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Free and Nearly Free Curves vs. Rational Cuspidal Plane Curves

by

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Abstract

We define a class of plane curves that are close to the free divisors in terms of the local cohomology of their Jacobian algebras and such that, conjecturally, any rational cuspidal curve C is either free or belongs to this class. We prove this conjecture when the degree of C is either even or a prime power, or when the group of C is abelian.

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§1. Introduction

Let f be a homogeneous polynomial in the polynomial ring $S = \mathbb{C}[x, y, z]$ and denote by f_x, f_y, f_z the corresponding partial derivatives. Let C be the plane curve in $\mathbb{P}^2$ defined by $f = 0$ and assume that C is reduced and not a union of lines passing through one point. We denote by J_f the Jacobian ideal of f, i.e., the homogeneous ideal of S spanned by the partial derivatives f_x, f_y, f_z, and let $M(f) = S/J_f$ be the corresponding graded ring, called the Jacobian (or Milnor) algebra of f. Let I_f denote the saturation of the ideal J_f with respect to the maximal ideal $\mathbf{m} = (x, y, z)$ in S and recall the relation with the 0-degree local cohomology $I_f/J_f = H^0_{\mathbf{m}}(M(f))$. Consider the graded S-submodule $AR(f) \subset S^3$ of *all Jacobian relations* involving the derivatives of f, namely

$$\rho = (a, b, c) \in AR(f)_m,$$

if and only if $af_x + bf_y + cf_z = 0$ and a, b, c are in S_m.

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One says that $C : f = 0$ is a free curve, and uses the notation $f \in \mathcal{F}$, if the graded S-module $AR(f)$ is free; see Definition 2.2. Freeness in the local analytic setting was introduced by K. Saito in [29] and has attracted a lot of interest in recent decades; see for instance [5], [27] or the long reference list in [35]. For a discussion of free curves in the projective setting, see for instance [33], [34], [35], [14]. Note that the projective curve $C : f = 0$ is free if and only if the surface singularity given by the cone over C at the origin, i.e., $(f^{-1}(0), 0)$, is free as a divisor in $\mathbb{C}^3$.

We say that $C : f = 0$ is a nearly free curve, and use the notation $f \in \mathcal{N}F$, if the Hilbert function of the graded module I_f/J_f takes only the values 0 and 1; see Definition 2.4 for an equivalent property stated in terms of minimal resolutions for the S-module $AR(f)$. The nearly free curves have many properties similar to the free ones. Here are some examples.

(1) If we fix the minimal degree $r = \mathrm{mdr}(f)$ of a Jacobian syzygy for f (see Definition 2.1(iii) below), then the global Tjurina number $\tau(C)$ satisfies the inequality
$$\tau(C) \leq \phi(d, r) := (d-1)^2 - r(d-1-r),$$
and equality holds if and only if $f \in \mathcal{F}$; see [21]. Moreover, one has
$$\tau(C) = \phi(d, r) - 1$$
if and only if $f \in \mathcal{N}F$; see [9].

(2) Both the free curves and the nearly free curves can be constructed using pencils of curves or points of high multiplicity on a given curve; see [23], [10]. These constructions can lead to families of nearly free curves degenerating to a free curve, exactly as a secant degenerates to a tangent.

(3) There is an efficient algorithm computing the Alexander polynomial of a plane curve $C : f = 0$, as soon as $f \in \mathcal{F} \cup \mathcal{N}F$; see [16].

The study of free and nearly free surfaces in $\mathbb{P}^3$ can be done along similar lines, and has led to a number of interesting results, in spite of additional technical difficulties; see [8], [17].

In this paper we consider the relation between the free and nearly free curves, and the rational cuspidal curves, i.e., rational curves having only unibranch singularities. This latter class of curves is extremely rich, as the classification results show, both from the algebraic point of view (see [25], [26], [31]) and from the topological viewpoint (see [24]). This richness makes the following conjecture rather surprising in our opinion.

Conjecture 1.1. Any rational cuspidal curve C in the plane is either free or nearly free.

In [19] we showed that a large number of the rational cuspidal curves as classified in [24], [25], [26], [31] give examples of irreducible free divisors. The class of nearly free curves seems to be the smallest class defined by homological properties similar to those enjoyed by a free curve, such that Conjecture 1.1 holds. Indeed, we have checked that this class contains all the nonfree rational cuspidal curves listed in the papers mentioned above; see [20]. Using deep Hodge-theoretic results by M. Saito [30] and Walther [39], we prove Conjecture 1.1 for all curves of even degree; see Theorem 3.1, which is our main result. The proof also implies that this conjecture holds for a curve C with an abelian fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ or having a prime power as degree; see Corollary 3.2.

Moreover, any unicuspidal rational curve with a unique Puiseux pair is shown to be either free or nearly free (see Corollary 3.5), except the curves of odd degree in one case of the classification of such unicuspidal curves obtained in [24] and recalled in Theorem 3.4 below. In this case our methods do not apply, since we need, when the degree is odd, an additional topological assumption on the cusps which is not always fulfilled. A very interesting discussion of Conjecture 1.1 can be found in [30].

