*Publ. RIMS Kyoto Univ.* **54** (2018), 105–139
DOI 10.4171/PRIMS/54-1-4

# Mather Discrepancy as an Embedding Dimension in the Space of Arcs

by

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**Abstract**

Let $X$ be a variety over a field $k$ and let $X_\infty$ be its space of arcs. We study the complete local ring $\widehat{A} := \widehat{\mathcal{O}_{X_\infty,P_{eE}}}$, where $P_{eE}$ is the stable point defined by an integer $e \geq 1$ and a divisorial valuation $\nu_E$ on $X$. Assuming $\operatorname{char} k = 0$, we prove that $\operatorname{embdim} \widehat{A} = e(\widehat{k}_E + 1)$, where $\widehat{k}_E$ is the Mather discrepancy of $X$ with respect to $\nu_E$. We also obtain that $\dim \widehat{A}$ has as lower bound $e(a_{\mathrm{MJ}}(E;X))$, where $a_{\mathrm{MJ}}(E;X)$ is the Mather–Jacobian log-discrepancy of $X$ with respect to $\nu_E$. For $X$ normal and a complete intersection, we prove as a consequence that if $P_E$ has codimension 1 in $X_\infty$ then the discrepancy $k_E \leq 0$.

*2010 Mathematics Subject Classification:* Primary14B05; Secondary13A18,14J17,14E15.
*Keywords:* Space of arcs, divisorial valuations, embedding dimension, Mather discrepancy.

## §1. Introduction

In 1968, Nash introduced the space of arcs $X_\infty$ of an algebraic variety $X$ in order to study the singularities of $X$. More precisely, he wanted to understand what the various *resolutions of singularities* have in common, his work being established just after the proof of resolution of singularities in characteristic zero by Hironaka. Nash's work was popularized by Hironaka and later by Lejeune-Jalabert.

The development of *motivic integration* gave powerful tools for studying finiteness properties in the (not of finite type) $k$-scheme $X_\infty$. Two main ideas in Denef and Loeser's article [2] appear in this work: the change of variables formula in motivic integration, due to Kontsevich for smooth $X$, and the stability property, which had already appeared in Kolchin's work on differential algebra. More pre-

Communicated by S. Mukai. Received March 23, 2017. Revised June 27, 2017.

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cisely, based on this stability property, in [16] and [17] (see also [18]) we introduced stable points of $X_\infty$, which are certain fat points of finite codimension in $X_\infty$. We proved that, if $P$ is stable then the complete local ring $\widehat{\mathcal{O}_{X_\infty,P}}$ is a Noetherian ring. From this result we proved a curve selection lemma ending at stable points of $X_\infty$. Stable points form a natural framework whenever induced morphisms $\eta_\infty : Y_\infty \to X_\infty$ are considered, where $\eta : Y \to X$ is of finite type and locally dominant ([17] and [18]).

*Mori theory* is also related to the study of the space of arcs. The recent work of de Fernex and Docampo [7] has confirmed this relationship. In fact, a divisorial valuation $\nu = \nu_E$ on $X$ defines a stable point $P_E$ on $X_\infty$ and, assuming the existence of a resolution of singularities and applying the previous curve selection lemma, we can characterize $\dim \mathcal{O}_{X_\infty,P_E} = 1$ in terms of a property of lifting wedges centered at $P_E$ ([18]). Then, de Fernex and Docampo's result, which gives an approach to Nash's project, can be understood as follows: assuming $\operatorname{char} k = 0$, we have that if $\nu_E$ is a terminal valuation then $\dim \mathcal{O}_{X_\infty,P_E} = \dim \widehat{\mathcal{O}_{X_\infty,P_E}} = 1$. On the other hand, several examples of a normal hypersurface $X$ and an essential valuation $\nu_E$, for which the property of lifting wedges centered at $P_E$ does not hold, have been studied ([11], [6], [12]). One of the key points in producing such examples is to require $k_E \geq 1$, where $k_E$ is the discrepancy of $X$ with respect to $E$. This suggests a connection between $\dim \mathcal{O}_{X_\infty,P_E}$, or $\dim \widehat{\mathcal{O}_{X_\infty,P_E}}$, and geometric invariants of $(X, \nu_E)$.

Understanding the algebraic properties of the rings $\widehat{\mathcal{O}_{X_\infty,P}}$ and $\mathcal{O}_{X_\infty,P}$, where $P$ is stable, is an important problem; it leads towards the study of nonconstant families of arcs in $X_\infty$. In particular, one of our main goals is to compute $\dim \mathcal{O}_{X_\infty,P}$. In general, for any stable point $P$, an *upper bound* on the dimension of $\mathcal{O}_{X_\infty,P}$ follows from the stability property: expressed in terms of cylinders, stable points are precisely the generic points of the irreducible cylinders in $X_\infty$, and $\dim \mathcal{O}_{X_\infty,P}$ is bounded above by the codimension as cylinder of the closure of $P$ in $X_\infty$ (see (2.2)). If $X$ is nonsingular at the center of $P$ in $X$, then the ring $\mathcal{O}_{X_\infty,P}$ is regular and the dimension is equal to the above upper bound, but in general the inequality in the bound is strict. From the change of variables formula in motivic integration, it follows that the codimension as cylinder of the set $N_{eE}$ of arcs with contact $e \geq 1$ with an exceptional divisor $E$ is equal to $e(\widehat{k}_E + 1)$. Here $\widehat{k}_E$ is the Mather discrepancy of $X$ with respect to $E$, introduced in [8] (see also [10]). Hence, for the generic point $P_{eE}$ of $N_{eE}$ we have $\dim \mathcal{O}_{X_\infty,P_{eE}} \leq e(\widehat{k}_E + 1)$.

In this article we study the embedding dimension of $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P_{eE}}$. We prove that, assuming $\operatorname{char} k = 0$, we have

$$\operatorname{embdim} \widehat{\mathcal{O}_{X_\infty,P_{eE}}} = \operatorname{embdim} \mathcal{O}_{(X_\infty)_{\mathrm{red}},P_{eE}} = e(\widehat{k}_E + 1), \tag{1.1}$$

that is, the embedding dimension of $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P_{eE}}$ is equal to the codimension as cylinder of $N_{eE}$. Moreover, we describe explicitly a minimal system of coordinates of $(X_\infty)_{\mathrm{red}}$ at $P_{eE}$ (Theorem 3.4 in this article), that is, we give a finite set of elements of the prime ideal $P_{eE}$ of $\mathcal{O}_{(X_\infty)_{\mathrm{red}}}$ whose classes modulo $(P_{eE})^2$ define a basis of $\kappa(P_{eE})$-vector space $P_{eE}/(P_{eE})^2$. Here $\kappa(P_{eE})$ is the residue field of $P_{eE}$ on $X_\infty$ and the reason why the system is finite is our finiteness property of stable points ([17, Thm. 4.1]; see (viii) below). Applying this description of minimal coordinates, we obtain the *lower bound*

$$\dim \widehat{\mathcal{O}_{X_\infty,P_{eE}}} \geq e\big(\widehat{k}_E - \nu_E(\mathrm{Jac}_X) + 1\big), \tag{1.2}$$

where $\mathrm{Jac}_X$ is the Jacobian ideal of $X$ (Theorem 4.1). In particular, if $X$ is normal and a complete intersection then $\dim \widehat{\mathcal{O}_{X_\infty,P_E}} \geq e(k_E+1)$. Hence, in this case, $\dim \mathcal{O}_{X_\infty,P_E} = 1$, or $\dim \widehat{\mathcal{O}_{X_\infty,P_E}} = 1$, implies $k_E \leq 0$ (Corollary 4.2).

The graded algebra associated to the divisorial valuation $\nu_E$ plays an essential role in this study. The natural coordinates of $(X_\infty)_{\mathrm{red}}$ at $P_{eE}$ are obtained by specialization techniques to the graded algebra of $\nu_E$ adapted from Teissier ([22], [9], [21]). These techniques are applied to a general projection $X \to \mathbb{A}^d$ and the induced valuation on $\mathbb{A}^d$. Such coordinates are introduced in [19]. In Section 3 of this paper we prove that they also provide minimal coordinates of $(X_\infty)_{\mathrm{red}}$ at $P_{eE}$ and we conclude (1.1). The way we obtain this proof is, with the language in [21], by embedding $X$ in a complete intersection $X'$ which is an overweight deformation of an affine toric variety associated to the divisorial valuation $\nu_E$. In Section 4 we prove the lower bound for $\dim \widehat{\mathcal{O}_{X_\infty,P_{eE}}}$ in (1.2); for this we embed $X$ in a general complete intersection $X'$. The important fact used here is that $X$ can be substituted by $X'$ in order to compute the local rings $\widehat{\mathcal{O}_{X_\infty,P_{eE}}}$ ([18]; cf. (ii) and (ix) of Section 2 in this paper). All these results extend to arbitrary stable points $P$ of $X_\infty$ (Remark 3.5).

## §2. Preliminaries

In this section we will set the notation and recall some properties of the space of arcs and their stable points. For more details see [2], [5], [11], [18].

Let $k$ be a perfect field and let $X$ be a $k$-scheme. Given a field extension $k \subseteq K$, a $K$-*arc* on $X$ is a $k$-morphism $\operatorname{Spec} K[[t]] \to X$. The $K$-arcs on $X$ are the $K$-rational points of a $k$-scheme $X_\infty$ called the *space of arcs* of $X$. More precisely, $X_\infty = \varprojlim X_n$, where, for $n \in \mathbb{N}$, $X_n$ is the $k$-scheme of $n$-jets whose $K$-rational points are the $k$-morphisms $\operatorname{Spec} K[t]/(t)^{n+1} \to X$. In fact, the projective limit is a $k$-scheme because the natural morphisms $X_{n'} \to X_n$, for $n' \geq n$, are affine morphisms. We denote by $j_n : X_\infty \to X_n$, $n \geq 0$ the natural projections.

For every $k$-algebra $A$, we have a natural isomorphism

$$\mathrm{Hom}_k(\mathrm{Spec}\, A, X_\infty) \cong \mathrm{Hom}_k(\mathrm{Spec}\, A[[t]], X). \tag{2.1}$$

Given $P \in X_\infty$, with residue field $\kappa(P)$, we denote by $h_P : \mathrm{Spec}\, \kappa(P)[[t]] \to X$ the $\kappa(P)$-arc on $X$ corresponding by (2.1) to the $\kappa(P)$-rational point of $X_\infty$ defined by $P$. The image in $X$ of the closed point of $\mathrm{Spec}\, \kappa(P)[[t]]$, or equivalently, the image $P_0$ of $P$ by $j_0 : X_\infty \to X = X_0$ is called the *center* of $P$. Then, we denote by $\nu_P$ the *order function* $\mathrm{ord}_t h_P^\sharp : \mathcal{O}_{X,P_0} \to \mathbb{N} \cup \{\infty\}$. It also follows from (2.1) that a $K$-arc on $X_\infty$ is equivalent to a $K$-wedge, i.e., a $k$-morphism $\Phi : \mathrm{Spec}\, K[[\xi, t]] \to X$.

The space of arcs of the affine space $\mathbb{A}_k^N = \mathrm{Spec}\, k[x_1, \ldots, x_N]$ is $(\mathbb{A}_k^N)_\infty = \mathrm{Spec}\, k[\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_n, \ldots]$ where for $n \geq 0$, $\underline{X}_n = (X_{1;n}, \ldots, X_{N;n})$ is an $N$-tuple of variables. For any $f \in k[x_1, \ldots, x_N]$, let $\sum_{n=0}^\infty F_n t^n$ be the Taylor expansion of $f(\sum_n \underline{X}_n t^n)$, hence $F_n \in k[\underline{X}_0, \ldots, \underline{X}_n]$. Equivalently, $\sum_{n=0}^\infty F_n t^n$ is the image of $f$ by the morphism of $k$-algebras $\mathcal{O}_{\mathbb{A}_k^N} \to \mathcal{O}_{(\mathbb{A}_k^N)_\infty}[[t]]$ induced in (2.1) by the identity map in $(\mathbb{A}_k^N)_\infty$. If $X \subseteq \mathbb{A}_k^N$ is affine, and $I_X \subset k[x_1, \ldots, x_N]$ is the ideal defining $X$ in $\mathbb{A}_k^N$, then we have

$$X_\infty = \mathrm{Spec}\, k[\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_n, \ldots]/(\{F_n\}_{n \geq 0, f \in I_X}).$$

Analogously, if $X = \mathrm{Spec}\, k[[x_1, \ldots, x_N]]/I_X$ then we have

$$X_\infty = \mathrm{Spec}\, k[[\underline{X}_0]][\underline{X}_1, \ldots, \underline{X}_n, \ldots]/(\{F_n\}_{n \geq 0, f \in I_X}).$$

Let $X$ be a separated $k$-scheme that is locally of finite type over some Noetherian complete local ring $R_0$ with residue field $k$. Note that $X$ may be a reduced separated $k$-scheme of finite type, and it may also be a $k$-scheme $\mathrm{Spec}\, \widehat{R}$, where $\widehat{R}$ is the completion of a local ring $R$ which is a $k$-algebra of finite type. In [18] the *stable points* of $X_\infty$ were defined as follows:

First, if $X$ is affine and irreducible and $P$ is a point of $X_\infty$, i.e., a prime ideal of $\mathcal{O}_{X_\infty}$, then the following conditions are equivalent:

(a) There exist $n_1 \in \mathbb{N}$ and $G \in \mathcal{O}_{X_\infty} \setminus P$, $G \in \mathcal{O}_{X_{n_1}}$ such that, for $n \geq n_1$, the map $X_{n+1} \longrightarrow X_n$ induces a trivial fibration

$$\overline{j_{n+1}(Z(P))} \cap (X_{n+1})_G \longrightarrow \overline{j_n(Z(P))} \cap (X_n)_G$$

with fiber $\mathbb{A}_k^d$, where $d = \dim X$, $(X_n)_G$ is the open subset $X_n \setminus Z(G)$ of $X_n$ and $\overline{j_n(Z(P))}$ is the closure of $j_n(Z(P))$ in $X_n$ with the reduced structure.

(b) There exists $G \in \mathcal{O}_{X_\infty} \setminus P$ such that the ideal $P(\mathcal{O}_{X_\infty})_G$ is the radical of a finitely generated ideal of $(\mathcal{O}_{X_\infty})_G$.

We say that the point $P$ is stable if the above conditions hold ([16, Lem. 3.1], [17, Def. 3.1] and [18, Def. 3.6]). For the stability property on the maps $j_{n+1}(X_\infty) \to j_n(X_\infty)$, see Denef–Loeser [2, Lem. 4.1] and Lejeune-Jalabert [13].

In general, i.e., for $X$ not necessarily irreducible, the set of stable points of $X_\infty$ is the union of the sets of stable points of the irreducible components of $X$. Besides, this union is disjoint (see (i) below).

Recall that a subset $C$ of $X_\infty$ is a *cylinder* if it is of the form $C = j_n^{-1}(S)$ for some $n$ and some constructible subset $S \subseteq X_n$ ([5, Sect. 5]). Hence, from (b) above it follows that the stable points of $X_\infty$ are precisely the generic points of the irreducible cylinders.

The following properties of stable points will be used in the next sections. The first ones, (i) to (iv), are direct consequences of the definition of stable points and of the stability property in [2]. Property (v) uses well-known facts of the theory of valuations:

**[18, Prop. 3.7].** Let $P$ be a stable point of $X_\infty$; then the following properties hold:

- **(i)** Let $X_0$ be an irreducible component of $X$ such that $P \in (X_0)_\infty$. Then, the arc $h_P : \operatorname{Spec} \kappa(P)[[t]] \to X_0$ defined by $P$ is a dominant morphism.
- **(ii)** Let $U$ be any irreducible open affine subscheme of $X$ that contains the generic point of the image of $h_P$; then
$$\mathcal{O}_{(X_\infty)_{\mathrm{red}},P} = \mathcal{O}_{(\overline{U}_\infty)_{\mathrm{red}},P}.$$
Moreover, there exists $X' \subseteq \mathbb{A}_k^N$ a complete intersection scheme that contains $U$ and of dimension $\dim U$ and, for any such $X'$, we have
$$\mathcal{O}_{(X_\infty)_{\mathrm{red}},P} \cong \mathcal{O}_{(U_\infty)_{\mathrm{red}},P} \cong \mathcal{O}_{(X'_\infty)_{\mathrm{red}},P},$$
where we also denote by $P$ the point induced by $P$ in $(X_\infty)_{\mathrm{red}}$ and in $(X'_\infty)_{\mathrm{red}}$. Therefore $X_\infty$ is irreducible at $P$, i.e., the nilradical of the ring $\mathcal{O}_{X_\infty,P}$ is a prime ideal.
- **(iii)** The residue field $\kappa(P)$ of $P$ on $X_\infty$ is a countably pure transcendental extension of a finite extension of $k$. This implies that $\kappa(P)$ is a separably generated field extension of $k$ ([18, Prop. 3.7(v)]).
- **(iv)** The quantity $\dim \mathcal{O}_{\overline{j_n(X_\infty)},P_n}$ is constant for $n \gg 0$, where $\overline{j_n(X_\infty)}$ is the closure of $j_n(X_\infty)$ in $X_n$, with the reduced structure, and $P_n$ is the prime ideal $P \cap \mathcal{O}_{\overline{j_n(X_\infty)}}$. Since
$$\dim \mathcal{O}_{X_\infty,P} \leq \sup_n \dim \mathcal{O}_{\overline{j_n(X_\infty)},P_n}, \tag{2.2}$$
this implies that $\dim \mathcal{O}_{X_\infty,P} < \infty$.

**(v)** Let $\nu_P$ be the valuation on the function field $K(X_0)$ of $X_0$ defined by the arc $h_P$, where $X_0$ is the irreducible component of $X$ such that $P \in (X_0)_\infty$. Then, either $P_0$ is the generic point of $X_0$ and in this case $\nu_P$ is trivial, or $\nu_P$ is a divisorial valuation.

Property (i) is equivalent to the statement in [5, Lem. 5.1] for cylinders. In property (iv), the right-hand-side term in (2.2) is the definition of the *codimension of the cylinder* $Z(P)$ (see [5, Sect. 5]); but the inequality in (2.2) may be strict. For property (v) in the setting of cylinders, see [8] and also [4]. The next property compares the local rings at stable points of the space of arcs of $X = \operatorname{Spec} R$, where $R$ is a local ring which is a $k$-algebra of finite type, and of $\widehat{X} = \operatorname{Spec} \widehat{R}$, where $\widehat{R}$ is the completion of $R$:

**(vi)** Let $P$ be a stable point of $X_\infty$, where $X = \operatorname{Spec} R$ is as before, whose center in $X$ is the maximal ideal of $R$. Then $P$ induces a stable point in $\widehat{X}_\infty$, that we also denote by $P$, and we have

$$\widehat{\mathcal{O}_{X_\infty,P}} = \widehat{\mathcal{O}_{\widehat{X}_\infty,P}}.$$

The following finiteness property of the stable points, which is the main result in [17], is expressed in terms of the local ring $\mathcal{O}_{X_\infty,P}$, or more precisely, its formal completion. It implies a curve selection lemma in $X_\infty$ ending at a stable point $P$ ([17, Cor. 4.8]). Property (viii) below helps understand this local ring.

