Publ. RIMS Kyoto Univ. **54** (2018), 213–243
DOI 10.4171/PRIMS/54-2-1

The Core of an Extended Affine Lie Superalgebra (A Characterization)

by

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Abstract

We characterize a class of root-graded Lie superalgebras containing a locally finite basic classical simple Lie superalgebra. As a by-product, we characterize the core and the core modulo the center of an extended affine Lie superalgebra whose root system is of finite type $X \neq A(m,n)$, $B(m,n)$, $BC(m,n)$, $C(m,n)$.

2010 Mathematics Subject Classification: 17B67, 17B70, 17B60.
Keywords: Extended affine Lie superalgebras, root-graded Lie superalgebras.

§1. Introduction

In 1990, Høegh-Krohn and Torresani [14] introduced irreducible quasi-simple Lie algebras as a generalization of both affine Lie algebras and finite-dimensional simple Lie algebras over complex numbers. In 1997, the authors in [1] systematically studied irreducible quasi-simple Lie algebras under the name "extended affine Lie algebras". Since then, different generalizations of extended affine Lie algebras have been studied: toral-type extended affine Lie algebras [7], locally extended affine Lie algebras [19] and invariant affine reflection algebras [22], which is also a generalization of the latter two stated classes, are examples of these generalizations. The most important question after introducing a new class of Lie (super)algebras is how to realize and construct the members of that class. There have been several attempts to realize and construct extended affine Lie algebras; see for example

Communicated by H. Nakajima. Received March 23, 2016. Revised November 9, 2016; January 9, 2017; April 16, 2017; June 2, 2017; June 23, 2017.

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[3], [4], [5] and [22]. The main ingredient in constructing extended affine Lie algebras and invariant affine reflection algebras is a root-graded Lie algebra [22, §6]. Root-graded Lie algebras have been studied in [12], [21], [2], [11] and [26].

Root-graded Lie superalgebras were defined in [21] and [8] in two different approaches. One works with Lie superalgebras that are graded by the root lattice of a locally finite root system and satisfy the modified properties of a root-graded Lie algebra [21]; the other works with a Lie superalgebra $\mathcal{L}$ containing a finite-dimensional basic classical simple Lie superalgebra with a Cartan subalgebra with respect to which $\mathcal{L}$ has a weight space decomposition satisfying certain properties [8]. In [29], root-graded Lie superalgebras were defined in a general setting including both mentioned classes of root-graded Lie superalgebras. The characterization of root-graded Lie (super)algebras is of great importance due to some interaction of these Lie superalgebras with some other classes of Lie (super)algebras; e.g., they play a crucial role in constructing invariant affine reflection algebras as well as extended affine Lie superalgebras; see [22], [27] and [29]. Locally finite basic classical simple Lie superalgebras, i.e., direct unions of finite-dimensional basic classical simple Lie superalgebras, are examples of extended affine Lie superalgebras. Extended affine Lie superalgebras were introduced in [27] and the properties of their root systems have been studied. Now a natural question comes up: What is the realization of extended affine Lie superalgebras? In [29], the second author of this article proves that starting from a specific root-graded Lie superalgebra, one can get an extended affine Lie superalgebra using a prescribed way, but an important question is how we can realize all extended affine Lie superalgebras. From a work under preparation, we know that to realize extended affine Lie superalgebras, one needs a set of data whose main part forms root-graded Lie superalgebras satisfying certain properties. These specific root-graded Lie superalgebras appear as the cores modulo the center of extended affine Lie superalgebras. So to realize extended affine Lie superalgebras, we need to realize their cores modulo the center and for this, we need to know the characterization of the specific root-graded Lie superalgebras mentioned above. To this end, we first characterize root-graded Lie superalgebras and then figure out those appearing as the centerless cores of extended affine Lie superalgebras.

The intrinsic features of root-graded Lie superalgebras strongly depend on the type of the root systems involved. This offers a type-by-type approach in order to get the characterization of root-graded Lie superalgebras. In [29], the author gives the characterization for type $BC(I,J)$. In a series of papers, Benkart and Elduque studied root-graded Lie superalgebras with finite-dimensional basic classical simple Lie superalgebras of types $C(n)$, $D(m,n)$, $D(2,1;\alpha)$, $F(4)$, $G(3)$, $A(m,n)$ and $B(m,n)$ as the grading subalgebras; see [8], [9] and [10]. In this work, we study root-graded Lie superalgebras graded by $C(J)$ and $D(I,J)$; recognition theorems for types $\dot{A}(I,J)$ and $B(I,J)$ need separate work.

More precisely, in this paper, we characterize a class of root-graded Lie superalgebras containing a locally finite basic classical simple Lie superalgebra. Moreover, for an additive abelian group Λ, we give the structure of a root-graded Lie superalgebra $\mathcal{L}$ of this class which is equipped with a Λ-grading compatible with the root grading on $\mathcal{L}$. We also prove that the core and the core modulo the center of an extended affine Lie superalgebra with a self-centralizing Cartan subalgebra whose root system is of finite type $X \neq A(m,m)$, $BC(m,n)$, $C(m,n)$, contains a finite-dimensional basic classical simple Lie superalgebra; this in particular allows us to characterize the core and the core modulo the center of all such extended affine Lie superalgebras.

§2. Preliminaries

In this paper, *we assume that the reader is familiar with the theory of finite-dimensional basic classical simple Lie superalgebras.* Throughout the paper, for a set S, by $|S|$ we mean the cardinality of S and for a subset S of an abelian group, by $\langle S\rangle$ we mean the subgroup generated by S. We also denote the center of a Lie superalgebra $\mathcal{L}$ by $Z(\mathcal{L})$ and for two symbols i,j, by $\delta_{i,j}$ we mean the Kronecker delta. Moreover, we always assume $\mathbb{F}$ is an algebraically closed field of characteristic 0 and unless otherwise mentioned, we consider all vector spaces over $\mathbb{F}$. We take $\mathbb{Z}_2 := \{0,1\}$ to be the unique abelian group of order 2. We denote the dual space of a vector space V by V^*. We denote the degree of a homogenous element v of a superspace by $|v|$ and make a convention that if in an expression, we use $|v|$ for an element v of a superspace, by default we have assumed v is homogeneous. We denote the set of all linear automorphisms of a vector space V by $\mathrm{GL}(V)$ and the set of all linear endomorphisms of a superspace $\mathcal{V}$ by $\mathrm{end}_{\mathbb{F}}(\mathcal{V})$. We denote the ends of proofs and definitions by $\square$ and $\diamond$ respectively. Also, the ends of remarks and examples are denoted by $\diamondsuit$.

There is more standard notation used in this paper, which we collect in the last section as an appendix. In the Appendix, we also gather some known terminology and facts needed throughout the paper.

We recall that a basic classical simple Lie superalgebra is either a finite-dimensional simple Lie algebra or one of the Lie superalgebras $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1,\alpha)$, $F(4)$ or $G(3)$. By the usual convention, we denote the root system of a finite-dimensional basic classical simple Lie superalgebra of type X again by X. Now if in types $A(m,n)$, $B(m,n)$, $C(n)$ and $D(m,n)$, we let the integers m, n be extended to each cardinal number, we will get some new Lie superalgebras as well as new root systems. In what follows we will discuss it in detail.

Suppose U is a vector space with a basis $\{\eta_1,\eta_2,\eta_3\}$. For $\lambda\in\mathbb{F}\setminus\{0,-1\}$, define the symmetric nondegenerate bilinear form $(\cdot,\cdot)$ on U by the linear extension

$$\begin{aligned}(\eta_1,\eta_1):=\lambda,\quad(\eta_2,\eta_2):=-1-\lambda,\quad(\eta_3,\eta_3):=1,\\ (\eta_i,\eta_j):=0\quad(1\leq i\neq j\leq 3).\end{aligned}\tag{2.1}$$

Define

$$D(2,1;\lambda):=\{0,\pm2\eta_i,\pm\eta_1\pm\eta_2\pm\eta_3\mid 1\leq i\leq 3\}.\tag{2.2}$$

Next suppose I and J are two index sets with $I\cup J\neq\emptyset$ and F is a vector space with a basis $\{\epsilon_i,\delta_j\mid i\in I,\ j\in J\}$. Define a symmetric bilinear form $(\cdot,\cdot):F\times F\longrightarrow\mathbb{F}$ by

$$(\epsilon_i,\epsilon_r):=\delta_{i,r},\quad(\delta_j,\delta_s):=-\delta_{j,s}\quad\text{and}\quad(\epsilon_i,\delta_j):=0;\quad i,r\in I,\ j,s\in J.$$

Set

$$\begin{aligned}
\dot A(I,I)&:=\pm\big\{\epsilon_i-\epsilon_r,\delta_i-\delta_r,\epsilon_i-\delta_r-\tfrac{1}{\ell}\textstyle\sum_{k\in I}(\epsilon_k-\delta_k)\mid i,r\in I\big\}\\
&\qquad(\ell:=|I|\in\mathbb{Z}^{\geq2}),\\
\dot A(I,J)&:=\pm\big\{\epsilon_i-\epsilon_r,\delta_j-\delta_s,\epsilon_i-\delta_j\mid i,r\in I,\ j,s\in J\big\}\\
&\qquad(|I|\neq|J|\text{ if }I,J\text{ are finite sets}),\\
B(I,J)&:=\pm\big\{\epsilon_i,\delta_j,\epsilon_i\pm\epsilon_r,\delta_j\pm\delta_s,\epsilon_i\pm\delta_j\mid i,r\in I,\ j,s\in J,\ i\neq r\big\},\\
C(I,J)&:=\pm\big\{\epsilon_i\pm\epsilon_r,\delta_j\pm\delta_s,\epsilon_i\pm\delta_j\mid i,r\in I\ ,j,s\in J\big\},\\
D(I,J)&:=\pm\big\{\epsilon_i\pm\epsilon_r,\delta_j\pm\delta_s,\epsilon_i\pm\delta_j\mid i,r\in I,\ j,s\in J,\ i\neq r\big\},\\
BC(I,J)&:=\pm\big\{\epsilon_i,\delta_j,\epsilon_i\pm\epsilon_r,\delta_j\pm\delta_s,\epsilon_i\pm\delta_j\mid i,r\in I,\ j,s\in J\big\},\\
F(4)&:=\pm\big\{0,\epsilon_1,\delta_i\pm\delta_j,\delta_i,\tfrac12(\epsilon_1\pm\delta_1\pm\delta_2\pm\delta_3)\mid 1\leq i\neq j\leq 3\big\}\\
&\qquad(I=\{1\},\ J=\{1,2,3\}),\\
G(3)&:=\pm\big\{0,\delta_1,2\delta_1,\epsilon_i-\epsilon_j,2\epsilon_i-\epsilon_j-\epsilon_k,\delta_1\pm(\epsilon_i-\epsilon_j)\mid\{i,j,k\}=\\
&\qquad\{1,2,3\}\big\}\quad(I=\{1,2,3\},\ J=\{1\}),
\end{aligned}\tag{2.3}$$

in which if I or J is empty, the corresponding indices disappear. We mention that the $\mathbb{F}$-spans of all these sets are F except for $\dot A(I,J)$, so to denote this type, we use $\dot A$ instead of A. When $I=\{1\}$ and J is a nonempty index set, we denote $D(I,J)$ by $C(J)$, so we have

$$C(J)=\{0,\pm\delta_j\pm\delta_s,\pm\epsilon_1\pm\delta_j\mid j,s\in J\}.\tag{2.4}$$

In the sequel if either I or J is a finite set, we may replace it by its cardinality in each type, e.g., we may denote $B(I,J)$ by $B(|I|,|J|)$ if I and J are finite sets.

Using this convention, $C(1)$ can be identified with $\dot{A}(1,2)$ and $D(2,1)$ is nothing but the root system of the finite-dimensional basic classical simple Lie superalgebra $D(2,1;\alpha)$ for $\alpha = 1$. We draw the attention of readers to the point that our notation has a minor difference from the notation in the literature: more precisely, $C(n)$ for $n \in \mathbb{Z}^{\geq 1}$ and $\dot{A}(m,n)$ for $m,n \in \mathbb{Z}^{\geq 1}$ in our sense are denoted by $C(n+1)$ and $A(m-1,n-1)$ respectively in the literature. We choose this notation in order to switch smoothly from the finite case to the infinite case.

Although $\dot{A}(I,J)$ does not span F, the restriction of the form $(\cdot,\cdot)$ to $\mathrm{span}_{\mathbb{F}}\, \dot{A}(I,J)$ is nondegenerate, i.e.,

(2.5) each of the sets $\dot{A}(I,J)$, $B(I,J)$, $C(J)$, $D(I,J)$, $BC(I,J)$, $D(2,1;\lambda)$, $F(4)$ or $G(3)$ is a spanning subset of a vector space equipped with a nondegenerate symmetric bilinear form.

Moreover, if R is one of these sets with the corresponding bilinear form $(\cdot,\cdot)$ on its $\mathbb{F}$-span, we set

$$R_{\mathrm{re}} := \{\alpha \in R \mid (\alpha,\alpha) \neq 0\} \cup \{0\} \quad \text{and} \quad R_{\mathrm{ns}} := (R \setminus R_{\mathrm{re}}) \cup \{0\}.$$

One can easily check that R_{re} is a locally finite root system (see the Appendix for the terminology); more precisely, we have Table 1.

Definition 2.1. Suppose that V is a vector space equipped with a symmetric bilinear form $(\cdot,\cdot)$.

(i) Assume S is one of the sets in the first column of Table 1, V_S is its $\mathbb{F}$-span and $(\cdot,\cdot)_S$ is its corresponding nondegenerate symmetric bilinear form. A subset R of V is called an *irreducible locally finite root supersystem of type S* if there is a nonzero scalar r and a linear isomorphism $\varphi : V \longrightarrow V_S$ such that $\varphi(R) = S$, $(u,v) = r(\varphi(u),\varphi(v))_S$ for all $u,v \in V$.

(ii) A subset R of V is called a *locally finite root supersystem* if either it is a locally finite root system in V or there is a class $\{V_i \mid i \in I\}$ of mutually orthogonal subspaces of V with respect to the form $(\cdot,\cdot)$ such that for $R_i := R \cap V_i$ and $(\cdot,\cdot)_i := (\cdot,\cdot)\mid_{V_i \times V_i}$ $(i \in I)$, we have the following:

- $V = \oplus_{i\in I} V_i$;
- $R = \cup_{i\in I} R_i$;
- for $i \in I$, R_i together with V_i and $(\cdot,\cdot)_i$ is an irreducible root supersystem of one of the types mentioned above.

A locally finite root supersystem that is finite is called a *finite root supersystem*. We mention that each locally finite root supersystem is in fact a direct union of finite root supersystems.

Table 1.