The present paper is a major revision and updating of our preprint [20], which contains more examples and additional conjectures, disproved in the recent paper by Artal Bartolo, Gorrochategui, Luengo and Melle-Hernández [3]. This paper [3] contains many interesting new examples of free and nearly free curves, with arbitrarily high genera and a large numbers of singularities. A code written in SINGULAR, which can be used to decide whether a given homogeneous polynomial defines a free or a nearly free curve in the projective plane $\mathbb{P}^2$, is available at http://math1.unice.fr/~dimca/singular.html.

§2. The definition and the first properties of nearly free divisors

With the notation from introduction, we set $m(f)_k = \dim M(f)_k$ for any integer k. We recall the definition of some invariants associated with a Milnor algebra $M(f)$; see [15].

Definition 2.1. For a reduced plane curve $C : f = 0$ of degree d, three integers are defined as follows:

(i) the *coincidence threshold*

$$\mathrm{ct}(f) = \max\{q : \dim M(f)_k = \dim M(f_s)_k \text{ for all } k \le q\},$$

with f_s a homogeneous polynomial in S of degree d such that $C_s : f_s = 0$ is a smooth curve in $\mathbb{P}^2$;

(ii) the *stability threshold* $\mathrm{st}(f) = \min\{q : \dim M(f)_k = \tau(C) \text{ for all } k \geq q\}$;

(iii) the *minimal degree of a nontrivial (or essential) syzygy*,

$$\mathrm{mdr}(f) = \min\{q : H^2(K^*(f))_{q+2} \neq 0\},$$

where $K^*(f)$ is the Koszul complex of f_x, f_y, f_z with natural grading.

Note that one has, for $j < d-1$, the equality

$$AR(f)_j = H^2(K^*(f))_{j+2}. \tag{2.1}$$

It is known that one has

$$\mathrm{ct}(f) = \mathrm{mdr}(f) + d - 2. \tag{2.2}$$

Let $T = 3(d-2)$ denote the degree of the socle of the Gorenstein ring $M(f_s)$.

Definition 2.2. The reduced curve $C : f = 0$ is a free divisor if the graded S-module $AR(f)$ is free of rank 2, i.e., there is an isomorphism of graded S-modules

$$AR(f) = S(-d_1) \oplus S(-d_2)$$

for some positive integers $d_1 \leq d_2$.

One has the following well-known result.

Proposition 2.3. For a reduced curve $C : f = 0$ in $\mathbb{P}^2$, the following conditions are equivalent:

(1) The curve $C : f = 0$ is a free divisor.

(2) The minimal resolution of the Milnor algebra $M(f)$ has the form

$$0 \to S(-d_1 - d + 1) \oplus S(-d_2 - d + 1) \to S^3(-d+1) \xrightarrow{(f_x, f_y, f_z)} S, \tag{2.3}$$

for some positive integers d_1, d_2.

(3) The Jacobian ideal J_f is saturated, i.e.,

$$I_f/J_f = H^0_{\mathbf{m}}(M(f)) = 0. \tag{2.4}$$

Proof. Since $AR(f)$ is the kernel of the morphism $S^3(-d+1) \xrightarrow{(f_x, f_y, f_z)} S$, it follows that (1) is equivalent to (2). Let $T\langle C\rangle$ (resp. $\mathcal{J}$) be the coherent sheaf on $\mathbb{P}^2$ associated to the graded S-module $AR(f)(1)$ (resp. to the ideal J_f). Then we have an exact sequence of sheaves on $\mathbb{P}^2$ as in [14, Sect. 3], induced by the sequence (2.3) above,

$$0 \to T\langle C\rangle \to \mathcal{O}^3_{\mathbb{P}^2}(1) \to \mathcal{J}(d) \to 0.$$

If C is free, then the vector bundle $T\langle C\rangle$ is the sum of two line bundles, and it follows as in [34, Prop. 2.1] that

$$(I_f/J_f)_k = H^0_{\mathbf{m}}(M(f))_k = H^1(\mathbb{P}^2, T\langle C\rangle(k-d)) = 0,$$

for any integer k. Conversely, the above vanishing implies that $T\langle C\rangle$ is the sum of two line bundles using Horrocks' theorem as explained in [14, Rem. 4.7]. Since $T\langle C\rangle$ is the sum of two line bundles exactly when $AR(f)$ is a free S-module, this shows that (2) is equivalent to (3). □

When C is a free divisor, the integers $d_1 \le d_2$ in Definition 2.2 are called the exponents of C, and clearly $\mathrm{mdr}(f) = d_1$. They satisfy the relations

$$d_1 + d_2 = d - 1 \text{ and } \tau(C) = (d-1)^2 - d_1 d_2, \tag{2.5}$$

where $\tau(C)$ is the total Tjurina number of C, i.e., $\tau(C) = \sum_{i=1}^p \tau(C, x_i)$, the x_i's being the singular points of C; see for instance [14], [19].

The S-graded quotient module $N(f) = I_f/J_f$ is called the Jacobian module in [39]. The class of curves introduced in this article is defined by imposing the condition that the Jacobian module $N(f)$ is nonzero, but as small as possible degree by degree.