**Finiteness property of the stable points ([17, Thm. 4.1]).** Let $P$ be a stable point of $X_\infty$; then the following properties hold:

**(vii)** The formal completion $\widehat{\mathcal{O}_{(X_\infty)_{\mathrm{red}},P}}$ of the local ring of $(X_\infty)_{\mathrm{red}}$ at a stable point $P$ is a Noetherian ring.

**(viii)** Moreover, if $X$ is affine, then there exists $G \in \mathcal{O}_{X_\infty} \setminus P$ such that the ideal $P\left(\mathcal{O}_{(X_\infty)_{\mathrm{red}}}\right)_G$ is a finitely generated ideal of $\left(\mathcal{O}_{(X_\infty)_{\mathrm{red}}}\right)_G$.

**(ix)** ([18, Thm. 3.13]. if $\operatorname{char} k = 0$) Moreover, we have $\widehat{\mathcal{O}_{X_\infty,P}} \cong \widehat{\mathcal{O}_{(X_\infty)_{\mathrm{red}},P}}$.

From this it follows that, if $P$ is a stable point of $X_\infty$, then the maximal ideal of $\widehat{\mathcal{O}_{X_\infty,P}}$ is $P\widehat{\mathcal{O}_{X_\infty,P}}$, and even more,

$$\operatorname{embdim} \widehat{\mathcal{O}_{X_\infty,P}} = \operatorname{embdim} \mathcal{O}_{(X_\infty)_{\mathrm{red}},P} \tag{2.3}$$

(see [1, Cap. III, Sect. 2, No. 12, Cor. 2]).

Stable points behave well under birational proper $k$-morphisms and, if we assume that $\operatorname{char} k = 0$, then also under $k$-morphisms locally of finite type which are locally dominant:

**(x)** ([18, Prop. 4.1]). Let $\pi : Y \to X$ be a birational and proper $k$-morphism; then the morphism $\pi_\infty : Y_\infty \to X_\infty$ induces a one-to-one map between the stable points of $Y_\infty$ and the stable points of $X_\infty$. Besides, if $Q$ is a stable point of $Y_\infty$ and $P$ its image, then the induced morphism $\widehat{\mathcal{O}_{X_\infty,P}} \to \widehat{\mathcal{O}_{Y_\infty,Q}}$ is surjective and induces an isomorphism on the residue fields $\kappa(P) \cong \kappa(Q)$.

**(xi)** ([18, Prop. 4.5]). Suppose that $\operatorname{char} k = 0$. Let $\eta : Y \to X$ be a $k$-morphism of finite type that is locally dominant; then the morphism $\eta_\infty : Y_\infty \to X_\infty$ induces a map from the set of stable points of $Y_\infty$ to the set of stable points of $X_\infty$. Besides, if $Q$ is a stable point of $Y_\infty$ and $P$ its image by the above map, then the induced morphism $(\mathcal{O}_{X_\infty,P})_{\mathrm{red}} \to (\mathcal{O}_{Y_\infty,Q})_{\mathrm{red}}$ is an injective local morphism.
Moreover, if $\eta$ is finite and dominant, then $\widehat{\mathcal{O}_{X_\infty,P}} \to \widehat{\mathcal{O}_{Y_\infty,Q}}$ is unramified at $Q\widehat{\mathcal{O}_{Y_\infty,Q}}$, that is, $P\widehat{\mathcal{O}_{Y_\infty,Q}} = Q\widehat{\mathcal{O}_{Y_\infty,Q}}$, and it induces a finite extension $\kappa(P) \subseteq \kappa(Q)$ on the residue fields.

**(xii)** ([19, Prop. 2.5]). Let $\eta : Y \to X$ be an étale $k$-morphism. Then $Y_\infty$ is étale over $X_\infty$ and, if $Q$ is a stable point of $Y_\infty$ and $P$ its image, then $\widehat{\mathcal{O}_{Y_\infty,Q}} \cong \widehat{\mathcal{O}_{X_\infty,P}} \otimes_{\kappa(P)} \kappa(Q)$.

Suppose that there exists a resolution of singularities $\pi : Y \to X$ of $X$, i.e., a proper, birational $k$-morphism, with $Y$ smooth, such that the induced morphism $Y \setminus \pi^{-1}(\operatorname{Sing} X) \to X \setminus \operatorname{Sing} X$ is an isomorphism. Let $E$ be a divisor on $Y$ and let $Y_\infty^E$ be the inverse image of $E$ by the natural projection $j_0^Y : Y_\infty \to Y$. Then $Y_\infty^E$ is an irreducible subset of $Y_\infty$ whose generic point $P_E^Y$ is a stable point of $Y_\infty$. Besides, the image $P_E^X$ of $P_E^Y$ by the morphism $\pi_\infty : Y_\infty \to X_\infty$ is a stable point of $X_\infty$ (see (x) above). We will denote $P_E^X$ by $P_E$ if there is no possible ambiguity. Note that $P_E$ depends only on the divisorial valuation $\nu_E$ defined by $E$; more precisely, if $\pi' : Y' \to X$ is another resolution of singularities such that the center $E'$ of $\nu_E$ in $Y'$ is a divisor, then the stable point $P_{E'}$ defined by $E'$ coincides with $P_E$. Note also that the order function $\nu_{P_E}$ is equal to the restriction of the divisorial valuation $\nu_E$ to the local ring of $X$ at the generic point of $\pi(E)$.

The set $Y_\infty^E$ is also denoted by $\operatorname{Cont}^{\geq 1}(E)$. More generally,

$$\operatorname{Cont}^e(E) := \{Q' \in Y_\infty / \nu_{Q'}(I_E) = e\} \quad \text{for every } e \geq 1,$$

where $I_E$ is the ideal defining $E$ in an open affine subset of $Y$. We also have that the closure of $\operatorname{Cont}^e(E)$ is an irreducible subset of $Y_\infty$ whose generic point $P_{eE}^Y$ is a stable point of $Y_\infty$, and the image $P_{eE}^X$ (also denoted by $P_{eE}$) of $P_{eE}^Y$ by $\pi_\infty$ is a stable point of $X_\infty$.

**Example 2.1.** Note that there are stable points that are not of the type $P_{eE}$ where $\nu_E$ is a divisorial valuation on $X$. For instance, let $X = \mathbb{A}^1$ and let $P$ be the prime ideal $(X_0, X_2)$ of $\mathcal{O}_{X_\infty} = k[X_0, X_1, \ldots]$. Then, given a polynomial $q(x)$ in $k[x]$ of multiplicity $m$, we have $\nu_P(q(x)) = \mathrm{ord}_t(q(X_1 t + X_3 t^3 + X_4 t^4 + \cdots)) = m$. That is, $\nu_P$ is the multiplicity in $k[x]$, i.e., the divisorial valuation $\nu_E$ defined by $\nu_E(x) = 1$. But $P_E = (X_0)$, hence $P \neq P_E$.

If $\pi : Y \to X$ is a resolution of singularities dominating the Nash blowing up of $X$, then the image of the canonical homomorphism $d\pi : \pi^*(\wedge^d \Omega_X) \to \wedge^d \Omega_Y$ is an invertible sheaf (recall that $d = \dim X$). That is, there exists an effective divisor $\widehat{K}_{Y/X}$ with support in the exceptional locus of $\pi$ such that $d\pi(\pi^*(\wedge^d \Omega_X)) = \mathcal{O}_Y(-\widehat{K}_{Y/X})(\wedge^d \Omega_Y)$. For any prime divisor $E$ on $Y$, we define the *Mather discrepancy* to be

$$\widehat{k}_E := \mathrm{ord}_E(\widehat{K}_{Y/X}).$$

Note that $\widehat{k}_E \neq 0$ implies that $E$ is contained in the exceptional locus of $\pi$, and that $\widehat{k}_E$ depends only on the divisorial valuation $\nu_E$ defined by $E$. We have $\sup_n \dim \mathcal{O}_{\overline{j_n(X_\infty)},(P_{eE})_n} = e(\widehat{k}_E + 1)$ ([2, Lem. 3.1], [8, Thm. 3.9]). Hence the inequality (2.2) states that

$$\dim \mathcal{O}_{X_\infty, P_{eE}} \leq e(\widehat{k}_E + 1).$$

On the other hand, if $X$ is normal and $\mathbb{Q}$-Gorenstein (for instance, $X$ is a normal complete intersection), the *discrepancy* of $X$ with respect to $E$ is defined to be the coefficient of $E$ in the divisor $K_{Y/X}$ with exceptional support that is linearly equivalent to $K_Y - \pi^*(K_X)$. If $X$ is nonsingular then $\widehat{k}_E = k_E$ ([5, Appendix]). Moreover, we have the following property:

**(xiii)** ([18, Prop. 4.2] and [19, Cor. 2.9]). If $X$ is nonsingular at the center $P_0$ of a stable point $P$ of $X_\infty$, then $\mathcal{O}_{X_\infty,P}$ is a regular ring of dimension $\dim \mathcal{O}_{X_\infty,P} = \sup_n \dim \mathcal{O}_{\overline{j_n(X_\infty)},P_n}$. In particular, taking $P = P_{eE}$, we have $\dim \mathcal{O}_{X_\infty,P_{eE}} = e(k_E + 1)$.

In Theorem 3.4, we will prove that, also in the case that $X$ is singular at $P_0$, we have that $e(\widehat{k}_E + 1)$ is the embedding dimension of $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P_{eE}}$.

**Example 2.2.** Let $X$ be an irreducible formal plane curve over a field $k$ of characteristic zero. Let us consider a (primitive) Puiseux parametrization

$$x = u^{\beta_0},$$
$$y = \sum_{\beta_0 \leq i} \lambda_i u^i,$$

where $\lambda_i \in k$ for every $i \geq \beta_0$. Set $e_0 := \beta_0$ and

$$\begin{aligned}\beta_{r+1} &:= \min\{i/\lambda_i \neq 0 \quad \text{and} \quad \gcd\{\beta_0, \dots, \beta_r, i\} < e_r\},\\ e_{r+1} &:= \gcd\{\beta_0, \dots, \beta_{r+1}\},\end{aligned}$$

for $1 \leq r \leq g-1$, where $g$ is such that $e_g = 1$. Let $n_0 = 1$ and $n_r := e_{r-1}/e_r$ for $1 \leq r \leq g$, and let $\overline{\beta}_0 = \beta_0$ and $\overline{\beta}_r$, $1 \leq r \leq g+1$ be defined by

$$\overline{\beta}_r - n_{r-1}\overline{\beta}_{r-1} = \beta_r - \beta_{r-1}; \tag{2.4}$$

hence we have

$$\begin{gathered}\overline{\beta}_r > n_{r-1}\overline{\beta}_{r-1} \quad \text{for } 1 \leq r \leq g, \quad \text{and} \quad \overline{\beta}_{g+1} \geq n_g\overline{\beta}_g;\\ n_r\overline{\beta}_r \text{ belongs to the semigroup generated by } \overline{\beta}_0, \dots, \overline{\beta}_{r-1}, \quad 1 \leq r \leq g+1.\end{gathered}$$

Let us consider $q_0, q_1, \dots, q_g \in k[x,y]$ and $q_{g+1} \in k[[x,y]]$ such that $q_{g+1}$ defines an equation of the branch, i.e., $X = \operatorname{Spec} k[[x,y]]/(q_{g+1})$, and $q_1, \dots, q_g$ are its approximate roots (see [22, Appendix]). More precisely, $q_0, \dots, q_{g+1}$ can be defined as

$$q_0 = x, \qquad q_1 = y - \sum_{i<\overline{\beta}_1} \lambda_i q_0^{i/\overline{\beta}_0},$$

with $\operatorname{ord}_u(q_1) = \overline{\beta}_1$ and, for $1 \leq r \leq g$,

$$q_{r+1} = q_r^{n_r} - c_r q_0^{b_{r,0}} \cdots q_{r-1}^{b_{r,r-1}} - \sum_{\gamma=(\gamma_0,\dots,\gamma_r)} c_\gamma q_0^{\gamma_0} \cdots q_r^{\gamma_r}, \quad 1 \leq r \leq g, \tag{2.5}$$

with $\operatorname{ord}_u(q_{r+1}) = \overline{\beta}_{r+1}$ (resp. $\infty$) for $1 \leq r < g$ (resp. $r = g$), where $\{b_{r,i}\}_{i=0}^{r-1}$ are the unique nonnegative numbers satisfying $b_{r,i} < n_i$ for $1 \leq i \leq r-1$ and $n_r\overline{\beta}_r = \sum_{0 \leq i < r} b_{r,i}\overline{\beta}_i$, for each sequence $\gamma$ of nonnegative integers in the right-hand side we have $n_r\overline{\beta}_r < \sum_{i=0}^r \gamma_i\overline{\beta}_i < \overline{\beta}_{r+1}$ (resp. $n_r\overline{\beta}_r < \sum_{i=0}^r \gamma_i\overline{\beta}_i$) if $1 \leq r < g$ (resp. if $r = g+1$) and $c_r$, $c_\gamma \in k$ and $c_r \neq 0$. For more details on approximate roots and the space of arcs of a plane branch see [15] and [14].

Let $\nu = \nu_E$ be the divisorial valuation on $X$ given by $\operatorname{ord}_u$, and let $P = P_E$ be the stable point in $X_\infty$ defined by $\nu$. Considering the projection $\eta : X \to \mathbb{A}^1_k$, $(x,y) \mapsto x$, and applying [18, Prop. 4.5] ((xi) above), we conclude that

$$P\widehat{\mathcal{O}_{X_\infty,P}} = (X_0, \dots, X_{\beta_0-1})\widehat{\mathcal{O}_{X_\infty,P}}.$$

We will next describe the ring $\widehat{\mathcal{O}_{X_\infty,P}}$, and we will see that embdim $\widehat{\mathcal{O}_{X_\infty,P}} = \beta_0$, which is equal to the multiplicity of $X$ (see [18, Cor. 5.7]).

First note that $P\mathcal{O}_{X_\infty,P}$ is generated by $\mathcal{Q} := \{Q_{r;n}\}_{0 \leq r \leq g, n_{r-1}\overline{\beta}_{r-1} \leq n < \overline{\beta}_r}$; moreover, there exists $G \in \mathcal{O}_{X_\infty} \setminus P$ such that $P(\mathcal{O}_{X_\infty})_G = (\mathcal{Q})(\mathcal{O}_{X_\infty})_G$ (we may take $G := \prod_{0\le r\le g} Q_{r;\overline{\beta}_r}$). More precisely, $(\mathcal{Q})$ defines a prime ideal in $(\mathcal{O}_{(\mathbb{A}^2)_\infty})_G$ (see [19, Prop. 4.5]) whose extension to $(\mathcal{O}_{X_\infty})_G$ is $P(\mathcal{O}_{X_\infty})_G$. Note that, setting $f := q_{g+1} \in k[[x,y]]$, the following hold:

(i) We have $\nu(\mathrm{Jac}(f)) = \nu(\frac{\partial f}{\partial y}) = n_g\overline{\beta}_g - \beta_g$. Set $\epsilon := n_g\overline{\beta}_g - \beta_g$.

(ii) For all $n \ge 0$, the class of $\frac{\partial F_{\epsilon+n}}{\partial Y_n}$ in $\mathcal{O}_{X_\infty,P}$ is a unit and, for $n' > n$, the class of $\frac{\partial F_{\epsilon+n}}{\partial Y_{n'}}$ in $\mathcal{O}_{X_\infty,P}$ belongs to $P\mathcal{O}_{X_\infty,P}$.

(iii) $F_0, \ldots, F_{\epsilon-1}$ belong to $(\mathcal{Q})^2\mathcal{O}_{(\mathbb{A}^2_k)_\infty}$.

From this it follows that

$$\kappa(P) \cong k(X_{\beta_0+1}, \ldots, X_n, \ldots)\left[\{W_r\}_{r=0}^g\right] \Big/ \left(\left\{W_r^{n_r} - c_r W_0^{b_{r,0}} \cdots W_{r-1}^{b_{r,r-1}}\right\}_{r=1}^g\right),$$

where $W_r$ is the class of $Q_{r;\overline{\beta}_r}$. We consider the embedding $\kappa(P) \hookrightarrow \widehat{\mathcal{O}_{X_\infty,P}}$, which sends $X_n$, $n \ge \beta_0$ (resp. $W_0$) to $X_n \in \widehat{\mathcal{O}_{X_\infty,P}}$ (resp. $X_{\beta_0} \in \widehat{\mathcal{O}_{X_\infty,P}}$) and recursively, for $1 \le r \le g$, sends $W_r$ to a $n_r$th root of the image in $\widehat{\mathcal{O}_{X_\infty,P}}$ of $c_r W_0^{b_{r,0}} \cdots W_{r-1}^{b_{r,r-1}}$, that exists by Hensel's lemma. In particular, for each $n \ge 0$ we have defined $Y_n^{(0)} \in \kappa(P)$ such that $Y_n - Y_n^{(0)} \in (\mathcal{Q})$. Arguing recursively on $m \ge 1$ and $n \ge 0$, with the lexicographical order on $(m,n)$, from $\{F_{\epsilon+n}\}_{n\ge 0}$, applying (ii) and Hensel's lemma, and reasoning as in [18, Cor. 5.6], it follows that, for $m,n \ge 0$, there exists $Y_n^{(m)} \in \kappa(P)[X_0, \ldots, X_{\beta_0-1}]$ such that

$$F_{\epsilon+n} \equiv L_\epsilon(Y_n - Y_n^{(m)}) \mod (\mathcal{Q})^{m+1}$$

in the ring $\mathcal{O}_{(\mathbb{A}^2)_\infty,(\mathcal{Q})}$, where $l := \frac{\partial f}{\partial y}$; hence $L_\epsilon$ is a unit. Therefore, the above equalities define series $\widetilde{Y}_n \in \kappa(P)[[X_0, \ldots, X_{\beta_0-1}]]$, $n \ge 0$, and we conclude that

$$\widehat{\mathcal{O}_{X_\infty,P}} \cong \kappa(P)\left[[X_0, \ldots, X_{\beta_0-1}]\right] \Big/ \left(\{\widetilde{F}_n\}_{0\le n\le \epsilon-1}\right),$$

where, for $0 \le n \le \epsilon - 1$, $\widetilde{F}_n$ is obtained from $F_n$ by substituting $Y_{n'}$ by $\widetilde{Y}_{n'}$, $0 \le n' \le n$. Since, for $0 \le r \le g, n_{r-1}\overline{\beta}_{r-1} \le n < \overline{\beta}_r$, the series obtained from $Q_n$ by substituting $Y_{n'}$ by $\widetilde{Y}_{n'}$, $0 \le n' \le n$ belongs to $(X_0, \ldots, X_{\beta_0-1})$, then from (iii) it follows that $\widetilde{F}_n \in (X_0, \ldots, X_{\beta_0-1})^2$ for $0 \le n \le \epsilon - 1$. Therefore $\mathrm{embdim}\, \widehat{\mathcal{O}_{X_\infty,P}} = \beta_0$.

