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# Inverse Scattering at Fixed Energy on Three-Dimensional Asymptotically Hyperbolic Stäckel Manifolds

by

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## Abstract

In this paper, we study an inverse scattering problem at fixed energy on three-dimensional asymptotically hyperbolic Stäckel manifolds having the topology of toric cylinders and satisfying the Robertson condition. On these manifolds the Helmholtz equation can be separated into a system of a radial ODE and two angular ODEs. We can thus decompose the full scattering operator into generalized harmonics and the resulting partial scattering matrices consist of a countable set of $2 \times 2$ matrices whose coefficients are the so-called transmission and reflection coefficients. It is shown that the reflection coefficients are nothing but generalized Weyl–Titchmarsh functions associated with the radial ODE. Using a novel multivariable version of the complex angular momentum method, we show that knowledge of the scattering operator at a fixed nonzero energy is enough to determine uniquely the metric of the three-dimensional Stäckel manifold up to natural obstructions.

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## §1. Introduction and statement of the main result

In this work we are interested in an inverse scattering problem at fixed energy for the Helmholtz equation on three-dimensional Stäckel manifolds satisfying the Robertson condition. The Stäckel manifolds were introduced in 1891 by Stäckel in [60] and are mainly of interest in the theory of variable separation. Indeed, it is known that the additive separability of the Hamilton–Jacobi equation for the geodesic flow on a riemannian manifold is equivalent to the fact that the metric is in Stäckel form. However, to obtain the multiplicative separability of

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the Helmholtz equation, we also have to assume that the Robertson condition is satisfied. As we will see in a brief review of the theory of variable separation, there also exist some intrinsic characterizations of the separability of the Hamilton–Jacobi and Helmholtz equations in terms of Killing tensors (which correspond to hidden symmetries) or symmetry operators. We refer to [4, 5, 14, 26, 27, 39, 38, 40, 61] for important contributions in this domain and to [2, 53] for surveys on these questions. We emphasize that the description of the riemannian manifolds given by Stäckel is local. We shall obtain a global description of these manifolds by choosing a global Stäckel system of coordinates and we shall thus use the name of "Stäckel manifold" in our study. We choose to work on a Stäckel manifold $(\mathcal{M}, g)$ having the topology of a toric cylinder and in order to define the scattering matrix on this manifold, we add an asymptotically hyperbolic structure, introduced in [35] (see also [32, 37, 58]), at the two radial ends of our cylinder. We can then define the scattering operator $S_g(\lambda)$ at a fixed energy $\lambda \neq 0$ associated with the Helmholtz equation on $(\mathcal{M}, g)$ which is the object of main interest in this paper. The question we address is the following:

> *Does the scattering operator $S_g(\lambda)$ at a fixed energy $\lambda \neq 0$ uniquely determine the metric $g$ of the Stäckel manifold?*

We recall that inverse scattering problems at fixed energy on asymptotically hyperbolic manifolds are closely related to the anisotropic Calderón problem on compact riemannian manifolds with boundary. We refer to the surveys [34, 35, 41, 43, 59, 62] for the current state of the art on this question. One of the aims of this paper is thus to give examples of manifolds on which we can solve the inverse scattering problem at fixed energy but we do not impose one of the particular structures for which the uniqueness for the anisotropic Calderón problem on compact manifolds with boundary is known. Note that the result we prove here is a uniqueness result. We are also interested in the stability result, i.e., in the study of the continuity of the mapping $g \mapsto S_g(\lambda)$. This question will be the object of future work.

The main result of this paper is the following:

**Theorem 1.1.** *Let $(\mathcal{M}, g)$ and $(\mathcal{M}, \tilde{g})$, where $\mathcal{M} = (0, A)_{x^1} \times \mathcal{T}^2_{x^2,x^3}$, be two three-dimensional Stäckel toric cylinders. We assume that these manifolds satisfy the Robertson condition and are endowed with asymptotically hyperbolic structures at the two ends $\{x^1 = 0\}$ and $\{x^1 = A\}$. We denote by $S_g(\lambda)$ and $S_{\tilde{g}}(\lambda)$ the corresponding scattering operators at an arbitrarily fixed energy $\lambda \neq 0$. Assume that $S_g(\lambda) = S_{\tilde{g}}(\lambda)$. Then, there exists a diffeomorphism $\Psi : \mathcal{M} \to \mathcal{M}$, equal to the identity at the compactified ends $\{x^1 = 0\}$ and $\{x^1 = A\}$, such that $\tilde{g}$ is the pull back of $g$ by $\Psi$, i.e., $\tilde{g} = \Psi^\star g$.*

The main tool of this work consists in complexifying the coupled angular momenta that appear in the variable separation procedure. Indeed, thanks to variable separation, the scattering operator at a fixed energy can be decomposed into scattering coefficients indexed by a discrete set of *two* angular momenta that correspond to the *two* constants of separation. Roughly speaking, the aim of the complexification of the angular momentum method is the following: from a discrete set of data (here the equality of the reflection coefficients on the coupled spectrum) we want to obtain a continuous regime of information (here the equality of these functions on $\mathbb{C}^2$). This method consists of three steps. We first allow the angular momentum to be a complex number. We then use uniqueness results for functions in certain analytic classes to obtain the equality of the non-physical corresponding data on the complex plane $\mathbb{C}$. Finally, we use this new information to solve our inverse problem thanks to the Börg–Marchenko theorem. The general idea of considering complex angular momentum originates from a paper by Regge (see [56]) and uses it as a tool in the analysis of the scattering matrix of Schrödinger operators in $\mathbb{R}^3$ with spherically symmetric potentials. We also refer to [1, 54] for books dealing with this method. This tool has already been used in the field of inverse problems for one angular momentum in [18, 19, 20, 22, 21, 23, 30, 55] and we note that this method is also used in high energy physics (see [17]). In this work we use a novel multivariable version of the complexification of the angular momentum method for *two angular momenta* which correspond to the constants of separation of the Helmholtz equation. Note that we have to use this multivariable version since these two angular momenta (which are also coupled eigenvalues of two commuting operators) are not independent and cannot be considered separately. This work is a continuation of the paper [19] by Daudé, Kamran and Nicoleau in which the authors treat the same question on Liouville surfaces which correspond to Stäckel manifolds in two dimensions.

### §1.1. Review of variable separation theory

In this subsection, we propose a brief review of variable separation theory. We refer to [2, 53] and to the introduction of [4] for surveys of this theory. Let $(\mathcal{M}, g)$ be a riemannian manifold of dimension $n$. On $(\mathcal{M}, g)$, we are first interested in the Hamilton–Jacobi equation

$$\nabla W \cdot \nabla W = E, \tag{1.1}$$

where $E$ is a constant parameter and $\nabla$ is the gradient operator $(\nabla W)^i = g^{ij}\partial_j W$, where we use the Einstein summation convention. We are also interested in the

Helmholtz equation

$$-\Delta_g \psi = E\psi, \tag{1.2}$$

where $\Delta_g$ is the Laplace–Beltrami operator $\Delta_g \psi = g^{ij}\nabla_i\nabla_j\psi$, where $\nabla_j$ is the covariant derivative with respect to the Levi-Civita connection. We note that, as is shown in [4], we can add a potential $V$ satisfying suitable conditions on these equations without more complications in the theory we describe here. It is known that, in many interesting cases, these equations admit local separated solutions. The reason why we are interested in such solutions is that it happens that for solutions of this kind, equations (1.1)–(1.2) become equivalent to a system of ordinary differential separated equations, each one involving a single coordinate. In this work we study the particular case of the orthogonal separation, i.e., when $g^{ij} = 0$ for $i \neq j$. We now recall the definition of separation of variables for the Hamilton–Jacobi and the Helmholtz equations.

**Definition 1.1** ([4]). The Hamilton–Jacobi equation is separable in the coordinates $\underline{x} = \{x^i\}$ if it admits a complete separated solution, i.e., a solution of the kind $W(\underline{x}, \underline{c}) = \sum_{i=1}^n W_i(x^i, \underline{c})$, depending on $n$ parameters $\underline{c} = (c_j)$ satisfying the completeness condition $\det(\frac{\partial p_i}{\partial c_j}) \neq 0$, where $p_i = \partial_i W$.

**Definition 1.2** ([4, Def. 4.1]). The Helmholtz equation is separable in the coordinates $\underline{x} = \{x^i\}$ if it admits a complete separated solution, i.e., a solution of the form $\psi(\underline{x}, \underline{c}) = \prod_{i=1}^n \psi_i(x^i, \underline{c})$, depending on $2n$ parameters $\underline{c} = (c_j)$ satisfying the completeness condition $\det\left(\begin{smallmatrix} \partial u_i/\partial c_j \\ \partial v_i/\partial c_j \end{smallmatrix}\right) \neq 0$, where $u_i = \frac{\psi_i'}{\psi_i}$ and $v_i = \frac{\psi_i''}{\psi_i}$.

We now recall the results proved by Stäckel, Robertson and Eisenhart at the end of the 19th century and at the beginning of the 20th century which

(1) characterize the riemannian manifolds admitting orthogonal variable separation;

(2) make the link between the variable separation for Hamilton–Jacobi and Helmholtz equations.

**Definition 1.3** (Stäckel matrix). A Stäckel matrix is a regular $n \times n$ matrix $S(\underline{x}) = (s_{ij}(x^i))$ whose components $s_{ij}(x^i)$ are functions depending on the variable corresponding to the row number only.

**Theorem 1.2** (Stäckel 1893, [61]). *The Hamilton–Jacobi equation is separable in orthogonal coordinates $\underline{x}$ if and only if the metric $g$ is of the form*

$$g = \sum_{i=1}^n H_i^2 (dx^i)^2,$$

*where $H_i^2$ is written using a Stäckel matrix $S$ as*

$$H_i^2 = \frac{\det(S)}{s^{i1}} \quad \forall\, i \in \{1, \ldots, n\},$$

*where $s^{i1}$ is the minor associated with the coefficient $s_{i1}$ for all $i \in \{1, \ldots, n\}$.*

**Theorem 1.3** (Robertson 1927, [57]). *The Helmholtz equation is separable in orthogonal coordinates $\underline{x}$ if and only if in these coordinates the Hamilton–Jacobi equation is separable and, moreover, the following condition is satisfied:*

$$\frac{\det(S)^2}{|g|} = \frac{\det(S)^2}{\prod_{i=1}^n H_i^2} = \prod_{i=1}^n f_i(x^i), \tag{1.3}$$

*by some Stäckel matrix $S$, where $f_i(x^i)$ are arbitrary functions of the corresponding coordinate only and $|g|$ is the determinant of the metric $g$.*

Thanks to this theorem we see that a full understanding of separation theory for the Helmholtz equation depends on an understanding of the corresponding problem for the Hamilton–Jacobi equation and we note that the separability of the Helmholtz equation is more demanding. The additional condition (1.3) in Theorem 1.3 is called the *Robertson condition*. This condition has a geometrical meaning given by the following characterization due to Eisenhart.

**Theorem 1.4** (Eisenhart 1934, [26]). *The Robertson condition (1.3) is satisfied if and only if in the orthogonal coordinates system $\underline{x}$ the Ricci tensor is diagonal: $R_{ij} = 0$ for all $i \neq j$.*

We note that the Robertson condition is satisfied for Einstein manifolds. Indeed, an Einstein manifold is a riemannian manifold whose Ricci tensor is proportional to the metric which is diagonal in the orthogonal case we study.

As shown by Eisenhart in [26, 27] and by Kalnins and Miller in [38], the separation of the Hamilton–Jacobi equation for the geodesic flow is related to the existence of Killing tensors of order 2 (whose presence highlights the presence of hidden symmetries). The reader is referred to [2] for the state of the art on this point and to [3, 4, 5, 6, 38, 44, 49] for important contributions and reviews.

We also mentioned that there exists an intrinsic characterization of separability for the Hamilton–Jacobi and the Helmholtz equations using symmetry operators given in [40, Thm. 3].

We finally note that there exists a more general notion of separability called the R-separation (see for instance [40, 4, 5]). Our notion of separability corresponds to the case $R = 1$. The study of R-separability in our framework will be the object of future work.

### §1.2. Description of the framework

We define a Stäckel matrix which is a $3 \times 3$ matrix of the form

$$S = \begin{pmatrix} s_{11}(x^1) & s_{12}(x^1) & s_{13}(x^1) \\ s_{21}(x^2) & s_{22}(x^2) & s_{23}(x^2) \\ s_{31}(x^3) & s_{32}(x^3) & s_{33}(x^3) \end{pmatrix},$$

where the coefficients $s_{ij}$ are smooth functions. Let $\mathcal{M}$ be endowed with the riemannian metric

$$g = H_1^2(dx^1)^2 + H_2^2(dx^2)^2 + H_3^2(dx^3)^2, \tag{1.4}$$

with

$$H_i^2 = \frac{\det(S)}{s^{i1}} \quad \forall\, i \in \{1,2,3\},$$

where $s^{i1}$ is the minor (with sign) associated with the coefficient $s_{i1}$ for all $i \in \{1,2,3\}$. The metric $g$ is riemannian if and only if the determinant of the Stäckel matrix $S$ and the minors $s^{11}$, $s^{21}$ and $s^{31}$ have the same sign. Moreover, if we develop the determinant with respect to the first column, we note that if we assume that $s_{11}$, $s_{21}$ and $s_{31}$ are positive functions and if the minors $s^{11}$, $s^{21}$ and $s^{31}$ have the same sign, then the sign of the determinant of $S$ is necessarily the same as the sign of these minors.

We emphasize that the mapping $S \mapsto g$ is not one-to-one. Indeed, we describe here two invariances of the metric which will be useful in solving our inverse problem.

**Proposition 1.5** (Invariances of the metric). *Let $S$ be a Stäckel matrix.*

*(1) Let $G$ be a $2 \times 2$ constant invertible matrix. The Stäckel matrix*

$$\hat{S} = \begin{pmatrix} s_{11}(x^1) & \hat{s}_{12}(x^1) & \hat{s}_{13}(x^1) \\ s_{21}(x^2) & \hat{s}_{22}(x^2) & \hat{s}_{23}(x^2) \\ s_{31}(x^3) & \hat{s}_{32}(x^3) & \hat{s}_{33}(x^3) \end{pmatrix},$$

*satisfying*

$$\begin{pmatrix} s_{i2} & s_{i3} \end{pmatrix} = \begin{pmatrix} \hat{s}_{i2} & \hat{s}_{i3} \end{pmatrix} G \quad \forall\, i \in \{1,2,3\},$$

*leads to the same metric as $S$.*

*(2) The Stäckel matrix*

$$\hat{S} = \begin{pmatrix} \hat{s}_{11}(x^1) & s_{12}(x^1) & s_{13}(x^1) \\ \hat{s}_{21}(x^2) & s_{22}(x^2) & s_{23}(x^2) \\ \hat{s}_{31}(x^3) & s_{32}(x^3) & s_{33}(x^3) \end{pmatrix},$$

*where*

$$\text{(1.5)} \qquad \begin{cases} \hat{s}_{11}(x^1) = s_{11}(x^1) + C_1 s_{12}(x^1) + C_2 s_{13}(x^1), \\ \hat{s}_{21}(x^2) = s_{21}(x^2) + C_1 s_{22}(x^2) + C_2 s_{23}(x^2), \\ \hat{s}_{31}(x^3) = s_{31}(x^3) + C_1 s_{32}(x^3) + C_2 s_{33}(x^3), \end{cases}$$

*where $C_1$ and $C_2$ are real constants, leads to the same metric as $S$.*

Since we are interested only in recovering the metric $g$ of the Stäckel manifold, we can choose any representative of the equivalence class described by the invariances given in the previous proposition. This fact allows us to make some assumptions on the Stäckel matrix we consider as we can see in the following proposition.

**Proposition 1.6.** *Let $S$ be a Stäckel matrix with corresponding metric $g_S$. There exists a Stäckel matrix $\hat{S}$ with $g_{\hat{S}} = g_S$ and such that*

$$\text{(C)} \qquad \begin{cases} \hat{s}_{12}(x^1) > 0 \text{ and } \hat{s}_{13}(x^1) > 0 \quad \forall x^1, \\ \hat{s}_{22}(x^2) < 0 \text{ and } \hat{s}_{23}(x^2) > 0 \quad \forall x^2, \\ \hat{s}_{32}(x^3) > 0 \text{ and } \hat{s}_{33}(x^3) < 0 \quad \forall x^3, \\ \lim\limits_{x^1 \to 0} s_{12}(x^1) = \lim\limits_{x^1 \to 0} s_{13}(x^1) = 1. \end{cases}$$

*Proof.* See Appendix A. □

**Remark 1.1.** Condition (C) has some interesting consequences which will be useful in our later analysis.

(1) We note that under condition (C), $s^{21} = s_{13}s_{32} - s_{12}s_{33}$ and $s^{31} = s_{12}s_{23} - s_{13}s_{22}$ are strictly positive. Thus, since the metric $g$ has to be a riemannian metric we must also have $\det(S) > 0$ and $s^{11} > 0$.
(2) We note that, since $s_{22}$, $s_{33} < 0$ and $s_{23}$, $s_{32} > 0$,

$$s^{11} > 0 \quad \Leftrightarrow \quad s_{22}s_{33} > s_{23}s_{32} \quad \Leftrightarrow \quad \frac{s_{22}}{s_{23}} < \frac{s_{32}}{s_{33}}.$$

We will use these facts later in the study of the coupled spectrum of the operators $H$ and $L$ corresponding to the symmetry operators of $\Delta_g$ introduced in Section 1.1.

From now on and without loss of generality, we assume that the Stäckel matrix $S$ we consider satisfies condition (C).

On the Stäckel manifold $(\mathcal{M}, g)$ we are interested in studying the Helmholtz equation

$$-\Delta_g f = -\lambda^2 f.$$

As mentioned in Section 1.1 the Stäckel structure is not enough to obtain the multiplicative separability of the Helmholtz equation. Indeed, we have to assume that the Robertson condition is satisfied. We recall that this condition can be defined as follows: for all $i \in \{1,2,3\}$ there exists $f_i(x^i)$, a function of $x^i$ alone, such that

$$\frac{s^{11}s^{21}s^{31}}{\det(S)} = f_1 f_2 f_3. \tag{1.6}$$

We can easily reformulate this condition into the form

$$\frac{\det(S)^2}{H_1^2 H_2^2 H_3^2} = f_1 f_2 f_3. \tag{1.7}$$

We note that the functions $f_i$, $i \in \{1,2,3\}$ are defined up to positive multiplicative constants whose product is equal to 1. In the following we will choose, without loss of generality, these constants equal to 1.

### §1.3. Asymptotically hyperbolic structure and examples

We say that a riemannian manifold $(\mathcal{M}, g)$ with boundary $\partial\mathcal{M}$ is asymptotically hyperbolic if its sectional curvature tends to $-1$ at the boundary. In this paper, we put an asymptotically hyperbolic structure at the two radial ends of our Stäckel cylinders in the sense given by Isozaki and Kurylev in [35, Sect. 3.2].[1] Now we give the definition of this structure in our framework.

**Definition 1.4** (Asymptotically hyperbolic Stäckel manifold). A Stäckel manifold with two asymptotically hyperbolic ends having the topology of a toric cylinder is a Stäckel manifold $(\mathcal{M}, g)$ whose Stäckel matrix $S$ satisfies condition (C) with a global chart

$$\mathcal{M} = (0, A)_{x^1} \times \mathcal{T}^2_{x^2,x^3},$$

where $x^1 \in (0, A)_{x^1}$ corresponds to a boundary defining function for the two asymptotically hyperbolic ends $\{x^1 = 0\}$ and $\{x^1 = A\}$, and $(x^2, x^3) \in [0, B]_{x^2} \times [0, C]_{x^3}$ are angular variables on the 2-torus $\mathcal{T}^2_{x^2,x^3}$, satisfying the following conditions:

(1) The Stäckel metric $g$ has the form (1.4).
(2) The coefficients $s_{ij}$, $(i,j) \in \{1,2,3\}^2$ of the Stäckel matrix are smooth functions.
(3) The coefficients of the Stäckel matrix satisfy

[1] Note that the asymptotically hyperbolic structure introduced in [35] is slightly more general than the one used by Melrose, Guillarmou, Joshi and Sá Barreto in [32, 37, 52, 58].

(a) $H_i^2 > 0$ for $i \in \{1,2,3\}$ (riemannian metric);

(b) $s_{2j}(0) = s_{2j}(B)$, $s'_{2j}(0) = s'_{2j}(B)$, $s_{3j}(0) = s_{3j}(C)$ and $s'_{3j}(0) = s'_{3j}(C)$ for $j \in \{1,2,3\}$ (periodic conditions in angular variables);

(c) asymptotically hyperbolic ends at $\{x^1 = 0\}$ and $\{x^1 = A\}$:

(i) at $\{x^1 = 0\}$ there exist $\epsilon_0 > 0$ and $\delta > 0$ such that for all $n \in \mathbb{N}$ there exists $C_n > 0$ such that $\forall\, x^1 \in (0, A-\delta)$,

$$\|(x^1\partial_{x^1})^n((x^1)^2 s_{11}(x^1) - 1)\| \leq C_n(1 + |\log(x^1)|)^{-\min(n,1)-1-\epsilon_0},$$
$$\|(x^1\partial_{x^1})^n(s_{12}(x^1) - 1)\| \leq C_n(1 + |\log(x^1)|)^{-\min(n,1)-1-\epsilon_0},$$
$$\|(x^1\partial_{x^1})^n(s_{13}(x^1) - 1)\| \leq C_n(1 + |\log(x^1)|)^{-\min(n,1)-1-\epsilon_0};$$

(ii) at $\{x^1 = A\}$ there exist $\epsilon_1 > 0$ and $\delta > 0$ such that for all $n \in \mathbb{N}$ there exists $C_n > 0$ such that $\forall\, x^1 \in (\delta, A)$,

$$\|((A - x^1)\partial_{x^1})^n((A - x^1)^2 s_{11}(x^1) - 1)\| \leq C_n(1 + |\log(A - x^1)|)^{-\min(n,1)-1-\epsilon_1},$$
$$\|((A-x^1)\partial_{x^1})^n(s_{12}(x^1)-1)\| \leq C_n(1 + |\log(A - x^1)|)^{-\min(n,1)-1-\epsilon_1},$$
$$\|((A-x^1)\partial_{x^1})^n(s_{13}(x^1)-1)\| \leq C_n(1 + |\log(A - x^1)|)^{-\min(n,1)-1-\epsilon_1}.$$

**Remark 1.2.** We know that, thanks to condition (C), $s_{12}$ and $s_{13}$ tend to 1 when $x^1$ tends to 0. However, at the end $\{x^1 = A\}$, we can just say that there exist two positive constants $\alpha$ and $\beta$ such that $s_{12}$ and $s_{13}$ tend to $\alpha$ and $\beta$ respectively. Thus, at the end $\{x^1 = A\}$, we should assume that

$$(A - x^1)^2 s_{11}(x^1) = [1]_{\epsilon_1}, \quad s_{12}(x^1) = \alpha[1]_{\epsilon_1} \quad \text{and} \quad s_{13}(x^1) = \beta[1]_{\epsilon_1},$$

where

$$[1]_{\epsilon_1} = 1 + O((1 + |\log(A - x^1)|)^{-1-\epsilon_1}).$$

However, we can show (see the last point of Remark 1.4) that, if $s_{22}$ or $s_{33}$ are not constant functions, then $\alpha = \beta = 1$.

Let us explain the meaning of asymptotically hyperbolic ends for Stäckel manifolds.[2] Since the explanation is similar at the end $\{x^1 = A\}$ we study just the end $\{x^1 = 0\}$. We first write the metric (1.4) in a neighbourhood of $\{x^1 = 0\}$ in the form

$$g = \sum_{i=1}^{3} H_i^2 (dx^i)^2 = \frac{\sum_{i=1}^{3} (x^1)^2 H_i^2 (dx^i)^2}{(x^1)^2}.$$

[2]We refer to [35, Sect. 3, p.99–101] for a justification of the name "asymptotically hyperbolic".

By definition,

$$
(1.8)\qquad \begin{cases}
(x^1)^2H_1^2 = (x^1)^2 s_{11} + (x^1)^2\left(s_{12}\frac{s^{12}}{s^{11}} + s_{13}\frac{s^{13}}{s^{21}}\right),\\
(x^1)^2H_2^2 = (x^1)^2 s_{11}\frac{s^{11}}{s_{32}s_{13}-s_{33}s_{12}}\\
\qquad\qquad + (x^1)^2\left(s_{12}\frac{s^{12}}{s_{32}s_{13}-s_{33}s_{12}} + s_{13}\frac{s^{13}}{s_{32}s_{13}-s_{33}s_{12}}\right),\\
(x^1)^2H_3^2 = (x^1)^2 s_{11}\frac{s^{11}}{s_{23}s_{12}-s_{22}s_{13}}\\
\qquad\qquad + (x^1)^2\left(\frac{s^{12}}{s_{23}s_{12}-s_{22}s_{13}} + \frac{s_{13}}{s_{12}}\frac{s^{13}}{s_{23}s_{12}-s_{22}s_{13}}\right).
\end{cases}
$$

As is shown in [51], we know that at the end $\{x^1 = 0\}$, the sectional curvature of $g$ approaches $-|dx^1|_h$ where $h = \sum_{i=1}^{3}(x^1)^2H_i^2(dx^i)^2$. In other words, the sectional curvature with opposite sign at the end $\{x^1 = 0\}$ is equivalent to

$$
(x^1)^2H_1^2 = (x^1)^2 s_{11} + (x^1)^2\left(s_{12}\frac{s^{12}}{s^{11}} + s_{13}\frac{s^{13}}{s^{21}}\right).
$$

Thus, since an asymptotically hyperbolic structure corresponds to a sectional curvature that tends to $-1$, we want this last quantity to tend to 1. This is ensured by the third assumption of Definition 1.4 which entails that (for $n = 0$)

$$
(1.9)\qquad (x^1)^2 s_{11}(x^1) = [1]_{\epsilon_0}, \quad s_{12}(x^1) = [1]_{\epsilon_0} \quad \text{and} \quad s_{13}(x^1) = [1]_{\epsilon_0},
$$

where

$$
[1]_{\epsilon_0} = 1 + O((1 + |\log(x^1)|)^{-1-\epsilon_0}).
$$

We also note that under these conditions we can write, thanks to (1.8), the metric $g$, in a neighbourhood of $\{x^1 = 0\}$, in the form

$$
(1.10)\qquad g = \frac{(dx^1)^2 + d\Omega^2_{\mathcal{T}^2} + P(x^1, x^2, x^3, dx^1, dx^2, dx^3)}{(x^1)^2},
$$

where

$$
d\Omega^2_{\mathcal{T}^2} = \frac{s^{11}}{s_{32} - s_{33}}(dx^2)^2 + \frac{s^{11}}{s_{23} - s_{22}}(dx^3)^2
$$

is a riemannian metric on the 2-torus $\mathcal{T}^2$ (since $s^{11}$, $s^{21}$ and $s^{31}$ have the same sign) and $P$ is a remainder term which is, roughly speaking, small as $x^1 \to 0$. Hence, in the limit $x^1 \to 0$, we see that

$$
g \sim \frac{(dx^1)^2 + d\Omega^2_{\mathcal{T}^2}}{(x^1)^2},
$$

that is, $g$ is a small perturbation of a hyperbolic-like metric.

**Remark 1.3.** (1) According to the previous definition, we also need conditions on the derivatives of $s_{1j}$, $j \in \{1, 2, 3\}$ to be in the framework of [35].

(2) By symmetry, we can do the same analysis at the end $\{x^1 = A\}$.

