Hartogs q -convex domain with values in a coherent sheaf. Andreotti and Grauert [1] introduced the notion of q -complete complex spaces and proved that they are cohomologically q -complete. In their setting, 1-complete spaces are precisely the Stein spaces. In general the intersection of finitely many q -complete domains is not q -complete.

Communicated by T. Mochizuki. Received November 8, 2016. Revised April 10, 2017.

M. Colțoiu: Simion Stoilow Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, Bucharest 014700, Romania;

e-mail: Mihnea.Coltoiu@imar.ro

C. Joița: Simion Stoilow Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, Bucharest 014700, Romania;

e-mail: Cezar.Joita@imar.ro

Therefore, for $q > 1$, we consider *decreasing* sequences of q -complete open subsets of a Stein space and we want to study the convexity properties of the interior of their intersection.

We prove, by means of a counterexample, that for a decreasing sequence $\{D_\nu\}$ of q -complete domains in \mathbb{C}^n , $\text{Int}(\bigcap D_\nu)$ is not necessarily q -complete (Theorem 5).

On the other hand, because for domains in \mathbb{C}^n , or more generally in Stein manifolds, Hartogs q -convexity is equivalent to q -completeness with corners (see [14]), it follows that, in the above setting, $\text{Int}(\bigcap D_\nu)$ is q -complete with corners.

We show that a similar statement does not hold for singular complex spaces. Namely, for each $q \geq 2$, we give an example of a normal Stein X space with only one singular point and a decreasing sequence $\{D_\nu\}$ of 2-complete domains in X , such that $\text{Int}(\bigcap D_\nu)$ is not q -complete with corners (Theorem 6).

As a dual statement, in the concave case, we show that for every integer $n > 1$ there exists a connected complex manifold M of dimension n such that M is an increasing union of 1-concave open subsets and is not weakly $(n-1)$ -concave (Theorem 8).

§2. Decreasing sequences of q -complete domains

Definition 1. Suppose that D is an open subset of \mathbb{C}^n . A smooth function $\varphi : D \rightarrow \mathbb{R}$ is called weakly q -convex if its Levi form $\sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(p) \xi_j \bar{\xi}_k$ has at least $n - q + 1$ nonnegative eigenvalues at every point $p \in D$. The function φ is called strictly q -convex if its Levi form has at least $n - q + 1$ positive eigenvalues at every point $p \in D$.

Using local embeddings these notions can be extended to complex spaces.

Definition 2. Suppose that X is a complex space and q a positive integer:

- (a) The space X is called q -convex if there exists a continuous exhaustion function $\varphi : X \rightarrow \mathbb{R}$ (i.e., $\{x \in X : \varphi(x) < c\} \Subset X$ for every $c \in \mathbb{R}$) and a compact set $K \subset X$ such that φ is strictly q -convex on $X \setminus K$.
- (b) If we can choose $K = \emptyset$ in the above definition, X is called q -complete.

Definition 3. If $H^p(X, \mathcal{F}) = 0$ for every coherent sheaf \mathcal{F} on a complex space X and every $p \geq q$, then X is called cohomologically q -complete.

By the results of Andreotti and Grauert [1] we have the following theorem.

Theorem 1. *Every q -complete complex space is cohomologically q -complete.*

Definition 4. (a) A continuous function $\varphi : X \rightarrow \mathbb{R}$ defined on a complex space is called q -convex with corners if, for every $x \in X$, there exists a neighborhood

U of x and finitely many strictly q -convex C^∞ functions $\varphi_1, \dots, \varphi_l$, defined on U , such that $\varphi|_U = \max\{\varphi_1, \dots, \varphi_l\}$.

- (b) A complex space X is called q -complete with corners if there exists a q -convex with-corners exhaustion function $\varphi : X \rightarrow \mathbb{R}$.

The next result is a particular case of a theorem due to Diederich and Fornaess [7]. It was generalized to the singular case in [8].

Theorem 2. *If M is an n -dimensional q -complete with-corners complex manifold, then M is $\tilde{q} = (n - \lfloor \frac{n}{q} \rfloor + 1)$ -complete.*

For $q \geq 1$ and $r > 0$, we denote by $P^q(r) \subset \mathbb{C}^q$ the polydisk centered at the origin with multiradius (r, \dots, r) . For the following definition, see, e.g., [17].