**Remark 2.3.** Let $X$ be an algebraic plane curve over a field $k$ of characteristic zero, and suppose that it is analytically irreducible. Then, there exists an étale morphism $X' \to X$ such that the curve $X'$ has a Puiseux parametrization

$$x' = (u')^{\beta_0},$$
$$y' = \sum_{\beta_0 \le i \le m} \lambda'_i (u')^i, \tag{2.6}$$

where $\lambda'_i \in k$ for $\beta_0 \leq i \leq m$, i.e., the image of $y'$ has a finite number of terms. Equivalently, the element $q'_{g+1}$ obtained as in (2.5) from the above parametrization, which defines an equation of the curve $X'$, is a polynomial.

Indeed, consider a Puiseux parametrization $x = u^{\beta_0}$, $y = \sum_{\beta_0 \leq i} \lambda_i u^i$ of $X$ and keep the notation in Example 2.2. Note that the series $\sum_{\beta_0 \leq i} \lambda_i u^i$ belongs to the henselization $k\langle u\rangle$ of $k[u]_{(u)}$ and also that the element $q_{g+1}$ in (2.5) belongs to $k\langle x, y\rangle$. Since $X$ is analytically irreducible, there exists a unit $\gamma \in k\langle x, y\rangle$ such that $\gamma q_{g+1}$ is a polynomial in $k[x, y]$. Then taking $x' = (\gamma)^{1/\overline{\beta}_1} x$, $y' = (\gamma)^{1/\overline{\beta}_0} y$ and $u' = (\gamma)^{1/\overline{\beta}_0\overline{\beta}_1} u$, we obtain (2.6). Recall that $n_1\overline{\beta}_1$ is the least common multiple of $\overline{\beta}_0$ and $\overline{\beta}_1$. Since $\operatorname{char} k = 0$, adding a $n_1\overline{\beta}_1$th root of $\gamma$ defines an étale morphism $X' \to X$.

**Example 2.4.** Let $X \subset \mathbb{A}^5_k$ be the hypersurface singularity in [11], defined by $x_1^3+x_2^3+x_3^3+x_4^3+x_5^6=0$ over a field of characteristic $\neq 2, 3$. The blowing up $X'$ of $X$ at the origin has a unique singular point, and its exceptional locus $E_\beta$ is irreducible and defines an *essential* valuation $\nu_\beta$ (i.e., the center of $\nu_\beta$ on any resolution of singularities $p : \widetilde{X} \to X$ is an irreducible component of the exceptional locus of $p$). The blowing up $Y$ of $X'$ at its singular point is nonsingular, and its exceptional locus is irreducible and defines an essential valuation $\nu_\alpha$, $\nu_\alpha \neq \nu_\beta$. Let $\pi : Y \to X$ be the induced resolution of singularities. Let $P_\alpha$, $P_\beta$ be the stable points of $X_\infty$ defined by $\nu_\alpha$ and $\nu_\beta$ respectively, and set $N_\alpha := \overline{\{P_\alpha\}}$, $N_\beta := \overline{\{P_\beta\}}$ and $X_\infty^{\mathrm{Sing}}$ the inverse image of $\operatorname{Sing} X$ by $j_0 : X_\infty \to X$. We have $N_\alpha \subset N_\beta = X_\infty^{\mathrm{Sing}}$ ([11, Thm. 4.3]).

Let $\Pi : \widetilde{\mathcal{Z}} \to \mathbb{A}^5_k$ be the embedded resolution of singularities of $X$ whose restriction to $X$ is $\pi$. There exists a divisor $\widetilde{E}$ on $\widetilde{\mathcal{Z}}$ whose intersection with $Y$ is $E_\beta$. Note that $b_{\widetilde{E}} := \operatorname{ord}_{\widetilde{E}} K_{\widetilde{\mathcal{Z}}/\mathbb{A}^5}$ is equal to 4 and $a_{\widetilde{E}} := \operatorname{ord}_{\widetilde{E}} \Pi^*(X)$ is equal to 3. Since, by the adjunction formula, $k_{E_\beta} = b_{\widetilde{E}} - a_{\widetilde{E}}$, we have $k_{E_\beta} = 1$. Hence, $\widehat{k}_{E_\beta} = k_{E_\beta} + \nu_\beta(\mathrm{Jac}_X) = 1 + 2 = 3$ (see [5, Rem. 9.6]).

On the other hand, we have

$$P_\beta\,(\mathcal{O}_{X_\infty})_{X_{1;1}} = (X_{1;0}, X_{2;0}, X_{3;0}, X_{4;0}, X_{5;0})\,(\mathcal{O}_{X_\infty})_{X_{1;1}}\,.$$

In fact, $(X_{1;0}, \ldots, X_{5;0})$ is the prime ideal in $\mathcal{O}_{(\mathbb{A}^5)_\infty}$ defined by $\nu_{\widetilde{E}}$, hence its minimal number of generators is $b_{\widetilde{E}} + 1 = 5$ (see (xiii)). In addition, the ring $\widehat{\mathcal{O}}_{X_\infty, P_\beta}$ has been described in [18, Rem. 5.16] as

$$\widehat{\mathcal{O}}_{X_\infty, P_\beta} \cong \kappa(P_\beta)[[X_{1;0}, X_{2;0}, X_{3;0}, X_{4;0}, X_{5;0}]]/(\widetilde{F}_0, \widetilde{F}_1, \widetilde{F}_2),$$

where, letting $f = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^6$ and letting $\overline{F}_n$ be the class of $F_n$ modulo $(X_{1;0}, \ldots, X_{5;0})$, we have that $3 = a_{\widetilde{E}}$ is the minimal $n$ such that $\overline{F}_n \neq 0$; in fact $\overline{F}_3 = X_{1;1}^3 + X_{1;2}^3 + X_{1;3}^3 + X_{1;4}^3$ and

$$\kappa(P_\beta) \cong k(\{X_{i;1}, \ldots, X_{i;n}, \ldots\}_{2 \leq i \leq 4})[X_{1;1}]/(\overline{F}_3).$$

In addition, we have $\widetilde{F}_0, \widetilde{F}_1 \in (X_{1;0}, \dots, X_{5;0})^2$ and the initial form $\mathrm{in}(\widetilde{F}_2)$ of $\widetilde{F}_2$ in $\kappa(P_\beta)[[X_{1;0}, \dots, X_{5;0}]]$ is $3\overline{X}_{1;1}^2 X_{1;0} + 3X_{2;1}^2 X_{2;0} + 3X_{3;1}^2 X_{3;0} + 3X_{4;1}^2 X_{4;0}$, where $\overline{X}_{1;1}$ is the class of $X_{1;1}$ in $\kappa(P_\beta)$. Note that $\nu_\beta(\mathrm{Jac}_X) = 2$; furthermore, for $1 \le i \le 4$, if $f^i := \frac{\partial f}{\partial x_i}$ then $\nu_\beta(f^i) = 2$, i.e., $F_0^i$, $F_1^i \in P_\beta$, $F_2^i \notin P_\beta$, and the coefficient in $X_{i;0}$ of $\mathrm{in}(\widetilde{F}_2)$ is the class of $F_2^i$ in $\kappa(P_\beta)$. From this it follows that

$$\begin{aligned}\operatorname{embdim} \widehat{\mathcal{O}_{X_\infty, P_\beta}} &= b_{\widetilde{E}} + 1 - (a_{\widetilde{E}} - \nu_\beta(\mathrm{Jac}_X)) \\ &= k_{E_\beta} + 1 + \nu_\beta(\mathrm{Jac}_X) = \widehat{k}_{E_\beta} + 1,\end{aligned}$$

which equals 4. Moreover, in this case,

$$\dim \widehat{\mathcal{O}_{X_\infty, P_\beta}} = b_{\widetilde{E}} + 1 - a_{\widetilde{E}} = k_{E_\beta} + 1 = 2.$$

The argument to compute $\operatorname{embdim} \widehat{\mathcal{O}_{X_\infty, P_\beta}}$ given in Example 2.4 can be generalized to monomial valuations restricted to a normal hypersurface over a perfect field of any characteristic. But, although, given a variety $X$ and a divisorial valuation $\nu_E$, there always exists a complete intersection $X'$ containing $X$ of the same dimension and we have $\widehat{\mathcal{O}_{X_\infty, P_E}} \cong \widehat{\mathcal{O}_{X'_\infty, P_E}}$ (see (ii) and (ix)), $X'$ is not normal in general. So, there is no hope of extending the result $\operatorname{embdim} \widehat{\mathcal{O}_{X_\infty, P_E}} = \widehat{k}_E + 1$ by applying this argument. For $\dim \widehat{\mathcal{O}_{X_\infty, P_E}}$, even if $X$ is a normal hypersurface, it is not true in general that $\dim \widehat{\mathcal{O}_{X_\infty, P_E}}$ equals $k_E + 1$, but we will show that $\dim \widehat{\mathcal{O}_{X_\infty, P_E}} \ge k_E + 1$.

## §3. Defining minimal coordinates at stable points of the space of arcs

Let $X$ be a (singular) reduced separated scheme of finite type over a field $k$ of characteristic zero. Let $\nu$ be a divisorial valuation on an irreducible component $X_0$ of $X$ whose center lies in $\operatorname{Sing} X$ and let $e \in \mathbb{N}$.

Let us consider the stable point $P_{eE}$ of $X_\infty$ defined by $\nu$ and $e$, i.e., we consider any resolution of singularities $\pi : Y \to X$ such that the center of $\nu$ on $Y$ is a divisor $E$, and define $P_{eE} = P_{eE}^X$ to be the image by $\pi_\infty$ of the generic point $P_{eE}^Y$ of the closure of $\mathrm{Cont}^e(E)$ (see Section 2). In order to study the ring $\mathcal{O}_{X_\infty, P_{eE}}$, or its completion $\widehat{\mathcal{O}_{X_\infty, P_{eE}}}$, we may suppose that $X$ is affine; let $X \subseteq \mathbb{A}_k^N = \operatorname{Spec} k[y_1, \dots, y_N]$. We may also suppose that $\pi : Y \to X$ dominates the Nash blowing up of $X$ and that, if $x_i$ denotes the class of $y_i$ in $\mathcal{O}_X$, $1 \le i \le N$, then, after reordering the $x_i$'s, we have

$$\mathrm{ord}_E \pi^*(dx_1 \wedge \dots \wedge dx_d) = \widehat{k}_E, \tag{3.1}$$

where $d = \dim X_0$.

Let $\rho : X \to \mathbb{A}^d_k$ be the projection on the first $d$ coordinates, let $\eta : Y \to \mathbb{A}^d$ be the composition $\eta = \rho \circ \pi$ and let $P^{\mathbb{A}^d}_{eE}$ be the image of $P^Y_{eE}$ by $\eta_\infty$. Then the discrepancy $k_E(\mathbb{A}^d_k)$ of $\mathbb{A}^d_k$ with respect to the valuation induced by $\nu_E$ is equal to $\widehat{k}_E$ by (3.1). Besides, we know that the local ring $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$ is a regular ring of dimension $e(k_E(\mathbb{A}^d_k)+1)$ (see (xiii) in Section 2). From this, and applying [18, Prop. 4.5] (see (xi) in Section 2), it follows that, if $\mathcal{Q}$ is a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$ (hence $\sharp\mathcal{Q} = e(\widehat{k}_E + 1)$) then we have

$$
\begin{aligned}
&P_{eE}\widehat{\mathcal{O}_{X_\infty, P_{eE}}} = (\mathcal{Q})\widehat{\mathcal{O}_{X_\infty, P_{eE}}} \\
\text{and}\quad &P_{eE}\mathcal{O}_{(X_\infty)_{\mathrm{red}}, P_{eE}} = (\mathcal{Q})\mathcal{O}_{(X_\infty)_{\mathrm{red}}, P_{eE}};
\end{aligned}
\tag{3.2}
$$

in fact, the last assertion follows from the first one by Nakayama's lemma. Therefore, $\operatorname{embdim} \mathcal{O}_{(X_\infty)_{\mathrm{red}}, P_{eE}} = \operatorname{embdim} \widehat{\mathcal{O}_{X_\infty, P_{eE}}} \le e(\widehat{k}_E + 1)$ ([19, Cor. 4.10]).

**Remark 3.1.** The above reasoning does not ensure an analogous statement to (3.2) for $P^X_{eE}\mathcal{O}_{X_\infty, P^X_{eE}}$ since, in general, the $P^X_{eE}$-adic topology on $\mathcal{O}_{X_\infty, P^X_{eE}}$ is not separated (see [18, Exa. 3.16 and Thm. 3.13]).

**The regular case.** Moreover, in [19] we described a regular system of parameters $\mathcal{Q}$ of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$. We will next recall how we proceeded.

**(I)** First, since char $k = 0$, there exists an open subset $U$ of $Y$ with nonempty intersection with $E$, an étale morphism $\widetilde{U} \to U$ and $\{u_1, \dots, u_d\} \subset \mathcal{O}_{\widetilde{U}}$, $\{x_1, \dots, x_d\} \subset \mathcal{O}_V$, where $V$ is an open subset of $X$, such that the following holds: for all closed points $y_0$ in an open subset of the strict transform $\widetilde{E}$ of $E$ in $\widetilde{U}$, after a possible replacement of $u_i$ by $u_i + c_i$, $c_i \in k$, $2 \le i \le d$, we may suppose that $\{u_1, \dots, u_d\}$ and $\{x_1, \dots, x_d\}$ are regular systems of parameters in $y_0$ and in $\eta \circ \varphi(y_0)$. In addition, the local morphism $\eta^\sharp : \mathcal{O}_{V, \eta(y_0)} \to \mathcal{O}_{\widetilde{U}, y_0}$ is given by

$$
\begin{aligned}
x_1 &\mapsto u_1^{m_1}, \\
x_2 &\mapsto \sum_{1 \le i \le m_2} \lambda_{2,i} u_1^i + u_1^{m_2} u_2, \\
x_3 &\mapsto \sum_{1 \le i \le m_3} \lambda_{3,i}(u_2) u_1^i + u_1^{m_3} u_3, \\
\cdots &\qquad \cdots \\
x_\delta &\mapsto \sum_{1 \le i \le m_\delta} \lambda_{\delta,i}(u_2, \dots, u_{\delta-1}) u_1^i + u_1^{m_\delta} u_\delta, \\
x_{\delta+1} &\mapsto u_{\delta+1}, \\
\cdots &\qquad \cdots \\
x_d &\mapsto u_d,
\end{aligned}
\tag{3.3}
$$

where $\delta = \operatorname{codim}_{\mathbb{A}^d}\overline{\eta(\xi_E)}$, $m_1 \leq \operatorname{ord}_{u_1} x_j$, $2 \leq j \leq d$, $0 < m_1 \leq m_2 \leq \cdots \leq m_d$, and, for $2 \leq j \leq \delta$ and $0 \leq i \leq m_j$, $\lambda_{j,i}(u_2, \ldots, u_{j-1})$ belongs to the *henselization* $k\langle u_2, \ldots, u_{j-1}\rangle$ of the local ring $k[u_2, \ldots, u_{j-1}]_{(u_2,\ldots,u_{j-1})}$, and, if $i < m_{j'}$, $j' < j$, then $\lambda_{j,i}$ belongs to $k\langle u_2, \ldots, u_{j'-1}\rangle$. Moreover, with no loss of generality, we may also suppose that $\lambda_{j,m_j}(u_2, \ldots, u_{j-1})$ is a unit for $2 \leq j \leq \delta$ ([19, (4)]; see also [18, Proof of Prop. 4.5]).

Recall that $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ is a regular local ring of dimension $e(k_E(\mathbb{A}^d_k)+1)$ (see (xiii)). Note that $e(k_E(\mathbb{A}^d_k)+1) = e\sum_{j=1}^{\delta} m_j$. Thus, if in (3.3) we have $\lambda_{j,i} = 0$ for $2 \leq j \leq \delta$, $1 \leq i \leq m_j$, then the set $\{X_{j;n}\}_{1\leq j\leq \delta, 0\leq n<m_j}$ generates $P_{eE}^{\mathbb{A}^d}$ and has cardinal $e(k_E(\mathbb{A}^d_k)+1)$; hence it is a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$. On the other hand, if $\delta = 2$, we may consider a generating sequence $\{q_i\}_{i=0}^{g+1}$ for the valuation $\nu$ and $\overline{\beta}_i = \nu(q_i)$, $0 \leq i \leq g+1$, which define the minimal generating sequence for the semigroup of $\nu$ (see [20]). Then $\mathcal{Q} = \{Q_{i;n}\}_{0\leq i\leq g+1, en_{i-1}\overline{\beta}_{i-1}\leq n<e\overline{\beta}_i}$ is a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$. Here the $n_i$'s are defined as in Example 2.2; hence the cardinal of $\mathcal{Q}$ is $e(m_1+m_2)$ by (2.4). Next we will use these techniques of the theory of plane curves to determine a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ in a similar way as before. Indeed, we have that the local ring at the generic point of $E$ has dimension one. We will consider plane projections of the curve it defines.

**(II)** Now we consider the following situation: let $j$, $2 \leq j \leq d+1$, let $v_2, \ldots, v_{j-1}$ be such that $u_1, v_2, \ldots, v_{j-1}, u_j, \ldots, u_d \in \mathcal{O}_{\widetilde{U}}$ define a regular system of parameters of $\mathcal{O}_{\widetilde{U},y_0}$ for all closed points $y_0$ in an open subset of $\widetilde{E}$ (more precisely, there exist $(c_i)_i \in k^{d-1}$ such that $(u_1, \{v_i + c_i\}_{i=2}^{\delta}, \{v_i + c_i\}_{i=\delta+1}^{d})$ is a regular system of parameters of $\mathcal{O}_{\widetilde{U},y_0}$). Let $\theta : \widetilde{U} \to \operatorname{Spec} k[v_2, \ldots, v_{j-1}]_h[x_1, y]$ be the $k$-morphism given by

$$\begin{aligned} x_1 &\mapsto u_1^{m_1}, \\ y &\mapsto \sum_{m_1\leq i\leq m} \lambda_i(v_2, \ldots, v_{j-1})u_1^i + u_1^m \varrho \mod (u_1)^{m+1}, \end{aligned}$$

where $h \in k[v_2, \ldots, v_{j-1}] \setminus (v_2, \ldots, v_{j-1})$, $m \geq m_1$, $\lambda_i(v_2, \ldots, v_{j-1}) \in R_{j-1} := k\langle v_2, \ldots, v_{j-1}\rangle$, $\varrho \in \mathcal{O}_{Y,y_0}$ and one of the following conditions holds:

(a) $\varrho$ is transcendental over $k(u_1, v_2, \ldots, v_{j-1})$,

(b) $\varrho = 0$.

Set $\mathfrak{e} := \gcd(\{m_1\} \cup \{i/\lambda_i \neq 0\})$ and define $\beta_0 := e_0 := m_1$, and $\beta_{r+1} := \min\{i/\lambda_i \neq 0 \text{ and } \gcd\{\beta_0, \ldots, \beta_r, i\} < e_r\}$, $e_{r+1} := \gcd\{\beta_0, \ldots, \beta_{r+1}\}$ for $1 \leq r <$

$g$, where $g$ is such that $e_g = \mathfrak{e}$, and $\beta_{g+1} := m$. Let $n_r = e_{r-1}/e_r$, $1 \le r \le g-1$. We define $\{\overline{\beta}_r\}_{r=0}^{g+1}$ from $\{\beta_r\}_{r=0}^{g+1}$ as in (2.4).

