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Almost Sure Well-Posedness of Fractional Schrödinger Equations with Hartree Nonlinearity

by

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Abstract

We consider a Cauchy problem of an energy-critical fractional Schrödinger equation with Hartree nonlinearity below the energy space. Using randomization of functions on $\mathbb{R}^d$ associated with the Wiener decomposition, we prove that the Cauchy problem is almost surely locally well posed. Our result includes the Hartree Schrödinger equation ($\alpha = 2$).

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§1. Introduction

In this paper we consider the following Cauchy problem of fractional nonlinear Schrödinger equations (FNLS):

$$\begin{cases} i\partial_t u = |\nabla|^\alpha u + F(u) & \text{in } \mathbb{R}^{1+d}, \\ u(x,0) = \phi(x) \in H^s & \text{in } \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $|\nabla| = (-\Delta)^{\frac{1}{2}}$, $d \geq 3$, $1 < \alpha \leq 2$ (with $\alpha < \frac{d}{2}$), and $F(u)$ is the nonlinear term of Hartree type given by

$$F(u) = \mu(|\cdot|^{-2\alpha} * |u|^2)u, \quad \mu \in \mathbb{R} \setminus \{0\}.$$

The fractional Schrödinger equation appears in fractional quantum mechanics (see [28, 29, 30]), where Laskin generalized the Brownian-like quantum mechanical path, in the Feynman path integral approach to quantum mechanics and to the α-stable Lévy-like quantum mechanical path.

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The solution u of (1.1) formally satisfies the mass and energy conservation laws:

$$\begin{aligned} m(u) &= \|u(t)\|_{L^2}^2, \\ E(u) &= K(u) + P(u), \end{aligned} \tag{1.2}$$

where

$$K(u) = \tfrac{1}{2}\langle u, |\nabla|^\alpha u\rangle, \qquad P(u) = \tfrac{1}{4}\langle u, \mu(|x|^{-2\alpha} * |u|^2)u\rangle.$$

Here $\langle\ ,\rangle$ is the complex inner product in L^2. Hence $H^{\frac{\alpha}{2}}$ is referred to energy space.

The equation (1.1) has a scaling invariance. In fact, if u is a solution of (1.1), then the scaled function u_λ given by

$$u_\lambda(t,x) = \lambda^{-\frac{\alpha}{2}+\frac{d}{2}} u(\lambda^\alpha t, \lambda x) \qquad \text{for any } \lambda > 0$$

is also a solution. Since the $\dot{H}^{\frac{\alpha}{2}}$-norm is preserved under the scaling $u \mapsto u_\lambda$, (1.1) is said to be energy-critical if $s = \frac{\alpha}{2}$. It is also said to be supercritical (subcritical) if $s < \frac{\alpha}{2}$ ($s > \frac{\alpha}{2}$, respectively).

By Duhamel's formula, (1.1) is written as an integral equation

$$u = U(t)\phi - i\mu \int_0^t U(t-t')((|\cdot|^{-2\alpha} * |u(t')|^2)u(t'))\, dt'. \tag{1.3}$$

Here we define the linear propagator $U(t)f$ to be the solution to the linear problem $i\partial_t z = |\nabla|^\alpha z$ with initial datum f. Then it is formally given by

$$U(t)f = \mathcal{F}^{-1} e^{-it|\xi|^\alpha} \mathcal{F} f = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x\cdot\xi - t|\xi|^\alpha)} \widehat{f}(\xi)\, d\xi, \tag{1.4}$$

where $\widehat{f}$=$\mathcal{F}f$ denotes the Fourier transform of f such that $\mathcal{F}f(\xi)$=$\int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx$ and we denote its inverse Fourier transform by $\mathcal{F}^{-1}g(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} g(\xi)\, d\xi$.

For the linear propagator $U(t)$, the Strichartz estimate is known to hold

Lemma 1.1 ([19, Thm. 2]). *Let $d \geq 2$ and (q,r) satisfy $2/q + d/r = d/2$, $2 \leq q, r \leq \infty$ and $(d,q,r) \neq (2,2,\infty)$. Then*

$$\| |\nabla|^{-\frac{2-\alpha}{q}} U(t)f\|_{L_t^q L^r([0,T]\times\mathbb{R}^d)} \lesssim \|f\|_{L_x^2}.$$

The implicit constant does not depend on $T > 0$.

Due to the weak dispersion of $U(t)$, the estimate accompanies a derivative loss of order $(2-\alpha)/q$. But if one imposes radial assumptions or an angularly regular condition on f, then a derivative loss can be recovered and a regularity gain can even be obtained (see [17, 24]).

Using Lemma 1.1, the local well-posedness of (1.1) can be shown in the subcritical case ($s > \frac{\alpha}{2}$). Actually a small revision of [13, Prop. 4.1] gives the following result:

Proposition 1.2. *Let $s > \frac{\alpha}{2}$. If $\phi \in H^s$ then there exists a positive time T such that (1.1) has a unique solution $u \in C([-T,T]; H^s) \cap L_T^2(H_{2d/(d-2)}^{s-(2-\alpha)/2})$. Here $H_r^s = (1-\Delta)^{-s/2} L^r$.*

On the other hand, when $s = \frac{\alpha}{2}$, by using the radial Strichartz estimate, the local and small data global well-posedness to (1.1) are proven under the radial assumption of ϕ as follows:

Proposition 1.3 ([13, Thm. 5.2]). *Let $\frac{2d}{2d-1} \le \alpha \le 2$ (with $\alpha < \frac{d}{2}$) and $\phi \in H^{\frac{\alpha}{2}}$ be radially symmetric; then there exists a positive time T such that (1.1) has a unique solution $u \in C([-T,T]; H^{\frac{\alpha}{2}}) \cap L_T^3 H_r^{\frac{\alpha}{2}}$, $r = 2n/(n - \frac{2\alpha}{3})$. If $\|\phi\|_{\dot H^{\frac{\alpha}{2}}}$ is sufficiently small, then (1.1) is globally well posed.*

Recently the author and collaborators of [15] obtained global well-posedness for $\frac{2d}{2d-1} < \alpha < 2$ without smallness when $\mu > 0$ and with $\|\phi\|_{\dot H^{\frac{\alpha}{2}}} < \|W_\alpha\|_{\dot H^{\frac{\alpha}{2}}}$ when $\mu < 0$, where W_α is a steady state solution of (1.1). Also see [23] for the power type. In [26], a power-type case was treated in some critical regularity without a radial assumption. When $\alpha = 2$, the equation is much easier to handle, so there exist numerous well-posedness and ill-posedness results (see [11, 33, 12, 32]).

In this paper, we focus on the supercritical case ($s < \frac{\alpha}{2}$). Many dispersive equations are known to be ill posed in the supercritical regime (see [1, 6, 7, 10, 20]). For the fractional Schrödinger equation, we also observe some negative results. One can readily show the following with a slight modification of ill-posedness in [16, 25].

Proposition 1.4. *If $s < \alpha(1 - \frac{\alpha}{2})$ and the flow map $\phi \mapsto u$ exists in a small neighborhood of the origin as a map from $H^s(\mathbb{R}^d)$ to $C([-T,T]; H^s)$, then it fails to be C^3 at the origin.*

Nonetheless, using probabilistic arguments, Bourgain [4], Burq–Tzvetkov [8, 9], Colliander–Oh [21], Lührmann–Mendelson [31] and Bényi–Oh–Pocovinicu [2, 3] established positive results on subsets of H^s for the supercritical case (see also [38, 37, 22, 34, 5, 35, 36]). In particular, in [38, 31, 2, 3], the authors introduced a randomization for functions in the usual Sobolev space on $\mathbb{R}^d$.

Many of these works are on the dispersive equation with power-type nonlinearity. So we are concerned with the Cauchy problem with random initial data of the equation with Hartree nonlinearity. Because of the nonlocal nonlinearity, the problem is more complicated. More precisely, we cannot apply the Hölder inequality and bilinear Strichartz estimates directly. In order to overcome the difficulty, we decompose functions with respect to frequency as in [12].

Before giving a precise description of the main theorem, we introduce randomization adapted to the Wiener decomposition in [38, 31, 2, 3]. Let $\psi \in \mathcal{S}$ be a function satisfying

$$\operatorname{supp}\psi \subset [-1,1]^d \quad \text{and} \quad \sum_{n\in\mathbb{Z}^d} \psi(\xi - n) = 1.$$

We define a pseudo-differential operator $\psi(D-n)$ as a Fourier multiplier,

$$\psi(D-n)f(x) = \mathcal{F}^{-1}\psi(\xi-n)\mathcal{F}f.$$

Then given a function $f \in L^2(\mathbb{R}^d)$, we have

$$f = \sum_{n\in\mathbb{Z}^d} \psi(D-n)f.$$

Let $\{g_n\}_{n\in\mathbb{Z}^d}$ be a sequence of independent mean-zero complex-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where the real and imaginary parts of g_n are independent and endowed with probability distributions μ_n^1 and μ_n^2. We assume there exists $c > 0$ such that

$$\Big| \int_{\mathbb{R}} e^{\gamma x}\, d\mu_n^j \Big| \le e^{c\gamma^2} \tag{1.5}$$

for all $\gamma \in \mathbb{R}$, $n \in \mathbb{Z}^d$, $j = 1, 2$. The condition (1.5) means that the exponential moment is bounded. It gives a large deviation estimate. For the detail, see [8]. Note that (1.5) is satisfied by standard Gaussian random variables, standard Bernoulli random variables and any random variables with compactly supported distributions. Actually, we have

1. Gaussian:
$$\int_{-\infty}^{\infty} e^{\gamma x}\, d\mu_n(x) = \int_{-\infty}^{\infty} e^{\gamma x} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = e^{\gamma^2/2};$$
2. Bernoulli:
$$\int_{-\infty}^{\infty} e^{\gamma x}\, d\mu_n(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{\gamma x}(\delta_1(x) + \delta_{-1}(x))\, dx \le e^{\gamma^2/2};$$
3. random variables with compactly supported distributions:
$$\Big| \int_{-\infty}^{\infty} e^{\gamma x}\, d\mu_n(x) \Big| \le \int_{-A}^{A} e^{\gamma x} C\, dx = \frac{C}{\gamma}(e^{\gamma A} - e^{-\gamma A}) \le e^{\widehat{A}^2\gamma^2}.$$

Thereafter we define the Wiener randomization of f by

$$f^\omega := \sum_{n\in\mathbb{Z}^d} g_n(\omega)\psi(D-n)f. \tag{1.6}$$

Now we state our main theorem.

Theorem 1.5. *Let $d \geq 3$, $1 < \alpha \leq 2$ (with $\alpha < \frac{d}{2}$), $\max(\frac{2\alpha-1}{4\alpha-3} \cdot \frac{\alpha}{2}, \frac{1}{2}) < s < \frac{\alpha}{2}$ and $\phi \in H^s$. Consider randomization ϕ^ω defined in (1.6) with a probability space $(\Omega, \mathcal{F}, P)$ satisfying the condition (1.5). Then (1.1) is almost surely locally well posed in the sense that there exist positive C, c, γ and $\sigma = \frac{\alpha}{2}+$ such that for each $T \ll 1$, there exists a set $\Omega_T \subset \Omega$ with the following properties:*

1. *$P(\Omega \backslash \Omega_T) \leq C \exp\big(-\frac{c}{T^\gamma \|\phi\|_{H^s}^2}\big)$.*
2. *For each $\omega \in \Omega_T$, there exists a unique solution $u \in C([0,T]; H^s)$ to (1.1) with initial data ϕ^ω.*
3. *The Duhamel part of the solution is smoother than the initial data, i.e.,*

$$u - U(t)\phi^\omega \in C([0,T]; H^\sigma).$$

The rest of the paper is organized as follows: In Section 2, we briefly review randomization adapted to the Wiener decomposition. In Section 3, we introduce the Bourgain space $X^{s,b}$ and show bilinear Strichartz estimates. Lastly, in Section 4, we prove Theorem 1.1.

As usual, different positive constants depending only on d, α are denoted by the same letter C, if not specified. The expressions $A \lesssim B$ and $A \gtrsim B$ mean $A \leq CB$ and $A \geq C^{-1}B$, respectively, for some $C > 0$, and $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$.

§2. Randomization

We briefly review randomization adapted to the Wiener decomposition.

Lemma 2.1 ([8, Lem. 3.1]). *For given $\{c_n\} \in \ell^2(\mathbb{Z}^d)$ and $p \geq 2$, there exists $C > 0$ such that*

$$\Big\| \sum_{n \in \mathbb{Z}^d} g_n(\omega) c_n \Big\|_{L^p(\Omega)} \leq C\sqrt{p} \|c_n\|_{l_n^2(\mathbb{Z}^d)}.$$

Lemma 2.2 ([2, Lem. 2.2]). *Given $f \in H^s(\mathbb{R}^d)$, we have for any $\lambda > 0$,*

$$P\Big(\|f^\omega\|_{H^s(\mathbb{R}^d)} > \lambda\Big) \leq Ce^{-c\lambda^2 \|f\|_{H^s}^{-2}}.$$

Lemma 2.3 ([2, Lem. 2.3]). *Given $f \in L^2(\mathbb{R}^d)$ and finite $p \geq 2$, there exist C, $c > 0$ such that for any $\lambda > 0$,*

$$P\Big(\|f^\omega\|_{L^p(\mathbb{R}^d)} > \lambda\Big) \leq Ce^{-c\lambda^2 \|f\|_{L^2}^{-2}}.$$

In particular, f^ω is in $L^p(\mathbb{R}^d)$ almost surely.

Exactly the same arguments for the Schrödinger equation in [2, 3] give probabilistic Strichartz estimates for the fractional Schrödinger equation. Actually, the only property of a linear propagator used in those papers is that the L^2-norm of a linear propagator is conserved in time (see [2, Prop. 1.3]).

Proposition 2.4. *Given $f \in L^2(\mathbb{R}^d)$, let f^ω be its randomization. Then, given $2 \le q,\ r < \infty$, for all $T > 0$ and $\lambda > 0$ there exist $C,\ c > 0$ such that*

$$P\big(\|U(t)f^\omega\|_{L_t^q L_x^r([0,T]\times\mathbb{R}^d)} > \lambda\big) \le C \exp\bigg(- c\frac{\lambda^2}{T^{\frac{2}{q}}\|f\|_{L^2}^2}\bigg). \tag{2.1}$$

§3. Bourgain space

We introduce the Bourgain space $X^{s,b}$ defined as follows: for $s,\ b \in \mathbb{R}$,

$$X^{s,b} = \big\{\varphi \in \mathcal{S}' : \|\varphi\|_{X^{s,b}} := \|\langle\xi\rangle^s \langle\tau + |\xi|^\alpha\rangle^b \widetilde{\varphi}(\tau,\xi)\|_{L^2(\mathbb{R}\times\mathbb{R}^d)} < \infty\big\},$$

where $\langle a\rangle = 1+|a|$ and $\widetilde{\varphi}$ denotes the time-space Fourier transform. In what follows we mention a few well-known properties of the $X^{s,b}$ space. Let η be a smooth cutoff function supported on $[-2,2]$, $\eta = 1$ on $[-1,1]$ and let $\eta_T(t) = \eta(t/T)$.

Lemma 3.1. *Let $T \in (0,1)$ and $b \in \big(\frac{1}{2}, \frac{3}{2}\big)$. Then for $s \in \mathbb{R}$ and $\theta \in \big[0, \frac{3}{2} - b\big)$ the following hold:*

$$\|\eta_T(t)U(t)f\|_{X^{s,b}(\mathbb{R}\times\mathbb{R}^d)} \lesssim T^{\frac{1}{2}-b}\|f\|_{H^s(\mathbb{R}^d)},$$

$$\bigg\|\eta_T(t)\int_0^t U(t-t')\eta_T(t')F(t')\,dt'\bigg\|_{X^{s,b}(\mathbb{R}\times\mathbb{R}^d)} \lesssim T^\theta\|F\|_{X^{s,b-1+\theta}(\mathbb{R}\times\mathbb{R}^d)}.$$

Lemma 3.2. *Let $d \ge 2$ and (q,r) satisfy $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, and $(d,q,r) \ne (2,2,\infty)$. Then for $b > \frac{1}{2}$ we have*

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u\|_{X^{\frac{2-\alpha}{q},b}(\mathbb{R}\times\mathbb{R}^d)}.$$

The above lemma follows from the Strichartz estimates (Lemma 1.1). By interpolation with trivial estimate $\|u\|_{L^2_{t,x}} \lesssim \|u\|_{X^{0,0}}$, we have the following lemma.

Lemma 3.3. *Let $q \ge 2$. Then for $b > \frac{1}{2}$ we have*

$$\|u\|_{L_t^q L_x^2} \lesssim \|u\|_{X^{0,b(1-\frac{2}{q})}(\mathbb{R}\times\mathbb{R}^d)}.$$

Because of scaling symmetry, the Strichartz estimate is optimal. But if one considers the interaction of two different frequency localized data, one can obtain a bilinear Strichartz estimate. Throughout the paper we denote by $A(N)$ the set $\{\xi : \xi \sim N\}$.

Lemma 3.4 ([14, Lem. 2.2]). *Let $d \geq 2$. Suppose that* $\operatorname{supp} \widehat{f} \subset A(N_1)$ *and* $\operatorname{supp} \widehat{g} \subset A(N_2)$ *with $N_1 \leq N_2$. Then,*

$$\begin{aligned}\|U(t)fU(t)g\|_{L^2_{t,x}} &\lesssim \left(\frac{N_1}{N_2}\right)^{\frac{d+\alpha-2}{4}} (N_1N_2)^{\frac{d-\alpha}{4}} \|f\|_{L^2_x}\|g\|_{L^2_x}\\ &= N_1^{\frac{d-1}{2}} N_2^{\frac{1-\alpha}{2}} \|f\|_{L^2_x}\|g\|_{L^2_x}.\end{aligned}$$

Moreover, we prove bilinear estimates for data whose Fourier transform is supported in a small ball, not necessarily centered at the origin.

