On the Non-vanishing Conjecture and Existence of Log Minimal Models

by

Kenta Hashizume

Abstract

We prove that the non-vanishing conjecture and the log minimal model conjecture for projective log canonical pairs can be reduced to the non-vanishing conjecture for smooth projective varieties such that the boundary divisor is zero.

§1. Introduction

Throughout this paper we will work over the complex number field, and we denote Conjecture • with $\dim X = n$ (resp. $\dim X \le n$) by Conjecture $\bullet_n$ (resp. Conjecture $\bullet_{\le n}$).

In this paper we deal with the following two conjectures.

Conjecture 1.1 (Non-vanishing). *Let (X, Δ) be a projective log canonical pair. If $K_X + \Delta$ is pseudo-effective, then there is an effective $\mathbb{R}$-divisor D such that $K_X + \Delta \sim_{\mathbb{R}} D$.*

Conjecture 1.2 (Existence of a log minimal model). *Let (X, Δ) be a projective log canonical pair. If $K_X + \Delta$ is pseudo-effective, then (X, Δ) has a log minimal model.*

Birkar [B1] proved that Conjecture 1.1_n implies Conjecture 1.2_n. On the other hand, Gongyo [G] proved that Conjecture 1.1_n for Kawamata log terminal pairs with boundary $\mathbb{Q}$-divisors implies Conjecture 1.1_n for log canonical pairs with

K. Hashizume: Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan;
e-mail: `hkenta@math.kyoto-u.ac.jp`

boundary $\mathbb{R}$-divisors, assuming the abundance conjecture for d-dimensional log canonical pairs with $d \leq n-1$. Today, Conjectures $1.1_{\leq 3}$ and $1.2_{\leq 4}$ are proved (Conjecture 1.2_4 was first proved by Shokurov [S] and another proof was given by Birkar [B3]) but Conjectures 1.1 and 1.2 are still open in higher dimensions.

In this paper we study the relation between the above two conjectures and the following special case of Conjecture 1.1.

Conjecture 1.3 (Non-vanishing for smooth varieties). *Let X be a smooth projective variety. If K_X is pseudo-effective, then there is an effective $\mathbb{Q}$-divisor D such that $K_X \sim_{\mathbb{Q}} D$.*

The following theorem is the main result of this paper.

Theorem 1.4. *Conjecture 1.3_n implies Conjectures $1.1_{\leq n}$ and $1.2_{\leq n}$.*

We remark that in Theorem 1.4 we do not have any assumptions about the abundance conjecture. The proof of Theorem 1.4 heavily depends on the arguments in [H]. A key ingredient is construction of fibrations with relatively trivial log canonical divisors. More precisely, for a given log canonical pair (X, Δ) and a general ample divisor A, we observe behavior of the pseudo-effective threshold $\tau(X, t\Delta; A)$ as a function of t. When $K_X + \Delta$ is pseudo-effective and t moves in $[\epsilon, 1]$ for an $\epsilon > 0$ sufficiently close to 1, we see that $\tau(X, t\Delta; A)$ is a linear function of t (see Remark 3.3). By using this observation, we construct the fibrations mentioned above. For details, see the proof of Theorem 1.4 in Section 3.

From Theorem 1.4 we immediately obtain the following corollaries.

Corollary 1.5. *Conjectures 1.1_n and 1.3_n are equivalent.*

Corollary 1.6. *Conjecture 1.3_n implies Conjecture 1.2_n.*

Corollary 1.5 is a generalization of [G, Thm. 1.5] and [DHP, Thm. 8.8], and Corollary 1.6 is a generalization of [B1, Thm. 1.4]. We emphasize that by Corollary 1.6 we can reduce the log minimal model conjecture for log canonical pairs to the non-vanishing conjecture in a very simple situation.

The contents of this paper are as follows: In Section 2, we collect some notation and definitions, and we recall two important theorems (i.e., Theorems 2.7 and 2.8). In Section 3 we prove Theorem 1.4 and Corollaries 1.5 and 1.6.

§2. Preliminaries

In this section we collect some notation and definitions. We will freely use the notation and definitions in [BCHM].

Definition 2.1 (Maps). Let $f : X \to Y$ be a morphism of normal projective varieties. Then f is a *contraction* if f is surjective and it has connected fibers.

Let $f : X \dashrightarrow Y$ be a birational map of normal projective varieties. Then f is a *birational contraction* if f^{-1} does not contract any divisors. Let D be an $\mathbb{R}$-divisor on X. Unless otherwise stated, we mean f_*D by denoting D_Y.

Definition 2.2 (Singularities of pairs). A *pair* (X, Δ) consists of a normal variety X and a boundary $\mathbb{R}$-divisor Δ, that is, an $\mathbb{R}$-divisor whose coefficients belong to $[0, 1]$, on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier.

Let (X, Δ) be a pair and $f : Y \to X$ be a log resolution of (X, Δ). Then we can write
$$K_Y = f^*(K_X + \Delta) + \sum_i a(E_i, X, \Delta)E_i,$$
where E_i are prime divisors on Y and $a(E_i, X, \Delta)$ is a real number for any i. Then we call $a(E_i, X, \Delta)$ the *discrepancy* of E_i with respect to (X, Δ). The pair (X, Δ) is called *Kawamata log terminal* (*klt*, for short) if $a(E_i, X, \Delta) > -1$ for any log resolution f of (X, Δ) and any E_i on Y. We call (X, Δ) *log canonical* (*lc*, for short) if $a(E_i, X, \Delta) \geq -1$ for any log resolution f of (X, Δ) and any E_i on Y. Finally, (X, Δ) is called *divisorial log terminal* (*dlt*, for short) if there exists a log resolution $f : Y \to X$ of (X, Δ) such that $a(E, X, \Delta) > -1$ for any f-exceptional prime divisor E on Y.

Next we introduce the definition of some models. For some remarks on the models, see [H, Rem. 2.4].

Definition 2.3 (Weak lc models and log minimal models). Let (X, Δ) be a projective log canonical pair and $\phi : X \dashrightarrow X'$ be a birational map to a normal projective variety X'. Let E be the reduced ϕ^{-1}-exceptional divisor on X', that is, $E = \sum E_j$ where E_j are ϕ^{-1}-exceptional prime divisors on X'. Then the pair $(X', \Delta' = \phi_*\Delta + E)$ is called a *log birational model* of (X, Δ). A log birational model (X', Δ') of (X, Δ) is a *weak log canonical model* (*weak lc model*, for short) if

- $K_{X'} + \Delta'$ is nef, and
- for any prime divisor D on X which is exceptional over X', we have

$$a(D, X, \Delta) \leq a(D, X', \Delta').$$

A weak lc model (X', Δ') of (X, Δ) is a *log minimal model* if

- X' is $\mathbb{Q}$-factorial, and
- the above inequality on discrepancies is strict.