Definition 5. (a) For $1 \leq q < n$ and $0 < r, r_1 < 1$, we let $H^q \subset \mathbb{C}^n$ be defined by $H^q := P^q(1) \times P^{n-q}(r) \cup \left[P^q(1) \setminus \overline{P^q(r_1)} \right] \times P^{n-q}(1)$. The pair $(H^q, P^n(1))$ is called a standard Hartogs q -figure.

- (b) If M is an n -dimensional complex manifold and $V \subset U \subset M$ are open subsets, the pair (V, U) is called a Hartogs q -figure if there exists a standard Hartogs q -figure $(H^q, P^n(1))$ and a biholomorphism $F : P^n(1) \rightarrow U$ such that $F(H^q) = V$.

Definition 6. Let $\Omega \subset \mathbb{C}^n$ be an open set. If, for every Hartogs q -figure (V, U) , we have that $V \subset \Omega$ implies $U \subset \Omega$, then Ω is called Hartogs q -convex.

As we mentioned in the introduction, it was proved in [14] that a domain in \mathbb{C}^n is Hartogs q -convex if and only if it is q -complete with corners.

The following result is Satz 2.3 in [15].

Proposition 3. *If X is a complex space and U and V are open subsets of X such that U is p -complete and V is q -complete, then $U \cup V$ is $(p+q)$ -complete.*

Proposition 4 was proved in [18] in the smooth case and in [10] and [13] in the singular case.

Proposition 4. *Suppose that X is a complex space of dimension n . If X is cohomologically q -complete then $H_{n+i}(X, \mathbb{C}) = 0$ for every $i \geq q$.*

Our first result is the following theorem.

Theorem 5. *There exists a sequence $\{D_\nu\}$ of 3-complete open subsets of \mathbb{C}^5 such that $D_{\nu+1} \subset D_\nu$ for every ν and $\text{Int}(\bigcap D_\nu)$ is not cohomologically 3-complete.*

Proof.

We consider the following two planes in \mathbb{C}^5 :

$$\begin{aligned} L_1 &= \{z = (z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : z_1 = z_2 = z_3 = 0\}, \\ L_2 &= \{z = (z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : z_1 = z_4 = z_5 = 0\}. \end{aligned}$$

Let $U_1 = \mathbb{C}^5 \setminus L_1$, $U_2 = \mathbb{C}^5 \setminus L_2$. It follows that U_1 and U_2 are 3-complete. At the same time, since $L_1 \cap L_2 = \{0\}$, we have $U_1 \cup U_2 = \mathbb{C}^5 \setminus \{0\}$. Then we have $H_8(U_1, \mathbb{C}) = H_8(U_2, \mathbb{C}) = H_9(U_1, \mathbb{C}) = H_9(U_2, \mathbb{C}) = 0$ and $H_9(U_1 \cup U_2, \mathbb{C}) = \mathbb{C}$ since $H_9(S^9, \mathbb{C}) = \mathbb{C}$. From the Mayer–Vietoris exact sequence

$$\begin{aligned} H_9(U_1, \mathbb{C}) \oplus H_9(U_2, \mathbb{C}) &\rightarrow H_9(U_1 \cup U_2, \mathbb{C}) \\ &\rightarrow H_8(U_1 \cap U_2, \mathbb{C}) \\ &\rightarrow H_8(U_1, \mathbb{C}) \oplus H_8(U_2, \mathbb{C}), \end{aligned}$$

it follows that $H_8(U_1 \cap U_2, \mathbb{C}) = \mathbb{C}$.

Let W be a relatively compact open subset of $U_1 \cap U_2$ such that the inclusion $W \hookrightarrow U_1 \cap U_2 = \mathbb{C}^5 \setminus (L_1 \cup L_2)$ induces an isomorphism $H_8(W, \mathbb{C}) \rightarrow H_8(U_1 \cap U_2, \mathbb{C})$. In fact, we have an exhaustion $\{W_k\}$ of $U_1 \cap U_2$ such that the inclusion $W_k \hookrightarrow U_1 \cap U_2$ induces an isomorphism at all homology and homotopy groups.