Lemma 3.5. *Let $d \geq 2$. Suppose that* $\operatorname{supp} \widehat{f} \subset B(\xi_0, \rho_1)$, *with ρ_1, $|\xi_0| \ll 1$ and* $\operatorname{supp} \widehat{g} \subset A(1)$. *Then we have*

$$\|U(t)fU(t)g\|_{L^2_{t,x}} \lesssim \rho_1^{\frac{d-1}{2}} \|f\|_{L^2_x}\|g\|_{L^2_x}.$$

Proof. By decomposing the support of $\widehat{g}$ into a finite number of sets, rotation and mild dilation, it suffices to prove the estimates when $\operatorname{supp} \widehat{g} \subset B(e_1, \delta)$ for some $0 < \delta \ll 1$. By definition of $U(t)$, we have

$$U(t)f(x)U(t)g(x) = (2\pi)^{-2d} \int e^{i(x\cdot(\xi+\eta)-t(|\xi|^\alpha+|\eta|^\alpha))} \widehat{f}(\xi)\widehat{g}(\eta)\, d\xi\, d\eta.$$

For each $\bar{\xi} = (\xi_2, \ldots, \xi_d)$, we define a bilinear operator

$$B_{\bar{\xi}}(f,g) = \int_{\mathbb{R}^{1+d}} e^{i(x\cdot(\xi+\eta)-t(|\xi|^\alpha+|\eta|^\alpha))} \widehat{f}(\xi_1, \bar{\xi})\widehat{g}(\eta)\, d\xi_1\, d\eta.$$

We make the change of variable $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{d+1}) = (\xi + \eta, |\xi|^\alpha + |\eta|^\alpha)$ with the observation $\left|\frac{\partial \zeta}{\partial(\xi_1, \eta)}\right| = \alpha\left|\, \xi_1|\xi|^{\alpha-2} - \eta_1|\eta|^{\alpha-2}\,\right| \sim 1$. Then applying Plancherel's theorem and reversing the change of variables $(\zeta \to (\xi_1, \eta))$, we get

$$\|B_{\bar{\xi}}(f,g)\|_{L^2_tL^2_x} \lesssim \|\widehat{f}(\xi_1, \bar{\xi})\widehat{g}(\eta)\|_{L^2_{\xi_1,\eta}}.$$

Hence by Minkowski's inequality, we have

$$\begin{aligned}\|U(t)fU(t)g\|_{L^2_tL^2_x} &= \left\| \int B_{\bar{\xi}}(f,g)\, d\bar{\xi} \right\|_{L^2_tL^2_x}\\ &\lesssim \int \|\widehat{f}(\xi_1, \bar{\xi})\widehat{g}(\eta)\|_{L^2_{\xi_1,\eta}}\, d\bar{\xi} \lesssim \rho_1^{\frac{d-1}{2}} \|f\|_{L^2_x}\|g\|_{L^2_x}.\end{aligned}$$

The last inequality follows from the fact that the support of $\widehat{f}$ is in $B(\xi_0, \rho_1)$. $\square$

From Lemmata 3.4, 3.5 and the definition of the $X^{s,b}$ space, one can prove the following lemma.

Lemma 3.6. *Let $d \geq 2$. Consider $u,\ v \in X^{0,b}$ for $b > \frac{1}{2}$. Then we have the following properties:*

1. *If* $\operatorname{supp}\widehat{u} \subset A(N_1)$ *and* $\operatorname{supp}\widehat{v} \subset A(N_2)$ *with* $N_1 \leq N_2$, *then*
$$\|uv\|_{L^2_{t,x}} \lesssim N_1^{\frac{d-1}{2}} N_2^{\frac{1-\alpha}{2}} \|u\|_{X^{0,b}} \|v\|_{X^{0,b}}.$$
2. *If* $\operatorname{supp}\widehat{u} \subset B(\xi_0, N)$ *and* $\operatorname{supp}\widehat{v} \subset A(N_2)$ *with* $N, |\xi_0| \ll N_2$, *then*
$$\|uv\|_{L^2_{t,x}} \lesssim N^{\frac{d-1}{2}} N_2^{\frac{1-\alpha}{2}} \|u\|_{X^{0,b}} \|v\|_{X^{0,b}}.$$

Furthermore, interpolation with the trivial inequality $\|uv\|_{L^2_{t,x}} \lesssim \|u\|_{L^\infty_{t,x}} \|v\|_{L^2_{t,x}} \lesssim \|u\|_{X^{\frac{d}{2}+,\frac{1}{2}+}} \|v\|_{X^{0,0}}$ yields the following.

Lemma 3.7. *Let $d \geq 2$. Then, for given small $\varepsilon > 0$, we have the following properties:*

1. *If* $\operatorname{supp}\widehat{u} \subset A(N_1)$ *and* $\operatorname{supp}\widehat{v} \subset A(N_2)$ *with* $N_1 \leq N_2$, *then*
$$\|uv\|_{L^2_{t,x}} \lesssim N_1^{\frac{d-1}{2}+2\varepsilon} N_2^{\frac{1-\alpha}{2}+2\varepsilon} \|u\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{0,\frac{1}{2}-\varepsilon}}.$$
2. *If* $\operatorname{supp}\widehat{u} \subset B(\xi_0, N)$ *and* $\operatorname{supp}\widehat{v} \subset A(N_2)$ *with* $|\xi_0| \sim N_1$ *and* $N,\ N_1 \ll N_2$, *then*
$$\|uv\|_{L^2_{t,x}} \lesssim N^{\frac{d-1}{2}-\varepsilon_1} N_1^{\varepsilon_2} N_2^{\frac{1-\alpha}{2}+2\varepsilon} \|u\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{0,\frac{1}{2}-\varepsilon}},$$
where $\varepsilon_1 = \frac{2(d-1)\varepsilon}{1+2\varepsilon}$ *and* $\varepsilon_2 = \frac{2(d+2\varepsilon)\varepsilon}{1+2\varepsilon}$ *so that* $\varepsilon_2 - \varepsilon_1 = 2\varepsilon$.

§4. Almost sure local well-posedness

We will prove Theorem 1.5. Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its randomization. We concern (1.1) with initial data ϕ^ω. Let $z(t) := U(t)\phi^\omega$ and $v(t) := u(t) - U(t)\phi^\omega$. Then (1.1) becomes

$$\begin{cases} i\partial_t v = |\nabla|^\alpha v + F(v+z) & \text{in } \mathbb{R}\times\mathbb{R}^d, \\ v(0,x) = 0. \end{cases} \tag{4.1}$$

By Duhamel's principle, (4.1) is written as the integral equation

$$\begin{aligned} v(t) &= \int_0^t U(t-t')F(v+z)(t')\,dt' \\ &= \eta_T(t)\int_0^t U(t-t')\eta_T(t')F(\eta_T(t')v + \eta_T(t')z)(t')\,dt'. \end{aligned}$$

So we define $\mathcal{D}$ by

$$\mathcal{D}v(t) = \eta_T(t)\int_0^t U(t-t')\eta_T(t')F(\eta_T(t')v+\eta_T(t')z)(t')\,dt'.$$

Now it suffices to prove $\mathcal{D}$ has a fixed point in closed subset of $C_tH_x^s([0,T]\times\mathbb{R}^d)$,

$$B=\{u\in X^{\sigma,b}:\|u\|_{X^{\sigma,b}}\le 1\},\quad \sigma=(\alpha/2)+,\quad b=(1/2)+$$

outside a set of probability $\le C\exp\big(-\frac{c}{T^\gamma\|\phi\|_{H^s}^2}\big)$. For that purpose, we show a contraction inequality (Proposition 4.1) for $\mathcal{D}$. Then by choosing $R=R(T)\sim T^{-\frac{\gamma}{2}}$ for some $\gamma\in(0,\frac{2\theta}{3})$ such that

$$C_1T^\theta(1+R^3)\le 1 \quad\text{and}\quad C_2T^\theta(2+R^2)\le 1/2,$$

one can prove Theorem 1.5. Precisely, for $v,w\in B$, we have

$$\|\mathcal{D}v\|_{X^{\sigma,b}}\le 1 \quad\text{and}\quad \|\mathcal{D}v-\mathcal{D}w\|_{X^{\sigma,b}}\le \tfrac12\|v-w\|_{X^{\sigma,b}}$$

outside $\widetilde{\Omega_T}$, where $|\widetilde{\Omega_T}|\le C\exp\big(-\frac{c}{T^\gamma\|\phi\|_{H^s}^2}\big)$. So for $\omega\in\Omega_T=(\widetilde{\Omega_T})^c$, there exists a unique v^ω such that $\mathcal{D}v^\omega=v^\omega$.

Proposition 4.1. *Let $d\ge 3$, $1<\alpha\le 2$ (with $\alpha<\frac{d}{2}$) and $\max(\frac{2\alpha-1}{4\alpha-3}\cdot\frac{\alpha}{2},\frac12)<s<\frac{\alpha}{2}$. Given $\phi\in H^s(\mathbb{R}^d)$, let ϕ^ω be its randomization. Then, there exist $\sigma=\frac{\alpha}{2}+$, $b=\frac12+$ and $\theta=0+$ such that for each small $T\ll 1$ and $R>0$, we have*

$$\begin{aligned}\|\mathcal{D}v\|_{X^{\sigma,b}}&\le C_1T^\theta\big(\|v\|_{X^{\sigma,b}}^3+R^3\big),\\ \|\mathcal{D}v-\mathcal{D}w\|_{X^{\sigma,b}}&\le C_2T^\theta\big(\|v\|_{X^{\sigma,b}}^2+\|w\|_{X^{\sigma,b}}^2+R^2\big)\|v-w\|_{X^{\sigma,b}},\end{aligned}$$

outside a set of probability at most $C\exp\big(-c\frac{R^2}{\|\phi\|_{H^s}^2}\big)$.

Proof. For a given s satisfying $\max(\frac{2\alpha-1}{4\alpha-3}\cdot\frac{\alpha}{2},\frac12)<s<\frac{\alpha}{2}$, we choose σ and b such that

$$\begin{aligned}&\frac{\alpha}{2}<\sigma<\min\left(\frac{4\alpha-3}{2\alpha-1}s,\ \frac{\alpha}{2}+s-\frac12\right),\\ &\frac12<b<\frac12+\frac15\min\Bigg(\Big(\frac{4\alpha-3}{2\alpha-1}s-\sigma\Big)\frac{2\alpha-1}{6\alpha-4},\ \sigma-\frac58\alpha+\frac14,\ s-\sigma\frac{\alpha}{2(3\alpha-2)},\\ &\qquad\qquad \frac{\alpha}{2}-\sigma+s-\frac12,\ \frac{1}{\alpha}\Big(\sigma-\frac{\alpha}{2}\Big)\Bigg).\end{aligned}$$

Let $\theta = b - \frac{1}{2}$. We show only the first estimate, and then the second one can be proved similarly. Lemma 3.1 and duality give

$$\begin{aligned}\|\mathcal{D}v(t)\|_{X^{\sigma,b}} &\lesssim T^{\widetilde{\theta}}\|F(\eta_T v + \eta_T z)\|_{X^{\sigma,b-1+\widetilde{\theta}}} \\ &= T^{\widetilde{\theta}} \sup_{\|v_4\|_{X^{0,1-b-\widetilde{\theta}}}\le 1} \Big| \int\int_{\mathbb{R}\times\mathbb{R}^d} \langle\nabla\rangle^\sigma [F(\eta_T v + \eta_T z)] v_4 \, dx\, dt\Big|,\end{aligned}$$

where $\widetilde{\theta} = 4\theta$. So there exist six terms to be considered:

1. $\big|\int\int \langle\nabla\rangle^\sigma((|x|^{-2\alpha} * |\eta_T v|^2)\eta_T v)v_4\, dx\, dt\big|$;
2. $\big|\int\int \langle\nabla\rangle^\sigma((|x|^{-2\alpha} * |\eta_T z|^2)\eta_T z)v_4\, dx\, dt\big|$;
3. $\big|\int\int \langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (\eta_T v\overline{\eta_T z} + \overline{\eta_T v}\eta_T z))\eta_T v)v_4\, dx\, dt\big|$;
4. $\big|\int\int \langle\nabla\rangle^\sigma((|x|^{-2\alpha} * |\eta_T v|^2)\eta_T z)v_4\, dx\, dt\big|$;
5. $\big|\int\int \langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (\eta_T v\overline{\eta_T z} + \overline{\eta_T v}\eta_T z))\eta_T z)v_4\, dx\, dt\big|$;
6. $\big|\int\int \langle\nabla\rangle^\sigma((|x|^{-2\alpha} * |\eta_T z|^2)\eta_T v)v_4\, dx\, dt\big|$.

We will estimate each term by using Strichartz estimates, bilinear Strichartz estimates and probabilistic estimates. In what follows, we let $\widetilde{\varepsilon} = -\frac{1}{2} + b + \widetilde{\theta}$, ε be any number satisfying $\widetilde{\varepsilon} < \varepsilon < \min(\frac{1}{\alpha}(\sigma - \frac{\alpha}{2}), \frac{1}{\alpha}(d - 2\alpha))$ and $\widetilde{b} = \frac{1/2-\widetilde{\varepsilon}}{1-2\varepsilon}$. We note that $\widetilde{\varepsilon} = 5(-\frac{1}{2} + b)$ and $1 - b - \widetilde{\theta} = \frac{1}{2} - \widetilde{\varepsilon}$.

1st term: *vvv*

$$\Big|\int\int \langle\nabla\rangle^\sigma((|x|^{-2\alpha} * |\eta_T v|^2)\eta_T v)v_4\, dx\, dt\Big|. \tag{4.2}$$

By the Hölder inequality and Lemma 3.3, (4.2) is bounded by

$$\begin{aligned}&\|\langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (|\eta_T v|^2))\eta_T v)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \\ &\lesssim \|\langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (|\eta_T v|^2))\eta_T v)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{X^{0,\widetilde{b}(1-2\varepsilon)}} \\ &= \|\langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (|\eta_T v|^2))\eta_T v)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.\end{aligned}$$

In order to deal with the nonlocal term, we introduce some useful lemmas.

Lemma 4.2 ([27, Lems. A1 to A4]). *For any $s \ge 0$ we have*

$$\|\,|\nabla|^s(uv)\|_{L^r} \lesssim \|\,|\nabla|^s u\|_{L^{r_1}}\|v\|_{L^{q_2}} + \|u\|_{L^{q_1}}\|\,|\nabla|^s v\|_{L^{r_2}},$$

where $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{r_2}$, $r_i \in (1,\infty)$, $q_i \in (1,\infty]$, $i = 1, 2$.

Lemma 4.3 ([18, Lem. 3.2]). *For any* $0 < \varepsilon_1 < d - 2\alpha$ *we have*

$$\big\| |x|^{-2\alpha} * (|u|^2)\big\|_{L^\infty} \lesssim \|u\|_{L^{\frac{2d}{d-2\alpha-\varepsilon_1}}} \|u\|_{L^{\frac{2d}{d-2\alpha+\varepsilon_1}}}.$$

By using Lemma 4.2, we get

$$\begin{aligned}
&\|\langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (|\eta_T v|^2))\eta_T v)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \\
&\quad\lesssim \| |x|^{-2\alpha} * (|\eta_T v|^2)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|\langle\nabla\rangle^\sigma \eta_T v\|_{L_t^\infty L_x^2} \\
&\qquad + \|\langle\nabla\rangle^\sigma(|x|^{-2\alpha} * |\eta_T v|^2)\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \|\eta_T v\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}.
\end{aligned}$$

From Lemma 3.2, we obtain

$$\|\langle\nabla\rangle^\sigma \eta_T v\|_{L_t^\infty L_x^2} \lesssim \|v\|_{X^{\sigma,b}}.$$

For $\| |x|^{-2\alpha} * (|\eta_T v|^2)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty}$, we use Lemma 4.3 (with $0 < \varepsilon < \frac{d-2\alpha}{\alpha}$) and the Hölder inequality

$$\| |x|^{-2\alpha} * (|\eta_T v|^2)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \lesssim \|\eta_T v\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|\eta_T v\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}.$$

Then, from Sobolev embedding, we obtain

$$\|\eta_T v\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \lesssim \|\langle\nabla\rangle^{\sigma_1} v\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2(1-\varepsilon)}}}$$

and

$$\|\eta_T v\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \lesssim \|\langle\nabla\rangle^{\sigma_2} v\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2(1-\varepsilon)}}},$$

where $\sigma_1 = \frac{d-2(1-\varepsilon)}{2} - \frac{d-2\alpha-\alpha\varepsilon}{2}$ and $\sigma_2 = \frac{d-2(1-\varepsilon)}{2} - \frac{d-2\alpha+\alpha\varepsilon}{2}$. Lemma 3.2 yields

$$\|\langle\nabla\rangle^{\sigma_1} v\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2(1-\varepsilon)}}}, \quad \|\langle\nabla\rangle^{\sigma_2} v\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2(1-\varepsilon)}}} \lesssim \|v\|_{X^{\sigma,b}},$$

because $\varepsilon < \frac{1}{\alpha}(\sigma - \frac{\alpha}{2})$ gives $\sigma_2 + (2-\alpha)\cdot\frac{1-\varepsilon}{2} < \sigma_1 + (2-\alpha)\cdot\frac{1-\varepsilon}{2} < \sigma$.