A log minimal model (X', Δ') of (X, Δ) is called a *good minimal model* if $K_{X'} + \Delta'$ is semi-ample.

Definition 2.4 (Mori fiber spaces). Let (X, Δ) be a projective log canonical pair and (X', Δ') be a log birational model of (X, Δ).

Then (X', Δ') is called a *Mori fiber space* if X' is $\mathbb{Q}$-factorial and there is a contraction $X' \to W$ with $\dim W < \dim X'$ such that

- the relative Picard number $\rho(X'/W)$ is 1 and $-(K_{X'} + \Delta')$ is ample over W, and

- for any prime divisor D over X, we have

$$a(D, X, \Delta) \leq a(D, X', \Delta')$$

and strict inequality holds if D is a divisor on X and exceptional over X'.

Note that our definition of log minimal models is slightly different from that of [B2]. The difference is that we do not assume that log minimal models are dlt. But this difference is intrinsically not important (cf. [H, Rem. 2.4]). In our definition, any weak lc model (X', Δ') of a $\mathbb{Q}$-factorial lc pair (X, Δ) constructed with the $(K_X + \Delta)$-MMP is a log minimal model of (X, Δ) even though (X', Δ') may not be dlt.

Finally, we introduce the definition of log canonical thresholds and pseudo-effective thresholds, and two important theorems which are proved by Hacon, McKernan and Xu [HMX].

Definition 2.5 (Log canonical thresholds; cf. [HMX]). Let (X, Δ) be a log canonical pair and let $M \neq 0$ be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. Then the *log canonical threshold* of M with respect to (X, Δ), denoted by $\mathrm{lct}(X, \Delta; M)$, is

$$\mathrm{lct}(X, \Delta; M) = \sup\{t \in \mathbb{R} \mid (X, \Delta + tM) \text{ is log canonical}\}.$$

Definition 2.6 (Pseudo-effective thresholds). Let (X, Δ) be a projective log canonical pair and M be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor such that $K_X + \Delta + tM$ is pseudo-effective for some $t \geq 0$. Then the *pseudo-effective threshold* of M with respect to (X, Δ), denoted by $\tau(X, \Delta; M)$, is

$$\tau(X, \Delta; M) = \inf\{t \in \mathbb{R}_{\geq 0} \mid K_X + \Delta + tM \text{ is pseudo-effective}\}.$$

Theorem 2.7 (ACC for log canonical thresholds; cf. [HMX, Thm. 1.1]). *Fix a positive integer n, a set $I \subset [0, 1]$ and a set $J \subset \mathbb{R}_{>0}$, where I and J satisfy the DCC. Let $\mathfrak{T}_n(I)$ be the set of log canonical pairs (X, Δ), where X is a variety*

of dimension n and the coefficients of Δ belong to I. Then the set

$$\{\mathrm{lct}(X, \Delta; M) \mid (X, \Delta) \in \mathfrak{T}_n(I), \text{ the coefficients of } M \text{ belong to } J\}$$

satisfies the ACC.

Theorem 2.8 (ACC for numerically trivial pairs; cf. [HMX, Thm. D]). *Fix a positive integer n and a set $I \subset [0, 1]$ that satisfies the DCC.*

Then there is a finite set $I_0 \subset I$ with the following property: If (X, Δ) is a log canonical pair such that

(i) X is projective of dimension n,

(ii) the coefficients of Δ belong to I, and

(iii) $K_X + \Delta$ is numerically trivial,

then the coefficients of Δ belong to I_0.

§3. Proof of Theorem 1.4 and corollaries

In this section we prove Theorem 1.4 and Corollaries 1.5 and 1.6.

First we recall the following theorem proved by Birkar, which plays a crucial role in the proof of Theorem 1.4.

Theorem 3.1 (cf. [B1, Cor. 1.7]). *Fix a positive integer d, and assume Conjecture $1.2_{\leq d-1}$. Let (X, Δ) be a d-dimensional projective log canonical pair such that $K_X + \Delta \sim_{\mathbb{R}} D$ for an effective $\mathbb{R}$-divisor D. Then Conjecture 1.2 holds for (X, Δ).*

The following lemma is known to the experts, but we write details of the proof for the reader's convenience.

Lemma 3.2. *Conjecture 1.3_n implies Conjecture $1.3_{\leq n}$.*

Proof. Assume Conjecture 1.3_n and pick any $d \leq n$. Let X be a smooth projective variety of dimension d such that K_X is pseudo-effective. Let W be the product of X and an $(n - d)$-dimensional abelian variety, and let $f : W \to X$ be the projection. Then $K_W = f^* K_X$ and K_W is pseudo-effective. Since we assume Conjecture 1.3_n, Conjecture 1.3 holds for W, and therefore Conjecture 1.3 holds for X. So we are done. $\square$

From now on we prove Theorem 1.4. We fix n in Theorem 1.4.

Proof of Theorem 1.4. By Lemma 3.2, we may assume Conjecture $1.3_{\leq n}$. Pick any positive integer $d \leq n$. We prove that Conjectures 1.1_d and 1.2_d hold under the assumption that Conjectures $1.1_{\leq d-1}$ and $1.2_{\leq d-1}$ hold. If we can prove

this then Theorem 1.4 immediately follows. By Theorem 3.1, we see that Conjectures $1.2_{\leq d-1}$ and 1.1_d imply Conjecture 1.2_d. Therefore it is sufficient to prove that Conjectures $1.1_{\leq d-1}$ and $1.2_{\leq d-1}$ imply Conjecture 1.1_d.

Let (X, Δ) be a d-dimensional lc pair. By taking a dlt blow-up, we can assume that (X, Δ) is a $\mathbb{Q}$-factorial dlt pair. We can write $\Delta = S + B$, where S is the reduced part of Δ and $B = \Delta - S$. Then we have following two cases.

Case 1. $S \neq 0$ and $\tau(X, B\,; S) = 1$, where $\tau(X, B\,; S)$ is the pseudo-effective threshold of S with respect to (X, B).

Case 2. $S \neq 0$ and $\tau(X, B\,; S) < 1$, or $S = 0$.

Proof of Case 1. We prove it with several steps.

Step 1. From this step to step 5, we prove that Conjecture 1.2 holds for (X, Δ).
    We run the $(K_X + \Delta)$-MMP with scaling of an ample divisor H,

$$(X, \Delta) \dashrightarrow \cdots \dashrightarrow (X^i, \Delta_{X^i}) \dashrightarrow \cdots .$$

Then for any i, the birational map $X \dashrightarrow \cdots \dashrightarrow X^i$ is also a sequence of finitely many steps of the $(K_X + \Delta - tS)$-MMP for any sufficiently small $t > 0$. Since $K_X + \Delta - tS$ is not pseudo-effective by hypothesis, we see that $S_{X^i} \neq 0$ and $\tau(X^i, B_{X^i}\,; S_{X^i}) = 1$ for any i. Therefore we can replace (X, Δ) with (X^i, Δ_{X^i}) for some $i \gg 0$ and we may assume that there is a big divisor H such that $K_X + \Delta + \delta H$ is movable for any sufficiently small $\delta > 0$.