Next we will proceed as in Example 2.2. In case (b) (resp. case (a)) we deal with the formal plane curve (resp. the divisorial valuation) defined over the integral closure of $k(v_2, \ldots, v_{j-1})$ by the Puiseux expansion above. In fact, the polynomials we obtain (see (2.5)) belong to a suitable étale extension of $k[v_2, \ldots, v_{j-1}]_h$. More precisely, let $B$ be a domain that is an étale extension of $k[v_2, \ldots, v_{j-1}]_h$ and contains $\lambda_i(v_2, \ldots, v_{j-1})$, $m_1 \le i \le m$. Let $\widetilde{\nu}$ be the order function on $B[x_1, y]$ extending $\nu$ and such that $\widetilde{\nu}(\ell) = 0$ for all $\ell \in B$ (note that $\widetilde{\nu}$ is a valuation if there is no nonzero element $h$ with $\widetilde{\nu}(h) = \infty$, for instance in case (a)). As in Example 2.2, we define $\widetilde{q}_0, \ldots, \widetilde{q}_g \in B[x_1, y]$ such that $\widetilde{\nu}(\widetilde{q}_r) = \overline{\beta}_r$ for $0 \le r \le g+1$ as follows: let $\{b_{r,i}\}_{i=0}^{r-1}$ be the unique nonnegative integers satisfying $b_{r,i} < n_i$, $1 \le i \le r-1$, and $n_r\overline{\beta}_r = \sum_{0\le i<r} b_{r,i}\overline{\beta}_i$. Let $\widetilde{q}_0 = x_1, \widetilde{q}_1 = y - \sum_{i<\overline{\beta}_1} \lambda'_i(\widetilde{q}_0)^{i/\overline{\beta}_0}$ and, for $1 \le r \le g$,

$$\widetilde{q}_{r+1} = \widetilde{q}_r^{\,n_r} - \widetilde{c}_r \widetilde{q}_0^{\,b_{r,0}} \cdots \widetilde{q}_{r-1}^{\,b_{r,r-1}} - \sum_{\gamma=(\gamma_0,\ldots,\gamma_r)} \widetilde{c}_\gamma \widetilde{q}_0^{\,\gamma_0} \cdots \widetilde{q}_r^{\,\gamma_r}, \quad 1 \le r < g, \tag{3.4}$$

where $\widetilde{\nu}(\widetilde{q}_0^{\,\gamma_0} \cdots \widetilde{q}_r^{\,\gamma_r}) > n_r\overline{\beta}_r$ for each sequence $\gamma$ of nonnegative integers on the right-hand side, and $\widetilde{c}_r$, $\widetilde{c}_\gamma \in B$, $\widetilde{c}_r \ne 0$ and $\widetilde{c}_\gamma \ne 0$ only for a finite number of $\gamma$'s. In case (a), we also define $\widetilde{q}_{g+1}$ as in (3.4); then we have that $\{\overline{\beta}_r\}_{r=0}^{g+1}$ is the minimal generating sequence for the semigroup $\widetilde{\nu}(B[x_1, y] \setminus \{0\})$ and $\widetilde{q}_0, \ldots, \widetilde{q}_{g+1} \in B[x_1, y]$ is a minimal generating sequence for $\widetilde{\nu}$ ([20, Thm. 8.6]). In case (b), $\widetilde{q}_{g+1} \in B[x_1, y]$, also defined as in (3.4), defines the kernel of $B[x_1, y] \to \mathcal{O}_{\widetilde{U}}$.

In case (a), by induction on $r$, $1 \le r \le g+1$, we will define elements $\{q'_r\}_{r=1}^{g+1}$ in $k(v_2, \ldots, v_{j-1}, x_1, y)$; more precisely,

$$q'_r \in \prod_{r'=0}^{r-1} T_{r'}^{-1} k[v_2, \ldots, v_{j-1}, x_1, y],$$

where $T_{r'}$ is the multiplicative system generated by $q'_{r'}$, satisfying the following: $q'_0 := x_1$ and, for $1 \le r \le g+1$, the image of $q'_r$ in the fraction field $K(\mathcal{O}_{Y,y_0})$ of $\mathcal{O}_{Y,y_0}$ belongs to $\mathcal{O}_{Y,y_0}$ and, if we identify $q'_r$ with its image, then

$$\begin{aligned} q'_r &= \mu_r(v_2, \ldots, v_{j-1}) u^{\overline{\beta}_r} \mod (u)^{\overline{\beta}_r+1} \quad \text{for } 1 \le r \le g, \\ q'_{g+1} &= \mu_{g+1}(v_2, \ldots, v_{j-1}) u^{\overline{\beta}_{g+1}} \varrho \mod (u)^{\overline{\beta}_{g+1}+1}, \end{aligned} \tag{3.5}$$

where $\mu_r(v_2, \ldots, v_{j-1})$ is a unit in $R_{j-1}$. In fact, once $q'_0, \ldots, q'_r$ are defined, the element $q'_{r+1}$ is defined as follows: let

$$h_{r,1} := {q'_0}^{b_{r,0}} \cdots {q'_{r-1}}^{b_{r,r-1}} P_{r,1}\left( \frac{\overline{\mu}_{r,1}(q'_r)^{n_r}}{{q'_0}^{b_{r,0}} \cdots {q'_{r-1}}^{b_{r,r-1}}}, v_2, \ldots, v_{j-1} \right),$$

where the integers $\{b_{r,r'}\}_{r'=0}^{r-1}$ are as in (3.4), $\overline{\mu}_{r,1} := \mu_1^{b_{r,1}} \cdots \mu_{r-1}^{b_{r,r-1}}$ is a unit and $P_{r,1} \in k[z, v_2, \ldots, v_{j-1}]$ is such that

$$(3.6)\quad P_{r,1}(\mu_r^{n_r}, v_2, \ldots, v_{j-1}) = 0, \qquad \frac{\partial P_{r,1}}{\partial z}(\mu_r^{n_r}, v_2, \ldots, v_{j-1}) \text{ is a unit in } R_{j-1}.$$

Then we have $n_r\overline{\beta}_r < \nu(h_1) \leq \overline{\beta}_{r+1}$. If $\nu(h_1) = \overline{\beta}_{r+1}$, we set $q'_{r+1} := h_1$. If not, we define recursively

$$h_{r,s} := {q'}_0^{b_0^s} \cdots {q'}_{r-1}^{b_{r-}^s} P_{r,s}\left(\frac{\overline{\mu}_{r,s} h_{r,s-1}}{q_0^{b_0^s} \cdots q_{r-1}^{b_{r-1}^s}}, v_2, \ldots, v_{j-1}\right),$$

where $\{b_{r'}^s\}_{r'=0}^{r-1}$ are the unique nonnegative integers satisfying $b_{r'}^s < n_{r'}$, $1 \leq r' \leq r-1$, and $\nu(h_{r,s-1}) = \sum_{0 \leq r' \leq r-1} b_{j,r'}^s \overline{\beta}_{j,r'}$, $\overline{\mu}_{r,s} := \mu_1^{b_1^s} \cdots \mu_{r-1}^{b_{r-1}^s}$ is a unit, and $P_{r,s} \in k[z, v_2, \ldots, v_{j-1}]$ is such that

$$(3.7)\quad P_{r,s}(\lambda_{s-1}, v_2, \ldots, v_{j-1}) = 0, \quad \frac{\partial P_{r,s}}{\partial z}(\lambda_{s-1}, v_2, \ldots, v_{j-1}) \text{ is a unit in } R_{j-1},$$

where $\lambda_{s-1} \in R_{j-1}$ is the initial form of $h_{r,s-1}$. We have $\nu(h_{r,s-1}) < \nu(h_{r,s}) \leq \overline{\beta}_{r+1}$; hence, after a finite number of steps we obtain $s$ such that $\nu(h_{r,s}) = \overline{\beta}_{r+1}$ and we set $q_{r+1} := h_{r,s}$ (for more details see [19, Lem. 3.1]).

The elements $q'_r$ and $\widetilde{q}_r$ are related. In fact, for $0 \leq r \leq g+1$, $q'_r$ and $\widetilde{q}_r$ define the same initial form in an étale covering of a localization of the graded algebra $\mathrm{gr}_\nu k[v_2, \ldots, v_{j-1}, x_1, y]_{(x_1,y)}$. More precisely, there exist $\widetilde{\ell}, \widetilde{h} \in \prod_{0 \leq r' < r} T_{r'}^{-1} B[x_1, y]$, $\widetilde{\ell}$ a unit and $\widetilde{\nu}(\widetilde{h}) > \overline{\beta}_r$, such that $q'_r = \widetilde{q}_r \cdot \widetilde{\ell} + \widetilde{h}$.

**(III)** Recall the expression in (3.3). We fix $j$, $2 \leq j \leq \delta$ and apply the previous study to

$$\begin{aligned} x_1 &\mapsto u_1^{m_1}, \\ x_j &\mapsto \sum_{1 \leq i \leq m_j} \lambda_{j,i}(u_2, \ldots, u_{j-1}) u_1^i + u_1^{m_j} u_j. \end{aligned}$$

Let $B_{j-1}$ be a domain that is an étale extension of $k[u_2, \ldots, u_{j-1}]$ and contains $\lambda_{j,i}(u_2, \ldots, u_{j-1})$, $m_1 \leq i \leq m_j$. Let $\widetilde{\nu}_j$ be the valuation on $B_{j-1}[x_1, x_j]$ extending $\nu$ and let $\{\overline{\beta}_{j,r}\}_{r=0}^{g_j+1}$ be the minimal generating sequence for the semigroup $\widetilde{\nu}_j(B_{j-1}[x_1, x_j] \setminus \{0\})$. Let $\{\widetilde{q}_{j,r}\}_{r=0}^{g_j+1} \in B_{j-1}[x_1, x_j]$ be a minimal generating sequence for $\widetilde{\nu}_j$, and define $\{q'_{j,r}\}_{r=0}^{g_j+1} \in k(u_2, \ldots, u_{j-1}, x_1, x_j)$ as in (II).

Consider the following sets with the lexicographical order

$$\mathcal{J}^* := \{(1,0)\} \cup \{(j,r)/2 \leq j \leq \delta, 1 \leq r \leq g_j\}, \quad \mathcal{J} := \mathcal{J}^* \cup \{(j, g_j+1)/2 \leq j \leq \delta\}.$$

Applying the argument in (II) and arguing by induction on $(j,r) \in \mathcal{J}$, we can define elements $\{q_{j,r}\}_{(j,r)\in\mathcal{J}}$,

$$q_{j,r} \in \prod_{\substack{(j',r')\in\mathcal{J}^* \\ (j',r')<(j,r)}} T^{-1}_{j',r'} k[x_1,\dots,x_j], \tag{3.8}$$

where $T_{j',r'}$ is the multiplicative system generated by $q_{j',r'}$, satisfying the following: $q_{1,0} := x_1$ and, for $(j,r) \in \mathcal{J}$, the image of $q_{j,r}$ in the fraction field $K(\mathcal{O}_{Y,y_0})$ of $\mathcal{O}_{Y,y_0}$ belongs to $\mathcal{O}_{Y,y_0}$ and, if we identify $q_{j,r}$ with its image, then

$$\begin{aligned} q_{j,r} &= \mu_{j,r}(u_2,\dots,u_{j-1})u^{\overline{\beta}_{j,r}} \mod (u)^{\overline{\beta}_{j,r}+1} \quad \text{for } 1 \le r \le g_j, \\ q_{j,g_j+1} &= \mu_{j,g_j+1}(u_2,\dots,u_{j-1})u^{\overline{\beta}_{j,g_j+1}}u_j \mod (u)^{\overline{\beta}_{j,g_j+1}+1}, \end{aligned} \tag{3.9}$$

where $\mu_{j,r}(u_2,\dots,u_{j-1})$ is a unit in $k\langle u_2,\dots,u_{j-1}\rangle$. In addition, if $b_{j,0},\dots,b_{j,g_j}$ are the unique nonnegative integers satisfying $b_{j,r} < n_{j,r}$, $1 \le r \le g_j$, and $\overline{\beta}_{j,g_j+1} = \sum_{0\le i\le g_j} b_{j,r}\overline{\beta}_{j,r}$, and if we set $q_{j,0} := q_{1,0} = x_1$, then, identifying $q_{j,r}$ with its image in $\mathcal{O}_{Y,y_0}$, we have

$$\frac{q_{j,g_j+1}}{q_{j,0}^{b_{j,0}} \cdots q_{j,g_j}^{b_{j,g_j}}} = v_j \in \mathcal{O}_{Y,y_0}. \tag{3.10}$$

Here $v_j = \gamma_j u_j \bmod (u)$, where $\gamma_j$ is a unit in $k\langle u_2,\dots,u_{j-1}\rangle$. In particular, note that $k\langle u_2,\dots,u_j\rangle = k\langle v_2,\dots,v_j\rangle$. Note also that $q_{j,r}$ is obtained from $q'_{j,r}$ by replacing $v_{j'}$ by $q_{j',g_{j'}+1}/\big(q_{j',0}^{b_{j',0}} \cdots q_{j',g_{j'}}^{b_{j',g_{j'}}}\big)$, for $1 \le j' < j$. We will denote by $\{P_{j,r,s}\}_s$ the polynomials in $k[z,v_2,\dots,v_{j-1}]$ defined in order to obtain $q'_{j,r+1}$ from $q'_{j,r}$, hence satisfying (3.6) (resp. (3.7)) for $s = 1$ (resp. $s > 1$). The elements $\{q_{j,r}\}_{(j,r)\in\mathcal{J}}$ are called a *system of transverse generators for* $\eta : Y \to \mathbb{A}^d_k$ *with respect to* $E$.

**(IV)** Finally, for every element $q \in \mathcal{O}_{Y,y_0}$ that is the image of an element in the fraction field of $k[x_1,\dots,x_d]$, i.e., we can write $q = l/g$ where $l, g \in k[x_1,\dots,x_d]$, we can define $\{\overline{Q}_n\}_{n\ge 0}$ in $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$ such that, in the ring $\mathcal{O}_{Y_\infty, P^Y_{eE}}$, we have

$$Q_n \equiv \overline{Q}_n \mod P^Y_{eE}. \tag{3.11}$$

More precisely, since $P^Y_{eE}$ is a stable point and the image of $g$ in $\mathcal{O}_{Y,y_0}$ is nonzero, there exists $c \in \mathbb{N}$ such that $G_0,\dots,G_{c-1} \in P^Y_{eE}$, $G_c \notin P^Y_{eE}$. Hence we have

$$G_c Q_n + \cdots + G_{n+c} Q_e \equiv L_{n+c} \mod P^Y_{eE} \quad \text{for } n \ge 0$$

([18, Proof of Prop. 4.1, (14)]) and we can define recursively $\overline{Q}_n \in S^{-1}\mathcal{O}_{\mathbb{A}^d_\infty}$, where $S$ is the multiplicative system generated by $G_c$, satisfying (3.11) (see also [19, Lem. 4.1]).

Applying this to each $q_{j,r}$, we obtain $\overline{Q}_{j,r;n} \in \mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$, $n \geq 0$ such that $Q_{j,r;n} \equiv \overline{Q}_{j,r;n}$ modulo $P_{eE}^Y$. More precisely,

$$\overline{Q}_{j,r;n} \in \prod_{\substack{(j',r')\in\mathcal{J}^* \\ (j',r')<(j,r)}} \overline{T}_{j',r'}^{-1} k[x_1,\ldots,x_j]_\infty,$$

where $k[x_1,\ldots,x_j]_\infty$ denotes $\mathcal{O}_{(\operatorname{Spec} k[x_1,\ldots,x_j])_\infty}$ and $\overline{T}_{j',r'}$ is the multiplicative system generated by $\overline{Q}_{j',r';e\overline{\beta}_{j',r'}}$. Then, let

$$\mathcal{Q} := \{\overline{Q}_{j,r;n}\}_{(j,r)\in\mathcal{J}, en_{j,r-1}\overline{\beta}_{j,r-1}\leq n \leq e\overline{\beta}_{j,r}-1}.$$

It is clear (see (3.9)) that $(\mathcal{Q})\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}} \subseteq P_{eE}^{\mathbb{A}^d}\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$. In addition, note that, applying (2.4), (3.3) and, for the last equality, also (3.1), we have

$$\begin{aligned}
\sharp\mathcal{Q} &= em_1 + \sum_{j=2}^{\delta}\big(e\overline{\beta}_{j,1} + e(\overline{\beta}_{j,2} - n_{j,1}\overline{\beta}_{j,1}) + \cdots + e(\overline{\beta}_{j,g_j+1} - n_{j,g_j}\overline{\beta}_{j,g_j})\big) \\
&= em_1 + e\sum_{j=2}^{\delta}\big(\beta_{j,1} + (\beta_{j,2} - \beta_{j,1}) + \cdots + (\beta_{j,g_j+1} - \beta_{j,g_j})\big) \qquad (3.12) \\
&= em_1 + e\sum_{j=2}^{\delta}\beta_{j,g_j+1} = e\sum_{j=1}^{\delta} m_j = e(k_E(\mathbb{A}_k^d)+1) = e(\widehat{k}_E(X)+1).
\end{aligned}$$

Recall that $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ is a regular local ring of dimension $e(k_E(\mathbb{A}_k^d)+1)$ (see (xiii) in Section 2). In [19] we proved that $\mathcal{Q}$ is a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$; then $\mathcal{Q}$ is called a *regular system of parameters of* $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{E}^{\mathbb{A}^d}}$ *associated to* $\{q_{j,r}\}_{(j,r)\in\mathcal{J}}$. The proof is based on the study of the graded algebra $\operatorname{gr}_{\nu_E} k[x_1,\ldots,x_d]$. In fact, the main idea in the proof is to show that $(\mathcal{Q})\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ is a prime ideal and it follows from the following: it is proved that, modulo étale extension, $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}/(\mathcal{Q})$ is isomorphic to a polynomial ring in countably many variables over a certain localization of $\operatorname{gr}_{\nu_E} k[x_1,\ldots,x_d]$. Since $\operatorname{gr}_{\nu_E} k[x_1,\ldots,x_d]$ is a domain because $\nu_E$ is a valuation, it follows that $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}/(\mathcal{Q})$ is a domain (see [19, Thm. 4.8]).

More generally, let $\widetilde{q}_0,\ldots,\widetilde{q}_{g+1} \in B[x_1,y]$ be as in (II), and let us define $\widetilde{\mathcal{Q}} := \{\widetilde{Q}_{r;n}\}_{0\leq r\leq g, en_{j,r-1}\overline{\beta}_{j,r-1}\leq n\leq e\overline{\beta}_{j,r}-1}$, where $\widetilde{Q}_{r;n} \in B[x_1,y]_\infty$ and $\widetilde{L} := \prod_{r=0}^{g}\widetilde{Q}_{r;e\overline{\beta}_r}$. Then $(\widetilde{\mathcal{Q}})$ is a prime ideal of $(B[x_1,y]_\infty)_{\widetilde{L}}$ ([19, Prop. 4.5]).