For $\|\langle\nabla\rangle^\sigma(|x|^{-2\alpha} * |\eta_T v|^2)\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}}$, we use the Hardy–Littlewood–Sobolev inequality (HLS inequality) and Lemma 4.2,

$$\begin{aligned}
\|\langle\nabla\rangle^\sigma(|x|^{-2\alpha} * |\eta_T v|^2)\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} &\lesssim \|\langle\nabla\rangle^\sigma(|\eta_T v|^2)\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2d-2\alpha+\alpha\varepsilon}}} \\
&\lesssim \|\langle\nabla\rangle^\sigma v\|_{L_t^\infty L_x^2} \|v\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}.
\end{aligned}$$

By using the Sobolev inequality and Lemma 3.2, we have

$$\|v\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \lesssim \|\langle\nabla\rangle^{\sigma_2} v\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2(1-\varepsilon)}}} \lesssim \|v\|_{X^{\sigma,b}}.$$

In conclusion, we obtain that (4.2) is bounded by $\|v\|_{X^{\sigma,b}}^3 \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}$.

In order to handle the remaining terms, we make a dyadic decomposition and assume the Fourier transform of z_i, v_i is supported on the set $A(N_i) = \{\xi \sim N_i\}$. In dealing with 2nd, 4th and 6th terms, we may assume $N_1 \le N_2$.

2nd term: zzz

$$\left| \int\int \langle\nabla\rangle^\sigma ((|x|^{-2\alpha} * z_1 z_2) z_3) v_4 \, dx\, dt \right|. \tag{4.3}$$

We consider two cases separately:

(2.i) $\max(N_1, N_2, N_3) \sim \mathrm{med}(N_1, N_2, N_3)$;

(2.ii) $\max(N_1, N_2, N_3) \gg \mathrm{med}(N_1, N_2, N_3)$.

Case (2.i): $\max(N_1, N_2, N_3) \sim \mathrm{med}(N_1, N_2, N_3)$

By the Hölder inequality and Lemma 3.3, (4.3) is bounded by

$$\begin{aligned} &\|\langle\nabla\rangle^\sigma ((|x|^{-2\alpha} * (z_1 z_2)) z_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \\ &\lesssim \|\langle\nabla\rangle^\sigma ((|x|^{-2\alpha} * (z_1 z_2)) z_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}. \end{aligned}$$

By using Lemma 4.2, we get

$$\begin{aligned} &\|\langle\nabla\rangle^\sigma ((|x|^{-2\alpha} * (z_1 z_2)) z_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \\ &\lesssim \| \, |x|^{-2\alpha} * (z_1 z_2)\|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{2\alpha}}} \|\langle\nabla\rangle^\sigma z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \\ &\quad + \|\langle\nabla\rangle^\sigma (|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{2\alpha}}} \|z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}}. \end{aligned}$$

We first consider $\| \, |x|^{-2\alpha} * (z_1 z_2)\|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{2\alpha}}}$. The HLS and Hölder inequalities yield

$$\begin{aligned} \| \, |x|^{-2\alpha} * (z_1 z_2)\|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{2\alpha}}} &\lesssim \|z_1 z_2\|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{3d-4\alpha}}} \\ &\lesssim \|z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}}. \end{aligned}$$

Then from $\max(N_1, N_2, N_3) \sim \mathrm{med}(N_1, N_2, N_3)$, we obtain

$$\begin{aligned} &\| \, |x|^{-2\alpha} * (z_1 z_2)\|_{L_t^{\frac{3}{2-2\varepsilon}} L_x^{\frac{3d}{2\alpha}}} \|\langle\nabla\rangle^\sigma z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \\ &\lesssim \|z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|\langle\nabla\rangle^{\frac{\sigma}{2}} z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|\langle\nabla\rangle^{\frac{\sigma}{2}} z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \\ &\quad + \|\langle\nabla\rangle^{\frac{\sigma}{2}} z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|\langle\nabla\rangle^{\frac{\sigma}{2}} z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}}. \end{aligned}$$

For $\|\langle\nabla\rangle^{\sigma}(|x|^{-2\alpha}*(z_1z_2))\|_{L_t^{\frac{3}{2-2\varepsilon}}L_x^{\frac{3d}{2\alpha}}}$, we use the HLS inequality, the Leibniz rule and the Hölder inequality,

$$\begin{aligned}\|\langle\nabla\rangle^{\sigma}(|x|^{-2\alpha}*(z_1z_2))\|_{L_t^{\frac{3}{2-2\varepsilon}}L_x^{\frac{3d}{2\alpha}}} &\lesssim \|\langle\nabla\rangle^{\sigma}(z_1z_2)\|_{L_t^{\frac{3}{2-2\varepsilon}}L_x^{\frac{3d}{3d-4\alpha}}}\\ &\lesssim \|\langle\nabla\rangle^{\sigma}z_1\|_{L_t^{\frac{3}{1-\varepsilon}}L_x^{2d/(d-\frac{4\alpha}{3})}}\|z_2\|_{L_t^{\frac{3}{1-\varepsilon}}L_x^{2d/(d-\frac{4\alpha}{3})}}\\ &\quad+\|z_1\|_{L_t^{\frac{3}{1-\varepsilon}}L_x^{2d/(d-\frac{4\alpha}{3})}}\|\langle\nabla\rangle^{\sigma}z_2\|_{L_t^{\frac{3}{1-\varepsilon}}L_x^{2d/(d-\frac{4\alpha}{3})}}.\end{aligned}$$

Then from $\max(N_1,N_2,N_3)\sim\mathrm{med}(N_1,N_2,N_3)$, we have

$$\begin{aligned}&\|\langle\nabla\rangle^{\sigma}(|x|^{-2\alpha}*(z_1z_2))\|_{L_t^{\frac{3}{2-2\varepsilon}}L_x^{\frac{3d}{2\alpha}}}\|z_3\|_{L_t^{\frac{3}{1-\varepsilon}}L_x^{2d/(d-\frac{4\alpha}{3})}}\\ &\lesssim \|z_1\|_{L_t^{\frac{3}{1-\varepsilon}}L_x^{2d/(d-\frac{4\alpha}{3})}}\|\langle\nabla\rangle^{\frac{\sigma}{2}}z_2\|_{L_t^{\frac{3}{1-\varepsilon}}L_x^{2d/(d-\frac{4\alpha}{3})}}\|\langle\nabla\rangle^{\frac{\sigma}{2}}z_3\|_{L_t^{\frac{3}{1-\varepsilon}}L_x^{2d/(d-\frac{4\alpha}{3})}}\\ &\quad+\|\langle\nabla\rangle^{\frac{\sigma}{2}}z_1\|_{L_t^{\frac{3}{1-\varepsilon}}L_x^{2d/(d-\frac{4\alpha}{3})}}\|\langle\nabla\rangle^{\frac{\sigma}{2}}z_2\|_{L_t^{\frac{3}{1-\varepsilon}}L_x^{2d/(d-\frac{4\alpha}{3})}}\|z_3\|_{L_t^{\frac{3}{1-\varepsilon}}L_x^{2d/(d-\frac{4\alpha}{3})}}.\end{aligned}$$

Therefore, from Proposition 2.4 and $\frac{\sigma}{2}<s$, we conclude that

$$\sum_{\substack{\max(N_1,N_2,N_3)\\ \sim\mathrm{med}(N_1,N_2,N_3)}}(4.3)\lesssim R^3\|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}$$

outside a set of probability $\le C\exp\big(-c\frac{R^2}{T^{\frac{2-2\varepsilon}{3}}\|\phi\|_{H^s}^2}\big)$.

Case (2.ii): $\max(N_1,N_2,N_3)\gg\mathrm{med}(N_1,N_2,N_3)$

Since the case of $\max(N_1,N_2,N_3)\sim N_2$ can be similarly handled, we deal only with the case of $\max(N_1,N_2,N_3)\sim N_3$. Then we consider four cases separately:

(2.ii.a) $N_4\sim N_3\gg N_3^{\frac{\alpha-1}{2\alpha-1}}\gg N_2\gg N_1$;

(2.ii.b) $N_4\sim N_3\gg N_3^{\frac{\alpha-1}{2\alpha-1}}\gg N_2\sim N_1$;

(2.ii.c) $N_4\sim N_3\gg N_2\gg N_3^{\frac{\alpha-1}{2\alpha-1}}\gg N_1$;

(2.ii.d) $N_4\sim N_3\gg N_2,\ N_1\gg N_3^{\frac{\alpha-1}{2\alpha-1}}$.

Subcase (2.ii.a): $N_4\sim N_3\gg N_3^{\frac{\alpha-1}{2\alpha-1}}\gg N_2\gg N_1$

In this case, we need to apply the Hölder inequality and bilinear Strichartz estimates to (z_1,z_3) and (z_2,v_4). But because of convolution, it is not easy to apply directly. So we decompose the convolution with $|\cdot|^{-2\alpha}$. The convolution with $|\cdot|^{-2\alpha}$ can be considered a pseudo-differential operator $|\nabla|^{2\alpha-d}$ with symbol $|\xi|^{2\alpha-d}$. The Fourier support of z_1z_2 is contained in $A(2N_2)$. So we observe $|\nabla|^{2\alpha-d}\sim N_2^{2\alpha-d}$

on the Fourier support of $z_1 z_2$. Then we have

$$\frac{|\nabla|^{2\alpha-d}}{N_2^{2\alpha-d}}(z_1 z_2) = (2\pi)^{-d}\int\int e^{ix\cdot(\xi+\eta)}\chi\Big(\frac{\xi}{N_2}\Big)\Big(\frac{|\xi+\eta|}{N_2}\Big)^{2\alpha-d}\chi\Big(\frac{\eta}{N_2}\Big)\widehat{z_1}(\xi)\widehat{z_2}(\eta)\,d\xi\,d\eta,$$

where χ is supported in $B(0,1)$. Now we take the Fourier series expansion for $\Psi(\xi,\eta) = \chi(\xi)|\xi+\eta|^{2\alpha-d}\chi(\eta)$ on a cube of side length 2π that contains the support of Ψ to get

$$\chi(\xi)|\xi+\eta|^{2\alpha-d}\chi(\eta) = \sum_{k,l\in\mathbb{Z}^d} C_{k,l}e^{i(k\cdot\xi+l\cdot\eta)}$$

with $\sum_{k,l}|C_{k,l}| \le C$. Then we have the identity

$$\frac{|\nabla|^{2\alpha-d}}{N_2^{2\alpha-d}}(z_1 z_2) = \sum_{k,l\in\mathbb{Z}^d} C_{k,l}z_1^k z_2^l,$$

where $z_1^k = (2\pi)^{-d}\int e^{ix\cdot\xi}e^{ik\cdot\xi}\widehat{z_1}(\xi)\,d\xi$ and $z_2^l = (2\pi)^{-d}\int e^{ix\cdot\eta}e^{il\cdot\eta}\widehat{z_2}(\eta)\,d\eta$. So we need to estimate

$$\sum_{k,l\in\mathbb{Z}^d} C_{k,l}N_2^{2\alpha-d}\int\int\Big|\langle\nabla\rangle^\sigma(z_1^k z_2^l z_3)v_4\Big|\,dx\,dt.$$

By using the Hölder inequality and Lemmata 3.6 and 3.7, we get

$$\begin{aligned}
N_2^{2\alpha-d}&\int\int\big|z_1^k z_2^l\langle\nabla\rangle^\sigma z_3 v_4\big|\,dx\,dt\\
&\lesssim N_2^{2\alpha-d}\|z_1^k\langle\nabla\rangle^\sigma z_3\|_{L^2_{t,x}}\|z_2^l v_4\|_{L^2_{t,x}}\\
&\lesssim N_1^{\frac{d-1}{2}}N_2^{2\alpha-d+\frac{d-1}{2}+2\widetilde{\varepsilon}}N_3^{1-\alpha+\sigma+2\widetilde{\varepsilon}}\|z_1^k\|_{X^{0,b}}\|z_2^l\|_{X^{0,b}}\|z_3\|_{X^{0,b}}\|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}}.
\end{aligned}$$

Since $\|z_1^k\|_{X^{0,b}} = \|z_1\|_{X^{0,b}}$, $\|z_2^l\|_{X^{0,b}} = \|z_2\|_{X^{0,b}}$ and $\sum_{k,l}|C_{k,l}| \le C$, we have

$$\begin{aligned}
\sum_{k,l\in\mathbb{Z}^d} &C_{k,l}N_2^{2\alpha-d}\int\int\big|\langle\nabla\rangle^\sigma((z_1^k z_2^l)z_3)v_4\big|\,dx\,dt\\
&\lesssim N_1^{\frac{d-1}{2}}N_2^{2\alpha-d+\frac{d-1}{2}+2\widetilde{\varepsilon}}N_3^{1-\alpha+\sigma+2\widetilde{\varepsilon}}\|z_1\|_{X^{0,b}}\|z_2\|_{X^{0,b}}\|z_3\|_{X^{0,b}}\|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}}.
\end{aligned}$$

By using the Bernstein inequality and carrying out a sum in N_1, we get

$$\begin{aligned}
\sum_{N_1\ll N_2}\sum_{k,l\in\mathbb{Z}^d} &C_{k,l}N_2^{2\alpha-d}\int\int\big|\langle\nabla\rangle^\sigma((z_1^k z_2^l)z_3)v_4\big|\,dx\,dt\\
&\lesssim N_2^{2\alpha-1-2s+2\widetilde{\varepsilon}}N_3^{\alpha-1+\sigma-s+2\widetilde{\varepsilon}}\|z\|_{X^{s,b}}\|z_2\|_{X^{s,b}}\|z_3\|_{X^{s,b}}\|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}}.
\end{aligned}$$

Since $N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1$ and $2\alpha - 1 + s + 2\widetilde{\varepsilon} > 0$, from a sum in N_2, we obtain

$$\sum_{N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1} (4.3) \lesssim N_3^{-2s\frac{\alpha-1}{2\alpha-1}+\sigma-s+\frac{6\alpha-4}{2\alpha-1}\widetilde{\varepsilon}} \|z\|_{X^{s,b}} \|z\|_{X^{s,b}} \|z_3\|_{X^{s,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.$$

Since $-2s\frac{\alpha-1}{2\alpha-1} + \sigma - s + \frac{6\alpha-4}{2\alpha-1}\widetilde{\varepsilon} < 0$, we can carry out a sum in N_3. Then by applying Lemma 3.1, we have

$$\sum_{N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1} (4.3) \lesssim T^{-3\theta} \|\phi^\omega\|_{H^s}^3 \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.$$

Therefore, from Lemma 2.2, we conclude that

$$\sum_{N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1} (4.3) \lesssim T^{-3\theta} R^3 \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}$$

outside a set of probability $\le C \exp\big(-c\frac{R^2}{\|\phi\|_{H^s}^2}\big)$.

Subcase (2.ii.b): $N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \sim N_1$

This case is more delicate because $|\nabla|^{2\alpha-d}$ might be singular on the Fourier support of $z_1 z_2$. So we decompose $|\nabla|^{2\alpha-d}$ and z_1, z_2 so that $|\nabla|^{2\alpha-d}$ can be treated as a dyadic number N and the Fourier support of z_1, z_2 is placed on a set of size N. First we decompose $|\nabla|^{2\alpha-d}$ such that

$$|\nabla|^{2\alpha-d} = \sum_N N^{2\alpha-d}\psi(|\nabla|/N),$$

with a cut-off ψ supported in $A(1)$. Here $\psi(|\nabla|)$ is a pseudo-differential operator defined by $\psi(|\nabla|)f = \mathcal{F}^{-1}(\psi(|\cdot|)\mathcal{F}f$. Then we have

$$\begin{aligned}\int\int \langle\nabla\rangle^\sigma((|x|^{-2\alpha} * z_1 z_2)z_3)v_4\,dx\,dt \\ = \int\int \langle\nabla\rangle^\sigma\Big(\sum_{N\le N_2} N^{2\alpha-d}\psi(|\nabla|/N)(z_1 z_2)z_3\Big)v_4\,dx\,dt.\end{aligned}$$

After that we decompose z_1 and z_2 into functions having Fourier supports in cubes of side length $2^{-2}N$. Let $\{Q\}$ be a collection of essentially disjoint cubes of side length $2^{-2}N$ covering $A(N_2)$. Let us define z_{iQ} by $\widehat{z_{iQ}} = \chi_Q(\xi)\widehat{z_i}$ for $i = 1, 2$. Then we have $z_i = \sum_Q z_{iQ}$ for $i = 1, 2$. Since $N_1 \sim N_2$, we may restrict $Q \subset A(N_2)$. So, we have

$$\begin{aligned}&\int\int\big|\langle\nabla\rangle^\sigma(N^{2\alpha-d}\psi(|\nabla|/N)(z_1z_2)z_3)v_4\big|\,dx\,dt\\&\quad\lesssim\sum_{Q,Q'}\int\int\big|\langle\nabla\rangle^\sigma(N^{2\alpha-d}\psi(|\nabla|/N)(z_{1Q}z_{2Q'})z_3)v_4\big|\,dx\,dt\\&\quad=\sum_{\mathrm{dist}(Q,-Q')\le 4N}\int\int\big|\langle\nabla\rangle^\sigma(N^{2\alpha-d}\psi(|\nabla|/N)(z_{1Q}z_{2Q'})z_3)v_4\big|\,dx\,dt.\end{aligned}$$