From conditions (1.9) and the Robertson condition (1.6) we can obtain more information on the functions $f_1$, $f_2$ and $f_3$. We first remark that

$$f_1 f_2 f_3 = \frac{s^{11}s^{21}s^{31}}{\det(S)} = \frac{s^{11}(s_{13}s_{32} - s_{12}s_{33})(s_{12}s_{23} - s_{13}s_{22})}{s_{11}s^{11} + s_{12}s^{12} + s_{13}s^{13}}.$$

Thus, using conditions (1.9), we obtain

$$f_1 f_2 f_3 \sim (x^1)^2 (s_{23} - s_{22})(s_{32} - s_{33}) \quad \text{when } x^1 \to 0.$$

Hence, we can say that there exist three positive constants $c_1$, $c_2$ and $c_3$ such that $c_1 c_2 c_3 = 1$ and

$$f_1(x^1) = c_1 (x^1)^2 [1]_{\epsilon_0}, \quad f_2(x^2) = c_2(s_{23} - s_{22}) \text{ and } f_3(x^3) = c_3(s_{32} - s_{33}). \tag{1.11}$$

We thus note that the functions $f_i$, $i \in \{1, 2, 3\}$ are defined up to positive constants $c_1$, $c_2$ and $c_3$ whose product is equal to 1. However, as mentioned previously, we can choose these constants to be equal to 1. Of course, the corresponding result on $f_1$ at the end $\{x^1 = A\}$ is also true.

**Remark 1.4.** The previous analysis allows us to simplify the Robertson condition and thus the expression of the riemannian metric on the 2-torus.

(1) We first note that, if we make a Liouville change of variables in the $i$th variable,

$$X^i = \int_0^{x^i} \sqrt{g_i(s)}\, ds, \tag{1.12}$$

where $g_i$ is a positive function of the variable $x^i$, the corresponding coefficient $H_i^2$ of the metric is also divided by $g_i(x^i)$. The same modification of the metric happens when we divide the $i$th line of the Stäckel matrix by the function $g_i$. Thus, proceeding to a Liouville change of variables is equivalent to dividing the $i$th line of the Stäckel matrix by the corresponding function.

(2) We now remark that, if we divide the $i$th line of the Stäckel matrix by a function $g_i$ of the variable $x^i$, the quantity $\frac{s^{11}s^{21}s^{31}}{\det(S)}$ is divided by $g_i$. Thus, recalling the form of the Robertson condition (1.6), we can always assume that $f_2 = f_3 = 1$ by choosing appropriate coordinates on $\mathcal{T}^2$. However, we do not divide the first line by $f_1$ because it changes the description of the hyperbolic structure (i.e., condition (1.9)). Nevertheless, there remains a degree of freedom on the first line. For instance, we can divide the first line by $s_{12}$ or $s_{13}$ and we then obtain that the radial part depends only on the two scalar functions $\frac{s_{11}}{s_{13}}$ and $\frac{s_{12}}{s_{13}}$. As we will see at the end of Section 4, these quotients are exactly

the scalar functions we recover in our study of the radial part. However, since it does not simplify our study we do not use this reduction for the moment.

(3) From now on, $f_2 = 1$ and $f_3 = 1$ and we can thus rewrite (1.11) in the form $s_{23} - s_{22} = 1$ and $s_{32} - s_{33} = 1$. Thanks to these equalities, we can also write $d\Omega^2_{\mathcal{T}^2} = s^{11}((dx^2)^2 + (dx^3)^2)$ for the induced metric on the compactified boundary $\{x^1 = 0\}$.

(4) Generally, we know that $s_{12}$ and $s_{13}$ tend to 1 when $x^1$ tends to 0 but we do not know that this is also true when $x^1$ tends to $A$. However, the Stäckel structure allows us to show that the asymptotically hyperbolic structure has to be the same at the two ends (under a mild additional assumption). Assume that the behaviour of the first line at the two ends is the following: at the end $\{x^1 = 0\}$,

$$(x^1)^2 s_{11}(x^1) = [1]_{\epsilon_0}, \quad s_{12}(x^1) = [1]_{\epsilon_0} \quad \text{and} \quad s_{13}(x^1) = [1]_{\epsilon_0},$$

and at the end $\{x^1 = A\}$,

$$(A - x^1)^2 s_{11}(x^1) = [1]_{\epsilon_1}, \quad s_{12}(x^1) = \alpha[1]_{\epsilon_1} \quad \text{and} \quad s_{13}(x^1) = \beta[1]_{\epsilon_1},$$

where

$$[1]_{\epsilon_0} = 1 + O((1 + |\log(x^1)|)^{-1-\epsilon_0}) \quad \text{and} \quad [1]_{\epsilon_1} = 1 + O((1 + |\log(A - x^1)|)^{-1-\epsilon_1})$$

and $\alpha$ and $\beta$ are real positive constants. Using the Robertson condition at the end $\{x^1 = 0\}$ and the end $\{x^1 = A\}$ we obtain

$$1 = f_2 = s_{23} - s_{22} = \alpha s_{23} - \beta s_{22} \quad \text{and} \quad 1 = f_3 = s_{32} - s_{33} = \beta s_{32} - \alpha s_{33}.$$

Thus, using that $s_{23} = 1 + s_{22}$ and $s_{32} = 1 + s_{33}$, we obtain $(\alpha - \beta)s_{22} = 1 - \alpha$ and $(\beta - \alpha)s_{33} = 1 - \beta$. Hence, if we assume that $s_{22}$ or $s_{33}$ is not a constant function, we obtain $\alpha = \beta = 1$.

**Example 1.1.** We give here three examples of Stäckel manifolds that illustrate the diversity of the manifolds we consider.

(1) We can first choose the Stäckel matrix

$$S = \begin{pmatrix} s_{11}(x^1) & s_{12}(x^1) & s_{13}(x^1) \\ a & b & c \\ d & e & f \end{pmatrix},$$

where $a$, $b$, $c$, $d$, $e$ and $f$ are real constants. The metric $g$ can thus be written as $g = \sum_{i=1}^{3} H_i^2 (dx^i)^2$, where $H_i^2$, for $i \in \{1, 2, 3\}$ are functions of $x^1$ alone. Therefore, $g$ trivially satisfies the Robertson condition and we can add the asymptotically hyperbolic structure given in Definition 1.4. We note that, as explained in the previous remark, $g$ depends only on two arbitrary functions (after a Liouville change of variables in the variable $x^1$). Moreover, we can show that $\partial_{x^2}$ and $\partial_{x^3}$ are Killing vector fields and the existence of these Killing vector fields reflects the symmetries with respect to the translation in $x^2$ and $x^3$.

(2) We can also choose the Stäckel matrix

$$S = \begin{pmatrix} s_{11}(x^1) & s_{12}(x^1) & as_{12}(x^1) \\ 0 & s_{22}(x^2) & s_{23}(x^2) \\ 0 & s_{32}(x^3) & s_{33}(x^3) \end{pmatrix},$$

where $a$ is a real constant. We can add the asymptotically hyperbolic structure given in Definition 1.4 and the metric $g$ can be written as

$$g = s_{11}(dx^1)^2 + \frac{s_{11}}{s_{12}} \left( \frac{s^{11}}{as_{32} - s_{33}}(dx^2)^2 + \frac{s^{11}}{s_{23} - as_{22}}(dx^3)^2 \right).$$

Therefore, $g$ satisfies the Robertson condition. We note that, after Liouville transformations in the three variables, $g$ depends on three arbitrary functions. Moreover, thanks to the Liouville transformation $X^1 = \int_0^{x^1} \sqrt{s_{11}(s)}\, ds$, we see that there exists a system of coordinates in which the metric $g$ takes the form $g = (dx^1)^2 + f(x^1)g_0$, where $g_0$ is a metric on the 2-torus $\mathcal{T}^2$. In other words, $g$ is a warped product. In particular, $g$ is conformal to a metric that can be written as the sum of one euclidean direction and a metric on a compact manifold. We recall that in this case, under some additional assumptions on the compact part, the uniqueness of the anisotropic Calderón problem on compact manifolds with boundary has been proved in [24, 25].

(3) Finally, we can choose the Stäckel matrix

$$S = \begin{pmatrix} s_1(x^1)^2 & -s_1(x^1) & 1 \\ -s_2(x^2)^2 & s_2(x^2) & -1 \\ s_3(x^3)^2 & -s_3(x^3) & 1 \end{pmatrix}.$$

This model was studied in [2, 10] and is of main interest in the field of geodesically equivalent riemannian manifolds, i.e., of manifolds that share the same unparametrized geodesics (see [10]). The associated metric

$$g = (s_1 - s_2)(s_1 - s_3)(dx^1)^2 + (s_2 - s_3)(s_1 - s_2)(dx^2)^2 + (s_3 - s_2)(s_3 - s_1)(dx^3)^2$$

satisfies the Robertson condition and $g$ has no a priori symmetry, is not a warped product and depends on three arbitrary functions that satisfy $s_1 >$

$s_2 > s_3$. To form an asymptotically hyperbolic structure in the sense given in Definition 1.4 we first multiply the second and the third columns of the Stäckel matrix on the right by the invertible matrix

$$G = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix},$$

since it does not change the metric. We thus obtain the new Stäckel matrix

$$\begin{pmatrix} s_1(x^1)^2 & s_1(x^1) & s_1(x^1) - 1 \\ -s_2(x^2)^2 & -s_2(x^2) & -s_2(x^1) + 1 \\ s_3(x^3)^2 & s_3(x^3) & s_3(x^1) - 1 \end{pmatrix}.$$

Next, we use the Liouville change of variables in the first variable $X^1 = \int_0^{x^1} \sqrt{s_1(s)}\, ds$, and we obtain the Stäckel matrix

$$S = \begin{pmatrix} s_1(X^1) & 1 & 1 - \frac{1}{s_1(X^1)} \\ -s_2(x^2)^2 & -s_2(x^2) & -s_2(x^1) + 1 \\ s_3(x^3)^2 & s_3(x^3) & s_3(x^1) - 1 \end{pmatrix}.$$

Finally, to form the asymptotically hyperbolic structure on the first line, we assume that

$$s_1(X^1) = \frac{1}{(X^1)^2}(1 + O((1 + |\log(X^1)|)^{-1-\epsilon_0})) \quad \text{when } X^1 \to 0$$

and

$$s_1(X^1) = \frac{1}{(A^1 - X^1)^2}(1 + O((1 + |\log(A^1 - X^1)|)^{-1-\epsilon_1})) \quad \text{when } X^1 \to A^1,$$

where $A^1 = \int_0^A \sqrt{s_1(s)}\, ds$.

### §1.4. Scattering operator and statement of the main result

The construction of the scattering operator is given in [35, 36] for asymptotically hyperbolic manifolds and it was used in [19] in the case of asymptotically hyperbolic Liouville surfaces. In our particular model, there are two ends and so we introduce two cut-off functions $\chi_0$ and $\chi_1$, smooth on $\mathbb{R}$, defined by

$$\chi_0 = 1 \text{ on } \left(0, \frac{A}{4}\right), \quad \chi_1 = 1 \text{ on } \left(\frac{3A}{4}, A\right), \quad \chi_0 + \chi_1 = 1 \text{ on } (0, A), \tag{1.13}$$

in order to separate these two ends. We consider the shifted stationary Helmholtz equation

$$-(\Delta_g + 1)f = \lambda^2 f,$$

where $\lambda^2 \neq 0$ is a fixed energy, which is usually studied in the case of asymptotically hyperbolic manifolds (see [11, 35, 36, 37]). Indeed, it is known (see [36]) that the essential spectrum of $-\Delta_g$ is $[1,+\infty)$ and thus we shift the bottom of the essential spectrum in order that it becomes 0. It is known that the operator $-\Delta_g - 1$ has no eigenvalues embedded in the essential spectrum $[0,+\infty)$ (see [13, 35, 36]). It is shown in [36] that the solutions of the shifted stationary equation $-(\Delta_g + 1)f = \lambda^2 f$ are unique when we impose on $f$ some radiation conditions at infinities. To be precise, as in [19], we define some Besov spaces that encode these radiation conditions at infinities as follows. To motivate our definitions, we first recall that the compactified boundaries $\{x^1 = 0\}$ and $\{x^1 = A\}$ are endowed with the induced metric

$$d\Omega^2_{\mathcal{T}^2} = s^{11}((dx^2)^2 + (dx^3)^2).$$

**Definition 1.5.** Let $\mathcal{H}_{\mathcal{T}^2} = L^2(\mathcal{T}^2, s^{11}\, dx^2\, dx^3)$. Let the intervals $(0,+\infty)$ and $(-\infty, A)$ be decomposed as

$$(0,+\infty) = \cup_{k\in\mathbb{Z}} I_k \quad \text{and} \quad (-\infty, A) = \cup_{k\in\mathbb{Z}} J_k,$$

where

$$I_k = \begin{cases} (\exp(e^{k-1}), \exp(e^k)] & \text{if } k \geq 1, \\ (e^{-1}, e] & \text{if } k = 0, \\ (\exp(-e^{|k|}), \exp(-e^{|k|-1})] & \text{if } k \leq -1, \end{cases}$$

and

$$J_k = \begin{cases} (A - \exp(e^k), A - \exp(e^{k-1})] & \text{if } k \geq 1, \\ (A - e, A - e^{-1}] & \text{if } k = 0, \\ (A - \exp(-e^{|k|-1}), A - \exp(-e^{|k|})] & \text{if } k \leq -1. \end{cases}$$

We define the Besov spaces $\mathcal{B}_0 = \mathcal{B}_0(\mathcal{H}_{\mathcal{T}^2})$ and $\mathcal{B}_1 = \mathcal{B}_1(\mathcal{H}_{\mathcal{T}^2})$ to be the Banach spaces of $\mathcal{H}_{\mathcal{T}^2}$-valued functions on $(0,+\infty)$ and $(-\infty, A)$ satisfying, respectively,

$$\|f\|_{\mathcal{B}_0} = \sum_{k\in\mathbb{Z}} e^{\frac{|k|}{2}} \left( \int_{I_k} \|f(x)\|^2_{\mathcal{H}_{\mathcal{T}^2}} \frac{dx}{x^2} \right)^{\frac{1}{2}} < \infty$$

and

$$\|f\|_{\mathcal{B}_1} = \sum_{k\in\mathbb{Z}} e^{\frac{|k|}{2}} \left( \int_{J_k} \|f(x)\|^2_{\mathcal{H}_{\mathcal{T}^2}} \frac{dx}{(A-x)^2} \right)^{\frac{1}{2}} < \infty.$$

The dual spaces $\mathcal{B}_0^\star$ and $\mathcal{B}_1^\star$ are then identified with the spaces equipped with the norms

$$\|f\|_{\mathcal{B}_0^\star} = \left( \sup_{R>e} \frac{1}{\log(R)} \int_{\frac{1}{R}}^{R} \|f(x)\|^2_{\mathcal{H}_{\mathcal{T}^2}} \frac{dx}{x^2} \right)^{\frac{1}{2}} < \infty$$

and

$$\|f\|_{\mathcal{B}_1^\star} = \left( \sup_{R>e} \frac{1}{\log(R)} \int_{A-R}^{A-\frac{1}{R}} \|f(x)\|^2_{\mathcal{H}_{\mathcal{T}^2}} \frac{dx}{(A-x)^2} \right)^{\frac{1}{2}} < \infty.$$

**Remark 1.5.** As is shown in [35], we can compare the Besov spaces $\mathcal{B}_0$ and $\mathcal{B}_0^\star$ to weighted $L^2$-spaces. Indeed, if we define $L_0^{2,s}((0,+\infty),\mathcal{H}_{\mathcal{T}^2})$ for $s \in \mathbb{R}$ by

$$\|f\|_s = \left( \int_0^{+\infty} (1+|\log(x)|)^{2s} \|f(x)\|^2_{\mathcal{H}_{\mathcal{T}^2}} \frac{dx}{x^2} \right)^{\frac{1}{2}} < \infty,$$

then for $s > \frac{1}{2}$,

$$L_0^{2,s} \subset \mathcal{B}_0 \subset L_0^{2,\frac{1}{2}} \subset L_0^2 \subset L_0^{2,-\frac{1}{2}} \subset \mathcal{B}_0^\star \subset L_0^{2,-s}.$$

There is a similar result for the Besov spaces $\mathcal{B}_1$ and $\mathcal{B}_1^\star$.

**Definition 1.6.** We define the Besov spaces $\mathcal{B}$ and $\mathcal{B}^\star$ as the Banach spaces of $\mathcal{H}_{\mathcal{T}^2}$-valued functions on $(0,A)$ with norms

$$\|f\|_{\mathcal{B}} = \|\chi_0 f\|_{\mathcal{B}_0} + \|\chi_1 f\|_{\mathcal{B}_1} \quad \text{and} \quad \|f\|_{\mathcal{B}^\star} = \|\chi_0 f\|_{\mathcal{B}_0^\star} + \|\chi_1 f\|_{\mathcal{B}_1^\star}.$$

We also define the Hilbert space of scattering data:

$$\mathcal{H}_\infty = \mathcal{H}_{\mathcal{T}^2} \otimes \mathbb{C}^2 \simeq \mathcal{H}_{\mathcal{T}^2} \oplus \mathcal{H}_{\mathcal{T}^2}.$$

In [35, Thm. 3.15] the following theorem is proved.

**Theorem 1.7** (Stationary construction of the scattering matrix).

*(1) For any solution $f \in \mathcal{B}^\star$ of the shifted stationary Helmholtz equation at nonzero energy $\lambda^2$,*

$$-(\Delta_g + 1)f = \lambda^2 f, \tag{1.14}$$

*there exists a unique $\psi^{(\pm)} = (\psi_0^{(\pm)}, \psi_1^{(\pm)}) \in \mathcal{H}_\infty$ such that*

$$\begin{aligned} f \simeq\ & \omega_-(\lambda) \left( \chi_0\, (x^1)^{\frac{1}{2}+i\lambda} \psi_0^{(-)} + \chi_1\, (A-x^1)^{\frac{1}{2}+i\lambda} \psi_1^{(-)} \right) \\ & - \omega_+(\lambda) \left( \chi_0\, (x^1)^{\frac{1}{2}-i\lambda} \psi_0^{(+)} + \chi_1\, (A-x^1)^{\frac{1}{2}-i\lambda} \psi_1^{(+)} \right), \end{aligned} \tag{1.15}$$

*where*

$$\omega_\pm(\lambda) = \frac{\pi}{(2\lambda \sinh(\pi\lambda))^{\frac{1}{2}} \Gamma(1 \mp i\lambda)}. \tag{1.16}$$

*(2) For any $\psi^{(-)} \in \mathcal{H}_\infty$, there exists a unique $\psi^{(+)} \in \mathcal{H}_\infty$ and $f \in \mathcal{B}^\star$ satisfying (1.14) for which the decomposition (1.15) above holds. This defines*

*uniquely the scattering operator $S_g(\lambda)$ as the $\mathcal{H}_\infty$-valued operator such that for all $\psi^{(-)} \in \mathcal{H}_\infty$,*

$$\psi^{(+)} = S_g(\lambda)\psi^{(-)}. \tag{1.17}$$

*(3) The scattering operator $S_g(\lambda)$ is unitary on $\mathcal{H}_\infty$.*

Note that in our model with two asymptotically hyperbolic ends the scattering operator has the structure of a $2 \times 2$ matrix whose components are $\mathcal{H}_{\mathcal{T}^2}$-valued operators. To be precise, we write

$$S_g(\lambda) = \begin{pmatrix} L(\lambda) & T_R(\lambda) \\ T_L(\lambda) & R(\lambda) \end{pmatrix},$$

where $T_L(\lambda)$ and $T_R(\lambda)$ are the transmission operators and $L(\lambda)$ and $R(\lambda)$ are the reflection operators from the right and from the left respectively. The transmission operators measure what is transmitted from one end to the other end in a scattering experiment, while the reflection operators measure the part of a signal sent from one end that is reflected to itself.

As mentioned in the introduction, the main result of this paper is the following:

**Theorem 1.8.** *Let $(\mathcal{M}, g)$ and $(\mathcal{M}, \tilde{g})$, where $\mathcal{M} = (0, A)_{x^1} \times \mathcal{T}^2_{x^2,x^3}$, be two three-dimensional Stäckel toric cylinders, i.e., endowed with the metrics $g$ and $\tilde{g}$ defined in (1.4) respectively. We assume that these manifolds satisfy the Robertson condition and are endowed with asymptotically hyperbolic structures at the two ends $\{x^1 = 0\}$ and $\{x^1 = A\}$ defined as in Definition 1.4. We denote by $S_g(\lambda)$ and $S_{\tilde{g}}(\lambda)$ the corresponding scattering operators at a fixed energy $\lambda \neq 0$ as defined in Theorem 1.7. Assume that $S_g(\lambda) = S_{\tilde{g}}(\lambda)$. Then, there exists a diffeomorphism $\Psi : \mathcal{M} \to \mathcal{M}$, equal to the identity at the compactified ends $\{x^1 = 0\}$ and $\{x^1 = A\}$, such that $\tilde{g}$ is the pull back of $g$ by $\Psi$, i.e., $\tilde{g} = \Psi^\star g$.*

**Remark 1.6.** In fact, the proof of Theorem 1.8 will show that it suffices to know one of the reflection operators at a fixed energy $\lambda \neq 0$, i.e., it is enough to assume that $R_g(\lambda) = R_{\tilde{g}}(\lambda)$ or $L_g(\lambda) = L_{\tilde{g}}(\lambda)$ to conclude to the uniqueness of $g$ modulo isometries.

For general asymptotically hyperbolic manifolds (AHMs in short) with no particular (hidden) symmetry, direct and inverse scattering results for scalar waves have been proved by Joshi and Sá Barreto in [37], by Sá Barreto in [58], by Guillarmou and Sá Barreto in [32, 31] and by Isozaki and Kurylev in [35]. In [37], it is shown that the asymptotics of the metric of an AHM are uniquely determined (up to isometries) by the scattering matrix $S_g(\lambda)$ at a fixed energy $\lambda$ off a discrete subset of $\mathbb{R}$. In [58], it is proved that the metric of an AHM is uniquely determined (up to isometries) by the scattering matrix $S_g(\lambda)$ for every $\lambda \in \mathbb{R}$ off an exceptional subset. Similar results have been obtained recently in [35] for even more general classes of AHMs. In [32], it is proved that, for connected conformally compact Einstein manifolds of even dimension $n+1$, the scattering matrix at energy $n$ on an open subset of its conformal boundary determines the manifold up to isometries. In [31], the authors study direct and inverse scattering problems for asymptotically complex hyperbolic manifolds and show that the topology and the metric of such a manifold are determined (up to invariants) by the scattering matrix at all energies. We also mention the work [50] of Marazzi in which the author studies inverse scattering for the stationary Schrödinger equation with smooth potential not vanishing at the boundary on a conformally compact manifold with sectional curvature $-\alpha^2$ at the boundary. The author then shows that the scattering matrix at two fixed energies $\lambda_1$ and $\lambda_2$, $\lambda_1 \neq \lambda_2$ in a suitable subset of $\mathbb{C}$, determines $\alpha$ and the Taylor series of both the potential and the metric at the boundary. Finally, we also mention [12] where related inverse problems — inverse resonance problems — are studied in certain subclasses of AHMs.

This work must also be put into perspective with the anisotropic Calderón problem on compact manifolds with boundary. We recall here the definition of this problem. Let $(\mathcal{M}, g)$ be a riemannian compact manifold with smooth boundary $\partial\mathcal{M}$. We denote by $-\Delta_g$ the Laplace–Beltrami operator on $(\mathcal{M}, g)$ and we recall that this operator with Dirichlet boundary conditions is self-adjoint on $L^2(\mathcal{M}, d\mathrm{Vol}_g)$ and has a pure point spectrum $\{\lambda_j^2\}_{j\geq 1}$. We are interested in the solutions $u$ of

$$\begin{cases} -\Delta_g u = \lambda^2 u & \text{on } \mathcal{M}, \\ u = \psi & \text{on } \partial\mathcal{M}. \end{cases} \tag{1.18}$$

It is known (see for instance [59]) that for any $\psi \in H^{\frac{1}{2}}(\partial\mathcal{M})$ there exists a unique weak solution $u \in H^1(\mathcal{M})$ of (1.18) when $\lambda^2$ does not belong to the Dirichlet spectrum $\{\lambda_i^2\}$ of $-\Delta_g$. This allows us to define the Dirichlet-to-Neumann (DN) map as the operator $\Lambda_g(\lambda^2)$ from $H^{\frac{1}{2}}(\partial\mathcal{M})$ to $H^{-\frac{1}{2}}(\partial\mathcal{M})$ defined for all $\psi \in H^{\frac{1}{2}}(\partial\mathcal{M})$ by

$$\Lambda_g(\lambda^2)(\psi) = (\partial_\nu u)_{|\partial\mathcal{M}},$$

where $u$ is the unique solution of (1.18) and $(\partial_\nu u)_{|\partial\mathcal{M}}$ is its normal derivative with respect to the unit outer normal vector $\nu$ on $\partial\mathcal{M}$. The anisotropic Calderón problem can be stated as follows:

*Does knowledge of the DN map $\Lambda_g(\lambda^2)$ at a frequency $\lambda^2$ determine uniquely the metric $g$?*

We refer for instance to [24, 25, 32, 31, 33, 42, 46, 47, 48] for important contributions to this subject and to the surveys [34, 43, 59, 62] for the current state of the art. One of the aims of this paper is thus to give an example of manifolds on which we can solve the inverse scattering problem at fixed energy but which do not have one of the particular structures we have just described for which the uniqueness for the anisotropic Calderón problem on compact manifolds with boundary is known (see Example 1.1(3)).

### §1.5. Overview of the proof

The proof of Theorem 1.8 is divided into four steps which we describe here.

**Step 1** (Section 2): The first step of the proof consists in solving the direct problem. This will be done in Section 2. In this section we first use the structure of Stäckel manifold satisfying the Robertson condition to proceed to the separation of variables for the Helmholtz equation. We obtain that the shifted Helmholtz equation

$$-(\Delta_g + 1)f = \lambda^2 f$$

can be rewritten as

$$A_1 f + s_{12} L f + s_{13} H f = 0,$$

where $A_1$ is a differential operator in the variable $x^1$ alone and $L$ and $H$ are commuting, elliptic, semibounded self-adjoint operators on $L^2(\mathcal{T}^2, s^{11}\, dx^2\, dx^3)$ with discrete spectra. We consider generalized harmonics $\{Y_m\}_{m\geq 1}$ which form a Hilbertian basis of $L^2(\mathcal{T}^2, s^{11}\, dx^2\, dx^3)$ associated with the coupled spectrum $(\mu_m^2, \nu_m^2)$ of $(H, L)$. We decompose the solutions $f = \sum_{m\geq 1} u_m(x^1) Y_m(x^2, x^3)$ of the Helmholtz equation on the common basis of harmonics $\{Y_m\}_{m\geq 1}$ and we then conclude that the Helmholtz equation separates into a system of three ordinary differential equations:

$$\begin{cases} -u_m''(x^1) + \frac{1}{2}(\log(f_1)(x^1))' u_m'(x^1) + [-(\lambda^2+1)s_{11}(x^1) + \mu_m^2 s_{12}(x^1) \\ \qquad + \nu_m^2 s_{13}(x^1)] u_m(x^1) = 0, \\ -v_m''(x^2) + [-(\lambda^2+1)s_{21}(x^2) + \mu_m^2 s_{22}(x^2) + \nu_m^2 s_{23}(x^2)] v_m(x^2) = 0, \\ -w_m''(x^3) + [-(\lambda^2+1)s_{31}(x^3) + \mu_m^2 s_{32}(x^3) + \nu_m^2 s_{33}(x^3)] w_m(x^3) = 0, \end{cases}$$

where $f_1$ is the function appearing in the Robertson condition and $Y_m(x^2, x^3) = v_m(x^2) w_m(x^3)$. In this system of ODEs there is one ODE in the radial variable $x^1$ and two ODEs in the angular variables $x^2$ and $x^3$. We emphasize that the angular momenta $\mu_m^2$ and $\nu_m^2$, which are the separation constants, correspond also to the coupled spectrum of the two angular operators $H$ and $L$. The fact that the angular momenta $(\mu_m^2, \nu_m^2)$ are coupled has an important consequence in the use of the complexification of the angular momentum method. Indeed, we cannot work separately with one angular momentum and we thus have to use a multivariable version of this method.