Definition 2.4. A reduced plane curve $C : f = 0$ is said to be *nearly free* if $N(f) \neq 0$ and $\dim N(f)_k \le 1$ for any k.

Remark 2.5. In the local analytic setting, Damon introduced the notion of an almost free divisor; see [5]. It seems that there is no direct relation between this notion and nearly free curves. Indeed, the line arrangement $C : f = xyz(x+y+z) = 0$ discussed in Example 2.14 is nearly free, and the corresponding cone singularity $(f^{-1}(0), 0)$ is almost free; see [27]. On the other hand, a smooth curve of degree > 2 is not nearly free, but the corresponding surface singularity $(f^{-1}(0), 0)$ is almost free by [27, Defn. 2.1].

The first result says that a nearly free divisor has a minimal resolution of $M(f)$ slightly more complicated than that of a free divisor.

Theorem 2.6. *The reduced curve $C : f = 0$ is a nearly free divisor if and only if the minimal resolution of the Milnor algebra $M(f)$ has the form*

$$0 \to S(-d-d_2) \to S(-d-d_1+1) \oplus S^2(-d-d_2+1) \to S^3(-d+1) \xrightarrow{(f_x, f_y, f_z)} S$$

for some integers $1 \le d_1 \le d_2$, called the exponents of C.

Proof. As a first step, we show that a nearly free curve $C : f = 0$ has a minimal resolution of the Milnor algebra $M(f)$ of the form

$$(2.6) \qquad 0 \to S(-b-2(d-1)) \to \oplus_{i=1,3} S(-d_i-(d-1)) \to S^3(-d+1) \to S,$$

for some integers $1 \le d_1 \le d_2 \le d_3$ and b.

By the Hilbert syzygy theorem (see [22, p. 3]), we know that the minimal resolution has the form

$$0 \to F_3 \to F_2 \to F_1 = S^3(-d+1) \to F_0 = S,$$

where $F_3 = \oplus S(b_j)$ and $F_2 = \oplus S(a_i)$ are graded free S modules, with a_i, b_j finite sets of integers. It is easy to see that $\operatorname{rank} F_2 = \operatorname{rank} F_3 + 2$, since $M(f)$ has a constant Hilbert function, $m(f)_k = \tau(C)$ for k large enough, by [4]. Hence we have to show only that $\operatorname{rank} F_3 = 1$.

If $F_i = \oplus S(-j)^{\beta_{i,j}}$, then the positive integers $\beta_{i,j}$ are called the graded Betti numbers of $M(f)$. They are denoted by $\beta_{i,j}(M(f))$ and they satisfy

$$\beta_{i,j}(M(f)) = \dim \operatorname{Tor}_i(M(f), \mathbb{C})_j;$$

see [22, p. 8]. Since $\operatorname{Tor}(M(f), \mathbb{C}) = \operatorname{Tor}(\mathbb{C}, M(f))$ and the graded S-module $\mathbb{C}$ has a minimal resolution given by the homological Koszul complex $C(x, y, z; S)$ of x, y, z in S, it follows that

$$\operatorname{Tor}_i(M(f), \mathbb{C}) = H_i(C(x, y, z; M(f))),$$

where $C(x, y, z; M(f))$ denotes the homological Koszul complex of x, y, z in $M(f)$. The third differential in the complex $C(x, y, z; M(f))$ being given essentially by multiplication by x, y and z, it follows that $H_3(C(x, y, z; M(f)))$ is exactly the socle $s(M(f))$ of the module $M(f)$, i.e.,

$$s(M(f)) = \{m \in M(f) : xm = ym = zm = 0\}.$$

It follows that $\beta_{3,j}(M(f)) = \dim s(M(f))_j$. On the other hand, it is clear that $s(M(f)) = s(N(f))$, since we have the equality $N(f) = H^0_{\mathbf{m}}(M(f))$, the local cohomology of $M(f)$ with respect to the maximal ideal $\mathbf{m} = (x, y, z)$; see [34]. The module $N(f)$ is a module with a self-dual resolution; see [36]. If we set $\beta_i(M(f)) = \sum_j \beta_{i,j}(M(f))$ and similarly for $N(f)$, this implies

$$\beta_3(M(f)) = \beta_3(N(f)) = \beta_0(N(f)).$$

In other words, $\beta_3(M(f)) = 1$, which is our claim, is equivalent to $\beta_0(N(f)) = 1$, i.e., $N(f)$ is a cyclic S-module. But this last claim is a direct consequence of [12, Cor. 4.3], which says that $N(f)$ has a Lefschetz-type property in the following sense: if $\ell \in S_1$ is a generic linear form, then the morphism $N(f)_k \to N(f)_{k+1}$ induced by the multiplication by ℓ has maximal rank for any k. By our assumption, $\dim N(f)_j \leq 1$ for any j, and it follows that $N(f)$ is generated as an S-module by any nonzero homogeneous element of minimal degree in $N(f)$. This completes the first step in the proof.