Step 2. Fix $A \geq 0$, a general ample $\mathbb{Q}$-divisor such that $(X, \Delta + A)$ is lc, $(X, B + A)$ is klt and $(1/2)A + K_X + B$ and $(1/2)A + S$ are both nef. Then

$$K_X + tS + B + A = \left(\tfrac{1}{2}A + K_X + B\right) + t\left(\tfrac{1}{2}A + S\right) + \tfrac{1}{2}(1 - t)A$$

is nef for any $0 \leq t \leq 1$. Let $\tau_t = \tau(X, tS + B\,; A)$ be the pseudo-effective threshold of A with respect to $(X, tS + B)$ for any $0 \leq t < 1$. By construction, we have $0 < \tau_t \leq 1$ for any t. In this step we prove that there is $0 < \epsilon < 1$ such that the divisor

$$K_X + (1 - t(1 - \epsilon))S + B + t\tau_\epsilon A = (1 - t)(K_X + \Delta) + t(K_X + \epsilon S + B + \tau_\epsilon A)$$

is not big for any $0 \leq t \leq 1$.

The idea is similar to [H, steps 2 and 3 in the proof of Prop. 5.3] (see also the proof of [DHP, Prop. 8.7] or [G, Lem. 3.1]). Let $\{u_k\}_{k \geq 1}$ be a strictly increasing infinite sequence of positive real numbers such that $u_k < 1$ for any k and $\lim_{k \to \infty} u_k = 1$. For each k, we run the $(K_X + u_k S + B)$-MMP with scaling of A.

Then we get a Mori fiber space $(X, u_k S + B) \dashrightarrow (X_k, u_k S_{X_k} + B_{X_k}) \to Z_k$. By the basic property of the log MMP with scaling, $K_{X_k} + u_k S_{X_k} + B_{X_k} + \tau_{u_k} A_{X_k}$ is trivial over Z_k. By replacing $\{u_k\}_{k \geq 1}$ with its subsequence, we may assume that $\dim Z_k$ is constant.

We note that $S_{X_k} \neq 0$ and S_{X_k} is ample over Z_k by construction. Since $\lim_{k \to \infty} u_k = 1$, by the ACC for log canonical thresholds (cf. Theorem 2.7), there are infinitely many indices k such that $\mathrm{lct}(X_k, B_{X_k}; S_{X_k}) = 1$. Therefore, by replacing $\{u_k\}_{k \geq 1}$ with its subsequence, we may assume that (X_k, Δ_{X_k}) is lc for any k. Moreover, by applying the ACC for numerically trivial pairs (cf. Theorem 2.8) to the general fibers of Mori fiber spaces $(X_k, u_k S_{X_k} + B_{X_k}) \to Z_k$, we can find an index k such that $K_{X_k} + \Delta_{X_k}$ is numerically trivial over Z_k. In particular, $K_{X_k} + \Delta_{X_k}$ is trivial over Z_k.

Set $\epsilon = u_k$ for this k. Then

$$(1-t)(K_{X_k} + \Delta_{X_k}) + t(K_{X_k} + \epsilon S_{X_k} + B_{X_k} + \tau_\epsilon A_{X_k})$$

is trivial over Z_k for any $0 \leq t \leq 1$. Since $\dim Z_k < \dim X_k$, we see that $K_X + (1 - t(1 - \epsilon))S + B + t\tau_\epsilon A$ is not big. Note that $0 < \tau_\epsilon \leq 1$.

Step 3. Set $G = (1 - \epsilon)S - \tau_\epsilon A$. Then $(X, \Delta - tG)$ is klt and $\Delta - tG$ is big for any $0 < t \leq 1$ because $\Delta - tG = (1 - t(1 - \epsilon))S + B + t\tau_\epsilon A$. In this step we show that there is an infinite sequence $\{a_k\}_{k \geq 1}$ of positive real numbers such that

(i) $a_k < 1$ for any k and $\lim_{k \to \infty} a_k = 0$, and

(ii) there are sequences of finitely many steps of the $(K_X + \Delta - a_k G)$-MMP to a good minimal model

$$(X, \Delta - a_k G) \dashrightarrow (X'_k, \Delta_{X'_k} - a_k G_{X'_k})$$

such that $(X'_k, \Delta_{X'_k})$ is lc and there is a contraction $X'_k \to Y_k$ to a normal projective variety Y_k such that

(ii-a) $\dim Y_k < \dim X'_k$, and

(ii-b) $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k}$ is $\mathbb{R}$-linearly equivalent to the pullback of an ample $\mathbb{R}$-divisor on Y_k and $K_{X'_k} + \Delta_{X'_k} \sim_{\mathbb{R}, Y_k} 0$.

Since $\tau_\epsilon > 0$ and A is ample, by applying [BCHM, Cor. 1.1.5] (see also [DHP, Thm. 8.9]), there are countably many birational maps $X \dashrightarrow X'_k$ and contractions $X'_k \to Y_k$ $(k = 1, 2, \dots)$ to normal projective varieties Y_k such that for any $0 < \alpha \leq 1$, there is an index k satisfying the following:

(♣) $(X, \Delta - \alpha G) \dashrightarrow (X'_k, \Delta_{X'_k} - \alpha G_{X'_k})$ is a sequence of finitely many steps of the $(K_X + \Delta - \alpha G)$-MMP to a good minimal model and $K_{X'_k} + \Delta_{X'_k} - \alpha G_{X'_k}$ is $\mathbb{R}$-linearly equivalent to the pullback of an ample divisor on Y_k.

Renumbering X'_k and Y_k if necessary, we can find an infinite sequence $\{a_k\}_{k\geq1}$ of positive real numbers such that

(1) $a_k < 1$ for any k and $\lim_{k\to\infty} a_k = 0$, and
(2) for any k and any $\alpha > 0$ sufficiently close to a_k, the birational map $(X, \Delta - \alpha G) \dashrightarrow (X'_k, \Delta_{X'_k} - \alpha G_{X'_k})$ and the contraction $X'_k \to Y_k$ satisfy ($\clubsuit$).

By construction, $\mathrm{lct}(X'_k, B_{X'_k}; S_{X'_k}) \geq 1 - a_k(1 - \epsilon)$. Since $\lim_{k\to\infty} a_k = 0$, by the ACC for log canonical thresholds (cf. Theorem 2.7), there are infinitely many indices k such that $\mathrm{lct}(X'_k, B_{X'_k}; S_{X'_k}) = 1$. Therefore, by replacing $\{a_k\}_{k\geq1}$ with its subsequence, we may assume that $(X'_k, \Delta_{X'_k})$ is lc for any k.