Type of R	R_{re}	Type of R_{re}
$\dot{A}(\ell,\ell)$, $\ell \geq 2$	$\pm\{\epsilon_i - \epsilon_r, \delta_j - \delta_s \mid 1 \leq i, r, j, s \leq \ell\}$	$\dot{A}_\ell \oplus \dot{A}_\ell$
$\dot{A}(I,J)$, $\lvert I\rvert, \lvert J\rvert < \infty$, $\lvert I\rvert \neq \lvert J\rvert$	$\pm\{\epsilon_i - \epsilon_r, \delta_j - \delta_s \mid i, r \in I,\ j, s \in J\}$	$\dot{A}_I \oplus \dot{A}_J$ ($\dot{A}_I$ if $J = \emptyset$)
$\dot{A}(I,J)$, $\lvert I \cup J\rvert = \infty$	$\pm\{\epsilon_i - \epsilon_r, \delta_j - \delta_s \mid i, r \in I,\ j, s \in J\}$	$\dot{A}_I \oplus \dot{A}_J$ ($\dot{A}_I$ if $J = \emptyset$)
$B(I,J)$	$\pm\{\epsilon_i, \delta_j, \epsilon_i \pm \epsilon_r, \delta_j \pm \delta_s \mid i, r \in I,\ j, s \in J,\ i \neq r\}$	$B_I \oplus BC_J$
$C(I,J)$	$\pm\{\epsilon_i \pm \epsilon_r, \delta_j \pm \delta_s \mid i, r \in I,\ j, s \in J\}$	$C_I \oplus C_J$
$D(I,J)$, $\lvert I\rvert \geq 3$, $\lvert J\rvert \geq 1$	$\pm\{\epsilon_i \pm \epsilon_r, \delta_j \pm \delta_s \mid i, r \in I,\ j, s \in J,\ i \neq r\}$	$D_I \oplus C_J$
$C(J)$=$D(1,J)$, $\lvert J\rvert \geq 2$	$\pm\{\delta_j \pm \delta_s \mid j, s \in J\}$	C_J
$D(2,J)$, $\lvert J\rvert \geq 2$	$\pm\{\epsilon_1 \pm \epsilon_2, \delta_j \pm \delta_s \mid j, s \in J,\ i \neq r\}$	$A_1 \oplus A_1 \oplus C_J$
$BC(I,J)$	$\pm\{\epsilon_i, \delta_j, \epsilon_i \pm \epsilon_r, \delta_j \pm \delta_s \mid i, r \in I,\ j, s \in J\}$	$BC_I \oplus BC_J$
$F(4)$	$\pm\{0, \epsilon_1, \delta_i \pm \delta_j, \delta_i \mid 1 \leq i \neq j \leq 3\}$	$A_1 \oplus B_3$
$G(3)$	$\pm\{0, \delta_1, 2\delta_1, \epsilon_i - \epsilon_j, 2\epsilon_i - \epsilon_j - \epsilon_k \mid \{i,j,k\} = \{1,2,3\}\}$	$BC_1 \oplus G_2$
$D(2,1;\lambda)$	$\pm\{2\eta_1, 2\eta_2, 2\eta_3\}$	$A_1 \oplus A_1 \oplus A_1$

To emphasize the underlying vector space and the form, we sometimes may say that $(V, (\cdot,\cdot), R)$ is a locally finite root supersystem. We denote the radical of the form $(\cdot,\cdot)$ by V^0 and for a locally finite root supersystem $R \subseteq V$, set

$$R^0 := R \cap V^0, \quad R^\times := R \setminus R^0,$$
$$R_{\mathrm{re}}^\times := \{\alpha \in R \mid (\alpha,\alpha) \neq 0\}, \quad R_{\mathrm{re}} := R_{\mathrm{re}}^\times \cup \{0\},$$
$$R_{\mathrm{ns}}^\times := \{\alpha \in R \setminus R^0 \mid (\alpha,\alpha) = 0\}, \quad R_{\mathrm{ns}} := R_{\mathrm{ns}}^\times \cup \{0\}.$$

We define the *Weyl group* of a locally finite root supersystem R of V to be the subgroup of $\mathrm{GL}(V)$ generated by

$$r_\alpha : V \longrightarrow V,$$
$$v \mapsto v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha \quad (v \in V),$$

for $\alpha \in R_{\mathrm{re}}^\times$. ◇

Remark 2.2. For an irreducible locally finite root supersystem $(V, (\cdot,\cdot), R)$, one can easily check the following:

(1) The restriction of the elements of the Weyl group of R to R_{re} forms the Weyl group of the locally finite root system R_{re}.

(2) If $\alpha \in R_{\mathrm{ns}}^{\times}$ and S is an irreducible component of R_{re}, then $(\alpha, S) \neq \{0\}$; see (2.3) and Table 1.

(3) For $\alpha, \beta \in R$ and an element w of the Weyl group of R,

$$(w\alpha, w\beta) = (\alpha, \beta);$$

in particular, R_{re} and R_{ns} are invariant under the Weyl group.

(4) For $\alpha, \beta \in R_{\mathrm{ns}}$ with $(\alpha, \beta) \neq 0$, we have either $\alpha + \beta \in R$ or $\alpha - \beta \in R$; moreover, only for types $\dot{A}(2,2)$, $C(I,J)$ and $BC(I,J)$ are there are nonorthogonal roots α, $\beta \in R_{\mathrm{ns}}$ with $\alpha + \beta$, $\alpha - \beta \in R$. ♢

Definition 2.3. A *root base* for a locally finite root supersystem R is a subset Π of R such that each element of R can be uniquely written as a $\mathbb{Z}$-linear combination of the elements of Π whose coefficients are all nonnegative or all nonpositive. *Positive roots* (resp. *negative roots*) are those nonzero elements of R written as a $\mathbb{Z}$-linear combination of the elements of Π whose coefficients are all nonnegative (resp. nonpositive). We say that a root base Π satisfies the *partial sum property* if each positive root can be written as $\beta_1 + \beta_2 + \cdots + \beta_n$ with $\beta_1, \ldots, \beta_n \in \Pi$ and $\beta_1 + \cdots + \beta_t \in R$ for all $t \in \{1, \ldots, n\}$. ⋄

Remark 2.4. If S is an irreducible finite root supersystem, either of type $D(2,1,\alpha)$ or of type $G(3)$, a straightforward verification shows that the so-called *distinguished base* (in the sense of [13, Tables 3.52–3.58]) of S satisfies the partial sum property. Moreover, looking at a finite root supersystem S of type $X \neq D(2,1,\lambda)$, $G(3)$, $C(m,n)$, $BC(m,n)$ accurately, one gets that S is a subset of a finite root system $\mathfrak{s}$ whose standard base Π is a distinguished base for S as well; moreover, as bases for a finite root system satisfy the partial sum property [17, §10], one can easily check that Π as a subset of S, satisfies the partial sum property. ♢

In what follows we introduce some Lie superalgebras which are in fact direct unions of finite-dimensional basic classical simple Lie superalgebras; root systems of these Lie superalgebras are locally finite root supersystems. Let us first fix notation: for a superset $I = I_0 \cup I_1$, by $\mathfrak{pl}_{\mathbb{F}}(I_0, I_1)$ we mean the set of all $(I \times I)$-supermatrices with finitely many nonzero entries; see the Appendix.

- $\mathfrak{sl}(J_0, J_1)$: Suppose that J is an infinite superset with $J_0 \neq \emptyset$. Set

$$\mathcal{G} := \mathfrak{sl}(J_0, J_1) = \{X \in \mathfrak{pl}_{\mathbb{F}}(J_0, J_1) \mid \mathrm{str}(X) = 0\}$$

and $\mathcal{H} := \operatorname{span}_{\mathbb{F}}\{e_{i,i} - e_{r,r}, e_{j,j} - e_{s,s}, e_{i,i} + e_{j,j} \mid i, r \in J_0,\ j, s \in J_1\}$ in which the $e_{j,j} - e_{s,s}$'s and $e_{i,i} + e_{j,j}$'s disappear if $J_1 = \emptyset$. For $t \in J_0$, $k \in J_1$ (if $J_1 \neq \emptyset$) define

$$\begin{aligned}
&\epsilon_t : \mathcal{H} \longrightarrow \mathbb{F},\\
&e_{i,i} - e_{r,r} \mapsto \delta_{i,t} - \delta_{r,t}, \quad e_{j,j} - e_{s,s} \mapsto 0, \quad e_{i,i} + e_{j,j} \mapsto \delta_{i,t},\\
&\delta_k : \mathcal{H} \longrightarrow \mathbb{F},\\
&e_{i,i} - e_{r,r} \mapsto 0, \quad e_{j,j} - e_{s,s} \mapsto \delta_{j,k} - \delta_{k,s}, \quad e_{i,i} + e_{j,j} \mapsto \delta_{k,j}
\end{aligned}$$

$(i, r \in J_0,\ j, s \in J_1)$. Then $\mathcal{G} = \mathfrak{sl}(J_{\bar{0}}, J_{\bar{1}})$ is a subsuperalgebra of $\mathfrak{pl}_{\mathbb{F}}(J_0, J_1)$ having a weight space decomposition with respect to $\mathcal{H}$ with the set of weights

$$R = \{\epsilon_i - \epsilon_j, \delta_p - \delta_q, \pm(\epsilon_i - \delta_p) \mid i, j \in J_0,\ p, q \in J_1\},$$

which is an irreducible locally finite root supersystem of type $\dot{A}(J_0, J_1)$. We refer to $\mathcal{H}$ as the *standard Cartan subalgebra* of $\mathcal{L}$.

• $\mathfrak{osp}(2I, 2J)$, $\mathfrak{osp}(2I+1, 2J)$: Suppose that I, J are two disjoint index sets with $I \cup J \neq \emptyset$. We mention that in what follows, if I or J is an empty set, each expression corresponding to I or J respectively must be omitted. We consider $\{0, i, \bar{i} \mid i \in I \cup J\}$ as a superset with $|0| = |i| = |\bar{i}| = 0$ for $i \in I$ and $|j| = |\bar{j}| = 1$ for $j \in J$. We set

$$\begin{aligned}
&\bar{I} := \{\bar{i} \mid i \in I\}, \quad \bar{J} := \{\bar{j} \mid j \in J\}, \quad \dot{I} := I \cup \bar{I},\\
&\dot{I}^0 := \{0\} \cup I \cup \bar{I}, \quad \dot{J} := J \cup \bar{J}.
\end{aligned} \tag{2.6}$$

For

$$\mathcal{I} := \dot{I} \cup \dot{J} \quad \text{or} \quad \mathcal{I} := \dot{I}^0 \cup \dot{J},$$

we set

$$Q_{\mathcal{I}} := \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \tag{2.7}$$

in which

$$\begin{aligned}
M_1 &:= \begin{cases} -2e_{0,0} + \sum_{i\in I}(e_{i,\bar{i}} + e_{\bar{i},i}) & \text{if } \mathcal{I} = \dot{I}^0 \cup \dot{J},\\ \sum_{i\in I}(e_{i,\bar{i}} + e_{\bar{i},i}) & \text{if } I \neq \emptyset,\ \mathcal{I} = \dot{I} \cup \dot{J} \end{cases}\\
\text{and} \quad M_2 &:= \sum_{j\in J}(e_{j,\bar{j}} - e_{\bar{j},j}) \quad \text{if } J \neq \emptyset.
\end{aligned}$$

Now

$$\mathcal{G}_{\mathcal{I}} := \{X \in \mathfrak{pl}_{\mathbb{F}}(\mathcal{I}_0, \mathcal{I}_1) \mid X^{\mathrm{st}} Q_{\mathcal{I}} = -Q_{\mathcal{I}} X\}$$

is a Lie subsuperalgebra of $\mathfrak{pl}_{\mathbb{F}}(\mathcal{I}_0,\mathcal{I}_1)$. We denote $\mathcal{G}_{\mathcal{I}}$ by $\mathfrak{osp}(2I,2J)$ or $\mathfrak{osp}(2I+1,2J)$ depending on $\mathcal{I}=\dot{I}\cup\dot{J}$ or $\mathcal{I}=\dot{I}^0\cup\dot{J}$ respectively. We set

$$\mathcal{H}:=\mathrm{span}_{\mathbb{F}}\{h_t,d_k\mid t\in I,\ k\in J\} \tag{2.8}$$

in which for $t\in I$ and $k\in J$,

$$h_t:=e_{t,t}-e_{\bar{t},\bar{t}}\quad\text{and}\quad d_k:=e_{k,k}-e_{\bar{k},\bar{k}}.$$

We refer to $\mathcal{H}$ as the *standard Cartan subalgebra* of $\mathcal{G}_{\mathcal{I}}$. For $i\in I$ and $j\in J$, define

$$\begin{array}{ll}\epsilon_i:\mathcal{H}\longrightarrow\mathbb{F}, & \delta_j:\mathcal{H}\longrightarrow\mathbb{F},\\ h_t\mapsto\delta_{i,t},\quad d_k\mapsto 0, & h_t\mapsto 0,\quad d_k\mapsto\delta_{j,k}\end{array} \tag{2.9}$$

($t\in I$, $k\in J$). One sees that $\mathcal{G}_{\mathcal{I}}$ has a weight space decomposition with respect to $\mathcal{H}$. Taking $R(\mathcal{I})$ to be the corresponding set of weights, we have

$$\begin{aligned}R(\dot{I}^0\cup\dot{J})&=\{\pm\epsilon_r,\pm(\epsilon_r\pm\epsilon_s),\pm\delta_p,\pm(\delta_p\pm\delta_q),\pm(\epsilon_r\pm\delta_p)\mid\\ &\qquad r,s\in I,\ p,q\in J,\ r\neq s\},\\ R(\dot{I}\cup\dot{J})&=\{\pm(\epsilon_r\pm\epsilon_s),\pm(\delta_p\pm\delta_q),\pm(\epsilon_r\pm\delta_p)\mid\\ &\qquad r,s\in I,\ p,q\in J,\ r\neq s\}\end{aligned} \tag{2.10}$$

in which the $\pm(\epsilon_r\pm\epsilon_s)$'s disappear if $|I|=1$. Here $R(\mathcal{I})$ is an irreducible locally finite root supersystem of type $B(I,J)$ if $I,J\neq\emptyset$ and $\mathcal{I}=\dot{I}^0\cup\dot{J}$, and of type $D(I,J)$ if $I,J\neq\emptyset$ and $\mathcal{I}=\dot{I}\cup\dot{J}$.

It is not hard to see that the Lie superalgebras $\mathfrak{sl}(J_0,J_1)$ and $\mathfrak{osp}(2I,2J)$ (excluding the case $J=\emptyset$ and $|I|=2$), as well as $\mathfrak{osp}(2I+1,2J)$, are simple Lie superalgebras. We say that a Lie superalgebra $\mathcal{L}$ is a *locally finite basic classical simple Lie superalgebra* if $\mathcal{L}$ is isomorphic to one of these Lie superalgebras or to one of the finite-dimensional basic classical simple Lie superalgebras.

Root supersystems of type $BC(I,J)$ and $C(I,J)$ do not appear as the root systems of direct unions of finite-dimensional basic classical simple Lie superalgebras. In the following example we introduce Lie superalgebras having weight space decomposition with respect to some Lie subsuperalgebras and the corresponding root systems are of types $BC(I,J)$ and $C(I,J)$.