To construct a resolution (2.6) for a given polynomial f we need the following ingredients:

(A) three relations $r_i = (a_i, b_i, c_i) \in S^3_{d_i}$ for $i = 1, 2, 3$ among f_x, f_y, f_z, i.e.,

$$a_i f_x + b_i f_y + c_i f_z = 0,$$

necessary to construct the morphism

$$\oplus_{i=1,3} S(-d_i - (d-1)) \to S^3(-d+1)$$

by the formula

$$(u_1, u_2, u_3) \mapsto u_1 r_1 + u_2 r_2 + u_3 r_3;$$

(B) one relation $R = (v_1, v_2, v_3) \in \oplus_{i=1,3} S(-d_i - (d-1))_{b+2(d-1)}$ among r_1, r_2, r_3, i.e., $v_1 r_1 + v_2 r_2 + v_3 r_3 = 0$, necessary to construct the morphism

$$S(-b - 2(d-1)) \to \oplus_{i=1,3} S(-d_i - (d-1))$$

by the formula $w \mapsto wR$. Note that $v_i \in S_{b-d_i+d-1}$.

In the second step of the proof of Theorem 2.6, consider the graded dual of the resolution (2.6) twisted by (-3), namely,

$$S(-3) \to S^3(d-4) \to F_2' = \oplus S(-a_i - 3) \to F_3' = S(b + 2d - 5) \to 0.$$

Then the cokernel Q of the morphism

$$\delta : \oplus_i S(d_i + d - 4) \to S(b + 2d - 5)$$

is the graded dual of $N(f) = H^0_{\mathbf{m}}(M(f))$ by [22, Thm. A 1.9, p. 193]. It follows that

$$Q = S/(v_1, v_2, v_3)(b + 2d - 5),$$

where v_1, v_2, v_3 are the homogeneous polynomials from (B) above. This quotient is finite-dimensional if and only if v_1, v_2, v_3 is a regular sequence, and then the Hilbert function of the quotient depends only on the degrees of the v_i's. In particular, we can take $v_1 = x^p$, $v_2 = y^q$ and $v_3 = z^r$ for some integers p, q, $r > 0$, when the quotient is the tensor product of the rings $\mathbb{C}[x]/(x^p)$, $\mathbb{C}[y]/(y^q)$, $\mathbb{C}[z]/(z^r)$. Hence, to get at most one-dimensional homogeneous components, we need to have (up to a permutation) $\deg v_2 = \deg v_3 = 1$ and $\deg v_1 = k$ a positive integer. It follows that $b - d_i + d - 1 = 1$ for $i = 2, 3$, and hence $d_2 = d_3 = b + d - 2 = d_1 + 2d_2 - d$, which implies $d_1 + d_2 = d$. The equality $b - d_1 + d - 1 = k$ yields our first claim.

Conversely, if the minimal resolution of the Milnor algebra $M(f)$ has the form given in Theorem 2.6, then we get that C is nearly free as follows. It is known that $N(f)$ is a finite-dimensional $\mathbb{C}$-vector space in general, as it follows from the general discussion in [22, pp. 187–188]. Hence the forms v_1, v_2, v_3 above define a complete intersection. This implies $\dim N(f)_k \leq 1$ for any k. □

Remark 2.7. (i) It follows from [9, Thm. 4.1] that the reduced curve $C : f = 0$ is nearly free if and only if we can find three relations r_1, r_2 and r_3 as in (A) above such that $d_1 = r$, $d_2 = d_3 = d - r$ with $r = \mathrm{mdr}(f) \leq d - r$ and r_2, r_3 linearly independent in $(AR(f)/Sr_1)_{d-r}$.

(ii) If the S-module $AR(f)$ has a minimal set of homogeneous generators r_i such that for two of them, say r_1 and r_2, one has $\deg r_1 + \deg r_2 \leq d - 1$, then the degree-$d$ curve $C : f = 0$ is free; see [35, Lem. 1.1]. Such a result does not extend to nearly free curves: one may have a curve $C : f = 0$ that is not nearly free, and such that a pair (r_1, r_2) in a minimal set of generators for $AR(f)$ satisfies $\deg r_1 + \deg r_2 \leq d$. To have an example, consider the curve $C : x(xy(x + y) + z^3) = 0$, a smooth cubic together with an inflectional tangent. Then the S-module $AR(f)$ has a minimal set of generators consisting of 3 elements of degree $d_1 = d_2 = 2$ and $d_3 = 3$.

Theorem 2.8. *Suppose the curve $C : f = 0$ has a minimal resolution for $M(f)$ as in (2.6) with $d_1 \leq d_2 \leq d_3$. Then one has the following properties:*

(i) $d_1 + d_2 \geq d$, $b = \sum_{i=1,3} d_i - 2(d - 1)$ *and*

$$\tau(C) = (d - 1) \sum_{i=1,3} d_i - \sum_{i<j} d_i d_j.$$

Moreover, $\mathrm{mdr}(f) = d_1$, $\mathrm{ct}(f) = d_1 + d - 2$ *and* $\mathrm{st}(f) = b + 2d - 4$.