In order to study the ring $\widehat{\mathcal{O}_{X_\infty,P_{eE}}}$, we will embed $X_0$ in a complete intersection scheme $X' \subseteq \mathbb{A}_k^M$ of dimension $d = \dim X_0$ (recall the notation at the beginning of the section). For any such $X'$ we have

$$\mathcal{O}_{(X_\infty)_{\mathrm{red}},P_{eE}} \cong \mathcal{O}_{(X'_\infty)_{\mathrm{red}},P_{eE}} \quad \text{and} \quad \widehat{\mathcal{O}_{(X_\infty),P_{eE}}} \cong \widehat{\mathcal{O}_{(X'_\infty),P_{eE}}},$$

where we also denote by $P_{eE}$ the point induced by $P^X_{eE}$ in $X'_\infty$ or in $(X'_\infty)_{\mathrm{red}}$ (see (ii) and (x) in Section 2).

**Proposition 3.2.** *Assume that* $\operatorname{char} k = 0$*. Let* $X_0$ *be an integral separated* $k$*-scheme of finite type. Let* $\nu = \nu_E$ *be a divisorial valuation on* $X_0$ *and let* $e \in \mathbb{N}$*. Then, there exist a complete intersection scheme*

$$X' = \operatorname{Spec} k[y_1, \ldots, y_N]/(f_{d+1}, \ldots, f_N) \subseteq \mathbb{A}^N_k$$

*that contains* $X_0$*, and of dimension* $d = \dim X_0$*, and elements* $\{z_{l,s}\}_{d+1 \leq l \leq N, 1 \leq s \leq g_l}$ *in* $k[y_1, \ldots, y_N]$ *such that, if for an element* $g \in k[y_1, \ldots, y_N]$ *we denote by* $\nu(g)$ *the* $\nu$*-value of the class of* $g$ *in* $\mathcal{O}_{X_0}$*, then the following hold:*

*(a) For* $d+1 \leq l \leq N$*,* $1 \leq s \leq g_l$*, let* $\overline{\alpha}_{l,s} := \nu(z_{l,s})$ *and let*

$$\mathcal{Z} = \cup_{l=d+1}^{N} \mathcal{Z}_l \quad \text{where } \mathcal{Z}_l := \{Z_{l,s;n}\}_{\substack{1 \leq s \leq g_l \\ 0 \leq n < e\overline{\alpha}_{l,s}}},$$

*where* $Z_{j,r;n} \in k[y_1, \ldots, y_N]_\infty$*. Then there exists* $G \in \mathcal{O}_{(\mathbb{A}^N)_\infty}$ *such that* $(\mathcal{Q} \cup \mathcal{Z})(\mathcal{O}_{(\mathbb{A}^N)_\infty})_G$ *is a prime ideal and*

$$P^{X'}_{eE}\mathcal{O}_{X'_\infty, P^{X'}_{eE}} = (\mathcal{Q} \cup \mathcal{Z})\,\mathcal{O}_{X'_\infty, P^{X'}_{eE}}.$$

*(b) For* $d+1 \leq l \leq N$*,* $f_l = f_l(y_1, \ldots, y_d, y_l) \in k[y_1, \ldots, y_d, y_l]$ *satisfies the following:*

*(i)* $\nu(\mathrm{Jac}(f_l)) = \nu(\frac{\partial f_l}{\partial y_l})$*; set* $\epsilon_l := \nu(\mathrm{Jac}(f_l))$*.*

*(ii) For all* $n \geq 0$*, the class of* $\frac{\partial F_{l;e\epsilon_l+n}}{\partial Y_{l;n}}$ *in* $\mathcal{O}_{X'_\infty,P_{eE}}$ *is a unit and, for* $n' > n$*, the class of* $\frac{\partial F_{l;e\epsilon_l+n}}{\partial Y_{l;n'}}$ *in* $\mathcal{O}_{X'_\infty,P_{eE}}$ *belongs to* $P_{eE}\mathcal{O}_{X'_\infty,P_{eE}}$*. Besides, if we define* $f'_{l,l} := \frac{\partial f_l}{\partial y_l}$ *then the class of* $\frac{\partial F_{l;e\epsilon_l+n}}{\partial Y_{l;n}} - F'_{l,l;e\epsilon_l}$ *in* $\mathcal{O}_{X'_\infty,P_{eE}}$ *belongs to* $P_{eE}$*.*

*(iii) There exists* $L \in \mathcal{O}_{\mathbb{A}^d_\infty} = k[x_1, \ldots, x_d]_\infty$*,* $L \notin P^{\mathbb{A}^d}_{eE}$ *such that the elements* $F_{l;0}, \ldots, F_{l;e\epsilon_l-1}$ *belong to* $(\mathcal{Q} \cup \mathcal{Z}_l)^2(\mathcal{O}_{(\mathbb{A}^N_k)_\infty})_L$*.*

*Proof.* Let $\pi : Y \to X_0$, $\rho : X_0 \to \mathbb{A}^d_k$ and $\eta = \rho \circ \pi : Y \to \mathbb{A}^d_k$ be as in the beginning of this section. Let us consider an étale morphism $\widetilde{U} \to U$ as in (I) and keep the notation in (I). From the discussion in (I), (II) and (III), it follows that there exist $\{u, v_2, \ldots, v_d\} \in \mathcal{O}_{\widetilde{U}}$ and $\{x_1, \ldots, x_d, x_{d+1}, \ldots, x_N\} \in \mathcal{O}_X$ such that, after replacing $v_i$ by $v_i + c_i$, where $c_i \in k$, $2 \leq i \leq d$, the following property holds for the points $y_0$ in an open subset of $\widetilde{E}$: $\{u, v_2, \dots, v_d\}$ (resp. $\{x_1, \dots, x_d\}$) is a regular system of parameters of $\mathcal{O}_{\widetilde{U},y_0}$ (resp. $\mathcal{O}_{\mathbb{A}^d_k,\eta(y_0)}$) and $\{x_1, \dots, x_d, x_{d+1}, \dots, x_N\}$ generate the maximal ideal of $\mathcal{O}_{X_0,\pi(y_0)}$. In addition, we have

(i) the local expression for $\eta$ in (3.3) holds for the regular system of parameters $\{u, v_2, \dots, v_d\}$ of $\mathcal{O}_{\widetilde{U},y_0}$ and $\{x_1, \dots, x_d\}$ of $\mathcal{O}_{\mathbb{A}^d,\eta(y_0)}$ (i.e., in (3.3) replace $u_1$ by $u$, $u_i$ by $v_i$ for $2 \le i \le \delta$ and set $v_i = u_i$ for $\delta < i \le d$);

(ii) there exists a system of transverse generators $\{q_{j,r}\}_{(j,r)\in\mathcal{J}}$ for $\eta : Y \to \mathbb{A}^d_k$ with respect to $E$, hence satisfying (3.8), (3.9) and (3.10);

(iii) for $d+1 \le l \le N$, the image of $x_l$ in $\mathcal{O}_{Y,y_0}$ is expressed as

$$x_l = \sum_{m_1 \le i} \lambda_{l,i}(\underline{v}) u^i, \tag{3.13}$$

where $\underline{v} := (v_2, \dots, v_d)$ and

$$\begin{aligned} &\lambda_{l,i}(\underline{v}) \in k\langle \underline{v}\rangle \cap \mathcal{O}_{\widetilde{U},y_0}, \\ &\lambda_{l,i}(\underline{v}) \in k\langle v_2, \dots, v_{j-1}\rangle \cap \mathcal{O}_{\widetilde{U},y_0} \quad \text{if } i < m_j \text{ for } 2 \le j \le \delta \end{aligned} \tag{3.14}$$

(recall (3.1) for the second assertion in (3.14)).

Fix $l$, $d+1 \le l \le N$. Let $\overline{\beta}_{l,0}, \dots, \overline{\beta}_{l,g_l}$ be a minimal system of generators of the semigroup defined by the restriction $\nu_l$ of $\nu_E$ to $k(\underline{v})[x_1, x_l]_{(x_1,x_l)}$. Let $e_{l,r} = \gcd\{\overline{\beta}_{l,0}, \dots, \overline{\beta}_{l,r}\}$, $0 \le r \le g_l$, $n_{l,r} = e_{l,r-1}/e_{l,r}$, $1 \le r \le g_l$, and let $\beta_{l,0}, \dots, \beta_{l,g_l}$ be defined by $\overline{\beta}_{l,r} - n_{l,r-1}\overline{\beta}_{l,r-1} = \beta_{l,r} - \beta_{l,r-1}$ as in (2.4). Consider $h \in k[\underline{v}]$ such that $k[\underline{v}]_h$ is contained in the ring $\mathcal{O}_{\widetilde{U}}$ and consider the morphism $\theta_l : \widetilde{U} \to \operatorname{Spec} k[\underline{v}]_h[x_1, y]$ given by

$$\begin{aligned} x_1 &\mapsto u^{m_1}, \\ y &\mapsto \sum_{m_1 \le i} \lambda_{l,i}(\underline{v}) u^i. \end{aligned}$$

There exists a domain $B_l$ such that $B_l[x_1, y]$ is an étale extension of $k[\underline{v}]_h[x_1, y]$ and there exist $x_1'$, $y' \in B_l[x_1, y]$ with

$$x_1' = \gamma_1 x_1, \quad y' = \gamma_l y, \quad \text{where } \gamma_1, \gamma_l \in B_l[x_1, y] \text{ are units},$$

and $u' = \mu u$, where $\mu$ is a unit in an étale extension of $k[\underline{v}]_h[u]$, such that the induced morphism $\widetilde{\theta}_l : \widetilde{\widetilde{U}} \to \operatorname{Spec} B_l[x_1', y']$, where $\widetilde{\widetilde{U}} \to \widetilde{U}$ is étale, is given by

$$\begin{aligned} x_1' &\mapsto (u')^{m_1}, \\ y' &\mapsto \sum_{m_1 \le i \le m} \lambda'_{l,i} (u')^i, \end{aligned}$$

where $\lambda'_{l,i} \in B_l$ for $m_1 \leq i \leq m$ (see Remark 2.3). Let $\widetilde{q}_{l,0}, \dots, \widetilde{q}_{l,g_l}, \widetilde{q}_{l,g_l+1} \in B_l[x'_1, y']$ be the elements defined as in (II) applied to the above expression; hence we are in case (b) in (II). Hence $\widetilde{q}_{l,g_l+1}$ defines the kernel of $B_l[x_1, y] \to \mathcal{O}_{\widetilde{\widetilde{U}}}$, i.e.,

$$B_l[x_1, x_l] \cong B_l[x_1, y]/(\widetilde{q}_{l,g_l+1}).$$

Thus $\widetilde{q}_{l,g_l+1}$ defines the equation of a plane curve in $\operatorname{Spec} L_l[x'_1, y']$, where $L_l$ is a field extension of $k$ containing $\lambda'_{l,i}$ for $m_1 \leq i \leq m$, which is analytically irreducible, and $\widetilde{q}_{l,1}, \dots, \widetilde{q}_{l,g_l}$ are its approximate roots. Let us also consider the following elements in $k[\underline{v}]_h[x_1, y]$: let $f'_0 := \widetilde{q}_{l,0} = x_1$ and, for $1 \leq r \leq g+1$, let us define $f'_{l,r}$ to be an irreducible polynomial in $k[\underline{v}]_h[x_1, y]$ defining the contracted ideal of $(\widetilde{q}_{l,r})B_l[x_1, y]$ to $k[\underline{v}]_h[x_1, y]$. Set $f'_l := f_{l,g_l+1}$ and note that we have

$$f'_l(\underline{v}, x_1, y) = \widetilde{q}_{l,g_l+1} \cdot \widetilde{h}, \tag{3.15}$$

where $\widetilde{h} \in B_l[x_1, y]$ and $\widetilde{q}_{l,g_l+1}$ does not divide $\widetilde{h}$. Indeed, if $\widetilde{q}_{l,g_l+1}$ divides $\widetilde{h}$ then $\frac{\partial f'_l}{\partial y}$ belongs to the contracted ideal of $(\widetilde{q}_{l,g_l+1})B_l[x_1, y]$, contradicting the definition of $f'_l$. Let

$$C_l := \operatorname{Spec} k[\underline{v}]_h[x_1, y]/(f'_l), \qquad \widetilde{C}_l := \operatorname{Spec} B_l[x_1, y]/(\widetilde{q}_{l,g_l+1}).$$

We consider now the spaces of arcs of $C_l$, $\widetilde{C}_l$. Let $\widetilde{\nu}$ be a divisorial valuation on $B_l[x_1, y]/(\widetilde{q}_{l,g_l+1})$ extending $\nu_l$ (recall that $\nu_l(v_j) = 0$, $2 \leq j \leq d$) and let $P'_l$ (resp. $\widetilde{P}_l$) be the stable point of $\mathcal{O}_{(C_l)_\infty}$ (resp. $\mathcal{O}_{(\widetilde{C}_l)_\infty}$) defined by $\nu_l$ and $e$ (resp. $\widetilde{\nu}$ and $e$). Note that we have

$$\widehat{\mathcal{O}}_{(C_l)_\infty, P'_l} \prec \widehat{\mathcal{O}}_{(\widetilde{C}_l)_\infty, \widetilde{P}_l},$$

i.e., the ring on the right-hand side dominates the ring on the left-hand side. Following (IV), let $\widetilde{\mathcal{Q}}_l := \{\widetilde{Q}_{l,r;n}\}_{0 \leq r \leq g_l, en_{j,r-1}\overline{\beta}_{j,r-1} \leq n \leq e\overline{\beta}_{l,r}-1}$. Then $(\widetilde{\mathcal{Q}}_l)$ defines a prime ideal $\widetilde{\mathbb{P}}_l$ in $(B_l[x_1, y]_\infty)_{\widetilde{L}}$, where $\widetilde{L} = \prod_{r=0}^{g} \widetilde{Q}_{l,r;e\overline{\beta}_{l,r}}$, and we have

$$(\widetilde{\mathcal{Q}}_l)\mathcal{O}_{(\widetilde{C}_l)_\infty, \widetilde{P}_l} = \widetilde{P}_l \mathcal{O}_{(\widetilde{C}_l)_\infty, \widetilde{P}_l}$$

(this argument has already been applied in Example 2.2; it is based on [19, Prop. 4.5]; see also (IV)). In addition, $\widetilde{\mathbb{P}}_l$ is a stable point of $B_l[x_1, y]_\infty$, since $\widetilde{\mathcal{Q}}_l$ is a finite set. Let $\mathbb{P}'_l$ be the image of $\widetilde{\mathbb{P}}_l$ in $(\operatorname{Spec} k[\underline{v}]_h[x_1, y])_\infty$. Since the morphism $k[\underline{v}]_h[x_1, y]_{(x_1,y)} \to B_l[x_1, y]_{(x_1,y)}$ is étale, $\mathbb{P}'_l$ is a stable point and we have

$$(B_l\widehat{[x_1, y]_\infty})_{\widetilde{\mathbb{P}}_l} \cong (k[\underline{v}]_h\widehat{[x_1, y]_\infty})_{\mathbb{P}'_l} \otimes_{\kappa(\mathbb{P}'_l)} \kappa(\widetilde{\mathbb{P}}_l) \tag{3.16}$$

([19, Prop. 2.5]; see (xii)). Let $\mathcal{F}'_l := \{F'_{l,r;n}\}_{0 \leq r \leq g_l, 0 \leq n < e\nu(f'_{l,r})}$ and let $L' = H_0 \cdot \prod_{r=0}^{g} F'_{l,r;e\nu(f'_{l,r})}$. Then $(\mathcal{F}'_l)\,(k[\underline{v}, x_1, y]_\infty)_{L'}$ is a prime ideal ([19, Proof of

Prop. 4.5]; see (IV)) and we have

$$(\mathcal{F}'_l)\,(k[\underline{v}]_h[x_1,y]_\infty)_{L'} = \mathbb{P}'_l\,(k[\underline{v}]_h[x_1,y]_\infty)_{L'} \tag{3.17}$$

and

$$(\mathcal{F}'_l)\mathcal{O}_{(C_l)_\infty,P'_l} = P'_l\mathcal{O}_{(C_l)_\infty,P'_l}.$$

Now, for $\widetilde{q}_{l,g_l+1}$, the following properties hold:

(a.1) We have $\widetilde{\nu}(\mathrm{Jac}(\widetilde{q}_{l,g_l+1})) = \widetilde{\nu}(\frac{\partial \widetilde{q}_{l,g_l+1}}{\partial y}) = \widetilde{\nu}(\frac{\partial \widetilde{q}_{l,g_l+1}}{\partial y'}) = (n_{l,g_l}-1)\overline{\beta}_{l,g_l} + \cdots + (n_{l,1}-1)\overline{\beta}_{l,1} = n_{l,g_l}\overline{\beta}_{l,g_l} - \beta_{l,g_l}$. Set $\widetilde{\epsilon} := n_{l,g_l}\overline{\beta}_{l,g_l} - \beta_{l,g_l}$.

(b.1) For all $n \geq 0$, the class of $\frac{\partial \widetilde{Q}_{l,g_l+1;e\widetilde{\epsilon}+n}}{\partial Y_n}$ in $\mathcal{O}_{(\widetilde{C}_l)_\infty,\widetilde{P}_l}$ is equal to the class of $n_{l,g_l}\cdots n_{l,1}\widetilde{Q}^{n_{l,g_l}-1}_{l,g_l;e\overline{\beta}_{l,g_l}}\cdots\widetilde{Q}^{n_{l,1}-1}_{l,1;e\overline{\beta}_{l,1}}$ modulo $\widetilde{P}_l$; hence $\frac{\partial \widetilde{Q}_{l,g_l+1;e\widetilde{\epsilon}+n}}{\partial Y_n}$ is a unit in $\mathcal{O}_{(\widetilde{C}_l)_\infty,\widetilde{P}_l}$.

(c.1) For $n' > n$, the class of $\frac{\partial \widetilde{Q}_{l,g_l+1;e\widetilde{\epsilon}+n}}{\partial Y_{n'}}$ in $\mathcal{O}_{(\widetilde{C}_l)_\infty,\widetilde{P}_l}$ belongs to $\widetilde{P}_l\mathcal{O}_{(\widetilde{C}_l)_\infty,\widetilde{P}_l}$.

(d.1) $\widetilde{Q}_{l,g_l+1;0},\ldots,\widetilde{Q}_{l,g_l+1;e\widetilde{\epsilon}-1}$ belong to $(\widetilde{\mathcal{Q}}_l)^2B_l[[x_1,y]]_\infty$.

In fact, to prove (d.1) we argue by induction, and prove that, for $1 \leq r \leq g_l+1$,

$$\widetilde{Q}_{l,r;n} \in \Big(\{\widetilde{Q}_{l,r';n}\}_{0\leq r'\leq r-1, 0\leq n\leq e\overline{\beta}_{l,r'}-1}\Big)^2 B_l[x_1,y]_\infty \tag{3.18}$$

for $0 \leq n < e\big((n_{l,r-1}-1)\overline{\beta}_{l,r-1} + \cdots + (n_{l,1}-1)\overline{\beta}_{l,1}\big) = e(n_{l,r-1}\overline{\beta}_{l,r-1} - \beta_{l,r-1})$. Now, from (3.16) and (3.17) we obtain that $F'_{l;0},\ldots,F'_{l;e\epsilon'-1}$ belong to $(\mathcal{F}'_l)^2\,(k[\underline{v},x_1,y]_\infty)_{\mathbb{P}'_l}$, where $\epsilon' = \widetilde{\nu}(\widetilde{h}) + n_{l,g_l}\overline{\beta}_{l,g_l} - \beta_{l,g_l}$. Therefore (recall (3.15)), we obtain the following conclusions:

(a.2) We have $\nu_l(\mathrm{Jac}(f'_l)) = \nu_l(\frac{\partial f'_l}{\partial y}) = \widetilde{\epsilon} + \widetilde{\nu}(\widetilde{h})$. Let $\epsilon' := \widetilde{\epsilon} + \widetilde{\nu}(\widetilde{h})$.