Example 2.5. Suppose that I and J are two nonempty disjoint index sets and take $\mathcal{A}$ to be the commutative associative $\mathbb{F}$-algebra $\mathbb{F}\oplus\mathbb{F}$ whose product is given by $(\alpha,\beta)(\gamma,\eta) := (\alpha\gamma - \beta\eta, \alpha\eta + \beta\gamma)$ for $\alpha,\beta,\gamma,\eta \in \mathbb{F}$. Define the algebra homomorphism $^* : \mathcal{A} \longrightarrow \mathcal{A}$, of order 2, mapping $(\alpha,\beta) \in \mathbb{F}\oplus\mathbb{F}$ to $(\alpha,-\beta)$. Set

$$^{(t)}\mathcal{A} := \{a \in \mathcal{A} \mid a^* = (-1)^t a\} \quad (t = 0,1). \tag{2.11}$$

Then $\mathcal{A} = {}^{(0)}\mathcal{A} \oplus {}^{(1)}\mathcal{A}$. Next take $\bar{I}$, $\bar{J}$, $\dot{I}$, $\dot{I}_0$, $\mathcal{I}$, as well as $Q_\mathcal{I}$ (also M_1, M_2), as in (2.6)–(2.7). For $T = \sum_{\jmath,k} r_{\jmath,k} e_{\jmath,k} \in M_{I,J}(\mathcal{A})$, take $T^* := \sum_{\jmath,k} r^*_{\jmath,k} e_{\jmath,k}$ and set

$$\mathcal{L}_\mathcal{I} := \{X \in \mathfrak{pl}_\mathcal{I}(\mathcal{A}) \mid (X^{\mathrm{st}})^* Q_\mathcal{I} = -Q_\mathcal{I} X\}.$$

This is a Lie subsuperalgebra of $\mathfrak{pl}_\mathcal{I}(\mathcal{A})$. Next set

$$\mathcal{H} := \mathrm{span}_\mathbb{F}\{h_t := e_{t,t} - e_{\bar{t},\bar{t}},\ d_k := e_{k,k} - e_{\bar{k},\bar{k}} \mid t \in I,\ k \in J\}$$

and for $\imath \in I$ and $\jmath \in J$, define

$$\begin{array}{ll} \epsilon_\imath : \mathcal{H} \longrightarrow \mathbb{F}, & \delta_\jmath : \mathcal{H} \longrightarrow \mathbb{F}, \\ h_t \mapsto \delta_{\imath,t},\ d_k \mapsto 0, & h_t \mapsto 0,\ d_k \mapsto \delta_{\jmath,k} \quad (t \in I, k \in J). \end{array}$$

One can see that with respect to $\mathcal{H}$, $\mathcal{L}$ has a weight space decomposition $\mathcal{L} = \oplus_{\alpha\in R}\mathcal{L}^\alpha$ with the set of weights

$$R := \begin{cases} \{\pm\epsilon_r,\ \pm(\epsilon_r\pm\epsilon_s),\ \pm\delta_p,\ \pm(\delta_p\pm\delta_q),\ \pm(\epsilon_r\pm\delta_p) \mid r,s \in I,\ p,q \in J\}, & 0 \in \mathcal{I}, \\ \{\pm(\epsilon_r\pm\epsilon_s),\ \pm(\delta_p\pm\delta_q),\ \pm(\epsilon_r\pm\delta_p) \mid r,s \in I,\ p,q \in J\}, & 0 \notin \mathcal{I}. \end{cases}$$

For $r,s \in I$ and $p,q \in J$ with $r \neq s$ and $p \neq q$, we have

$$\begin{aligned}
\mathcal{L}_\mathcal{I}^{\epsilon_r} &= \mathrm{span}_\mathbb{F}\{a_t(e_{r,0} - (-1)^t e_{0,\bar{r}}) \mid a_t \in {}^{(t)}\mathcal{A},\ t = 0,1\} \quad \text{if } 0 \in \mathcal{I},\\
\mathcal{L}_\mathcal{I}^{-\epsilon_r} &= \mathrm{span}_\mathbb{F}\{a_t(e_{\bar{r},0} - (-1)^t e_{0,r}) \mid a_t \in {}^{(t)}\mathcal{A},\ t = 0,1\} \quad \text{if } 0 \in \mathcal{I},\\
\mathcal{L}_\mathcal{I}^{2\epsilon_r} &= {}^{(1)}\mathcal{A} e_{r,\bar{r}},\\
\mathcal{L}_\mathcal{I}^{-2\epsilon_r} &= {}^{(1)}\mathcal{A} e_{\bar{r},r},\\
\mathcal{L}_\mathcal{I}^{2\delta_p} &= {}^{(0)}\mathcal{A} e_{p,\bar{p}},\\
\mathcal{L}_\mathcal{I}^{-2\delta_p} &= {}^{(0)}\mathcal{A} e_{\bar{p},p},\\
\mathcal{L}_\mathcal{I}^{\epsilon_r+\epsilon_s} &= \mathrm{span}_\mathbb{F}\{a_t(e_{r,\bar{s}} - (-1)^t e_{s,\bar{r}}) \mid a_t \in {}^{(t)}\mathcal{A},\ t = 0,1\},\\
\mathcal{L}_\mathcal{I}^{-\epsilon_r-\epsilon_s} &= \mathrm{span}_\mathbb{F}\{a_t(e_{\bar{r},s} - (-1)^t e_{\bar{s},r}) \mid a_t \in {}^{(t)}\mathcal{A},\ t = 0,1\},\\
\mathcal{L}_\mathcal{I}^{\epsilon_r-\epsilon_s} &= \mathrm{span}_\mathbb{F}\{a_t(e_{r,s} - (-1)^t e_{\bar{s},\bar{r}}) \mid a_t \in {}^{(t)}\mathcal{A},\ t = 0,1\},\\
\mathcal{L}_\mathcal{I}^{\delta_p+\delta_q} &= \mathrm{span}_\mathbb{F}\{a_t(e_{p,\bar{q}} + (-1)^t e_{q,\bar{p}}) \mid a_t \in {}^{(t)}\mathcal{A},\ t = 0,1\},\\
\mathcal{L}_\mathcal{I}^{-\delta_p-\delta_q} &= \mathrm{span}_\mathbb{F}\{a_t(e_{\bar{p},q} + (-1)^t e_{\bar{q},p}) \mid a_t \in {}^{(t)}\mathcal{A},\ t = 0,1\},\\
\mathcal{L}_\mathcal{I}^{\delta_p-\delta_q} &= \mathrm{span}_\mathbb{F}\{a_t(e_{p,q} - (-1)^t e_{\bar{q},\bar{p}}) \mid a_t \in {}^{(t)}\mathcal{A},\ t = 0,1\},\\
\mathcal{L}_\mathcal{I}^{\delta_p} &= \mathrm{span}_\mathbb{F}\{a_t(e_{0,\bar{p}} + (-1)^t e_{p,0}) \mid a_t \in {}^{(t)}\mathcal{A},\ t = 0,1\} \quad \text{if } 0 \in \mathcal{I},
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{\mathcal{I}}^{-\delta_p} &= \mathrm{span}_{\mathbb{F}}\{a_t(e_{0,p} - (-1)^t e_{\bar{p},0}) \mid a_t \in {}^{(t)}\mathcal{A},\ t = 0,1\} \quad \text{if } 0 \in \mathcal{I},\\
\mathcal{L}_{\mathcal{I}}^{\epsilon_r+\delta_p} &= \mathrm{span}_{\mathbb{F}}\{a_t(e_{r,\bar{p}} + (-1)^t e_{p,\bar{r}}) \mid a_t \in {}^{(t)}\mathcal{A},\ t = 0,1\},\\
\mathcal{L}_{\mathcal{I}}^{-\epsilon_r-\delta_p} &= \mathrm{span}_{\mathbb{F}}\{a_t(e_{\bar{r},p} - (-1)^t e_{\bar{p},r}) \mid a_t \in {}^{(t)}\mathcal{A},\ t = 0,1\},\\
\mathcal{L}_{\mathcal{I}}^{\epsilon_r-\delta_p} &= \mathrm{span}_{\mathbb{F}}\{a_t(e_{r,p} - (-1)^t e_{\bar{p},\bar{r}}) \mid a_t \in {}^{(t)}\mathcal{A},\ t = 0,1\},\\
\mathcal{L}_{\mathcal{I}}^{-\epsilon_r+\delta_p} &= \mathrm{span}_{\mathbb{F}}\{a_t(e_{\bar{r},\bar{p}} + (-1)^t e_{p,r}) \mid a_t \in {}^{(t)}\mathcal{A},\ t = 0,1\};
\end{aligned}$$

moreover, setting

$$\begin{aligned}
K := \mathrm{span}_{\mathbb{F}}\{&a_t(e_{r,r} - (-1)^t e_{\bar{r},\bar{r}}),\ a_t(e_{p,p} - (-1)^t e_{\bar{p},\bar{p}}) \mid a_t \in {}^{(t)}\mathcal{A},\\
&t \in \{0,1\},\ r \in I,\ p \in J\},
\end{aligned}$$

we have

$$\mathcal{L}^0 = \begin{cases} {}^{(1)}\mathcal{A}e_{0,0} \oplus K & \text{if } 0 \in \mathcal{I},\\ K & \text{if } 0 \notin \mathcal{I}. \end{cases}$$

The set of weights R of $\mathcal{L}$ with respect to $\mathcal{H}$ is an irreducible locally finite root supersystem of type $BC(I,J)$ if $0 \in \mathcal{I}$ and of type $C(I,J)$ if $0 \notin \mathcal{I}$. ♢

§3. Extended affine Lie superalgebras

Definition 3.1. A triple $(\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1, \mathcal{H}, (\cdot,\cdot))$ (or $\mathcal{L}$ if there is no confusion) is called an *extended affine Lie superalgebra* if

- $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ is a nonzero Lie superalgebra, $\mathcal{H}$ is a nonzero subalgebra of $\mathcal{L}_0$ and $(\cdot,\cdot)$ is a nondegenerate invariant even supersymmetric bilinear form on $\mathcal{L}$;
- $\mathcal{L}$ has a weight space decomposition $\mathcal{L} = \bigoplus_{\alpha\in\mathcal{H}^*} \mathcal{L}^\alpha$ with respect to $\mathcal{H}$ via the adjoint representation, in particular $\mathcal{H} \subset \mathcal{L}^0$; see Section A.3;
- the restriction of the form $(\cdot,\cdot)$ to $\mathcal{H}$ is nondegenerate;
- for $0 \neq \alpha \in R_i := \{\alpha \in \mathcal{H}^* \mid \mathcal{L}^\alpha \cap \mathcal{L}_i \neq \{0\}\}$ $(i \in \mathbb{Z}_2)$, there are $x_\alpha \in \mathcal{L}_i^\alpha := \mathcal{L}^\alpha \cap \mathcal{L}_i$ and $x_{-\alpha} \in \mathcal{L}_i^{-\alpha} := \mathcal{L}^{-\alpha} \cap \mathcal{L}_i$ such that $0 \neq [x_\alpha, x_{-\alpha}] \in \mathcal{H}$;
- considering the root system $R := \{\alpha \in \mathcal{H}^* \mid \mathcal{L}^\alpha \neq \{0\}\}$ of $\mathcal{L}$ and the induced form $(\cdot,\cdot)$ on $\mathrm{span}_{\mathbb{F}}(R)$ (see Section A.3), for $\alpha \in R$ with $(\alpha,\alpha) \neq 0$ and $x \in \mathcal{L}^\alpha$, $\mathrm{ad}_x : \mathcal{L} \longrightarrow \mathcal{L}$, mapping $y \in \mathcal{L}$ to $[x,y]$, is a locally nilpotent linear transformation.

The subsuperalgebra $\mathcal{L}_c$ of $\mathcal{L}$ generated by $\cup_{\{\alpha\in R;(\alpha,R)\neq\{0\}\}}\mathcal{L}^\alpha$ is called the *core* of $\mathcal{L}$. ⋄

In this paper, we always assume that the root system of an extended affine Lie superalgebra contains some elements that are not self-orthogonal.

Suppose that $(\mathcal{L}, \mathcal{H}, (\cdot,\cdot))$ is an extended affine Lie superalgebra with root system R. Set $V := \operatorname{span}_{\mathbb{F}} R$ and denote the induced form on V again by $(\cdot,\cdot)$; see (A.1). Take V^0 to be the radical of the form on V. Suppose that $^{-}\colon V \longrightarrow \bar{V} := V/V^0$ is the canonical projection map and take $\bar{R}$ to be the image of R under the projection map "$^{-}$". Denote by $\bar{(\cdot,\cdot)}$, the induced form on $\bar{V}$, then by [27, §3] and [28, Prop. 1.11], $(\operatorname{span}_{\mathbb{F}} \bar{R}, \bar{(\cdot,\cdot)}, \bar{R})$ is a locally finite root supersystem. We say that R is *irreducible* if $\bar{R}$ is irreducible. We define the *type* of $\mathcal{L}$ to be the type of $\bar{R}$. So $\mathcal{L}$ is one of the following types:

- types A_n, B_n, C_n, D_n, BC_n, F_4, G_2 and $E_{6,7,8}$, which are types of finite root systems;
- types $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1,\alpha)$, $F(4)$ and $G(3)$, which are types of finite-dimensional basic classical simple Lie superalgebras that are not a Lie algebra;
- finite types $C(m,n)$ and $BC(m,n)$;
- infinite types $\dot{A}(I,J)$, $B(I,J)$, $C(I,J)$, $D(I,J)$ and $BC(I,J)$.

Example 3.2 (Affine Lie superalgebras). In this example, we show that an affine Lie superalgebra is an extended affine Lie superalgebra. We briefly recall what an affine Lie superalgebra is and we ask readers to see [23] for details.[1]

Suppose that n is a positive integer and $I = \{1, \ldots, n\}$. Let τ be a subset of I and A be a nonzero indecomposable symmetrizable[2] $(n \times n)$-matrix, with complex entries, satisfying

- if $a_{i,j} = 0$, then $a_{j,i} = 0$;
- if $a_{i,i} = 0$, then $i \in \tau$;
- if $a_{i,i} \neq 0$, then $2a_{i,j}/a_{i,i}$ (resp. $a_{i,j}/a_{i,i}$) is a nonpositive integer for $i \in I \setminus \tau$ (resp. $i \in \tau$) with $i \neq j$.

Fix a complex vector space $\mathcal{H}$ of dimension $n+\operatorname{corank}(A)$. Then there exist linearly independent subsets

$$\Pi := \{\alpha_i \mid i \in I\} \subseteq \mathcal{H}^* \quad \text{and} \quad \check{\Pi} := \{\check{\alpha}_i \mid i \in I\} \subseteq \mathcal{H}$$

such that

$$\alpha_j(\check{\alpha}_i) = a_{i,j} \quad (i, j \in I).$$

[1] See also [24], which is a short paper from [23].

[2] This means that A has a decomposition $A = DB$ in which D is an invertible diagonal matrix and B is a symmetric matrix.

Let $\tilde{\mathcal{G}}(A,\tau)$ be the Lie superalgebra generated by $\{e_i, f_i \mid i \in I\} \cup \mathcal{H}$ subject to the following relations:

$$
\begin{aligned}
&[e_i, f_j] = \delta_{i,j}\check{\alpha}_i, \quad [h,h'] = 0,\\
&[h, e_j] = \alpha_j(h)e_j, \quad [h, f_j] = -\alpha_j(h)f_j,\\
&\deg(h) = 0, \quad \deg(e_i) = \deg(f_i) = \begin{cases} 0 & \text{if } i \notin \tau,\\ 1 & \text{if } i \in \tau,\end{cases}
\end{aligned} \tag{3.1}
$$

for $i, j \in I$ and $h, h' \in \mathcal{H}$. Then $\tilde{\mathcal{G}}(A,\tau) = \tilde{\mathfrak{n}}^+ \oplus \mathcal{H} \oplus \tilde{\mathfrak{n}}^-$, where $\tilde{\mathfrak{n}}^+$ (resp. $\tilde{\mathfrak{n}}^-$) is the subsuperalgebra of $\tilde{\mathcal{G}}(A,\tau)$ generated by $\{e_1,\ldots,e_n\}$ (resp. $\{f_1,\ldots,f_n\}$). Moreover, there is a unique maximal ideal $\mathfrak{r}$ of $\tilde{\mathcal{G}}(A,\tau)$ intersecting $\mathcal{H}$ trivially. We make a convention that the images of e_i, f_i, h ($i \in I, h \in \mathcal{H}$) in $\mathcal{G}(A,\tau) := \tilde{\mathcal{G}}(A,\tau)/\mathfrak{r}$ under the canonical projection map are denoted again by e_i, f_i, h, respectively. The Lie superalgebra $\mathcal{G} := \mathcal{G}(A,\tau)$ has a weight space decomposition

$$
\mathcal{G} = \oplus_{\alpha \in \mathcal{H}^*} \mathcal{G}^\alpha \tag{3.2}
$$

with respect to $\mathcal{H}$ with $\mathcal{G}^0 = \mathcal{H}$. It also has a natural $\mathbb{Z}$-grading $\mathcal{G} = \oplus_{i\in\mathbb{Z}}\mathcal{G}^i$ with

$$
\deg(e_i) := 1, \quad \deg(f_i) := -1 \quad \text{and} \quad \deg(h) = 0 \quad (1 \le i \le n,\ h \in \mathcal{H}).
$$

The Lie superalgebra $\mathcal{G}$ is called an *affine Lie superalgebra* if it is not of finite dimension but of finite *growth*; see [23, §6.1]. Suppose that Δ is the root system of $\mathcal{G}$ with respect to $\mathcal{H}$. Then

(3.3)
- there exists an invariant even nondegenerate supersymmetric $\mathbb{C}$-valued bilinear form $(\cdot,\cdot)$ on $\mathcal{G}$;
- the restriction of the form $(\cdot,\cdot)$ to $\mathcal{G}^\alpha + \mathcal{G}^{-\alpha}$ ($\alpha \in \Delta$) is nondegenerate;
- $[x,y] = (x,y)t_\alpha$, where $\alpha \in \Delta$, $x \in \mathcal{G}^\alpha$ and $y \in \mathcal{G}^{-\alpha}$;

in which t_α is the unique element of $\mathcal{H}$ representing α through $(\cdot,\cdot)|_{\mathcal{H}\times\mathcal{H}}$.