(ii) If the curve $C : f = 0$ is nearly free, then one has $d_1 + d_2 = d$ and

$$\tau(C) = (d - 1)^2 - d_1(d_2 - 1) - 1.$$

Moreover, in this case $\mathrm{st}(f) = d_2 + d - 2$ *and* $\mathrm{ct}(f) + \mathrm{st}(f) = T + 2$.

Proof. First we prove the part (i). The claims for $\mathrm{mdr}(f)$ and $\mathrm{ct}(f)$ are obvious. The relation r_2 in (A) above is not a multiple of r_1 (otherwise the resolution (2.6) is not minimal). Then the claim $d_1 + d_2 \geq d$ follows from [35, Lem. 1.1].

For any integer j, the resolution (2.6) yields an exact sequence

$$(2.7) \qquad 0 \to S_{j-b-2(d-1)} \to \oplus_{i=1,3} S_{j-d_i-(d-1)} \to S^3_{j-d+1} \to S_j \to M(f)_j \to 0.$$

For large j, this gives the equality $\tau(C) = P(d, d_1, d_2, d_3, b; j)$, where the polynomial $P(j) = P(d, d_1, d_2, d_3, b; j) \in \mathbb{Q}[j]$ is defined by

$$P(j) = \binom{j+2}{2} - 3\binom{j-d+3}{2} + \sum_{i=1,2,3} \binom{j-d_i-d+3}{2} - \binom{j-b-2d+4}{2}.$$

The coefficient of j^2 in the right-hand side is zero, the vanishing of the coefficient of j gives the formula for b, while the constant term gives the value for $\tau(C)$, since $m(f)_j = \tau(C)$ for large j by [4]. For the claim on $\mathrm{st}(f)$, note that for $j \geq b+2d-4$ the above binomial coefficients give the dimensions of the corresponding spaces in the exact sequence (2.7), but not for $j = b+2d-5$, when the last term is 1 instead of being $0 = \dim S_{-1}$, due to the equality

$$\binom{-1}{2} = (-1)(-2)/2 = 1.$$

When C is nearly free, the equality $b = d_1 + d_2$ was obtained at the end of the proof of Theorem 2.6. The remaining claims follow by direct computation. □

Remark 2.9. Some people have suggested that a nearly free curve should be what we call either a free curve or a nearly free curve. Though this option is very reasonable, we have chosen our convention in view of the following situation (in fact, similar situations for other invariants, e.g., $\tau(C)$, occur again and again in this theory). If C is a free (or nearly free) curve of degree d with exponents $d_1 \leq d_2$, then $d = d_1+d_2+1$ when C is free, but $d = d_1+d_2$ when C is nearly free. Hence in order to know which formula to use, our convention seems simpler in our opinion.

Another illustration of our choice of terminology is the following.

Remark 2.10. It is shown in [9] that $f \in \mathcal{F}$ (resp. $f \in \mathcal{N}F$) if and only if $\mathrm{ct}(f) + \mathrm{st}(f) = T$, (resp. $\mathrm{ct}(f) + \mathrm{st}(f) = T + 2$).

The first natural question is whether such nearly free curves exist. The following examples show that the answer is positive. Consider first the simplest rational cuspidal curve of degree d (it is the unique rational curve up to projective equivalence with a unique cusp that is weighted homogeneous), namely $C_d : f_d = x^d + y^{d-1}z = 0$.

Proposition 2.11. The curve $C_d : f_d = x^d + y^{d-1}z = 0$ is nearly free for $d \geq 2$ and $\mu(C) = \tau(C) = (d-1)(d-2)$. Moreover, $(d_1, d_2) = (1, d-1)$, $\dim N(f)_k = 1$ for $d-2 \leq k \leq 2d-4$ and $N(f)_k = 0$ otherwise.

Proof. The polynomial f_d is of Sebastiani–Thom type, i.e., the variables are separated into two groups. It follows that $M(f) = M(x^d) \otimes M(y^{d-1}z)$. Moreover, we have $N(f) = N(x^d) \otimes N(y^{d-1}z)$, and hence the graded module $N(f)$ has the following monomial basis $y^{d-2}, xy^{d-2}, \ldots, x^{d-2}y^{d-2}$. This completes the proof. $\square$

Exactly the same approach gives a similar result for a family having two singularities.

Proposition 2.12. The curve $C_d : f_d = x^d - y^{d-k}z^k = 0$ for $d \geq 4$ and $2 \leq k \leq d/2$ is nearly free, has two singularities and $\tau(C) = (d-1)(d-2)$. In addition, $(d_1, d_2) = (1, d-1)$, $\dim N(f)_j = 1$ for $d-2 \leq j \leq 2d-4$ and $N(f)_j = 0$ otherwise. The number of irreducible components is given by the greatest common divisor $\gcd(d, k)$ and each of them is a rational curve.

Example 2.13. In degree $d = 3$, consider a conic plus a secant line, e.g., $C : x^3 + xyz = 0$. Then C is nearly free with the resolution for $M(f)$ of the form

$$0 \to S(-5) \to S(-3) \oplus S(-4)^2 \to S(-2)^3 \to S,$$

and hence $(d_1, d_2) = (1, 2)$. On the other hand, the nodal cubic $C : f = xyz + x^3 + y^3 = 0$ is not nearly free, since $\dim N(f)_1 = 2$.