Here, the last equality follows from $\psi(|\nabla|/N)(z_{1Q}z_{2Q'})=0$ if $\mathrm{dist}(Q,-Q')>4N$. We observe that

$$\begin{aligned}&\psi(|\nabla|/N)(z_{1Q}z_{2Q'})\\&\quad=\int\int e^{ix\cdot(\xi+\eta)}\chi(\xi/N-\xi_0)\psi((\xi+\eta)/N)\chi(\eta/N-\eta_0)\widehat{z_{1Q}}\widehat{z_{2Q'}}\,d\xi\,d\eta\end{aligned}$$

for some ξ_0, $\eta_0\in\mathbb{R}^d$ and χ supported in $B(0,1)$. Let us take the Fourier series expansion for $\Psi(\xi,\eta)=\chi(\xi-\xi_0)\psi(\xi+\eta)\chi(\eta-\eta_0)$ on the cube of side length 2π that contains the support of Ψ to get

$$\chi(\xi-\xi_0)\psi(\xi+\eta)\chi(\eta-\eta_0)=\sum_{k,l\in\mathbb{Z}^d}C_{k,l}e^{i(k\cdot\xi+l\cdot\eta)}$$

with $\sum_{k,l}|C_{k,l}|\le C$, independent of ξ_0, η_0. So, we have

$$\psi(|\nabla|/N)(z_{1Q}z_{2Q'})=\sum_{k,l\in\mathbb{Z}^d}C_{k,l}z_{1Q}^kz_{2Q'}^l,$$

where $z_{1Q}^k=\int e^{2\pi ix\cdot\xi}e^{2\pi ik\cdot\xi}\widehat{z_{1Q}}(\xi)\,d\xi$ and $z_{2Q'}^l=\int e^{2\pi ix\cdot\eta}e^{2\pi il\cdot\eta}\widehat{z_{2Q'}}(\eta)\,d\eta$. Hence we obtain

$$\begin{aligned}&\sum_{\mathrm{dist}(Q,-Q')\le 4N}\int\int\big|\langle\nabla\rangle^\sigma(N^{2\alpha-d}\psi(|\nabla|/N)(z_{1Q}z_{2Q'})z_3)v_4\big|\,dx\,dt\\&\quad\lesssim\sum_{\mathrm{dist}(Q,-Q')\le 4N}\sum_{k,l\in\mathbb{Z}^d}C_{k,l}N^{2\alpha-d}\int\int\big|\langle\nabla\rangle^\sigma((z_{1Q}^kz_{2Q'}^l)z_3)v_4\big|\,dx\,dt.\end{aligned}$$

So we need to handle $N^{2\alpha-d}\int\int\big|z_{1Q}^kz_{2Q'}^l\langle\nabla\rangle^\sigma z_3v_4\big|\,dx\,dt$. The Hölder inequality and Lemmata 3.6 and 3.7 give

$$\begin{aligned}&N^{2\alpha-d}\int\int\big|z_{1Q}^kz_{2Q'}^l\langle\nabla\rangle^\sigma z_3v_4\big|\,dx\,dt\\&\quad\lesssim N^{2\alpha-d}\|z_{1Q}^k\langle\nabla\rangle^\sigma z_3\|_{L^2_{t,x}}\|z_{2Q'}^lv_4\|_{L^2_{t,x}}\\&\quad\lesssim N^{2\alpha-1-\varepsilon_1}N_2^{\varepsilon_2}N_3^{1-\alpha+\sigma+2\widetilde{\varepsilon}}\|z_{1Q}^k\|_{X^{0,b}}\|z_{2Q'}^l\|_{X^{0,b}}\|z_3\|_{X^{0,b}}\|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}},\end{aligned}$$

where $\varepsilon_1 = \frac{2(d-1)\widetilde{\varepsilon}}{1+2\widetilde{\varepsilon}}$ and $\varepsilon_2 = \frac{2(d+2\widetilde{\varepsilon})\widetilde{\varepsilon}}{1+2\widetilde{\varepsilon}}$. Since $\|z_{1Q}^k\|_{X^{0,b}} = \|z_{1Q}\|_{X^{0,b}}$, $\|z_{2Q'}^l\|_{X^{0,b}} = \|z_{2Q'}\|_{X^{0,b}}$ and $\sum_{k,l}|C_{k,l}| \le C$, we have

$$\sum_{k,l\in\mathbb{Z}^d} C_{k,l}N^{2\alpha-d}\int\int\left|z_{1Q}^k z_{2Q'}^l\langle\nabla\rangle^\sigma z_3 v_4\right|dx\,dt$$
$$\lesssim N^{2\alpha-1-\varepsilon_1}N_2^{\varepsilon_2}N_3^{1-\alpha+\sigma+2\widetilde{\varepsilon}}\|z_{1Q}\|_{X^{0,b}}\|z_{2Q'}\|_{X^{0,b}}\|z_3\|_{X^{0,b}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.$$

Thereafter we use the Cauchy–Schwarz inequality, orthogonality and the Bernstein inequality to get

$$\begin{aligned}
&\sum_{\mathrm{dist}(Q,-Q')\le 4N}\ \sum_{k,l\in\mathbb{Z}^d} C_{k,l}N^{2\alpha-d}\int\int\left|z_{1Q}^k z_{2Q'}^l\langle\nabla\rangle^\sigma z_3 v_4\right|dx\,dt\\
&\quad\lesssim N^{2\alpha-1-\varepsilon_1}N_2^{\varepsilon_2}N_3^{1-\alpha+\sigma+2\widetilde{\varepsilon}}\|z_1\|_{X^{0,b}}\|z_2\|_{X^{0,b}}\|z_3\|_{X^{0,b}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}\\
&\quad\lesssim N^{2\alpha-1-\varepsilon_1}N_2^{-2s+\varepsilon_2}N_3^{1-\alpha+\sigma-s+2\widetilde{\varepsilon}}\|z_1\|_{X^{s,b}}\|z_2\|_{X^{s,b}}\|z_3\|_{X^{s,b}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.
\end{aligned}\tag{4.4}$$

Then a sum in $N \lesssim N_2$ gives

$$\sum_{N_2\ge N>0}(4.4) \lesssim N_2^{2\alpha-1-2s+2\widetilde{\varepsilon}}N_3^{1-\alpha+\sigma-s+2\widetilde{\varepsilon}}\|z_1\|_{X^{s,b}}\|z_2\|_{X^{s,b}}\|z_3\|_{X^{s,b}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.$$

Since $N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \sim N_1$ and $2\alpha-1-2s+2\widetilde{\varepsilon} > 0$, a sum in N_2 gives

$$\sum_{N_3^{\frac{\alpha-1}{2\alpha-1}}\gg N_2\sim N_1}\ \sum_{0<N\le N_2}(4.4)$$
$$\lesssim N_3^{-2s\frac{\alpha-1}{2\alpha-1}+\sigma-s+\frac{6\alpha-4}{2\alpha-1}\widetilde{\varepsilon}}\|z\|_{X^{s,b}}\|z\|_{X^{s,b}}\|z_3\|_{X^{s,b}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.$$

Thus, from Lemma 3.1, we obtain

$$\lesssim T^{-3\theta}N_3^{-2s\frac{\alpha-1}{2\alpha-1}+\sigma-s+\frac{6\alpha-4}{2\alpha-1}\widetilde{\varepsilon}}\|\phi^\omega\|_{H^s}^2\|P_{N_3}\phi^\omega\|_{H^s}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.$$

We can carry out a sum in N_3 because $-2s\frac{\alpha-1}{2\alpha-1}+\sigma-s+\frac{6\alpha-4}{2\alpha-1}\widetilde{\varepsilon}$ is negative.

Therefore, by using Lemma 2.2, we conclude that

$$\sum_{N_4\sim N_3\gg N_3^{\frac{\alpha-1}{2\alpha-1}}\gg N_2\sim N_1}(4.4) \lesssim T^{-3\theta}R^3\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}$$

outside a set of probability $\le C\exp\big(-c\frac{R^2}{\|\phi\|_{H^s}^2}\big)$.

Subcase (2.ii.c) $N_4 \sim N_3 \gg N_2 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$

Adopting the method in subcase (2.ii.a), it suffices to estimate

$$\sum_{k,l\in\mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int\int \left| z_1^k z_2^l \langle\nabla\rangle^\sigma z_3 v_4 \right| dx\, dt. \tag{4.5}$$

The Hölder inequality and Lemmata 3.6 and 3.7 give

$$\begin{aligned} N_2^{2\alpha-d} \int\int & \left| z_1^k z_2^l \langle\nabla\rangle^\sigma z_3 v_4 \right| dx\, dt \\ &\lesssim N_2^{2\alpha-d} \|z_2^l\|_{L^4_{t,x}} \|\langle\nabla\rangle^\sigma z_3\|_{L^4_{t,x}} \|z_1^k v_4\|_{L^2_{t,x}} \\ &\lesssim N_2^{2\alpha-d} N_1^{\frac{d-1}{2}+2\widetilde{\varepsilon}} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \|z_1^k\|_{X^{0,b}} \|z_2^l\|_{L^4_{t,x}} \|\langle\nabla\rangle^\sigma z_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}}. \end{aligned}$$

Thereafter, from sums in k, l, the Bernstein inequality and Lemma 3.1, we obtain

$$\begin{aligned} (4.5) &\lesssim T^{-\theta} N_2^{-s+2\alpha-d} N_3^{\sigma-s} N_1^{\frac{d-1}{2}+2\widetilde{\varepsilon}-s} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \\ &\quad \times \|P_{N_1}\phi^\omega\|_{H^s} \|\langle\nabla\rangle^s z_2\|_{L^4_{t,x}} \|\langle\nabla\rangle^s z_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}}. \end{aligned}$$

Then sums in N_1 and N_2 yield

$$\begin{aligned} \sum_{N_3\geq N_2\geq N_3^{\frac{\alpha-1}{2\alpha-1}}} \sum_{N_3^{\frac{\alpha-1}{2\alpha-1}}\geq N_1=1} & (4.5) \\ &\lesssim T^{-\theta} N_3^{-s\left(\frac{4\alpha-3}{2\alpha-1}\right)+\frac{\alpha-1}{2(2\alpha-1)}(2\alpha-d)+\sigma+\frac{6\alpha-4}{2\alpha-1}\widetilde{\varepsilon}} \\ &\quad \times \|\phi^\omega\|_{H^s} \|\langle\nabla\rangle^s z_2\|_{L^4_{t,x}} \|\langle\nabla\rangle^s z_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}}. \end{aligned}$$

Hence, from a sum in N_3 and Lemma 2.2 and Proposition 2.4, we conclude that

$$\sum_{N_4\sim N_3\gg N_2\gg N_3^{\frac{\alpha-1}{2\alpha-1}}\gg N_1} (4.5) \lesssim T^{-\theta} R^3 \|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}}$$

outside a set of probability $\leq C\exp\left(-c\frac{R^2}{\|\phi\|_{H^s}^2}\right) + C\exp\left(-c\frac{R^2}{T^{\frac12}\|\phi\|_{H^s}^2}\right)$.

Subcase (2.ii.d): $N_4 \sim N_3 \gg N_2 \gg N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}}$

The Hölder inequality and Lemma 3.3 yield that (4.3) is bounded by

$$\begin{aligned} &\|\langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (z_1 z_2)) z_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \\ &\quad\lesssim \|\langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (z_1 z_2)) z_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}}. \end{aligned}$$

Then the HLS and Hölder inequalities and Lemma 4.2 give

$$
\begin{aligned}
&(4.3)\\
&\lesssim \|\langle\nabla\rangle^\sigma z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}\\
&\quad + \|z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|\langle\nabla\rangle^\sigma z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}\\
&\quad + \|z_1\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|z_2\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|\langle\nabla\rangle^\sigma z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.
\end{aligned}
$$

Thereafter we use Bernstein's inequality and carry out sums in N_1, N_2 to get

$$
\begin{aligned}
&\sum_{N_3\geq N_2\geq 1}\ \sum_{N_2\geq N_1\geq N_3^{\frac{\alpha-1}{2\alpha-1}}} (4.3)\\
&\qquad\lesssim N_3^{\sigma-\frac{4\alpha-3}{2\alpha-1}s} \|\langle\nabla\rangle^s z\|^2_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|\langle\nabla\rangle^s z_3\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{2d/(d-\frac{4\alpha}{3})}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.
\end{aligned}
$$

Therefore, from Proposition 2.4, we conclude that

$$
\sum_{N_4\sim N_3\gg N_2\gg N_1\gg N_3^{\frac{\alpha-1}{2\alpha-1}}} (4.3) \lesssim R^3 \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}
$$

outside a set of probability $\leq C\exp\big(-c\frac{R^2}{T^{\frac{2-2\varepsilon}{3}}\|\phi\|^2_{H^s}}\big)$.

3rd term: vzv

$$
\Big|\int\!\!\int \langle\nabla\rangle^\sigma(|x|^{-2\alpha}*(v_1z_2)v_3)v_4\,dx\,dt\Big|. \tag{4.6}
$$

We consider two cases separately:

(3.i) $\max(N_1,N_3)\gtrsim N_2$;

(3.ii) $\max(N_1,N_3)\ll N_2\sim N_4$.

Case (3.i): $\max(N_1,N_3)\gtrsim N_2$

We assume $N_1\geq N_3$, because the other case can be handled similarly. The Hölder inequality and Lemmata 4.2 and 3.3 yield

$$
\begin{aligned}
(4.6) &\lesssim \|\langle\nabla\rangle^\sigma((|x|^{-2\alpha}*(v_1z_2))v_3)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}}L_x^2}\\
&\lesssim \|\,|x|^{-2\alpha}*(v_1z_2)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^\infty} \|\langle\nabla\rangle^\sigma v_3\|_{L_t^\infty L_x^2}\|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}\\
&\quad + \|\langle\nabla\rangle^\sigma(|x|^{-2\alpha}*(v_1z_2))\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \|v_3\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.
\end{aligned}
$$

Thereafter we use Lemma 4.3, the Hölder inequality, Sobolev embedding and Lemma 4.2 to get

$$\lesssim \Big(\|v_1\|^{\frac12}_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_1\|^{\frac12}_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|z_2\|^{\frac12}_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|z_2\|^{\frac12}_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} + \|\langle\nabla\rangle^\sigma (v_1 z_2)\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2d-2\alpha+\alpha\varepsilon}}} \Big) \times \|v_3\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.$$

Then Lemmata 4.2 and 3.2 give

$$\lesssim \Big(\|v_1\|_{X^{\sigma,b}} \|z_2\|^{\frac12}_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|z_2\|^{\frac12}_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} + \|\langle\nabla\rangle^\sigma v_1\|_{L_t^\infty L_x^2} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} + \|v_1\|_{L_t^\infty L_x^2} \|\langle\nabla\rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \Big) \times \|v_3\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.$$

Since $N_1 \gtrsim N_2$, we have

$$\|v_1\|_{L_t^\infty L_x^2} \|\langle\nabla\rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \lesssim \|\langle\nabla\rangle^\sigma v_1\|_{L_t^\infty L_x^2} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}.$$

Therefore, from Proposition 2.4, we conclude that

$$\sum_{\max(N_1,N_3)\gtrsim N_2} (4.6) \lesssim R\|v\|^2_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}$$

outside a set of probability $\le C\exp\big(-c\frac{R^2}{T^{1-\varepsilon}\|\phi\|_{H^{0+}}}\big)$.

Case (3.ii): $\max(N_1,N_3) \ll N_2 \sim N_4$

We assume $N_3 \ge N_1$, because the other case can be handled similarly. As in subcase (2.ii.a), we shall deal with

$$\sum_{k,l\in\mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int\int \big|v_1^k \langle\nabla\rangle^\sigma z_2^l v_3 v_4\big|\, dx\, dt. \tag{4.7}$$

The Hölder inequality and Lemmata 3.6 and 3.7 give

$$\begin{aligned} N_2^{2\alpha-d} \int\int & \big|v_1^k \langle\nabla\rangle^\sigma z_2^l v_3 v_4\big|\, dx\, dt \\ &\lesssim N_2^{2\alpha-d} \|\langle\nabla\rangle^\sigma z_2^l\|_{L^4_{t,x}} \|v_3\|_{L^4_{t,x}} \|v_1^k v_4\|_{L^2_{t,x}} \\ &\lesssim N_2^{2\alpha-d} N_1^{\frac{d-1}{2}+2\tilde\varepsilon} N_4^{\frac{1-\alpha}{2}+2\tilde\varepsilon} \|v_1^k\|_{X^{0,b}} \|\langle\nabla\rangle^\sigma z_2^l\|_{L^4_{t,x}} \|v_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}. \end{aligned}$$

Thereafter we carry out sums in k, l, then use the Bernstein inequality and Lemma 3.2 to obtain

$$\begin{aligned}(4.7) &\lesssim N_2^{\sigma-s+2\alpha-d} N_1^{\frac{d-1}{2}+2\widetilde{\varepsilon}-\sigma} N_3^{-\sigma+\frac{d-\alpha}{4}} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \\ &\quad \times \|v_1\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z_2\|_{L^4_{t,x}} \|v_3\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.\end{aligned}$$

Then we carry out a sum in N_1:

$$\begin{aligned}\sum_{N_3\geq N_1\geq 1} (4.7) &\lesssim N_2^{\sigma-s+2\alpha-d+\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} N_3^{\frac{d-1}{2}+2\widetilde{\varepsilon}-\sigma-\sigma+\frac{d-\alpha}{4}} \\ &\quad \times \|v\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z_2\|_{L^4_{t,x}} \|v_3\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.\end{aligned}$$

We observe that $\sigma - s + 2\alpha - d + \frac{1-\alpha}{2} + 2\widetilde{\varepsilon} < 0$. So if $\frac{d-1}{2} - 2\sigma + \frac{d-\alpha}{4} + 2\widetilde{\varepsilon} < 0$, then a sum can be carried out over N_2, N_3. Otherwise it should be checked that the sum of exponents of N_2 and N_3 is negative. Actually the following conditions hold: $-\sigma - s + \frac{5}{4}\alpha - \frac{d}{4} + 4\widetilde{\varepsilon} < -\big(\frac{6\alpha-4}{4\alpha-3}\big)\sigma + \frac{5}{4}\alpha - \frac{d}{4} + 4\widetilde{\varepsilon}$ (because $s > \frac{2\alpha-1}{4\alpha-3}\sigma$) $< -\big(\frac{6\alpha-4}{4\alpha-3}\big)\sigma + \frac{3}{4}\alpha + 4\widetilde{\varepsilon}$ (because $d > 2\alpha$) $< -\frac{1}{4(4\alpha-3)}\alpha + 4\widetilde{\varepsilon}$ (because $\sigma > \frac{\alpha}{2}$) < 0. Hence we have

$$\sum_{\max(N_1,N_3)\ll N_2\sim N_4} (4.7) \lesssim \|v\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z\|_{L^4_{t,x}} \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.$$

Therefore, from Lemma 2.3, we conclude that

$$\sum_{\max(N_1,N_3)\ll N_2\sim N_4} (4.7) \lesssim R\|v\|^2_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}$$

outside a set of probability $\leq C\exp\big(-c\frac{R^2}{T^{\frac{1}{2}}\|\phi\|^2_{H^s}}\big)$.