In [23], the author gives the classification of affine Lie superalgebras. To a quadruple $(\mathfrak{k}, \sigma, X, m)$, where $\mathfrak{k}$ is a finite-dimensional basic classical simple Lie superalgebra of type X and σ is a certain automorphism on $\mathfrak{k}$ of finite order m, there corresponds a specific extension $\widehat{\mathcal{L}}(\mathfrak{k},\sigma)$ of the fixed point subalgebra $\mathcal{L}(\mathfrak{k},\sigma)$ of the current Lie superalgebra $\mathcal{L}(\mathfrak{k}) := \mathbb{C}[t,t^{-1}] \otimes \mathfrak{k}$ under a particular automorphism induced from σ. He shows that $\widehat{\mathcal{L}}(\mathfrak{k},\sigma)$ is an affine Lie superalgebra and says that it is of *type* $X^{(m)}$. If $m = 1$ (resp. $m \neq 1$), $\widehat{\mathcal{L}}(\mathfrak{k},\sigma)$ is called a *nontwisted* (resp. *twisted*) affine Lie superalgebra. He proves that nontwisted and twisted affine Lie superalgebras cover all affine Lie superalgebras.

The root systems of affine Lie superalgebras are given in [23]. In what follows we explicitly determine these root systems in terms of finite root supersystems.[3] Suppose $(\mathfrak{k}, \sigma, X, m)$ is a quadruple as mentioned above.

• Nontwisted case ([23, §7.2–§7.4]). In this case $\sigma = \mathrm{id}$. Denote by Φ the root system of $\mathfrak{k}$ with respect to a fixed Cartan subalgebra $\mathfrak{h}$ and take the free abelian group $\mathbb{Z}\delta$ of rank 1. Then the root system Δ of $\widehat{\mathcal{L}}(\mathfrak{k}) := \widehat{\mathcal{L}}(\mathfrak{k}, \mathrm{id})$ is identified as

$$\Delta \subseteq \{\dot\alpha + s\delta \mid \dot\alpha \in \Phi,\ \ s \in \mathbb{Z}\} \subseteq \mathfrak{h}^* \oplus \mathbb{Z}\delta$$

and $\mathbb{Z}\delta$ lies in the radical of the bilinear form on $\widehat{\mathcal{L}}(\mathfrak{k})$. Since Φ is a finite set and for $\alpha, \beta \in \Delta$, $[\widehat{\mathcal{L}}(\mathfrak{k})_\alpha, \widehat{\mathcal{L}}(\mathfrak{k})_\beta] \subseteq \widehat{\mathcal{L}}(\mathfrak{k})_{\alpha+\beta}$, it is immediate that if $\dot\alpha + s\delta \in \Delta$, for some nonzero $\dot\alpha \in \Phi$ and $s \in \mathbb{Z}$, then weight vectors of $\widehat{\mathcal{L}}(\mathfrak{k})$ corresponding to $\dot\alpha + s\delta$ act locally nilpotently on $\widehat{\mathcal{L}}(\mathfrak{k})$. This, together with (3.2) and (3.3), implies that $\widehat{\mathcal{L}}(\mathfrak{k})$ is an extended affine Lie superalgebra of type X.

• Twisted case ([23, §7.5]). We take the free abelian group $\mathbb{Z}\delta$ of rank 1 and let F be a vector space with a basis $\{\epsilon_i \mid 1 \le i \le k\} \cup \{\delta_p \mid 1 \le p \le \ell\}$ equipped with a symmetric bilinear form given by

$$(\epsilon_i, \epsilon_r) := \delta_{i,r}, \quad (\delta_j, \delta_s) := -\delta_{j,s}, \quad (\epsilon_i, \delta_j) = 0 \quad (1 \le i, r \le k,\ 1 \le j, s \le \ell).$$

Suppose that Δ is the root system of a twisted affine Lie superalgebra of type $X^{(m)} \ne G(3)^{(2)}$. In [23, §7.5.6–§7.5.9], Δ is given as in the following table:

$X^{(m)}$	Δ
$A(2k, 2\ell-1)^{(2)}$	$\mathbb{Z}\delta \cup \mathbb{Z}\delta \pm \{\epsilon_i, \delta_j, \epsilon_i \pm \epsilon_r, \delta_j \pm \delta_s, \epsilon_i \pm \delta_j \mid i \ne r,\ j \ne s\}$ $\cup\, (2\mathbb{Z}+1)\delta \pm \{2\epsilon_i \mid 1 \le i \le k\}$ $\cup\, 2\mathbb{Z}\delta \pm \{2\delta_j \mid 1 \le j \le \ell\}$.
$A(2k-1, 2\ell-1)^{(2)},\ (k,\ell) \ne (1,1)$	$\mathbb{Z}\delta \cup \mathbb{Z}\delta \pm \{\epsilon_i \pm \epsilon_r, \delta_j \pm \delta_s, \delta_j \pm \epsilon_i \mid i \ne r,\ j \ne s\}$ $\cup\, (2\mathbb{Z}+1)\delta \pm \{2\epsilon_i \mid 1 \le i \le k\}$ $\cup\, 2\mathbb{Z}\delta \pm \{2\delta_j \mid 1 \le j \le \ell\}$
$A(2k, 2\ell)^{(4)}$	$\mathbb{Z}\delta \cup \mathbb{Z}\delta \pm \{\epsilon_i, \delta_j \mid 1 \le i \le k,\ 1 \le j \le \ell\}$ $\cup\, 2\mathbb{Z}\delta \pm \{\epsilon_i \pm \epsilon_r, \delta_j \pm \delta_s, \delta_j \pm \epsilon_i \mid i \ne r,\ j \ne s\}$ $\cup\, (4\mathbb{Z}+2)\delta \pm \{2\epsilon_i \mid 1 \le i \le k\}$ $\cup\, 4\mathbb{Z}\delta \pm \{2\delta_j \mid 1 \le j \le \ell\}$
$C(\ell+1)^{(2)}$ & $D(k+1,\ell)^{(2)}$	$\mathbb{Z}\delta \cup \mathbb{Z}\delta \pm \{\epsilon_i, \delta_j \mid 1 \le i \le k,\ 1 \le j \le \ell\}$ $\cup\, 2\mathbb{Z}\delta \pm \{2\delta_j, \epsilon_i \pm \epsilon_r, \delta_j \pm \delta_s, \delta_j \pm \epsilon_i \mid i \ne r,\ j \ne s\}$

As we see, the root system Δ is a subset of $\mathbb{Z}\delta + R_\Delta$ where R_Δ is a finite root supersystem as in the following table.

[3] Root systems in [23] do not contain 0 while 0 *is* an element of our root supersystems; here we include 0 in root systems given in [23].

$X^{(m)}$	R_Δ	Type of R_Δ
$A(2k,2\ell-1)^{(2)}$	$\pm\{\epsilon_i,\delta_j,\epsilon_i\pm\epsilon_r,\delta_j\pm\delta_s,\epsilon_i\pm\delta_j \mid 1\leq i\leq k,\ 1\leq j\leq \ell\}$	$BC(k,\ell)$
$A(2k-1,2\ell-1)^{(2)}$ $(k,\ell)\neq(1,1)$	$\pm\{\epsilon_i\pm\epsilon_r,\delta_j\pm\delta_s,\epsilon_i\pm\delta_j \mid 1\leq i\leq k,\ 1\leq j\leq \ell\}$	$C(k,\ell)$
$A(2k,2\ell)^{(4)}$	$\pm\{\epsilon_i,\delta_j,\epsilon_i\pm\epsilon_r,\delta_j\pm\delta_s,\epsilon_i\pm\delta_j \mid 1\leq i\leq k,\ 1\leq j\leq \ell\}$	$BC(k,\ell)$
$C(\ell+1)^{(2)}$& $D(k+1,\ell)^{(2)}$	$\pm\{\epsilon_i,\delta_j,\epsilon_i\pm\epsilon_r,\delta_j\pm\delta_s,\epsilon_i\pm\delta_j \mid 1\leq i\neq r\leq k,\ 1\leq j,s\leq \ell\}$	$B(k,\ell)$

Using the same argument as in the nontwisted case, we get that the affine Lie superalgebra of type $X^{(m)}\neq G(3)^{(2)}$ is an extended affine Lie superalgebra of type R_Δ.[4] ♢

Theorem 3.3. *Suppose that $(\mathcal{L},\mathcal{H},(\cdot,\cdot))$ is an extended affine Lie superalgebra with $\mathcal{L}^0=\mathcal{H}$. Keeping the same notation as above, we assume that the root system R of $\mathcal{L}$ is irreducible with $\bar{R}_{\rm ns}\neq\{0\}$. Let $\bar{R}$ be an irreducible finite root supersystem of finite type $X\neq A(\ell,\ell)$, $BC(m,n)$, $C(m,n)$. Then $\mathcal{L}_c$, as well as $\mathcal{L}_c/Z(\mathcal{L}_c)$, contains a finite-dimensional basic classical simple Lie subsuperalgebra, compatible with the root space decomposition, of type X.*

Proof. We first note that using [25, Props. 2.5, 2.14], we have

$$\dim(\mathcal{L}^\alpha)=1\quad(\alpha\in R^\times). \tag{3.4}$$

We next set $V:=\operatorname{span}_{\mathbb{F}}R$ and consider the canonical projection map $^-\colon V\longrightarrow \bar{V}=V/V^0$, where V^0 is the radical of the form. We recall that $\bar{R}$ is an irreducible finite root supersystem in $\bar{V}$. Consider Remark 2.4 and fix a subset $\dot{\Pi}$ of R such that $\bar{\dot{\Pi}}$ is the corresponding distinguished base for $\bar{R}$ as in [13, Tables 3.52–3.58]; in particular, $\bar{\dot{\Pi}}$ is a root base that is linearly independent and satisfies the partial sum property. Take $\dot{V}:=\operatorname{span}_{\mathbb{F}}\dot{\Pi}$, as well as $\dot{R}:=\{\dot{a}\in\dot{V}\mid \exists\,\eta\in V^0;\ \dot{a}+\eta\in R\}$. One can see that $V=\dot{V}\oplus V^0$. Moreover, $(\dot{V},(\cdot,\cdot),\dot{R})$ is a finite root supersystem, of the same type as $(\bar{V},\overline{(\cdot,\cdot)},\bar{R})$, with root base $\dot{\Pi}$; in particular, $\dot{\Pi}$ satisfies the partial sum property and the form on $\dot{V}$ is nondegenerate. We know that R is of type $X\neq A(\ell,\ell)$, $BC(I,J)$, $C(I,J)$, so for $\alpha,\beta\in R^\times_{\rm ns}$ with $(\alpha,\beta)\neq 0$, one and only one of $\alpha+\beta$ or $\alpha-\beta$ is an element of R; see Remark 2.2(4). Using this, together with the fact that $\dot{\Pi}\subseteq R$ and that R is invariant under its Weyl group, it follows that

$$\dot{R}\setminus\{0\}\subseteq R^\times. \tag{3.5}$$

[4] Using odd reflections, $G(3)^{(2)}$ is isomorphic to $G(3)^{(1)}$ which is a nontwisted affine Lie superalgebra.

Next suppose $\dot{\Pi}=\{\dot{\alpha}_1,\dots,\dot{\alpha}_n\}$ and set $\dot{\mathcal{H}}:=\text{span}_{\mathbb{F}}\{t_{\dot{\alpha}_i}|i\in\{1,\dots,n\}\}$ in which, for $i\in\{1,\dots,n\}$, $t_{\dot{\alpha}_i}$ is the unique element of $\mathcal{H}$ representing $\dot{\alpha}_i$ through the form. For each $i\in\{1,\dots,n\}$, we fix $e_{\dot{\alpha}_i}\in\mathcal{L}^{\dot{\alpha}_i}$ and $f_{\dot{\alpha}_i}\in\mathcal{L}^{-\dot{\alpha}_i}$ with $[e_{\dot{\alpha}_i},f_{\dot{\alpha}_i}]=t_{\dot{\alpha}_i}$; see [27, Lem. 3.1]. We claim that the subsuperalgebra $\mathcal{G}$ of $\mathcal{L}$ generated by $\{e_{\dot{\alpha}_i},f_{\dot{\alpha}_i},h_{\dot{\alpha}_i}\mid 1\le i\le n\}$ is a finite-dimensional basic classical simple Lie superalgebra of type X. Consider (3.4) and take $\tau\subseteq\{1,\dots,n\}$ to be such that

$$|e_{\dot{\alpha}_i}|=|f_{\dot{\alpha}_i}|=0\quad\text{if } i\notin\tau,$$
$$|e_{\dot{\alpha}_i}|=|f_{\dot{\alpha}_i}|=1\quad\text{if } i\in\tau.$$

Consider the $(n\times n)$-matrix $A=(a_{ij})$ with

$$a_{ij}=(\dot{\alpha}_i,\dot{\alpha}_j),\quad i,j\in\{1,\dots,n\};$$

we mention that as the form $(\cdot,\cdot)$ on $\dot{V}$ is nondegenerate, A is invertible. One knows that the contragredient Lie superalgebra $\mathcal{G}(A,\tau)$ is the finite-dimensional basic classical simple Lie superalgebra of type X; see [20, §5.1, Cor. 5.2.4] and [16, Prop. 2.5.6, Thm. 3]. We show that $\mathcal{G}\cong\mathcal{G}(A,\tau)$. We first note that we have

$$[t_{\dot{\alpha}_i},e_{\dot{\alpha}_j}]=a_{ij}e_{\dot{\alpha}_j},\qquad [t_{\dot{\alpha}_i},f_{\dot{\alpha}_j}]=-a_{ij}f_{\dot{\alpha}_j},$$
$$[e_{\dot{\alpha}_i},f_{\dot{\alpha}_j}]=\delta_{ij}t_{\dot{\alpha}_i},\qquad [t_{\dot{\alpha}_i},t_{\dot{\alpha}_j}]=0.$$