In degree $d = 4$ we consider first the quartic with 3 cusps, e.g.,

$$C : x^2y^2 + y^2z^2 + x^2z^2 - 2xyz(x + y + z) = 0.$$

Then C is nearly free with the minimal resolution for $M(f)$ of the form

$$0 \to S(-6) \to S(-5)^3 \to S(-3)^3 \to S,$$

and hence $(d_1, d_2) = (2, 2)$. The quartic $C : z^4 - xz^3 - 2xyz^2 + x^2y^2 = 0$ (resp. $C : y^4 - 2xy^2z + yz^3 + x^2z^2 = 0$) has 2 cusps of type A_2 and A_4 (resp. 1 cusp of type A_6), and it is nearly free with the same resolution for $M(f)$ as for the 3-cuspidal quartic.

Example 2.14. We give now an example of line arrangement that is nearly free. Let $C : f = xyz(x + y + z) = 0$ be the union of 4 lines in general position. Then C is a nearly free curve with the resolution for $M(f)$ given by

$$0 \to S(-6) \to S(-5)^3 \to S(-3)^3 \to S,$$

with $(d_1, d_2) = (2, 2)$, $\dim N(f)_3 = 1$ and $N(f)_k = 0$ for other k's.

Remark 2.15. In fact, we have shown that for all pairs (d_1, d_2) with $d_1 + d_2 = d$, there is a rational unicuspidal curve $C : f = 0$ (resp. a line arrangement $\mathcal{A} : f = 0$) such that f has degree d and C (resp. $\mathcal{A}$) is nearly free with exponents (d_1, d_2); see [18] for the precise statement.

Recall the formula

$$\dim N(f)_k = m(f)_k + m(f)_{T-k} - m(f_s)_k - \tau(C), \tag{2.8}$$

which is part of [13, Cor. 3], but can be also obtained as follows. Note that $\dim N(f)_k = m(f)_k - \dim(S/I_f)_k = m(f)_k - (\tau(C) - \mathrm{def}_k\Sigma_f)$, with $\mathrm{def}_k\Sigma_f$ the defect of the singular subscheme Σ_f with respect to degree-k polynomials, and $\mathrm{def}_k\Sigma_f = m(f)_{T-k} - m(f_s)_k$; for all this see [7]. Formula (2.8) and Theorem 2.8 also yield the following.

Corollary 2.16. *The fact that a reduced curve C is nearly free depends only on the dimensions $m(f)_k$ of the homogeneous components of the Milnor algebra $M(f)$. Conversely, two nearly free curves $C_1 : f_1 = 0$ and $C_2 : f_2 = 0$ with the same degrees and the same total Tjurina numbers have the same exponents. In particular, they satisfy $m(f_1)_k = m(f_2)_k$ for any k.*

Note that the other invariants of two such curves C_1 and C_2 can be quite different; see Proposition 2.12. Formula (2.8) and Theorem 2.8 also give the following related result. Recall that projective rigidity is equivalent to $N(f)_d = 0$; see [34].

Corollary 2.17. *Let $C : f = 0$ be a nearly free curve of degree d with exponents (d_1, d_2). Then $N(f)_k \neq 0$ for $d+d_1-3 \leq k \leq d+d_2-3$ and $N(f)_k = 0$ otherwise. The curve C is projectively rigid if and only if $d_1 \geq 4$.*

Proposition 2.18. Consider the reduced plane curve $C : f = 0$ of degree d. Then $\dim N(f)_{[T/2]} = 1$ if and only if $f \in \mathcal{N}F$.

Proof. If $N(f) \neq 0$, then there is a nonnegative integer $n(C)$, $0 \leq n(C) \leq [T/2]$, such that $N(f)_k = 0$ for $k < n(C)$ or $k > T - n(C)$, and the remaining dimensions $\dim N(f)_k$ form a unimodal sequence of strictly positive numbers (see [12, Cor. 4.3]), symmetric with respect to the middle point $[T/2]$; see [13] and [34]. □

We end this section with a remark on nearly free line arrangements. It is known that for a free arrangement, the characteristic polynomial has a nice factorization in terms of the exponents; see [40] for an excellent survey. For a nearly free line arrangement we have the following similar result.

Proposition 2.19. Let C be a nearly free arrangement of d lines in $\mathbb{P}^2$ with exponents (d_1, d_2) and denote by U its complement. Then the characteristic polynomial

$$\chi(U)(t) = t^2 - b_1(U)t + b_2(U)$$

is given by

$$\chi(U)(t) = (t - d_1)(t - (d_2 - 1)) + 1.$$

Proof. For any line arrangement one has $b_1(U) = d-1$; hence in our case $b_1(U) = d_1 + d_2 - 1$. Note that $E(U) = E(\mathbb{P}^2) - E(C) = 3 + d^2 - 3d - \mu(C)$. It follows that

$$b_2(U) = E(U) + d - 2 = (d-1)^2 - \mu(C).$$

Since all the singularities of C are homogeneous, it follows that $\mu(C) = \tau(C)$ and using the formula for $\tau(C)$ given in Theorem 2.8 we get $b_2(U) = d_1(d_2-1)+1$. □

Example 2.20. (i) For the line arrangement in Example 2.14 we have $d_1 = d_2 = 2$ and hence $\chi(U)(t) - 1 = (t-1)(t-2)$.