4th term: vvz

$$\Big|\int\int \langle\nabla\rangle^\sigma((|x|^{-2\alpha} * v_1v_2)z_3)v_4\,dx\,dt\Big|. \tag{4.8}$$

We consider three cases separately:

(4.i) $N_2 \gtrsim N_3$;

(4.ii) $N_1 \ll N_2 \ll N_3 \sim N_4$;

(4.iii) $N_1 \sim N_2 \ll N_3 \sim N_4$.

Case (4.i): $N_2 \gtrsim N_3$

The Hölder inequality and Lemma 3.3 yield

$$\begin{aligned}(4.8) &\lesssim \|\langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (v_1v_2))z_3)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^2}\|v_4\|_{L_t^{\frac{1}{\varepsilon}}L_x^2}\\ &\lesssim \|\,|x|^{-2\alpha} * (v_1v_2)\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}}\|\langle\nabla\rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}\\ &\quad + \|\langle\nabla\rangle^\sigma(|x|^{-2\alpha} * (v_1v_2))\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}}\|z_3\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.\end{aligned}$$

Then we use the HLS inequality to get

$$\begin{aligned}&\lesssim \|v_1v_2\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{2d-2\alpha+\alpha\varepsilon}}}\|\langle\nabla\rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}\\ &\quad + \|\langle\nabla\rangle^\sigma(v_1v_2)\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{2d-2\alpha+\alpha\varepsilon}}}\|z_3\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.\end{aligned}$$

Thereafter, from the Hölder inequality and Lemma 4.2, we obtain

$$\begin{aligned}&\lesssim \|v_1\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}\|v_2\|_{L_t^\infty L_x^2}\|\langle\nabla\rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}\\ &\quad + \|v_1\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}\|\langle\nabla\rangle^\sigma v_2\|_{L_t^\infty L_x^2}\|z_3\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}\\ &\quad + \|\langle\nabla\rangle^\sigma v_1\|_{L_t^\infty L_x^2}\|v_2\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}\|z_3\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.\end{aligned}$$

Therefore, from $N_2 \gtrsim N_3$, Lemma 3.2 and Proposition 2.4, we conclude that

$$\sum_{N_2\gtrsim N_3}(4.8) \lesssim R\|v\|_{X^{\sigma,b}}^2\|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}$$

outside a set of probability $\leq C\exp\big(-c\frac{R^2}{T^{1-\varepsilon}\|\phi\|_{H^{0+}}}\big)$.

Case (4.ii): $N_1 \ll N_2 \ll N_3 \sim N_4$

As in subcase (2.ii.a), we shall deal with

$$\sum_{k,l\in\mathbb{Z}^d} C_{k,l}N_2^{2\alpha-d}\int\int\big|v_1^kv_2^l\langle\nabla\rangle^\sigma z_3v_4\big|\,dx\,dt. \tag{4.9}$$

By using the Hölder inequality and Lemma 3.7, we get

$$\begin{aligned}&N_2^{2\alpha-d}\int\int\big|v_1^kv_2^l\langle\nabla\rangle^\sigma z_3v_4\big|\,dx\,dt\\ &\quad\lesssim N_2^{2\alpha-d}\|v_2^l\|_{L_{t,x}^4}\|\langle\nabla\rangle^\sigma z_3\|_{L_{t,x}^4}\|v_1^kv_4\|_{L_{t,x}^2}\\ &\quad\lesssim N_2^{2\alpha-d}N_1^{\frac{d-1}{2}+2\tilde\varepsilon}N_4^{\frac{1-\alpha}{2}+2\tilde\varepsilon}\|v_1^k\|_{X^{0,b}}\|v_2^l\|_{L_{t,x}^4}\|\langle\nabla\rangle^\sigma z_3\|_{L_{t,x}^4}\|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.\end{aligned}$$

Then sums in k, l, the Bernstein inequality and Lemma 3.2 yield

$$(4.9) \lesssim N_2^{-\sigma+\frac{d-\alpha}{4}+2\alpha-d} N_1^{\frac{d-1}{2}+2\tilde{\varepsilon}-\sigma} N_3^{\sigma-s} N_4^{\frac{1-\alpha}{2}+2\tilde{\varepsilon}} \times \|v_1\|_{X^{\sigma,b}} \|v_2\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.$$

Thereafter we carry out a sum in N_1 to get

$$\sum_{N_2\geq N_1\geq 1} (4.9) \lesssim N_2^{-2\sigma-\frac{d}{4}+\frac{7}{4}\alpha-\frac{1}{2}+2\tilde{\varepsilon}} N_3^{\sigma-s+\frac{1-\alpha}{2}+2\tilde{\varepsilon}} \times \|v\|_{X^{\sigma,b}} \|v_2\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.$$

Since the exponents of N_2 and N_3 are negative, one can carry out sums over N_2 and N_3.

Therefore, from Proposition 2.4, we conclude that

$$\sum_{N_1\ll N_2\ll N_3\sim N_4} (4.9) \lesssim \|v\|^2_{X^{\sigma,b}} \|\langle\nabla\rangle^s z\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}} \lesssim R\|v\|^2_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}$$

outside a set of probability $\leq C\exp\big(-c\frac{R^2}{T^{\frac{1}{2}}\|\phi\|^2_{H^s}}\big)$.

Case (4.iii): $N_1 \sim N_2 \ll N_3 \sim N_4$

As in case (2.ii.b), we consider

$$\sum_{\mathrm{dist}(Q,-Q')\leq 4N} \sum_{k,l\in\mathbb{Z}^d} C_{k,l} N^{2\alpha-d} \int\int \big|v^k_{1Q} v^l_{2Q'} \langle\nabla\rangle^\sigma z_3 v_4\big|\, dx\, dt. \tag{4.10}$$

The Hölder inequality and Lemmata 3.6 and 3.7 give

$$\begin{aligned} N^{2\alpha-d} &\int\int \big|v^k_{1Q} v^l_{2Q'} \langle\nabla\rangle^\sigma z_3 v_4\big|\, dx\, dt \\ &\lesssim N^{2\alpha-d} \|v^l_{2Q'}\langle\nabla\rangle^\sigma z_3\|_{L^2_{t,x}} \|v^k_{1Q} v_4\|_{L^2_{t,x}} \\ &\lesssim N^{2\alpha-d} N^{d-1-\varepsilon_1} N_1^{\varepsilon_2} N_3^{1-\alpha+2\tilde{\varepsilon}} \|v^k_{1Q}\|_{X^{0,b}} \|v^l_{2Q'}\|_{X^{0,b}} \|\langle\nabla\rangle^\sigma z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}, \end{aligned}$$

where $\varepsilon_1 = \frac{2(d-1)\tilde{\varepsilon}}{1+2\tilde{\varepsilon}}$ and $\varepsilon_2 = \frac{2(d+2\tilde{\varepsilon})\tilde{\varepsilon}}{1+2\tilde{\varepsilon}}$. By carrying out sums in k, l, we have

$$\begin{aligned} \sum_{k,l\in\mathbb{Z}^d} &C_{k,l} N^{2\alpha-d} \int\int \big|v^k_{1Q} v^l_{2Q'} \langle\nabla\rangle^\sigma z_3 v_4\big|\, dx\, dt \\ &\lesssim N^{2\alpha-d} N^{d-1-\varepsilon_1} N_1^{\varepsilon_2} N_3^{1-\alpha+2\tilde{\varepsilon}} \|v_{1Q}\|_{X^{0,b}} \|v_{2Q'}\|_{X^{0,b}} \|\langle\nabla\rangle^\sigma z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}. \end{aligned}$$

Thereafter we use the Cauchy–Schwarz inequality, orthogonality and the Bernstein inequality to obtain

$$\begin{aligned}(4.10) &\lesssim N^{2\alpha-d}N^{d-1-\varepsilon_1}N_1^{\varepsilon_2}N_3^{1-\alpha+2\widetilde{\varepsilon}}\|v_1\|_{X^{0,b}}\|v_2\|_{X^{0,b}}\|\langle\nabla\rangle^{\sigma}z_3\|_{X^{0,b}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}\\ &\lesssim N^{2\alpha-d}N^{d-1-\varepsilon_1}N_1^{-\sigma+\varepsilon_2}N_2^{-\sigma}N_3^{1-\alpha+\sigma-s+2\widetilde{\varepsilon}}\\ &\quad\times\|v_1\|_{X^{\sigma,b}}\|v_2\|_{X^{\sigma,b}}\|\langle\nabla\rangle^{s}z_3\|_{X^{0,b}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.\end{aligned}$$

Since $N\le N_1\sim N_2\ll N_4$, we have

$$\sum_{N_3\ge N_2\ge 1}\sum_{N_2\ge N>0}(4.10)\lesssim N_3^{\alpha-1-\sigma-s+4\widetilde{\varepsilon}}\|v\|_{X^{\sigma,b}}\|v\|_{X^{\sigma,b}}\|\langle\nabla\rangle^{s}z_3\|_{X^{0,b}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.$$

Then, from Lemma 3.1, we get

$$\lesssim T^{-\theta}N_3^{\alpha-1-\sigma-s+4\widetilde{\varepsilon}}\|P_{N_3}\phi^{\omega}\|_{H^s}\|v\|_{X^{\sigma,b}}\|v\|_{X^{\sigma,b}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.$$

Because $\alpha-1-\sigma-s+4\widetilde{\varepsilon}<0$, we have that a sum in N_3 is finite.

Hence, by using Lemma 2.1, we conclude that

$$\sum_{N_1\sim N_2\ll N_3\sim N_4}(4.10)\lesssim T^{-\theta}R\|v\|_{X^{\sigma,b}}^2\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}$$

outside a set of probability $\le C\exp\big(-c\frac{R^2}{\|\phi\|_{H^s}^2}\big)$.

5th term: vzz

$$\Big|\int\int\langle\nabla\rangle^{\sigma}((|x|^{-2\alpha}*(v_1z_2))z_3)v_4\,dx\,dt\Big|. \tag{4.11}$$

We consider four cases separately:

- (5.i) $N_1\gtrsim\max(N_2,N_3)$;
- (5.ii) $N_1\ll N_2\sim N_3$;
- (5.iii) $N_1,N_2\ll N_3$;
- (5.iv) $N_1,N_3\ll N_2$.

Case (5.i): $N_1\gtrsim\max(N_2,N_3)$

The Hölder inequality and Lemmata 4.2 and 3.3 give

$$\begin{aligned}(4.11) &\lesssim\|\langle\nabla\rangle^{\sigma}((|x|^{-2\alpha}*(v_1z_2))z_3)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^2}\|v_4\|_{L_t^{\frac{1}{\varepsilon}}L_x^2}\\ &\lesssim\|\,|x|^{-2\alpha}*(v_1z_2)\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}}\|\langle\nabla\rangle^{\sigma}z_3\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}\\ &\quad+\|\langle\nabla\rangle^{\sigma}(|x|^{-2\alpha}*(v_1z_2))\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}}\|z_3\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.\end{aligned}$$

Thereafter we use the HLS inequality to obtain

$$\lesssim \|v_1 z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2d-2\alpha+\alpha\varepsilon}}} \|\langle\nabla\rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}$$
$$+ \|\langle\nabla\rangle^\sigma (v_1 z_2)\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.$$

Then the Hölder inequality and Lemma 4.2 yield

$$\lesssim \|v_1\|_{L_t^\infty L_x^2} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|\langle\nabla\rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}$$
$$+ \|v_1\|_{L_t^\infty L_x^2} \|\langle\nabla\rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}$$
$$+ \|\langle\nabla\rangle^\sigma v_1\|_{L_t^\infty L_x^2} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.$$

Therefore, from Proposition 2.4, we conclude that

$$\sum_{N_1 \gtrsim \max(N_2,N_3)} (4.11) \lesssim R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}$$

outside a set of probability $\leq C \exp\big(-c\frac{R^2}{T^{1-\varepsilon}\|\phi\|_{H^{0+}}}\big)$.

Case (5.ii): $N_1 \ll N_2 \sim N_3$

We assume $N_4 \geq N_1$, because the other case can be handled similarly. As in subcase (2.ii.a), we shall deal with

$$\sum_{k,l\in\mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int\int \left|v_1^k \langle\nabla\rangle^\sigma z_2^l z_3 v_4\right| dx\, dt. \tag{4.12}$$

By using the Hölder inequality and Lemma 3.7, we get

$$N_2^{2\alpha-d} \int\int \left|v_1^k \langle\nabla\rangle^\sigma z_2^l z_3 v_4\right| dx\, dt$$
$$\lesssim N_2^{2\alpha-d} \|\langle\nabla\rangle^\sigma z_2^l\|_{L^4_{t,x}} \|z_3\|_{L^4_{t,x}} \|v_1^k v_4\|_{L^2_{t,x}}$$
$$\lesssim N_2^{2\alpha-d} N_1^{\frac{d-1}{2}+2\tilde{\varepsilon}} N_4^{\frac{1-\alpha}{2}+2\tilde{\varepsilon}} \|v_1^k\|_{X^{0,b}} \|\langle\nabla\rangle^\sigma z_2^l\|_{L^4_{t,x}} \|z_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.$$

Then sums in k, l, the Bernstein inequality and $N_1 \leq N_4$ give

$$(4.12) \lesssim N_2^{\sigma-s+2\alpha-d} N_1^{\frac{d-1}{2}+2\tilde{\varepsilon}-\sigma} N_3^{-s} N_4^{\frac{1-\alpha}{2}+2\tilde{\varepsilon}}$$
$$\times \|v_1\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z_2\|_{L^4_{t,x}} \|\langle\nabla\rangle^s z_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}$$
$$\lesssim N_2^{\sigma-s+2\alpha-d} N_1^{\frac{d-1}{2}-\sigma+\frac{1-\alpha}{2}+4\tilde{\varepsilon}} N_3^{-s}$$
$$\times \|v_1\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z_2\|_{L^4_{t,x}} \|\langle\nabla\rangle^s z_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.$$

Thereafter we carry out a sum in N_1:

$$\sum_{N_1=1}^{N_2} (4.12) \lesssim \begin{cases} N_2^{\frac{3\alpha-d}{2}-2s+4\widetilde{\varepsilon}} \|v\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z_2\|_{L^4_{t,x}} \|\langle\nabla\rangle^s z_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}} \\ \quad (\text{when } \frac{d-1}{2} - \sigma + \frac{1-\alpha}{2} + 4\widetilde{\varepsilon} > 0), \\ N_2^{\sigma-2s+2\alpha-d} \|v\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z_2\|_{L^4_{t,x}} \|\langle\nabla\rangle^s z_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}} \\ \quad (\text{when } \frac{d-1}{2} - \sigma + \frac{1-\alpha}{2} + 4\widetilde{\varepsilon} \leq 0). \end{cases}$$

Since $\frac{3\alpha-d}{2} - 2s + 4\widetilde{\varepsilon} \leq \frac{\alpha}{2} - 2s + 4\widetilde{\varepsilon} < 0$ and $\sigma - 2s + 2\alpha - d < 0$, we have

$$\sum_{N_1 \ll N_2 \sim N_3} (4.12) \lesssim \|v\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z\|^2_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.$$

Therefore, from Proposition 2.4, we conclude that

$$\sum_{N_1 \ll N_2 \sim N_3} (4.12) \lesssim R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}$$

outside a set of probability $\leq C \exp\big(-c\frac{R^2}{T^{\frac{1}{2}}\|\phi\|^2_{H^s}}\big)$.

Case (5.iii): $N_1, N_2 \ll N_3$

We consider five cases separately:

(5.iii.a) $N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1$;
(5.iii.b) $N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \sim N_1$;
(5.iii.c) $N_4 \sim N_3 \gg N_2 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$;
(5.iii.d) $N_4 \sim N_3 \gg N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2$;
(5.iii.e) $N_4 \sim N_3 \gg N_2, N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}}$.