So using the theory of Kac–Moody Lie superalgebras, we just need to show that $\{0\}$ is the only ideal of $\mathcal{G}$ intersecting $\dot{\mathcal{H}}$ trivially; see [20, Lem. 5.3.1]). We note that with respect to $\dot{\mathcal{H}}$, $\mathcal{G}$ has the weight space decomposition $\mathcal{G}=\oplus_{\dot{\alpha}\in\dot{A}}\mathcal{G}^{\dot{\alpha}}$ in which, for $\dot{\alpha}\in\dot{A}$ with $\mathcal{G}^{\dot{\alpha}}\neq\{0\}$, $\mathcal{G}^{\dot{\alpha}}$ is the linear span of the elements of the form $[x_1,\dots,x_t]$ where, for $r\in\{1,\dots,t\}$, $x_r\in\{e_{\dot{\alpha}_i},f_{\dot{\alpha}_i}\mid 1\le i\le n\}$ is a root vector corresponding to $\beta_i\in\{\pm\dot{\alpha}_1,\dots,\pm\dot{\alpha}_n\}$ with $\beta_1+\cdots+\beta_n=\dot{\alpha}$. Now set $\dot{A}:=\langle\dot{R}\rangle$ and to the contrary, suppose that I is a nontrivial ideal of $\mathcal{G}$ such that $I\cap\dot{\mathcal{H}}=\{0\}$, by [18, Prop. 2.1.1], $I=\oplus_{\dot{\alpha}\in\dot{A}\setminus\{0\}}I^{\dot{\alpha}}$ in which, for $\dot{\alpha}\in\dot{A}$, $I^{\dot{\alpha}}:=I\cap\mathcal{G}^{\dot{\alpha}}$. Since $I\cap\dot{\mathcal{H}}=\{0\}$ and $\mathcal{G}^0=\dot{\mathcal{H}}$, there is $\dot{\alpha}\in\dot{A}\setminus\{0\}$ such that $I^{\dot{\alpha}}\neq\{0\}$. So there are root vectors $x_1,\dots,x_t\in\{e_{\dot{\alpha}_i},f_{\dot{\alpha}_i}\mid 1\le i\le n\}$ corresponding to roots $\beta_1,\dots,\beta_t\in\{\pm\dot{\alpha}_1,\dots,\pm\dot{\alpha}_n\}$ such that $\beta_1+\cdots+\beta_n=\dot{\alpha}$ and $0\neq[x_1,\dots,x_t]\in I^{\dot{\alpha}}$. But $[x_1,\dots,x_t]\in\mathcal{L}^{\dot{\alpha}}$, so $\dot{\alpha}\in R$. Now as $\mathcal{L}$ is an extended affine Lie superalgebra, considering (3.4), one finds $x\in\mathcal{L}^{\dot{\alpha}}=I^{\dot{\alpha}}$ and $y\in\mathcal{L}^{-\dot{\alpha}}$ such that $0\neq[x,y]\in\mathcal{H}$. By [27, Lem. 3.1], $[x,y]=(x,y)t_{\dot{\alpha}}\in\dot{\mathcal{H}}\setminus\{0\}$. Therefore,

$$y\notin\mathcal{G}\tag{3.6}$$

since otherwise we get $0\neq t_{\dot{\alpha}}\in I\cap\dot{\mathcal{H}}$, which is a contradiction. On the other hand, since $\mathcal{L}^{\dot{\alpha}}\neq\{0\}$, $\dot{\alpha}\in R\cap\dot{R}$ and so using the partial sum property of $\dot{\Pi}$ and (3.5), there exist $\dot{\gamma}_1,\dots,\dot{\gamma}_p\in\dot{\Pi}$ or $\dot{\gamma}_1,\dots,\dot{\gamma}_p\in-\dot{\Pi}$ such that $\dot{\alpha}=\dot{\gamma}_1+\cdots+\dot{\gamma}_p$

and $\dot{\gamma}_1 + \cdots + \dot{\gamma}_l \in \dot{R} \setminus \{0\} \subseteq R^\times$ for all $l \in \{1, \ldots, p\}$ and so $-\dot{\gamma}_1 - \cdots - \dot{\gamma}_l \in R^\times$ for $1 \leq l \leq p$. Since by (3.4), $\mathcal{L}^{-\dot{\gamma}_i} = \mathcal{G}^{-\dot{\gamma}_i}$ for $1 \leq i \leq p$, [25, Lem. 2.4] together with (3.4), implies

$$\begin{aligned}\mathcal{L}^{-\dot{\alpha}} &= [\mathcal{L}^{-\dot{\gamma}_p}, [\mathcal{L}^{-\dot{\gamma}_{p-1}}, [\ldots, [\mathcal{L}^{-\dot{\gamma}_2}, \mathcal{L}^{-\dot{\gamma}_1}] \ldots]]] \\ &= [\mathcal{G}^{-\dot{\gamma}_p}, [\mathcal{G}^{-\dot{\gamma}_{p-1}}, [\ldots, [\mathcal{G}^{-\dot{\gamma}_2}, \mathcal{G}^{-\dot{\gamma}_1}] \ldots]]] \subseteq \mathcal{G};\end{aligned}$$

in particular, $y \in \mathcal{L}^{-\dot{\alpha}} \subseteq \mathcal{G}$. But this contradicts (3.6) and so $\mathcal{G}$ has no nontrivial ideal intersecting $\dot{\mathcal{H}}$ trivially. In particular, $\mathcal{G} \cong \mathcal{G}(A, \tau)$. As we mentioned before, $\mathcal{G}(A, \tau)$ is a finite-dimensional basic classical simple Lie superalgebra of type X and so is $\mathcal{G}$ ($\cong \mathcal{G}(A, \tau)$). Therefore as $\mathcal{G}$ is contained in $\mathcal{L}_c$, $\mathcal{L}_c$ has a finite-dimensional basic classical simple Lie superalgebra of type X. Also as $\mathcal{G}$ is a simple Lie subsuperalgebra of $\mathcal{L}_c$, $\mathcal{G} \cap Z(\mathcal{L}_c) = \{0\}$ and so the restriction of the canonical projection map $\mathfrak{pr} : \mathcal{L}_c \longrightarrow \mathcal{L}_c/Z(\mathcal{L}_c)$ to $\mathcal{G}$ is injective. So $\mathcal{G} \cong \mathfrak{pr}(\mathcal{G})$ and $\mathfrak{pr}(\mathcal{G}) \subseteq \mathcal{L}_c/Z(\mathcal{L}_c)$ is a finite-dimensional basic classical simple Lie superalgebra of type X. □

§4. Characterization theorem

The core and the core modulo the center of an extended affine Lie superalgebra is a so-called root-graded Lie superalgebra and conversely the main ingredient to construct an extended affine Lie superalgebra is a root-graded Lie superalgebra; see [6], [22], [27] and [29]. This means that to characterize extended affine Lie superalgebras, one needs first to characterize root-graded Lie superalgebras.

The structure of a root-graded Lie superalgebra $\mathcal{L}$ graded by an irreducible locally finite root supersystem R (see Definition 4.1) strongly depends on the corresponding locally finite basic classical simple Lie superalgebra appearing as its grading subalgebra. The intrinsic features of locally finite basic classical simple Lie superalgebras and their modules offer different approaches to get the recognition theorems for R-graded Lie superalgebras. In a series of papers ([8], [9] and [10]), Benkart and Elduque study Lie superalgebras graded by finite root supersystems $C(n)$, $D(m, n)$, $D(2, 1; \alpha)$, $F(4)$, $G(3)$, $A(m, n)$ and $B(m, n)$; see also [29] for definition and a recognition theorem for Lie superalgebras graded by the irreducible locally finite root supersystem of type $BC(I, J)$. In this section, we give the recognition theorems for root-graded Lie superalgebras graded by $C(J)$ and $D(I, J)$. Recognition theorems for types $\dot{A}(I, J)$ and $B(I, J)$ need separate works.

Among root-graded Lie superalgebras, those that are equipped with a Λ-grading, where Λ is an abelian group, are of great importance due to their roles in the structure of other kinds of Lie superalgebras. Using our recognition theorems, we give a characterization of a nice class of these Lie superalgebras for types $D(2,1,\alpha)$, $F(4)$, $G(3)$, $C(J)$ and $D(I,J)$.

In the Appendix, one will find some basic information and notation on (Λ-graded) superalgebras as well as supermatrices which we need in this section.

Definition 4.1. Suppose that $\mathfrak{g}$ is a locally finite basic classical simple Lie superalgebra with the standard Cartan subalgebra $\mathfrak{h}$ and root system R. We call a Lie superalgebra $\mathfrak{L}$, an *R-graded Lie superalgebra* with *grading subalgebra* $\mathfrak{g}$ and *grading pair* $(\mathfrak{g},\mathfrak{h})$ if

(4.1)
- $\mathfrak{L}$ contains $\mathfrak{g}$ as a subsuperalgebra;
- $\mathfrak{L}$ has a weight space decomposition $\mathfrak{L}=\oplus_{\alpha\in R}\mathfrak{L}^\alpha$, with respect to $\mathfrak{h}$;
- $\mathfrak{L}^0=\sum_{\alpha\in R\setminus\{0\}}[\mathfrak{L}^\alpha,\mathfrak{L}^{-\alpha}]$.

The R-graded Lie superalgebra $\mathfrak{L}$ is called an (R,Λ)*-graded Lie superalgebra* if $\mathfrak{L}$ is equipped with a Λ-grading $\{{}^\lambda\mathcal{L}\mid\lambda\in\Lambda\}$ that is compatible with the $\langle R\rangle$-grading on $\mathfrak{L}$. The (R,Λ)-graded Lie superalgebra $\mathcal{L}$ is called *predivision* if for each $\alpha\in R^\times$ and $\lambda\in\Lambda$ with ${}^\lambda\mathcal{L}^\alpha\neq\{0\}$, there are $x\in{}^\lambda\mathcal{L}^\alpha$ and $y\in{}^{-\lambda}\mathcal{L}^{-\alpha}$ with $[x,y]\in\mathfrak{h}\setminus\{0\}$. ⋄

One knows that Definition 4.1 is a generalization of the notion of a root-graded Lie superalgebra in the sense of [10]; roughly speaking, we just switch from finite-dimensional basic classical simple Lie superalgebras to locally finite basic classical simple Lie superalgebras. Lie superalgebras defined in Definition 4.1 are also special root-graded Lie superalgebras defined in [29, Def. 3.2].

Theorem 4.2. *Suppose that $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ is a locally finite basic classical simple Lie superalgebra with standard Cartan subalgebra $\mathfrak{h}$ and corresponding root system R of type $C(J)$ $(|J|\geq 2)$, $D(I,J)$ $(|I|,|J|\geq 2)$, $G(3), F(4)$ or $D(2,1,\alpha)$ $(\alpha\neq 0,-1)$. Suppose that $\mathfrak{L}$ is an R-graded Lie superalgebra with grading pair $(\mathfrak{g},\mathfrak{h})$. Then there are a subsuperalgebra $\mathcal{D}$ of $\mathfrak{L}$, a superspace $\mathcal{A}$ and even bilinear maps*

$$\cdot:\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{A}\quad\textit{and}\quad\langle\cdot,\cdot\rangle:\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{D}$$

such that $(\mathcal{A},\cdot)$ is a unital supercommutative associative superalgebra, and

(4.2)
- *$\langle\cdot,\cdot\rangle$ is superskewsymmetric and $\langle\mathcal{A},\mathcal{A}\rangle=\mathcal{D}$;*
- *for a_1, a_2, $a_3\in\mathcal{A}$, $\sum_{\circlearrowleft}(-1)^{|a_1||a_3|}\langle a_1,a_2\cdot a_3\rangle=0$, in which "$\circlearrowleft$" indicates the cyclic permutation.*

Moreover, the Lie superalgebra $\mathfrak{L}$ *is isomorphic to*

$$(\mathfrak{g} \otimes \mathcal{A}) \oplus \mathcal{D}$$

on which the Lie bracket is defined by

$$\begin{aligned} &[x \otimes a, y \otimes a'] = (-1)^{|a|\,|y|}([x, y]_{\mathfrak{g}} \otimes (a \cdot a') + \mathrm{str}(xy)\langle a, a'\rangle),\\ &[(\mathfrak{g} \otimes \mathcal{A}) \oplus \mathcal{D}, \mathcal{D}] = \{0\}, \end{aligned} \tag{4.3}$$

for $x, y \in \mathfrak{g}$ *and* $a, a' \in \mathcal{A}$; *we refer to* $\mathcal{A}$ *as the coordinate algebra of* $\mathfrak{L}$.

Proof. Using [8, Thm. 5.2], it is enough to prove the theorem for type $C(J)$ where J is an infinite index set and for type $D(I, J)$ where I and J are index sets with $|I|, |J| \geq 2$ and $|I \cup J| = \infty$. As the proofs for the two cases are similar, we consider just type $C(J)$. In this case, $\mathfrak{g}$ can be considered to be $\mathfrak{osp}(2\{1\}, 2J)$. Using the same notation as in Section 2, the root system of $\mathfrak{g}$, with respect $\mathfrak{h}$, is $\{\pm\delta_p \pm \delta_q, \pm\epsilon_1 \pm \delta_p \mid p, q \in J\}$. Fix a finite subset J_0 of J of cardinal number greater than 3. Then take Γ to be an index set containing 0 and $\{J_\gamma \mid \gamma \in \Gamma\}$ to be the class of all finite subsets of J of size greater than 3 that contains J_0. Now for $\gamma \in \Gamma$, take $R(\gamma)$ to be $\{\pm\delta_p \pm \delta_q, \pm\epsilon_1 \pm \delta_p \mid p, q \in J_\gamma\}$ and set

$$\mathfrak{L}(\gamma) := \sum_{\alpha \in R(\gamma)\setminus\{0\}} \mathfrak{L}^\alpha \oplus \sum_{\alpha \in R(\gamma)\setminus\{0\}} [\mathfrak{L}^\alpha, \mathfrak{L}^{-\alpha}]$$

and

$$\mathfrak{g}(\gamma) := \sum_{\alpha \in R(\gamma)\setminus\{0\}} \mathfrak{g}^\alpha \oplus \sum_{\alpha \in R(\gamma)\setminus\{0\}} [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}].$$

Fix $\gamma \in \Gamma \setminus \{0\}$. It is not hard to see that the above decomposition for $\mathfrak{L}(\gamma)$ is in fact the weight space decomposition of $\mathfrak{L}(\gamma)$ with respect to $\mathfrak{h} \cap \mathfrak{g}(\gamma)$. So using [10, Lem. 2.2] together with [8, §2, Prop. 3.1], $\mathfrak{L}(\gamma)$ is a completely reducible $\mathfrak{g}(\gamma)$-module such that each of the irreducible constituents is either isomorphic to $\mathfrak{g}(\gamma)$ or to the trivial $\mathfrak{g}(\gamma)$-module; in particular, there are a trivial $\mathfrak{g}(\gamma)$-module $\mathcal{D}_\gamma$ and a class $\{\mathfrak{g}(\gamma)^{(t)} \mid t \in T_\gamma\}$ of irreducible $\mathfrak{g}(\gamma)$-modules isomorphic to $\mathfrak{g}(\gamma)$ such that

$$\mathfrak{L}(\gamma) = \sum_{t \in T_\gamma} \mathfrak{g}(\gamma)^{(t)} \oplus \mathcal{D}_\gamma.$$

For $t \in T_\gamma$, set

$$\mathfrak{g}(0)^{(t)} := \sum_{\alpha \in R(0)\setminus\{0\}} (\mathfrak{g}(\gamma)^{(t)})^\alpha \oplus \sum_{\alpha \in R(0)\setminus\{0\}} [\mathfrak{g}^\alpha, (\mathfrak{g}(\gamma)^{(t)})^{-\alpha}]$$

in which for $\alpha \in R(\gamma)$, $(\mathfrak{g}(\gamma)^{(t)})^\alpha$ is the weight space of $\mathfrak{g}(\gamma)$-module $\mathfrak{g}(\gamma)^{(t)}$ of weight α. Then $\mathfrak{g}(0)^{(t)}$ is a $\mathfrak{g}(0)$-module isomorphic to $\mathfrak{g}(0)$. But using the same reasoning as above, one gets that $\mathfrak{L}(0)$ is a completely reducible $\mathfrak{g}(0)$-module which in turn implies that there is a $\mathfrak{g}(0)$-submodule $\mathcal{D}$ of $\mathfrak{L}(0)$ with $\mathfrak{L}(0) = \sum_{t\in T_\gamma} \mathfrak{g}(0)^{(t)} \oplus \mathcal{D}$. Now it follows that $\mathcal{D}$ is a trivial $\mathfrak{g}(0)$-module since for each $\alpha \in R(0) \setminus \{0\}$, $\mathfrak{L}(0)^\alpha = \mathfrak{L}(\gamma)^\alpha = \sum_{t\in T_\gamma} (\mathfrak{g}(\gamma)^{(t)})^\alpha \subseteq \sum_{t\in T_\gamma} \mathfrak{g}(0)^{(t)}$.