(ii) Consider the line arrangement $C : f = xyz(x-y)(x-2y)(x-3y)(x+5y+7z) = 0$, which occurs essentially in [28, Exa. 4.139] as an illustration of the fact that $\chi(U)(t)$ can factor even for nonfree arrangements. It turns out that this curve is a nearly free arrangement with exponents $(d_1, d_2) = (2,5)$ and hence

$$\chi(U)(t) = (t-2)(t-4) + 1 = (t-3)^2.$$

Remark 2.21. It is known that two line arrangements $\mathcal{A}_i : f_i = 0$, $i = 1,2$ with the same combinatorics can have the distinct invariants $r_i = \mathrm{mdr}(f_i)$; see [32], [42]. However, these arrangements are far from being free or nearly free. On the other hand, in many cases, one can determine the invariant $r = \mathrm{mdr}(f)$ from the combinatorics of a line arrangement, and prove Terao's conjecture in special cases; see [23], [10] and the references there. Moreover, a similar property to Terao's conjecture holds for nearly free line arrangements; see [10, Cor. 1.7].

§3. Local and global Milnor fiber monodromies

We state now the main result of our paper.

Theorem 3.1. *Let $C : f = 0$ be a rational cuspidal curve of degree d. Assume that either*

(1) d is even, or

(2) d is odd and for any singularity x of C, the order of any eigenvalue λ_x of the local monodromy operator h_x is not d.

Then C is either a free or a nearly free curve.

For examples of rational cuspidal curves not satisfying the assumption in this theorem, see case (3) with d odd in Theorem 3.4 below.

Proof. The results in Walther [39, Thm. 4.3] and M. Saito [30, Thm. A.1, Appendix] yield the inequality

$$\dim N(f)_{2d-2-j} \leq \dim H^2(F, \mathbb{C})_\lambda,$$

with $j = 1, 2, \ldots, d$, where $F : f(x, y, z) - 1 = 0$ is the Milnor fiber associated to the plane curve C or, equivalently, to the surface singularity $f = 0$, and the subscript λ indicates the eigenspace of the monodromy action corresponding to the eigenvalue $\lambda = \exp(2\pi i(d + 1 - j)/d)$.

Suppose first that d is even, say $d = 2d_1$. In view of Proposition 2.18, it is enough to show that $\dim H^2(F, \mathbb{C})_\lambda = 1$ for $\lambda = -1$, which corresponds to $j = d_1 + 1$. To show this, note first that [2, Prop. 2.1] implies $H^1(F, \mathbb{C})_\lambda = 0$. Denote again by U the complement $\mathbb{P}^2 \setminus C$, and note that its topological Euler characteristic is given by $E(U) = E(\mathbb{P}^2) - E(C) = 1$. Since F is a cyclic d-fold covering of U, it follows that $H^1(F)_1 = H^1(U)$ and also

$$\dim H^2(F, \mathbb{C})_\lambda - \dim H^1(F, \mathbb{C})_\lambda + \dim H^0(F, \mathbb{C})_\lambda = 1;$$

see for instance [6, Prop. 1.21, Chap. 4] or [11, Cor. 5.1 and Rem. 5.1]. Since clearly $H^0(F, \mathbb{C})_\lambda = 0$, we get $\dim H^2(F, \mathbb{C})_\lambda = 1$ as claimed. The last three equalities are valid for any eigenvalue $\neq 1$ of prime power order, but in view of Proposition 2.18 it is enough to consider only the eigenvalue $\lambda = -1$.

Suppose now that d is odd, say $d = 2d_1 + 1$. By Proposition 2.18 again, it is enough to show that $\dim H^2(F, \mathbb{C})_\lambda = 1$, for $\lambda = \exp(2\pi i d_1/(2d_1 + 1))$, which corresponds to $j = d_1 + 2$. As this eigenvalue λ clearly has order d, we conclude as in the previous case. Indeed, it is known that the Alexander polynomial of C, which is the characteristic polynomial of the monodromy action on $H^1(F, \mathbb{C})$, divides the product of the Alexander polynomials of the singularities (C, x) of C; see for instance [6, Cor. (6.3.29)]. □

Corollary 3.2. *Let $C : f = 0$ be a rational cuspidal curve of degree d such that*

(1) either $d = p^k$ is a prime power, or

(2) $\pi_1(U)$ is abelian, where $U = \mathbb{P}^2 \setminus C$.

Then C is either a free or a nearly free curve.