We show that $\{a_k\}_{k\geq1}$, $(X, \Delta - a_k G) \dashrightarrow (X'_k, \Delta_{X'_k} - a_k G_{X'_k})$ and $X'_k \to Y_k$ satisfy all the conditions stated at the start of this step. By construction, it is sufficient to check that the contraction $X'_k \to Y_k$ satisfies conditions (ii-a) and (ii-b) for any k. First we recall that $K_X + \Delta - a_k G$ is not big by step 2. Therefore $\dim Y_k < \dim X'_k$ and thus the contraction $X'_k \to Y_k$ satisfies condition (ii-a) for any k. Next, by condition (2), we can find a positive real number $\widetilde{\alpha} \neq a_k$ sufficiently close to a_k such that $K_{X'_k} + \Delta_{X'_k} - \widetilde{\alpha} G_{X'_k} \sim_{\mathbb{R}, Y_k} 0$. Because $K_{X'_k} + \Delta_{X'_k}$ is represented by a linear combination of $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k}$ and $K_{X'_k} + \Delta_{X'_k} - \widetilde{\alpha} G_{X'_k}$, we have $K_{X'_k} + \Delta_{X'_k} \sim_{\mathbb{R}, Y_k} 0$. Therefore $X'_k \to Y_k$ satisfies condition (ii-b) for any k.

In this way we see that $\{a_k\}_{k\geq1}$, $(X, \Delta - a_k G) \dashrightarrow (X'_k, \Delta_{X'_k} - a_k G_{X'_k})$ and $X'_k \to Y_k$ satisfy all the conditions stated at the start of this step. Thus we complete this step.

Step 4. In this step we prove that Conjecture 1.2 holds for $(X'_k, \Delta_{X'_k})$ for any k, where $(X'_k, \Delta_{X'_k})$ was constructed in step 3. By Theorem 3.1, we may show that Conjecture 1.1 holds for $(X'_k, \Delta_{X'_k})$ for any k. In this step we fix k.

Since $K_{X'_k} + \Delta_{X'_k} \sim_{\mathbb{R}, Y_k} 0$ and $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k} \sim_{\mathbb{R}, Y_k} 0$, we have $G_{X'_k} = (1 - \epsilon) S_{X'_k} - \tau_\epsilon A_{X'_k} \sim_{\mathbb{R}, Y_k} 0$. Since $A_{X'_k}$ is big, we see that $S_{X'_k}$ is big over Y_k. Therefore $S_{X'_k} \neq 0$ and some component of $S_{X'_k}$ dominates Y_k because $\dim Y_k < \dim X'_k$ by condition (ii-a) in step 3. Let $f : (V, \Gamma) \to (X'_k, \Delta_{X'_k})$ be a dlt blow-up and let T be a component of $f_*^{-1} S_{X'_k}$ dominating Y_k. Then we have $K_V + \Gamma \sim_{\mathbb{R}, Y_k} 0$. Let M be an $\mathbb{R}$-divisor on Y_k such that $K_V + \Gamma$ is $\mathbb{R}$-linearly equivalent to the pullback of M. Then M is pseudo-effective. By the adjunction, $(T, \mathrm{Diff}(\Gamma - T))$ is dlt, and $K_T + \mathrm{Diff}(\Gamma - T)$ is pseudo-effective. Since we assume Conjecture $1.1_{\leq d-1}$, Conjecture 1.1 holds for $(T, \mathrm{Diff}(\Gamma - T))$. Then there is an effective $\mathbb{R}$-divisor E on Y_k such that $M \sim_{\mathbb{R}} E$. Therefore Conjecture 1.1 holds for (V, Γ), and hence Conjecture 1.1 holds for $(X'_k, \Delta_{X'_k})$.

In this way we see that Conjecture 1.2 holds for $(X'_k, \Delta_{X'_k})$.

Step 5. In this step we show that Conjecture 1.2 holds for (X, Δ).

We continue the track of [H, step 4 in the proof of Prop. 5.1]. By replacing $\{a_k\}_{k\geq 1}$ with its subsequence, we can assume that X_i' and X_j' are isomorphic in codimension one for any i and j. Indeed, any prime divisor P contracted by the birational map $X \dashrightarrow X_k'$ is a component of $N_\sigma(K_X + \Delta - a_kG)$. But since we have

$$N_\sigma(K_X + \Delta - a_kG) \leq (1 - a_k)N_\sigma(K_X + \Delta) + a_kN_\sigma(K_X + \Delta - G),$$

P is also a component of $N_\sigma(K_X + \Delta) + N_\sigma(K_X + \Delta - G)$, which does not depend on k. Thus we can assume that X_i' and X_j' are isomorphic in codimension one by replacing $\{a_k\}_{k\geq 1}$ with its subsequence.

By step 4, $(X_1', \Delta_{X_1'})$ has a log minimal model. Therefore, by [B2, Thm. 4.1 (iii)], we can run the $(K_{X_1'} + \Delta_{X_1'})$-MMP with scaling of an ample divisor and get a log minimal model $(X_1', \Delta_{X_1'}) \dashrightarrow (X'', \Delta_{X''})$. Then we can check that $(X'', \Delta_{X''} - tG_{X''})$ is klt and $\Delta_{X''} - tG_{X''}$ is big for any sufficiently small $t > 0$. Fix a sufficiently small positive real number $t \ll a_1$. By [BCHM, Cor. 1.4.2] and running the $(K_{X''} + \Delta_{X''} - tG_{X''})$-MMP with scaling of an ample divisor, we can get a log minimal model $(X'', \Delta_{X''} - tG_{X''}) \dashrightarrow (X''', \Delta_{X'''} - tG_{X'''})$. Since $t > 0$ is sufficiently small, by the standard argument of the length of extremal rays (cf. [B1, Prop. 3.2]), $K_{X'''} + \Delta_{X'''}$ is nef. Now we get the following sequence of birational maps

$$X \dashrightarrow X_1' \dashrightarrow X'' \dashrightarrow X''',$$

where $X \dashrightarrow X_1'$ (resp. $X_1' \dashrightarrow X''$, $X'' \dashrightarrow X'''$) is a sequence of finitely many steps of the $(K_X + \Delta - a_1G)$-MMP (resp. the $(K_{X_1'} + \Delta_{X_1'})$-MMP, the $(K_{X''} + \Delta_{X''} - tG_{X''})$-MMP) to a log minimal model.

