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On the Cauchy Problem for Differential Operators with Double Characteristics, A Transition from Non-effective to Effective Characteristics

by

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Abstract

We discuss the well-posedness of the Cauchy problem for hyperbolic operators with double characteristics which changes from non-effectively hyperbolic to effectively hyperbolic, on the double characterisitic manifold, across a submanifold of codimension 1. We assume that there is no bicharacteristic tangent to the double characteristic manifold and the spatial dimension is 2. Then we prove the well-posedness of the Cauchy problem in all Gevrey classes assuming, on the double characteristic manifold, that the ratio of the imaginary part of the subprincipal symbol to the real eigenvalue of the Hamilton map is bounded and that the sum of the real part of the subprincipal symbol and the modulus of the imaginary eigenvalue of the Hamilton map is strictly positive.

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§1. Introduction

This paper is a continuation of our previous papers [N3, N4]. Let

$$P(x, D) = -D_0^2 + A_1(x, D')D_0 + A_2(x, D')$$

be a differential operator of order 2 in D_0 with coefficients $A_j(x, D')$, classical pseudodifferential operator of order j on $\mathbb{R}^n$ depending smoothly on x_0 where $x = (x_0, x') = (x_0, x_1, \ldots, x_n)$. We put $P(x_0, x', \xi_0, \xi') = p(x, \xi) + P_1(x, \xi) + P_0(x, \xi)$, where p, P_1 and P_0, respectively, are the principal symbol, the first-order and the zeroth-order parts of the symbol of $P(x, D)$. We assume that the principal symbol

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$p(x,\xi)$ of $P(x,D)$ vanishes exactly of order 2 on a C^∞ manifold Σ and

$$\text{rank}\left(\sum_{j=0}^{n} d\xi_j \wedge dx_j\big|_{\Sigma}\right) = \text{constant}. \tag{1.1}$$

As in [N3, N4] we assume that $\operatorname{codim}\Sigma = 3$ and

$$\begin{cases} \text{the spectral structure of } F_p \text{ changes simply} \\ \text{across a submanifold } S \text{ of codimension 1 of } \Sigma. \end{cases} \tag{1.2}$$

By conjugation with a Fourier integral operator one can assume $A_1 = 0$, and then, near any point $\rho \in \Sigma$, one can write

$$p(x,\xi) = -\xi_0^2 + \phi_1(x,\xi')^2 + \phi_2(x,\xi')^2$$

where $d\phi_1$ and $d\phi_2$ are linearly independent at ρ and $\Sigma = \{\xi_0 = 0, \phi_1 = 0, \phi_2 = 0\}$. Under assumptions (1.1) and (1.2) without restrictions we can assume (see [N3])

$$\{\xi_0, \phi_2\} > 0, \quad \{\xi_0, \phi_1\} = O(|(\phi_1, \phi_2)|)$$

near ρ. Here and in what follows $f = O(|(\phi_1,\phi_2)|)$ means that f is a linear combination of ϕ_1 and ϕ_2 near the reference point. We first recall the following lemma.

Lemma 1.1 ([N4, Lem. 1.2]). *If the spectral structure of F_p changes across S then we have $\{\xi_0,\phi_2\}^2 - \{\phi_1,\phi_2\}^2 = 0$ on S and one of the following cases occurs:*

(i) $\{\xi_0,\phi_2\}^2 - \{\phi_1,\phi_2\}^2 < 0$ in $\Sigma \setminus S$ so that p is non-effectively hyperbolic in Σ with $\operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2 = \{0\}$ in $\Sigma \setminus S$ and $\operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2 \neq \{0\}$ on S,

(ii) $\{\xi_0,\phi_2\}^2 - \{\phi_1,\phi_2\}^2 > 0$ in $\Sigma \setminus S$ so that p is effectively hyperbolic in $\Sigma \setminus S$ and non-effectively hyperbolic on S with $\operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2 \neq \{0\}$,

(iii) $\{\xi_0,\phi_2\}^2 - \{\phi_1,\phi_2\}^2$ changes the sign across S, that is, p is non-effectively hyperbolic on one side of $\Sigma \setminus S$ with $\operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2 = \{0\}$, non-effectively hyperbolic on S with $\operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2 \neq \{0\}$ and effectively hyperbolic on the other side.