Subcases (5.iii.a), (5.iii.c) and (5.iii.d) are similar to subcase (2.ii.a), so we need to estimate

$$\sum_{k,l\in\mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int\int \big|v_1^k z_2^l \langle\nabla\rangle^\sigma z_3 v_4\big| \, dx\, dt. \tag{4.13}$$

And subcase (5.iii.b) is similar to subcase (2.ii.b), so we have to deal with

$$\sum_{\text{dist}(Q,-Q')\leq 4N} \sum_{k,l\in\mathbb{Z}^d} C_{k,l} N^{2\alpha-d} \int\int \big|v_{1Q}^k z_{2Q'}^l \langle\nabla\rangle^\sigma z_3 v_4\big| \, dx\, dt. \tag{4.14}$$

Subcase (5.iii.a): $N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1$

The Hölder inequality and Lemmata 3.6 and 3.7 give

$$\begin{aligned}
&N_2^{2\alpha-d} \int\int \left|v_1^k z_2^l \langle\nabla\rangle^\sigma z_3 v_4\right| dx\, dt \\
&\quad\lesssim N_2^{2\alpha-d} \|v_1^k \langle\nabla\rangle^\sigma z_3\|_{L^2_{t,x}} \|z_2^l v_4\|_{L^2_{t,x}} \\
&\quad\lesssim N_2^{2\alpha-d} N_1^{\frac{d-1}{2}} N_3^{\frac{1-\alpha}{2}} N_2^{\frac{d-1}{2}+2\tilde{\varepsilon}} N_4^{\frac{1-\alpha}{2}+2\tilde{\varepsilon}} \\
&\qquad \times \|v_1^k\|_{X^{0,b}} \|\langle\nabla\rangle^\sigma z_3\|_{X^{0,b}} \|z_2^l\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.
\end{aligned}$$

Then, from sums in k, l, the Bernstein inequality and Lemma 3.1, we have

$$\begin{aligned}
(4.13) &\lesssim N_1^{\frac{d-1}{2}-\sigma} N_3^{\frac{1-\alpha}{2}+\sigma-s} N_2^{\frac{d-1}{2}+2\alpha-d-s+2\tilde{\varepsilon}} N_4^{\frac{1-\alpha}{2}+2\tilde{\varepsilon}} \\
&\quad \times \|v_1\|_{X^{\sigma,b}} \|z_2\|_{X^{s,b}} \|z_3\|_{X^{s,b}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}} \\
&\lesssim T^{-2\theta} N_1^{\frac{d-1}{2}-\sigma} N_3^{\frac{1-\alpha}{2}+\sigma-s} N_2^{\frac{d-1}{2}+2\alpha-d-s+2\tilde{\varepsilon}} N_4^{\frac{1-\alpha}{2}+2\tilde{\varepsilon}} \\
&\quad \times \|v_1\|_{X^{\sigma,b}} \prod_{j=2}^{3} \|P_{N_j}\phi^\omega\|_{H^s} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.
\end{aligned}$$

Thereafter we carry out sums in N_1 and N_2:

$$\begin{aligned}
&\sum_{N_3^{\frac{\alpha-1}{2\alpha-1}} \geq N_2 \geq 1} \; \sum_{N_2 \geq N_1 \geq 1} (4.13) \\
&\quad \lesssim T^{-2\theta} N_3^{\sigma(\frac{\alpha}{2\alpha-1})-s(\frac{3\alpha-2}{2\alpha-1})+2\tilde{\varepsilon}(\frac{3\alpha-2}{2\alpha-1})} \|v\|_{X^{\sigma,b}} \|\phi^\omega\|_{H^s} \|P_{N_3}\phi^\omega\|_{H^s} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.
\end{aligned}$$

Since $s > \sigma\frac{2\alpha-1}{4\alpha-3} > \sigma\frac{\alpha}{3\alpha-2}$, a sum in N_3 can be also carried out.

Therefore, from Lemma 2.2, we conclude that

$$\sum_{N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \gg N_1} (4.13) \lesssim T^{-2\theta} R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}$$

outside a set of probability $\leq C\exp(-c\frac{R^2}{\|\phi\|^2_{H^s}})$.

Subcase (5.iii.b): $N_4 \sim N_3 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2 \sim N_1$

The Hölder inequality and Lemmata 3.6 and 3.7 give

$$\begin{aligned}
&N^{2\alpha-d} \int\int \left|v_{1Q}^k z_{2Q'}^l \langle\nabla\rangle^\sigma z_3 v_4\right| dx\, dt \\
&\quad\lesssim N^{2\alpha-d} \|v_{1Q}^k \langle\nabla\rangle^\sigma z_3\|_{L^2_{t,x}} \|z_{2Q'}^l v_4\|_{L^2_{t,x}} \\
&\quad\lesssim N^{2\alpha-d} N^{d-1-\varepsilon_1} N_2^{\varepsilon_2} N_3^{1-\alpha+2\tilde{\varepsilon}} \|v_{1Q}^k\|_{X^{0,b}} \|z_{2Q'}^l\|_{X^{0,b}} \|\langle\nabla\rangle^\sigma z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}},
\end{aligned}$$

where $\varepsilon_1 = \frac{2(d-1)\widetilde{\varepsilon}}{1+2\widetilde{\varepsilon}}$ and $\varepsilon_2 = \frac{2(d+2\widetilde{\varepsilon})\widetilde{\varepsilon}}{1+2\widetilde{\varepsilon}}$. By carrying out sums in k, l, we have

$$\begin{aligned}\sum_{k,l\in\mathbb{Z}^d} C_{k,l} \int\int \left|v_{1Q}^k z_{2Q'}^l \langle\nabla\rangle^\sigma z_3 v_4\right| dx\, dt \\ \lesssim N^{2\alpha-d} N^{d-1-\varepsilon_1} N_2^{\varepsilon_2} N_3^{1-\alpha+2\widetilde{\varepsilon}} \\ \times \|v_{1Q}\|_{X^{0,b}} \|z_{2Q'}\|_{X^{0,b}} \|\langle\nabla\rangle^\sigma z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.\end{aligned}$$

Then the Cauchy–Schwarz inequality, orthogonality and the Bernstein inequality yield

$$\begin{aligned}\sum_{N_2\geq N>0} (4.14) &\lesssim \sum_{N_2\geq N>0} N^{2\alpha-d} N^{d-1-\varepsilon_1} N_2^{\varepsilon_2} N_3^{1-\alpha+2\widetilde{\varepsilon}} \\ &\quad \times \|v_1\|_{X^{0,b}} \|z_2\|_{X^{0,b}} \|\langle\nabla\rangle^\sigma z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}} \\ &\lesssim N_2^{2\alpha-1+2\widetilde{\varepsilon}} N_3^{1-\alpha+2\widetilde{\varepsilon}} \|v_1\|_{X^{0,b}} \|z_2\|_{X^{0,b}} \|\langle\nabla\rangle^\sigma z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}} \\ &\lesssim N_2^{2\alpha-1-\sigma-s+2\widetilde{\varepsilon}} N_3^{1-\alpha+\sigma-s+2\widetilde{\varepsilon}} \\ &\quad \times \|v_1\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z_2\|_{X^{0,b}} \|\langle\nabla\rangle^s z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.\end{aligned}$$

Since $N_2 \ll N_3^{\frac{\alpha-1}{2\alpha-1}}$ and $2\alpha - 1 - \sigma - s + 2\widetilde{\varepsilon} > 0$, we can carry out a sum in N_2 so that

$$\begin{aligned}\sum_{N_3^{\frac{\alpha-1}{2\alpha-1}}\geq N_2\geq 1} \sum_{N_2\geq N>0} (4.14) \\ \lesssim N_3^{\sigma(\frac{\alpha}{2\alpha-1})-s(\frac{3\alpha-2}{2\alpha-1})+2\widetilde{\varepsilon}(\frac{3\alpha-2}{2\alpha-1})} \|v\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z\|_{X^{0,b}} \|\langle\nabla\rangle^s z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.\end{aligned}$$

Then from Lemma 3.1, we obtain

$$\lesssim T^{-2\theta} N_3^{\sigma(\frac{\alpha}{2\alpha-1})-s(\frac{3\alpha-2}{2\alpha-1})+2\widetilde{\varepsilon}(\frac{3\alpha-2}{2\alpha-1})} \|v\|_{X^{\sigma,b}} \|\phi^\omega\|_{H^s} \|P_{N_3}\phi^\omega\|_{H^s} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.$$

Since $s > \sigma\frac{2\alpha-1}{4\alpha-3} > \sigma\frac{\alpha}{3\alpha-2}$, a sum in N_3 can be carried out.

Therefore, from Lemma 2.2, we conclude that

$$\sum_{N_4\sim N_3\gg N_3^{\frac{\alpha-1}{2\alpha-1}}\gg N_2\sim N_1} (4.14) \lesssim T^{-2\theta} R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}$$

outside a set of probability $\leq C\exp(-c\frac{R^2}{\|\phi\|_{H^s}^2})$.

Subcase (5.iii.c): $N_4 \sim N_3 \gg N_2 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$

By using the Hölder inequality and Lemma 3.7, we have

$$\begin{aligned} N_2^{2\alpha-d} \int\int \left|v_1^k z_2^l \langle\nabla\rangle^\sigma z_3 v_4\right| dx\, dt \\ &\lesssim N_2^{2\alpha-d} \|z_2^l\|_{L^4_{t,x}} \|\langle\nabla\rangle^\sigma z_3\|_{L^4_{t,x}} \|v_1^k v_4\|_{L^2_{t,x}} \\ &\lesssim N_1^{\frac{d-1}{2}-\sigma+2\widetilde{\varepsilon}} N_2^{2\alpha-d} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \|v_1^k\|_{X^{\sigma,b}} \|z_2^l\|_{L^4_{t,x}} \|\langle\nabla\rangle^\sigma z_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}. \end{aligned}$$

Then sums in k, l and the Bernstein inequality yield

$$(4.13) \lesssim N_1^{\frac{d-1}{2}-\sigma+2\widetilde{\varepsilon}} N_3^{\sigma-s} N_2^{2\alpha-d-s} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \|v_1\|_{X^{\sigma,b}} \prod_{j=2}^{3} \|\langle\nabla\rangle^s z_j\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.$$

Since $N_4 \sim N_3 \gg N_2 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$, $\frac{d-1}{2} - \sigma + 2\widetilde{\varepsilon} > 0$ and $2\alpha - d - s < 0$, we can carry out sums in N_1 and N_2:

$$\begin{aligned} \sum_{N_3 \geq N_2 \geq N_3^{\frac{\alpha-1}{2\alpha-1}}} \sum_{N_3^{\frac{\alpha-1}{2\alpha-1}} \geq N_1 \geq 1} (4.13) \lesssim N_3^{\frac{(2\alpha-d)(\alpha-1)}{2(2\alpha-1)}+\sigma(\frac{\alpha}{2\alpha-1})-s(\frac{3\alpha-2}{2\alpha-1})+2\widetilde{\varepsilon}(\frac{3\alpha-2}{2\alpha-1})} \\ \times \|v\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z\|_{L^4_{t,x}} \|\langle\nabla\rangle^s z_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}. \end{aligned}$$

Since $\frac{(2\alpha-d)(\alpha-1)}{2(2\alpha-1)} + \sigma(\frac{\alpha}{2\alpha-1}) - s(\frac{3\alpha-2}{2\alpha-1}) + 2\widetilde{\varepsilon}(\frac{3\alpha-2}{2\alpha-1}) < 0$, a sum in N_3 can be also carried out.

Therefore, by using Proposition 2.4, we conclude that

$$\sum_{N_4 \sim N_3 \gg N_2 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_1} (4.13) \lesssim R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}$$

outside a set of probability $\leq C\exp(-c\frac{R^2}{T^{\frac{1}{2}}\|\phi\|_{H^s}^2})$.

Subcase (5.iii.d): $N_4 \sim N_3 \gg N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2$

By using the Hölder inequality and Lemma 3.7, we have

$$\begin{aligned} N_1^{2\alpha-d} \int\int \left|v_1^k z_2^l \langle\nabla\rangle^\sigma z_3 v_4\right| dx\, dt \\ &\lesssim N_1^{2\alpha-d} \|v_1^k\|_{L^q_t L^r_x} \|\langle\nabla\rangle^\sigma z_3\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|z_2^l v_4\|_{L^2_{t,x}} \\ &\lesssim N_1^{2\alpha-d} N_2^{\frac{d-1}{2}-s+2\widetilde{\varepsilon}} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \|v_1^k\|_{L^q_t L^r_x} \|\langle\nabla\rangle^\sigma z_3\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|z_2^l\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}, \end{aligned}$$

where $q = \frac{2(2-\alpha)}{3-2\alpha}$ when $d = 3$, $q = 3$ when $d \geq 4$ and $d/r = d/2 - 2/q$.

Then, from sums in k, l and Lemma 3.2, we obtain

$$\begin{aligned}(4.13) \lesssim N_1^{2\alpha-d+\frac{2-\alpha}{q}} N_2^{\frac{d-1}{2}+2\widetilde{\varepsilon}} N_3^{\sigma} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \\ \times \|v_1\|_{X^{0,b}} \|z_2\|_{X^{0,b}} \|\langle\nabla\rangle^{\sigma} z_3\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.\end{aligned}$$

Thereafter we use the Bernstein inequality and Lemma 3.1 to get

$$\begin{aligned}&\lesssim N_1^{2\alpha-d+\frac{2-\alpha}{q}-\sigma} N_2^{\frac{d-1}{2}-s+2\widetilde{\varepsilon}} N_3^{\sigma-s} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \\ &\quad\times \|v_1\|_{X^{\sigma,b}} \|z_2\|_{X^{s,b}} \|\langle\nabla\rangle^{s} z_3\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}} \\ &\lesssim T^{-\theta} N_1^{2\alpha-d+\frac{2-\alpha}{q}-\sigma} N_2^{\frac{d-1}{2}-s+2\widetilde{\varepsilon}} N_3^{\sigma-s} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \\ &\quad\times \|v_1\|_{X^{\sigma,b}} \|P_{N_2}\phi^{\omega}\|_{H^s} \|\langle\nabla\rangle^{s} z_3\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.\end{aligned}$$

Since $N_4 \sim N_3 \gg N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2$, $2\alpha - d + \frac{2-\alpha}{q} - \sigma < 0$ and $\frac{d-1}{2} - s + 2\widetilde{\varepsilon} > 0$, we can carry out sums in N_1 and N_2:

$$\begin{aligned}&\sum_{N_3 \geq N_1 \geq N_3^{\frac{\alpha-1}{2\alpha-1}}} \sum_{N_3^{\frac{\alpha-1}{2\alpha-1}} \geq N_2 \geq 1} (4.13) \\ &\quad\lesssim T^{-\theta} N_3^{\frac{\alpha-1}{2\alpha-1}(\frac{2-\alpha}{q}+\alpha-\frac{d}{2})+\frac{\alpha}{2\alpha-1}\sigma-\frac{3\alpha-2}{2\alpha-1}s+2\frac{3\alpha-2}{2\alpha-1}\widetilde{\varepsilon}} \\ &\qquad\times \|v\|_{X^{\sigma,b}} \|\phi^{\omega}\|_{H^s} \|\langle\nabla\rangle^{s} z_3\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.\end{aligned}$$

After carrying out a sum in N_3, we apply Lemma 2.2 and Proposition 2.4 to get

$$\sum_{N_4 \sim N_3 \gg N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}} \gg N_2} (4.13) \lesssim T^{-\theta} R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}$$

outside a set of probability $\leq C\exp(-c\frac{R^2}{\|\phi\|_{H^s}^2}) + C\exp(-c\frac{R^2}{T^{\frac{q-2}{q}}\|\phi\|_{H^s}^2})$.

Subcase (5.iii.e): $N_4 \sim N_3 \gg N_2, N_1 \gg N_3^{\frac{\alpha-1}{2\alpha-1}}$

From the Hölder inequality and Lemmata 4.2 and 3.3, we obtain

$$\begin{aligned}(4.11) &\lesssim \|\langle\nabla\rangle^{\sigma}((|x|^{-2\alpha} * (v_1 z_2)) z_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \\ &\lesssim \| |x|^{-2\alpha} * (v_1 z_2)\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \|\langle\nabla\rangle^{\sigma} z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}} \\ &\quad+ \|\langle\nabla\rangle^{\sigma}(|x|^{-2\alpha} * (v_1 z_2))\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.\end{aligned}$$

By using the HLS inequality, we have

$$\begin{aligned}
&\lesssim \|v_1 z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2d-2\alpha+\alpha\varepsilon}}} \|\langle\nabla\rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}} \\
&\quad + \|\langle\nabla\rangle^\sigma (v_1 z_2)\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.
\end{aligned}$$

Then the Hölder inequality and Lemma 4.2 yield

$$\begin{aligned}
&\lesssim \|v_1\|_{L_t^\infty L_x^2} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|\langle\nabla\rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}} \\
&\quad + \|v_1\|_{L_t^\infty L_x^2} \|\langle\nabla\rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}} \\
&\quad + \|\langle\nabla\rangle^\sigma v_1\|_{L_t^\infty L_x^2} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.
\end{aligned}$$

Thereafter we use the Bernstein inequality and Lemma 3.2 to get

$$\begin{aligned}
&\lesssim N_1^{-\sigma} N_2^{-s} N_3^{\sigma-s} \\
&\quad \times \|v_1\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|\langle\nabla\rangle^s z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.
\end{aligned}$$

Now we carry out sums in N_1, N_2 and N_3:

$$\begin{aligned}
&\sum_{N_4\sim N_3\gg N_2,\, N_1\gg N_3^{\frac{\alpha-1}{2\alpha-1}}} (4.11) \\
&\qquad \lesssim \|v\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|\langle\nabla\rangle^s z\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.
\end{aligned}$$

Therefore, from Proposition 2.4, we conclude that

$$\sum_{N_4\sim N_3\gg N_2,\, N_1\gg N_3^{\frac{\alpha-1}{2\alpha-1}}} (4.11) \lesssim R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}$$

outside a set of probability $\le C\exp\big(-c\frac{R^2}{T^{1-\varepsilon}\|\phi\|_{H^s}}\big)$.

Case (5.iv): $N_1, N_3 \ll N_2$

We consider four cases separately:

(5.iv.a) $N_4 \sim N_2 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_1, N_3$;

(5.iv.b) $N_4 \sim N_2 \gg N_3 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$;

(5.iv.c) $N_4 \sim N_2 \gg N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3$;

(5.iv.d) $N_4 \sim N_2 \gg N_3, N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}}$.