Next, we define the characteristic and Weyl–Titchmarsh functions following the construction given in [19, 29, 45]. We briefly recall here the definition of these objects and the reason why we use them. Using a Liouville change of variables $X^1 = g(x^1)$, $X^1 \in (0, A^1)$ where $A^1 = \int_0^A g(x^1) dx^1$, we can write the radial equation as

$$- \ddot{U} + q_{\nu_m^2} U = -\mu_m^2 U, \tag{1.19}$$

where $-\mu_m^2$ is now the spectral parameter and $q_{\nu_m^2}$ satisfies at the end $\{X^1 = 0\}$,

$$q_{\nu_m^2}(X^1, \lambda) = -\frac{\lambda^2 + \frac{1}{4}}{(X^1)^2} + q_{0,\nu_m^2}(X^1, \lambda),$$

where $X^1 q_{0,\nu_m^2}(X^1, \lambda)$ is integrable at the end $\{X^1 = 0\}$ (the potential $q_{\nu_m^2}$ also has the same property at the other end). We are thus in the framework of [29].. We can then define the characteristic and Weyl–Titchmarsh functions associated with this singular non-self-adjoint Schrödinger equation. To do this, we follow the method given in [19]. We thus define two fundamental systems of solutions $\{S_{10}, S_{20}\}$ and $\{S_{11}, S_{21}\}$ having the following properties:

(1) When $X^1 \to 0$,

$$S_{10}(X^1, \mu^2, \nu^2) \sim (X^1)^{\frac{1}{2} - i\lambda} \quad \text{and} \quad S_{20}(X^1, \mu^2, \nu^2) \sim \frac{1}{2i\lambda}(X^1)^{\frac{1}{2} + i\lambda}$$

and when $X^1 \to A^1$,

$$S_{11}(X^1, \mu^2, \nu^2) \sim (A^1 - X^1)^{\frac{1}{2} - i\lambda} \quad \text{and} \quad S_{21}(X^1, \mu^2, \nu^2) \sim -\frac{1}{2i\lambda}(A^1 - X^1)^{\frac{1}{2} + i\lambda}.$$

(2) $W(S_{1n}, S_{2n}) = 1$ for $n \in \{0, 1\}$.

(3) For all $X^1 \in (0, A^1)$, $\mu \mapsto S_{jn}(X^1, \mu^2, \nu^2)$ is an entire function for $j \in \{1, 2\}$ and $n \in \{0, 1\}$.

We add some singular separated boundary conditions at the two ends (see (2.24)) and we consider the new radial equation as an eigenvalue problem. Finally, we define the two characteristic functions of this radial equation as Wronskians of functions of the fundamental systems of solutions

$$\Delta_{q_{\nu_m^2}}(\mu_m^2) = W(S_{11}, S_{10}) \quad \text{and} \quad \delta_{q_{\nu_m^2}}(\mu_m^2) = W(S_{11}, S_{20})$$

and we also define the Weyl–Titchmarsh function by

$$M_{q_{\nu_m^2}}(\mu_m^2) = -\frac{\delta_{q_{\nu_m^2}}(\mu_m^2)}{\Delta_{q_{\nu_m^2}}(\mu_m^2)}. \tag{1.20}$$

The above definition generalizes the usual definition of classical Weyl–Titchmarsh functions for regular Sturm–Liouville differential operators. We refer to [45] for the theory of self-adjoint singular Sturm–Liouville operators and the definition and main properties of Weyl–Titchmarsh functions. In our case the boundary conditions make the Sturm–Liouville equation non-self-adjoint. The generalized Weyl–Titchmarsh function can nevertheless be defined by the same recipe as shown in [19, 29] and recalled above.

We note that the characteristic and generalized Weyl–Titchmarsh functions obtained for each one-dimensional equation (1.19) can be summed over the span of each of the harmonics $Y_m$, $m \geq 1$ in order to define operators from $L^2(\mathcal{T}^2, s^{11}\, dx^2\, dx^3)$ onto itself. To be precise, recalling that

$$L^2(\mathcal{T}^2, s^{11}\, dx^2\, dx^3) = \bigoplus_{m\geq 1} \langle Y_m \rangle,$$

we have the following definition:

**Definition 1.7.** Let $\lambda \neq 0$ be a fixed energy. The characteristic operator $\Delta(\lambda)$ and the generalized Weyl–Titchmarsh operator $M(\lambda)$ are defined as operators from $L^2(\mathcal{T}^2, s^{11}\, dx^2\, dx^3)$ onto itself that are diagonalizable on the Hilbert basis of eigenfunctions $\{Y_m\}_{m\geq 1}$ associated with the eigenvalues $\Delta_{q_{\nu_m^2}}(\mu_m^2)$ and $M_{q_{\nu_m^2}}(\mu_m^2)$. More precisely, for all $v \in L^2(\mathcal{T}^2, s^{11}\, dx^2\, dx^3)$, $v$ can be decomposed as

$$v = \sum_{m\geq 1} v_m Y_m, \quad v_m \in \mathbb{C}$$

and we have

$$\Delta(\lambda) v = \sum_{m\geq 1} \Delta_{q_{\nu_m^2}}(\mu_m^2) v_m Y_m \quad \text{and} \quad M(\lambda) v = \sum_{m\geq 1} M_{q_{\nu_m^2}}(\mu_m^2) v_m Y_m.$$

We emphasize that the separation of the variables allows us to “diagonalize” the reflection and the transmission operators into a countable family of multiplication operators by numbers $R_g(\lambda, \mu_m^2, \nu_m^2)$, $L_g(\lambda, \mu_m^2, \nu_m^2)$ and $T_g(\lambda, \mu_m^2, \nu_m^2)$, called reflection and transmission coefficients respectively. We will show (see equations (2.34)–(2.36)) that the characteristic and Weyl–Titchmarsh functions are nothing but the transmission and the reflection coefficients respectively. The aim of this identification is to use the Börg–Marchenko theorem from the equality of the scattering matrix at fixed energy.

**Step 2** (Section 3): The second step of the proof consists in solving the inverse problem for the angular part of the Stäckel matrix. We begin our proof with a reduction of our problem. Indeed, our main assumption is

$$S_g(\lambda) = S_{\tilde{g}}(\lambda)$$

and these operators act on $L^2(\mathcal{T}^2, s^{11}\, dx^2\, dx^3)$ and $L^2(\mathcal{T}^2, \tilde{s}^{11}\, dx^2\, dx^3)$ respectively. To compare these objects we thus must have

$$s^{11} = \tilde{s}^{11}.$$

Using the invariance of the metric $g$ under the choice of the Stäckel matrix $S$ mentioned in the introduction and the particular structure of the operators $H$ and $L$, we then prove that

$$\begin{pmatrix} s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} = \begin{pmatrix} \tilde{s}_{21} & \tilde{s}_{22} & \tilde{s}_{23} \\ \tilde{s}_{31} & \tilde{s}_{32} & \tilde{s}_{33} \end{pmatrix}.$$

We conclude Section 3 by noticing that, thanks to these results, $H = \tilde{H}$ and $L = \tilde{L}$. As a consequence, since the generalized harmonics depend only on $H$ and $L$, we can choose $Y_m = \tilde{Y}_m$ and

$$\begin{pmatrix} \mu_m^2 \\ \nu_m^2 \end{pmatrix} = \begin{pmatrix} \tilde{\mu}_m^2 \\ \tilde{\nu}_m^2 \end{pmatrix} \quad \forall\, m \geq 1.$$

**Step 3** (Section 4): In the third step, we solve in Section 4 the inverse problem for the radial part of the Stäckel matrix. The main tool of this section is a multivariable version of the complex angular momentum method. The main assumption of Theorem 1.8 implies

$$M(\mu_m^2, \nu_m^2) = \tilde{M}(\mu_m^2, \nu_m^2) \quad \forall\, m \geq 1$$

and as explained in the introduction we deduce from this equality, thanks to this method, that

$$M_{q_{\nu^2}}(\mu^2) = M_{\tilde{q}_{\nu^2}}(\mu^2) \quad \forall\, (\mu, \nu) \in \mathbb{C}^2 \setminus P,$$

where $P$ is the set of points $(\mu, \nu) \in \mathbb{C}^2$ such that the Weyl–Titchmarsh functions do not exist, i.e., such that the denominator vanishes.

**Step 4** (Section 5): We now conclude the proof. We use the celebrated Börg–Marchenko theorem (see [19, 29]) to deduce from the previous equality that

$$q_{\nu_m^2} = \tilde{q}_{\nu_m^2} \quad \forall\, m \geq 1.$$

Since this equality is true for all $m \geq 1$, we can "decouple" the potential

$$q_{\nu_m^2} = -(\lambda^2+1)\frac{s_{11}}{s_{12}} + \nu_m^2 \frac{s_{13}}{s_{12}} + \frac{1}{16}\left(\left(\log\left(\frac{f_1}{s_{12}}\right)\right)^{\cdot}\right)^2 - \frac{1}{4}\left(\log\left(\frac{f_1}{s_{12}}\right)\right)^{\cdot\cdot}$$

and we thus obtain the uniqueness of the ratio $\frac{s_{13}}{s_{12}}$ and one ODE on the ratios $\frac{f_1}{s_{12}}$, $\frac{\tilde{f}_1}{\tilde{s}_{12}}$ and $\frac{s_{11}}{s_{12}}$, $\frac{\tilde{s}_{11}}{\tilde{s}_{12}}$. Finally, using the Robertson condition and the uniqueness of a Cauchy problem, we conclude that

$$\frac{s_{11}}{s_{12}} = \frac{\tilde{s}_{11}}{\tilde{s}_{12}} \quad \text{and} \quad \frac{s_{11}}{s_{13}} = \frac{\tilde{s}_{11}}{\tilde{s}_{13}}.$$

This finishes the proof of Step 4 and together with the previous steps, the proof of Theorem 1.8.

This paper is organized as follows. In Section 2 we solve the direct problem. In this section we study the separation of variables for the Helmholtz equation, we define the characteristic and Weyl–Titchmarsh functions for different choices of spectral parameters and we make the link between these different functions and the scattering coefficients. In Section 3 we solve the inverse problem for the angular part of the Stäckel matrix. In Section 4 we solve the inverse problem for the radial part of the Stäckel matrix using a multivariable version of the complex angular momentum method. Finally, in Section 5, we finish the proof of our main Theorem 1.8.

## §2. The direct problem

In this section we will study the direct scattering problem for the shifted Helmholtz equation (2.1). We first study the separation of the Helmholtz equation. Second, we define several characteristic and generalized Weyl–Titchmarsh functions associated with unidimensional Schrödinger equations in the radial variable corresponding to different choices of spectral parameters and we study the link between these functions and the scattering operator associated with the Helmholtz equation.

### §2.1. Separation of variables for the Helmholtz equation

We consider (see [11, 35, 36, 37]) the shifted stationary Helmholtz equation

$$-(\Delta_g + 1)f = \lambda^2 f, \tag{2.1}$$

where $\lambda \neq 0$ is a fixed energy, which is usually studied in the case of asymptotically hyperbolic manifolds (see [11, 35, 36, 37]). Indeed, it is known (see [36]) that the essential spectrum of $-\Delta_g$ is $[1, +\infty)$ and thus we shift the bottom of the essential spectrum to 0. It is known that the operator $-\Delta_g - 1$ has no eigenvalues embedded in the essential spectrum $[0,+\infty)$ (see [13, 35, 36]). We know that there exists a coordinate system separable for the Helmholtz equation (2.1) if and only if the metric (1.4) is in Stäckel form and furthermore if the Robertson condition (1.6) is satisfied. We emphasize that, contrary to the case $n=2$ studied in [19], we really need the Robertson condition in the case $n=3$.

**Lemma 2.1.** *The Helmholtz equation (2.1) can be rewritten as*

$$A_1 f + s_{12} L f + s_{13} H f = 0, \tag{2.2}$$

*where*

$$A_i = -\partial_i^2 + \frac{1}{2}\partial_i\left(\log(f_i)\right)\partial_i - (\lambda^2+1)s_{i1} \quad \textit{for } i \in \{1,2,3\} \tag{2.3}$$

*and*

$$L = -\frac{s_{33}}{s^{11}}A_2 + \frac{s_{23}}{s^{11}}A_3 \quad \textit{and} \quad H = \frac{s_{32}}{s^{11}}A_2 - \frac{s_{22}}{s^{11}}A_3. \tag{2.4}$$

*Proof.* Letting $|g|$ be the determinant of the metric and $(g^{ij})$ the inverse of the metric $(g_{ij})$, $g^{ii} = \frac{1}{H_i^2}$ and $\sqrt{|g|} = H_1 H_2 H_3$. Using the Robertson condition (1.7), we easily show that

$$\Delta_g = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\right) = \sum_{i=1}^{3}\frac{1}{H_i^2}\left(\partial_i^2 - \frac{1}{2}\partial_i\left(\log(f_i)\right)\partial_i\right). \tag{2.5}$$

Hence, from (2.5) we immediately obtain that the Helmholtz equation (2.1) can be written as

$$\sum_{i=1}^{3}\frac{1}{H_i^2}A_i^0 f = (\lambda^2+1)f, \tag{2.6}$$

where

$$A_i^0 = -\partial_i^2 + \frac{1}{2}\partial_i\left(\log(f_i)\right)\partial_i \quad \text{for } i \in \{1,2,3\}. \tag{2.7}$$

If we multiply equation (2.6) by $H_1^2$ and if we use that

$$H_1^2 = s_{11} + s_{21}\frac{s^{21}}{s^{11}} + s_{31}\frac{s^{31}}{s^{11}}, \quad \frac{H_1^2}{H_2^2} = \frac{s^{21}}{s^{11}} \quad \text{and} \quad \frac{H_1^2}{H_3^2} = \frac{s^{31}}{s^{11}},$$

we obtain

$$A_1 f + \frac{s^{21}}{s^{11}}A_2 f + \frac{s^{31}}{s^{11}}A_3 f = 0. \tag{2.8}$$

Finally, using the equalities

$$\frac{s^{21}}{s^{11}} = -s_{12}\frac{s_{33}}{s^{11}} + s_{13}\frac{s_{32}}{s^{11}} \quad \text{and} \quad \frac{s^{31}}{s^{11}} = s_{12}\frac{s_{23}}{s^{11}} - s_{13}\frac{s_{22}}{s^{11}},$$

from (2.8) we obtain equation (2.2). □

**Remark 2.1.** (1) We note that the Robertson condition is equivalent to the existence of three functions $f_i = f_i(x^i)$ such that

$$\partial_i \log\left(\frac{H_i^4}{H_1^2 H_2^2 H_3^2}\right) = \partial_i \log(f_i) \quad \forall\, i \in \{1,2,3\}.$$

This equality is interesting since it gives us an expression of the Robertson condition directly in terms of the coefficients $H_i^2$ of the metric $g$.

(2) Since we assumed that $f_2$ and $f_3$ are constant functions equal to 1 (see Remark 1.4) we know that

$$A_2^0 = -\partial_2^2 \quad \text{and} \quad A_3^0 = -\partial_3^2.$$

**Lemma 2.2.** *The operators $L$ and $H$ are self-adjoint, semibounded and elliptic on $L^2(\mathcal{T}^2, s^{11}\, dx^2\, dx^3)$. Moreover, they commute.*

**Remark 2.2.** Since the operators $L$ and $H$ commute, there exists a common Hilbertian basis of eigenfunctions $\{Y_m\}_{m\geq 1}$ of $H$ and $L$, i.e.,

$$HY_m = \mu_m^2 Y_m \quad \text{and} \quad LY_m = \nu_m^2 Y_m, \quad \forall\, m \geq 1, \tag{2.9}$$

and

$$L^2(\mathcal{T}^2, s^{11} dx^2 dx^3) = \bigoplus_{m\geq 1} \langle Y_m \rangle.$$

Here, we arrange the coupled spectrum $(\mu_m^2, \nu_m^2)$ such that

(1) counting multiplicity,

$$\mu_1^2 < \mu_2^2 \leq \mu_3^2 \leq \mu_4^2 \leq \cdots \leq \mu_n^2 \leq \cdots \to \infty;$$

(2) starting from $n = 1$ and by induction on $n$, for each $n \geq 1$ such that $\mu_n^2$ has multiplicity $k$, i.e., $\mu_n^2 = \mu_{n+1}^2 = \cdots = \mu_{n+k-1}^2$, we order the corresponding $(\nu_j^2)_{n\leq j\leq n+k-1}$ in increasing order, i.e., counting multiplicity,

$$\nu_n^2 \leq \nu_{n+1}^2 \leq \cdots \leq \nu_{n+k-1}^2.$$

The toric cylinder topology implies that the boundary conditions are compatible with the decomposition on the common harmonics $\{Y_m\}_{m\geq 1}$ of $H$ and $L$. We thus look for solutions of (2.1) of the form

$$f(x^1, x^2, x^3) = \sum_{m\geq 1} u_m(x^1) Y_m(x^2, x^3). \tag{2.10}$$

We use (2.10) in (2.2) and we obtain that $u_m$ satisfies, for all $m \geq 1$,

$$\begin{aligned} &- u''(x^1) + \frac{1}{2}(\log(f_1)(x^1))' u'(x^1) \\ &\quad + \left[-(\lambda^2+1)s_{11}(x^1) + \mu_m^2 s_{12}(x^1) + \nu_m^2 s_{13}(x^1)\right] u(x^1) = 0. \end{aligned}$$

Finally, inverting (2.9), we obtain

$$(2.11)\qquad \begin{cases} A_2 Y_m = -(s_{22}\mu_m^2 + s_{23}\nu_m^2) Y_m, \\ A_3 Y_m = -(s_{32}\mu_m^2 + s_{33}\nu_m^2) Y_m. \end{cases}$$

**Remark 2.3.** The harmonics $Y_m(x^2, x^3)$, $m \geq 1$ can be written as a product of a function of the variable $x^2$ and a function of the variable $x^3$. Let $(f_2, g_2)$ and $(f_3, g_3)$ be periodic fundamental systems of solutions associated with the operators $A_2$ and $A_3$ respectively. We can thus write $Y_m(x^2, x^3)$ as

$$Y_m(x^2, x^3) = a(x^3) f_2(x^2) + b(x^3) g_2(x^2).$$

We then apply the operator $A_3$ on this equality and we obtain

$$A_3(Y_m)(x^2, x^3) = A_3(a)(x^3) f_2(x^2) + A_3(b)(x^3) g_2(x^2).$$

Thus, using that $A_3 Y_m = -(s_{32}\mu_m^2 + s_{33}\nu_m^2) Y_m$ and the fact that $(f_2, g_2)$ is a fundamental system of solutions we obtain

$$Y_m(x^2, x^3) = a f_2(x^2) f_3(x^3) + b f_2(x^2) g_3(x^3) + c g_2(x^2) f_3(x^3) + d g_2(x^2) g_3(x^3),$$

where $a$, $b$, $c$ and $d$ are real constants. Thus, for each coupled eigenvalue $(\mu_m^2, \nu_m^2)$, $m \geq 1$, the corresponding eigenspace for the couple of operators $(H, L)$ is at most of dimension 4. However, the diagonalization of the scattering matrix $S_g(\lambda)$ does not depend on the choice of the harmonics in each eigenspace associated with a coupled eigenvalue $(\mu_m^2, \nu_m^2)$ and we can thus choose as harmonics $Y_m = f_2 f_3$, $Y_m = f_2 g_3$, $Y_m = g_2 f_3$ and $Y_m = g_2 g_3$. We can then assume that $Y_m(x^2, x^3)$ is a product of a function of the variable $x^2$ and a function of the variable $x^3$.

**Lemma 2.3.** *Any solution $u \in H^1(\mathcal{M})$ of $-(\Delta_g + 1)u = \lambda^2 u$ can be written as*

$$u = \sum_{m \geq 1} u_m(x^1) Y_m(x^2, x^3),$$

*where $Y_m(x^2, x^3) = v_m(x^2) w_m(x^3)$ and*

$$\begin{cases} -u_m''(x^1) + \frac{1}{2}(\log(f_1)(x^1))' u_m'(x^1) + [-(\lambda^2 + 1) s_{11}(x^1) + \mu_m^2 s_{12}(x^1) \\ \qquad + \nu_m^2 s_{13}(x^1)] u_m(x^1) = 0, \\ -v_m''(x^2) + [-(\lambda^2 + 1) s_{21}(x^2) + \mu_m^2 s_{22}(x^2) + \nu_m^2 s_{23}(x^2)] v_m(x^2) = 0, \\ -w_m''(x^3) + [-(\lambda^2 + 1) s_{31}(x^3) + \mu_m^2 s_{32}(x^3) + \nu_m^2 s_{33}(x^3)] w_m(x^3) = 0. \end{cases}$$

From Lemma 2.3 we can deduce more information on the eigenvalues $(\mu_m^2)_{m \geq 1}$ and $(\nu_m^2)_{m \geq 1}$. Indeed, we can prove the following lemma which will be useful in the sequel.

**Lemma 2.4.** *There exist real constants $C_1$, $C_2$, $D_1$ and $D_2$ such that for all $m \geq 1$,*

$$C_1\mu_m^2 + D_1 \leq \nu_m^2 \leq C_2\mu_m^2 + D_2,$$

*where*

$$C_1 = \min\left(-\frac{s_{32}}{s_{33}}\right) > 0 \quad \text{and} \quad C_2 = -\min\left(\frac{s_{22}}{s_{23}}\right) > 0.$$

*Proof.* We first recall the angular equations of Lemma 2.3:

$$-v''(x^2) + [-(\lambda^2+1)s_{21}(x^2) + \mu_m^2 s_{22}(x^2) + \nu_m^2 s_{23}(x^2)]v(x^2) = 0 \tag{2.12}$$

and

$$-w''(x^3) + [-(\lambda^2+1)s_{31}(x^3) + \mu_m^2 s_{32}(x^3) + \nu_m^2 s_{33}(x^3)]w(x^3) = 0. \tag{2.13}$$

We use a Liouville change of variables in (2.12) to transform this equation into a Schrödinger equation in which $-\nu_m^2$ is the spectral parameter. Thus, we define the diffeomorphism $X^2 = g_2(x^2) = \int_0^{x^2} \sqrt{s_{23}(t)}\,dt$ and we define $v(X^2,\mu_m^2,\nu_m^2) = v(h_2(X^2),\mu_m^2,\nu_m^2)$, where $h_2 = g_2^{-1}$ is the inverse function of $g_2$. We also introduce a weight function to cancel the first-order term. We thus define

$$V(X^2,\mu_m^2,\nu_m^2) = \left(\frac{1}{s_{23}(h_2(X^2))}\right)^{-\frac{1}{4}} v(h_2(X^2),\mu_m^2,\nu_m^2).$$

After calculation, we obtain that $V(X^2,\mu_m^2,\nu_m^2)$ satisfies, in the variable $X^2$, the Schrödinger equation

$$-\ddot{V}(X^2,\mu_m^2,\nu_m^2) + p_{\mu_m^2,2}(X^2,\lambda)V(X^2,\mu_m^2,\nu_m^2) = -\nu_m^2 V(X^2,\mu_m^2,\nu_m^2), \tag{2.14}$$

where

$$p_{\mu_m^2,2}(X^2,\lambda) = -(\lambda^2+1)\frac{s_{21}(X^2)}{s_{23}(X^2)} + \mu_m^2\frac{s_{22}(X^2)}{s_{23}(X^2)}, \tag{2.15}$$

with $s_{21}(X^2) := s_{21}(h_2(X^2))$, $s_{22}(X^2) := s_{22}(h_2(X^2))$ and $s_{23}(X^2) := s_{23}(h_2(X^2))$. We follow the same procedure for (2.13) putting

$$X^3 = g_3(x^3) = \int_0^{x^3} \sqrt{-s_{33}(t)}\,dt$$
$$\text{and} \quad W(X^3,\mu_m^2,\nu_m^2) = \left(\frac{1}{-s_{33}(h_3(X^3))}\right)^{-\frac{1}{4}} w(h_3(X^3),\mu_m^2,\nu_m^2)$$

and we obtain that $W(X^3)$ satisfies, in the variable $X^3$, the Schrödinger equation

$$-\ddot{W}(X^3,\mu_m^2,\nu_m^2) + p_{\mu_m^2,3}(X^3,\lambda)W(X^3,\mu_m^2,\nu_m^2) = \nu_m^2 W(X^3,\mu_m^2,\nu_m^2), \tag{2.16}$$

where

$$p_{\mu_m^2,3}(X^3,\lambda) = (\lambda^2+1)\frac{s_{31}(X^3)}{s_{33}(X^3)} - \mu_m^2\frac{s_{32}(X^3)}{s_{33}(X^3)}, \tag{2.17}$$

with $s_{31}(X^3) := s_{31}(h_3(X^3))$, $s_{32}(X^3) := s_{32}(h_3(X^3))$ and $s_{33}(X^3) := s_{33}(h_3(X^3))$. Assume now that $\mu_m^2$ is fixed and look at (2.14) and (2.16) as eigenvalue problems in $\pm\nu_m^2$. We suppose that $\mu_m^2$ has multiplicity $k \geq 1$ and we use the notation given in Remark 2.2, i.e., that we want to show that

$$C_1\mu_m^2 + D_2 \leq \nu_j^2 \leq C_2\mu_m^2 + D_2 \quad \forall\, m \leq j \leq m+k-1,$$

where $\nu_j^2 \leq \nu_{j+1}^2$ for all $j \in \{m,\dots,m+k-1\}$. We know that the spectra of the operators

$$P_2 = -\frac{d^2}{(dX^2)^2} + p_{\mu_m^2,2} \quad \text{and} \quad P_3 = -\frac{d^2}{(dX^3)^2} + p_{\mu_m^2,3}$$

are included in $[\min(p_{\mu_m^2,2}), +\infty)$ and $[\min(p_{\mu_m^2,3}), +\infty)$ respectively. The first condition gives us that

$$-\nu_j^2 \geq -C_2\mu_m^2 - D_2, \quad \text{where } -C_2 = \min\left(\frac{s_{22}}{s_{23}}\right) \text{ and } -D_2 = (\lambda^2+1)\min\left(-\frac{s_{21}}{s_{23}}\right)$$

and the second one tells us that

$$\nu_j^2 \geq C_1\mu_m^2 + D_1, \quad \text{where } C_1 = \min\left(-\frac{s_{32}}{s_{33}}\right) \text{ and } D_1 = (\lambda^2+1)\min\left(\frac{s_{31}}{s_{33}}\right).$$

Since $(\nu_j^2)_{m\leq j\leq m+k-1}$ is the set of eigenvalues of (2.12) and (2.13), for a fixed $\mu_m^2$ of multiplicity $k$ we obtain from these estimates that

$$C_1\mu_m^2 + D_1 \leq \nu_j^2 \leq C_2\mu_m^2 + D_2 \quad \forall\, m \leq j \leq m+k-1.$$

In other words, thanks to our numbering of the coupled spectrum explained in Remark 2.2,

$$C_1\mu_m^2 + D_1 \leq \nu_m^2 \leq C_2\mu_m^2 + D_2 \quad \forall\, m \geq 1.$$

□

**Remark 2.4.** (1) Using the condition given in Remark 1.1, $C_1 = \min\left(-\frac{s_{32}}{s_{33}}\right) < -\min\left(\frac{s_{22}}{s_{23}}\right) = C_2$.

(2) The previous lemma says that the coupled spectrum $\{(\mu_m^2, \nu_m^2),\ m \geq 1\}$ lives in a cone contained in the quadrant $(\mathbb{R}^+)^2$ (up to a possible shift due to the presence of the constants $D_1$ and $D_2$). Moreover, since the multiplicity of $\mu_m^2$ is finite for all $m \geq 1$, there is a finite number of points of the coupled spectrum on each vertical line. We can summarize these facts with the generic picture in Figure 1.

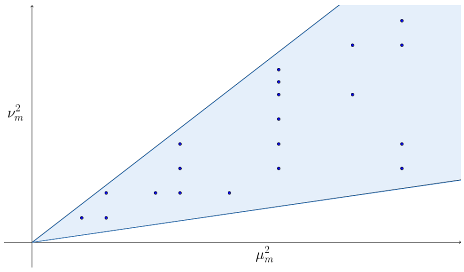


Figure 1. Coupled spectrum.

(3) The Weyl law (see [41, Thm. 2.21]) which says (in two dimensions) that there exists a constant $C$ such that the eigenvalues are equivalent for large $m$ to $Cm$, is satisfied by the eigenvalues $\{\mu_m^2\}_m$ and $\{\nu_m^2\}_m$ if we arrange them in increasing order. However, we labelled the coupled spectrum in such a way that the order for the $(\nu_m^2)$ is not necessarily increasing.

(4) An eigenvalue of the coupled spectrum $(\mu_m^2, \nu_m^2)$ has at most multiplicity 4, as was mentioned in Remark 2.3.