(b.2) For all $n \geq 0$, the class of $\frac{\partial F'_{l;e\epsilon'+n}}{\partial Y_n}$ in $\mathcal{O}_{(C_l)_\infty,P'_l}$ is a unit. In addition, if $h_l := \frac{\partial f'_l}{\partial y}$ then the class of $\frac{\partial F'_{l;e\epsilon'+n}}{\partial Y_n} - H_{l;e\epsilon'}$ in $\mathcal{O}_{(C_l)_\infty,P'_l}$ belongs to $P'_l$.

(c.2) For $n' > n$, the class of $\frac{\partial F'_{l;e\epsilon+n}}{\partial Y_{n'}}$ in $\mathcal{O}_{(C_l)_\infty,P'_l}$ belongs to $P'_l\mathcal{O}_{(C_l)_\infty,P'_l}$.

(d.2) $F'_{l;0},\ldots,F'_{l;e\epsilon'-1}$ belong to $(\mathcal{F}'_l)^2\,(k[\underline{v},x_1,y]_\infty)_{H_0}$.

Now, let $b$ be the smallest nonnegative integer such that $g'_l := h^b f'_l$ belongs to $k[\underline{v},x_1,y]$ and let $\{b_{j,r}\}_{(j,r)\in\mathcal{J}^*}$ be a minimal sequence of nonnegative integers such that

$$\begin{aligned} &f_l(x_1,\ldots,x_d,y_l)\\ &\quad := \prod_{(j,r)\in\mathcal{J}^*} q_{j,r}^{b_{j,r}}\, g'_l\left(\frac{q_{2,g_2+1}}{q_{1,0}^{b_{2,0}}\cdots q_{2,g_2}^{b_{2,g_2}}},\ldots,\frac{q_{\delta,g_\delta+1}}{q_{\delta,0}^{b_{\delta,0}}\cdots q_{\delta,g_\delta}^{b_{\delta,g_\delta}}},x_{\delta+1},\ldots,x_d,x_1,y_l\right)\end{aligned}$$

belongs to $k[x_1, \ldots, x_d, y_l]$, where $y_l$ is an indeterminate (recall (3.8) and (3.10)). Therefore we have

$$f_l(x_1, \ldots, x_d, x_l) = 0. \tag{3.19}$$

From (3.1) and (a.2) it follows that

$$\epsilon_l := \nu(\mathrm{Jac}(f_l)) = \nu\left(\frac{\partial f_l}{\partial y_l}\right) = \nu\left(\prod_{(j,r)\in\mathcal{J}^*} q_{j,r}^{b_{j,r}} h^b\right) + \epsilon', \tag{3.20}$$

i.e., (i) in the statement of the proposition holds. From (b.2) and (c.2) we obtain that (ii) also holds.

For $0 \leq s \leq g_l + 1$, let $b(l,s)$ be the smallest nonnegative integer such that $g'_{l,s} := h^{b(l,s)} f'_{l,s}$ belongs to $k[\underline{v}, x_1, y]$ and let $\{b_{j,r}(l,s)\}_{(j,r)\in\mathcal{J}^*}$ be a minimal sequence of nonnegative integers such that

(3.21)

$$z_{l,s} := \prod_{(j,r)\in\mathcal{J}^*} q_{j,r}^{b_{j,r}(l,s)} \cdot g'_{l,s}\left(\frac{q_{2,g_2+1}}{q_{2,0}^{b_{2,0}} \cdots q_{2,g_2}^{b_{2,g_2}}}, \ldots, \frac{q_{\delta,g_\delta+1}}{q_{\delta,0}^{b_{\delta,0}} \cdots q_{\delta,g_\delta}^{b_{\delta,g_\delta}}}, x_{\delta+1}, \ldots, x_d, x_1, y_l\right)$$

belongs to $k[x_1, \ldots, x_d, y_l]$. Set $\overline{\alpha}_{l,s} := \nu(x_{l,s})$, where $x_{l,s}$ is the class of $z_{l,s}$ in $\mathcal{O}_{X_0}$, and $\mathcal{Z}_l := \{Z_{l,s;n}\}_{1\leq s\leq g_l, 0\leq n< e\overline{\alpha}_{l,s}}$. Then, from (d.2) and applying the second assertion in (3.14), we conclude that

$$F_{l;0}, \ldots, F_{l;e\epsilon_l-1} \in (\mathcal{Q}\cup\mathcal{Z}_l)^2\left(\prod_{(j,r)\in\mathcal{J}^*} \overline{T}_{j,r}^{-1} k[x_1, \ldots, x_d, y_l]_\infty\right)_{\overline{H}_0},$$

where, if we consider $h$ as an element of $k(x_1, \ldots, x_d)$, i.e., we replace $v_j$ by $q_{j,g_j+1}/(q_{j,0}^{b_{j,0}} \cdots q_{j,g_j}^{b_{j,g_j}})$ (resp. $x_j$), for $2 \leq j \leq \delta$ (resp. $\delta+1 \leq j \leq d$), then $\overline{H}_0 \in \prod_{(j,r)\in\mathcal{J}^*} \overline{T}_{j,r}^{-1} k[x_1, \ldots, x_d]_\infty$ satisfies $H_0 \equiv \overline{H}_0 \bmod P_{eE}^Y$, as in (IV). In particular, if $L := \overline{H}_0 \cdot \prod_{(j,r)\in\mathcal{J}^*} \overline{Q}_{j,r;e\overline{\beta}_{j,r}}$, we obtain that $F_{l;0}, \ldots, F_{l;e\epsilon_l-1} \in (\mathcal{Q}\cup\mathcal{Z}_l)^2\,(k[x_1, \ldots, x_d, y_l]_\infty)_L$. Setting $G_l = L \cdot \prod_{s=1}^{g_l} Z_{l,s;e\overline{\alpha}_{l,s}}$, and applying (3.17) and that $\mathcal{Q}$ is a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$, we have that $(\mathcal{Q}\cup\mathcal{Z}_l)(k[x_1, \ldots, x_d, y_l]_\infty)_{G_l}$ is a prime ideal.

Finally, applying (3.19) we conclude that

$$X' = \operatorname{Spec} k[x_1, \ldots, x_d, y_{d+1}, \ldots, y_N]/(f_{d+1}, \ldots, f_N)$$

is a $d$-dimensional complete intersection scheme in $\mathbb{A}_k^N$ containing $X_0$ and satisfying (i) to (iii) in (b). If we set $G = L \cdot \prod_{l=d+1}^{N} \prod_{s=1}^{g_l} Z_{l,s;e\overline{\alpha}_{l,s}}$ then we conclude that $(\mathcal{Q}\cup\mathcal{Z}_l)(k[x_1, \ldots, x_d, y_{d+1}, \ldots, y_N]_\infty)_G$ is a prime ideal such that

$$(\mathcal{Q}\cup\mathcal{Z}_l)\mathcal{O}_{X'_\infty, P_{eE}^{X'}} = P_{eE}^{X'}\mathcal{O}_{X'_\infty, P_{eE}^{X'}}.$$

Thus, the proposition is proved. □

**Remark 3.3.** Keep the notation in Proposition 3.2, and fix $l$, $d+1 \leq l \leq N$. Define $\underline{Y}_n^{(l)} := (Y_{1;n}, \ldots, Y_{d;n}, Y_{l;n})$, $n \geq 0$, and $f'_{l,j} := \frac{\partial f_l}{\partial y_j}$, $j \in \{1, \ldots, d, l\}$. Then, applying Taylor's formula it follows that, for $n \geq e\epsilon_l$,

$$F_{l;n+e\epsilon_l+1} = H_{l;n+e\epsilon_l+1} + \sum_{j=1}^{d} \sum_{i=0}^{e\epsilon_l} F'_{l,j;i} Y_{j;n+e\epsilon_l+1-i} + \sum_{i=0}^{e\epsilon_l} F'_{l,l;i} Y_{l;n+e\epsilon_l+1-i}, \tag{3.22}$$

where $H_{l;n+e+1} \in k[\underline{Y}_0^{(l)}, \ldots, \underline{Y}_n^{(l)}]$ is the coefficient of $t^{n+e\epsilon_l+1}$ in $f_l(\sum_{i=0}^{n} \underline{Y}_i^{(l)} t^i)$ (see [16, Proof of Lem. 3.2]). In particular, since $\epsilon_l := \nu(\mathrm{Jac}(f_l)) = \nu(\frac{\partial f_l}{\partial y_l})$, it follows that, for $n \geq e\epsilon_l$,

$$\frac{\partial F_{l;n+e\epsilon_l+1}}{Y_{l;n+1}} = F'_{l,l;e\epsilon_l} \notin P_{eE},$$

$$\frac{\partial F_{l;n+e\epsilon_l+1}}{Y_{l;n'+1}} = \begin{cases} F'_{l,l;e\epsilon_l-(n'-n)} \in P_{eE} & \text{for } n+1 \leq n' \leq n+e\epsilon_l, \\ 0 & \text{for } n+e\epsilon_l < n'. \end{cases}$$

This idea, generalized to complete intersection schemes (see [17, Proof of Lem. 4.2]) is a key point in [17, Proof of Thm. 4.1] (see (vii) and (viii) in Section 2). The statement in Proposition 3.2(b)(ii) is an improvement of the previous assertion. Indeed, it states that for $n \geq 0$ (in particular, also for $0 \leq n < e\epsilon_l$) in the ring $\mathcal{O}_{X'_\infty, P_{eE}}$, we have $\frac{\partial F_{l;n+e\epsilon_l+1}}{Y_{l;n+1}} \equiv F'_{l,l;e\epsilon_l} \bmod P_{eE}$ and $\frac{\partial F_{l;n+e\epsilon_l+1}}{Y_{l;n'+1}} \equiv 0 \bmod P_{eE}$ for $n' > n$.

**Theorem 3.4.** *Assume that* $\operatorname{char} k = 0$. *Let $X$ be a reduced separated $k$-scheme of finite type, let $\nu = \nu_E$ be a divisorial valuation on an irreducible component $X_0$ of $X$, and let $e \in \mathbb{N}$. Then*

$$\operatorname{embdim} \mathcal{O}_{(X_\infty)_{red}, P_{eE}} = \operatorname{embdim} \widehat{\mathcal{O}_{(X_\infty)_{red}, P_{eE}}} = e(\widehat{k}_E + 1), \tag{3.23}$$

*where $\widehat{k}_E$ is the Mather discrepancy of $X$ with respect to $E$.*

*Moreover, if $\rho : X \to \mathbb{A}_k^d$, where $d = \dim X_0$, is a general projection, more precisely a projection that satisfies (3.1), and $\mathcal{Q} = \{\overline{Q}_{j,r;n}\}_{(j,r) \in \mathcal{J}, en_{j,r-1}\overline{\beta}_{j,r-1} \leq n \leq e\overline{\beta}_{j,r}-1}$ is a regular system of parameters of $\mathcal{O}_{(\mathbb{A}_k^d)_\infty, P_{eE}^{\mathbb{A}^d}}$, then $\mathcal{Q}$ is a minimal system of coordinates of $((X_\infty)_{red}, P_{eE}^X)$, that is, we have $\sharp\mathcal{Q} = e(\widehat{k}_E + 1)$ and*

$$P_{eE}^X \mathcal{O}_{(X_\infty)_{red}, P_{eE}^X} = (\mathcal{Q})\, \mathcal{O}_{(X_\infty)_{red}, P_{eE}^X}.$$

*Proof.* First recall that, since $\mathcal{Q}$ is a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ ([19, Thm. 4.8]) and $\rho : X \to \mathbb{A}_k^d$ is a dominant morphism, we have

$$P_{eE}^X \widehat{\mathcal{O}_{X_\infty, P_{eE}^X}} = (\mathcal{Q})\, \widehat{\mathcal{O}_{X_\infty, P_{eE}^X}}$$

([18, Prop. 4.5]; see (xi)). From this and Nakayama's lemma, the second assertion of the theorem follows (see also (3.12)). Therefore, we have to prove (3.23) only, or equivalently, the independence of the elements of $\mathcal{Q}$ in $P_{eE}^X/(P_{eE}^X)^2$.

Let $X'$ be the $d$-dimensional complete intersection scheme containing $X_0$, defined in Proposition 3.2, and keep the notation in that proposition. We have $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P_{eE}} \cong \mathcal{O}_{(X'_\infty)_{\mathrm{red}},P_{eE}^{X'}}$ and $\widehat{\mathcal{O}_{(X_\infty),P_{eE}}} \cong \widehat{\mathcal{O}_{(X'_\infty),P_{eE}^{X'}}}$ (see (ii) and (ix)). Therefore, in order to prove (3.23) we may suppose that $X = X'$. We will next describe the ring $\widehat{\mathcal{O}_{X_\infty,P_{eE}}}$, where $X = X'$ and $P_{eE} = P_{eE}^X$. We will follow the ideas in Example 2.2 (or [18, Cor. 4.6]), where an analogous description is given.

The residue field of $P_{eE}^{\mathbb{A}^d}$ is

$$\kappa(P_{eE}^{\mathbb{A}^d}) \cong k\left(\{X_{1;n}\}_{n>em_1} \cup \{X_{j;n}\}_{\substack{2\leq j\leq d\\ n\geq em_j}}\right)\left[\{W_{j,r}\}_{(j,r)\in\mathcal{J}^*}\right]/J,$$

where we set $m_j := 0$ for $\delta+1 \leq j \leq d$ (see (3.3)), $W_{j,r}$ is the class of $\overline{Q}_{j,r;e\overline{\beta}_{j,r}}$ and $J$ is the ideal generated by

$$P_{j,r,1}\left(\frac{\overline{\mu}_{j,r,1}(W_{j,r})^{n_{j,r}}}{W_{1,0}^{b_{j,0}}\cdots W_{j,r-1;}^{b_{j,r-1}}}, \frac{W_{2,g_2+1}}{W_{1,0}^{b_{2,0}}\cdots W_{2,g_2}^{b_{2,g_2}}}, \ldots, \frac{W_{j-1,g_{j-1}+1}}{W_{1,0}^{b_{j-1,0}}\cdots W_{j-1,g_{j-1}}^{b_{j-1,g_{j-1}}}}\right) \tag{3.24}$$

(recall (II) and (III)). From property (3.6) satisfied by $P_{j,r,1}$ and Hensel's lemma, it follows that we can define an embedding $\kappa(\mathbb{A}_k^d) \hookrightarrow \widehat{\mathcal{O}_{(\mathbb{A}^d)_\infty,P_{eE}^{\mathbb{A}^d}}}$ sending $X_{j;n}$ to $X_{j;n} \in \widehat{\mathcal{O}_{(\mathbb{A}^d)_\infty,P_{eE}^{\mathbb{A}^d}}}$, for $j = 1, n > em_1$, and $2 \leq j \leq d, n \geq em_j$, sending $W_{1,0}$ to $X_{1;em_1}$ and, recursively, for $(j,r) \in \mathcal{J}^* \setminus \{(1,0)\}$, sending $W_{j,r}$ to a root of the polynomial obtained from (3.24) by replacing $W_{j',r'}$, $(j',r') < (j,r)$ by its image in $\widehat{\mathcal{O}_{(\mathbb{A}^d)_\infty,P_{eE}^{\mathbb{A}^d}}}$; this root exists by Hensel's lemma. Then we have

$$\widehat{\mathcal{O}_{(\mathbb{A}^d)_\infty,P_{eE}^{\mathbb{A}^d}}} \cong \kappa(P_{eE}^{\mathbb{A}^d})\left[\left[\{X_{j,r;n}\}_{\substack{(j,r)\in\mathcal{J}\\ en_{j,r-1}\overline{\beta}_{j,r-1}\leq n<e\overline{\beta}_{j,r}}}\right]\right],$$

where the image of $X_{j,r;n}$ in $\widehat{\mathcal{O}_{(\mathbb{A}^d)_\infty,P_{eE}^{\mathbb{A}^d}}}$ is $\overline{Q}_{r,j;n}$. Besides, $\widehat{\mathcal{O}_{X_\infty,P_{eE}^X}}$ is a quotient of $\kappa(P_{eE}^X)\left[\left[\{X_{j,r;n}\}_{(j,r)\in\mathcal{J},en_{j,r-1}\overline{\beta}_{j,r-1}\leq n<e\overline{\beta}_{j,r}}\right]\right]$, where the residue field $\kappa(P_{eE}^X)$ of $P_{eE}^X$ is a finite field extension of $\kappa(P_{eE}^{\mathbb{A}^d})$.

Now, fix $l$, $d+1 \leq l \leq N$. Arguing analogously we obtain

$$\kappa_l := \kappa(P_{eE}^{\mathbb{A}^d})\left[\{W_{l,s}\}_{s=1}^{g_l}\right]/J_l \hookrightarrow \kappa(P_{eE}^X),$$

where $W_{l,s}$ is the class of $Z_{l,s;e\overline{\alpha}_{l,s}}$ and $J_l$ is the ideal generated by the relations on $\{W_{l,s}\}_{s=1}^{g_l}$ induced by $G'_{l,s;e\nu(f'_{l,s})-e(\overline{\beta}_{l,s}-n_{l,s-1}\overline{\beta}_{l,s-1})}$, $2 \leq s \leq g_l$ (see (3.21)). Applying Hensel's lemma recursively to these relations, we can define an embedding $\kappa_l \hookrightarrow \widehat{\mathcal{O}_{X_\infty,P_{eE}^X}}$ sending $X_{j;n}$ to $X_{j;n} \in \widehat{\mathcal{O}_{X_\infty,P_{eE}^X}}$, for $j = 1, n > em_1$ and

$2 \leq j \leq d, n \geq em_j$, and sending $W_{1,0}$ to $X_{1;em_1} \in \widehat{\mathcal{O}_{X_\infty, P^X_{eE}}}$. In particular, for each $n \geq 0$ we have defined $Y^{(0)}_{l;n} \in \kappa_l$ such that $Y_{l;n} - Y^{(0)}_{l;n} \in (\mathcal{Q} \cup \mathcal{Z}_l)$. Arguing recursively on $m \geq 1$ and $n \geq 0$, with the lexicographical order on $(m, n)$, from $\{F_{l;e\epsilon_l+n}\}_{n \geq 0}$, applying property (ii) in Proposition 3.2(b) and Hensel's lemma, and reasoning as in [18, Cor. 5.6], it follows that, for $m, n \geq 0$, there exists $Y^{(m)}_{l;n} \in \kappa_l[\{X_{j,r;n}\}_{(j,r)\in\mathcal{J}, en_{j,r-1}\overline{\beta}_{j,r-1} \leq n < e\overline{\beta}_{j,r}}]$ such that

$$F_{e\epsilon_l+n} \equiv L^{(m,n)}_{e\epsilon_l}(Y_{l;n} - Y^{(m)}_{l;n}) \mod (\mathcal{Q} \cup \mathcal{Z}_l)^{m+1} \tag{3.25}$$

in the ring $(k[x_1, \ldots, x_d, y_l]_\infty)_{(\mathcal{Q}\cup\mathcal{Z}_l)}$, where $L^{(m,n)}_{e\epsilon_l}$ is a unit. More precisely, $L^{(m,n)}_{e\epsilon_l}$ $- F'_{l,l;e\epsilon_l} \in \big(\mathcal{Q} \cup \mathcal{Z}_l\big)$ where we recall that $f'_{l,l} := \frac{\partial f_l}{\partial y_l}$.