Let us denote

$$\Sigma^{\pm} = \{(x,\xi) \in \Sigma \mid \pm(\{\xi_0,\phi_2\}^2 - \{\phi_1,\phi_2\}^2) > 0\}.$$

The eigenvalues of F_p are 0 and $\pm\sqrt{\{\xi_0,\phi_2\}^2 - \{\phi_1,\phi_2\}^2}$ on Σ so that F_p has non-zero real eigenvalues on Σ^+ and non-zero purely imaginary eigenvalues on Σ^- in case (iii). Let us set

$$2\kappa(\rho)^2 = |\{\xi_0,\phi_2\}^2 - \{\phi_1,\phi_2\}^2|$$

and we make precise the meaning of "simply" in (1.2), namely we assume that

(1.3) $\kappa(\rho) \approx \mathrm{dist}_\Sigma(\rho, S)$ (case (i) or case (ii)), $\kappa^2(\rho) \approx \mathrm{dist}_\Sigma(\rho, S)$ (case (iii))

on Σ where $\mathrm{dist}_\Sigma(\rho, S)$ denotes the distance from ρ to S on Σ. Our aim in this paper is to complete the proof of the following result:

Theorem 1.2. *Assume* (1.2) *and that there is no bicharacteristic tangent to* Σ *and that there exist* $\epsilon > 0$, $C > 0$ *such that*

(1.4) $(1-\epsilon)\mu(\rho) + \mathsf{Re}\, P_{\rm sub}(\rho) \geq \epsilon, \quad |\,\mathsf{Im}\, P_{\rm sub}(\rho)| \leq Ce(\rho), \quad \rho \in \Sigma \cap \{|\xi| = 1\}$

where $\pm e(\rho)$ $(e(\rho) \geq 0)$ *are real eigenvalues and* $\pm i\mu(\rho)$ $(\mu(\rho) \geq 0)$ *are purely imaginary eigenvalues of* $F_p(\rho)$*. We also assume* $n = 2$ *in case (iii). Then the Cauchy problem for* P *is well posed in any Gevrey class* $\gamma^{(s)}$ *for* $s > 1$*.*

Case (i), namely $e(\rho) \equiv 0$ on Σ: Theorem 1.2 was proved in [BPP] while in [N3], it was proved under a less restrictive assumption of the non-existence of bicharacteristics tangent to S. Case (ii) and hence $\mu(\rho) \equiv 0$ on Σ: Theorem 1.2 was proved in [N4]. Some transition cases from non-effectively hyperbolic to effectively hyperbolic are studied in [BB, BE, E]. In particular in [BE, E] a typical case of (iii) was studied but condition (1.4) was not investigated. In this paper we give a proof of Theorem 1.2 for case (iii) assuming $n = 2$, while if $n = 1$ the case $\mathrm{Ker}\, F_p^2 \cap \mathrm{Im}\, F_p^2 \neq \{0\}$ never occurs.

Remark 1.3. For differential operators, condition (1.4) with $\epsilon = 0$ can be expressed as

$$\mathrm{dist}_{\mathbb{C}}\left(P_{\rm sub}(\rho), [-\mu(\rho), \mu(\rho)]\right) \leq Ce(\rho), \quad \rho \in \Sigma,$$

which generalizes the Ivrii–Petkov–Hörmander condition ([IP, H1]) and Melrose conjectured in [Me] that this condition is necessary for the Cauchy problem to be C^∞ well posed, but little is known about necessary conditions for well-posedness when the spectral structure of F_p changes.

Remark 1.4. With $X^\pm = \{\xi_0, \phi_2\}H_{\xi_0} - \{\phi_1, \phi_2\}H_{\phi_1} \pm \sqrt{2}\kappa(\rho)H_{\phi_2}$ it is easy to see that

$$F_p(\rho)X^\pm = \pm e(\rho)X^\pm, \quad \rho \in \Sigma^+$$

and that there exist exactly two bicharacteristics passing ρ transversally to Σ^+ with tangents $X^\pm$ (see [KoN]). We note that the surface $\phi_2 = 0$ is spacelike on Σ^+ because $d\phi_2(X^\pm) = \{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 = 2\kappa(\rho)^2 > 0$. On the other hand there is no bicharacteristic reaching Σ^-(see [Iv1]).