**Example 2.1.** We can illustrate the notion of coupled spectrum on the examples given in Example 1.1.

(1) We define the Stäckel matrix

$$S_1 = \begin{pmatrix} s_{11}(x^1) & s_{12}(x^1) & s_{13}(x^1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this case $H = -\partial_3^2$ and $L = -\partial_2^2$ and we note that these operators can be obtained by derivation of the Killing vector fields $\partial_2$ and $\partial_3$. The coupled spectrum of these operators is $\{(m^2, n^2), (m,n) \in \mathbb{Z}^2\}$ and we can decompose the space $L^2(\mathcal{T}^2, s^{11}\, dx^2\, dx^3)$ on the basis of generalized harmonics $Y_{mn} = e^{imx^2 + inx^3}$. We note that this coupled spectrum is not included in a cone strictly contained in $(\mathbb{R}^+)^2$ but there is no contradiction with Lemma 2.4 since the Stäckel matrix $S$ does not satisfy condition (C). However, we can use the invariances of Proposition 1.5 to come down to our framework (this transformation modifies the coupled spectrum). Indeed, we can obtain the Stäckel matrix

$$S_2 \begin{pmatrix} s_{11}(x^1) & s_{12}(x^1) & s_{13}(x^1) \\ a & b & c \\ d & e & f \end{pmatrix},$$

where $s_{11}$, $s_{12}$ and $s_{13}$ are smooth functions of $x^1$ and $a$, $b$, $c$, $d$, $e$ and $f$ are real constants such that $b, f < 0$ and $c, e > 0$. In this case we have

$$H = -\frac{s_{32}}{s^{11}}\partial_2^2 + \frac{s_{22}}{s^{11}}\partial_3^2 = -\frac{e}{bf-ce}\partial_2^2 + \frac{b}{bf-ce}\partial_3^2$$

and

$$L = \frac{s_{33}}{s^{11}}\partial_2^2 - \frac{s_{23}}{s^{11}}\partial_3^2 = \frac{f}{bf-ce}\partial_2^2 - \frac{c}{bf-ce}\partial_3^2.$$

Thus, the coupled spectrum of the operators $H$ and $L$ can be computed using the same procedure as the one used for $S_1$.

We emphasize that in the case of the Stäckel matrix $S_1$ the coupled spectrum is in fact uncoupled. We can thus freeze one angular momentum and let the other one move on the integers. After the use of the invariance to come down to our framework these vertical or horizontal half-lines are transformed into half-lines contained in our cone of $(\mathbb{R}^+)^2$. This allows us to use the usual complexification of the angular momentum method in one dimension on a half-line contained in our cone.

(2) We define the Stäckel matrix

$$S = \begin{pmatrix} s_{11}(x^1) & s_{12}(x^1) & as_{12}(x^1) \\ 0 & s_{22}(x^2) & s_{23}(x^2) \\ 0 & s_{32}(x^3) & s_{33}(x^3) \end{pmatrix},$$

where $s_{11}$ and $s_{12}$ are smooth functions of $x^1$, $s_{22}$ and $s_{23}$ are smooth functions of $x^2$, $s_{32}$ and $s_{33}$ are smooth functions of $x^3$ and $a$ is a real constant. In this case, the Helmholtz equation (2.1) can be rewritten as $A_1 f + s_{12}(L + aH)f = 0$. Therefore, the separation of variables depends only on a single angular operator given by $L + aH$ whose properties can be easily derived from the ones for $H$ and $L$. In particular, the set of angular momenta is given by $\omega_m^2 = \mu_m^2 + \nu_m^2$, $m \geq 1$, and could be used to apply the Complexification of the Angular Momentum method. Note that, even though the spectra $\{\mu_m^2, \nu_m^2\}$ are coupled, only the spectrum $\omega_m^2$ appears in the separated radial equation.

(3) In the case of the Stäckel matrix

$$S = \begin{pmatrix} s_1(x^1)^2 & -s_1(x^1) & 1 \\ -s_2(x^2)^2 & s_2(x^2) & -1 \\ s_3(x^3)^2 & -s_3(x^3) & 1 \end{pmatrix},$$

where $s_1$ is a smooth function of $x^1$, $s_2$ is a smooth function of $x^2$ and $s_3$ is a smooth function of $x^3$, there is no trivial symmetry. We are thus in the general case and we have to use the general method we develop in this paper.

## §2.2. A first construction of characteristic and Weyl–Titchmarsh functions

The aim of this section is to define the characteristic and Weyl–Titchmarsh functions for the radial equation choosing $-\mu_m^2$ as the spectral parameter. We recall that the radial equation is

$$-u'' + \frac{1}{2}(\log(f_1))'u' + \left[-(\lambda^2+1)s_{11} + \mu_m^2 s_{12} + \nu_m^2 s_{13}\right] u = 0, \tag{2.18}$$

where the functions depend only on $x^1$. We choose $-\mu^2 := -\mu_m^2$ to be the spectral parameter. As mentioned in the introduction, to do this we make a Liouville change of variables:

$$X^1 = g(x^1) = \int_0^{x^1} \sqrt{s_{12}(t)}\, dt$$

and we define $u(X^1, \mu^2, \nu^2) = u(h(X^1), \mu^2, \nu^2)$, where $h = g^{-1}$ is the inverse function of $g$ and $\nu^2 := \nu_m^2$. Next, to cancel the resulting first-order term and obtain a Schrödinger equation, we introduce a weight function. To be precise, we define

$$U(X^1, \mu^2, \nu^2) = \left(\frac{f_1}{s_{12}}(h(X^1))\right)^{-\frac{1}{4}} u(h(X^1), \mu^2, \nu^2). \tag{2.19}$$

After calculation, we obtain that $U(X^1, \mu^2, \nu^2)$ satisfies, in the variable $X^1$, the Schrödinger equation

$$-\ddot{U}(X^1, \mu^2, \nu^2) + q_{\nu^2}(X^1, \lambda)U(X^1, \mu^2, \nu^2) = -\mu^2 U(X^1, \mu^2, \nu^2), \tag{2.20}$$

where

$$\begin{aligned} q_{\nu^2}(X^1, \lambda) &= -(\lambda^2+1)\frac{s_{11}(X^1)}{s_{12}(X^1)} + \nu^2 \frac{s_{13}(X^1)}{s_{12}(X^1)} \\ &\quad + \frac{1}{16}\left(\left(\log\left(\dot{\frac{f_1(X^1)}{s_{12}(X^1)}}\right)\right)\right)^2 - \frac{1}{4}\left(\log\left(\ddot{\frac{f_1(X^1)}{s_{12}(X^1)}}\right)\right) \end{aligned} \tag{2.21}$$

with $\dot{f} = \frac{df}{dX^1}$, $f_1(X^1) := f_1(h_1(X^1))$, $s_{11}(X^1) := s_{11}(h_1(X^1))$, $s_{12}(X^1) := s_{12}(h_1(X^1))$ and $s_{13}(X^1) := s_{13}(h_1(X^1))$.

**Lemma 2.5.** *The potential $q_{\nu^2}$ satisfies, at the end $\{X^1 = 0\}$,*

$$q_{\nu^2}(X^1, \lambda) = -\frac{\lambda^2 + \frac{1}{4}}{(X^1)^2} + q_{0,\nu^2}(X^1, \lambda),$$

*where* $X^1 q_{0,\nu^2}(X^1,\lambda) \in L^1\left(0, \frac{A^1}{2}\right)$ *with* $A^1 = g(A)$*. Similarly, at the end* $\{X^1 = A^1\}$,

$$q_{\nu^2}(X^1,\lambda) = -\frac{\lambda^2 + \frac{1}{4}}{(A^1 - X^1)^2} + q_{A^1,\nu^2}(X^1,\lambda),$$

*where* $(A^1 - X^1) q_{A^1,\nu^2}(X^1,\lambda) \in L^1\left(\frac{A^1}{2}, A^1\right)$.

*Proof.* We first note that since $s_{12}(x^1) \sim 1$ when $x^1 \to 0$ we obtain by definition of $X^1$ that $X^1 \sim x^1$ as $x^1 \to 0$. Thus we can use the hyperbolicity conditions (1.9) directly in the variable $X^1$. The lemma is then a consequence of these conditions. □

We now follow the paper [19] to define the characteristic and the Weyl–Titchmarsh functions associated with the equation (2.20). To do that, we introduce two fundamental systems of solutions $S_{jn}$, $j \in \{1,2\}$ and $n \in \{0,1\}$ defined by

(1) when $X^1 \to 0$,

(2.22) $$S_{10}(X^1,\mu^2,\nu^2) \sim (X^1)^{\frac{1}{2}-i\lambda} \quad \text{and} \quad S_{20}(X^1,\mu^2,\nu^2) \sim \frac{1}{2i\lambda}(X^1)^{\frac{1}{2}+i\lambda}$$

and when $X^1 \to A^1$,

(2.23) $$\begin{aligned} &S_{11}(X^1,\mu^2,\nu^2) \sim (A^1 - X^1)^{\frac{1}{2}-i\lambda} \\ &\text{and} \quad S_{21}(X^1,\mu^2,\nu^2) \sim -\frac{1}{2i\lambda}(A^1 - X^1)^{\frac{1}{2}+i\lambda}; \end{aligned}$$

(2) $W(S_{1n}, S_{2n}) = 1$ for $n \in \{0,1\}$;

(3) for all $X^1 \in (0, A^1)$, $\mu \mapsto S_{jn}(X^1,\mu^2,\nu^2)$ is an entire function for $j \in \{1,2\}$ and $n \in \{0,1\}$.

As in [19, 29], we add singular boundary conditions at the ends $\{X^1 = 0\}$ and $\{X^1 = A^1\}$ and we consider (2.20) as an eigenvalue problem. To be precise, we consider the conditions

(2.24) $$U(u) := W(S_{10}, u)_{|X^1=0} = 0 \quad \text{and} \quad V(u) := W(S_{11}, u)_{|X^1=A^1} = 0,$$

where $W(f,g) = fg' - f'g$ is the Wronskian of $f$ and $g$. Finally, we can define the characteristic functions

(2.25) $$\Delta_{q_{\nu^2}}(\mu^2) = W(S_{11}, S_{10}) \quad \text{and} \quad \delta_{q_{\nu^2}}(\mu^2) = W(S_{11}, S_{20})$$

and the Weyl–Titchmarsh function

(2.26) $$M_{q_{\nu^2}}(\mu^2) = -\frac{W(S_{11}, S_{20})}{W(S_{11}, S_{10})} = -\frac{\delta_{q_{\nu^2}}(\mu^2)}{\Delta_{q_{\nu^2}}(\mu^2)}.$$

**Remark 2.5.** The Weyl–Titchmarsh function is the unique function such that the solution of (2.20) given by $\Phi(X^1, \mu^2, \nu^2) = S_{10}(X^1, \mu^2, \nu^2) + M_{q_{\nu^2}}(\mu^2) S_{20}(X^1, \mu^2, \nu^2)$ satisfies the boundary condition at the end $\{X^1 = A^1\}$.

An immediate consequence of the third condition in the definition of the fundamental systems of solutions is the following lemma.

**Lemma 2.6.** *For any fixed $\nu$, the maps $\mu \mapsto \Delta_{q_{\nu^2}}(\mu^2)$ and $\mu \mapsto \delta_{q_{\nu^2}}(\mu^2)$ are entire.*

In the following (see Sections 2.4 and 2.5), we will define other characteristic and Weyl–Titchmarsh functions that correspond to other choices of spectral parameters which are $-\nu_m^2$ and $-(\mu_m^2 + \nu_m^2)$. We study here the influence of these other choices.

**Proposition 2.7.** *Let $\check{X}^1 = \check{g}(x^1)$ be a change of variables and $\check{p}$ be a weight function; then*

$$U(X^1, \mu^2, \nu^2) = \frac{p(h(X^1))}{\check{p}(\check{h}(\check{X}^1))} \check{U}((\check{g} \circ h)(X^1), \mu^2, \nu^2)$$
$$\text{and} \quad V(X^1, \mu^2, \nu^2) = \frac{p(h(X^1))}{\check{p}(\check{h}(\check{X}^1))} \check{V}((\check{g} \circ h)(X^1), \mu^2, \nu^2),$$

*where* $p(h(X^1)) = \left(\frac{f_1}{s_{12}}(h(X^1))\right)^{-\frac{1}{4}}$,

$$\check{U}(\check{X}^1, \mu^2, \nu^2) = \check{p}(\check{h}(\check{X}^1)) u(\check{h}(\check{X}^1), \mu^2, \nu^2)$$
$$\text{and} \quad \check{V}(\check{X}^1, \mu^2, \nu^2) = \check{p}(\check{h}(\check{X}^1)) v(\check{h}(\check{X}^1), \mu^2, \nu^2).$$

*Moreover,*

$$W_{X^1}(U, V) = \left(\frac{p(h(X^1))}{\check{p}(\check{h}(\check{X}^1))}\right)^2 \partial_{X^1}(\check{g} \circ h)(X^1) W_{\check{X}^1}(\check{U}, \check{V}).$$

**Corollary 2.8.** *Let $\hat{X}^1$ and $\check{X}^1$ be two Liouville changes of variables defined by*

$$\hat{X}^1 = \hat{g}(x^1) = \int_0^{x^1} \sqrt{s_{13}(t)}\, dt \quad \text{and} \quad \check{X}^1 = \check{g}(x^1) = \int_0^{x^1} \sqrt{r_{\mu^2,\nu^2}(t)}\, dt,$$

*where*

$$r_{\mu^2,\nu^2}(x^1) := \frac{\mu^2 s_{12}(x^1) + \nu^2 s_{13}(x^1)}{\mu^2 + \nu^2},$$

*and let $\hat{p}$ and $\check{p}$ be two weight functions defined by*

$$\hat{p}(\hat{h}(\hat{X}^1)) = \left(\frac{f_1}{s_{13}}(\hat{h}(\hat{X}^1))\right)^{-\frac{1}{4}} \quad \text{and} \quad \check{p}(\check{h}(\check{X}^1)) = \left(\frac{f_1}{r_{\mu^2,\nu^2}}(\check{h}(\check{X}^1))\right)^{-\frac{1}{4}}.$$

*Let $\hat{U}$ and $\hat{V}$ be defined as*

$$\hat{U}(\hat{X}^1,\mu^2,\nu^2) = \hat{p}(\hat{h}(\hat{X}^1))u(\hat{h}(\hat{X}^1),\mu^2,\nu^2)$$
$$\text{and} \quad \hat{V}(\hat{X}^1,\mu^2,\nu^2) = \hat{p}(\hat{h}(\hat{X}^1))v(\hat{h}(\hat{X}^1),\mu^2,\nu^2)$$

*and $\check{U}$ and $\check{V}$ be defined as*

$$\check{U}(\check{X}^1,\mu^2,\nu^2) = \check{p}(\check{h}(\check{X}^1))u(\check{h}(\check{X}^1),\mu^2,\nu^2)$$
$$\text{and} \quad \check{V}(\check{X}^1,\mu^2,\nu^2) = \check{p}(\check{h}(\check{X}^1))v(\check{h}(\check{X}^1),\mu^2,\nu^2).$$

*Then,*

$$W_{X^1}(U,V) = W_{\hat{X}^1}(\hat{U},\hat{V}) = W_{\check{X}^1}(\check{U},\check{V}).$$

We will use $\hat{X}^1$ and $\hat{p}$ in Section 2.4 to obtain holomorphic properties and good estimates in the variable $\nu^2$. Second, we will use $\check{X}^1$ and $\check{p}$ in Section 2.5 to show that the characteristic functions are bounded for $(\mu,\nu) \in (i\mathbb{R})^2$.

### §2.3. Link between the scattering coefficients and the Weyl–Titchmarsh and characteristic functions

In this section, we follow [19, Sect. 3.3] and we make the link between the transmission and the reflection coefficients, corresponding to the restriction of the transmission and the reflection operators on each generalized harmonics, and the characteristic and Weyl–Titchmarsh functions defined in Section 2.2. First, we observe that the scattering operator defined in Theorem 1.7 leaves invariant the span of each generalized harmonic $Y_m$. Hence, it suffices to calculate the scattering operator on each vector space generated by the $Y_m$'s. To do this, we recall from Theorem 1.7 that, given any solution $f = u_m(x^1)Y_m(x^2,x^3) \in \mathcal{B}^\star$ of (2.1), there exists a unique $\psi_m^{(\pm)} = (\psi_{0m}^{\pm}, \psi_{1m}^{(\pm)}) \in \mathbb{C}^2$ such that

$$\begin{aligned} u_m(x^1) \simeq \omega_-(\lambda)\left(\chi_0\,(x^1)^{\frac{1}{2}+i\lambda}\psi_{0m}^{(-)} + \chi_1\,(A-x^1)^{\frac{1}{2}+i\lambda}\psi_{1m}^{(-)}\right) \\ - \omega_+(\lambda)\left(\chi_0\,(x^1)^{\frac{1}{2}-i\lambda}\psi_{0m}^{(+)} + \chi_1\,(A-x^1)^{\frac{1}{2}-i\lambda}\psi_{1m}^{(+)}\right), \end{aligned} \tag{2.27}$$

where $\omega_\pm$ are given by (1.16) and the cut-off functions $\chi_0$ and $\chi_1$ are defined in (1.13). As in [19], we would like to apply this result to the fundamental systems of solutions (FSS) $\{S_{jn},\ j=1,2,\ n=0,1\}$. However we recall that $S_{jn}$ are solutions of the equation (2.20) and this Schrödinger equation was obtained from the Helmholtz equation (2.1) by a change a variables and the introduction of a weight function (see (2.19)). We thus apply the previous result to the functions

$$u_{jn}(x^1,\mu^2,\nu^2) = \left(\frac{f_1}{s_{12}}(x^1)\right)^{\frac{1}{4}} S_{jn}(g(x^1),\mu^2,\nu^2), \quad j\in\{1,2\},\ n\in\{0,1\}. \tag{2.28}$$

We first study the behaviour of the weight function at the two ends in the following lemma.

**Lemma 2.9.** *When* $x^1 \to 0$,

$$\left(\frac{f_1}{s_{12}}(x^1)\right)^{\frac{1}{4}} = \sqrt{x^1}[1]_{\epsilon_0} \quad \textit{where } [1]_{\epsilon_0} = 1 + O((1+|\log(x^1)|)^{-1-\epsilon_0}).$$

*The corresponding result at the end* $\{x^1 = A\}$ *is also true.*

*Proof.* We first divide the Robertson condition (1.6) by $s_{12}$ and we obtain

$$\frac{f_1}{s_{12}} = \frac{\left(\frac{\det(S)}{s_{12}}\right)^2}{\frac{H_1^2}{s_{12}} H_2^2 H_3^2}.$$

We use the hyperbolicity conditions given in (1.8)–(1.9) and Remark 1.4 to obtain

$$\begin{cases} \frac{\det(S)}{s_{12}} = \frac{s^{11}}{(x^1)^2}[1]_{\epsilon_0}, \\ \frac{H_1^2}{s_{12}} = \frac{1}{(x^1)^2}[1]_{\epsilon_0}, \\ H_2^2 = \frac{s^{11}}{(x^1)^2}[1]_{\epsilon_0}, \\ H_3^2 = \frac{s^{11}}{(x^1)^2}[1]_{\epsilon_0}. \end{cases}$$

The lemma is then a direct consequence of these estimates. □

Thanks to (2.22)–(2.23), (2.28) and Lemma 2.9, we obtain that when $x^1 \to 0$,

$$u_{10}(x^1, \mu^2, \nu^2) \sim (x^1)^{1-i\lambda} \quad \text{and} \quad u_{20}(x^1, \mu^2, \nu^2) \sim \frac{1}{2i\lambda}(x^1)^{1+i\lambda}$$

and when $x^1 \to A$,

$$u_{11}(x^1, \mu^2, \nu^2) \sim (A - x^1)^{1-i\lambda} \quad \text{and} \quad u_{21}(x^1, \mu^2, \nu^2) \sim -\frac{1}{2i\lambda}(A - x^1)^{1+i\lambda}. \tag{2.29}$$

We denote by $\psi^{(-)} = (\psi_0^{(-)}, \psi_1^{(-)})$ and $\psi^{(+)} = (\psi_0^{(+)}, \psi_1^{(+)})$ the constants appearing in Theorem 1.7 corresponding to $u_{10}$. Since $u_{10} \sim (x^1)^{1-i\lambda}$ when $x^1 \to 0$, we obtain

$$\psi_0^{(-)} = 0 \quad \text{and} \quad \psi_0^{(+)} = -\frac{1}{\omega_+(\lambda)}.$$

We now write $S_{10}$ as a linear combination of $S_{11}$ and $S_{21}$, i.e., $S_{10} = a_1(\mu_m^2, \nu_m^2) S_{11} + b_1(\mu_m^2, \nu_m^2) S_{21}$, where $a_1(\mu_m^2, \nu_m^2) = W(S_{10}, S_{21})$ and $b_1(\mu_m^2, \nu_m^2) = W(S_{11}, S_{10})$. Thus, thanks to (2.28),

$$u_{10} = a_1(\mu_m^2, \nu_m^2) u_{11} + b_1(\mu_m^2, \nu_m^2) u_{21}.$$

We then obtain, thanks to (2.29), that

$$\psi_1^{(-)} = -\frac{b_1(\mu_m^2, \nu_m^2)}{2i\lambda\omega_-(\lambda)} \quad \text{and} \quad \psi_1^{(+)} = -\frac{a_1(\mu_m^2, \nu_m^2)}{\omega_+(\lambda)}.$$

Finally, we have shown that $u_{10}$ satisfies the decomposition of Theorem 1.7 with

$$\psi^{(-)} = \begin{pmatrix} 0 \\ -\frac{b_1(\mu_m^2, \nu_m^2)}{2i\lambda\omega_-(\lambda)} \end{pmatrix} \quad \text{and} \quad \psi^{(+)} = \begin{pmatrix} -\frac{1}{\omega_+(\lambda)} \\ -\frac{a_1(\mu_m^2, \nu_m^2)}{\omega_+(\lambda)} \end{pmatrix}. \tag{2.30}$$

We follow the same procedure for $u_{11}$ and we obtain the corresponding vectors

$$\phi^{(-)} = \begin{pmatrix} \frac{b_0(\mu_m^2, \nu_m^2)}{2i\lambda\omega_-(\lambda)} \\ 0 \end{pmatrix} \quad \text{and} \quad \phi^{(+)} = \begin{pmatrix} -\frac{a_0(\mu_m^2, \nu_m^2)}{\omega_+(\lambda)} \\ -\frac{1}{\omega_+(\lambda)} \end{pmatrix}, \tag{2.31}$$

where $a_0(\mu_m^2, \nu_m^2) = W(S_{11}, S_{20})$ and $b_0(\mu_m^2, \nu_m^2) = W(S_{10}, S_{11})$. We now recall that for all $\psi_m^{(-)} \in \mathbb{C}^2$ there exists a unique vector $\psi_m^{(+)} \in \mathbb{C}^2$ and $u_m(x)Y_m \in \mathcal{B}^\star$ satisfying (2.1) for which the expansion (1.15) holds. This defines the scattering matrix $S_g(\lambda, \mu_m^2, \nu_m^2)$ as the $2 \times 2$ matrix such that for all $\psi_m^{(-)} \in \mathbb{C}^2$,

$$\psi_m^{(+)} = S_g(\lambda, \mu_m^2, \nu_m^2)\psi_m^{(-)}. \tag{2.32}$$

Using the notation

$$S_g(\lambda, \mu_m^2, \nu_m^2) = \begin{pmatrix} L(\lambda, \mu_m^2, \nu_m^2) & T_L(\lambda, \mu_m^2, \nu_m^2) \\ T_R(\lambda, \mu_m^2, \nu_m^2) & R(\lambda, \mu_m^2, \nu_m^2) \end{pmatrix},$$

and using definition (2.32) of the partial scattering matrix, together with (2.30)–(2.31), we find

$$S_g(\lambda, \mu_m^2, \nu_m^2) = \begin{pmatrix} -\frac{2i\lambda\omega_-(\lambda)}{\omega_+(\lambda)} \frac{a_0(\mu_m^2, \nu_m^2)}{b_0(\mu_m^2, \nu_m^2)} & \frac{2i\lambda\omega_-(\lambda)}{\omega_+(\lambda)} \frac{1}{b_1(\mu_m^2, \nu_m^2)} \\ -\frac{2i\lambda\omega_-(\lambda)}{\omega_+(\lambda)} \frac{1}{b_0(\mu_m^2, \nu_m^2)} & \frac{2i\lambda\omega_-(\lambda)}{\omega_+(\lambda)} \frac{a_1(\mu_m^2, \nu_m^2)}{b_1(\mu_m^2, \nu_m^2)} \end{pmatrix}. \tag{2.33}$$

In this expression of the partial scattering matrix, we recognize the usual transmission coefficients $T_L(\lambda, \mu_m^2, \nu_m^2)$ and $T_R(\lambda, \mu_m^2, \nu_m^2)$ and the reflection coefficients $L(\lambda, \mu_m^2, \nu_m^2)$ and $R(\lambda, \mu_m^2, \nu_m^2)$ from the left and the right respectively. Since they are written in terms of Wronskians of the $S_{jn}$, $j = 1, 2$, $n = 0, 1$, we can make the link between the characteristic function (2.25) and the generalized Weyl–Titchmarsh function (2.26) as follows. Noting that

$$\Delta_{q_{\nu_m^2}}(\mu_m^2) = b_1(\mu_m^2, \nu_m^2) = -b_0(\mu_m^2, \nu_m^2) \quad \text{and} \quad M_{q_{\nu_m^2}}(\mu_m^2) = \frac{a_0(\mu_m^2, \nu_m^2)}{b_0(\mu_m^2, \nu_m^2)},$$

we get

$$L(\lambda, \mu_m^2, \nu_m^2) = -\frac{2i\lambda\omega_-(\lambda)}{\omega_+(\lambda)} M_{q_{\nu_m^2}}(\mu_m^2) \tag{2.34}$$

and

$$T(\lambda,\mu_m^2,\nu_m^2)=T_L(\lambda,\mu_m^2,\nu_m^2)=T_R(\lambda,\mu_m^2,\nu_m^2)=\frac{2i\lambda\omega_-(\lambda)}{\omega_+(\lambda)}\frac{1}{\Delta_{q_{\nu_m^2}}(\mu_m^2)}. \tag{2.35}$$

Finally, using the fact that the scattering operator is unitary (see Theorem 1.7) we obtain as in [19] the equality

$$R(\lambda,\mu_m^2,\nu_m^2)=\frac{2i\lambda\omega_-(\lambda)}{\omega_+(\lambda)}\frac{\overline{\Delta_{q_{\nu_m^2}}(\mu_m^2)}}{\Delta_{q_{\nu_m^2}}(\mu_m^2)}\overline{M_{q_{\nu_m^2}}(\mu_m^2)}. \tag{2.36}$$

## §2.4. A second construction of characteristic and Weyl–Titchmarsh functions

In Section 2.2 we defined the characteristic and the Weyl–Titchmarsh functions when $-\mu_m^2$ is the spectral parameter. We can also define these functions when we put $-\nu_m^2$ as the spectral parameter. We recall that the radial equation is given by (2.18). To choose $-\nu^2:=-\nu_m^2$ as the spectral parameter we make the Liouville change of variables

$$\hat{X}^1=\hat{g}(x^1)=\int_0^{x^1}\sqrt{s_{13}(t)}\,dt,$$

and we define $\hat{u}(\hat{X}^1,\mu^2,\nu^2)=u(\hat{h}(\hat{X}^1),\mu^2,\nu^2)$, where $\hat{h}=\hat{g}^{-1}$ and $\mu^2:=\mu_m^2$. As in Section 3.1 we introduce a weight function and we define

$$\hat{U}(\hat{X}^1,\mu^2,\nu^2)=\left(\frac{f_1}{s_{13}}(\hat{h}(\hat{X}^1))\right)^{-\frac{1}{4}}u(\hat{h}(\hat{X}^1),\mu^2,\nu^2).$$

After calculation, we obtain that $\hat{U}(\hat{X}^1,\mu^2,\nu^2)$ satisfies, in the variable $\hat{X}^1$, the Schrödinger equation

$$-\ddot{\hat{U}}(\hat{X}^1,\mu^2,\nu^2)+\hat{q}_{\mu^2}(\hat{X}^1,\lambda)U(\hat{X}^1,\mu^2,\nu^2)=-\nu^2U(\hat{X}^1,\mu^2,\nu^2), \tag{2.37}$$

where

$$\begin{aligned}\hat{q}_{\mu^2}(\hat{X}^1,\lambda)&=-(\lambda^2+1)\frac{s_{11}(\hat{X}^1)}{s_{13}(\hat{X}^1)}+\mu^2\frac{s_{12}(\hat{X}^1)}{s_{13}(\hat{X}^1)}\\&\quad+\frac{1}{16}\left(\left(\log\left(\frac{f_1(\hat{X}^1)}{s_{13}(\hat{X}^1)}\right)\right)^{\cdot}\right)^2-\frac{1}{4}\left(\log\left(\frac{f_1(\hat{X}^1)}{s_{13}(\hat{X}^1)}\right)\right)^{\cdot\cdot}.\end{aligned} \tag{2.38}$$

As in Section 2.2 we can prove the following lemma.