For $\nu \geq 1$ we define

$$\begin{aligned} L_{1,\nu} &= \{z \in \mathbb{C}^5 : z_1 = \frac{1}{\nu}, z_2 = z_3 = 0\}, \\ L_{2,\nu} &= \{z \in \mathbb{C}^5 : z_1 = \frac{\sqrt{2}}{\nu}, z_4 = z_5 = 0\}. \end{aligned}$$

It follows that $L_{i,\mu} \cap L_{j,\nu} = \emptyset$ if $(i, \mu) \neq (j, \nu)$. Because W is relatively compact in $\mathbb{C}^5 \setminus (L_1 \cup L_2)$, it follows that there exists $\nu_0 \geq 1$ such that, for $\nu \geq \nu_0$, $L_{1,\nu} \cap \overline{W} = \emptyset$ and $L_{2,\nu} \cap \overline{W} = \emptyset$.

For $\nu \geq \nu_0$, let $D_\nu = \mathbb{C}^5 \setminus \bigcup_{j=\nu_0}^\nu (L_{1,j} \cup L_{2,j})$. Since $L_{i,\mu} \cap L_{j,\nu} = \emptyset$, it follows that D_ν are 3-complete. Let $D = \text{Int}(\bigcap_{\nu \geq \nu_0} D_\nu)$. It follows that $W \subset D \subset \mathbb{C}^5 \setminus (L_1 \cup L_2)$. Hence we have

$$H_8(W, \mathbb{C}) \rightarrow H_8(D, \mathbb{C}) \rightarrow H_8(\mathbb{C}^5 \setminus (L_1 \cup L_2), \mathbb{C}),$$

where the morphisms are induced by inclusions. Since $H_8(W, \mathbb{C}) \rightarrow H_8(\mathbb{C}^5 \setminus (L_1 \cup L_2), \mathbb{C})$ is surjective, it follows that $H_8(D, \mathbb{C}) \rightarrow H_8(\mathbb{C}^5 \setminus (L_1 \cup L_2), \mathbb{C})$ is surjective as well. In particular, we have $H_8(D, \mathbb{C}) \neq 0$. Proposition 4 implies that D is not cohomologically 3-complete. \square

As we mentioned in the introduction, if $\{D_\nu\}$ is a decreasing sequence of q -complete open subsets of \mathbb{C}^n , it follows that $\text{Int}(\bigcap D_\nu)$ is q -complete with corners. This is not the case for singular complex spaces, as the following result shows.

Theorem 6. *For every integer $q \geq 2$, there exists a normal Stein complex space X with only one isolated singularity, and $\{D_\nu\}$ a decreasing sequence of open subsets of X such that each D_ν is 2-complete and $\text{Int}(\bigcap D_\nu)$ is not q -complete with corners.*

Proof. Let q be an integer, $q \geq 2$. Let $\pi : F \rightarrow \mathbb{P}^1$ be a negative vector bundle of rank $r \geq 3q - 1$ and let S be the zero section of F (hence S is biholomorphic to \mathbb{P}^1). Let X be the blow-down of $S \subset F$ and $\tau : F \rightarrow X$ be the contraction map. We let $x_0 = \tau(S)$. We fix a point $a \in S$ and we set $U = S \setminus \{a\}$ (hence U is biholomorphic to \mathbb{C}) and $W = \pi^{-1}(U)$. We have that $\pi : W \rightarrow U$ is a trivial holomorphic vector bundle and therefore W is biholomorphic to $U \times \mathbb{C}^r$ (in particular W is Stein). We consider $W_\nu \subset F$ a fundamental system of Stein open neighborhoods of a and we define

$$D_\nu = \tau(W \cup W_\nu).$$

Note that D_ν are open neighborhoods of x_0 in X and, since $\bigcap W_\nu = \{a\}$, we have that $\text{Int}(\bigcap D_\nu) = \tau(W \setminus S)$. Hence $\text{Int}(\bigcap D_\nu)$ is biholomorphic to $W \setminus S$ and therefore to $U \times (\mathbb{C}^r \setminus \{0\})$.