§2. Idea of the proof of Theorem 1.2

From Lemma 1.1 we have $\mathrm{Ker}\, F_p^2 \cap \mathrm{Im}\, F_p^2 \neq \{0\}$ on Σ^- and there is no bicharacteristic tangent to Σ^- by assumption. Then thanks to [N2, Thm. 3.3] p admits an elementary decomposition microlocally at every point on Σ^-. As in [BPP, N3] we try to decompose $p = -(\xi_0 + \phi_1 - \psi)(\xi_0 - \phi_1 + \psi) + q$ with $\psi = o(|\phi_1|)$ and non-negative q verifying $\{\xi_0 - \phi_1 + \psi, q\} = O(q)$ in Σ^-. These requirements essentially determine ψ and actually the non-existence of a tangent bicharacteristic ensures that $\xi_0 - \psi_1 + \psi$ commutes against q better than the usual case. On the other hand, as checked in Remark 1.4, the surface $\hat{\phi}_2 = 0$ is spacelike on Σ^+; then [N1, N4] suggest the use of a pseudodifferential weight $T \approx e^{\zeta \log \hat{\phi}_2}$ where ζ is a cutoff symbol to Σ^+. Our strategy for proving Theorem 1.2 is rather naive so that we make such a decomposition and derive weighted energy estimates with the cutoff weight T. But the decomposition should be compatible with the cutoff weights and to achieve this goal we must be careful in choosing cutoff symbols and in estimating errors caused by them. The assumption $n = 2$ enables us to choose all symbols that we need, including cutoff symbols, in $S_{3/4,1/2}$ and we carry out a pseudodifferential calculus within the framework of $S_{3/4,1/2}$ though we often need the calculus in smaller and more specific classes than $S_{3/4,1/2}$.

In the rest of this section we rewrite the assumptions in more explicit forms. In what follows we assume $n = 2$ and we work in a conic neighborhood of $\bar{\rho} \in S$. Without restrictions we may assume $\bar{\rho} = (0, \mathbf{e}_3)$, $\mathbf{e}_3 = (0, 0, 1) \in \mathbb{R}^3$ with a system of local coordinates $x = (x_0, x') = (x_0, x_1, x_2)$. From (1.3) and Lemma 1.1 one can write

$$\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 = \theta|\xi'| + c_1\phi_1 + c_2\phi_2 \tag{2.1}$$

in a neighborhood of $\bar{\rho}$ where S is defined by $\{\theta = 0\} \cap \Sigma$ and $d\theta \neq 0$ on S and hence $\Sigma^{\pm} = \Sigma \cap \{\pm\theta > 0\}$. Compare this to cases (i) and (ii) where we have $\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 = \mp\theta^2 + c_1\phi_1 + c_2\phi_2$ respectively ([N3, N4]). Here note that

$$e(\rho) = \begin{cases} \sqrt{2}\kappa(\rho), & \rho \in \Sigma^+, \\ 0, & \rho \in \Sigma^-, \end{cases} \qquad \mu(\rho) = \begin{cases} 0, & \rho \in \Sigma^+, \\ \sqrt{2}\kappa(\rho), & \rho \in \Sigma^-. \end{cases}$$

Since $\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 = \{\xi_0 - \phi_1, \phi_2\}\{\xi_0 + \phi_1, \phi_2\} = 0$ on S we may assume without restrictions that

$$\{\xi_0 - \phi_1, \phi_2\} = 0 \quad \text{on } S \tag{2.2}$$

and $\{\xi_0, \phi_2\} = \{\phi_1, \phi_2\} > 0$ on S (see [N3, N4]).

Lemma 2.1. *In a conic neighborhood of* $\bar\rho' = (0, \mathbf{e}_2)$ *one can assume that*

$$\phi_2(x,\xi') = \hat\phi_2(x)e(x,\xi'), \qquad \theta(x,\xi')|\xi_2|^{-1} = \psi(x') + f(x,\xi')\hat\phi_2(x),$$

where $0 \neq e(x,\xi') \in S^1_{1,0}$ *and* $f(x,\xi') \in S^0_{1,0}$. *Moreover we have* $\{\theta,\phi_j\} = c_j\phi_2$ *with* $c_j \in S^0_{1,0}$.