**Lemma 2.10.** *The potential $\hat{q}_{\mu^2}$ satisfies, at the end $\{\hat{X}^1=0\}$,*

$$\hat{q}_{\mu^2}(\hat{X}^1,\lambda)=-\frac{\lambda^2+\frac{1}{4}}{(\hat{X}^1)^2}+\hat{q}_{0,\mu^2}(\hat{X}^1,\lambda),$$

*where* $\hat{X}^1 \hat{q}_{0,\mu^2}(\hat{X}^1, \lambda) \in L^1\big(0, \frac{\hat{A}^1}{2}\big)$ *with* $\hat{A}^1 = \hat{g}(A)$*. Similarly, at the end* $\{\hat{X}^1 = \hat{A}^1\}$*,*

$$\hat{q}_{\mu^2}(\hat{X}^1, \lambda) = -\frac{\lambda^2 + \frac{1}{4}}{(\hat{A}^1 - \hat{X}^1)^2} + \hat{q}_{\hat{A}^1,\mu^2}(\hat{X}^1, \lambda),$$

*where* $(\hat{A}^1 - \hat{X}^1)\hat{q}_{\hat{A}^1,\mu^2}(\hat{X}^1, \lambda) \in L^1\big(\frac{\hat{A}^1}{2}, \hat{A}^1\big)$*.*

We can now follow the procedure of Section 2.2 to define the characteristic and Weyl–Titchmarsh functions corresponding to equation (2.37) using two fundamental systems of solutions. Thus, we can define the characteristic functions

$$\hat{\Delta}_{\hat{q}_{\mu^2}}(\nu^2) = W(\hat{S}_{11}, \hat{S}_{10}) \quad \text{and} \quad \hat{\delta}_{\hat{q}_{\mu^2}}(\nu^2) = W(\hat{S}_{11}, \hat{S}_{20}) \tag{2.39}$$

and the Weyl–Titchmarsh function

$$\hat{M}_{\hat{q}_{\mu^2}}(\nu^2) = -\frac{W(\hat{S}_{11}, \hat{S}_{20})}{W(\hat{S}_{11}, \hat{S}_{10})} = -\frac{\hat{\delta}_{\hat{q}_{\mu^2}}(\nu^2)}{\hat{\Delta}_{\hat{q}_{\mu^2}}(\nu^2)}. \tag{2.40}$$

Thanks to Corollary 2.8 we immediately obtain the following lemma.

**Lemma 2.11.**

$$\Delta_{q_{\nu_m^2}}(\mu_m^2) = \hat{\Delta}_{\hat{q}_{\mu_m^2}}(\nu_m^2) \quad \textit{and} \quad M_{q_{\nu_m^2}}(\mu_m^2) = \hat{M}_{\hat{q}_{\mu_m^2}}(\nu_m^2), \quad \forall\, m \geq 1.$$

As in Section 2.2 the characteristic functions satisfy the following lemma.

**Lemma 2.12.** *For any fixed* $\mu$ *the maps*

$$\nu \mapsto \hat{\Delta}_{\hat{q}_{\mu^2}}(\nu^2) = \Delta_{q_{\nu^2}}(\mu^2) \quad \textit{and} \quad \nu \mapsto \hat{\delta}_{\hat{q}_{\mu^2}}(\nu^2) = \delta_{q_{\nu^2}}(\mu^2)$$

*are entire.*

### §2.5. A third construction of characteristic and Weyl–Titchmarsh functions and application

The aim of this subsection is to show that, if we allow the angular momenta to be complex numbers, the characteristic functions $\Delta$ and $\delta$ are bounded on $(i\mathbb{R})^2$. Thus, in this subsection $\mu_m$ and $\nu_m$ are assumed to be in $i\mathbb{R}$. In Sections 2.2 and 2.4 we defined the characteristic and the Weyl–Titchmarsh functions with $-\mu_m^2$ and $-\nu_m^2$ as the spectral parameter respectively. We now make a third choice of spectral parameter. We recall that the radial equation is given by (2.18) and we rewrite this equation as

$$-u'' + \frac{1}{2}(\log(f_1))'u' - (\lambda^2 + 1)s_{11}u = -(\mu_m^2 + \nu_m^2)\left(\frac{\mu_m^2 s_{12} + \nu_m^2 s_{13}}{\mu_m^2 + \nu_m^2}\right)u.$$

We put, for $(y, y') \in \mathbb{R}^2$,

$$\mu := \mu_m = iy, \quad \nu := \nu_m = iy', \quad \omega^2 := \mu^2 + \nu^2$$
$$\text{and} \quad r_{\mu^2,\nu^2}(x^1) := \frac{\mu^2 s_{12}(x^1) + \nu^2 s_{13}(x^1)}{\mu^2 + \nu^2}.$$

**Remark 2.6.** There exist some positive constants $c_1$ and $c_2$ such that for all $(\mu, \nu) \in (i\mathbb{R})^2$ and $x^1 \in (0, A)$,

$$0 < c_1 \leq r_{\mu^2,\nu^2}(x^1) \leq c_2 < +\infty.$$

To choose $-\omega^2$ as the spectral parameter we make a Liouville change of variables (that depends on $\mu^2$ and $\nu^2$ and is a kind of average of the previous ones):

$$\check{X}^1_{\mu^2,\nu^2} = \check{g}_{\mu^2,\nu^2}(x^1) = \int_0^{x^1} \sqrt{r_{\mu^2,\nu^2}(t)}\, dt.$$

For the sake of clarity, we put $\check{X}^1 := \check{X}^1_{\mu^2,\nu^2}$ and $\check{g}(x^1) := \check{g}_{\mu^2,\nu^2}(x^1)$. We define $\check{u}(\check{X}^1, \mu^2, \nu^2) = u(\check{h}(\check{X}^1), \mu^2, \nu^2)$, where $\check{h} = \check{g}^{-1}$. As in Section 3.1, we introduce a weight function and we define

$$\check{U}(\check{X}^1, \mu^2, \nu^2) = \left( \frac{f_1}{r_{\mu^2,\nu^2}}(\check{h}(\check{X}^1)) \right)^{-\frac{1}{4}} u(\check{h}(\check{X}^1), \mu^2, \nu^2).$$

After calculation, we obtain that $\check{U}(\check{X}^1, \mu^2, \nu^2)$ satisfies, in the variable $\check{X}^1$, the Schrödinger equation

$$-\ddot{\check{U}}(\check{X}^1, \mu^2, \nu^2) + \check{q}_{\mu^2,\nu^2}(\check{X}^1, \lambda)\check{U}(\check{X}^1, \mu^2, \nu^2) = -\omega^2 \check{U}(\check{X}^1, \mu^2, \nu^2), \tag{2.41}$$

where

$$\begin{aligned} \check{q}_{\mu^2,\nu^2}(\check{X}^1, \lambda) = -(\lambda^2 + 1)\frac{s_{11}(\check{X}^1)}{r_{\mu^2,\nu^2}(\check{X}^1)} + \frac{1}{16}\left( \left( \log\left( \frac{f_1(\check{X}^1)}{r_{\mu^2,\nu^2}(\check{X}^1)} \right) \right)^{\cdot} \right)^2 \\ - \frac{1}{4}\left( \log\left( \frac{f_1(\check{X}^1)}{r_{\mu^2,\nu^2}(\check{X}^1)} \right) \right)^{\cdot\cdot}. \end{aligned} \tag{2.42}$$

**Lemma 2.13.** *The potential $\check{q}_{\mu^2,\nu^2}$ satisfies, at the end $\{\check{X}^1 = 0\}$,*

$$\check{q}_{\mu^2,\nu^2}(\check{X}^1, \lambda) = -\frac{\lambda^2 + \frac{1}{4}}{(\check{X}^1)^2} + \check{q}_{0,\mu^2,\nu^2}(\check{X}^1, \lambda),$$

*where* $\check{X}^1 \check{q}_{0,\mu^2,\nu^2}(\check{X}^1,\lambda) \in L^1\left(0, \frac{\check{A}^1}{2}\right)$ *with* $\check{A}^1 = \check{g}(A)$ *and* $\check{q}_{0,\mu^2,\nu^2}$ *is uniformly bounded for* $(\mu,\nu) \in (i\mathbb{R})^2$*. Similarly, at the end* $\{\check{X}^1 = \check{A}^1\}$*,*

$$\check{q}_{\mu^2,\nu^2}(\check{X}^1,\lambda) = -\frac{\lambda^2 + \frac{1}{4}}{(\check{A}^1 - \check{X}^1)^2} + \check{q}_{\check{A}^1,\mu^2,\nu^2}(\check{X}^1,\lambda),$$

*where* $(\check{A}^1 - \check{X}^1)\check{q}_{\check{A}^1,\mu^2,\nu^2}(\check{X}^1,\lambda) \in L^1\left(\frac{\check{A}^1}{2}, \check{A}^1\right)$ *with* $\check{A}^1 = \check{g}(A)$ *and* $\check{q}_{\check{A}^1,\mu^2,\nu^2}$ *is uniformly bounded for* $(\mu,\nu) \in (i\mathbb{R})^2$*.*

**Remark 2.7.** Thanks to Remark 2.6, we immediately obtain that there exist some positive constants $A^-$ and $A^+$ such that for all $(\mu,\nu) \in (i\mathbb{R})^2$,

$$A^- \leq \check{A}^1 =: \check{A}^1_{\mu^2,\nu^2} \leq A^+.$$

Once more, we follow the procedure of Section 2.2 to define the characteristic and Weyl–Titchmarsh functions corresponding to equation (2.41) using two fundamental systems of solutions $\{\check{S}_{j0}\}_{j=1,2}$ and $\{\check{S}_{j1}\}_{j=1,2}$ satisfying the asymptotics (2.22)–(2.23). Thus, we define the characteristic function

$$\check{\Delta}_{\check{q}_{\mu^2,\nu^2}}(\omega^2) = W(\check{S}_{11},\check{S}_{10}), \tag{2.43}$$

and the Weyl–Titchmarsh function

$$\check{M}_{\check{q}_{\mu^2,\nu^2}}(\omega^2) = -\frac{W(\check{S}_{11},\check{S}_{20})}{W(\check{S}_{11},\check{S}_{10})} =: -\frac{\check{\delta}_{\check{q}_{\mu^2,\nu^2}}(\omega^2)}{\check{\Delta}_{\check{q}_{\mu^2,\nu^2}}(\omega^2)}. \tag{2.44}$$

As in Section 2.4, Lemma 2.11, we can use Corollary 2.8 to prove the following lemma.

**Lemma 2.14.**

$$\Delta_{q_{\nu^2}}(\mu^2) = \check{\Delta}_{\check{q}_{\mu^2,\nu^2}}(\omega^2) \quad \textit{and} \quad M_{q_{\nu^2}}(\mu^2) = \check{M}_{\check{q}_{\mu^2,\nu^2}}(\omega^2), \quad \forall\, (\mu,\nu) \in (i\mathbb{R})^2.$$

We finish this subsection following the proof of [19, Prop. 3.2] and proving the following proposition.

**Proposition 2.15.** *For* $\omega = iy$*, where* $\pm y \geq 0$*, when* $|\omega| \to \infty$*,*

$$\begin{aligned}
\check{\Delta}_{\check{q}_{\mu^2,\nu^2}}(\omega^2) &= \frac{\Gamma(1-i\lambda)^2}{\pi 2^{2i\lambda}}\omega^{2i\lambda}e^{\pm\lambda\pi}2\cosh\left(\omega\check{A}^1 \mp \lambda\pi\right)[1]_\epsilon,\\
\check{\delta}_{\check{q}_{\mu^2,\nu^2}}(\omega^2) &= \frac{\Gamma(1-i\lambda)\Gamma(1+i\lambda)}{2i\lambda\pi}2\cosh\left(\omega\check{A}^1\right)[1]_\epsilon,\\
\check{M}_{\check{q}_{\mu^2,\nu^2}}(\omega^2) &= -\frac{\Gamma(1+i\lambda)^2 e^{\mp\lambda\pi}2^{2i\lambda}}{2i\lambda\Gamma(1-i\lambda)}\omega^{-2i\lambda}\frac{\cosh\left(\omega\check{A}^1\right)}{\cosh\left(\omega\check{A}^1 \mp \lambda\pi\right)}[1]_\epsilon,
\end{aligned}$$

*where* $[1]_\epsilon = O\left(\frac{1}{(\log|\omega|)^\epsilon}\right)$ *when* $|\omega| \to \infty$ *with* $\epsilon = \min(\epsilon_0,\epsilon_1)$*.*

*Proof.* The only difference from [19, Prop. 3.2] is the fact that our potential $q_{\mu^2,\nu^2}$ depends on the angular momenta $\mu^2$ and $\nu^2$. However, since $\check{q}_{0,\mu^2,\nu^2}$ is uniformly bounded for $(\mu,\nu) \in (i\mathbb{R})^2$, we obtain Proposition 2.15 without additional complication. ☐

**Corollary 2.16.** *There exists $C > 0$ such that for all $(\mu,\nu) \in (i\mathbb{R})^2$,*

$$|\Delta_{q_{\nu^2}}(\mu^2)| = |\check{\Delta}_{\check{q}_{\mu^2,\nu^2}}(\omega^2)| \leq C \quad \textit{and} \quad |\delta_{q_{\nu^2}}(\mu^2)| = |\check{\delta}_{\check{q}_{\mu^2,\nu^2}}(\omega^2)| \leq C.$$

*Proof.* This corollary is an immediate consequence of Proposition 2.15, Remark 2.7 and the definition of $\omega^2 = \mu^2 + \nu^2 \leq 0$ when $(\mu,\nu) \in (i\mathbb{R})^2$. ☐

## §3. The inverse problem at fixed energy for the angular equations

The aim of this section is to show the uniqueness of the angular part of the Stäckel matrix, i.e., of the second and the third lines. First, we prove that the block $\begin{pmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{pmatrix}$ is uniquely determined by knowledge of the scattering matrix at a fixed energy using the fact that the scattering matrices act on the same space and the first invariance described in Proposition 1.5. Second, we use the decomposition on the generalized harmonics and the second invariance described in Proposition 1.5 to prove the uniqueness of the coefficients $s_{21}$ and $s_{31}$. We finally show the uniqueness of the coupled spectrum which will be useful in the study of the radial part.

### §3.1. A first reduction and a first uniqueness result

We first recall that (see (1.10))

$$g = \frac{(dx^1)^2 + d\Omega^2_{\mathcal{T}^2} + P(x^1,x^2,x^3,dx^1,dx^2,dx^3)}{(x^1)^2}$$

and

$$\tilde{g} = \frac{(dx^1)^2 + d\tilde{\Omega}^2_{\mathcal{T}^2} + \tilde{P}(x^1,x^2,x^3,dx^1,dx^2,dx^3)}{(x^1)^2}.$$

Our main assumption is

$$S_g(\lambda) = S_{\tilde{g}}(\lambda),$$

where the equality holds as operators acting on $L^2(\mathcal{T}^2, d\mathrm{Vol}_{d\Omega_{\mathcal{T}^2}}; \mathbb{C}^2)$ with

$$d\mathrm{Vol}_{d\Omega_{\mathcal{T}^2}} = \sqrt{\det(d\Omega^2_{\mathcal{T}^2})}\, dx^2\, dx^3.$$

Thus, $\sqrt{\det(d\Omega^2_{\mathcal{T}^2})} = \sqrt{\det(d\tilde{\Omega}^2_{\mathcal{T}^2})}$, since these operators have to act on the same space. Since

$$d\Omega^2_{\mathcal{T}^2} = s^{11}((dx^2)^2 + (dx^3)^2) \quad \text{and} \quad d\tilde{\Omega}^2_{\mathcal{T}^2} = \tilde{s}^{11}((dx^2)^2 + (dx^3)^2),$$

this equality implies

$$s^{11} = \tilde{s}^{11}. \tag{3.1}$$

Using Remark 1.4, we can obtain more information from this equality. Indeed, we first note that

$$s^{11} = s_{22}s_{33} - s_{23}s_{32} = s_{22}s_{33} - (1 + s_{22})(1 + s_{33}) = -1 - s_{22} - s_{33}.$$

Thus, equality (3.1) allows us to obtain

$$s_{22} - \tilde{s}_{22} = \tilde{s}_{33} - s_{33}.$$

Since the left-hand side depends only on the variable $x^2$ and the right-hand side depends only on the variable $x^3$, we can deduce that there exists a constant $c \in \mathbb{R}$ such that

$$s_{22} - \tilde{s}_{22} = c = \tilde{s}_{33} - s_{33}.$$

Using Remark 1.4 again, we can thus write

$$\begin{pmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{pmatrix} = \begin{pmatrix} \tilde{s}_{22} & \tilde{s}_{23} \\ \tilde{s}_{32} & \tilde{s}_{33} \end{pmatrix} + c \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},$$

or equivalently,

$$\begin{pmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{pmatrix} = \begin{pmatrix} \tilde{s}_{22} & \tilde{s}_{23} \\ \tilde{s}_{32} & \tilde{s}_{33} \end{pmatrix} G,$$

where

$$G = \begin{pmatrix} 1 - c & -c \\ c & 1 + c \end{pmatrix}$$

is a constant matrix with determinant equal to 1. Moreover, as was mentioned in Proposition 1.5, if $\hat{S}$ is a second Stäckel matrix such that

$$\begin{pmatrix} s_{i2} & s_{i3} \end{pmatrix} = \begin{pmatrix} \hat{s}_{i2} & \hat{s}_{i3} \end{pmatrix} G \quad \forall\, i \in \{1, 2, 3\},$$

then $g = \hat{g}$, since $s^{i1} = \hat{s}^{i1}$ for all $i \in \{1, 2, 3\}$. The presence of the matrix $G$ is then due to the invariance of the metric $g$. We can thus assume that $G = I_2$. We conclude that

$$\begin{pmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{pmatrix} = \begin{pmatrix} \tilde{s}_{22} & \tilde{s}_{23} \\ \tilde{s}_{32} & \tilde{s}_{33} \end{pmatrix}. \tag{3.2}$$

### §3.2. End of the inverse problem for the angular part

The aim of this subsection is to show that the coefficients $s_{21}$ and $s_{31}$ are uniquely defined.

First, since $\{\tilde{Y}_m\}_{m\geq 1}$ is a Hilbertian basis of $L^2(\mathcal{T}^2, s^{11}\, dx^2\, dx^3)$, we can deduce that, for all $m \in \mathbb{N}\setminus\{0\}$, there exists a subset $E_m \subset \mathbb{N}\setminus\{0\}$ such that

$$Y_m = \sum_{p\in E_m} c_p \tilde{Y}_p.$$

We recall that, thanks to (2.4),

$$\begin{pmatrix} H \\ L \end{pmatrix} = \frac{1}{s^{11}} \begin{pmatrix} s_{32} & -s_{22} \\ -s_{33} & s_{23} \end{pmatrix} \begin{pmatrix} A_2 \\ A_3 \end{pmatrix},$$

where $A_j$, $j \in \{2,3\}$, were defined in (2.3). Clearly,

$$\begin{pmatrix} A_2 \\ A_3 \end{pmatrix} = T \begin{pmatrix} H \\ L \end{pmatrix}, \tag{3.3}$$

where

$$T = -\begin{pmatrix} s_{23} & s_{22} \\ s_{33} & s_{32} \end{pmatrix}.$$

We recall that

$$\tilde{T} = -\begin{pmatrix} \tilde{s}_{23} & \tilde{s}_{22} \\ \tilde{s}_{33} & \tilde{s}_{32} \end{pmatrix} = T.$$

We finally deduce from (3.3) that

$$-\begin{pmatrix} \partial_2^2 \\ \partial_3^2 \end{pmatrix} = T \begin{pmatrix} H \\ L \end{pmatrix} + (\lambda^2+1) \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}$$

and we then obtain

$$T \begin{pmatrix} H \\ L \end{pmatrix} + (\lambda^2+1) \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix} = T \begin{pmatrix} \tilde{H} \\ \tilde{L} \end{pmatrix} + (\lambda^2+1) \begin{pmatrix} \tilde{s}_{21} \\ \tilde{s}_{31} \end{pmatrix}. \tag{3.4}$$

**Lemma 3.1.** *For all $m \geq 1$,*

$$\tilde{H}\left(\sum_{p\in E_m} c_p \tilde{Y}_p\right) = \sum_{p\in E_m} c_p \tilde{H}(\tilde{Y}_p) \quad \text{and} \quad \tilde{L}\left(\sum_{p\in E_m} c_p \tilde{Y}_p\right) = \sum_{p\in E_m} c_p \tilde{L}(\tilde{Y}_p).$$

*Proof.* We first recall that $\tilde{H}$ is self-adjoint and we note that the sum $\sum_{p\in E_m} c_p \tilde{H}(\tilde{Y}_p)$ converges because the coefficients $c_p$ are decaying sufficiently rapidly. Indeed, we note that $c_p = \langle Y_m, \tilde{Y}_p\rangle$ and we can use integration by parts with the help of the operator $H$ and the Weyl law on the eigenvalues to obtain the decay we want. We can then conclude the lemma using the closedness of $\tilde{H}$. □

**Remark 3.1.** If $E_m$, $m \in \mathbb{N} \setminus \{0\}$, are finite then Lemma 3.1 is obvious. In fact, following the idea of [19, Prop. 4.1], we could obtain that these sets are finite using asymptotics of the Weyl–Titchmarsh function.

Applying (3.4) to the vector of generalized harmonics

$$\begin{pmatrix} Y_m \\ Y_m \end{pmatrix} = \begin{pmatrix} \sum_{p \in E_m} c_p \tilde{Y}_p \\ \sum_{p \in E_m} c_p \tilde{Y}_p \end{pmatrix}$$

we obtain, thanks to Lemma 3.1 and (2.9), that

$$\begin{aligned} \left( T \begin{pmatrix} H \\ L \end{pmatrix} + (\lambda^2+1) \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix} \right) \begin{pmatrix} Y_m \\ Y_m \end{pmatrix} &= \left( T \begin{pmatrix} \mu_m^2 \\ \nu_m^2 \end{pmatrix} + (\lambda^2+1) \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix} \right) \begin{pmatrix} Y_m \\ Y_m \end{pmatrix} \\ &= \sum_{p \in E_m} c_p \left( T \begin{pmatrix} \mu_m^2 \\ \nu_m^2 \end{pmatrix} + (\lambda^2+1) \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix} \right) \begin{pmatrix} \tilde{Y}_p \\ \tilde{Y}_p \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \left( T \begin{pmatrix} H \\ L \end{pmatrix} + (\lambda^2+1) \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix} \right) \begin{pmatrix} Y_m \\ Y_m \end{pmatrix} &= \sum_{p \in E_m} c_p \left( T \begin{pmatrix} \tilde{H} \\ \tilde{L} \end{pmatrix} + (\lambda^2+1) \begin{pmatrix} \tilde{s}_{21} \\ \tilde{s}_{31} \end{pmatrix} \right) \begin{pmatrix} \tilde{Y}_p \\ \tilde{Y}_p \end{pmatrix} \\ &= \sum_{p \in E_m} c_p \left( T \begin{pmatrix} \tilde{\mu}_p^2 \\ \tilde{\nu}_p^2 \end{pmatrix} + (\lambda^2+1) \begin{pmatrix} \tilde{s}_{21} \\ \tilde{s}_{31} \end{pmatrix} \right) \begin{pmatrix} \tilde{Y}_p \\ \tilde{Y}_p \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} &\sum_{p \in E_m} c_p \left( T \begin{pmatrix} \mu_m^2 \\ \nu_m^2 \end{pmatrix} + (\lambda^2+1) \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix} \right) \begin{pmatrix} \tilde{Y}_p \\ \tilde{Y}_p \end{pmatrix} \\ &\quad = \sum_{p \in E_m} c_p \left( T \begin{pmatrix} \tilde{\mu}_p^2 \\ \tilde{\nu}_p^2 \end{pmatrix} + (\lambda^2+1) \begin{pmatrix} \tilde{s}_{21} \\ \tilde{s}_{31} \end{pmatrix} \right) \begin{pmatrix} \tilde{Y}_p \\ \tilde{Y}_p \end{pmatrix}. \end{aligned}$$

Since $\{\tilde{Y}_p\}_{p \geq 1}$ is a Hilbertian basis we deduce from this equality that for all $m \geq 1$, for all $p \in E_m$,

$$T \begin{pmatrix} \mu_m^2 \\ \nu_m^2 \end{pmatrix} + (\lambda^2+1) \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix} = T \begin{pmatrix} \tilde{\mu}_p^2 \\ \tilde{\nu}_p^2 \end{pmatrix} + (\lambda^2+1) \begin{pmatrix} \tilde{s}_{21} \\ \tilde{s}_{31} \end{pmatrix}. \tag{3.5}$$

We deduce from (3.5) that

$$\begin{pmatrix} \tilde{\mu}_p^2 - \mu_m^2 \\ \tilde{\nu}_p^2 - \nu_m^2 \end{pmatrix} = (\lambda^2+1) T^{-1} \begin{pmatrix} s_{21} - \tilde{s}_{21} \\ s_{31} - \tilde{s}_{31} \end{pmatrix}.$$

Since the right-hand side is independent of $m$ and $p$, we can deduce from this equality that there exists a constant vector $\binom{c_1}{c_2}$ such that

$$\begin{pmatrix} \tilde{\mu}_p^2 \\ \tilde{\nu}_p^2 \end{pmatrix} = \begin{pmatrix} \mu_m^2 \\ \nu_m^2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

and

$$\begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix} = \begin{pmatrix} \tilde{s}_{21} \\ \tilde{s}_{31} \end{pmatrix} + \frac{1}{\lambda^2+1} T \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{3.6}$$

From (3.6), we immediately deduce that

$$\begin{cases} s_{21}(x^2) = \tilde{s}_{21}(x^2) - C_1 s_{23}(x^2) - C_2 s_{22}(x^2), \\ s_{31}(x^3) = \tilde{s}_{31}(x^3) - C_1 s_{33}(x^3) - C_2 s_{32}(x^3), \end{cases} \tag{3.7}$$

where $C_i = \frac{c_i}{\lambda^2+1}$ for $i \in \{1,2\}$. We recall that

$$g = \sum_{i=1}^{3} H_i^2 (dx^i)^2 \quad \text{with } H_i^2 = \frac{\det(S)}{s^{i1}} \quad \forall\, i \in \{1,2,3\}.$$

Since the minors $s^{i1}$ depend only on the second and the third columns, they do not change under the transformation given in (3.7). Thus, as mentioned in the introduction in Proposition 1.5, recalling that the determinant of a matrix does not change if we add to the first column a linear combination of the second and the third columns, we conclude that equalities (3.7) describe an invariance of the metric $g$ under the definition of the Stäckel matrix $S$. We can then choose $C_i = 0$, $i \in \{1,2\}$, i.e., $c_1 = c_2 = 0$. Finally, we have shown that

$$\begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix} = \begin{pmatrix} \tilde{s}_{21} \\ \tilde{s}_{31} \end{pmatrix}. \tag{3.8}$$

From the definition of the operators $L$ and $H$ given by (2.4), we deduce from (3.2) and (3.8) that

$$H = \tilde{H} \quad \text{and} \quad L = \tilde{L}.$$

We then conclude that these operators have the same eigenfunctions, i.e., we can choose

$$Y_m = \tilde{Y}_m \quad \forall\, m \geq 1 \tag{3.9}$$

and the same coupled spectrum

$$\begin{pmatrix} \mu_m^2 \\ \nu_m^2 \end{pmatrix} = \begin{pmatrix} \tilde{\mu}_m^2 \\ \tilde{\nu}_m^2 \end{pmatrix} \quad \forall\, m \geq 1. \tag{3.10}$$

## §4. The inverse problem at fixed energy for the radial equation

The aim of this section is to show that the radial part of the Stäckel matrix, i.e., the first line, is uniquely determined by knowledge of the scattering matrix. We first use a multivariable version of the complex angular momentum method to extend the equality of the Weyl–Titchmarsh functions (valid on the coupled spectrum) to complex angular momenta. Next, we use the Börg–Marchenko theorem (see for instance [29, 45]) to obtain the uniqueness of the quotients $\frac{s_{11}}{s_{12}}$ and $\frac{s_{11}}{s_{13}}$.