Subcases (5.iv.a), (5.iv.b) and (5.iv.c) are similar to subcase (2.ii.a). We need to estimate

$$\sum_{k,l\in\mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int\int \left|v_1^k \langle\nabla\rangle^\sigma z_2^l z_3 v_4\right| dx\,dt. \tag{4.15}$$

Subcase (5.iv.a): $N_4 \sim N_2 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_1, N_3$

We assume $N_1 \leq N_3$, because the other case can be handled similarly. The Hölder inequality and Lemmata 3.6 and 3.7 give

$$\begin{aligned} N_2^{2\alpha-d} \int\int \left|v_1^k z_3 \langle\nabla\rangle^\sigma z_2^l v_4\right| dx\,dt &\lesssim N_2^{2\alpha-d} \|v_1^k \langle\nabla\rangle^\sigma z_2^l\|_{L^2_{t,x}} \|z_3 v_4\|_{L^2_{t,x}} \\ &\lesssim N_2^{2\alpha-d} N_1^{\frac{d-1}{2}} N_2^{\frac{1-\alpha}{2}} N_3^{\frac{d-1}{2}+2\widetilde{\varepsilon}} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \\ &\quad\times \|v_1^k\|_{X^{0,b}} \|\langle\nabla\rangle^\sigma z_2^l\|_{X^{0,b}} \|z_3\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}. \end{aligned}$$

Then from sums in k, l, the Bernstein inequality and Lemma 3.1, we obtain

$$\begin{aligned} &(4.15) \\ &\lesssim N_1^{\frac{d-1}{2}-\sigma} N_3^{\frac{d-1}{2}-s+2\widetilde{\varepsilon}} N_2^{\frac{1-\alpha}{2}+\sigma+2\alpha-d-s} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \\ &\quad\times \|v_1\|_{X^{\sigma,b}} \|z_3\|_{X^{s,b}} \|z_2\|_{X^{s,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}} \\ &\lesssim T^{-2\theta} N_1^{\frac{d-1}{2}-\sigma} N_3^{\frac{d-1}{2}-s+2\widetilde{\varepsilon}} N_2^{\frac{1-\alpha}{2}+\sigma+2\alpha-d-s} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \\ &\quad\times \|v_1\|_{X^{\sigma,b}} \prod_{j=2}^{3} \|P_{N_j}\phi^\omega\|_{H^s} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}. \end{aligned}$$

Thereafter we carry out sums in N_1, N_3:

$$\begin{aligned} &\sum_{N_2^{\frac{\alpha-1}{2\alpha-1}} \geq N_3 \geq 1} \sum_{N_3 \geq N_1 \geq 1} (4.15) \\ &\lesssim T^{-2\theta} N_3^{\sigma(\frac{\alpha}{2\alpha-1})-s(\frac{3\alpha-2}{2\alpha-1})+2\widetilde{\varepsilon}(\frac{3\alpha-2}{2\alpha-1})} \|v\|_{X^{\sigma,b}} \|\phi^\omega\|_{H^s} \|P_{N_3}\phi^\omega\|_{H^s} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}. \end{aligned}$$

Since $s > \sigma\frac{2\alpha-1}{4\alpha-3} > \sigma\frac{\alpha}{3\alpha-2}$, a sum in N_2 can be also carried out.

Therefore, from Lemma 2.2, we conclude that

$$\sum_{N_4 \sim N_2 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3 \geq N_1} (4.15) \lesssim T^{-2\theta} R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}$$

outside a set of probability $\leq C\exp(-c\frac{R^2}{\|\phi\|_{H^s}^2})$.

Subcase (5.iv.b): $N_4 \sim N_2 \gg N_3 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$

By using the Hölder inequality and Lemma 3.7, we have

$$\begin{aligned} N_2^{2\alpha-d}\int\int \left|v_1^k z_3 \langle\nabla\rangle^\sigma z_2^l v_4\right| dx\, dt &\lesssim N_2^{2\alpha-d}\|z_3\|_{L^4_{t,x}}\|\langle\nabla\rangle^\sigma z_2^l\|_{L^4_{t,x}}\|v_1^k v_4\|_{L^2_{t,x}} \\ &\lesssim N_1^{\frac{d-1}{2}-\sigma+2\widetilde{\varepsilon}} N_3^{-s} N_2^{\sigma+2\alpha-d-s} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \\ &\quad\times \|v_1^k\|_{X^{\sigma,b}}\|z_3\|_{L^4_{t,x}}\|\langle\nabla\rangle^\sigma z_2^l\|_{L^4_{t,x}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}. \end{aligned}$$

Then sums in k, l and the Bernstein inequality give

$$(4.15) \lesssim N_1^{\frac{d-1}{2}-\sigma+2\widetilde{\varepsilon}} N_3^{-s} N_2^{\sigma+2\alpha-d-s} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}}\|v_1\|_{X^{\sigma,b}}\prod_{j=2}^{3}\|\langle\nabla\rangle^s z_j\|_{L^4_{t,x}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.$$

Since $N_4 \sim N_2 \gg N_3 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$, $\frac{d-1}{2} - \sigma + 2\widetilde{\varepsilon} > 0$ and $-s < 0$, we can carry out sums in N_1 and N_3 to get

$$\begin{aligned} &\sum_{N_2 \geq N_3 \geq N_2^{\frac{\alpha-1}{2\alpha-1}}} \sum_{N_2^{\frac{\alpha-1}{2\alpha-1}} \geq N_1 \geq 1} (4.15) \\ &\lesssim N_2^{\frac{(2\alpha-d)(3\alpha-1)}{2(2\alpha-1)}+\sigma(\frac{\alpha}{2\alpha-1})-s(\frac{3\alpha-2}{2\alpha-1})+2\widetilde{\varepsilon}(\frac{3\alpha-2}{2\alpha-1})} \\ &\quad\times \|v\|_{X^{\sigma,b}}\|\langle\nabla\rangle^s z\|_{L^4_{t,x}}\|\langle\nabla\rangle^s z_2\|_{L^4_{t,x}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}. \end{aligned}$$

Since $\frac{(2\alpha-d)(3\alpha-1)}{2(2\alpha-1)} + \sigma(\frac{\alpha}{2\alpha-1}) - s(\frac{3\alpha-2}{2\alpha-1}) + 2\widetilde{\varepsilon}(\frac{3\alpha-2}{2\alpha-1}) < 0$, a sum in N_2 can be also carried out.

Therefore, by using Proposition 2.4, we conclude that

$$\sum_{N_4 \sim N_2 \gg N_3 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_1} (4.15) \lesssim R^2\|v\|_{X^{\sigma,b}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}$$

outside a set of probability $\leq C\exp(-c\frac{R^2}{T^{\frac{1}{2}}\|\phi\|^2_{H^s}})$.

Subcase (5.iv.c): $N_4 \sim N_2 \gg N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3$

By using the Hölder inequality and Lemma 3.7, we have

$$\begin{aligned} &N_2^{2\alpha-d}\int\int \left|v_1^k z_3 \langle\nabla\rangle^\sigma z_2^l v_4\right| dx\, dt \\ &\lesssim N_2^{2\alpha-d}\|v_1^k\|_{L^q_t L^r_x}\|\langle\nabla\rangle^\sigma z_2^l\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}}\|z_3 v_4\|_{L^2_{t,x}} \\ &\lesssim N_2^{2\alpha-d} N_3^{\frac{d-1}{2}+2\widetilde{\varepsilon}} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}}\|v_1^k\|_{L^q_t L^r_x}\|\langle\nabla\rangle^\sigma z_2^l\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}}\|z_3\|_{X^{0,b}}\|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}, \end{aligned}$$

where $q = \frac{2(2-\alpha)}{3-2\alpha}$ when $d = 3$, $q = 3$ when $d \geq 4$ and $d/r = d/2 - 2/q$.

Then, from sums in k, l and Lemma 3.2, we obtain

$$\begin{aligned}(4.15) &\lesssim N_2^{2\alpha-d} N_1^{\frac{2-\alpha}{q}} N_3^{\frac{d-1}{2}+2\widetilde{\varepsilon}} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \\ &\quad \times \|v_1\|_{X^{0,b}} \|z_3\|_{X^{0,b}} \|\langle\nabla\rangle^\sigma z_2\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}}.\end{aligned}$$

Thereafter we use the Bernstein inequality and Lemma 3.1 to get

$$\begin{aligned}&\lesssim N_2^{2\alpha-d} N_1^{\frac{2-\alpha}{q}-\sigma} N_3^{\frac{d-1}{2}-s+2\widetilde{\varepsilon}} N_2^{\sigma-s} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \\ &\quad \times \|v_1\|_{X^{\sigma,b}} \|z_3\|_{X^{s,b}} \|\langle\nabla\rangle^s z_2\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}} \\ &\lesssim T^{-\theta} N_2^{2\alpha-d} N_1^{\frac{2-\alpha}{q}-\sigma} N_3^{\frac{d-1}{2}-s+2\widetilde{\varepsilon}} N_2^{\sigma-s} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \\ &\quad \times \|v_1\|_{X^{\sigma,b}} \|P_{N_3}\phi^\omega\|_{H^s} \|\langle\nabla\rangle^s z_2\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}}.\end{aligned}$$

Since $N_4 \sim N_2 \gg N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3$, we can carry out sums in N_1 and N_3:

$$\begin{aligned}&\sum_{N_2 \geq N_1 \geq N_2^{\frac{\alpha-1}{2\alpha-1}}} \sum_{N_2^{\frac{\alpha-1}{2\alpha-1}} \geq N_3 \geq 1} (4.15) \\ &\lesssim T^{-\theta} N_2^{\frac{1}{2\alpha-1}\left(\frac{(2-\alpha)(\alpha-1)}{q} + \frac{(2\alpha-d)(3\alpha-1)}{2}\right) + \frac{\alpha}{2\alpha-1}\sigma - \frac{3\alpha-2}{2\alpha-1}s + 2\frac{3\alpha-2}{2\alpha-1}\widetilde{\varepsilon}} \\ &\quad \times \|v\|_{X^{\sigma,b}} \|\phi^\omega\|_{H^s} \|\langle\nabla\rangle^s z_2\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}}.\end{aligned}$$

After carrying out a sum in N_2, we apply Lemma 2.2 and Proposition 2.4 to get

$$\sum_{N_4 \sim N_2 \gg N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3} (4.15) \lesssim T^{-\theta} R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}}$$

outside a set of probability $\leq C\exp(-c\frac{R^2}{\|\phi\|_{H^s}^2}) + C\exp(-c\frac{R^2}{T^{\frac{q-2}{q}}\|\phi\|_{H^s}^2})$.

Subcase (5.iv.d): $N_4 \sim N_2 \gg N_3, N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}}$

From the Hölder inequality and Lemmata 4.2 and 3.3, we obtain

$$\begin{aligned}(4.11) &\lesssim \|\langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (v_1 z_2)) z_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \\ &\lesssim \| |x|^{-2\alpha} * (v_1 z_2)\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \|\langle\nabla\rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}} \\ &\quad + \|\langle\nabla\rangle^\sigma(|x|^{-2\alpha} * (v_1 z_2))\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\widetilde{\varepsilon}}}.\end{aligned}$$

By using the HLS inequality, we have

$$\lesssim \|v_1 z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2d-2\alpha+\alpha\varepsilon}}} \|\langle\nabla\rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}$$
$$+ \|\langle\nabla\rangle^\sigma (v_1 z_2)\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{2d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.$$

Then the Hölder inequality and Lemma 4.2 yield

$$\lesssim \|v_1\|_{L_t^\infty L_x^2} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|\langle\nabla\rangle^\sigma z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}$$
$$+ \|v_1\|_{L_t^\infty L_x^2} \|\langle\nabla\rangle^\sigma z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}$$
$$+ \|\langle\nabla\rangle^\sigma v_1\|_{L_t^\infty L_x^2} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.$$

Thereafter we use the Bernstein inequality and Lemma 3.2 to get

$$\lesssim N_1^{-\sigma} N_2^{\sigma-s} N_3^{-s}$$
$$\times \|v_1\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|\langle\nabla\rangle^s z_3\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.$$

Now we carry out sums in N_1, N_3 and N_2:

$$\sum_{N_4\sim N_2\gg N_3, N_1\gg N_2^{\frac{\alpha-1}{2\alpha-1}}} (4.11)$$
$$\lesssim \|v\|_{X^{\sigma,b}} \|\langle\nabla\rangle^s z\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|\langle\nabla\rangle^s z\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}.$$

Therefore, from Proposition 2.4, we conclude that

$$\sum_{N_4\sim N_2\gg N_3, N_1\gg N_2^{\frac{\alpha-1}{2\alpha-1}}} (4.11) \lesssim R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac12-\tilde\varepsilon}}$$

outside a set of probability $\leq C\exp\big(-c\frac{R^2}{T^{1-\varepsilon}\|\phi\|_{H^s}}\big)$.

6th term: zzv

$$\Big|\int\int \langle\nabla\rangle^\sigma((|x|^{-2\alpha} * z_1 z_2)v_3)v_4\, dx\, dt\Big|. \tag{4.16}$$

We consider three cases separately:

(6.i) $N_3 \gtrsim N_2$;

(6.ii) $N_3 \ll N_2 \sim N_1$;

(6.iii) $N_3, N_1 \ll N_2$.

Case (6.i): $N_3 \gtrsim N_2$

The Hölder inequality and Lemma 3.3 yield that (4.16) is bounded by

$$\begin{aligned}
&\|\langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (z_1 z_2))v_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{L_t^{\frac{1}{\varepsilon}} L_x^2} \\
&\quad \lesssim \|\langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (z_1 z_2))v_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.
\end{aligned}$$

Then we use Lemmata 4.2 and 3.2 to get

$$\begin{aligned}
&\|\langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (z_1 z_2))v_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2} \\
&\quad \lesssim \| |x|^{-2\alpha} * (z_1 z_2)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|\langle\nabla\rangle^\sigma v_3\|_{L_t^\infty L_x^2} \\
&\qquad + \|\langle\nabla\rangle^\sigma(|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|v_3\|_{L_t^\infty L_x^2} \\
&\quad \lesssim \| |x|^{-2\alpha} * (z_1 z_2)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|v_3\|_{X^{\sigma,b}} \\
&\qquad + \|\langle\nabla\rangle^\sigma(|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \|v_3\|_{X^{0,b}}.
\end{aligned}$$

Thereafter, from Lemma 4.3, the Hölder inequality, the Bernstein inequality and $N_3 \gtrsim N_2$, we obtain

$$\| |x|^{-2\alpha} * (z_1 z_2)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \lesssim \|z_1\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}$$

and

$$\begin{aligned}
&\|\langle\nabla\rangle^\sigma(|x|^{-2\alpha} * (z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \\
&\quad = \| |x|^{-2\alpha} * (\langle\nabla\rangle^\sigma(z_1 z_2))\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^\infty} \\
&\quad \lesssim \|\langle\nabla\rangle^\sigma(z_1 z_2)\|^{\frac{1}{2}}_{L_t^{\frac{1}{1-\varepsilon}} L_x^{\frac{d}{d-2\alpha-\alpha\varepsilon}}} \|\langle\nabla\rangle^\sigma(z_1 z_2)\|^{\frac{1}{2}}_{L_t^{\frac{1}{1-\varepsilon}} L_x^{\frac{d}{d-2\alpha+\alpha\varepsilon}}} \\
&\quad \lesssim N_3^\sigma \|z_1\|^{\frac{1}{2}}_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|z_2\|^{\frac{1}{2}}_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|z_1\|^{\frac{1}{2}}_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|z_2\|^{\frac{1}{2}}_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}.
\end{aligned}$$

Therefore, from Proposition 2.4, we conclude that

$$\sum_{N_3 \gtrsim N_2} (4.16) \lesssim R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}$$

outside a set of probability $\leq C \exp\big(-c\frac{R^2}{T^{1-\varepsilon}\|\phi\|_{H^{0+}}}\big)$.

Case (6.ii): $N_3 \ll N_2 \sim N_1$

We assume $N_4 \geq N_3$, because the other case can be handled similarly. From the Hölder inequality and Lemma 3.3, we get

$$\begin{aligned}(4.16) &\lesssim \|\langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (z_1z_2))v_3)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^2}\|v_4\|_{L_t^{\frac{1}{\varepsilon}}L_x^2} \\ &\lesssim \|\langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (z_1z_2))v_3)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^2}\|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.\end{aligned}$$

Lemmata 4.2 and 3.2 give

$$\begin{aligned}&\|\langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (|\eta_T v|^2))\eta_T v)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^2} \\ &\quad\lesssim \|\,|x|^{-2\alpha} * (z_1z_2)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^\infty}\|\langle\nabla\rangle^\sigma v_3\|_{L_t^\infty L_x^2} \\ &\qquad + \|\langle\nabla\rangle^\sigma(|x|^{-2\alpha} * (z_1z_2))\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^\infty}\|v_3\|_{L_t^\infty L_x^2} \\ &\quad\lesssim \|\,|x|^{-2\alpha} * (z_1z_2)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^\infty}\|v_3\|_{X^{\sigma,b}} \\ &\qquad + \|\langle\nabla\rangle^\sigma(|x|^{-2\alpha} * (z_1z_2))\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^\infty}\|v_3\|_{X^{0,b}}.\end{aligned}$$

Thereafter we use Lemma 4.3, the Hölder inequality, the Bernstein inequality and $N_1 \sim N_2$ to get

$$\|\,|x|^{-2\alpha} * (z_1z_2)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^\infty} \lesssim \|z_1\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|z_2\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}$$

and

$$\begin{aligned}&\|\langle\nabla\rangle^\sigma(|x|^{-2\alpha} * (z_1z_2))\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^\infty} \\ &\quad= \|\,|x|^{-2\alpha} * (\langle\nabla\rangle^\sigma(z_1z_2))\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^\infty} \\ &\quad\lesssim \|\langle\nabla\rangle^\sigma(z_1z_2)\|^{\frac{1}{2}}_{L_t^{\frac{1}{1-\varepsilon}}L_x^{\frac{d}{d-2\alpha-\alpha\varepsilon}}}\|\langle\nabla\rangle^\sigma(z_1z_2)\|^{\frac{1}{2}}_{L_t^{\frac{1}{1-\varepsilon}}L_x^{\frac{d}{d-2\alpha+\alpha\varepsilon}}} \\ &\quad\lesssim N_1^{\sigma-2s}\prod_{j=1}^{2}\|\langle\nabla\rangle^s z_j\|^{\frac{1}{2}}_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|\langle\nabla\rangle^s z_j\|^{\frac{1}{2}}_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}.\end{aligned}$$

Therefore, from Proposition 2.4, we conclude that

$$\sum_{N_3 \ll N_2 \sim N_1}(4.16) \lesssim R^2\|v\|_{X^{\sigma,b}}\|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}$$

outside a set of probability $\leq C\exp\big(-c\frac{R^2}{T^{1-\varepsilon}\|\phi\|_{H^s}}\big)$.