Next contemplating Remark A.3, we take $\mathcal{A}$ to be a superspace with a basis $\{a_t \mid t \in T_\gamma\}$ in which for $t \in T_\gamma$, a_t is of degree $i \in \mathbb{Z}_2$ if the isomorphism between $\mathfrak{g}(\gamma)$ and $\mathfrak{g}(\gamma)^t$ is of parity i. Using [8, Thm. 5.2], there are bilinear maps " $\cdot$ " and " $*$ ", as well as bilinear maps $\langle\cdot,\cdot\rangle_\gamma$ and $\langle\cdot,\cdot\rangle_0$ on $\mathcal{A}\times\mathcal{A}$, such that $(\mathcal{A}, *)$ and $(\mathcal{A}, \cdot)$ are unital supercommutative associative superalgebras, (4.2) is satisfied for $\langle\cdot,\cdot\rangle_0$ and $\langle\cdot,\cdot\rangle_\gamma$ in place of $\langle\cdot,\cdot\rangle$ and $\mathfrak{L}(\gamma)$ can be identified with $(\mathfrak{g}(\gamma)\otimes\mathcal{A})\oplus\mathcal{D}_\gamma$ with the Lie bracket

$$[x\otimes a, y\otimes a'] = (-1)^{|a|\,|y|}([x,y]_{\mathfrak{g}} \otimes (a * a') + \mathrm{str}(xy)\langle a, a'\rangle_\gamma),$$
$$[(\mathfrak{g}(\gamma)\otimes\mathcal{A})\oplus\mathcal{D}_\gamma, \mathcal{D}_\gamma] = \{0\},$$

for $x, y \in \mathfrak{g}(\gamma)$ and $a, a' \in \mathcal{A}$. Similarly, $\mathfrak{L}(0)$ can be identified with $(\mathfrak{g}(0)\otimes\mathcal{A})\oplus\mathcal{D}$ with the Lie bracket

$$[x\otimes a, y\otimes a'] = (-1)^{|a|\,|y|}([x,y]_{\mathfrak{g}} \otimes (a \cdot a') + \mathrm{str}(xy)\langle a, a'\rangle_0),$$
$$[(\mathfrak{g}(0)\otimes\mathcal{A})\oplus\mathcal{D}, \mathcal{D}] = \{0\},$$

for $x, y \in \mathfrak{g}(0)$ and $a, a' \in \mathcal{A}$. Now taking $x, y \in \mathfrak{g}(0)$ to be such that $[x,y]_{\mathfrak{g}} \neq 0$ and $\mathrm{str}(xy) = 0$, we have for all $a, a' \in \mathcal{A}$ that

$$[x,y]_{\mathfrak{g}} \otimes (a * a') = [x\otimes a, y\otimes a'] = [x,y]_{\mathfrak{g}} \otimes (a\cdot a')$$

which in turn implies that

(4.4) the coordinate algebra $(\mathcal{A}, \cdot)$ of $\mathfrak{L}(\gamma)$ coincides with the coordinate algebra $(\mathcal{A}, *)$ of $\mathfrak{L}(0)$.

Also for $a, a' \in \mathcal{A}$ and $x, y \in \mathfrak{g}(0)$ with $\mathrm{str}(xy) \neq 0$, we have

$$\begin{aligned}[x,y]_{\mathfrak{g}} \otimes (a\cdot a') + \mathrm{str}(xy)\langle a, a'\rangle_\gamma &= [x\otimes a, y\otimes a'] \\ &= [x,y]_{\mathfrak{g}} \otimes (a\cdot a') + \mathrm{str}(xy)\langle a, a'\rangle_0.\end{aligned}$$

This implies $\langle a, a'\rangle_\gamma = \langle a, a'\rangle_0$ for all $a, a' \in \mathcal{A}$. Therefore,

$$Z(\mathfrak{L}(0)) = \mathcal{D} = \langle\mathcal{A},\mathcal{A}\rangle_0 = \langle\mathcal{A},\mathcal{A}\rangle_\gamma = \mathcal{D}_\gamma = Z(\mathfrak{L}(\gamma)) \quad (\gamma\in\Gamma). \tag{4.5}$$

Now (4.4) and (4.5) together with the fact that $\mathfrak{L}$ is the direct union of $\mathfrak{L}(\gamma)$'s, implies that we can identify $\mathfrak{L}$ with $(\mathfrak{g}\otimes\mathcal{A})\oplus\mathcal{D}$ and $\mathfrak{L}(\gamma)$ $(\gamma\in\Gamma)$ with $(\mathfrak{g}(\gamma)\otimes\mathcal{A})\oplus\mathcal{D}$. Moreover, the above discussion shows that all assertions of the theorem are fulfilled. □

As we proved in Theorem 4.2, if R is an irreducible locally finite root supersystem of type $D(2,1,\alpha)$, $F(4)$, $G(3)$, $C(J)$ and $D(I,J)$ and $\mathfrak{L}$ is an R-graded Lie superalgebra with grading pair $(\mathfrak{g},\mathfrak{h})$, then there are a central subsuperalgebra $\mathcal{D}$ of $\mathfrak{L}$ and a unital supercommutative associative superalgebra $\mathcal{A}$ such that $\mathcal{L}$ can be identified with $(\mathfrak{g}\otimes\mathcal{A})\oplus\mathcal{D}$. Now if $\mathcal{L}$ is centerless, we get that $\mathcal{D}=\{0\}$ and so $\mathcal{L}$ is nothing but $\mathfrak{g}\otimes\mathcal{A}$ with the natural Lie bracket. The most interesting examples of root-graded Lie (super)algebras are predivision (R,Λ)-graded Lie (super)algebras for an abelian group Λ. In the following theorem, we give the structure of some centerless predivision (R,Λ)-graded Lie superalgebra $\mathcal{L}$ where Λ is an abelian group and R is an irreducible locally finite root supersystem of type $D(2,1,\alpha)$, $F(4)$, $G(3)$, $C(J)$ and $D(I,J)$. As we mentioned above, $\mathcal{L}$ can be identified with $\mathfrak{g}\otimes\mathcal{A}$ where $\mathfrak{g}$ is its grading subalgebra and $\mathcal{A}$ is a unital supercommutative associative superalgebra, but as $\mathcal{L}$ is predivision, we will prove that $\mathcal{A}$ is a unital commutative associative algebra.

Theorem 4.3. *Suppose that Λ is an additive abelian group and $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ is a locally finite basic classical simple Lie superalgebra with the standard Cartan subalgebra $\mathfrak{h}$ and root system R which is of one of the types $C(J)$ $(|J|\geq 2)$, $D(I,J)$ $(|I|,|J|\geq 2)$, $G(3), F(4)$ or $D(2,1,\alpha)$ $(\alpha\neq 0,-1)$. Suppose that*

- *$\mathfrak{L}$ is a predivision centerless (R,Λ)-graded Lie superalgebra with grading pair $(\mathfrak{g},\mathfrak{h})$ such that $\mathfrak{g}\subseteq {}^0\mathcal{L}$;*
- *for each $\dot\alpha\in R\setminus\{0\}$ and $\lambda\in\Lambda$, $\dim({}^\lambda\mathfrak{L}^{\dot\alpha})\leq 1$.*

Then there is a Λ-graded unital commutative associative algebra $\mathcal{A}=\oplus_{\lambda\in\Lambda}{}^\lambda\mathcal{A}$ such that

- *each nonzero homogeneous element of $\mathcal{A}$ is invertible;*
- *Λ-homogeneous spaces are of dimension less than or equal to 1;*
- *$\mathfrak{L}$ is isomorphic to the Lie superalgebra $\mathfrak{g}\otimes\mathcal{A}$ whose Lie bracket is defined by*

$$[x\otimes a, y\otimes b]:=[x,y]_{\mathfrak{g}}\otimes ab$$

for $x,y\in\mathfrak{g}$ and $a,b\in\mathcal{A}$.

Proof. Since $\mathfrak{L}$ is centerless, using Theorem 4.2 one knows that there is a unital supercommutative associative superalgebra $\mathcal{A}$ such that $\mathfrak{L}$ is isomorphic to the Lie superalgebra $\mathfrak{g}\otimes\mathcal{A}$ whose Lie bracket is defined by

$$[x\otimes a, y\otimes b]=(-1)^{|a|\,|y|}[x,y]_{\mathfrak{g}}\otimes ab$$

for $x,y\in\mathfrak{g}$ and $a,b\in\mathcal{A}$. We mention that as vector spaces are flat, for a subspace X of $\mathfrak{g}$ and a subspace B of $\mathcal{A}$, we consider $X\otimes B$ as a subspace of $\mathfrak{g}\otimes\mathcal{A}$. Now we identify $\mathfrak{L}$ with $\mathfrak{g}\otimes\mathcal{A}$ and $\mathfrak{g}$ with $\mathfrak{g}\otimes\mathbb{F}$; in particular, for $\dot\alpha\in R$, we have $\mathfrak{L}^{\dot\alpha}=\mathfrak{g}^{\dot\alpha}\otimes\mathcal{A}$. Since for $\dot\alpha\in R\setminus\{0\}$ and $\lambda\in\Lambda$, we have $\dim({}^\lambda\mathfrak{L}^{\dot\alpha})\leq 1$, there is a subspace ${}^\lambda\mathcal{A}^{\dot\alpha}$ of $\mathcal{A}$ such that $\dim({}^\lambda\mathcal{A}^{\dot\alpha})\leq 1$ and ${}^\lambda\mathfrak{L}^{\dot\alpha}=\mathcal{G}^{\dot\alpha}\otimes{}^\lambda\mathcal{A}^{\dot\alpha}$. We claim that

$$\text{for } \lambda\in\Lambda \text{ and } \dot\alpha,\dot\beta\in R^\times, \text{ we have } {}^\lambda\mathcal{A}^{\dot\alpha}={}^\lambda\mathcal{A}^{\dot\beta}. \tag{4.6}$$

We first show that

$$\text{for } \lambda\in\Lambda \text{ and } \dot\alpha,\dot\beta\in R^\times \text{ with } \dot\alpha-\dot\beta\in R^\times, \text{ we have } {}^\lambda\mathcal{A}^{\dot\alpha}={}^\lambda\mathcal{A}^{\dot\beta}. \tag{4.7}$$

Indeed, if ${}^\lambda\mathcal{A}^{\dot\alpha}=\{0\}$ and ${}^\lambda\mathcal{A}^{\dot\beta}=\{0\}$, there is nothing to prove; otherwise, say for example ${}^\lambda\mathcal{A}^{\dot\alpha}\neq\{0\}$, then we have

$$\mathfrak{g}^{\dot\beta}\otimes{}^\lambda\mathcal{A}^{\dot\alpha}=[\mathfrak{g}^{\dot\beta-\dot\alpha}\otimes\mathbb{F}1,\mathfrak{g}^{\dot\alpha}\otimes{}^\lambda\mathcal{A}^{\dot\alpha}]\subseteq[{}^0\mathfrak{L}^{\dot\beta-\dot\alpha},{}^\lambda\mathfrak{L}^{\dot\alpha}]\subseteq{}^\lambda\mathfrak{L}^{\dot\beta}=\mathfrak{g}^{\dot\beta}\otimes{}^\lambda\mathcal{A}^{\dot\beta}.$$

This implies ${}^\lambda\mathcal{A}^{\dot\alpha}={}^\lambda\mathcal{A}^{\dot\beta}$.

Now we are ready to prove (4.6). We first assume R is of type $D(2,1,\alpha)$. So R_{re} is of type $A_1\oplus A_1\oplus A_1$ and we may assume

$$R_{re}=\{0\}\cup\{\pm2\epsilon\}\cup\{\pm2\delta\}\cup\{\pm2\gamma\}\quad\text{and}\quad R_{ns}^\times=\{\pm\epsilon\pm\delta\pm\gamma\}.$$

It is immediate that

$$\begin{aligned}
2\epsilon-(\epsilon\pm\delta\pm\gamma)&\in R^\times, & 2\delta-(\epsilon+\delta\pm\gamma)&\in R^\times,\\
-2\delta-(\epsilon-\delta\pm\gamma)&\in R^\times, & 2\gamma-(\epsilon\pm\delta+\gamma)&\in R^\times,\\
-2\gamma-(\epsilon\pm\delta-\gamma)&\in R^\times, & 2\gamma-(-\epsilon\pm\delta+\gamma)&\in R^\times,\\
-2\gamma-(-\epsilon\pm\delta-\gamma)&\in R^\times, & -2\epsilon-(-\epsilon\pm\delta\pm\gamma)&\in R^\times.
\end{aligned}$$

Therefore using (4.7), we get that (4.6) is satisfied in this case. Next assume R is not of type $D(2,1,\alpha)$; contemplating Table 1, we take $n\in\{1,2,3\}$ and $R_1,\ldots,R_n$ to be irreducible components of locally finite root system R_{re}. Since R is not of type $D(2,1,\alpha)$, there is $i^*\in\{1,\ldots,n\}$ such that R_{i^*} is an irreducible locally finite root system of type G_2, B_T or C_T, for an index set T with $|T|\geq 2$, or D_T for an index set T with $|T|\geq 3$. Now we prove (4.6) through the following steps:

Step 1: $\dot\alpha,\dot\beta\in R_{i^*}^\times$. Since R_{i^*} is one of the types mentioned above, there is a sequence $\dot\alpha_1,\ldots,\dot\alpha_m\in R_{i^*}^\times$ such that $\dot\alpha_1=\dot\alpha$, $\dot\alpha_m=\dot\beta$ and $\dot\alpha_t-\dot\alpha_{t+1}\in R_{i^*}^\times$ for $t\in\{1,\ldots,m-1\}$. Now using (4.7), we are done.