### §4.1. Complexification of the angular momenta

We recall that, thanks to our main assumption in Theorem 1.8, (3.9)–(3.10) and (2.34), we know that

$$M(\mu_m^2, \nu_m^2) = \tilde{M}(\mu_m^2, \nu_m^2) \quad \forall\, m \geq 1, \tag{4.1}$$

where

$$\begin{aligned} &M(\mu_m^2, \nu_m^2) = M_{q_{\nu_m^2}}(\mu_m^2) = M_{\hat{q}_{\mu_m^2}}(\nu_m^2) \\ \text{and}\quad &\tilde{M}(\mu_m^2, \nu_m^2) = M_{\tilde{q}_{\nu_m^2}}(\mu_m^2) = M_{\tilde{\hat{q}}_{\mu_m^2}}(\nu_m^2). \end{aligned}$$

The aim of this subsection is to show that

$$M(\mu^2, \nu^2) = \tilde{M}(\mu^2, \nu^2) \quad \forall\, (\mu, \nu) \in \mathbb{C}^2 \setminus P, \tag{4.2}$$

where $P$ is the set of points $(\mu, \nu) \in \mathbb{C}^2$ such that $(\mu^2, \nu^2)$ is a pole of $M$ and $\tilde{M}$, or equivalently that is a zero of $\Delta$ and $\tilde{\Delta}$. Usually, in the complexification of the angular momentum method there is only one angular momentum that we complexify using uniqueness results for holomorphic functions in certain classes. In [20], there are two *independent* angular momenta and the authors are able to use the complexification of the angular momentum method for only one angular momentum. Here, we cannot complexify one angular momentum keeping the other fixed since the two angular momenta are not independent (see Lemma 2.4). We thus have to complexify simultaneously the two angular momenta and we then need to use uniqueness results for multivariable holomorphic functions. Therefore, to obtain (4.2) we want to use the following result given in [7, 8].

**Theorem 4.1.** *Let $K$ be an open cone in $\mathbb{R}^2$ with vertex the origin and $T(K) = \{z \in \mathbb{C}^2, \operatorname{Re}(z) \in K\}$. Suppose that $f$ is a bounded analytic function on $T(K)$. Let $E$ be a discrete subset of $K$ such that for some constant $h > 0$, $|e_1 - e_2| \geq h$ for all $(e_1, e_2) \in E$. Let $n(r) = \# E \cap B(0, r)$. Assume that $f$ vanishes on $E$. Then $f$ is identically zero if*

$$\overline{\lim} \frac{n(r)}{r^2} > 0, \quad r \to +\infty.$$

We first introduce the function

$$\psi : \quad \mathbb{C} \ni (\mu,\nu) \mapsto \tilde{\Delta}(\mu^2,\nu^2)\delta(\mu^2,\nu^2) - \Delta(\mu^2,\nu^2)\tilde{\delta}(\mu^2,\nu^2) \in \mathbb{C},$$

with $\delta(\mu^2,\nu^2) = \delta_{q_{\nu^2}}(\mu^2) = \delta_{\hat{q}_{\mu^2}}(\nu^2)$, where $\delta_{q_{\nu^2}}(\mu^2)$ and $\delta_{\hat{q}_{\mu^2}}(\nu^2)$ were defined in (2.25) and $\Delta(\mu^2,\nu^2) = \Delta_{q_{\nu^2}}(\mu^2) = \Delta_{\hat{q}_{\mu^2}}(\nu^2)$, where $\Delta_{q_{\nu^2}}(\mu^2)$ and $\Delta_{\hat{q}_{\mu^2}}(\nu^2)$ were defined in (2.39). Our aim is then to show that $\psi$ is identically zero.

**Lemma 4.2.** *The map $\psi$ is entire as a function of two complex variables.*

To use Theorem 4.1 we need the following estimate on the function $\psi$.

**Lemma 4.3.** *There exist some positive constants $C$, $A$ and $B$ such that*

$$|\psi(\mu,\nu)| \leq Ce^{A|\mathrm{Re}(\mu)|+B|\mathrm{Re}(\nu)|} \quad \forall\,(\mu,\nu) \in \mathbb{C}^2.$$

*Proof.* The proof of this lemma consists of four steps.

**Step 1:** We claim that for all fixed $\nu \in \mathbb{C}$ there exists a constant $C_1(\nu)$ such that for all $\mu \in \mathbb{C}$,

$$|\psi(\mu,\nu)| \leq C_1(\nu)e^{A|\mathrm{Re}(\mu)|}.$$

To obtain this estimate we study the solutions $S_{j0}$ and $S_{j1}$ defined in Section 2.2. First, we show that for $j \in \{1,2\}$,

$$\begin{aligned}
|S_{j0}(X^1,\mu^2,\nu^2)| &\leq C(\nu)\frac{e^{|\mathrm{Re}(\mu)|X^1}}{|\mu|^{\frac{1}{2}}},\\
|S_{j0}'(X^1,\mu^2,\nu^2)| &\leq C(\nu)|\mu|^{\frac{1}{2}}e^{|\mathrm{Re}(\mu)|X^1},\\
|S_{j1}(X^1,\mu^2,\nu^2)| &\leq C(\nu)\frac{e^{|\mathrm{Re}(\mu)|(A^1-X^1)}}{|\mu|^{\frac{1}{2}}},\\
|S_{j1}'(X^1,\mu^2,\nu^2)| &\leq C(\nu)|\mu|^{\frac{1}{2}}e^{|\mathrm{Re}(\mu)|(A^1-X^1)}.
\end{aligned}$$

As in [19], we can show by an iterative procedure that

$$|S_{j0}(X^1,\mu^2,\nu^2)| \leq C\left(\frac{X^1}{1+|\mu|X^1}\right)^{\frac{1}{2}} e^{|\mathrm{Re}(\mu)|X^1}\exp\left(\int_0^{X^1}\frac{t|q_{0,\nu^2}(t)|}{1+|\mu|t}\,dt\right). \tag{4.3}$$

Recall now that, thanks to the asymptotically hyperbolic structure, we have for all $X^1 \in (0,X_0^1)$, where $X_0^1 \in (0,A^1)$ is fixed,

$$t|q_{0,\nu^2}(t)| \leq \frac{C(1+\nu^2)}{t(1+|\log(t)|)^{1+\epsilon_0}} \quad \forall\, t \in (0,X).$$

Thus, as shown in [19, Sect. 3.1],

$$\int_0^{X^1}\frac{t|q_{0,\nu^2}(t)|}{1+|\mu|t}\,dt \leq (1+\nu^2)O\left(\frac{1}{(\log(|\mu|))^{\epsilon_0}}\right). \tag{4.4}$$

We can then conclude

$$|S_{j0}(X^1,\mu^2,\nu^2)| \le C(\nu)\frac{e^{|\mathrm{Re}(\mu)|X^1}}{|\mu|^{\frac{1}{2}}}.$$

The result on $S'_{j0}(X^1,\mu^2,\nu^2)$ is obtained similarly using the estimate on the derivative of the Green kernel given in [19, Prop. 3.1]. By symmetry, we also obtain the corresponding estimates on $S_{j1}(X^1,\mu^2,\nu^2)$ and $S'_{j1}(X^1,\mu^2,\nu^2)$. We can then conclude that

$$\Delta(\mu^2,\nu^2) = \Delta_{q_{\nu^2}}(\mu^2) = W(S_{11},S_{10}) \quad \text{and} \quad \delta(\mu^2,\nu^2) = \delta_{q_{\nu^2}}(\mu^2) = W(S_{11},S_{20})$$

satisfy

$$|\Delta(\mu^2,\nu^2)| \le C_1(\nu)e^{A|\mathrm{Re}(\mu)|} \quad \text{and} \quad |\delta(\mu^2,\nu^2)| \le C_1(\nu)e^{A|\mathrm{Re}(\mu)|}, \quad \forall\,(\mu,\nu)\in\mathbb{C}^2.$$

Finally, we have shown

$$|\psi(\mu,\nu)| \le C_1(\nu)e^{A|\mathrm{Re}(\mu)|} \quad \forall\,(\mu,\nu)\in\mathbb{C}^2.$$

**Step 2:** We can also show that for all fixed $\mu\in\mathbb{C}$ there exists a constant $C_2(\mu)$ such that for all $\nu\in\mathbb{C}$,

$$|\psi(\mu,\nu)| \le C_2(\mu)e^{\hat{A}|\mathrm{Re}(\nu)|}.$$

To obtain this estimate we use the strategy of the first step of the equation (2.37) with potential (2.38) introduced in Section 2.4.

**Step 3:** Thanks to Corollary 2.16, there exists a constant $C$ such that for all $(y,y')\in\mathbb{R}^2$,

$$|\psi(iy,iy')| \le C. \tag{4.5}$$

**Step 4:** We finish the proof of the lemma by the use of the Phragmén–Lindelöf theorem (see [9, Thm. 1.4.3]). We first fix $\nu\in i\mathbb{R}$. Thus, the mapping $\mu\mapsto\psi(\mu,\nu)$ satisfies

$$\begin{cases} |\psi(\mu,\nu)| \le C_1(\nu)e^{A|\mathrm{Re}(\mu)|} \quad \forall\,\mu\in\mathbb{C} & \text{(Step 1)},\\ |\psi(\mu,\nu)| \le C \quad \forall\,\mu\in i\mathbb{R} & \text{(Step 3)}.\end{cases}$$

Thanks to the Phragmén–Lindelöf theorem, we deduce from these equalities that

$$|\psi(\mu,\nu)| \le Ce^{A|\mathrm{Re}(\mu)|} \quad \forall\,(\mu,\nu)\in\mathbb{C}\times i\mathbb{R}.$$

We now fix $\mu \in \mathbb{C}$; then the mapping $\nu \mapsto \psi(\mu,\nu)$ satisfies

$$\begin{cases} |\psi(\mu,\nu)| \leq C_2(\mu) e^{B|\mathrm{Re}(\nu)|} & \forall\, \nu \in \mathbb{C} \quad \text{(Step 1)}, \\ |\psi(\mu,\nu)| \leq C e^{A|\mathrm{Re}(\mu)|} & \forall\, \nu \in i\mathbb{R}. \end{cases}$$

Thus, using the Phragmén–Lindelöf theorem once more, we obtain

$$|\psi(\mu,\nu)| \leq C e^{A|\mathrm{Re}(\mu)|+B|\mathrm{Re}(\nu)|} \quad \forall\, (\mu,\nu) \in \mathbb{C}^2.$$

□

We apply Theorem 4.1 with $K = (\mathbb{R}^+)^2$ and $F(\mu,\nu) = \psi(\mu,\nu)e^{-A\mu - B\nu}$.

**Lemma 4.4.** *The function $F$ is bounded and holomorphic on*

$$T((\mathbb{R}^+)^2) = \{(\mu,\nu) \in \mathbb{C}^2,\ (\mathrm{Re}(\mu), \mathrm{Re}(\nu)) \in \mathbb{R}^+ \times \mathbb{R}^+\}.$$

*Proof.* This lemma is an immediate consequence of Lemmas 4.2 and 4.3. □

We now recall that $(\mu_m^2, \nu_m^2)$, $m \geq 1$ denotes the coupled spectrum of the operators $H$ and $L$. We note that $\mu_m^2$ and $\nu_m^2$ tend to $+\infty$, as $m \to +\infty$. Therefore, there exists $M \geq 1$ such that $\mu_m^2 \geq 0$ and $\nu_m^2 \geq 0$ for all $m \geq M$. We then set

$$E_M = \{(|\mu_m|, |\nu_m|),\ m \geq M\}.$$

Thanks to equation (4.1), we note that the function $F$ satisfies

$$F(\mu_m, \nu_m) = 0 \quad \forall\, m \geq M$$

since

$$\psi(\mu_m, \nu_m) = 0 \quad \forall\, m \geq M. \tag{4.6}$$

Moreover, since the characteristic functions are, by definition, even functions with respect to $\mu$ and $\nu$, we obtain

$$F(|\mu_m|, |\nu_m|) = F(\mu_m, \nu_m) = 0 \quad \forall\, m \geq M,$$

i.e., $F$ vanishes on $E_M$.

**Remark 4.1.** We emphasize that $E_M$ denotes the set of eigenvalues counted with multiplicity (which is at most 4). Since we need a separation property, given in the following lemma, to apply Bloom's theorem, we have to consider a new set, also denoted by $E_M$, which corresponds to the previous set of eigenvalues counted without multiplicity. To obtain this separation property on the coupled spectrum $E_M$, we also need to restrict our analysis to a suitable cone given in the following lemma.

**Lemma 4.5.** *We set*

$$\mathcal{C} = \{(\mu^2, \theta^2\mu^2),\ \ c_1 + \epsilon \leq \theta^2 \leq c_2 - \epsilon\}, \quad 0 < \epsilon \ll 1, \tag{4.7}$$

*where*

$$c_1 = \max\left(-\frac{s_{32}}{s_{33}}\right) \quad \textit{and} \quad c_2 = \min\left(-\frac{s_{22}}{s_{23}}\right).$$

*In this case, there exists $h > 0$ such that $|e_1 - e_2| \geq h$ for all $(e_1, e_2) \in (E_M \cap \mathcal{C})^2$, $e_1 \neq e_2$.*

*Proof.* See Appendix B. □

**Remark 4.2.** We note that, as we showed in Lemma 2.4, there exist real constants $C_1$, $C_2$, $D_1$ and $D_2$ such that for all $m \geq 1$,

$$C_1\mu_m^2 + D_1 \leq \nu_m^2 \leq C_2\mu_m^2 + D_2,$$

where

$$C_1 = \min\left(-\frac{s_{32}}{s_{33}}\right) > 0 \quad \text{and} \quad C_2 = -\min\left(\frac{s_{22}}{s_{23}}\right) > 0.$$

We then easily obtain

$$0 < C_1 \leq c_1 < c_2 \leq C_2.$$

**Lemma 4.6.** *We set*

$$n(r) = \#E_M \cap B(0, r) \cap \mathcal{C},$$

*where $\mathcal{C}$ is defined in (4.7); then*

$$\overline{\lim}\, \frac{n(r)}{r^2} > 0, \quad r \to +\infty.$$

*Proof.* See Appendix C. □

**Remark 4.3.** We emphasize that the number of points of the coupled spectrum $n(r)$ we use to apply Bloom's theorem is not exactly the one we compute in the framework of Colin de Verdière. Indeed, Colin de Verdière computes the number of points of the coupled spectrum counting multiplicity whereas $n(r)$ denotes the number of points of the coupled spectrum counting without multiplicity. However, as we have seen before (see Remark 2.3) the multiplicity of a coupled eigenvalue is at most 4. Therefore, $n(r)$ is greater than a quarter of the number computed in the work of Colin de Verdière and is thus still of order $r^2$.

We can then use Theorem 4.1 on the cone $\mathcal{C}$ to conclude that

$$F(\mu, \nu) = 0 \quad \forall\, (\mu, \nu) \in \mathbb{C}^2.$$

From this equality we immediately deduce that

$$M(\mu^2,\nu^2) = \tilde{M}(\mu^2,\nu^2) \quad \forall\,(\mu,\nu)\in\mathbb{C}^2\setminus P, \tag{4.8}$$

where $P$ is the set of points $(\mu,\nu)\in\mathbb{C}^2$ such that $(\mu^2,\nu^2)$ is a zero of the characteristic functions $\Delta$ and $\tilde{\Delta}$.

## §4.2. Inverse problem for the radial part

By definition, formula (4.8) means that

$$M_{q_{\nu^2}}(\mu^2) = M_{\tilde{q}_{\nu^2}}(\mu^2) \quad \forall\,(\mu,\nu)\in\mathbb{C}^2\setminus P,$$

where $M_{q_{\nu^2}}(\mu^2)$ is defined in (2.26). We can thus use the celebrated Börg–Marchenko theorem in the form given in [19, 29] to obtain

$$q_{\nu_m^2}(X^1,\lambda) = \tilde{q}_{\nu_m^2}(X^1,\lambda) \quad \forall\, m\geq 1, \quad \forall\, X^1\in(0,A^1).$$

Thanks to (2.21), and since the previous equality is true for all $m\geq 1$, we then have, for all $X^1\in(0,A^1)$,

$$\frac{s_{13}(X^1)}{s_{12}(X^1)} = \frac{\tilde{s}_{13}(X^1)}{\tilde{s}_{12}(X^1)} \tag{4.9}$$

and

$$\begin{aligned} &-(\lambda^2+1)\frac{s_{11}(X^1)}{s_{12}(X^1)} + \frac{1}{16}\left(\left(\log\left(\frac{f_1(X^1)}{s_{12}(X^1)}\right)\right)^{\dot{}}\right)^2 - \frac{1}{4}\left(\log\left(\frac{f_1(X^1)}{s_{12}(X^1)}\right)\right)^{\ddot{}} \\ &= -(\lambda^2+1)\frac{\tilde{s}_{11}(X^1)}{\tilde{s}_{12}(X^1)} + \frac{1}{16}\left(\left(\log\left(\frac{\tilde{f}_1(X^1)}{\tilde{s}_{12}(X^1)}\right)\right)^{\dot{}}\right)^2 - \frac{1}{4}\left(\log\left(\frac{\tilde{f}_1(X^1)}{\tilde{s}_{12}(X^1)}\right)\right)^{\ddot{}}. \end{aligned} \tag{4.10}$$

We want to rewrite this equation as a Cauchy problem for a second-order nonlinear differential equation with boundary conditions at the end $\{X^1=0\}$. To do that, we put

$$f = \frac{s_{11}}{s_{12}}, \quad h = \frac{s_{12}}{f_1}, \quad l = \frac{s_{13}}{s_{12}} = \tilde{l} \quad \text{and} \quad u = \left(\frac{h}{\tilde{h}}\right)^{\frac{1}{4}}.$$

We can thus rewrite (4.10) in the form

$$u'' + \tfrac{1}{2}(\log(\tilde{h}))'u' + (\lambda^2+1)(\tilde{f}-f)u = 0. \tag{4.11}$$

Using the Robertson condition (1.6) we can write

$$\frac{s_{11}}{s_{12}} = f = -\frac{s^{12}}{s^{11}} - l\frac{s^{13}}{s^{11}} + h(ls_{32}-s_{33})(s_{23}-ls_{22}). \tag{4.12}$$

Thanks to (4.12), we see that we can write $\frac{s_{11}}{s_{12}}$ as a function of $\frac{s_{13}}{s_{12}}$ and $\frac{f_1}{s_{12}}$, i.e.,

$$\frac{s_{11}}{s_{12}} = \Phi\left(\frac{s_{13}}{s_{12}}, \frac{f_1}{s_{12}}\right) \quad \text{and} \quad \frac{\tilde{s}_{11}}{\tilde{s}_{12}} = \Phi\left(\frac{s_{13}}{s_{12}}, \frac{\tilde{f}_1}{\tilde{s}_{12}}\right), \tag{4.13}$$

where

$$\Phi(X,Y) = -\frac{s^{12}}{s^{11}} - X\frac{s^{13}}{s^{11}} + \frac{1}{Y}(Xs_{32} - s_{33})(s_{23} - Xs_{22}).$$

Thus, to show that $\frac{s_{11}}{s_{12}} = \frac{\tilde{s}_{11}}{\tilde{s}_{12}}$, it is sufficient by (4.9) to prove that $\frac{f_1}{s_{12}} = \frac{\tilde{f}_1}{\tilde{s}_{12}}$. From (4.12), we deduce that

$$\begin{aligned} f - \tilde{f} &= (h - \tilde{h})(ls_{32} - s_{33})(s_{23} - ls_{22}) \\ &= \tilde{h}(u^4 - 1)(ls_{32} - s_{33})(s_{23} - ls_{22}). \end{aligned}$$

Finally, using this last equality, we can rewrite (4.11) as

$$u'' + \tfrac{1}{2}(\log(\tilde{h}))'u' + (\lambda^2 + 1)\tilde{h}(ls_{32} - s_{33})(s_{23} - ls_{22})(u^5 - u) = 0. \tag{4.14}$$

**Lemma 4.7.** *The function $u$ defined by $u = \left(\frac{h}{\tilde{h}}\right)^{\frac{1}{4}}$ satisfies $u(0) = 1$ and $u'(0) = 0$.*

*Proof.* The proof is a consequence of the fact that the asymptotically hyperbolic structures given in Definition 1.4(3) are the same on the two manifolds. □

We thus study the Cauchy problem

$$\begin{cases} u'' + \frac{1}{2}(\log(\tilde{h}))'u' + (\lambda^2 + 1)\tilde{h}(ls_{32} - s_{33})(s_{23} - ls_{22})(u^5 - u) = 0, \\ u(0) = 1 \quad \text{and} \quad u'(0) = 0. \end{cases} \tag{4.15}$$

We immediately note that $u = 1$ is a solution of (4.15). By uniqueness of the Cauchy problem for the ODE (4.15) we conclude that $u = 1$. We then have shown that

$$\frac{f_1}{s_{12}} = \frac{\tilde{f}_1}{\tilde{s}_{12}}$$

and, using (4.9) and (4.13), we can conclude that

$$\frac{s_{11}}{s_{12}} = \frac{\tilde{s}_{11}}{\tilde{s}_{12}} \quad \text{and} \quad \frac{s_{11}}{s_{13}} = \frac{\tilde{s}_{11}}{\tilde{s}_{13}}. \tag{4.16}$$

## §5. Solution of the inverse problem

We can now finish the proof of our inverse problem. We first note that

$$g' = \frac{H_1^2}{s_{12}}(dX^1)^2 + H_2^2(dx^2)^2 + H_3^2(dx^3)^2,$$

where $\psi$ is the diffeomorphism (equal to the identity at the compactified ends $\{x^1 = 0\}$ and $\{x^1 = A\}$) corresponding to the Liouville change of variables in the first variable $X^1 = \int_0^{x^1} \sqrt{s_{12}(s)}\, ds$. Similarly, $\tilde{g} = \sum_{i=1}^3 \tilde{H}_i^2 (dx^i)^2 = \tilde{\psi}^\star \tilde{g}'$, where

$$\tilde{g}' = \frac{\tilde{H}_1^2}{\tilde{s}_{12}} (d\tilde{X}^1)^2 + \tilde{H}_2^2 (dx^2)^2 + \tilde{H}_3^2 (dx^3)^2,$$

where $\tilde{\psi}$ is the diffeomorphism (equal to the identity at the compactified ends $\{x^1 = 0\}$ and $\{x^1 = A\}$) corresponding to the same Liouville change of variables in the first variable for the second manifold. We note that, thanks to the Börg–Marchenko theorem, we can identify $A^1 = \tilde{A}^1$. We now note that, thanks to (4.16),

$$\frac{H_1^2}{s_{12}} = \frac{\det(S)}{s_{12} s^{11}} = \frac{s_{11}}{s_{12}} + \frac{s^{12}}{s^{11}} + \frac{s_{13}}{s_{12}} \frac{s^{13}}{s^{11}} = \frac{\tilde{H}_1^2}{\tilde{s}_{12}},$$
$$H_2^2 = \frac{\det(S)}{s^{21}} = \frac{\frac{s_{11}}{s_{12}} s^{11} + s^{12} + \frac{s_{13}}{s_{12}} s^{13}}{\frac{s_{13}}{s_{12}} s_{32} - s_{33}} = \tilde{H}_2^2$$

and

$$H_3^2 = \frac{\det(S)}{s^{31}} = \frac{\frac{s_{11}}{s_{12}} s^{11} + s^{12} + \frac{s_{13}}{s_{12}} s^{13}}{s_{23} - \frac{s_{13}}{s_{12}} s_{22}} = \tilde{H}_3^2.$$

We can then deduce from these equalities that

$$g' = \tilde{g}'.$$

Finally, we have shown that there exists a diffeomorphism $\Psi := \psi^{-1}\tilde{\psi}$ such that

$$\tilde{g} = \Psi^\star g,$$

where $\Psi$ is the identity at the two ends.

## Appendix A. Proof of Proposition 1.6

The proof of this proposition consists of three steps and uses the riemannian structure and the invariances of the metric described in Proposition 1.5. We first show that the coefficients of the second and the third columns are nonnegative or nonpositive functions. Second, we show that these coefficients can be assumed to be positive or negative functions. Finally, we show that we can find a Stäckel matrix with the same associated metric and satisfying condition (C).

**Step 1:** We claim that for all $(i, j) \in \{1, 2, 3\} \times \{2, 3\}$, $s_{ij} \geq 0$ or $s_{ij} \leq 0$. Since the proof is similar for the third column we give the proof just for the second one. First, if one of the functions $s_{12}$, $s_{22}$ and $s_{32}$ is identically zero, the two others cannot vanish on their intervals of definition since the minors $s^{11}$, $s^{21}$ and $s^{31}$

cannot vanish. Thus, in this case we immediately obtain $s_{i2} \geq 0$ or $s_{i2} \leq 0$ for all $i \in \{1,2,3\}$. We can thus assume that there exists a triplet $(x_0^1, x_0^2, x_0^3)$ such that $s_{12}(x_0^1) \neq 0$, $s_{22}(x_0^2) \neq 0$ and $s_{32}(x_0^3) \neq 0$. Without loss of generality, we assume that $\det(S) > 0$ and $s^{i1} > 0$ for all $i \in \{1,2,3\}$. From the positivity property of the minors we can deduce that, according to the sign of the quantities $s_{12}(x_0^1)$, $s_{22}(x_0^2)$ and $s_{32}(x_0^3)$,

- $s_{12}(x_0^1) > 0$, $s_{22}(x_0^2) > 0$ and $s_{32}(x_0^3) > 0$: This case is impossible since the minors $s^{11}$, $s^{21}$ and $s^{31}$ cannot all be positive.
- $s_{12}(x_0^1) > 0$, $s_{22}(x_0^2) < 0$ and $s_{32}(x_0^3) > 0$:

$$\text{(A.1)} \qquad \frac{s_{33}(x_0^3)}{s_{32}(x_0^3)} < \frac{s_{23}(x_0^2)}{s_{22}(x_0^2)} < \frac{s_{13}(x_0^1)}{s_{12}(x_0^1)}.$$

- $s_{12}(x_0^1) > 0$, $s_{22}(x_0^2) > 0$ and $s_{32}(x_0^3) < 0$:

$$\text{(A.2)} \qquad \frac{s_{13}(x_0^1)}{s_{12}(x_0^1)} < \frac{s_{33}(x_0^3)}{s_{32}(x_0^3)} < \frac{s_{23}(x_0^2)}{s_{22}(x_0^2)}.$$

- $s_{12}(x_0^1) > 0$, $s_{22}(x_0^2) < 0$ and $s_{32}(x_0^3) > 0$:

$$\text{(A.3)} \qquad \frac{s_{23}(x_0^2)}{s_{22}(x_0^2)} < \frac{s_{13}(x_0^1)}{s_{12}(x_0^1)} < \frac{s_{33}(x_0^3)}{s_{32}(x_0^3)}.$$

Since the four cases corresponding to the case $s_{12}(x_0^1) < 0$ are similar, we treat just the four cases above. Assume, for instance, that there exists $\alpha_0^2$ such that $s_{22}(\alpha_0^2) = 0$. We want to show that $s_{22}$ does not change its sign. We denote by $I$ the maximal interval (possibly reduced to $\alpha_0^2$) containing $\alpha_0^2$ such that $s_{22}(x^2) = 0$ for all $x^2 \in I$. Since the minors $s^{11}$ and $s^{31}$ are nonvanishing quantities, the functions $s_{12}$ and $s_{32}$ then cannot vanish. Thus, there exist two real constants $c_1$ and $c_2$ such that

$$\text{(A.4)} \qquad c_1 \leq \frac{s_{13}}{s_{12}} \leq c_2 \quad \text{and} \quad c_1 \leq \frac{s_{33}}{s_{32}} \leq c_2,$$

i.e., these quotients are bounded. Moreover, $s_{23}(x^2) \neq 0$ for all $x^2 \in I$ and by continuity there exists an interval $J$ such that $I \subsetneq J$ and $s_{23}(x^2) \neq 0$ for all $x^2 \in J$. If we assume that $s_{22}$ changes sign in a neighbourhood of $I$ we obtain that for all $\epsilon > 0$ there exist $y_0^2 \in J$ and $y_1^2 \in J$ such that

$$0 < s_{22}(y_0^2) < \epsilon \quad \text{and} \quad -\epsilon < s_{22}(y_1^2) < 0.$$

Thus, for all $M > 0$ there exist $y_0^2 \in J$ and $y_1^2 \in J$ such that

$$\frac{s_{23}(y_0^2)}{s_{22}(y_0^2)} > M \quad \text{and} \quad \frac{s_{23}(y_1^2)}{s_{22}(y_1^2)} < -M.$$

We thus obtain a contradiction between (A.4) and each of the equalities (A.1), (A.2) and (A.3). We can then conclude that $s_{22}(x^2) \geq 0$ or $s_{22}(x^2) \leq 0$. The proof is similar for $s_{12}$ and $s_{32}$.