Proof. When d is an odd prime power, the vanishing $H^1(F, \mathbb{C})_\lambda = 0$, with λ as above in the degree-d, odd case, goes back to Zariski [41]; see also [2, Prop. 2.1]. When $\pi_1(U)$ is abelian, which actually means $\pi_1(U) = H_1(U) = \mathbb{Z}/d\mathbb{Z}$, it follows that $\pi_1(F) = 0$, as the Milnor fiber F is a Galois degree-d cover of U. This implies $H^1(F, \mathbb{C}) = 0$, which completes the proof as above. □

A lot of examples of rational cuspidal curves C with an abelian fundamental group $\pi_1(U)$ can be found in the papers [1], [37], [38]. Here is one example of an infinite family of such curves.

Example 3.3. Let C be a rational cuspidal curve of type $(d, d-2)$ having 3 cusps. Then there exists a unique pair of integers a, b, $a \geq b \geq 1$ with $a+b = d-2$ such that up to projective equivalence the equation of C can be written in affine coordinates (x, y) as

$$f(x,y) = \frac{x^{2a+1}y^{2b+1} - ((x-y)^{d-2} - xyg(x,y))^2}{(x-y)^{d-2}} = 0,$$

where $d \geq 4$, $g(x,y) = y^{d-3}h(x/y)$ and

$$h(t) = \sum_{k=0,d-3} \frac{a_k}{k!}(t-1)^k,$$

with $a_0 = 1$, $a_1 = a - \frac{1}{2}$ and $a_k = a_1(a_1-1)\cdots(a_1-k+1)$ for $k > 1$; see [26]. Then it follows from [1, Cor. 1], that $\pi_1(U)$ is abelian if $2a+1$ and $2b+1$ are relatively prime. It follows by Corollary 3.2 that all these curves are either free or nearly free. In [19, Exa. 4.3] we have checked by direct computation that these curves (without any condition on $2a+1$ and $2b+1$) are actually free for $5 \leq d \leq 10$.

To get further examples, including curves of odd degree, we recall now the classification of unicuspidal rational curves with a unique Puiseux pair; see [24, Thm. 1.1].

Theorem 3.4. *Let a_i be the Fibonacci numbers with $a_0 = 0$, $a_1 = 1$, $a_{j+2} = a_{j+1} + a_j$. A Puiseux pair (a, b) can be realized by a unicuspidal rational curve of degree $d \geq 3$ if and only if the triple (a, b, d) occurs in the following list:*

(1) $(d-1, d, d)$*;*

(2) $(d/2, 2d-1, d)$ *with d even;*

(3) $(a_{j-2}^2, a_j^2, a_{j-2}a_j)$ *with $j \geq 5$ odd;*

(4) (a_{j-2}, a_{j+2}, a_j) *with $j \geq 5$ odd;*

(5) $(3, 22, 8)$*;*

(6) $(6, 43, 16)$.

Corollary 3.5. *A unicuspidal rational curve with a unique Puiseux pair not of the type (3) above, with $d = a_{j-2}a_j$ odd, is either free or nearly free.*

Numerical experiments suggest that unicuspidal rational curves with a unique Puiseux pair of type (3) above, with $d = a_{j-2}a_j$ odd, are also either free or nearly free, but our method of proof does not work in this case.

Proof. Recall that a cusp with a unique Puiseux pair (a, b) has the same monodromy eigenvalues as the cusp $u^a + v^b = 0$. It follows that the order k_x of any such eigenvalue λ_x should be a divisor of ab, but not a divisor of either a or b. This remark combined with Theorem 3.1 settles cases (1), (2), (5) and (6) in Theorem 3.4, as well as case (3) when the degree d is even. For case (4), we use Catalan's identity

$$a_j^2 - a_{j-2}a_{j+2} = (-1)^j,$$

and conclude that $k_x = 1$ for any eigenvalue λ_x that is a dth root of unity. □

The same idea as that in the proof of Theorem 3.1 gives the following, to be compared with Corollary 2.17.

Proposition 3.6. Let $C : f = 0$ be a rational cuspidal curve of degree d. Then $N(f)_k = 0$ for $k \leq d-3$ or $k \geq 2d-3$ and $\mathrm{st}(f) \leq 2d-3$.

Propositions 2.11 and 2.12 show that this vanishing result is sharp.

Proof. As in the proof of Theorem 3.1 we have an inequality

$$\dim N(f)_{2d-3} \leq \dim H^2(F, \mathbb{C})_1,$$

and $H^2(F, \mathbb{C})_1 = H^2(U, \mathbb{C})$, where $U = \mathbb{P}^2 \setminus C$. Since $E(U) = E(\mathbb{P}^2) - E(C) = 3 - 2 = 1$ and $b_0(U) = 1$, $b_1(U) = 0$ (since C is irreducible), it follows that $b_2(U) = 0$. Hence $N(f)_{2d-3} = 0$ and using the Lefschetz-type property of the Jacobian module $N(f)$ (see [12, Cor. 4.3]), it follows that $N(f)_k = 0$ for any $k \geq 2d-3$. To end the proof of the vanishing claim, it is enough to use the self-duality of the graded module $N(f)$ (see [13] or [34]), which yields $\dim N(f)_k = \dim N(f)_{T-k}$.

To prove the claim on the stability threshold $\mathrm{st}(f)$, one uses formula (2.8). □

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