Step 2: $\dot\alpha\in R_{ns}^\times$ and $\dot\beta=-\dot\alpha$. Using Remark 2.2(2), we take $\dot\gamma\in R_{i^*}$ to be such that $(\dot\alpha,\dot\gamma)\neq 0$, so there is $r\in\{\pm1\}$ such that $\dot\alpha+r\dot\gamma\in R^\times$. Therefore, by (4.7) and Step 1, we have

$${}^\lambda\mathcal{A}^{\dot\alpha}={}^\lambda\mathcal{A}^{-r\dot\gamma}={}^\lambda\mathcal{A}^{r\dot\gamma}={}^\lambda\mathcal{A}^{-\dot\alpha}.$$

Step 3: $\dot\alpha \in R_i^\times$ for some $i \in \{1, \dots, n\}$ and $\dot\beta = -\dot\alpha$. We first claim that there is $\dot\delta \in R_{\rm ns}$ such that $(\dot\alpha, \dot\delta) \neq 0$. To the contrary assume $(\dot\alpha, R_{\rm ns}) = \{0\}$. This, together with Remark 2.2(3), implies $(\dot\gamma, R_{\rm ns}) = \{0\}$ for all $\dot\gamma \in R_i$ of the same length as $\dot\alpha$. But one knows that R_i is a *direct union* of finite root systems, so it follows from finite root system theory (see e.g. [15, Lem. 10.4.B]) that $\mathrm{span}_{\mathbb{Q}}\{\dot\gamma \in R_i \mid (\dot\alpha, \dot\alpha) = (\dot\gamma, \dot\gamma)\} = \mathrm{span}_{\mathbb{Q}} R_i$, which in turn implies $(R_i, R_{\rm ns}) = \{0\}$. This contradicts Remark 2.2(2). Therefore, there is $\dot\delta \in R_{\rm ns}$ such that $(\dot\alpha, \dot\delta) \neq 0$. So $\dot\alpha + s\dot\delta \in R^\times$ for some $s \in \{\pm 1\}$. Now using (4.7) and Step 2, we have

$$^\lambda\mathcal{A}^{\dot\alpha} = {^\lambda\mathcal{A}^{-s\dot\delta}} = {^\lambda\mathcal{A}^{s\dot\delta}} = {^\lambda\mathcal{A}^{-\dot\alpha}}.$$

Step 4: $\dot\alpha, \dot\beta \in R^\times$. Since R is irreducible, there are $\dot\alpha_1, \dots, \dot\alpha_m$ such that for $t \in \{1, \dots, m-1\}$, $\dot\alpha_t \neq \pm\dot\alpha_{t+1}$ and $(\dot\alpha_t, \dot\alpha_{t+1}) \neq 0$ with $\dot\alpha_1 = \dot\alpha$, $\dot\alpha_m = \dot\beta$. So for each $t \in \{1, \dots, m-1\}$, there is $r \in \{\pm 1\}$ such that $\dot\alpha_t + r\dot\alpha_{t+1} \in R^\times$. Therefore, by (4.7) and Steps 2 and 3, we have

$$^\lambda\mathcal{A}^{\dot\alpha_t} = {^\lambda\mathcal{A}^{-r\dot\alpha_{t+1}}} = {^\lambda\mathcal{A}^{r\dot\alpha_{t+1}}};$$

in particular, ${^\lambda\mathcal{A}^{\dot\alpha}} = {^\lambda\mathcal{A}^{\dot\beta}}$.

This completes the proof of (4.6). Now for $\lambda \in \Lambda$, set

$$^\lambda\mathcal{A} := {^\lambda\mathcal{A}^{\dot\alpha}} \quad (\dot\alpha \in R^\times).$$

Since for $\dot\alpha \in R^\times$,

$$\mathfrak{g}^{\dot\alpha} \otimes \mathcal{A} = \mathcal{L}^{\dot\alpha} = \sum_{\lambda\in\Lambda} {^\lambda\mathcal{L}^{\dot\alpha}} = \sum_{\lambda\in\Lambda} (\mathfrak{g}^{\dot\alpha} \otimes {^\lambda\mathcal{A}^{\dot\alpha}}) = \sum_{\lambda\in\Lambda} (\mathfrak{g}^{\dot\alpha} \otimes {^\lambda\mathcal{A}}) = \mathfrak{g}^{\dot\alpha} \otimes \sum_{\lambda\in\Lambda} {^\lambda\mathcal{A}},$$

we have $\mathcal{A} = \oplus_{\lambda\in\Lambda} {^\lambda\mathcal{A}}$ with $\dim({^\lambda\mathcal{A}}) \leq 1$ for all $\lambda \in \Lambda$. Also, since $\mathcal{L}^0 = \sum_{\dot\alpha\in R^\times} [\mathcal{L}^{\dot\alpha}, \mathcal{L}^{-\dot\alpha}]$, it follows that

$$^\lambda\mathcal{L}^{\dot\alpha} = \mathfrak{g}^{\dot\alpha} \otimes {^\lambda\mathcal{A}}, \quad \lambda \in \Lambda,\ \dot\alpha \in R.$$

This in turn implies that $\mathcal{A} = \oplus_{\lambda\in\Lambda} {^\lambda\mathcal{A}}$ is a Λ-graded superalgebra.

Next suppose $\lambda \in \Lambda$ and $0 \neq a \in {^\lambda\mathcal{A}}$; we want to show that a is invertible. We first note that since $\dim({^\lambda A}) \leq 1$, a is homogeneous. Fix $\dot\alpha \in R \setminus \{0\}$ and a nonzero element $e \in \mathfrak{g}^{\dot\alpha}$. Since $\mathcal{L}$ is predivision, there is $f \in \mathfrak{g}^{-\dot\alpha}$ and $b \in {^{-\lambda}\mathcal{A}}$ such that

$$0 \neq [e, f]_{\mathfrak{g}} \otimes ab = [e \otimes a, f \otimes b] \in \mathcal{L}_0 = (\mathfrak{g}_0 \otimes \mathcal{A}_0) \oplus (\mathfrak{g}_1 \otimes \mathcal{A}_1).$$

This, together with the fact that $[e, f]_{\mathfrak{g}} \in \mathfrak{g}^0 = \mathfrak{h} \subseteq \mathfrak{g}_0$, implies $ab \in \mathcal{A}_0$. Since $\dim({^0\mathcal{A}}) = 1$, this implies $0 \neq ab \in \mathcal{A}_0 \cap {^0\mathcal{A}} = \mathbb{F}1_{\mathcal{A}}$. Similarly, $ba \in \mathbb{F}$, therefore a is invertible. But $\mathcal{A}$ is supercommutative and so invertibility of a guarantees that the homogeneous element a is of degree 0; in particular, $\mathcal{A}_1 = \{0\}$. This completes the proof. ∎

Theorem 4.4. *Suppose that* $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ *is an extended affine Lie superalgebra with* $\mathcal{L}^0 = \mathcal{H}$*. Suppose that the root system* R *of* $\mathcal{L}$ *is an irreducible extended affine root supersystem of finite type* $X = C(n)$ $(n \geq 2)$, $D(m,n)$ $(m,n \geq 2)$, $G(3), F(4)$ *or* $D(2,1,\alpha)$ $(\alpha \neq 0,-1)$*; then there is an additive abelian group* Λ*, a finite-dimensional basic classical simple Lie superalgebra* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *and a* Λ*-graded unital commutative associative algebra* $\mathcal{A} = \oplus_{\lambda\in\Lambda}{}^{\lambda}\mathcal{A}$ *such that each nonzero homogeneous element of* $\mathcal{A}$ *is invertible,* Λ*-homogeneous spaces of* $\mathcal{A}$ *are of dimension less than or equal to* 1 *and* $\mathcal{L}_c/Z(\mathcal{L}_c)$ *is isomorphic to the Lie superalgebra* $\mathfrak{g}\otimes\mathcal{A}$ *whose Lie bracket is defined by*

$$[x\otimes a, y\otimes b] = [x,y]_{\mathfrak{g}}\otimes ab$$

for $x, y \in \mathfrak{g}$ *and* $a, b \in \mathcal{A}$.

Proof. Keep the same notation as in the proof of Theorem 3.3 and for $\dot\alpha \in \dot R$, take $S_{\dot\alpha} := \{\sigma \in V^0 \mid \dot\alpha + \sigma \in R\}$. Set $\Lambda := \langle \cup_{\dot\alpha\in\dot R^\times} S_{\dot\alpha}\rangle$. Then using Theorem 3.3 and the same argument as in [29, Lem. 3.4], we get that $\mathcal{L}_c/Z(\mathcal{L}_c)$ satisfies the conditions of Theorem 4.3 and so we are done. □

Appendix A.

Appendix A.1. Preliminaries

For superspaces U, V and W, a bilinear map $f : U\times V \longrightarrow W$ is called *even* (or *homogeneous of degree* 0) if $f(U_r, V_s) \subseteq W_{r+s}$ for all $r, s \in \mathbb{Z}_2$ and is called *odd* or (*homogeneous of degree* 1) if $f(U_r, V_s) \subseteq W_{r+s+1}$ for all $r, s \in \mathbb{Z}_2$; moreover, if $U = V$, it is called *supersymmetric* (resp. *superskewsymmetric*) if $f(x,y) = (-1)^{|x|\,|y|}f(y,x)$ (resp. $f(x,y) = -(-1)^{|x|\,|y|}f(y,x)$) for all $x, y \in U$.

For an additive abelian group Λ and a superspace $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$, a class $\{\mathcal{V}^\lambda \mid \lambda \in \Lambda\}$ of subsuperspaces of $\mathcal{V}$ is called a Λ-*grading* on $\mathcal{V}$ if $\mathcal{V} = \oplus_{\lambda\in\Lambda}\mathcal{V}^\lambda$. In this case, we say that $\mathcal{V} = \oplus_{\lambda\in\Lambda}\mathcal{V}^\lambda$ is a Λ-*graded* superspace and note that, for $i \in \mathbb{Z}_2$, we have $\mathcal{V}_i = \bigoplus_{\lambda\in\Lambda}\mathcal{V}_i^\lambda$ in which $\mathcal{V}_i^\lambda := \mathcal{V}_i \cap \mathcal{V}^\lambda$ $(\lambda \in \Lambda)$. A bilinear form $(\cdot,\cdot)$ on $\mathcal{V}$ is called Λ-*graded* if $(\mathcal{V}^\lambda, \mathcal{V}^\mu) = \{0\}$ for all $\lambda, \mu \in \Lambda$ with $\lambda + \mu \neq 0$. A subsuperspace $\mathcal{W}$ of a Λ-graded vector space $\mathcal{V} = \oplus_{\lambda\in\Lambda}\mathcal{V}^\lambda$ is called a Λ-*graded subsuperspace* if $\{\mathcal{V}^\lambda \cap \mathcal{W} \mid \lambda \in \Lambda\}$ is a Λ-grading on $\mathcal{W}$.

Appendix A.2. Superalgebras

A superspace $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, together with a bilinear map

$$\cdot : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A},$$

referred to as the *product* of $\mathcal{A}$, is called a *superalgebra* if $\mathcal{A}_i \cdot \mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ for all $i, j \in \mathbb{Z}_2$. The superalgebra $\mathcal{A}$ is called *unital* if there is $1_{\mathcal{A}} \in \mathcal{A}$ with $a \cdot 1_{\mathcal{A}} =$

$1_{\mathcal{A}} \cdot a = a$ for all $a \in \mathcal{A}$. A superspace I of $\mathcal{A}$ is called a *subsuperalgebra* (resp. a *left ideal*) of $\mathcal{A}$ if $I \cdot I \subseteq I$ (resp. $\mathcal{A} \cdot I \subseteq I$). The superalgebra $\mathcal{A}$ is called *associative* if $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in \mathcal{A}$ and it is called *supercommutative* if $a \cdot b = (-1)^{|a|\,|b|} b \cdot a$ for all $a, b \in \mathcal{A}$. For simplicity, we denote the product of two elements a, b of an associative superalgebra by ab. An element a of a unital associative superalgebra $\mathcal{A}$ is called invertible if there is $b \in \mathcal{A}$ with $ab = ba = 1_{\mathcal{A}}$. For two superalgebra $\mathcal{A}$ and $\mathcal{B}$, a homogeneous linear transformation ϕ of degree 0 is called a (*superalgebra*) *homomorphism* if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{A}$; monomorphisms, epimorphisms, isomorphisms and endomorphisms are defined in the usual sense.

Supermatrices with entries in $\mathbb{F}$, as introduce in the following, are examples of superalgebras. For a nonempty superset $I = I_0 \uplus I_1$, we use $|i|$ to denote the degree of an element $i \in I$. Suppose that $\mathcal{A}$ is a unital associative superalgebra and I and J are nonempty supersets; by an $(I \times J)$-(*super*)*matrix* with entries in $\mathcal{A}$, we mean a map $A : I \times J \longrightarrow \mathcal{A}$. For $i \in I$, $j \in J$, we set $a_{ij} := A(i, j)$ and call it the (i, j)*th entry* of A. By convention, we denote the (super)matrix A by (a_{ij}). We also denote the set of all $(I \times J)$-(super)matrices with entries in $\mathcal{A}$ by $\mathcal{A}^{I\times J}$.

For $A = (a_{ij}) \in \mathcal{A}^{I\times J}$, the matrix $B = (b_{ij}) \in \mathcal{A}^{J\times I}$ with

$$b_{ij} := \begin{cases} a_{ji} & \text{if } |i| = |j|, \\ a_{ji} & \text{if } |i| = 1,\ |j| = 0, \\ -a_{ji} & \text{if } |i| = 0,\ |j| = 1 \end{cases}$$

is called the *supertransposition* of A and denoted by A^{st}. If $A = (a_{ij}) \in \mathcal{A}^{I\times J}$ and $B = (b_{ij}) \in \mathcal{A}^{J\times K}$ are such that for all $i \in I$ and $k \in K$, at most for finitely many $j \in J$, the $a_{ij}b_{jk}$'s are nonzero, we define the product AB of A and B to be the $(I \times K)$-matrix $C = (c_{ik})$ with $c_{ik} := \sum_{j\in J} a_{ij}b_{jk}$ for all $i \in I$, $k \in K$. We note that if A, B, C are three matrices such that AB, $(AB)C$, BC and $A(BC)$ are defined, then $A(BC) = (AB)C$. If I is a disjoint union of subsets $I^1, \ldots, I^t$ of I, then for an $(I \times I)$-matrix A we write

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,t} \\ A_{2,1} & \cdots & A_{2,t} \\ \vdots & \vdots & \vdots \\ A_{t,1} & \cdots & A_{t,t} \end{bmatrix},$$

in which for $1 \le r, s \le t$, $A_{r,s}$ is an $(I^r \times I^s)$-matrix whose (i, j)th entry coincides with the (i, j)th entry of A for all $i \in I^r$, $j \in I^s$. In this case, we say that $A \in \mathcal{A}^{I^1\uplus\cdots\uplus I^t}$ and note that the defined matrix product obeys the product of block matrices. If $\{a_i \mid i \in I\} \subseteq \mathcal{A}$, by $\mathrm{diag}(a_i)$ we mean an $(I \times I)$-matrix whose (i,i)th entry is a_i for all $i \in I$ and other entries are 0. If $\mathcal{A}$ is unital, we set $1_I := \mathrm{diag}(1_{\mathcal{A}})$. A matrix $A \in \mathcal{A}^{I\times I}$ is called *invertible* if there is a matrix $B \in \mathcal{A}^{I\times I}$ such that AB, as well as BA, is defined and $AB = BA = 1_I$; such a B is unique and denoted by A^{-1}. For $i \in I$, $j \in J$ and $a \in \mathcal{A}$, we define $E_{ij}(a)$ to be the matrix in $\mathcal{A}^{I\times J}$ whose (i,j)th entry is a and other entries are 0. If $\mathcal{A}$ is unital, we set

$$e_{i,j} := E_{i,j}(1_{\mathcal{A}}).$$

Take $M_{I\times J}(\mathcal{A})$ to be the subspace of $\mathcal{A}^{I\times J}$ spanned by $\{E_{ij}(a) \mid i \in I,\ j \in J,\ a \in A\}$. It is a superspace with $M_{I\times J}(\mathcal{A})_i := \mathrm{span}_{\mathbb{F}}\{E_{r,s}(a) \mid |r| + |s| + |a| = i\}$, for $i \in \mathbb{Z}_2$. Also with respect to the multiplication of matrices, the vector superspace $M_{I\times I}(\mathcal{A})$ is an associative $\mathbb{F}$-superalgebra and so it is a Lie superalgebra under the Lie bracket $[A,B] := AB - (-1)^{|A|\,|B|}BA$ for all $A, B \in M_{I\times I}(\mathcal{A})$. We denote this Lie superalgebra by $\mathfrak{pl}_I(\mathcal{A})$ and if $\mathcal{A} = \mathbb{F}$, we denote it also by $\mathfrak{pl}_{\mathbb{F}}(I_0, I_1)$. For an element $X \in \mathfrak{pl}_I(\mathcal{A})$, we define the *supertrace* of X to be $\mathrm{str}(X) := \sum_{i\in I}(-1)^{|i|}x_{i,i}$. We finally note that for $X, Y \in \mathfrak{pl}_I(\mathcal{A})$, we have $(XY)^{\mathrm{st}} = (-1)^{|X|\,|Y|}Y^{\mathrm{st}}X^{\mathrm{st}}$.