We can show that X_1' and X''' are isomorphic in codimension one. To see this, we may show that $X_1' \dashrightarrow X''$ and $X'' \dashrightarrow X'''$ contain only flips. Recall that there is a big divisor H such that $K_X + \Delta + \delta H$ is movable for any sufficiently small $\delta > 0$, which is stated in step 1. Since $X \dashrightarrow X_1'$ is a birational contraction, $K_{X_1'} + \Delta_{X_1'} + \delta H_{X_1'}$ is movable for any sufficiently small $\delta > 0$. Then $N_\sigma(K_{X_1'} + \Delta_{X_1'}) = 0$ and thus $X_1' \dashrightarrow X''$ contains only flips. Furthermore we see that $N_\sigma(K_{X''} + \Delta_{X''} - a_1G_{X''}) = 0$ since $K_{X_1'} + \Delta_{X_1'} - a_1G_{X_1'}$ is semi-ample, which is condition (ii) in step 3. Now we have $N_\sigma(K_{X''} + \Delta_{X''}) = 0$, which follows since $K_{X''} + \Delta_{X''}$ is nef. From these facts we have

$$N_\sigma(K_{X''} + \Delta_{X''} - tG_{X''})$$
$$\leq \left(1 - \frac{t}{a_1}\right)N_\sigma(K_{X''} + \Delta_{X''}) + \frac{t}{a_1}N_\sigma(K_{X''} + \Delta_{X''} - a_1G_{X''}) = 0$$

and hence $X'' \dashrightarrow X'''$ contains only flips. In this way we see that X_1' and X''' are isomorphic in codimension one.

Since $\lim_{k\to\infty} a_k = 0$, we have $t \geq a_k$ for any $k \gg 0$. Then $K_{X'''} + \Delta_{X'''} - a_k G_{X'''}$ is nef for any $k \gg 0$ because $K_{X'''} + \Delta_{X'''}$ and $K_{X'''} + \Delta_{X'''} - tG_{X'''}$ are nef. Moreover, X''' and X'_k are isomorphic in codimension one since X'_k and X'_1 are isomorphic in codimension one and X'_1 and X''' are isomorphic in codimension one. We recall that $(X'_k, \Delta_{X'_k} - a_k G_{X'_k})$ is a log minimal model of $(X, \Delta - a_k G)$, which is condition (ii) in step 3. From these facts, we see that $(X''', \Delta_{X'''} - a_k G_{X'''})$ is a log minimal model of $(X, \Delta - a_k G)$ for any $k \gg 0$. Let $p : W \to X$ and $q : W \to X'''$ be a common resolution of $X \dashrightarrow X'''$. Then for any $k \gg 0$, we have

$$p^*(K_X + \Delta - a_k G) - q^*(K_{X'''} + \Delta_{X'''} - a_k G_{X'''}) \geq 0.$$

By considering the limit $k \to \infty$, we have

$$p^*(K_X + \Delta) - q^*(K_{X'''} + \Delta_{X'''}) \geq 0.$$

Since $K_{X'''} + \Delta_{X'''}$ is nef, we see that $(X''', \Delta_{X'''})$ is a weak lc model of (X, Δ). Therefore, by [B2, Cor. 3.7], (X, Δ) has a log minimal model.

Step 6. Finally we prove that Conjecture 1.1 holds for (X, Δ). By running the $(K_X + \Delta)$-MMP with scaling of an ample divisor and replacing (X, Δ) with the resulting log minimal model, we can assume that $K_X + \Delta$ is nef. Note that after this process $S \neq 0$ and the equation $\tau(X, B\,;\,S) = 1$ still holds. Pick a sufficiently small positive real number t and run the $(K_X + \Delta - tS)$-MMP with scaling of an ample divisor. Then we get a Mori fiber space

$$(X, \Delta - tS) \dashrightarrow (X', \Delta_{X'} - tS_{X'}) \to Z.$$

Moreover, since t is sufficiently small, $K_{X'} + \Delta_{X'}$ is trivial over Z and Conjecture 1.1 for (X, Δ) is equivalent to Conjecture 1.1 for $(X', \Delta_{X'})$ (see [B1, Prop. 3.2]). We also see that there is a component of $S_{X'}$ dominating Z because $S_{X'}$ is ample over Z. By the same arguments as in step 4, we can prove that Conjecture 1.1 holds for $(X', \Delta_{X'})$ with the adjunction and Conjecture $1.1_{\leq d-1}$. Thus Conjecture 1.1 holds for (X, Δ) and so we are done.

$\square$

Proof of Case 2. In this case we can assume that (X, Δ) is klt since we have to prove only that Conjecture 1.1 holds for (X, Δ). Taking a log resolution of (X, Δ), we can assume that X is smooth. We put $\tau = \tau(X, 0\,;\,\Delta)$. Then we may assume that $\Delta \neq 0$ and $\tau > 0$ because otherwise Conjecture 1.1 for (X, Δ) is obvious from Conjecture $1.3_{\leq n}$. Moreover we may assume that $\tau = 1$ by replacing (X, Δ) with $(X, \tau\Delta)$.

We prove Case 2 with several steps. The proof is very similar to the proof of Case 1 except for step 4. In the rest of the proof we will use the fact that (X, Δ) is $\mathbb{Q}$-factorial klt but we will not use the assumption that X is smooth.

Step 1. From this step to step 5, we prove that Conjecture 1.2 holds for (X, Δ).
We run the $(K_X + \Delta)$-MMP with scaling of an ample divisor H,

$$(X, \Delta) \dashrightarrow \cdots \dashrightarrow (X^i, \Delta_{X^i}) \dashrightarrow \cdots .$$

By the same argument as in step 1 in the proof of Case 1, we can replace (X, Δ) with (X^i, Δ_{X^i}) for some $i \gg 0$ and we may assume that there is a big divisor H such that $K_X + \Delta + \delta H$ is movable for any sufficiently small $\delta > 0$. Note that $\Delta \neq 0$ and $\tau(X, 0\,; \Delta) = 1$ still hold after this process.

Step 2. Fix $A \geq 0$ a general ample $\mathbb{Q}$-divisor such that $(X, \Delta + A)$ is klt and $(1/2)A + K_X$ and $(1/2)A + \Delta$ are both nef. Then

$$K_X + t\Delta + A = \left(\tfrac{1}{2}A + K_X\right) + t\left(\tfrac{1}{2}A + \Delta\right) + \tfrac{1}{2}(1 - t)A$$

is nef for any $0 \leq t \leq 1$. We put $\tau_t = \tau(X, t\Delta\,; A)$ for any $0 \leq t < 1$. By construction, we have $0 < \tau_t \leq 1$ for any t. In this step we prove that there is $0 < \epsilon < 1$ such that the divisor

$$K_X + (1 - t(1 - \epsilon))\Delta + t\tau_\epsilon A = (1 - t)(K_X + \Delta) + t(K_X + \epsilon\Delta + \tau_\epsilon A)$$

is not big for any $0 \leq t \leq 1$.