Note the following points:

- $n = \dim X = r + 1 \geq 3q$.
- As W and W_ν are Stein, by Theorem 3 we have that $W \cup W_\nu$ is 2-complete. Therefore D_ν is 2-convex and since X is Stein, we deduce that D_ν is 2-complete.
- We have that $U \times (\mathbb{C}^r \setminus \{0\})$ is not cohomologically $(n - 2)$ complete since $\mathbb{C}^r \setminus \{0\}$ is not cohomologically $(n - 2)$ -complete.

Because $n \geq 3q$, we have $\tilde{q} = n - \left\lceil \frac{n}{q} \right\rceil + 1 \leq n - 2$. Using Theorems 1 and 2 we deduce that $U \times (\mathbb{C}^r \setminus \{0\})$ is not q -complete with corners.

Hence although each D_ν is 2-complete, the interior of their intersection is not q -complete with corners. □

Next we would like to say a few things about the intersection of Stein open subsets of a normal Stein space. Let X be a normal Stein complex space, and $\{D_\nu\}$ be a sequence of Stein open subsets of X . It is a completely open problem whether the interior of their intersection is Stein or not, even if X has dimension 2; see [2]. Of course, the problem is due to singularities. However, we have the following proposition.

Proposition 7. *Let X be a normal Stein complex space and $\{D_\nu\}$ be a sequence of Stein open subsets of X . If $D = \text{Int}(\bigcap D_\nu)$, then we have*

- (a) $\text{Reg}(X) \cap \partial D$ is dense in ∂D ;
- (b) D is a domain of holomorphy in X .

Proof.

- (a) Suppose that this is not the case and let $x_0 \in \partial D$ and W be a Stein neighborhood of x_0 such that $\partial D \cap W \subset \text{Sing}(X)$. As X is normal and therefore locally irreducible, we have that $W \setminus \partial D$ is connected. Since $W \setminus \partial D = (W \cap D) \cup (W \setminus \overline{D})$, we deduce that $W \setminus \partial D = W \cap D$. Therefore $W \setminus \text{Sing}(X) \subset D$. Using again the normality of X , the Riemann second extension theorem and the fact that each D_ν is Stein, we deduce that the inclusion $W \setminus \text{Sing}(X) \hookrightarrow D_\nu$ extends to W (with values in D_ν) and therefore $W \subset D_\nu$ for every ν . Hence $W \subset D$. In particular $x_0 \in D$, which contradicts our choice of x_0 .
- (b) Obviously, D is locally Stein at every point $x \in \partial D \cap \text{Reg}(X)$. Then for every sequence $\{x_k\}$, $x_k \in D$ such that $x_k \rightarrow x \in \partial D \cap \text{Reg}(X)$, there exists $f \in \mathcal{O}(D)$ which is unbounded on $\{x_k\}$. This was proved for relatively compact domains $D \Subset X$ in [11] and extended to arbitrary domains in [16]. From this fact and part (a), we deduce that D is a domain of holomorphy in X .

□

Remark 1. Using the method in [6] it can be proved that, in the same setting, if $\dim(X) = 2$ then D satisfies the disk property. This means that if $\overline{\Delta} = \{z \in \mathbb{C} : |z| \leq 1\}$ is the closed unit disk and $f_n : \overline{\Delta} \rightarrow X$ is a sequence of holomorphic functions converging uniformly to a holomorphic function $f : \overline{\Delta} \rightarrow X$ and if $f_n(\overline{\Delta}) \subset D$ and $f(\partial\Delta) \subset D$, then $f(\overline{\Delta}) \subset D$.

§3. Increasing sequences of q -concave domains

We want to discuss a dual question, namely concavity properties of a union of q -concave open subsets of a complex manifold.

For the next definition, see [1].

Definition 7. A complex space X is called q -concave if there exists a continuous function $\varphi : X \rightarrow (0, \infty)$ and a compact set $K \subset X$ such that φ is strictly q -convex on $X \setminus K$ and $\{x \in X : \varphi(x) > c\} \Subset X$ for every $c > 0$.

By analogy with the notion of weakly q -convex space, we introduce the following definition:

Definition 8. A complex space X is called weakly q -concave if there exists a continuous function $\varphi : X \rightarrow (0, \infty)$ and a compact set $K \subset X$ such that φ is weakly q -convex on $X \setminus K$ and $\{x \in X : \varphi(x) > c\} \Subset X$ for every $c > 0$.