Therefore, $Y^{(m+1)}_{l;n} - Y^{(m)}_{l;n} \in (\mathcal{Q} \cup \mathcal{Z}_l)^{m+1}$ by (3.25). Hence we have defined series $\widetilde{Y}_{l;n} \in \kappa_l\big[\big[\{X_{j,r;n}\}_{(j,r)\in\mathcal{J}, en_{j,r-1}\overline{\beta}_{j,r-1} \leq n < e\overline{\beta}_{j,r}}\big]\big]$, $\widetilde{Y}_{l;n} = \lim_m Y^{(m)}_{l;n}$. We conclude that

$$\kappa(P^X_{eE}) = \kappa(P^Z_{eE})\,[\{W_{l,s}\}_{d+1\leq l\leq N, 1\leq s\leq g_l}] \Big/ \sum_{l=d+1}^{N} J_l$$

and

$$\widehat{\mathcal{O}_{X_\infty, P^X_{eE}}} \cong \kappa(P^X_{eE})\left[\left[\{X_{j,r;n}\}_{\substack{(j,r)\in\mathcal{J} \\ en_{j,r-1}\overline{\beta}_{j,r-1} \leq n < e\overline{\beta}_{j,r}}}\right]\right] \Big/ \left(\{\widetilde{F}_{l;n}\}_{\substack{d+1\leq l\leq N \\ 0\leq n<e\epsilon_l}}\right), \tag{3.26}$$

where, for $d+1 \leq l \leq N$, $0 \leq n \leq e\epsilon_l - 1$, $\widetilde{F}_{l;n}$ is obtained from $F_{l;n}$ by substituting $Y_{l;n'}$ by $\widetilde{Y}_{l;n'}$, $0 \leq n' \leq n$ (see [18, (25)]). In fact, we have applied the definition $\widehat{\mathcal{O}_{X_\infty, P^X_{eE}}} := \varprojlim_m \mathcal{O}_{X_\infty, P^X_{eE}}/(P^X_{eE})^{m+1}$ and also that $P^X_{eE}\mathcal{O}_{X_\infty, P^X_{eE}} = \big(\mathcal{Q} \cup \mathcal{Z}\big)\mathcal{O}_{X_\infty, P^X_{eE}}$ and $\mathcal{O}_{X_\infty} = k[x_1, \ldots, x_d, y_{d+1}, \ldots, y_N]_\infty/\big(\{F_{l;n}\}_{d+1\leq l\leq N, n\geq 0}\big)$. If $\widetilde{Z}_{l,s;n}$ denotes the series obtained from $Z_{l,s;n}$ by substituting $Y_{l;n'}$ by $\widetilde{Y}_{l;n'}$, $0 \leq n' \leq n$, we have

$$\widetilde{Z}_{l,s;n} \in \left(\{X_{j,r;n}\}_{\substack{(j,r)\in\mathcal{J} \\ en_{j,r-1}\overline{\beta}_{j,r-1} \leq n < e\overline{\beta}_{j,r}}}\right) \text{ for } d+1 \leq l \leq N, 0 \leq n \leq e\overline{\alpha}_{l,s}. \tag{3.27}$$

Since $F_{l;0}, \ldots, F_{l;e\epsilon_l-1} \in (\mathcal{Q} \cup \mathcal{Z}_l)^2\kappa(P^{\mathbb{A}^d}_{eE})[[\{X_{j,r;n}\}_{(j,r)\in\mathcal{J}, en_{j,r-1}\overline{\beta}_{j,r-1} \leq n < e\overline{\beta}_{j,r}}]]$ by property (iii) in Proposition 3.2(b), applying (3.27) we conclude that

$$\widetilde{F}_{l;n} \in \left(\{X_{j,r;n}\}_{\substack{(j,r)\in\mathcal{J} \\ en_{j,r-1}\overline{\beta}_{j,r-1} \leq n < e\overline{\beta}_{j,r}}}\right)^2 \quad \text{for } d+1 \leq l \leq N, 0 \leq n \leq e\epsilon_l - 1.$$

Therefore, the images of $\{X_{j,r;n}\}_{(j,r)\in\mathcal{J}, en_{j,r-1}\overline{\beta}_{j,r-1} \leq n < e\overline{\beta}_{j,r}}$ define a basis of $P^X_{eE}\widehat{\mathcal{O}_{X_\infty, P^X_{eE}}}/(P^X_{eE}\widehat{\mathcal{O}_{X_\infty, P^X_{eE}}})^2$. Thus we obtain (3.23), and this finishes the proof. □

**Remark 3.5.** Let $X$ be a reduced separated scheme of finite type over a field $k$ of characteristic zero. Let $P$ be any stable point of $X_\infty$ and suppose that the center $P_0$ of $P$ is not the generic point of $X$. There exists a birational and proper morphism $\pi : Y \to X$ such that the center of $\nu_P$ on $Y$ is a divisor $E$, and $e \in \mathbb{N}$ such that $\nu_P = e\nu_E$ ([18, Prop. 3.7(vii)]; see (v) in Section 2). Let $P^Y \in Y_\infty$ whose image by $\pi_\infty$ is $P$, let $\rho : X \to \mathbb{A}^d_k$ be a general projection and let $P^{\mathbb{A}^d}$ be the image of $P$ in $(\mathbb{A}^d_k)_\infty$. Then $k_E(\mathbb{A}^d) = \widehat{k}_E$, where $\widehat{k}_E$ is the Mather discrepancy of $X$ with respect to $E$, and we have $\dim \mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}} = e\widehat{k}_E + \dim \mathcal{O}_{Y_\infty, P^Y}$ (see (xiii) in Section 2). Recall that $P \supseteq P^X_{eE}$, hence $P^{\mathbb{A}^d} \supseteq P^{\mathbb{A}^d}_{eE}$ and, if $\mathcal{Q}$ is a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$, then $\mathcal{Q} \subset P$. Note that, since $\nu_P = e\nu_E$, the proof of Proposition 3.2 extends to this case, and we obtain that the complete intersection scheme $X'$ and the set $\mathcal{Z}$ defined in Proposition 3.2 for the valuation $\nu_E$ and $e$ also satisfy the properties obtained replacing $P_{eE}$ by $P$ in (i) to (iii) in Proposition 3.2(b). Then from the proof of Theorem 3.4 it follows that

$$\operatorname{embdim} \mathcal{O}_{(X_\infty)_{\mathrm{red}}, P} = \operatorname{embdim} \widehat{\mathcal{O}_{(X_\infty)_{\mathrm{red}}, P}} = e\widehat{k}_E + \dim \mathcal{O}_{Y_\infty, P^Y}.$$

## §4. A lower bound for the dimension

Recall that, given a divisorial valuation $\nu = \nu_E$ on $X$, the Mather–Jacobian log-discrepancy of $X$ with respect to $E$ is defined to be

$$a_{\mathrm{MJ}}(E; X) := \widehat{k}_E - \nu_E(\mathrm{Jac}_X) + 1,$$

where $\mathrm{Jac}_X$ is the Jacobian ideal of $X$ (see [10], [3]).

**Theorem 4.1.** *Assume that* $\operatorname{char} k = 0$. *Let $X$ be a reduced separated $k$-scheme of finite type, let $\nu = \nu_E$ be a divisorial valuation on an irreducible component $X_0$ of $X$, and let $e \in \mathbb{N}$. Then we have*

$$\dim \widehat{\mathcal{O}_{X_\infty, P^X_{eE}}} \geq e a_{\mathrm{MJ}}(E; X).$$

*In particular, if $X$ is normal and a complete intersection then*

$$\dim \widehat{\mathcal{O}_{X_\infty, P^X_{eE}}} \geq e(k_E + 1).$$

*Proof.* It is always possible to embed $X$ in a complete intersection scheme $X'$ such that $\widehat{k}_E(X) = \widehat{k}_E(X')$ and $\nu_E(\mathrm{Jac}_X) = \nu_E(\mathrm{Jac}_{X'})$. Hence, since $\widehat{\mathcal{O}_{(X_\infty), P_{eE}}} \cong \widehat{\mathcal{O}_{(X'_\infty), P^{X'}_{eE}}}$ (see (ii) and (ix) in Section 2), it suffices to prove the result for $X'$. That is, we may assume that $X$ is a complete intersection; more precisely, we may suppose that

$$X = \operatorname{Spec} k[x_1, \dots, x_N]/(f_1, \dots, f_{N-d}).$$

We may also suppose that (3.1) holds, i.e.,

$$\mathrm{ord}_E \pi^*(dx_1 \wedge \cdots \wedge dx_d) = \widehat{k}_E. \tag{3.1}$$

For simplicity in the notation we will prove the result when $e = 1$; the proof when $e > 1$ follows in the same way. Let $\rho : X \to \mathbb{A}^d_k$ be the projection on the first $d$ coordinates, let $\eta : Y \to \mathbb{A}^d_k$ be the composition $\eta = \rho \circ \pi$, let $P_E^{\mathbb{A}^d}$ be the image of $P_E^Y$ by $\eta_\infty$ and let $\mathcal{Q} = \{\overline{Q}_{j,r;n}\}_{(j,r)\in\mathcal{J}, n_{j,r-1}\overline{\beta}_{j,r-1}\leq n\leq\overline{\beta}_{j,r}-1}$ be a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_E^{\mathbb{A}^d}}$ associated to $\{q_{j,r}\}_{(j,r)\in\mathcal{J}}$, as in (IV) in Section 3. We have

$$P_E^X \mathcal{O}_{(X_\infty)_{\mathrm{red}}, P_E^X} = \left(\{\overline{Q}_{j,r;n}\}_{(j,r)\in\mathcal{J}, n_{j,r-1}\overline{\beta}_{j,r-1}\leq n\leq\overline{\beta}_{j,r}-1}\right) \mathcal{O}_{(X_\infty)_{\mathrm{red}}, P_E^X} \tag{4.1}$$

(see Theorem 3.4).

Let us consider the following $(N-d)\times(N-d)$-matrix with coefficients in $k[x_1, \ldots, x_N]$:

$$\Delta := \left(\frac{\partial f_i}{\partial x_{d+j}}\right)_{1\leq i,j\leq N-d},$$

and let $d^{i_1,\ldots,i_r}_{j_1,\ldots,j_r}$ denote the determinant of the $r\times r$-minor of $\Delta$ defined by the rows $i_1, \ldots, i_r$ and the columns $j_1, \ldots, j_r$. After reordering $\{x_{d+j}\}_{j=1}^{N-d}$ we may assume that

$$\nu_E\left(d^{1,\ldots,i}_{1,\ldots,i}\right) = \inf\left\{\nu_E\left(d^{1,\ldots,i-1,i}_{1,\ldots,i-1,j}\right)\right\}_{j=i}^{N-d} \quad \text{for } 1 \leq i \leq N-d. \tag{4.2}$$

For $1 \leq i \leq N-d$ set

$$\delta_i := \nu_E\left(d^{1,\ldots,i}_{1,\ldots,i}\right), \qquad \epsilon_i := \inf\left\{\nu_E\left(\frac{\partial f_i}{\partial x_{d+j}}\right)\right\}_{j=1}^{N-d} = \inf\left\{\nu_E\left(d^i_j\right)\right\}_{j=1}^{N-d}$$

and note that $\delta_1 = \epsilon_1$ and $\delta_{N-d} := \nu_E(\mathrm{Jac}_X)$ by (3.1). It can be proved by induction that, for $1 \leq l \leq N-d$, $l \leq i, j \leq N-d$, we have

$$d^{1,\ldots,l-1,i}_{1,\ldots,l-1,j} \cdot d^{1,\ldots,l-2}_{1,\ldots,l-2} = d^{1,\ldots,l-2,i}_{1,\ldots,l-2,j} \cdot d^{1,\ldots,l-1}_{1,\ldots,l-1} - d^{1,\ldots,l-2,i}_{1,\ldots,l-2,l-1} \cdot d^{1,\ldots,l-2,l-1}_{1,\ldots,l-2,j}. \tag{4.3}$$

Let $f'_{1,i} := \frac{\partial f_1}{\partial x_i}$, $1 \leq i \leq N$; thus $f'_{1,d+i} = d^1_i$, $1 \leq i \leq N-d$. Let $\sum_{n\geq 0} F'_{1,i;n} t^n$ (resp. $\sum_{n\geq 0} D^{i_1,\ldots,i_r}_{j_1,\ldots,j_r;n} t^n$) denote the image of $f'_{1,i}$ (resp. $d^{i_1,\ldots,i_r}_{j_1,\ldots,j_r}$) in $k[x_1, \ldots, x_N]_\infty$. Given $a_1 > \epsilon_1$ and $n > (a_1 - \epsilon_1)$, if we apply Taylor's formula to $f_1(w_0 + t^{n-(a_1-\epsilon_1)} w_1)$, where $w_0 = \sum_{i=0}^{n-(a_1-\epsilon_1)-1} \underline{x}_i t^i$ and $w_1 = \sum_{i\geq n-(a_1-\epsilon_1)} \underline{x}_i t^{i-(n-(a_1-\epsilon_1))}$, we obtain that for $n > n_1 := 2a_1 - \epsilon_1$ (i.e., $2(n - (a_1 - \epsilon_1)) > n + \epsilon_1$) we have

$$F_{1;\epsilon_1+n} = H'_{1;n}(\underline{X}_0, \ldots, \underline{X}_{n-(a_1-\epsilon_1)-1}) + \sum_{i=1}^{N}\sum_{r=0}^{a_1} F'_{1,i;r} X_{i;n+\epsilon_1-r},$$

where $H'_{1,n} \in k[\underline{X}_0, \ldots, \underline{X}_{n-(a_1-\epsilon_1)-1}]$ (see [17, Proof of Thm. 4.1,] or equality (3.22) in Remark 3.3, where the same argument is applied). Hence, there exists a polynomial $H_{1;n} \in k\big[\underline{X}_0, \ldots, \underline{X}_{n-(a_1-\epsilon_1)-1}, \{X_{j;n'}\}_{1\leq j\leq d, n-(a_1-\epsilon_1)\leq n'\leq n+\epsilon_1}\big]$ such that

$$(4.4)\qquad \begin{aligned} F_{1;\epsilon_1+n} = H_{1;n}\Big(&\underline{X}_0, \ldots, \underline{X}_{n-(a_1-\epsilon_1)-1}, \{X_{j;n'}\}_{\substack{1\leq j\leq d\\ n'\leq n+\epsilon_1}}\Big)\\ &+ \sum_{i=1}^{N-d}\sum_{r=\epsilon_1}^{a_1} D^1_{i;r}X_{d+i;n+\epsilon_1-r} \mod \Big(\{D^1_{i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\epsilon_1}}\Big). \end{aligned}$$

It follows that, for $n > n_1$, there exists

$$X^{(1)}_{d+1;n} \in k\left[\{X_{j;n'}\}_{\substack{1\leq j\leq d\\ 0\leq n'\leq n+\epsilon_1}} \cup \{X_{d+1;n'}\}_{0\leq n'\leq n_1} \cup \{X_{d+i;n'}\}_{\substack{2\leq i\leq N-d\\ 0\leq n'\leq n}}\right]_{D^1_{1;\epsilon_1}}$$

such that

$$F_{1;\epsilon_1+n} = D^1_{1;\epsilon_1}(X_{d+1;n} - X^{(1)}_{d+1;n}) \mod \Big(\{D^1_{i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\epsilon_1}} \cup \{F_{1;\epsilon_1+n'}\}_{n_1<n'<n}\Big)$$

in the ring $(k[x_1, \ldots, x_N]_\infty)_{D^1_{1;\epsilon_1}}$. It can be proved by induction that, for $n > n_1 + a_1 - \epsilon_1$, $2 \leq i \leq N-d$ and $0 \leq r \leq a_1 - \epsilon_1$ we have

$$(4.5)\qquad \frac{\partial X^{(1)}_{d+1;n}}{\partial X_{d+i;n-r}} = -\sum_{s=0}^{r} \frac{D^1_{i;\epsilon_1+s}}{D^1_{1;\epsilon_1}}B^1_{r-s} \mod \Big(\{D^1_{i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\epsilon_1}}\Big),$$

where

$$B^1_{r-s} := \sum_{k_1,\ldots,k_m,b_1,\ldots,b_m} (-1)^b \frac{b!}{b_1!\cdots b_m!}\frac{(D^1_{1;\epsilon_1+k_1})^{b_1}\cdots(D^1_{1;\epsilon_1+k_m})^{b_m}}{(D^1_{1;\epsilon_1})^b},$$

with $k_1, \ldots, k_m, b_1, \ldots, b_m$ running over all positive integers satisfying $k_1 < k_2 < \cdots < k_m$ and $\sum_{i=1}^m b_ik_i = r - s$, and $b := \sum_{i=1}^m b_i$.