Pick a strictly increasing infinite sequence $\{u_k\}_{k \geq 1}$ of positive real numbers such that $u_k < 1$ for any k and $\lim_{k \to \infty} u_k = 1$. For each k, run the $(K_X + u_k\Delta)$-MMP with scaling of A. Then we get a Mori fiber space $(X, u_k\Delta) \dashrightarrow (X_k, u_k\Delta_{X_k}) \to Z_k$. By the basic property of the log MMP with scaling, $K_{X_k} + u_k\Delta_{X_k} + \tau_{u_k}A_{X_k}$ is trivial over Z_k. Now we follow the same arguments as in step 2 in the proof of Case 1, and we can find an index k such that (X_k, Δ_{X_k}) is lc and $K_{X_k} + \Delta_{X_k}$ is numerically trivial over Z_k by the ACC for log canonical thresholds (cf. Theorem 2.7) and the ACC for numerically trivial pairs (cf. Theorem 2.8). Set $\epsilon = u_k$ for this k. Then we see that

$$(1 - t)(K_{X_k} + \Delta_{X_k}) + t(K_{X_k} + \epsilon\Delta_{X_k} + \tau_\epsilon A_{X_k})$$

is not big, and therefore $K_X + (1 - t(1 - \epsilon))\Delta + t\tau_\epsilon A$ is not big. Note that $0 < \tau_\epsilon \leq 1$.

Step 3. Set $G = (1 - \epsilon)\Delta - \tau_\epsilon A$. Then $(X, \Delta - tG)$ is klt and $\Delta - tG$ is big for any $0 < t \leq 1$ because $\Delta - tG = (1 - t(1 - \epsilon))\Delta + t\tau_\epsilon A$. In this step we show that there is an infinite sequence $\{a_k\}_{k \geq 1}$ of positive real numbers such that

(i) $a_k < 1$ for any k and $\lim_{k\to\infty} a_k = 0$, and

(ii) there are sequences of finitely many steps of the $(K_X + \Delta - a_k G)$-MMP to a good minimal model

$$(X, \Delta - a_k G) \dashrightarrow (X'_k, \Delta_{X'_k} - a_k G_{X'_k})$$

such that $(X'_k, \Delta_{X'_k})$ is lc, and there is a contraction $X'_k \to Y_k$ to a normal projective variety Y_k such that

(ii-a) $\dim Y_k < \dim X'_k$, and

(ii-b) $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k}$ is $\mathbb{R}$-linearly equivalent to the pullback of an ample $\mathbb{R}$-divisor on Y_k and $K_{X'_k} + \Delta_{X'_k} \sim_{\mathbb{R},Y_k} 0$.

The arguments are very similar to step 3 in the proof of Case 1. Since $\tau_\epsilon > 0$ and A is ample, by applying [BCHM, Cor. 1.1.5] (see also [DHP, Thm. 8.9]), there are countably many birational maps $X \dashrightarrow X'_k$ and contractions $X'_k \to Y_k$ ($k = 1, 2, \ldots$) to normal projective varieties Y_k such that for any $0 < \alpha \le 1$, there is an index k satisfying the following:

(♣) $(X, \Delta - \alpha G) \dashrightarrow (X'_k, \Delta_{X'_k} - \alpha G_{X'_k})$ is a sequence of finitely many steps of the $(K_X + \Delta - \alpha G)$-MMP to a good minimal model and $K_{X'_k} + \Delta_{X'_k} - \alpha G_{X'_k}$ is $\mathbb{R}$-linearly equivalent to the pullback of an ample divisor on Y_k.

Renumbering X'_k and Y_k if necessary, we can find an infinite sequence $\{a_k\}_{k\ge 1}$ of positive real numbers such that

(1) $a_k < 1$ for any k and $\lim_{k\to\infty} a_k = 0$, and

(2) for any k and any $\alpha > 0$ sufficiently close to a_k, the birational map $(X, \Delta - \alpha G) \dashrightarrow (X'_k, \Delta_{X'_k} - \alpha G_{X'_k})$ and the contraction $X'_k \to Y_k$ satisfy (♣).

By construction, $\mathrm{lct}(X'_k, 0; \Delta_{X'_k}) \ge 1 - a_k(1 - \epsilon)$. Since $\lim_{k\to\infty} a_k = 0$, by the ACC for log canonical thresholds (cf. Theorem 2.7), there are infinitely many indices k such that $\mathrm{lct}(X'_k, 0; \Delta_{X'_k}) = 1$. Therefore, by replacing $\{a_k\}_{k\ge 1}$ with its subsequence, we may assume that $(X'_k, \Delta_{X'_k})$ is lc for any k.

Furthermore, by the same arguments as in step 3 in the proof of Case 1, we can check that the contraction $X'_k \to Y_k$ satisfies conditions (ii-a) and (ii-b) for any k. Note that $K_X + \Delta - a_k G$ is not big by step 2. In this way we see that the sequence $\{a_k\}_{k\ge 1}$, the birational maps $(X, \Delta - a_k G) \dashrightarrow (X'_k, \Delta_{X'_k} - a_k G_{X'_k})$ and contractions $X'_k \to Y_k$ satisfy all the conditions stated at the start of this step. Thus we complete this step.

Step 4. In this step we prove that Conjecture 1.2 holds for $(X'_k, \Delta_{X'_k})$ for any k, where $(X'_k, \Delta_{X'_k})$ was constructed in step 3. We note that (X, Δ) is klt but

$(X'_k, \Delta_{X'_k})$ may not be klt. By Theorem 3.1, we have to show only that Conjecture 1.1 holds for $(X'_k, \Delta_{X'_k})$ for any k. In this step we fix k.

If $(X'_k, \Delta_{X'_k})$ is klt, by applying Ambro's canonical bundle formula (cf. [FG, Cor. 3.2]) to $X'_k \to Y_k$ and since we assume Conjecture $1.1_{\leq d-1}$, Conjecture 1.1 holds for $(X'_k, \Delta_{X'_k})$. Therefore we may assume that $(X'_k, \Delta_{X'_k})$ is not klt.

Let $f : (V, \Gamma) \to (X'_k, \Delta_{X'_k})$ be a dlt blow-up of $(X'_k, \Delta_{X'_k})$ and we write $\Gamma = S_V + B_V$, where S_V is the reduced part of Γ and $B_V = \Gamma - S_V$. Then $S_V \neq 0$ by our assumption. We may prove that Conjecture 1.1 holds for (V, Γ). If $\tau(V, B_V\,; S_V) = 1$, then Conjecture 1.1 holds for (V, Γ) by Case 1. Therefore we may assume that $\tau(V, B_V\,; S_V) < 1$. Note that $K_V + \Gamma \sim_{\mathbb{R}, Y_k} 0$ by construction.