Remark 2. A proper modification of a q -concave manifold is weakly q -concave.

Example. The following example appears in [3]. Let $a \in \mathbb{P}^2$ and $\{x_n\}_{n \geq 1}$ be a sequence in $\mathbb{P}^2 \setminus \{a\}$ converging to a , and M be the blow-up of $\mathbb{P}^2 \setminus \{a\}$ at this sequence. Then M is weakly 1-concave but it is not 1-concave. Moreover, M is an increasing sequence of 1-concave open subsets. Indeed, we let $M_k, k \geq 1$ be the blow-up of $\mathbb{P}^2 \setminus (\{a\} \cup \{x_n : n \geq k + 1\})$ at x_1, \dots, x_k . Then M_k is an open subset of $M, M_k \subset M_{k+1}$ and $\bigcup_{k \geq 1} M_k = M$. It was noticed in [9] that if $\{A_n\}$ is a countable set of closed, completely pluripolar subsets of a complex manifold Ω such that $A := \bigcup A_n$ is closed in Ω , and Ω' is an open subset of Ω such that $\Omega' \Subset \Omega$ then $A \cap \Omega'$ is completely pluripolar in Ω' . It follows then that $\mathbb{P}^2 \setminus (\{a\} \cup \{x_n : n \geq k + 1\})$ is 1-concave and hence M_k is 1-concave.

Theorem 8. For every integer $n > 1$ there exists a connected complex manifold M of dimension n such that M is an increasing union of 1-concave open subsets and M is not weakly $(n-1)$ -concave.

Proof. The following construction was used in [4] and [5]. We start with $\Omega_0 := \mathbb{P}^n$ (or any compact complex manifold of dimension n) and we choose $a_0 \in \Omega_0$ to be any point. We set $M_0 := \Omega_0 \setminus \{a_0\}$ and let $p_0 : \Omega_1 \rightarrow \Omega_0$ be the blow-up of Ω_0 at a_0 . Let a_1 be a point on the exceptional divisor of $p_0, M_1 = \Omega_1 \setminus \{a_1\}$ and $p_1 : \Omega_2 \rightarrow \Omega_1$ be the blow-up of Ω_1 at a_1 . Suppose now that we have defined inductively

- Ω_j for $j = 0, \dots, k;$
- $a_j \in \Omega_j, p_j : \Omega_{j+1} \rightarrow \Omega_j$ and $M_j = \Omega_j \setminus \{a_j\}$ for $j = 0, \dots, k - 1.$

We choose a point a_k on the exceptional divisor of $p_{k-1} : \Omega_k \rightarrow \Omega_{k-1}$ such that a_k is not on the proper transform of the exceptional divisor of $p_{k-2} : \Omega_{k-1} \rightarrow \Omega_{k-2}$. We let $M_k = \Omega_k \setminus \{a_k\}$ and $p_{k+1} : \Omega_{k+1} \rightarrow \Omega_k$ be the blow-up of Ω_k at a_k .

Note that M_k is an open subset of M_{k+1} for every $k \geq 0$. We set $\tilde{M} := \bigcup_{k \geq 0} M_k$. Each M_k is 1-concave since it is the complement of a point in a compact complex manifold. At the same time, \tilde{M} contains a noncompact connected $(n-1)$ -dimensional complex subspace X such that all irreducible components of X are

compact. This subspace is the union of the (proper transforms of the) exceptional divisors of all the blow-ups defined above.

If $\varphi : M \rightarrow (0, \infty)$ is weakly $(n-1)$ -convex outside a compact subset K of M , since each irreducible component of X has dimension $n-1$, we have, by the maximum principle, that φ must be constant on each irreducible component of X that does not intersect K . Therefore it is constant on at least one noncompact connected component of $X \setminus K$. Hence M cannot be weakly $(n-1)$ -concave. \square

Remark 3. It was noticed in [3] that if a complex manifold is an increasing union of 1-concave open subsets then its cohomology with values in any locally free coherent sheaf is separated. It is an open question raised by R. Hartshorne [12] whether a complex connected manifold such that its cohomology with values in any locally free coherent sheaf is finite-dimensional is necessarily a compact manifold.