Case (6.iii): $N_3, N_1 \ll N_2$

We consider four cases separately:

(6.iii.a) $N_4 \sim N_2 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3, N_1$;

(6.iii.b) $N_4 \sim N_2 \gg N_3 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$;

(6.iii.c) $N_4 \sim N_2 \gg N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3$;

(6.iii.d) $N_4 \sim N_2 \gg N_3, N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}}$.

Subcases (6.iii.a), (6.iii.b) and (6.iii.c) are similar to subcase (2.ii.a). We need to estimate

$$\sum_{k,l\in\mathbb{Z}^d} C_{k,l} N_2^{2\alpha-d} \int\int \left| z_1^k \langle\nabla\rangle^\sigma z_2^l v_3 v_4 \right| dx\, dt. \tag{4.17}$$

Subcase (6.iii.a): $N_4 \sim N_2 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3, N_1$

The Hölder inequality and Lemmata 3.6 and 3.7 give

$$\begin{aligned}
&N_2^{2\alpha-d} \int\int \left| z_1^k \langle\nabla\rangle^\sigma z_2^l v_3 v_4 \right| dx\, dt \\
&\quad\lesssim N_2^{2\alpha-d} \|\langle\nabla\rangle^\sigma z_2^l v_3\|_{L^2_{t,x}} \|z_1^k v_4\|_{L^2_{t,x}} \\
&\quad\lesssim N_3^{\frac{d-1}{2}} N_2^{\frac{1-\alpha}{2}+2\alpha-d} N_1^{\frac{d-1}{2}+2\widetilde{\varepsilon}} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \\
&\qquad\times \|\langle\nabla\rangle^\sigma z_2^l\|_{X^{0,b}} \|v_3\|_{X^{0,b}} \|z_1^k\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.
\end{aligned}$$

Thereafter, from sums in k, l, the Bernstein inequality and Lemma 3.1, we obtain

$$\begin{aligned}
(4.17) &\lesssim N_3^{\frac{d-1}{2}} N_2^{\frac{1-\alpha}{2}+2\alpha-d} N_1^{\frac{d-1}{2}+2\widetilde{\varepsilon}} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \\
&\quad\times \|\langle\nabla\rangle^\sigma z_2\|_{X^{0,b}} \|v_3\|_{X^{0,b}} \|z_1\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}} \\
&\lesssim T^{-2\theta} N_3^{\frac{d-1}{2}-\sigma} N_2^{\frac{1-\alpha}{2}+2\alpha-d+\sigma-s} N_1^{\frac{d-1}{2}-s+2\widetilde{\varepsilon}} N_4^{\frac{1-\alpha}{2}+2\widetilde{\varepsilon}} \\
&\quad\times \prod_{j=1}^{2} \|P_{N_j}\phi^\omega\|_{H^s} \|v_3\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.
\end{aligned}$$

Then we carry out sums in N_1 and N_3 so that

$$\begin{aligned}
&\sum_{N_2^{\frac{\alpha-1}{2\alpha-1}} \geq N_3 \geq} \sum_{N_2^{\frac{\alpha-1}{2\alpha-1}} \geq N_1 \geq 1} (4.17) \\
&\quad\lesssim T^{-2\theta} N_2^{\alpha\frac{2\alpha-d}{2\alpha-1}+\sigma(\frac{\alpha}{2\alpha-1})-s(\frac{3\alpha-2}{2\alpha-1})+2\widetilde{\varepsilon}(\frac{3\alpha-2}{2\alpha-1})} \\
&\qquad\times \|\phi^\omega\|_{H^s} \|P_{N_3}\phi^\omega\|_{H^s} \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\widetilde{\varepsilon}}}.
\end{aligned}$$

Since $s > \sigma \frac{2\alpha-1}{4\alpha-3} > \sigma \frac{\alpha}{3\alpha-2}$, a sum in N_2 can be also carried out.

Hence, from Lemma 2.2, we have

$$\sum_{N_4 \sim N_2 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3, N_1} (4.17) \lesssim T^{-2\theta} R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}$$

outside a set of probability $\leq C \exp(-c \frac{R^2}{\|\phi\|_{H^s}^2})$.

Subcase (6.iii.b): $N_4 \sim N_2 \gg N_3 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_1$

The Hölder inequality and Lemma 3.7 yield

$$\begin{aligned}
N_2^{2\alpha-d} & \int\int \left| z_1^k \langle\nabla\rangle^\sigma z_2^l v_3 v_4 \right| dx\, dt \\
&\lesssim N_2^{2\alpha-d} \|v_3\|_{L_t^q L_x^r} \|\langle\nabla\rangle^\sigma z_2^l\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|z_1^k v_4\|_{L^2_{t,x}} \\
&\lesssim N_2^{2\alpha-d} N_1^{\frac{d-1}{2}+2\tilde{\varepsilon}} N_4^{\frac{1-\alpha}{2}+2\tilde{\varepsilon}} \|v_3\|_{L_t^q L_x^r} \|\langle\nabla\rangle^\sigma z_2^l\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|z_1^k\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}},
\end{aligned}$$

where $q = \max(3, \frac{3(\alpha-1)(2-\alpha)}{(3\alpha-1)(d-2\alpha)})$ and $d/r = d/2 - 2/q$.

Thereafter we carry out sums in k, l, then use the Bernstein inequality and Lemma 3.2 to obtain

$$\begin{aligned}
(4.17) \lesssim\ & N_3^{\frac{2-\alpha}{q}-\sigma} N_2^{2\alpha-d+\sigma-s} N_1^{\frac{d-1}{2}-s+2\tilde{\varepsilon}} N_4^{\frac{1-\alpha}{2}+2\tilde{\varepsilon}} \|v_3\|_{X^{\sigma,b}} \\
& \times \|z_1\|_{X^{s,b}} \|\langle\nabla\rangle^s z_2\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.
\end{aligned}$$

After that, we carry out sums in N_1 and N_3 and apply Lemma 3.1:

$$\begin{aligned}
\sum_{N_2 \geq N_3 \geq N_2^{\frac{\alpha-1}{2\alpha-1}}} & \sum_{N_2^{\frac{\alpha-1}{2\alpha-1}} \geq N_1 \geq 1} (4.17) \\
&\lesssim N_2^{\frac{\alpha-1}{2\alpha-1}(\frac{2-\alpha}{q}+\frac{d-2\alpha}{2(\alpha-1)}(1-3\alpha))+\frac{\alpha}{2\alpha-1}\sigma-\frac{3\alpha-2}{2\alpha-1}s+2\frac{3\alpha-2}{2\alpha-1}\tilde{\varepsilon}} \\
&\quad \times \|v\|_{X^{\sigma,b}} \|z\|_{X^{s,b}} \|\langle\nabla\rangle^s z_2\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}} \\
&\lesssim T^{-\theta} N_2^{\frac{\alpha-1}{2\alpha-1}(\frac{2-\alpha}{q}+\frac{d-2\alpha}{2(\alpha-1)}(1-3\alpha))+\frac{\alpha}{2\alpha-1}\sigma-\frac{3\alpha-2}{2\alpha-1}s+2\frac{3\alpha-2}{2\alpha-1}\tilde{\varepsilon}} \\
&\quad \times \|v\|_{X^{\sigma,b}} \|\phi^\omega\|_{H^s} \|\langle\nabla\rangle^s z_2\|_{L_t^{\frac{2q}{q-2}} L_x^{\frac{2r}{r-2}}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.
\end{aligned}$$

Since the exponent of N_2 is negative, a sum in N_2 can also be carried out.

Therefore, from Lemma 2.2 and Proposition 2.4, we conclude that

$$\sum_{N_4\sim N_2\gg N_3\gg N_2^{\frac{\alpha-1}{2\alpha-1}}\gg N_1} (4.17) \lesssim T^{-\theta}R^2\|v\|_{X^{\sigma,b}}\|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}$$

outside a set of probability $\leq C\exp(-c\frac{R^2}{\|\phi\|_{H^s}^2}) + C\exp(-c\frac{R^2}{T^{\frac{q-2}{q}}\|\phi\|_{H^s}^2})$.

Case(6.iii.c): $N_4 \sim N_2 \gg N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3$

The Hölder inequality and Lemma 3.7 give

$$\begin{aligned}
&N_2^{2\alpha-d}\int\int \left|z_1^k\langle\nabla\rangle^\sigma z_2^l v_3 v_4\right| dx\, dt \\
&\quad\lesssim N_2^{2\alpha-d}\|z_1^k\|_{L^4_{t,x}}\|\langle\nabla\rangle^\sigma z_2^l\|_{L^4_{t,x}}\|v_3v_4\|_{L^2_{t,x}} \\
&\quad\lesssim N_2^{2\alpha-d}N_3^{\frac{d-1}{2}+2\tilde{\varepsilon}}N_4^{\frac{1-\alpha}{2}+2\tilde{\varepsilon}}\|z_1^k\|_{L^4_{t,x}}\|\langle\nabla\rangle^\sigma z_2^l\|_{L^4_{t,x}}\|v_3\|_{X^{0,b}}\|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.
\end{aligned}$$

Then, from sums in k, l and the Bernstein inequality, we obtain

$$\begin{aligned}
(4.17) \lesssim\ & N_1^{-s}N_2^{2\alpha-d+\sigma-s}N_3^{\frac{d-1}{2}-\sigma+2\tilde{\varepsilon}}N_4^{\frac{1-\alpha}{2}+2\tilde{\varepsilon}} \\
&\times\prod_{j=1}^{2}\|\langle\nabla\rangle^s z_j\|_{L^4_{t,x}}\|v_3\|_{X^{\sigma,b}}\|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.
\end{aligned}$$

Since $N_4 \sim N_2 \gg N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}} \gg N_3$, $\frac{d-1}{2} - \sigma + 2\tilde{\varepsilon} > 0$ and $-s < 0$, we can carry out sums in N_1 and N_3 such that

$$\begin{aligned}
&\sum_{N_2\geq N_1\geq N_2^{\frac{\alpha-1}{2\alpha-1}}}\ \sum_{N_2^{\frac{\alpha-1}{2\alpha-1}}\geq N_3\geq 1} (4.17) \\
&\quad\lesssim N_2^{\frac{(d-2\alpha)(1-3\alpha)}{2(2\alpha-1)}+\frac{\alpha}{2\alpha-1}\sigma-\frac{3\alpha-2}{2\alpha-1}s+\frac{3\alpha-2}{2\alpha-1}\tilde{\varepsilon}} \\
&\qquad\times\|\langle\nabla\rangle^s z\|_{L^4_{t,x}}\|\langle\nabla\rangle^s z_2\|_{L^4_{t,x}}\|v\|_{X^{\sigma,b}}\|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.
\end{aligned}$$

After carrying out a sum in N_2, from Proposition 2.4, we conclude that

$$\sum_{N_4\sim N_2\gg N_1\gg N_2^{\frac{\alpha-1}{2\alpha-1}}\gg N_3} (4.17) \lesssim R^2\|v\|_{X^{\sigma,b}}\|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}$$

outside a set of probability $\leq C\exp(-c\frac{R^2}{T^{\frac{1}{2}}\|\phi\|_{H^s}^2})$.

Subcase (6.iii.d): $N_4 \sim N_2 \gg N_3, N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}}$

The Hölder inequality and Lemma 3.3 yield that (4.16) is bounded by

$$\begin{aligned}&\|\langle\nabla\rangle^\sigma((|x|^{-2\alpha}*(z_1z_2))v_3)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^2}\|v_4\|_{L_t^{\frac{1}{\varepsilon}}L_x^2}\\ &\lesssim \|\langle\nabla\rangle^\sigma((|x|^{-2\alpha}*(z_1z_2))v_3)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^2}\|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}.\end{aligned}$$

Then we use Lemmata 4.2 and 3.2 to obtain

$$\begin{aligned}\|\langle\nabla\rangle^\sigma((|x|^{-2\alpha}*(z_1z_2))v_3)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^2} &\lesssim \|\,|x|^{-2\alpha}*(z_1z_2)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^\infty}\|\langle\nabla\rangle^\sigma v_3\|_{L_t^\infty L_x^2}\\ &\quad+\|\langle\nabla\rangle^\sigma(|x|^{-2\alpha}*(z_1z_2))\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^\infty}\|v_3\|_{L_t^\infty L_x^2}\\ &\lesssim \|\,|x|^{-2\alpha}*(z_1z_2)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^\infty}\|v_3\|_{X^{\sigma,b}}\\ &\quad+\|\langle\nabla\rangle^\sigma(|x|^{-2\alpha}*(z_1z_2))\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^\infty}\|v_3\|_{X^{0,b}}.\end{aligned}$$

Thereafter, from Lemma 4.3, the Hölder inequality, the Bernstein inequality and $N_2 \gtrsim N_1$, we obtain

$$\begin{aligned}\|\,|x|^{-2\alpha}*(z_1z_2)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^\infty} &\lesssim \|z_1\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|z_2\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}\\ &\lesssim N_1^{-s}N_2^{-s}\|\langle\nabla\rangle^s z_1\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|\langle\nabla\rangle^s z_2\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}\end{aligned}$$

and

$$\begin{aligned}&\|\langle\nabla\rangle^\sigma(|x|^{-2\alpha}*(z_1z_2))\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^\infty}\\ &=\|\,|x|^{-2\alpha}*(\langle\nabla\rangle^\sigma(z_1z_2))\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^\infty}\\ &\lesssim \|\langle\nabla\rangle^\sigma(z_1z_2)\|^{\frac{1}{2}}_{L_t^{\frac{1}{1-\varepsilon}}L_x^{\frac{d}{d-2\alpha-\alpha\varepsilon}}}\|\langle\nabla\rangle^\sigma(z_1z_2)\|^{\frac{1}{2}}_{L_t^{\frac{1}{1-\varepsilon}}L_x^{\frac{d}{d-2\alpha+\alpha\varepsilon}}}\\ &\lesssim N_2^{\sigma-s}N_1^{-s}\prod_{j=1}^{2}\|\langle\nabla\rangle^s z_j\|^{\frac{1}{2}}_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|\langle\nabla\rangle^s z_j\|^{\frac{1}{2}}_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}.\end{aligned}$$

Thus, from $N_2 \gtrsim N_3$, we have

$$\begin{aligned}&\|\langle\nabla\rangle^\sigma((|x|^{-2\alpha}*(z_1z_2))v_3)\|_{L_t^{\frac{1}{1-\varepsilon}}L_x^2}\\ &\lesssim N_2^{\sigma-s}N_1^{-s}N_3^{-\sigma}\|\langle\nabla\rangle^s z_1\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|\langle\nabla\rangle^s z_2\|_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}\|v_3\|_{X^{\sigma,b}}\\ &\quad+N_2^{\sigma-s}N_1^{-s}N_3^{-\sigma}\\ &\quad\times\left(\prod_{j=1}^{2}\|\langle\nabla\rangle^s z_j\|^{\frac{1}{2}}_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}}\|\langle\nabla\rangle^s z_j\|^{\frac{1}{2}}_{L_t^{\frac{2}{1-\varepsilon}}L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}\right)\|v_3\|_{X^{\sigma,b}}.\end{aligned}$$

Now we carry out sums in N_1 and N_3:

$$\sum_{N_2 \geq N_3 \geq N_2^{\frac{\alpha-1}{2\alpha-1}}} \sum_{N_2 \geq N_1 \geq N_2^{\frac{\alpha-1}{2\alpha-1}}} \|\langle\nabla\rangle^\sigma((|x|^{-2\alpha} * (z_1 z_2))v_3)\|_{L_t^{\frac{1}{1-\varepsilon}} L_x^2}$$
$$\lesssim N_2^{\frac{\alpha}{2\alpha-1}\sigma - \frac{3\alpha-2}{2\alpha-1}s}$$
$$\times \Bigg(\|\langle\nabla\rangle^s z\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|\langle\nabla\rangle^s z_2\|_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|v\|_{X^{\sigma,b}}$$
$$+ \Big(\|\langle\nabla\rangle^s z\|^{\frac{1}{2}}_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|\langle\nabla\rangle^s z\|^{\frac{1}{2}}_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}}$$
$$\times \|\langle\nabla\rangle^s z_2\|^{\frac{1}{2}}_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha-\alpha\varepsilon}}} \|\langle\nabla\rangle^s z_2\|^{\frac{1}{2}}_{L_t^{\frac{2}{1-\varepsilon}} L_x^{\frac{2d}{d-2\alpha+\alpha\varepsilon}}} \|v\|_{X^{\sigma,b}} \Big) \Bigg).$$

Therefore, from a sum in N_2 and Proposition 2.4, we conclude that

$$\sum_{N_4 \sim N_2 \gg N_3, N_1 \gg N_2^{\frac{\alpha-1}{2\alpha-1}}} (4.16) \lesssim R^2 \|v\|_{X^{\sigma,b}} \|v_4\|_{X^{0,\frac{1}{2}-\tilde{\varepsilon}}}$$

outside a set of probability $\leq C \exp\big(-c\frac{R^2}{T^{1-\varepsilon}\|\phi\|_{H^s}}\big)$. □

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