**Step 2:** We show, thanks to the first invariance given in Proposition 1.5, that there exists a Stäckel matrix having the same associated metric as $S$ and such that for all $(i,j) \in \{1,2,3\} \times \{2,3\}$, $s_{ij} > 0$ or $s_{ij} < 0$. We recall that there is at most one vanishing function $s_{ij}$, $(i,j) \in \{1,2,3\} \times \{2,3\}$ per column since the minors $s^{11}$, $s^{21}$ and $s^{31}$ are nonzero quantities. We assume that one coefficient of the second column vanishes. By symmetry, we can assume that this is $s_{12}$, i.e., that $s_{12}(x_0^1) = 0$ at one point $x_0^1$. We first assume that $s_{23}$ and $s_{33}$ do not vanish. In this case, there exists a real $a \geq 1$ such that

$$|s_{23}| < a|s_{22}| \quad \text{and} \quad |s_{33}| < a|s_{32}|$$

and a real constant $b \geq 1$ such that

$$|s_{22}| < b|s_{23}| \quad \text{and} \quad |s_{32}| < b|s_{33}|.$$

We now search a $2\times 2$ constant invertible matrix $G$ such that the coefficients of the new Stäckel matrix, obtained by the transformation given in Proposition 1.5(1), are positive or negative. For instance, if $s_{12}$ and $s_{13}$ have the same sign, we put

$$G = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$$

and we thus obtain a new Stäckel matrix whose second and third columns are

$$\begin{pmatrix} as_{12} + s_{13} & s_{12} + bs_{13} \\ as_{22} + s_{23} & s_{22} + bs_{23} \\ as_{32} + s_{33} & s_{32} + bs_{33} \end{pmatrix}.$$

We can easily show that these six components are positive or negative (we recall that $s_{12}$ and $s_{13}$ cannot vanish simultaneously). However, if $s_{12}$ and $s_{13}$ have different signs, we put

$$G = \begin{pmatrix} a & -1 \\ -1 & b \end{pmatrix}$$

and we also obtain positive or negative components. If $s_{23}$ or $s_{33}$ vanish we have just to choose suitable constants $a$ and $b$ using the fact there is at most one vanishing function in the third column.

**Step 3:** Finally, we show, thanks to the first invariance given in Proposition 1.5 and the riemannian structure, that there exists a Stäckel matrix having the same associated metric as $S$ and satisfying condition (C) of Proposition 1.6. We recall that thanks, to the second step, we can assume that the Stäckel matrix $S$ satisfies $s_{ij} > 0$ or $s_{ij} < 0$ for all $(i,j) \in \{1,2,3\} \times \{2,3\}$. We recall that the metric $g$ is riemannian if and only if $\det(S)$, $s^{11}$, $s^{21}$ and $s^{31}$ have the same sign. Without loss of generality, we assume that these quantities are all positive. We recall that according to the sign of the functions $s_{12}$, $s_{22}$ and $s_{32}$, inequalities (A.1)–(A.3) are satisfied. We thus have to treat different cases according to the sign of the components of the Stäckel matrix. First we want to obtain the sign conditions in (C). Since the proof is similar in the other cases, we give the proof just in the case

$$s_{12} > 0, \quad s_{22} < 0 \quad \text{and} \quad s_{32} > 0.$$

We give below, in each case, the matrix $G \in \mathrm{GL}_2(\mathbb{R})$ such that the transformation given in the first invariance of Proposition 1.5 provides us with the signs we want.

- If $s_{13} > 0$, $s_{23} < 0$ and $s_{33} > 0$ we put

  $$G = \begin{pmatrix} 1 & -1 \\ 0 & b \end{pmatrix},$$

  where

  $$\frac{s_{13}}{s_{12}} < b < \frac{s_{23}}{s_{22}} < \frac{s_{33}}{s_{32}},$$

  and we obtain the required signs. Indeed, we obtain that the second and the third columns of the new Stäckel matrix are given by

  $$\begin{pmatrix} s_{12} & -s_{12} + bs_{13} \\ s_{22} & -s_{22} + bs_{23} \\ s_{32} & -s_{32} + bs_{33} \end{pmatrix}$$

  which has the desired signs thanks to our choice of constant $b$.
- If $s_{13} > 0$, $s_{23} > 0$ and $s_{33} < 0$ we put $G = I_2$.
- If $s_{13} > 0$, $s_{23} < 0$ and $s_{33} < 0$ we put

  $$G = \begin{pmatrix} 1 & -1 \\ 0 & b \end{pmatrix},$$

  where

  $$\frac{s_{33}}{s_{32}} < \frac{s_{13}}{s_{12}} < b < \frac{s_{23}}{s_{22}}.$$

As previously, the case $s_{13} < 0$ is similar and we thus omit its proof. Up to this point, we proved that we can assume that

$$\text{(A.5)} \qquad \begin{cases} s_{12}(x^1) > 0 \ \text{ and } \ s_{13}(x^1) > 0 \quad \forall\, x^1, \\ s_{22}(x^2) < 0 \ \text{ and } \ s_{23}(x^2) > 0 \quad \forall\, x^2, \\ s_{32}(x^3) > 0 \ \text{ and } \ s_{33}(x^3) < 0 \quad \forall\, x^3. \end{cases}$$

Finally, we have to use just once more the invariance with respect to the multiplication of the second and the third columns by an invertible constant $2 \times 2$ matrix $G$ to obtain that we can assume

$$\lim_{x^1 \to 0} s_{12}(x^1) = \lim_{x^1 \to 0} s_{13}(x^1) = 1.$$

Indeed, we have just to set

$$G = \begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix},$$

where

$$\alpha = \lim_{x^1 \to 0} s_{12}(x^1) > 0 \quad \text{and} \quad \beta = \lim_{x^1 \to 0} s_{13}(x^1) > 0.$$

The result then follows.

## Appendix B. Proof of Lemma 4.5

We recall that the coupled spectrum was defined in Remark 2.2 by

$$\text{(B.1)} \qquad HY_m = \mu_m^2 Y_m \quad \text{and} \quad LY_m = \nu_m^2 Y_m, \quad \forall\, m \geq 1,$$

where $H$ and $L$ are commuting, elliptic and self-adjoint operators of order 2. Writing $Y_m(x^2, x^3) = v_m(x^2) w_m(x^3)$, we obtain that (B.1) is equivalent to

$$\text{(B.2)} \qquad -v_m''(x^2) + \left[ -(\lambda^2+1) s_{21}(x^2) + \mu_m^2 s_{22}(x^2) + \nu_m^2 s_{23}(x^2) \right] v_m(x^2) = 0$$

and

$$\text{(B.3)} \qquad -w_m''(x^3) + \left[ -(\lambda^2+1) s_{31}(x^3) + \mu_m^2 s_{32}(x^3) + \nu_m^2 s_{33}(x^3) \right] w_m(x^3) = 0,$$

where $v_m$ and $w_m$ are periodic functions, i.e.,

$$\text{(B.4)} \qquad \begin{cases} v_m(0) = v_m(B) \quad \text{and} \quad v_m'(0) = v_m'(B), \\ w_m(0) = w_m(C) \quad \text{and} \quad w_m'(0) = w_m'(C). \end{cases}$$

We first consider equation (B.2) which we rewrite as

$$-v'' - (\lambda^2+1) s_{21} v = \mu^2 \left[ -s_{22} - \theta^2 s_{23} \right] v,$$

where $v := v_m$, $\mu^2 := \mu_m^2$, $\nu^2 := \nu_m^2$ and $\theta^2 := \frac{\nu^2}{\mu^2}$. In the following we will consider Schrödinger equations associated with (B.2)–(B.3) whose spectral parameter is $\mu^2$ which tends to $+\infty$. Moreover, these equations depend on the parameter $\theta^2$ which is always bounded in a suitable cone that we introduce now. We recall that, as we showed in Lemma 2.4, there exist real constants $C_1$, $C_2$, $D_1$ and $D_2$ such that for all $m \geq 1$,

$$C_1\mu_m^2 + D_1 \leq \nu_m^2 \leq C_2\mu_m^2 + D_2,$$

where $C_1 = \min\left(-\frac{s_{32}}{s_{33}}\right) > 0$ and $C_2 = -\min\left(\frac{s_{22}}{s_{23}}\right) > 0$. Let $\epsilon > 0$ be fixed; we then consider $\theta^2$ such that

$$c_1 + \frac{D_1}{\mu^2} + \epsilon \leq \theta^2 \leq c_2 + \frac{D_2}{\mu^2} - \epsilon, \tag{B.5}$$

where $c_1 = \max\left(-\frac{s_{32}}{s_{33}}\right)$ and $c_2 = \min\left(-\frac{s_{22}}{s_{23}}\right)$. We note that $0 < C_1 \leq c_1 < c_2 \leq C_2$. This implies that, for sufficiently large $\mu^2$, there exists $\delta > 0$ such that

$$-s_{22} - \theta^2 s_{23} \geq \left(\epsilon - \frac{D_2}{\mu^2}\right) s_{23} \geq \delta > 0 \tag{B.6}$$

and

$$-s_{32} - \theta^2 s_{33} \geq \left(\epsilon + \frac{D_1}{\mu^2}\right)(-s_{33}) \geq \delta > 0. \tag{B.7}$$

For such a $\theta^2$, we can thus proceed to the Liouville change of variables $X^2 = \int_0^{x^2} \sqrt{-s_{22}(t) - \theta^2 s_{23}(t)}\, dt$, in equation (B.2). This new variable thus satisfies $X^2 \in [0, \tilde{B}(\theta^2))$, where

$$\tilde{B}(\theta^2) = \int_0^B \sqrt{-s_{22}(t) - \theta^2 s_{23}(t)}\, dt. \tag{B.8}$$

Finally, we set

$$V(X^2) = \left[-s_{22}(x^2(X^2)) - \theta^2 s_{23}(x^2(X^2))\right]^{\frac{1}{4}} v(x^2(X^2)).$$

This new function then satisfies in the variable $X^2$ the Schrödinger equation

$$-\ddot{V}^2(X^2) + Q_{\theta^2}(X^2)V(X^2) = \mu^2 V(X^2), \tag{B.9}$$

where $\mu^2$ is the spectral parameter, $Q_{\theta^2}(X^2)$ is uniformly bounded with respect to $\theta^2$ satisfying (B.5) and for such a $\theta^2$, $Q_{\theta^2}(X^2) = O(1)$.

We now search the couples $(\mu^2, \theta^2)$ such that (B.9) admits periodic solutions. We define $\{C_0, S_0\}$ and $\{C_1, S_1\}$ to be the usual fundamental systems of solutions of (B.9), i.e.,

$$C_0(0) = 1, \quad \dot{C}_0(0) = 0, \quad S_0(0) = 0 \quad \text{and} \quad \dot{S}_0(0) = 1$$

and

$$C_1(\tilde{B}) = 1, \quad \dot{C}_1(\tilde{B}) = 0, \quad S_1(\tilde{B}) = 0 \quad \text{and} \quad \dot{S}_1(\tilde{B}) = 1.$$

We recall that these functions are analytic and even with respect to $\mu$. We write the solution $V$ of (B.9) as

$$V = \alpha C_0 + \beta S_0 = \gamma C_1 + \delta S_1,$$

where $\alpha$, $\beta$, $\gamma$ and $\delta$ are real constants. Thus, $V(0) = \alpha$, $\dot{V}(0) = 0$, $V(\tilde{B}) = \gamma$ and $\dot{V}(\tilde{B}) = \delta$. Then $V$ is a periodic function if and only if

$$V(0) = V(\tilde{B}) \quad \Leftrightarrow \quad \alpha = \gamma \quad \Leftrightarrow \quad W(V, S_0) = W(V, S_1)$$

and

$$\dot{V}(0) = \dot{V}(\tilde{B}) \quad \Leftrightarrow \quad \beta = \delta \quad \Leftrightarrow \quad W(C_0, V) = W(C_1, V),$$

where $W(f, g) = fg' - f'g$ denotes the Wronskian of two functions $f$ and $g$. In other words, $V$ is a periodic solution of (B.9) if and only if

$$W(V, S_0 - S_1) = W(C_0 - C_1, V) = 0. \tag{B.10}$$

We thus add to equation (B.9) the boundary conditions (B.10) and we define the corresponding characteristic functions. In other words, we define

$$\Delta_1(\mu^2, \theta^2) = W(C_0 - C_1, S_0 - S_1) = 2 - W(C_0, S_1) - W(C_1, S_0).$$

We emphasize that $\Delta_1(\mu^2, \theta^2)$ vanishes if and only if there exists a periodic solution of (B.9) for $(\mu^2, \theta^2)$. The asymptotics of $W(C_0, S_1)$ and $W(C_1, S_0)$ are well known (see for instance [20, 28]). Indeed, we know

$$W(C_0, S_1) = \cos\left(\mu \tilde{B}(\theta^2)\right) \times \left(1 + O\left(\frac{1}{\mu}\right)\right) \tag{B.11}$$

and

$$W(C_1, S_0) = \cos\left(\mu \tilde{B}(\theta^2)\right) \times \left(1 + O\left(\frac{1}{\mu}\right)\right), \tag{B.12}$$

where $\mu = \sqrt{\mu^2}$ (we do not have to make the sign of $\mu$ precise since the characteristic functions are even functions). We then obtain

$$\Delta_1(\mu^2, \theta^2) = 0 \quad \Leftrightarrow \quad 2 - 2\cos\left(\mu \tilde{B}(\theta^2)\right) + O\left(\frac{1}{\mu}\right) = 0. \tag{B.13}$$

Using Rouché's theorem (see for instance [28]) we can then deduce that the couples $(\mu^2, \theta^2)$ satisfying (B.13) are close for large $\mu$ to the couples $(\mu^2, \theta^2)$ satisfying

$$2 - 2\cos\left(\mu \tilde{B}(\theta^2)\right) = 0 \quad \Leftrightarrow \quad \cos\left(\mu \tilde{B}(\theta^2)\right) = 1.$$

The solutions of this last equation are $\mu = \frac{2m\pi}{\tilde{B}(\theta^2)}$, $m \in \mathbb{Z}$ for $\theta^2$ satisfying (B.5) and $m$ sufficiently large. Finally, we recall that $\tilde{B}(\theta^2)$ is given by (B.8). Thus, since $s_{23}$ is a positive function, the map $\tilde{B}$ is strictly decreasing with respect to $\theta^2 \in [c_1 + \epsilon, c_2 - \epsilon]$. The map $\frac{1}{\tilde{B}(\theta^2)}$ is then strictly increasing. We can summarize these facts in Figure 2.

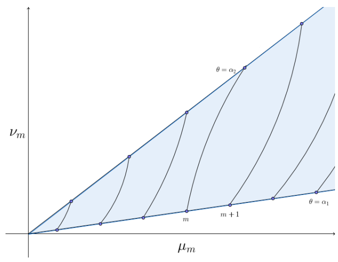


Figure 2. First approximation of the coupled spectrum.

We do the same analysis on equation (B.3). We recall that if $\theta^2$ satisfies (B.5) then the inequality (B.7) is satisfied for $\mu^2$ sufficiently large. We can thus set $X^3 = \int_0^{x^3} \sqrt{-s_{32}(t) - \theta^2 s_{33}(t)}\, dt$. This new variable satisfies $X^3 \in [0, \tilde{C}(\theta^2)]$, where

$$\tilde{C}(\theta^2) = \int_0^C \sqrt{-s_{32}(t) - \theta^2 s_{33}(t)}\, dt. \tag{B.14}$$

We then set

$$W(X^3) = \left[-s_{32}(x^3(X^3)) - \theta^2 s_{33}(x^3(X^3))\right]^{\frac{1}{4}} w(x^3(X^3)).$$

This function then satisfies, in the variable $X^3$, the Schrödinger equation

$$-\ddot{W}^2(X^3) + \tilde{Q}_{\theta^2}(X^3)W(X^3) = \mu^2 W(X^3), \quad \text{where } \tilde{Q}_{\theta^2}(X^3) = O(1), \tag{B.15}$$

for $\theta^2$ satisfying (B.5) and $\mu^2$ sufficiently large. As previously, we obtain that (B.15) has a periodic solution if and only if

$$\Delta_2(\mu^2, \theta^2) := 2 - W(C_0, S_1) - W(C_1, S_0) = 0.$$

Thanks to the asymptotics (B.11)–(B.12) we obtain

$$\Delta_2(\mu^2, \theta^2) = 0 \quad \Leftrightarrow \quad 2 - 2\cos\left(\mu\tilde{C}(\theta^2)\right) + O\left(\frac{1}{\mu}\right) = 0.$$

Using Rouché's theorem once more, we obtain that the couple $(\mu^2, \theta^2)$ satisfying the previous equality are close for large $\mu$ to the couple satisfying $\cos\left(\mu\tilde{C}(\theta^2)\right) = 1$, i.e., $\mu^2 = \frac{2\pi k}{\tilde{C}(\theta^2)}$, $k \in \mathbb{Z}$, where $k$ is sufficiently large and $\theta^2$ satisfies equation (B.5). We recall that $\tilde{C}(\theta^2)$ is given by (B.14). Since $s_{33}$ is a negative function, the map $\tilde{C}$ is strictly increasing for $\theta^2 \in [c_1 + \epsilon, c_2 - \epsilon]$. The map $\frac{1}{\tilde{C}(\theta^2)}$ is then strictly decreasing. We can summarize these facts in Figure 3.

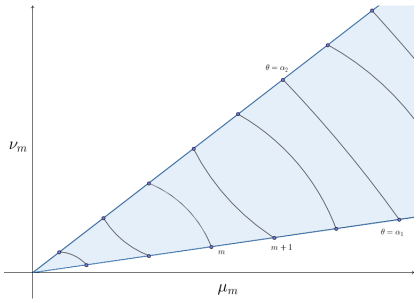


Figure 3. Second approximation of the coupled spectrum.

The coupled spectrum $\Lambda = \{(\mu_m^2, \nu_m^2),\ m \geq 1\}$, or equivalently the coupled spectrum $(\mu_m^2, \theta_m^2)$, is then given by

$$\Lambda = \{\Delta_1(\mu^2, \theta^2) = 0\} \cap \{\Delta_2(\mu^2, \theta^2) = 0\},$$

since for all $(\mu_m^2, \nu_m^2) \in \Lambda$, there exist simultaneously a periodic solution of (B.9) and a periodic solution of (B.15). Using Figures 2 and 3 we obtain Figure 4 in which the coupled spectrum corresponds to the intersection between the previous curves.

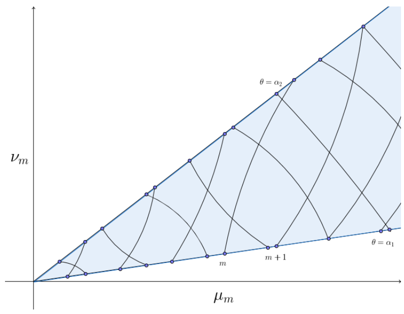


Figure 4. The coupled spectrum.

We now want to use this particular structure of the coupled spectrum to prove Lemma 4.5. We work on the plane $(\mu, \theta)$ and we set $\nu = \theta\mu$, with $0 < \alpha_1 \leq \theta \leq \alpha_2$, where $\alpha_1 = \sqrt{c_1^2 + \epsilon}$ and $\alpha_2 = \sqrt{c_2^2 - \epsilon}$, with $\epsilon > 0$. We recall that for large $m$ we can approximate $\mu_m$ by $\mu_m = \frac{2m\pi}{\tilde{B}(\theta^2)}$, where $\tilde{B}(\theta^2) = \int_0^B \sqrt{-s_{22}(t) - \theta^2 s_{23}(t)}\, dt$. We first want to show that the curves drawn in Figure 2 are uniformly separated. In other words, we show that there exists $\delta > 0$ such that the distance between two successive curves is greater than $\delta$. To be precise, we want to show that there exists $\delta > 0$ such that for large $m$ and for all $(\theta_1, \theta_2) \in [\alpha_1, \alpha_2]^2$,

$$|\mu_{m+1}(\theta_2) - \mu_m(\theta_1)| + |\theta_2\mu_{m+1}(\theta_2) - \theta_1\mu_m(\theta_1)| \geq \delta. \tag{B.16}$$

If we put $d = |\mu_{m+1}(\theta_2) - \mu_m(\theta_1)|$ we immediately obtain that (B.16) is equivalent to

$$d + |d\theta_2 + (\theta_2 - \theta_1)\mu_m(\theta_1)| \geq \delta. \tag{B.17}$$

We now use the mean value theorem on the map $\frac{1}{\tilde{B}(\theta^2)}$ and we thus obtain

$$\frac{1}{\tilde{B}(\theta_2^2)} = \frac{1}{\tilde{B}(\theta_1^2)} + e(\xi)(\theta_2^2 - \theta_1^2),$$

where

$$e(\xi) = -\frac{\tilde{B}'(\xi^2)}{\tilde{B}(\xi^2)^2} > 0,$$

with $\xi \in (\theta_1, \theta_2)$. Actually, we can show that there exist two positive constants $e_1$ and $e_2$ such that

$$0 < e_1 \leq e(\xi) \leq e_2 \quad \forall\, \xi \in [\alpha_1, \alpha_2].$$

We then easily obtain

$$d = \left| \frac{2\pi}{\tilde{B}(\theta_1^2)} + 2(m+1)\pi e(\xi)(\theta_1 + \theta_2)(\theta_1 - \theta_2) \right|.$$

Using the triangle inequality we thus obtain

$$2(m+1)\pi e(\xi)(\theta_1 + \theta_2)|\theta_1 - \theta_2| \geq \frac{2\pi}{\tilde{B}(\theta_1^2)} - d. \tag{B.18}$$

We thus have to study different cases.

**Case 1:** If $d \geq \frac{2\pi}{\tilde{B}(\theta_1^2)}$, we easily obtain

$$|\mu_{m+1}(\theta_2) - \mu_m(\theta_1)| + |\theta_2\mu_{m+1}(\theta_2) - \theta_1\mu_m(\theta_1)| \geq d \geq \frac{2\pi}{\tilde{B}(\theta_1^2)}.$$

**Case 2:** If $d < \frac{2\pi}{\tilde{B}(\theta_1^2)}$, then (B.18) gives us

$$|\theta_1 - \theta_2| > \frac{2\pi - d\tilde{B}(\theta_1^2)}{2(m+1)\pi e(\xi)(\theta_1 + \theta_2)\tilde{B}(\theta_1^2)}.$$

Thus,

$$\begin{aligned}\mu_m(\theta_1)|\theta_1-\theta_2| &= \frac{2m\pi}{\tilde{B}(\theta_1^2)}|\theta_1-\theta_2|\\ &> \frac{m}{m+1}\frac{2\pi-d\tilde{B}(\theta_1^2)}{e(\xi)(\theta_1+\theta_2)\tilde{B}(\theta_1^2)^2}\\ &> \frac{2\pi-d\tilde{B}(\alpha_1^2)}{4e_2\alpha_2\tilde{B}(\alpha_1^2)^2}.\end{aligned}$$

We note that

$$d\theta_2 < \frac{2\pi-d\tilde{B}(\alpha_1^2)}{4e_2\alpha_2\tilde{B}(\alpha_1^2)^2} \quad \Leftrightarrow \quad d < \frac{2\pi}{(4\theta_2e_2\alpha_2\tilde{B}(\alpha_1^2)+1)\tilde{B}(\alpha_1^2)}.$$

If

$$d > \frac{2\pi}{(4\theta_2e_2\alpha_2\tilde{B}(\alpha_1^2)+1)\tilde{B}(\alpha_1^2)},$$

then as in Case 1, we easily obtain

$$|\mu_{m+1}(\theta_2)-\mu_m(\theta_1)|+|\theta_2\mu_{m+1}(\theta_2)-\theta_1\mu_m(\theta_1)| \geq d \geq \delta.$$

If

$$d < \frac{2\pi}{(4\theta_2e_2\alpha_2\tilde{B}(\alpha_1^2)+1)\tilde{B}(\alpha_1^2)},$$

we then obtain

$$\begin{aligned}&|\mu_{m+1}(\theta_2)-\mu_m(\theta_1)|+|\theta_2\mu_{m+1}(\theta_2)-\theta_1\mu_m(\theta_1)|\\ &\quad = d+|d\theta_2+(\theta_2-\theta_1)\mu_m(\theta_1)|\\ &\quad = d+|\theta_2-\theta_1|\mu_m(\theta_1)-d\theta_2\\ &\quad > d+\frac{2\pi-d\tilde{B}(\alpha_1^2)}{4e_2\alpha_2\tilde{B}(\alpha_1^2)^2}-d\theta_2\\ &\quad = \frac{\pi}{2e_2\alpha_2\tilde{B}(\alpha_1^2)^2}+d\left(1-\frac{1}{4e_2\alpha_2\tilde{B}(\alpha_1^2)}-\theta_2\right).\end{aligned}$$

We note that there exists $d_0 > 0$ such that for all $d < d_0$,

$$d\left(1-\frac{1}{4e_2\alpha_2\tilde{B}(\alpha_1^2)}-\theta_2\right) > -\frac{\pi}{4e_2\alpha_2\tilde{B}(\alpha_1^2)^2}.$$

Thus, for all $d < d_0$, we immediately obtain

$$|\mu_{m+1}(\theta_2)-\mu_m(\theta_1)|+|\theta_2\mu_{m+1}(\theta_2)-\theta_1\mu_m(\theta_1)| \geq \frac{\pi}{4e_2\alpha_2\tilde{B}(\alpha_1^2)^2} \geq \delta.$$

Moreover, if $d \geq d_0$ we conclude as in Case 1.

We thus have shown that the curves of Figure 2 are uniformly separated. Since, the same analysis is also true for Figure 3 we have shown Lemma 4.5.

## Appendix C. Proof of Lemma 4.6

To prove the lemma we use the work of Colin de Verdière on the coupled spectrum of commuting pseudodifferential operators in [15, 16]. We recall that the operators $L$ and $H$ are defined by (2.4) and satisfy (2.9). Since $L$ and $H$ are semibounded operators by Lemma 2.2, there exists $M \in \mathbb{R}$ such that $L + M$ and $H + M$ are positive operators. We set

$$P_1 = \sqrt{L+M} \quad \text{and} \quad P_2 = \sqrt{H+M}.$$

The operators $P_1$ and $P_2$ are commuting, self-adjoint pseudodifferential operators of order 1 such that $P_1^2 + P_2^2$ is an elliptic operator. These operators are thus in the framework of [15]. The principal symbols of $P_1$ and $P_2$ are given by

$$p_1(x,\xi) = \sqrt{-\frac{s_{33}}{s^{11}}\xi_2^2 + \frac{s_{23}}{s^{11}}\xi_3^2} \quad \text{and} \quad p_2(x,\xi) = \sqrt{\frac{s_{32}}{s^{11}}\xi_2^2 - \frac{s_{22}}{s^{11}}\xi_3^2}, \tag{C.1}$$

respectively. We put $p(x,\xi) = (p_1(x,\xi), p_2(x,\xi))$, where $x := (x^2, x^3)$, $\xi := (\xi_2, \xi_3)$ and $(x,\xi)$ is a point on the cotangent bundle of $\mathcal{T}^2$, i.e., $T^\star\mathcal{T}^2$. We will apply [15, Thm. 0.7] to $P_1$ and $P_2$. We recall here this result adapted to our framework.