For an additive abelian group Λ, a superalgebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is said to be *equipped with a* Λ*-grading* $\mathcal{A} = \oplus_{\lambda\in\Lambda}\mathcal{A}^\lambda$ if $\{\mathcal{A}^\lambda \mid \lambda \in \Lambda\}$ is a Λ-grading on the superspace $\mathcal{A}$ with $\mathcal{A}^\lambda \cdot \mathcal{A}^\mu \subseteq \mathcal{A}^{\lambda+\mu}$ for $\lambda, \mu \in \Lambda$; we usually say that $\mathcal{A}$ is Λ-graded. For two additive abelian groups Λ and Γ, we say that a superalgebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is a (Λ,Γ)*-graded* superalgebra if $\mathcal{A}$ is simultaneously equipped with a Λ-grading $\mathcal{A} = \bigoplus_{\lambda\in\Lambda}\mathcal{A}^\lambda$ and a Γ-grading $\mathcal{A} = \bigoplus_{\xi\in\Gamma}{}^{\xi}\mathcal{A}$ such that these two gradings are *compatible* in the sense that for $\lambda \in \Lambda$, $\mathcal{A}^\lambda = \bigoplus_{\xi\in\Gamma}{}^{\xi}\mathcal{A}^\lambda$ in which for $\xi \in \Gamma$, ${}^{\xi}\mathcal{A}^\lambda := {}^{\xi}\mathcal{A} \cap \mathcal{A}^\lambda$.

Appendix A.3. Lie superalgebras

A superalgebra $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ with the product $[\cdot,\cdot]$ is called a *Lie superalgebra* if for all $a, b, c \in \mathcal{A}$,

(i) $[a,b] = -(-1)^{|a|\,|b|}[b,a]$ (anti-supercommutativity);
(ii) $[\,[a,b],c] = [a,[b,c]\,] - (-1)^{|a|\,|b|}[b,[a,c]\,]$ (the Jacobi superidentity).

A Lie superalgebra $\mathcal{L}$ is called *simple* if $[\mathcal{L},\mathcal{L}] := \mathrm{span}_{\mathbb{F}}\{[x,y] \mid x, y \in \mathcal{L}\} \neq \{0\}$ and the only left ideals of $\mathcal{L}$ are $\{0\}$ and $\mathcal{L}$. A bilinear form on a Lie superalgebra $(\mathcal{L},[\cdot,\cdot])$ is called *invariant* if $([x,y],z) = (x,[y,z])$ for $x, y, z \in \mathcal{L}$.

Suppose that $\mathfrak{L} = \mathfrak{L}_0 \oplus \mathfrak{L}_1$ is a Lie superalgebra equipped with a nondegenerate invariant even supersymmetric bilinear form $(\cdot,\cdot)$ and $\mathfrak{h}$ is a nontrivial subalgebra of $\mathcal{L}_0$. Suppose that

- $\mathfrak{L}$ has a weight space decomposition $\mathfrak{L} = \bigoplus_{\alpha\in\mathfrak{h}^*}\mathfrak{L}^\alpha$ with respect to $\mathfrak{h}$ via the adjoint representation;

- the restriction of the form $(\cdot,\cdot)$ to $\mathfrak{h}$ is nondegenerate.

We call $R := \{\alpha \in \mathfrak{h}^* \mid \mathfrak{L}^\alpha \neq \{0\}\}$, the *root system* of $\mathfrak{L}$ (with respect to $\mathfrak{h}$). Each element of R is called a *root*. We mention that since $\mathfrak{L}_0$, as well as $\mathfrak{L}_1$, is an $\mathfrak{h}$-submodule of $\mathfrak{L}$, we have using [18, Prop. 2.1.1] that $\mathfrak{L}_0 = \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{L}_0^\alpha$ and $\mathfrak{L}_1 = \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{L}_1^\alpha$ with $\mathfrak{L}_i^\alpha := \mathfrak{L}_i \cap \mathfrak{L}^\alpha$, $i = 0, 1$. We refer to the elements of $R_0 := \{\alpha \in \mathfrak{h}^* \mid \mathfrak{L}_0^\alpha \neq \{0\}\}$ (resp. $R_1 := \{\alpha \in \mathfrak{h}^* \mid \mathfrak{L}_1^\alpha \neq \{0\}\}$) as *even roots* (resp. *odd roots*). We note that $R = R_0 \cup R_1$.

Next suppose $h_1, h_2 \in \mathfrak{h}$ and $x \in \mathfrak{L}_0^\beta$ for an arbitrary element $\beta \in R_0$. Then

$$[\,[h_1, h_2], x] = \beta(h_1)\beta(h_2)x - \beta(h_2)\beta(h_1)x = 0;$$

in other words, $[h_1, h_2] \in Z(\mathfrak{L}_0)$. Now consider the decomposition $h_2 = \sum_{\alpha\in R_0} h^\alpha$ where, for $\alpha \in R_0$, $h^\alpha \in \mathfrak{L}_0^\alpha$, we have

$$\sum_{\alpha\in R_0} \alpha(h_1)h^\alpha = [h_1, h_2] \in Z(\mathfrak{L}_0).$$

So if $\alpha \in R_0$ and $\alpha(h_1) \neq 0$, we have $h^\alpha \in Z(\mathfrak{L}_0)$. Therefore, we have $[h_1, h_2] = \sum_{\alpha\in R_0}[h_1, h^\alpha] = 0$. This means that $\mathfrak{h}$ is abelian and so $\mathfrak{h} \subseteq \mathfrak{L}^0$.

The map $\mathfrak{p} : \mathfrak{h} \longrightarrow \mathfrak{h}^*$ mapping $h \in \mathfrak{h}$ to $(h, \cdot)$ is injective. So for each element α of the image $\mathfrak{h}^\mathfrak{p}$ of $\mathfrak{h}$ under $\mathfrak{p}$, there is a unique $t_\alpha \in \mathfrak{h}$ representing α through the form $(\cdot,\cdot)$. Now we can transfer the form on $\mathfrak{h}$ to a form on $\mathfrak{h}^\mathfrak{p}$, denoted again by $(\cdot,\cdot)$, and defined by

$$(\alpha, \beta) := (t_\alpha, t_\beta) \quad (\alpha, \beta \in \mathfrak{h}^\mathfrak{p}). \tag{A.1}$$

If $\mathfrak{h}$ is finite-dimensional, $\mathfrak{p}$ is a bijection. This means that in the case $\mathfrak{h}$ is finite-dimensional, we can transfer the form on $\mathfrak{h}$ to a form on $\mathrm{span}_\mathbb{F}(R) \subseteq \mathfrak{h}^\mathfrak{p}$ but in general we cannot transfer the form on $\mathfrak{h}$ naturally to a form on $\mathrm{span}_\mathbb{F}(R)$. But if for each $\alpha \in R$, there are $x \in \mathcal{L}^\alpha$ and $y \in \mathcal{L}^{-\alpha}$ with $0 \neq [x, y] \in \mathfrak{h}$, then α is an element of $\mathfrak{h}^\mathfrak{p}$ (see [27, Lem. 3.1]) and so we can restrict the form on $\mathfrak{h}^\mathfrak{p}$ to $\mathrm{span}_\mathbb{F} R$ to get a natural bilinear form on $\mathrm{span}_\mathbb{F}(R)$ via $\mathfrak{p}$.

Definition A.1. Suppose that $\mathcal{L}$ is a Lie superalgebra and M is a superspace; we say that M together with a bilinear map $\cdot : M \times \mathcal{L} \longrightarrow M$ is a *right $\mathcal{L}$-module* if

$$M_i \cdot \mathcal{L}_j \subseteq M_{i+j},$$
$$a \cdot [x, y] = (a \cdot x) \cdot y - (-1)^{|x|\,|y|}(a \cdot y) \cdot x$$

for $x, y \in \mathcal{L}$, $a \in M$ and $i, j \in \mathbb{Z}_2$. We also say that M together with a bilinear map $* : \mathcal{L} \times M \longrightarrow M$ is a *left $\mathcal{L}$-module* if

$$\mathcal{L}_i * M_j \subseteq M_{i+j},$$
$$[x, y] * a = x * (y * a) - (-1)^{|x|\,|y|} y * (x * a)$$

for $x, y \in \mathcal{L}$, $a \in M$ and $i, j \in \mathbb{Z}_2$. ⋄

If $\mathcal{L}$ is a Lie superalgebra and $(M, \cdot)$ is a right $\mathcal{L}$-module, then M together with the action

$$x * a := -(-1)^{|x|\,|a|} a \cdot x \quad (x \in \mathcal{L},\ a \in M) \tag{A.2}$$

is a left $\mathcal{L}$-module and conversely, if $(M, *)$ is a left $\mathcal{L}$-module, then M together with the action $a \cdot x := -(-1)^{|x|\,|a|} x * a$ ($x \in \mathcal{L}$, $a \in M$) is a right $\mathcal{L}$-module. In this text, when we say that M is an $\mathcal{L}$-module, we simultaneously consider M as a left and a right module whose actions interact as (A.2) and for simplicity, we denote both actions by juxtaposition. Submodules and irreducible modules are naturally defined.

Definition A.2. Suppose that $\mathcal{L}$ is a Lie superalgebra. A linear transformation $f : M \longrightarrow N$ from an $\mathcal{L}$-module M to an $\mathcal{L}$-module N is called an $\mathcal{L}$-module homomorphism if $f(vx) = f(v)x$ for all $x \in \mathcal{L}$ and $v \in M$. ⋄

Remark A.3. Suppose that $\mathcal{L}$ is a Lie superalgebra and $\mathcal{M}$ and $\mathcal{N}$ are two irreducible isomorphic $\mathcal{L}$-modules, say via $\phi : \mathcal{M} \longrightarrow \mathcal{N}$, such that as $\mathcal{L}_0$-modules $\mathcal{M}_0 \not\simeq \mathcal{M}_1$. Take $\phi = \phi_0 + \phi_1$ in which ϕ_0 and ϕ_1 are homogeneous components of ϕ. Since $\ker \phi_i$ is an $\mathcal{L}$-submodule of $\mathcal{M}$ and $\mathcal{M}$ is irreducible, we get that either ϕ_i is an isomorphism or it is 0. If $\phi_i \neq 0$ for both $i = 0$ and $i = 1$, we get that as $\mathcal{L}_0$-modules, $\mathcal{M}_0 \simeq \mathcal{N}_0$, $\mathcal{M}_1 \simeq \mathcal{N}_1$ (via ϕ_0) and $\mathcal{M}_0 \simeq \mathcal{N}_1$, $\mathcal{M}_1 \simeq \mathcal{N}_0$ (via ϕ_1). This in turn implies that as $\mathcal{L}_0$-modules, $\mathcal{M}_0 \simeq \mathcal{M}_1$, a contradiction, so at least one of ϕ_0 and ϕ_1 is 0; in particular, ϕ is homogeneous. ♢

Appendix A.4. Locally finite root systems

Definition A.4 ([17, Def. 3.3]). Suppose that V is a vector space. A subset R of V is called a *locally finite root system* if

- $0 \in R$ and R spans V;
- R is locally finite in the sense that the intersection of R with each finite-dimensional subspace W of V is a finite set;
- for $\alpha \in R^\times := R \setminus \{0\}$, there is $\check{\alpha} \in \mathcal{V}^*$ such that
  - $\check{\alpha}(\alpha) = 2$;

– $\check{\alpha}(\beta) \in \mathbb{Z}$, $\beta \in R$;
– R is invariant under the reflection r_α of $\mathcal{V}$ mapping $v \in \mathcal{V}$ to $v - \check{\alpha}(v)\alpha$.

A locally finite root system R is called *irreducible* if $R^\times$ cannot be written as a union of two orthogonal nonempty subsets of R. The *Weyl group* of R is defined to be the subgroup of $\mathrm{GL}(V)$ generated by $\{r_\alpha \mid \alpha \in R^\times\}$. A subset S of R is called a *subsystem* of R if $0 \in S$ and $r_\alpha(\beta) \in S$ for all $\alpha \in S \setminus \{0\}$ and $\beta \in S$. Two locally finite root systems (V_1, R_1) and (V_2, R_2) are called *isomorphic* if there is a linear isomorphism $\varphi : V_1 \longrightarrow V_2$ such that $\varphi(R_1) = R_2$. ◇

Lemma A.5 ([17, Prop. 3.13]). *Suppose that (V, R) is a locally finite root system. Then there is a nonempty index set I and a class $\{R_i \mid i \in I\}$ of irreducible subsystems of R such that $R = \cup_{i\in I} R_i$ and for $i, j \in I$ with $i \neq j$, $\alpha \in R_i^\times$ and $\beta \in R_j^\times$, we have $\check{\alpha}(\beta) = \check{\beta}(\alpha) = 0$. In this case we write $R = \oplus_{i\in I} R_i$.*

Suppose that T is a nonempty index set with $|T| \geq 2$, and $\mathcal{U} := \oplus_{i\in T}\mathbb{F}\epsilon_i$ is the free $\mathbb{F}$-module over the set T. Define the form

$$
\begin{aligned}
&(\cdot,\cdot) : \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{F},\\
&(\epsilon_i, \epsilon_j) \mapsto \delta_{i,j} \quad \text{for } i, j \in T,
\end{aligned}
$$

and set

$$
\begin{aligned}
\dot{A}_T &:= \{\epsilon_i - \epsilon_j \mid i, j \in T\},\\
D_T &:= \dot{A}_T \cup \{\pm(\epsilon_i + \epsilon_j) \mid i, j \in T,\ i \neq j\},\\
B_T &:= D_T \cup \{\pm\epsilon_i \mid i \in T\},\\
C_T &:= D_T \cup \{\pm 2\epsilon_i \mid i \in T\},\\
BC_T &:= B_T \cup C_T.
\end{aligned}
\tag{A.3}
$$

One can see that these are irreducible locally finite root systems in their $\mathbb{F}$-spans, which we refer to as *types* A, D, B, C and BC respectively. Moreover, every irreducible locally finite root system is either an irreducible finite root system or isomorphic to one of these root systems (see [17, §4.14, §8]). Now we suppose that R is an irreducible locally finite root system; one can define

$$
\begin{aligned}
R_{\mathrm{sh}} &:= \{\alpha \in R^\times \mid (\alpha,\alpha) \leq (\beta,\beta) \text{ for all } \beta \in R\},\\
R_{\mathrm{ex}} &:= R \cap 2R_{\mathrm{sh}},\\
R_{\mathrm{lg}} &:= R^\times \setminus (R_{\mathrm{sh}} \cup R_{\mathrm{ex}}).
\end{aligned}
$$

The elements of R_{sh} (resp. $R_{\mathrm{lg}}, R_{\mathrm{ex}}$) are called *short roots* (resp. *long roots, extra-long roots*) of R. It is well known that roots of the same length are conjugate with respect to the Weyl group. We point out that following the usual notation in the literature, we use " $\dot{}$ " on the top of A in the list of locally finite root systems above as all of them span $\mathcal{U}$, other than the first one which spans a subspace of codimension 1.

Acknowledgements

This research was in part supported by a grant from IPM (No. 94170415) and partially carried out at IPM–Isfahan Branch.

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