Analogously, taking $a_2 > \epsilon_2$, applying Taylor's formula to $f_2$ and then replacing $X_{d+1;n'}$ by $X^{(1)}_{d+1;n'}$ for $n' > n_1$, i.e., considering the image $F^{(1)}_{2;\epsilon_2+n}$ of $F_{2;\epsilon_2+n}$ in $k\left[\{X_{j;n'}\}_{1\leq j\leq d, 0\leq n'\leq\epsilon_2+n} \cup \{X_{d+1;n'}\}_{0\leq n'\leq n_1} \cup \{X_{d+i;n'}\}_{2\leq i\leq N-d, 0\leq n'\leq\epsilon_2+n}\right]_{D^1_{1;\epsilon_1}}$, we obtain that for $n \gg 0$, $2 \leq i \leq N-d$, $0 \leq r \leq \inf\{(a_1-\epsilon_1), (a_2-\epsilon_2)\}$, we have

$$(4.6)\qquad \frac{\partial F^{(1)}_{2;\epsilon_2+n}}{\partial X_{d+i;n-r}} = \sum_{s=0}^{r}\frac{D^{1,2}_{1,i;\epsilon_1+\epsilon_2+s}}{D^1_{1;\epsilon_1}}B^1_{r-s} \mod \Big(\{D^1_{i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\epsilon_1}} \cup \{D^2_{i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\epsilon_2}}\Big).$$

In fact, to conclude (4.6), we have to apply Taylor's expansion as in (4.4) and also the identities (4.5). Hence, if $(a_1-\epsilon_1)$ and $(a_2-\epsilon_2)$ are bigger than $(\delta_2-\delta_1-\epsilon_2)$, for $n \gg 0$, $0 \leq r \leq \inf\{(a_1-\epsilon_1)-(\delta_2-\delta_1-\epsilon_2),(a_2-\epsilon_2)-(\delta_2-\delta_1-\epsilon_2)\}$ and $2 \leq i \leq N-d$, we have

$$\frac{\partial F^{(1)}_{2;\delta_2-\delta_1+n}}{\partial X_{d+i,n-r}} = \sum_{s=0}^{r} \frac{D^{1,2}_{1,i;\delta_2+s}}{D^1_{1;\epsilon_1}} B^1_{r-s} \mod \left( \{D^1_{i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\epsilon_1}} \cup \{D^2_{i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\epsilon_2}} \cup \{D^{1,2}_{1,i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\delta_2}} \right).$$

In particular,

$$\frac{\partial F^{(1)}_{2;\delta_2-\delta_1+n}}{\partial X_{d+i,n}} \equiv \frac{D^{1,2}_{1,i;\delta_2}}{D^1_{1;\epsilon_1}} \quad \text{and} \quad \frac{\partial F^{(1)}_{2;\delta_2-\delta_1+n}}{\partial X_{d+i,n'}} \equiv 0 \quad \text{for } n' > n.$$

This implies that there exists $n_2$ such that for $n > n_2$ there exists

$$X^{(1)}_{d+2;n} \in k\left[ \{X_{j;n'}\}_{\substack{1\leq j\leq d\\ n'\leq n+\delta_2-\delta_1}} \cup \{X_{d+i;n'}\}_{\substack{1\leq i\leq 2\\ n'\leq n_i}} \cup \{X_{d+i;n'}\}_{\substack{3\leq i\leq N-d\\ n'\leq n}} \right]_{D^1_{1;\epsilon_1}\cdot D^{1,2}_{1,2;\delta_2}}$$

such that

$$F_{2;\delta_2-\delta_1+n} = \frac{D^{1,2}_{1,2;\delta_2}}{D^1_{1;\epsilon_1}} (X_{d+2;n} - X^{(1)}_{d+2;n}) \mod \Bigg( \{D^j_{i;s}, D^{1,2}_{1,i;s_2}\}_{\substack{1\leq i\leq N-d\\ 1\leq j\leq 2\\ s<\epsilon_j, s_2<\delta_2}} \cup \{F_{1;\epsilon_1+n'}\}^{n+(\delta_2-\delta_1-\epsilon_2)}_{n'=n_1+1} \cup \{F_{2;\delta_2-\epsilon_1+n'}\}_{n_2<n'<n} \Bigg)$$

in the ring $(k[x_1,\ldots,x_N]_\infty)_{D^1_{1;\epsilon_1}\cdot D^{1,2}_{1,2;\delta_2}}$ and

$$\frac{\partial X^{(1)}_{d+2;n}}{\partial X_{d+i;n-r}} = -\sum_{s=0}^{r} \frac{D^{1,2}_{1,i;\delta_2+s}}{D^{1,2}_{1,2;\delta_2}} B^2_{r-s} \mod \left( \{D^j_{i;s}\}_{\substack{1\leq i\leq N-d\\ 1\leq j\leq 2\\ 0\leq s<\epsilon_j}} \cup \{D^{1,2}_{1,i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\delta_2}} \right)$$

for $2 \leq i \leq N-d$ and $0 \leq r \leq \inf\{(a_l-\epsilon_l)-(\delta_l-\delta_{l-1}-\epsilon_l)-\cdots-(\delta_2-\delta_1-\epsilon_2)\}_{1\leq l\leq 2}$, where we set $\delta_0 := 0$ and

$$B^2_{r-s} := \sum_{k_1,\ldots,k_m,b_1,\ldots,b_m} (-1)^b \frac{b!}{b_1!\cdots b_m!} \frac{(D^{1,2}_{1,2;\delta_2+k_1})^{b_1}\cdots(D^{1,2}_{1,2;\delta_2+k_m})^{b_m}}{(D^{1,2}_{1,2;\delta_2})^b},$$

with $k_1,\ldots,k_m,b_1,\ldots,b_m$ positive integers such that $k_1<\cdots<k_m$ and $\sum_{i=1}^m b_ik_i = r-s$, and $b := \sum_{i=1}^m b_i$.

Now let

$$\mathcal{D} := \{D^j_{i;s}\}_{\substack{1\le i,j\le N-d\\ 0\le s<\epsilon_j}} \cup \{D^{1,2}_{1,i;s}\}_{\substack{1\le i\le N-d\\ 0\le s<\delta_2}} \cup \cdots \cup \{D^{1,2,\dots,N-d-1,N-d}_{1,2,\dots,N-d-1,i;s}\}_{\substack{1\le i\le N-d\\ 0\le s<\delta_{N-d}}}$$

and $D_0 := D^1_{1;\epsilon_1} \cdot D^{1,2}_{1,2;\delta_2} \cdot \dots \cdot D^{1,2,\dots,N-d}_{1,2,\dots,N-d;\delta_{N-d}}$. Recall that, by (4.2) and since $\delta_i = \nu_E(d^{1,\dots,i}_{1,\dots,i})$, we have that, for each element in $\mathcal{D}$, its class in $\mathcal{O}_{X_\infty,P^X_E}$ is in $P^X_E$ and also that the class of $D_0$ is a unit in $\mathcal{O}_{X_\infty,P^X_E}$. Proceeding as before, we obtain that, for $1 \le i \le N-d$, given $a_i > \epsilon_i$, there exists $n_i$ such that for $n > n_i$ there exists

$$X^{(1)}_{d+i;n} \in k\left[\{X_{j;n'}\}_{\substack{1\le j\le d\\ 0\le n'\le n+\delta_i-\delta_{i-1}}} \cup \{X_{d+j;n'}\}_{\substack{1\le j\le i\\ 0\le n'\le n_j}} \cup \{X_{d+j;n'}\}_{\substack{i+1\le j\le N-d\\ 0\le n'\le n}}\right]_{D_0}$$

satisfying

$$\begin{aligned} F_{i;\delta_i-\delta_{i-1}+n} = {} & \frac{D^{1,\dots,i}_{1,\dots,i;\delta_i}}{D^{1,\dots,i-1}_{1,\dots,i-1;\delta_{i-1}}}(X_{d+i;n} - X^{(1)}_{d+i;n}) \\ & \operatorname{mod} (\mathcal{D} \cup \{F_{j;\delta_j-\delta_{j-1}+n'}\}_{\substack{1\le j<i\\ n_j<n'<+n+(\delta_i-\delta_{i-1}-\epsilon_i)}} \\ & \cup \{F_{i;\delta_i-\delta_{i-1}+n'}\}_{n_i<n'<n}) \end{aligned} \tag{4.7}$$

in the ring $(k[x_1,\dots,x_N]_\infty)_{D_0}$. In addition, we have

$$\frac{\partial X^{(1)}_{d+i;n}}{\partial X_{d+j;n-r}} = -\sum_{s=0}^{r} \frac{D^{1,\dots,i-1,i}_{1,\dots,i-1,j;\delta_i+s}}{D^{1,\dots,i}_{1,\dots,i;\delta_i}} B^i_{r-s} \quad \operatorname{mod} (\mathcal{D}), \tag{4.8}$$

for $i \le j \le N-d$ and $r \le \inf\{(a_l-\epsilon_l)-(\delta_l-\delta_{l-1}-\epsilon_l)-\cdots-(\delta_i-\delta_{i-1}-\epsilon_i)\}_{1\le l\le i}$, where

$$B^i_{r-s} := \sum_{k_1,\dots,k_m,b_1,\dots,b_m} (-1)^b \frac{b!}{b_1!\cdots b_m!} \frac{(D^{1,\dots,i}_{1,\dots,i;\delta_i+k_1})^{b_1}\cdots(D^{1,\dots,i}_{1,\dots,i;\delta_i+k_m})^{b_m}}{(D^{1,\dots,i}_{1,\dots,i;\delta_i})^b}.$$

Here $k_1,\dots,k_m,b_1,\dots,b_m$ run over all positive integers such that $k_1 < \cdots < k_m$ and $\sum_{i=1}^m b_ik_i = r-s$, and $b := \sum_{i=1}^m b_i$. Note that from (4.8) and applying the equalities (4.3) it follows that for $n \gg 0$, the image $F^{(1)}_{i+1;\delta_{i+1}-\delta_i+n}$ of $F_{i+1;\delta_{i+1}-\delta_i+n}$ in

$$k\left[\{X_{j;n'}\}_{\substack{1\le j\le d\\ 0\le n'\le\epsilon_{i+1}+n}} \cup \{X_{d+j;n'}\}_{\substack{1\le j\le i\\ 0\le n'\le n_j}} \cup \{X_{d+j;n'}\}_{\substack{i+1\le j\le N-d\\ 0\le n'\le n}}\right]_{D_0}$$

satisfies

$$\frac{\partial F^{(1)}_{i+1;\delta_{i+1}-\delta_i+n}}{\partial X_{d+j,n-r}} = \sum_{s=0}^{r} \frac{D^{1,\dots,i,i+1}_{1,\dots,i,j;\delta_{i+1}+s}}{D^{1,\dots,i}_{1,\dots,i;\delta_i}} B^i_{r-s} \quad \operatorname{mod} (\mathcal{D}),$$

for $i+1 \leq j \leq N-d$ and $r \leq \inf\{(a_l-\epsilon_l)-(\delta_l-\delta_{l-1}-\epsilon_l)-\cdots-(\delta_{i+1}-\delta_i-\epsilon_i)\}_{1\leq l\leq i+1}$. This is used in the inductive reasoning. Therefore, taking $a_l > \epsilon_l+(\delta_l-\delta_{l-1}-\epsilon_l)+\cdots+(\delta_{N-d}-\delta_{N-d-1}-\epsilon_{N-d})$ for $1\leq l\leq N-d$, we conclude the existence of $n_i$, $1\leq i\leq N-d$, and $X^{(1)}_{d+i;n}$, $1\leq i\leq N-d$, $n>n_i$, satisfying (4.7) and (4.8).

From the above discussion and arguing by induction on $(m,i,n)$, $m\geq 1, 1\leq i\leq N-d, n\geq n_i+1$, with the lexicographical order, we obtain

$$X^{(m)}_{d+i;n} \in k\left[\{X_{j;n'}\}_{\substack{1\leq j\leq d\\ n'\geq 0}} \cup \{X_{d+j;n'}\}_{\substack{1\leq j\leq N-d\\ 0\leq n'\leq n_j}}\right]_{D_0}$$

satisfying

$$F_{i;\delta_i-\delta_{i-1}+n} = \frac{D^{1,\ldots,i}_{1,\ldots,i;\delta_i}}{D^{1,\ldots,i-1}_{1,\ldots,i-1;\delta_{i-1}}}(X_{d+i;n}-X^{(m)}_{d+i;n})$$
$$\mod (\mathcal{D})^m + \left(\{F_{j;\delta_j-\delta_{j-1}+n'}\}_{\substack{1\leq j\leq N-d\\ n_j<n'}}\right)$$

in $(k[x_1,\ldots,x_N]_\infty)_{D_0}$. Thus we have

$$X^{(m+1)}_{d+i;n} - X^{(m)}_{d+i;n} \in (\mathcal{D})^m + \left(\{F_{j;\delta_j-\delta_{j-1}+n'}\}_{\substack{1\leq j\leq N-d\\ n_j<n'}}\right).$$

Recall (4.1) and that the image of $\mathcal{D}$ in $\mathcal{O}_{X_\infty,P^X_E}$ is in $P^X_E$. Fix an embedding $\kappa(P^X_E)\hookrightarrow \widehat{\mathcal{O}_{X_\infty,P^X_E}}$ sending $X_{j;n}$ to $X_{j;n}\in \widehat{\mathcal{O}_{X_\infty,P^X_E}}$, for $1\leq j\leq d, n\geq m_j$ (see the proof of Theorem 3.4). Then, for $1\leq i\leq N-d$ and $n>n_i$, the polynomials $\{X^{(m)}_{d+i;n}\}_{m\geq 1}$ define a series

$$\widetilde{X}_{d+i;n} \in \kappa(P)\left[\left[\{X_{j,r;n}\}_{\substack{(j,r)\in\mathcal{J}\\ n_{j,r-1}\overline{\beta}_{j,r-1}\leq n<\overline{\beta}_{j,r}}} \cup \{X_{d+j;n'}-\overline{X}_{d+j;n'}\}_{\substack{1\leq j\leq N-d\\ 0\leq n'\leq n_j}}\right]\right],$$

where we identify $X_{j,r;n}$ with $\overline{Q}_{j,r;n}$, as in the proof of Theorem 3.4, and where $\overline{X}_{d+j;n'}\in\widehat{\mathcal{O}_{X_\infty,P^X_E}}$ is the image of the class of $X_{d+j;n'}$ in $\kappa(P^X_E)$, for $1\leq j\leq N-d, 0\leq n'\leq n_j$. Setting $Y_{d+j;n'} := X_{d+j;n'}-\overline{X}_{d+j;n'}$, $1\leq j\leq N-d, 0\leq n'\leq n_j$, we conclude that $\widehat{\mathcal{O}_{X_\infty,P^X_E}}$ is isomorphic to

$$\kappa(P^X_{eE})\left[\left[\{X_{j,r;n}\}_{\substack{(j,r)\in\mathcal{J}\\ n_{j,r-1}\overline{\beta}_{j,r-1}\leq n<\overline{\beta}_{j,r}}} \cup \{Y_{d+j;n'}\}_{\substack{1\leq j\leq N-d\\ n'\leq n_j}}\right]\right] \Big/ \left(\{\widetilde{F}_{j;n}\}_{\substack{1\leq j\leq N-d\\ n\leq\delta_j-\delta_{j-1}+n_j}}\right),$$

where for $1\leq j\leq N-d$, $0\leq n\leq \delta_j-\delta_{j-1}+n_j$, $\widetilde{F}_{j;n}$ is obtained from $F_{j;n}$ by substituting $X_{d+i;n'}$ by $\widetilde{X}_{d+i;n'}$, for $1\leq i\leq N-d$ and $n_i<n'\leq n$, and $X_{d+j;n'}$

by $\overline{X}_{d+j;n'} + Y_{d+j;n'}$ for $1 \leq j \leq N-d$, $0 \leq n' \leq n_j$. Applying Krull's theorem we obtain that

$$\begin{aligned}\dim \widehat{\mathcal{O}_{X_\infty, P^X_{eE}}} &\geq \widehat{k}_E + 1 + \sum_{i=1}^{N-d}(n_i+1) - \sum_{i=1}^{N-d}(\delta_i - \delta_{i-1} + n_i + 1) \\ &= \widehat{k}_E + 1 - \delta_{N-d} = a_{\mathrm{MJ}}(E).\end{aligned}$$

Finally, if $X$ is normal and a complete intersection, we have $a_{\mathrm{MJ}}(E) = k_E + 1$ ([5, Appendix]). Hence we conclude the result. □

Recall that, given an extension of fields $k \subseteq K$, a $K$-*wedge* on $X$ is a $k$-morphism $\operatorname{Spec} K[[\xi, t]] \to X$; equivalently, it is a $K$-arc on $X_\infty$ (see (2.1)). Given a birational and proper $k$-morphism $p : Y \to X$ and a stable point $P$ of $X_\infty$, we say that $p$ *satisfies the property of lifting wedges centered at* $P$ if, for any field extension $K$ of the residue field $\kappa(P)$ of $P$ in $X_\infty$, and for any $K$-wedge $\phi : \operatorname{Spec} K[[\xi, t]] \to X$ on $X$ whose special arc is $P$ (i.e., $P$ is the image in $X_\infty$ of the closed point of $\operatorname{Spec} K[[\xi]]$), there exists a $K$-wedge $\widetilde{\phi} : \operatorname{Spec} K[[\xi, t]] \to Y$ on $Y$ such that $p \circ \widetilde{\phi} = \phi$.

In [18, Cor. 5.12], it is proved that, if $\nu = \nu_E$ is an essential divisorial valuation on $X$, then, the following are equivalent:

(i) $\dim \widehat{\mathcal{O}_{X_\infty, P^X_E}} = 1$ and $\operatorname{Spec} \widehat{\mathcal{O}_{X_\infty, P^X_E}}$ is irreducible.

(ii) $\dim \mathcal{O}_{X_\infty, P^X_E} = 1$.

(iii) For every resolution of singularities $p : Y \to X$, $p$ satisfies the property of lifting wedges centered at $P^X_E$.

(iii′) There exists a resolution of singularities $p : Y \to X$ that satisfies the condition in (iii), and such that the center of $\nu$ on $Y$ has codimension 1.

De Fernex and Docampo [7] have proved that, if $\nu_E$ is a terminal valuation then condition (iii) above holds. In fact, this follows from [7, Proof of Thm. 1.1]. Note that their statement in Thm. 1.1 is weaker than condition (iii) (see [17, Thm. 5.1] or [18, Sect. 5]). Terminal valuations are the divisorial valuations defined by the exceptional divisors of a minimal model of $X$, hence they are essential (see [7]).

From this and Theorem 4.1 above, Corollaries 4.2 and 4.4 below follow.

**Corollary 4.2.** *Let $X$ be a reduced separated scheme of finite type over a field $k$ of* $\operatorname{char} k = 0$. *Let $\nu = \nu_E$ be an essential divisorial valuation on an irreducible component $X_0$ of $X$. Consider the following conditions:*

*(1)* $\nu_E$ *is a terminal valuation.*

*(2)* $\dim \widehat{\mathcal{O}_{X_\infty, P_E^X}} = 1$.

*(3)* $a_{\mathrm{MJ}}(E;X) \leq 1$, *in particular* $k_E(X) \leq 0$ *if* $X$ *is normal and a complete intersection.*

*We have that (1) implies (2) and (2) implies (3).*

The following example shows that (2) does not imply (1). It has been pointed out to us by M. Mustata.

**Remark 4.3.** In [7, Exa. 6.3], the toric variety $X$ defined by the cone $\sigma$ in $\mathbb{R}^3$ spanned by the vectors $(1,0,0)$, $(0,1,0)$ and $(1,1,2)$ is considered, and the divisorial valuation $\nu_E$ defined by $(1,1,1)$, which is not a terminal valuation. It can be proved that $\dim \widehat{\mathcal{O}_{X_\infty, P_E^X}} = 1$. In this case we have $\widehat{k}_E(X) = 2$ and $\nu_E(\mathrm{Jac}_X) = 3$, hence $a_{\mathrm{MJ}}(E;X) = 0$.

**Corollary 4.4.** *Let* $X$ *be a reduced separated scheme of finite type over a field* $k$ *of* $\operatorname{char} k = 0$. *Suppose that* $X$ *is normal and a complete intersection. Let* $\nu = \nu_E$ *be an essential divisorial valuation on an irreducible component* $X_0$ *of* $X$ *and suppose that* $k_E \geq 1$. *Then, for every resolution of singularities* $p : Y \to X$ *such that the center of* $\nu$ *on* $Y$ *has codimension* 1, $p$ *does not satisfy the property of lifting wedges centered at* $P_E$, *i.e., there exist a field extension* $K$ *of* $\kappa(P_E)$ *and a* $K$*-wedge* $\phi : \operatorname{Spec} K[[\xi, t]] \to X$ *on* $X$ *whose special arc is* $P_E$ *and which does not lift to* $Y$.

## Acknowledgements

We are grateful to Monique Lejeune-Jalabert for so many enlightening discussions over so many years. We thank O. Piltant for his suggestions and comments.

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