**Theorem C.1.** *Let $C$ be a cone of $\dot{\mathbb{R}}^2 = \mathbb{R}^2 \setminus \{(0,0)\}$, with piecewise $C^1$ boundary such that $\partial C \cap W = \emptyset$, where $\partial C$ is the boundary of $C$ and $W$ is the set of critical values of $p$. We then have*

$$\#\{\lambda \in C \cap \Lambda,\ |\lambda| \le r\} = \frac{1}{4\pi^2}\mathrm{vol}_\Omega\left(p^{-1}(C \cap B(0,r))\right) + O(r),$$

*where $\Lambda$ is the coupled spectrum of $P_1$ and $P_2$ and $\Omega = dx^2 \wedge dx^3 \wedge d\xi_2 \wedge d\xi_3$.*

Thus, to use Theorem C.1, we have to determine the set $W$ of critical values of $p$. We first have to determine the critical points of $p$, i.e., the points for which the differential of $p$ is not onto. The differential of $p$ is given by (we omit the variables)

$$Dp(x,\xi) = \frac{-1}{4p_1p_2}\begin{pmatrix} \partial_2\left(\frac{s_{33}}{s^{11}}\right)\xi_2^2 - \partial_2\left(\frac{s_{23}}{s^{11}}\right)\xi_3^2 & \partial_3\left(\frac{s_{33}}{s^{11}}\right)\xi_2^2 - \partial_3\left(\frac{s_{23}}{s^{11}}\right)\xi_3^2 & 2\frac{s_{33}}{s^{11}}\xi_2 & -2\frac{s_{23}}{s^{11}}\xi_3 \\ -\partial_2\left(\frac{s_{32}}{s^{11}}\right)\xi_2^2 + \partial_2\left(\frac{s_{22}}{s^{11}}\right)\xi_3^2 & -\partial_3\left(\frac{s_{32}}{s^{11}}\right)\xi_2^2 + \partial_3\left(\frac{s_{22}}{s^{11}}\right)\xi_3^2 & -2\frac{s_{32}}{s^{11}}\xi_2 & 2\frac{s_{22}}{s^{11}}\xi_3 \end{pmatrix}.$$

We compute the six $2 \times 2$ minors of this matrix and we search the points $(x,\xi)$ for which all these minors vanish. After calculation, we obtain that $(x,\xi)$ is a critical point of $p$ if and only if the following four conditions are satisfied:

$$\begin{cases} \xi_2\xi_3 = 0, \\ \xi_3\partial_2(s_{22})(\xi_2^2+\xi_3^2) = 0, \\ \xi_2\partial_3(s_{33})(\xi_2^2+\xi_3^2) = 0, \\ \partial_2(s_{22})\partial_3(s_{33})(\xi_2^2+\xi_3^2)^2 = 0. \end{cases}$$

Thus, there are four cases to study according to the vanishing of $\partial_2(s_{22})$ and $\partial_3(s_{33})$. We finally obtain

$$W = \begin{cases} (0,0) & \text{if } \partial_2(s_{22}) \neq 0 \text{ and } \partial_3(s_{33}) \neq 0, \\ \mathcal{D}_1 & \text{if } \partial_2(s_{22}) = 0 \text{ and } \partial_3(s_{33}) \neq 0, \\ \mathcal{D}_2 & \text{if } \partial_2(s_{22}) \neq 0 \text{ and } \partial_3(s_{33}) = 0, \\ \mathcal{D}_1 \cup \mathcal{D}_2 & \text{if } \partial_2(s_{22}) = 0 \text{ and } \partial_3(s_{33}) = 0, \end{cases}$$

where

$$\mathcal{D}_1 = \{t(\sqrt{s_{23}}, \sqrt{-s_{22}}),\ t \geq 0\} \quad \text{and} \quad \mathcal{D}_2 = \{t(\sqrt{-s_{33}}, \sqrt{s_{32}}),\ t \geq 0\},$$

where $s_{22}$, $s_{23} = s_{22}+1$, $s_{33}$ and $s_{32} = s_{33}+1$ are constants according to the case we study. We now recall that in Theorem C.1, we have to choose a cone $C$ such that $\partial C \cap W = \emptyset$ and we want to study the set $p^{-1}(C \cap B(0,r)) = p^{-1}(C) \cap p^{-1}(B(0,r))$. Letting $r > 0$, we first study the set $p^{-1}(B(0,r))$. We recall that there exists a constant $c_1 > 0$ such that $\max\left(-\frac{s_{33}}{s^{11}}, \frac{s_{23}}{s^{11}}, \frac{s_{32}}{s^{11}}, -\frac{s_{22}}{s^{11}}\right) \leq c_1$. Thus, if $(\xi_2,\xi_3) \in B\big(0, \frac{r}{\sqrt{2c_1}}\big)$ and $(x^2,x^3) \in \mathcal{T}^2$, then $\|p(x,\xi)\| = \sqrt{p_1(x,\xi)+p_2(x,\xi)} \leq \sqrt{2c_1(\xi_2^2+\xi_3^2)} \leq r$. We deduce from this fact that

$$\mathcal{T}^2 \times B\left(0, \frac{r}{\sqrt{2c_1}}\right) \subset p^{-1}(B(0,r)). \tag{C.2}$$

We now study the set $p^{-1}(C)$. We have to divide our study in four cases as we have seen before.

**Case 1:** $\partial_2(s_{22}) \neq 0$ and $\partial_3(s_{33}) \neq 0$. In this case (see Figure 5) we have just to avoid the point $\{(0,0)\}$. We consider the cone $C = \{(x,y) \in \mathbb{R}^2$ such that $\epsilon \leq x,\ \epsilon \leq y\}$, $\epsilon > 0$.

By definition

$$p^{-1}(C) = \{(x,\xi) \in \mathcal{T}^2 \times \mathbb{R}^2,\ \epsilon \leq p_1(x,\xi),\ \epsilon \leq p_2(x,\xi)\}$$

and since there exists $c_2 > 0$ such that $c_2 \leq \min\left(-\frac{s_{33}}{s^{11}}, \frac{s_{23}}{s^{11}}, \frac{s_{32}}{s^{11}}, -\frac{s_{22}}{s^{11}}\right)$, there exists $\eta > 0$ such that $\mathcal{T}^2 \times (\mathbb{R}^2 \setminus B(0,\eta)) \subset p^{-1}(C)$.

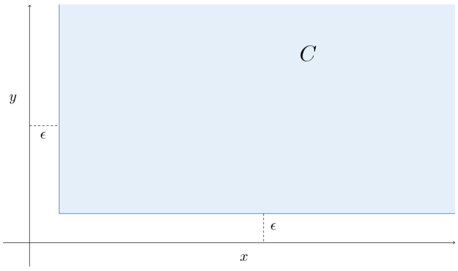


Figure 5. Case 1.

**Case 2:** $\partial_2(s_{22}) = 0$ and $\partial_3(s_{33}) \neq 0$. We have to avoid the half-line $\mathcal{D}_1$ which has slope $\beta_1 = \sqrt{\frac{-s_{22}}{s_{23}}}$ (see Figure 6). We consider the cone $C = \{(x, y) \in \mathbb{R}^2$ such that $\epsilon \leq x,\ \epsilon \leq y \leq \beta_1 x - \epsilon\}$, $\epsilon > 0$.

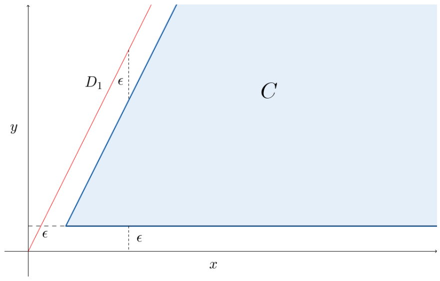


Figure 6. Case 2.

As in the first case, there is $\eta > 0$ such that

$$p_1(x,\xi) \geq \epsilon \quad \text{and} \quad p_2(x,\xi) \geq \epsilon, \quad \forall (x,\xi) \in \mathcal{T}^2 \times (\mathbb{R}^2 \setminus B(0,\eta)).$$

The last condition can be rewritten as

$$\begin{aligned} p_2(x,\xi) \leq \beta_1 p_1(x,\xi) - \epsilon \Leftrightarrow \sqrt{\frac{s_{32}}{s^{11}}\xi_2^2 - \frac{s_{22}}{s^{11}}\xi_3^2} &\leq \sqrt{\frac{-s_{22}}{s_{23}}}\sqrt{-\frac{s_{33}}{s^{11}}\xi_2^2 + \frac{s_{23}}{s^{11}}\xi_3^2} - \epsilon \\ \Leftrightarrow \sqrt{\frac{s_{32}}{s^{11}}\xi_2^2 - \frac{s_{22}}{s^{11}}\xi_3^2} &\leq \sqrt{\frac{s_{22}s_{33}}{s_{23}s^{11}}\xi_2^2 - \frac{s_{22}}{s^{11}}\xi_3^2} - \epsilon. \end{aligned}$$

We recall that, thanks to the condition given in Remark 1.1, $\frac{s_{22}s_{33}}{s_{23}} > s_{32}$. Thus, there exists $\epsilon > 0$ small enough such that

$$p_2(x,\xi) \leq \beta_1 p_1(x,\xi) - \epsilon \quad \forall (x,\xi) \in \mathcal{T}^2 \times \mathbb{R}^2.$$

Finally, we have shown that for such an $\epsilon$, there exists $\eta > 0$ such that $\mathcal{T}^2 \times (\mathbb{R}^2 \setminus B(0,\eta)) \subset p^{-1}(C)$.

**Case 3:** $\partial_2(s_{22}) \neq 0$ and $\partial_3(s_{33}) = 0$. We have to avoid the half-line $\mathcal{D}_2$ which has slope $\beta_2 = \sqrt{-\frac{s_{32}}{s_{33}}}$ (see Figure 7). We consider the cone $C = \{(x,y) \in \mathbb{R}^2$ such that $\epsilon \leq x,\ \beta_2 x + \epsilon \leq y\}$, $\epsilon > 0$, and we show, as in the second case, that

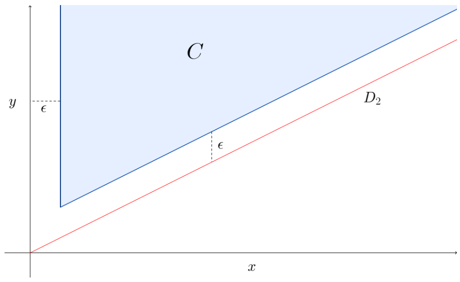


Figure 7. Case 3.

for $\epsilon > 0$ small enough there exists $\eta > 0$ such that $\mathcal{T}^2 \times (\mathbb{R}^2 \setminus B(0,\eta)) \subset p^{-1}(C)$.

**Case 4:** $\partial_2(s_{22}) = 0$ and $\partial_3(s_{33}) = 0$. We have to avoid $\mathcal{D}_1 \cup \mathcal{D}_2$ which have slopes $\alpha_1$ and $\alpha_2$ respectively (see Figure 8). We consider the cone $C = \{(x,y) \in \mathbb{R}^2$ such that $\epsilon \leq x,\ \beta_2 x + \epsilon \leq y \leq \beta_1 x - \epsilon\}$, $\epsilon > 0$. As in the first case, there is

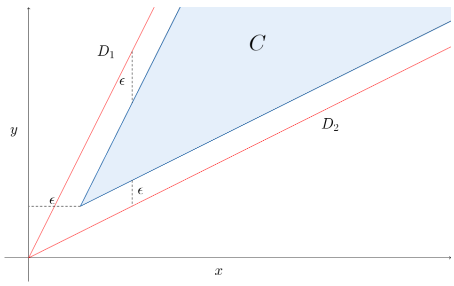


Figure 8. Case 4.

$\eta > 0$ such that

$$p_1(x,\xi) \geq \epsilon \quad \forall\,(x,\xi) \in \mathcal{T}^2 \times (\mathbb{R}^2 \setminus B(0,\eta))$$

and, as in the second and the third cases, there exists $\epsilon > 0$ small enough such that

$$p_2(x,\xi) \leq \beta_1 p_1(x,\xi) - \epsilon \quad \text{and} \quad \beta_2 p_1(x,\xi) + \epsilon \leq p_2(x,\xi), \quad \forall\,(x,\xi) \in \mathcal{T}^2 \times \mathbb{R}^2.$$

Thus, for $\epsilon > 0$ small enough, there exists $\eta > 0$ such that $\mathcal{T}^2 \times (\mathbb{R}^2 \setminus B(0,\eta)) \subset p^{-1}(C)$.

In conclusion, we have shown that in any cases there exists $\eta > 0$ such that

$$\mathcal{T}^2 \times (\mathbb{R}^2 \setminus B(0,\eta)) \subset p^{-1}(C). \tag{C.3}$$

Moreover, in each case the cone $\mathcal{C}$ defined in (4.7) is, by definition, included in the cone $C$ we considered and we can thus apply Theorem C.1 to this cone. Therefore, thanks to (C.2)–(C.3) we thus have shown that for $r > 0$ large enough,

$$\mathcal{T}^2 \times \left( B\left(0, \frac{r}{\sqrt{2c_1}}\right) \setminus B(0,\eta) \right) \subset p^{-1}(\mathcal{C}) \cap p^{-1}(B(0,r)) = p^{-1}(\mathcal{C} \cap B(0,r)). \tag{C.4}$$

From the inclusion (C.4) we can deduce that there exists a constant $c > 0$ such that

$$cr^2 \leq \frac{1}{4\pi^2} \mathrm{vol}_\Omega \left( B\left(0, \frac{r}{\sqrt{2c_1}}\right) \setminus B(0,\eta) \right) \leq \frac{1}{4\pi^2} \mathrm{vol}_\Omega \left( p^{-1}(\mathcal{C} \cap B(0,r)) \right).$$

Thanks to Theorem C.1, we can then conclude that there exists $c > 0$ such that

$$\#\{\lambda \in \mathcal{C} \cap \Lambda,\ |\lambda| \leq r\} \geq cr^2.$$

Finally, we recall that $\Lambda = \{(\sqrt{\mu_m^2 + M}, \sqrt{\nu_m^2 + M}),\ m \geq 1\}$ and we note that, thanks to the fact that $\mu_m^2 \to +\infty$ and $\nu_m^2 \to +\infty$ as $m \to +\infty$,

$$\sqrt{\mu_m^2 + M} \sim |\mu_m| \quad \text{and} \quad \sqrt{\nu_m^2 + M} \sim |\nu_m|, \quad m \to +\infty.$$

We recall that $n(r) = \#\{\lambda \in \mathcal{C} \cap E_M,\ |\lambda| \leq r\}$, without multiplicity, whereas the result obtained before was computed counting multiplicity. However, the multiplicity of the coupled eigenvalues is at most 4 (see Remark 2.3). Taking account of this fact, we can conclude that

$$\overline{\lim}\, \frac{n(r)}{r^2} > 0, \quad r \to +\infty.$$

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## References

[1] V. de Alfaro and T. Regge, *Potential scattering*, North-Holland, Amsterdam; Interscience Publishers Division John Wiley and Sons, 1965. Zbl 0141.23202 MR 0191316

[2] S. Benenti, Separability in Riemannian manifolds, arXiv:1512.07833.

[3] S. Benenti, Orthogonal separable dynamical systems, in *Differential geometry and its applications*, Mathematical Publications, Silesian University Opava 1, 1993, 163–184. Zbl 0817.70012 MR 1255538

[4] S. Benenti, C. Chanu and G. Rastelli, Remarks on the connection between the additive separation of the Hamilton-Jacobi equation and the multiplicative separation of the Schrödinger equation. I. The completeness and Robertson condition, J. Math. Phys. **43**, (2002), 5183–5222. Zbl 1060.35116 MR 1935158

[5] S. Benenti, C. Chanu and G. Rastelli, Remarks on the connection between the additive separation of the Hamilton-Jacobi equation and the multiplicative separation of the Schrödinger equation. II. First integrals and symmetry operators, J. Math. Phys. **43** (2002), 5223–5253. Zbl 1060.35117 MR 1935159

[6] S. Benenti and M. Francaviglia, The theory of separability of the Hamilton-Jacobi equation and its applications to general relativity, in *General relativity and gravitation* vol. 1, Plenum, New York-London, 1980, 393–439. MR 0583725

[7] B. Berndtsson, Zeros of analytic functions of several variables, Ark. Mat. **16** (1978), 251–262. Zbl 0409.32001 MR 0524753

[8] T. Bloom, A spanning set for $\mathcal{C}(I^n)$, Trans. Amer. Math. Soc. **321**, (1990), 741–759. Zbl 0723.41031 MR 0984854

[9] R. P. Boas, *Entire functions*, Academic Press, New York, 1954. Zbl 0058.30201 MR 0068627

[10] V. Bolsinov A. and V. S. Matveev, Local normal forms for geodesically equivalent pseudo-Riemannian metrics, Trans. Amer. Math. Soc. **9** (2015), 6719–6749. Zbl 1320.53024 MR 3356952

[11] D. Borthwick, *Spectral theory for infinite-area hyperbolic surfaces*, Birkhäuser, Boston-Basel-Berlin, 2007. Zbl 1130.58001 MR 3497464

[12] D. Borthwick and P. A. Perry, Inverse scattering results for manifolds hyperbolic near infinity, J. Geom. Anal. **21** (2011), 305–333. Zbl 1229.58024 MR 2772075

[13] J. M. Bouclet, Absence of eigenvalue at the bottom of the continuous spectrum on asymptotically hyperbolic manifolds, Ann. Global Anal. Geom. **44** (2013), 115–136. Zbl 1286.58019 MR 3073583

[14] C. Chanu and G. Rastelli, Fixed energy R-separation for Schrödinger equation, Int. J. Geom. Methods Mod. Phys. **3** (2006), 489–508. Zbl 1114.81027 MR 2232869

[15] Y. Colin de Verdière, Spectre conjoint d'opérateurs pseudo-différentiels qui commutent. I. Le cas non intégrable, Duke Math. J. **46** (1979), 169–182. Zbl 0411.35073 MR 0523605

[16] Y. Colin de Verdière, Spectre conjoint d'opérateurs pseudo-différentiels qui commutent. II. Le cas intégrable, Math. Z. 171 (1980), 51–73. Zbl 0478.35073 MR 0566483

[17] P. D. B. Collins, *An introduction to Regge theory and high energy physics*, Cambridge Monographs on Mathematical Physics 4, Cambridge University Press, 1977. MR 0522334

[18] T. Daudé, D. Gobin and F. Nicoleau, Local inverse scattering results at fixed energy in spherically symmetric asymptotically hyperbolic manifolds, Inverse Probl. Imaging **10** (2016), 659–688. Zbl 1348.81441 MR 3562266

[19] T. Daudé, N. Kamran and F. Nicoleau, Inverse scattering at fixed energy on asymptotically hyperbolic Liouville surfaces, Inverse Problems **31** (2015), 125009. Zbl 1339.58016 MR 3540316

[20] T. Daudé, N. Kamran and F. Nicoleau, Non-uniqueness results for the anisotropic Calderon problem with data measured on disjoint sets, arXiv:1510.06559, Preprint (2015).

[21] T. Daudé and F. Nicoleau, Inverse scattering at fixed energy in de Sitter-Reissner-Nordström black holes, Ann. Henri Poincaré **12** (2011), 1–47. Zbl 1243.81218 MR 2770088

[22] T. Daudé and F. Nicoleau, *Direct and inverse scattering at fixed energy for massless charged Dirac fields by Kerr-Newman-De Sitter black holes*, Mem. Am. Math. Soc. 1170, 2017. Zbl 06724389 MR 3633460

[23] T. Daudé and F. Nicoleau, Local inverse scattering at a fixed energy for radial Schrödinger operators and localization of the Regge poles, Ann. Henri Poincaré **17** (2016), 2849–2904. Zbl 1355.81072 MR 3546990

[24] D. Dos Santos Ferreira, C. E. Kenig, M. Salo and G. Uhlmann, Limiting Carleman weights and anisotropic inverse problems, Invent. Math. **178** (2009), 119–171. Zbl 1181.35327 MR 2534094

[25] D. Dos Santos Ferreira, Y. Kurylev, M. Lassas and M. Salo, The Calderón problem in transversally anisotropic geometries, J. Eur. Math. Soc. (JEMS) **18** (2016), 2579–2626. Zbl 1359.35224 MR 3562352

[26] L. P. Eisenhart, Separable systems of Stäckel, Ann. Math. **35** (1934), 284D305. Zbl 0009.38002 MR 1503163

[27] L. P. Eisenhart, *Riemannian geometry*, 8th printing, Princeton University Press, Princeton, NJ, 1997. MR 1487892

[28] G. Freiling and V. Yurko, *Inverse Sturm-Liouville problems and their applications*, NOVA Science Publishers New York, 2001. Zbl 1037.34005 MR 2094651

[29] G. Freiling and V. Yurko, Inverse problems for differential operators with singular boundary conditions, Math. Nachr. **278** (2005), 1561–1578. Zbl 1090.34007 MR 2169700

[30] D. Gobin, Inverse scattering at fixed energy for massive charged Dirac fields in de Sitter-Reissner-Nordström black holes, Inverse Problems **31** (2015), 1561–1578. Zbl 1327.81210 MR 3346142

[31] C. Guillarmou and A. Sá Barreto, Scattering and inverse scattering on ACH manifolds, J. reine angew. Math. **622** (2008), 1–55. Zbl 1159.58016 MR 2433611

[32] C. Guillarmou and A. Sá Barreto, Inverse problems for Einstein manifolds, Inverse Probl. Imaging **3** (2009), 1–15. Zbl 1229.58025 MR 2558301

[33] C. Guillarmou and L. Tzou, *Calderón inverse problem with partial data on Riemann surfaces*, Duke Math. J. **158** (2011), 83–120. Zbl 1222.35212 MR 2794369

[34] C. Guillarmou and L. Tzou, The Calderon inverse problem in two dimensions, in *Inverse problems and applications: Inside out. II*, Mathematical Sciences Research Institute Publications 60, Cambridge University Press, Cambridge, 2013, 119–166. Zbl 1317.35294 MR 3098657

[35] H. Isozaki and J. Kurylev, *Introduction to spectral theory and inverse problems on asymptotically hyperbolic manifolds*, MSJ Memoirs 32, Mathematical Society of Japan, 2014. Zbl 06317422 MR 3222614

[36] H. Isozaki, Y. Kurylev and M. Lassas, Conic singularities, generalized scattering matrix, and inverse scattering on asymptotically hyperbolic surfaces, J. reine angew. Math. 724 (2017), 53–103. Zbl 06695051 MR 3619104

[37] M. S. Joshi and A. Sá Barreto, Inverse scattering on asymptotically hyperbolic manifolds, Acta Math. **184** (2000), 41–86. Zbl 1142.58309 MR 1756569

[38] E. G. Kalnins and W. Miller Jr, Killing tensors and variable separation for Hamilton-Jacobi and Helmholtz equations, SIAM J. Math. Anal. **11** (1980), 1011–1026. Zbl 0454.70023 MR 0595827

[39] E. G. Kalnins and W. Miller Jr, Intrinsic characterisation of orthogonal R separation for Laplace equations, J. Phys. A **15** (1982), 2699–2709. Zbl 0504.58039 MR 0673477

[40] E. G. Kalnins and W. Miller Jr, The theory of orthogonal R-separation for Helmholtz equations, Adv. Math. **51** (1984), 91–106. MR 0729000

[41] A. Katchalov, Y. Kurylev and M. Lassas, *Inverse boundary spectral problems*, Chapman and Hall/CRC, 2001. Zbl 1037.35098 MR 1889089

[42] C. Kenig and M. Salo, The Calderón problem with partial data on manifolds and applications, Anal. PDE **6** (2013), 2003–2048. Zbl 1335.35301 MR 3198591

[43] C. Kenig and M. Salo, Recent progress in the Calderon problem with partial data, Contemp. Math. **615** (2014), 193–222. Zbl 1330.35534 MR 3221605

[44] T. H. Koornwinder, A precise definition of separation of variables, in *Proc. Fourth Scheveningen Conf., Scheveningen (1979)*, Lecture Notes in Mathematics 810, Springer, Berlin,1980, 240–263. Zbl 0459.35005 MR 0589212

[45] A. Kostenko, A. Sakhnovich and G. Teschl, Weyl-Titchmarsh theory for Schrödinger operators with strongly singular potentials, Int. Math. Res. Not. **2012** (2012), 1699–1747. Zbl 1248.34027 MR 2920828

[46] M. Lassas, M. Taylor and G. Uhlmann, The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary, Comm. Anal. Geom. **11** (2003), 207–221. Zbl 1077.58012 MR 2014876

[47] M. Lassas and G. Uhlmann, On determining a Riemannian manifold from the Dirichlet-to-Neumann map, Ann. Sci. École Norm. Sup. (4) **34** (2001), 771–787. Zbl 0992.35120 MR 1862026

[48] J. M. Lee and G. Uhlmann, Determining anisotropic real-analytic conductivities by boundary measurements, Comm. Pure Appl. Math. **42** (1989), 1097–1112. Zbl 0702.35036 MR 1029119

[49] T. Levi-Civita, Sulla integrazione della equazione di Hamilton-Jacobi per separazione di variabili, Math. Ann. **59** (1904), 383–397. Zbl 35.0362.02 MR 1511275

[50] L. Marazzi, Inverse scattering on conformally compact manifolds, Inverse Probl. Imaging **3** (2009), 537–550. Zbl 1183.35270 MR 2557918

[51] R. Mazzeo and R. B. Melrose, Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature, J. Funct. Anal. **75** (1987), 260–310. Zbl 0636.58034 MR 0916753

[52] R. B. Melrose, *Geometric scattering theory*, Cambridge University Press, Cambridge, 1995. Zbl 0849.58071 MR 1350074

[53] W. Miller Jr, Mechanisms for variable separation in partial differential equations and their relationship to group theory, in *Symmetries and nonlinear phenomena*, CIF Series 9, 1988, 188–221. Zbl 0788.35017 MR 1119857

[54] R. G. Newton, *Scattering theory of waves and particles*, reprint of the 1982 2nd edition (Springer, New York) with list of errata prepared for this edition by the author, Dover Publications, Mineola, 2002. Zbl 1079.81001 MR 1947260

[55] A. G. Ramm, An inverse scattering problem with part of the fixed-energy phase shifts, Comm. Math. Phys. **207** (1999), 231–247. Zbl 0943.35106 MR 1724847

[56] T. Regge, *Introduction to complex orbital momenta*, Nuevo Cimento XIV(5) **14** (1959), 951–976. Zbl 0087.42903 MR 0143532

[57] H. P. Robertson, Bemerkung über separierbare Systeme in der Wellenmechanik, Math. Ann. **98** (1928), 749–752. Zbl 54.0963.05 MR 1512435

[58] A. Sá Barreto, Radiation fields, scattering, and inverse scattering on asymptotically hyperbolic manifolds, Duke Math. J. **129** (2005), 407–480. Zbl 1154.58310 MR 2169870

[59] M. Salo, The Calderón problem on Riemannian manifolds, in *Inverse problems and applications: Inside out. II*, Mathematical Sciences Research Institute Publications 60, Cambridge University Press, Cambridge, 2013, 167–247. Zbl 1316.35301 MR 3098658

[60] P. Stäckel, Über die integration der Hamilton-Jacobischen differentialgleichung mittelst separation der variabel, Habilitationsschirft, PhD thesis, Halle, 1891.

[61] P. Stäckel, Ueber die Bewegung eines Punktes in einer $n$-fachen Mannigfaltigkeit, Math. Ann. **42** (1893), 537–563. Zbl 25.1380.01 MR 1510794

[62] G. Uhlmann, Electrical impedance tomography and Calderon's problem, Inverse Problems **25** (2009), 123011. Zbl 1181.35339 MR 3460047