Since $K_{X'_k} + \Delta_{X'_k} \sim_{\mathbb{R}, Y_k} 0$ and $K_{X'_k} + \Delta_{X'_k} - a_k G_{X'_k} \sim_{\mathbb{R}, Y_k} 0$, we have $G_{X'_k} = (1 - \epsilon)\Delta_{X'_k} - \tau_\epsilon A_{X'_k} \sim_{\mathbb{R}, Y_k} 0$. Since $A_{X'_k}$ is big, we see that $\Delta_{X'_k}$ is big over Y_k. Then Γ is also big over Y_k because Γ contains $f_*^{-1}\Delta_{X'_k}$ and all f-exceptional prime divisors. We pick a sufficiently small positive real number $t < 1$ so that $\tau(V, B_V\,; S_V) \leq 1 - t$ and $\Gamma - tS_V$ is big over Y_k. Then $(V, \Gamma - tS_V)$ is klt and $K_V + \Gamma - tS_V$ is pseudo-effective. Moreover, we see that $K_V + \Gamma - tS_V$ is not big over Y_k because $K_V + \Gamma - tS_V \sim_{\mathbb{R}, Y_k} -tS_V$ and $\dim Y_k < \dim V$. By construction, it is sufficient to prove that Conjecture 1.1 holds for $(V, \Gamma - tS_V)$.

We run the $(K_V + \Gamma - tS_V)$-MMP over Y_k with scaling of an ample divisor. By [BCHM, Cor. 1.4.2], we get a good minimal model $(V, \Gamma - tS_V) \dashrightarrow (V', \Gamma_{V'} - tS_{V'})$ over Y_k, where $S_{V'}$ is the birational transform of S_V on V'. Then there is a contraction $V' \to \widetilde{Y}$ to a normal projective variety $\widetilde{Y}$ over Y_k such that $K_{V'} + \Gamma_{V'} - tS_{V'} \sim_{\mathbb{R}, \widetilde{Y}} 0$. We can check that $(V', \Gamma_{V'} - tS_{V'})$ is klt, and furthermore $\dim \widetilde{Y} < \dim V'$ since $K_{V'} + \Gamma_{V'} - tS_{V'}$ is not big over Y_k. Therefore, applying Ambro's canonical bundle formula (cf. [FG, Cor. 3.2]) to $V' \to \widetilde{Y}$ and since we assume Conjecture $1.1_{\leq d-1}$, Conjecture 1.1 holds for $(V', \Gamma_{V'} - tS_{V'})$. Then Conjecture 1.1 holds for $(V, \Gamma - tS_V)$, and thus Conjecture 1.1 holds for (V, Γ_V).

In this way we see that Conjecture 1.2 holds for $(X'_k, \Delta_{X'_k})$ for any k, and we complete this step.

Step 5. In this step we show that Conjecture 1.2 holds for (X, Δ). The arguments are very similar to step 5 in the proof of Case 1.

By replacing $\{a_k\}_{k \geq 1}$ with its subsequence, we can assume that X'_i and X'_j are isomorphic in codimension one for any i and j. Indeed, any prime divisor P contracted by the birational map $X \dashrightarrow X'_k$ is a component of $N_\sigma(K_X + \Delta - a_k G)$. But since we have

$$N_\sigma(K_X + \Delta - a_k G) \leq (1 - a_k)N_\sigma(K_X + \Delta) + a_k N_\sigma(K_X + \Delta - G),$$

P is also a component of $N_\sigma(K_X + \Delta) + N_\sigma(K_X + \Delta - G)$, which does not depend

on k. Thus we can assume that X_i' and X_j' are isomorphic in codimension one by replacing $\{a_k\}_{k \geq 1}$ with its subsequence.

Since $(X_1', \Delta_{X_1'})$ has a log minimal model, by [B2, Thm. 4.1(iii)], we can run the $(K_{X_1'} + \Delta_{X_1'})$-MMP with scaling of an ample divisor and get a log minimal model $(X_1', \Delta_{X_1'}) \dashrightarrow (X'', \Delta_{X''})$. Then we can check that $(X'', \Delta_{X''} - tG_{X''})$ is klt and $\Delta_{X''} - tG_{X''}$ is big for any sufficiently small $t > 0$. Fix a sufficiently small positive real number $t \ll a_1$. By [BCHM, Cor. 1.4.2] and running the $(K_{X''} + \Delta_{X''} - tG_{X''})$-MMP with scaling of an ample divisor, we can get a log minimal model $(X'', \Delta_{X''} - tG_{X''}) \dashrightarrow (X''', \Delta_{X'''} - tG_{X'''})$. Since $t > 0$ is sufficiently small, by the standard argument of the length of extremal rays (cf. [B1, Prop. 3.2]), we see that $K_{X'''} + \Delta_{X'''}$ is nef. Now we get the following sequence of birational maps:

$$X \dashrightarrow X_1' \dashrightarrow X'' \dashrightarrow X''',$$

where $X \dashrightarrow X_1'$ (resp. $X_1' \dashrightarrow X''$, $X'' \dashrightarrow X'''$) is a sequence of finitely many steps of the $(K_X + \Delta - a_1 G)$-MMP (resp. the $(K_{X_1'} + \Delta_{X_1'})$-MMP, the $(K_{X''} + \Delta_{X''} - tG_{X''})$-MMP) to a log minimal model.

We can show that X_1' and X''' are isomorphic in codimension one. To see this, we may show that $X_1' \dashrightarrow X''$ and $X'' \dashrightarrow X'''$ contain only flips. Recall that there is a big divisor H such that $K_X + \Delta + \delta H$ is movable for any sufficiently small $\delta > 0$, which is stated in step 1 in this proof. Since $X \dashrightarrow X_1'$ is a birational contraction, $K_{X_1'} + \Delta_{X_1'} + \delta H_{X_1'}$ is movable for any sufficiently small $\delta > 0$. Then $N_\sigma(K_{X_1'} + \Delta_{X_1'}) = 0$ and thus $X_1' \dashrightarrow X''$ contains only flips. Furthermore we see that $N_\sigma(K_{X''} + \Delta_{X''} - a_1 G_{X''}) = 0$ since $K_{X_1'} + \Delta_{X_1'} - a_1 G_{X_1'}$ is semi-ample, which is condition (ii) in step 3. Now we have $N_\sigma(K_{X''} + \Delta_{X''}) = 0$, which follows since $K_{X''} + \Delta_{X''}$ is nef. From these facts, we have

$$N_\sigma(K_{X''} + \Delta_{X''} - tG_{X''})$$
$$\leq \left(1 - \frac{t}{a_1}\right) N_\sigma(K_{X''} + \Delta_{X''}) + \frac{t}{a_1} N_\sigma(K_{X''} + \Delta_{X''} - a_1 G_{X''}) = 0$$

and hence $X'' \dashrightarrow X'''$ contains only flips. In this way we see that X_1' and X''' are isomorphic in codimension one.