References

- [1] A. Andreotti and H. Grauert, Théorèmes de finitude pour la cohomologie des espaces complexes, *Bull. Soc. Math. France* **90** (1962), 193–259. [Zbl 0106.05501](#) [MR 0150342](#)
- [2] M. Colțoiu, Some open problems concerning Stein spaces. *Analyse complexe* (Bucharest, 1989), *Rev. Roumaine Math. Pures Appl.* **36** (1991), 225–229. [Zbl 0754.32007](#) [MR 1144554](#)
- [3] M. Colțoiu, On the separation of cohomology groups of increasing unions of $(1, 1)$ convex-concave manifolds, *J. Math. Kyoto Univ.* **45** (2005), 405–409. [Zbl 1115.32007](#) [MR 2161700](#)
- [4] M. Colțoiu and C. Joița, Convexity properties of coverings of 1-convex surfaces, *Math. Z.* **275** (2013), 781–792. [Zbl 1287.32006](#) [MR 3127037](#)
- [5] M. Colțoiu and C. Joița, On the separation of the cohomology of universal coverings of 1-convex surfaces, *Adv. Math.* **265** (2014), 362–370. [Zbl 1300.32005](#) [MR 3255464](#)
- [6] M. Colțoiu and M. Tibăr, On the disk theorem, *Math. Ann.* **345** (2009), 175–183. [Zbl 1175.32004](#) [MR 2520056](#)
- [7] K. Diederich and J. E. Fornaess, Smoothing q -convex functions and vanishing theorems, *Invent. Math.* **82** (1985), 291–305. [Zbl 0586.32022](#) [MR 0809717](#)
- [8] K. Diederich and J. E. Fornaess, Smoothing q -convex functions in the singular case, *Math. Ann.* **273** (1986), 665–671. [Zbl 0586.32023](#) [MR 0826465](#)
- [9] H. El Mir, Sur le prolongement des courants positifs fermés, *Acta Math.* **153** (1984), 1–45. [Zbl 0557.32003](#) [MR 0744998](#)
- [10] A. Ferrari, Coomologia e forme differenziali sugli spazi analitici complessi, *Ann. Scuola Norm. Sup. Pisa* **25** (1971), 469–480. [Zbl 0233.32005](#) [MR 0414922](#)
- [11] J. E. Fornaess and R. Narasimhan, The Levi problem on complex spaces with singularities, *Math. Ann.* **248** (1980), 47–72. [Zbl 0411.32011](#) [MR 0569410](#)
- [12] R. Hartshorne, Ample subvarieties of algebraic varieties. Notes written in collaboration with C. Musili. *Lecture Notes in Mathematics* 156, Springer, Berlin-New York, 1970. [Zbl 0208.48901](#) [MR 0282977](#)

- [13] J. Le Potier, Une propriété topologique des espaces q -convexes, *Bull. Soc. Math. France* **98** (1970), 319–328. [Zbl 0204.56501](#) [MR 0281950](#)
- [14] K. Matsumoto, Boundary distance functions and q -convexity of pseudoconvex domains of general order in Kähler manifolds, *J. Math. Soc. Japan* **48** (1996), 85–107. [Zbl 0860.32003](#) [MR 1361549](#)
- [15] M. Peternell, Algebraische Varietäten und q -vollständige komplexe Räume, *Math. Z.* **200** (1989), 547–581. [Zbl 0675.32014](#) [MR 0987586](#)
- [16] O. Preda, Locally Stein open subsets in normal Stein spaces, *J. Geom. Anal.* **25** (2015), 2759–2766. [Zbl 1337.32021](#) [MR 3427147](#)
- [17] O. Riemenschneider, Über den Flächeninhalt analytischer Mengen und die Erzeugung k -pseudokonvexer Gebiete, *Invent. Math.* **2** (1967), 307–331. [Zbl 0148.32101](#) [MR 0212210](#)
- [18] G. Sorani and V. Villani, q -complete spaces and cohomology, *Trans. Amer. Math. Soc.* **125** (1966) 432–448. [Zbl 0155.40203](#) [MR 0206335](#)