Since $\lim_{k \to \infty} a_k = 0$, we have $t \geq a_k$ for any $k \gg 0$. Then $K_{X'''} + \Delta_{X'''} - a_k G_{X'''}$ is nef for any $k \gg 0$ because $K_{X'''} + \Delta_{X'''}$ and $K_{X''} + \Delta_{X''} - tG_{X''}$ are nef. Moreover, X''' and X_k' are isomorphic in codimension one since X_k' and X_1' are isomorphic in codimension one and X_1' and X''' are isomorphic in codimension one. We recall that $(X_k', \Delta_{X_k'} - a_k G_{X_k'})$ is in particular a log minimal model of $(X, \Delta - a_k G)$, which is condition (ii) in step 3. From these facts, we see that $(X''', \Delta_{X'''} - a_k G_{X'''})$ is a log minimal model of $(X, \Delta - a_k G)$ for any $k \gg 0$. Let $p : W \to X$ and $q : W \to X'''$ be a common resolution of $X \dashrightarrow X'''$. Then for any

$k \gg 0$ we have

$$p^*(K_X + \Delta - a_k G) - q^*(K_{X'''} + \Delta_{X'''} - a_k G_{X'''}) \geq 0.$$

By considering the limit $k \to \infty$, we have

$$p^*(K_X + \Delta) - q^*(K_{X'''} + \Delta_{X'''}) \geq 0.$$

Since $K_{X'''} + \Delta_{X'''}$ is nef, we see that $(X''', \Delta_{X'''})$ is a weak lc model of (X, Δ). Therefore, by [B2, Cor. 3.7], (X, Δ) has a log minimal model.

Step 6. Finally we prove that Conjecture 1.1 holds for (X, Δ). By running the $(K_X + \Delta)$-MMP with scaling of an ample divisor and replacing (X, Δ) with the resulting log minimal model, we can assume that $K_X + \Delta$ is nef. Pick a sufficiently small positive real number t and run the $(K_X + (1 - t)\Delta)$-MMP with scaling of an ample divisor. Since $K_X + (1 - t)\Delta$ is not pseudo-effective we get a Mori fiber space

$$(X, (1 - t)\Delta) \dashrightarrow (X', (1 - t)\Delta_{X'}) \to Z.$$

Moreover, since t is sufficiently small, $K_{X'} + \Delta_{X'}$ is trivial over Z and Conjecture 1.1 for (X, Δ) is equivalent to Conjecture 1.1 for $(X', \Delta_{X'})$ (see [B1, Prop. 3.2]). Now we can easily check that Conjecture 1.1 for $(X', \Delta_{X'})$ holds by Ambro's canonical bundle formula (cf. [FG, Cor. 3.2]) and Conjecture $1.1_{\leq d-1}$. So we are done. $\qquad\square$

Therefore we complete the proof of Theorem 1.4. $\qquad\square$

Remark 3.3. Let (X, Δ) be a projective $\mathbb{Q}$-factorial log canonical pair such that $(X, 0)$ is Kawamata log terminal, and let A be a general sufficiently ample divisor. Suppose that $K_X + \Delta$ is pseudo-effective and $K_X + t\Delta$ is not pseudo-effective for any $t < 1$. Then as in step 2 in the proof of Case 1 (or Case 2), we see that the pseudo-effective threshold $\tau(X, t\Delta; A)$ is a linear function of t when $t \in [\epsilon, 1]$ for some $\epsilon > 0$ sufficiently close to 1. In the proof of Theorem 1.4 we construct lc-trivial fibrations by using this property.

We introduce a simple application of the above property. With notation as above, assume that Δ and A are $\mathbb{Q}$-divisors. Then

$$\mathcal{E} = \{(a, b) \in [0, 1] \times [0, 1] \mid K_X + a\Delta + bA \text{ is pseudo-effective}\}$$

is a rational polytope in $[0, 1] \times [0, 1]$. Indeed, if we set $\tau_\epsilon = \tau(X, \epsilon\Delta; A)$, where $\epsilon \in \mathbb{Q}$ is as above, then $\tau_\epsilon \in \mathbb{Q}$ (cf. [DHP, Prop. 8.7]) and

$$\mathcal{E} \cap \big([\epsilon, 1] \times [0, 1]\big) = \{(a, b) \mid (1 - \epsilon)b \geq \tau_\epsilon(1 - a)\}$$

by the above property. On the other hand, if we pick a positive rational number $\epsilon' < \tau_\epsilon$, then $K_X + \epsilon\Delta + \epsilon'A$ is not pseudo-effective. Thus

$$\mathcal{E} \cap \big([0,\epsilon] \times [0,1]\big) = \{(a, b + \epsilon') \mid K_X + (a\Delta + bA) + \epsilon'A \text{ is pseudo-effective}\}$$

is a rational polytope by [BCHM, Cor. 1.1.5]. In this way we see that $\mathcal{E}$ is a rational polytope in $[0,1] \times [0,1]$.

Proof of Corollary 1.5. It immediately follows from Theorem 1.4. $\qquad\square$

Proof of Corollary 1.6. It immediately follows from Theorem 1.4. $\qquad\square$

Acknowledgements

References

[B1] C. Birkar, On existence of log minimal models II, J. reine angew Math. **658** (2011), 99–113. Zbl 1226.14021 MR 2831514

[B2] C. Birkar, Existence of log canonical flips and a special LMMP, Publ. Math. Inst. Hautes Études Sci. **115** (2012), 325–368. Zbl 1256.14012 MR 2929730

[B3] C. Birkar, On existence of log minimal models and weak Zariski decompositions, Math. Ann. **354** (2012), 787–799. Zbl 1255.14016 MR 2965261

[BCHM] C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. **23** (2010), 405–468. Zbl 1210.14019 MR 2601039

[DHP] J.-P. Demailly, C. D. Hacon and M. Păun, Extension theorems, non-vanishing and the existence of good minimal models, Acta Math. **210** (2013), 203–259. Zbl 1278.14022 MR 3070567

[FG] O. Fujino and Y. Gongyo, On canonical bundle formulas and subadjunctions, Michigan Math. J. **61** (2012), 255–264. Zbl 1260.14010 MR 2944479

[G] Y. Gongyo, Remarks on the non-vanishing conjecture, in *Algebraic geometry in East Asia—Taipei 2011*, Advanced Studies in Pure Mathematics 65, Math. Soc. Japan, Tokyo, 2015, 107–116. Zbl 1360.14054 MR 3380777

[HMX] C. D. Hacon, J. McKernan, C. Xu, ACC for log canonical thresholds, Ann. of Math. (2) **180** (2014), 523–571. Zbl 1320.14023 MR 3224718

[H] K. Hashizume, Minimal model theory for relatively trivial log canonical pairs, to appear in Ann. Inst. Fourier (Grenoble).

[S] V. V. Shokurov, Letters of a bi-rationalist. VII. Ordered termination, Tr. Mat. Inst. Steklova **264** (2009), Mnogomernaya Algebraicheskaya Geometriya, 184–208; translation in Proc. Steklov Inst. Math **264** (2009), 178–200. Zbl 1312.14041 MR 2590847