Proof. Since $\{\xi_0,\phi_2\} \neq 0$ then one can write $\phi_2 = (x_0 - \psi_2(x,\xi'))b_2$ where ψ_2 is independent of x_0 and $b_2 \neq 0$. From $\{\phi_1,\phi_2\} \neq 0$ we see $\{\psi_2,\phi_1\} \neq 0$. This shows that $d\psi_2$ is not proportional to $\sum_{j=0}^2 \xi_j\, dx_j$ at $\bar\rho$ because otherwise we would have $\phi_1(0,\mathbf{e}_2) = \partial\phi_1(0,\mathbf{e}_2)/\partial\xi_2 \neq 0$. Since $\Xi_0 = \xi_0$, $X_0 = x_0$, $X_1 = \psi_2$ verifies the commutation relations and $d\Xi_0$, dX_0, dX_1, $\sum_{j=0}^2 \xi_j\, dx_j$ are linearly independent at $\bar\rho$, as just observed above, these coordinates extend to homogeneous symplectic coordinates (X,Ξ) (see [H1, Thm. 21.1.9]). Switching the notation to (x,ξ) we can assume that $\phi_2 = (x_0 - x_1)e$. Since $\{\phi_2,\phi_1\} \neq 0$ one can write $\phi_1 = (\xi_1 - \psi_1)b_1$ where ψ_1 is independent of ξ_0 and ξ_1. Writing $\psi_1(x,\xi_2) = \bar\psi_1(x',\xi_2) + e_1\phi_2$ and $\theta(x,\xi') = \tilde\theta(x',\xi_2) + (x_0-x_1)\theta_1 + (\xi_1-\bar\psi_1)\theta_2$ so that S is given by $\xi_0 = 0$, $x_0 - x_1 = 0$, $\xi_1 - \bar\psi_1(x',\xi_2) = 0$, $\tilde\theta(x',\xi_2) = 0$ where $\tilde\theta(x',\xi_2) = \theta(x_1,x_1,x_2,\bar\psi_1(x',\xi_2),\xi_2)$. Since $\tilde\theta$ is homogeneous of degree 1 in ξ_2 one can write

$$\tilde\theta(x',\xi_2) = \tilde\theta(x',1)\xi_2 = \psi(x')\xi_2$$

in a conic neighborhood of $(0,\mathbf{e}_2)$, *where we have used the assumption* $n = 2$. Let us set $\theta = \psi(x')\xi_2 + (\{\psi(x')\xi_2,\phi_1\}/\{\phi_1,\phi_2\})\phi_2$; then it is clear that $\{\theta,\phi_j\} = c_j\phi_2$ and hence this θ is the desired one. □

Remark 2.2. Since the restriction $n = 2$ is used only to prove Lemma 2.1 then Theorem 1.2 is still true if we can choose homogeneous symplectic coordinates such that Lemma 2.1 holds.

We now assume that ϕ_2 and θ satisfy Lemma 2.1 and set

$$\hat\theta = \theta|\xi_2|^{-1}, \quad \hat\phi_1 = \phi_1|\xi'|^{-1}$$

so that $\hat\theta$ and $\hat\phi_1$ are homogeneous of degree 0 in ξ'. From (2.2) we can write

$$\{\xi_0 - \phi_1, \hat\phi_2\} = \hat c\,\hat\theta + c_1'\hat\phi_1 + c_2'\hat\phi_2 \tag{2.3}$$

near $\bar\rho$ where $\hat c > 0$, which follows from (2.1). Since we have $\{\xi_0+\phi_1,\phi_2\}|\hat c\,\hat\theta||e| = 2\kappa^2$ on Σ and $\{\xi_0+\phi_1,\phi_2\}/2\{\phi_1,\phi_2\} = 1$ on S then for any $\epsilon > 0$ there is a neighborhood of $\bar\rho$ where we have

$$(1-\epsilon)\kappa^2(\rho) \le \{\phi_1,\phi_2\}|\hat c\,\hat\theta||e| \le (1+\epsilon)\kappa^2(\rho). \tag{2.4}$$

Here we examine how the non-existence of tangent bicharacteristics reflects on the Poisson brackets of symbols.

Proposition 2.3. *[N4, Prop. 2.1] Assume* $\{\theta,\phi_j\}=O(|(\phi_1,\phi_2)|)$ *and that there is no bicharacteristic tangent to* Σ*. Then we have*

$$\{\xi_0,\theta\}(\rho)=0,\quad \{\{\xi_0-\phi_1,\phi_2\},\phi_2\}(\rho)=0,\quad \rho\in S.$$

Lemma 2.4. *Assume that* $\{\{\xi_0-\phi_1,\phi_2\},\phi_2\}=0$ *on* S*. Then one can write* $\{\xi_0-\phi_1,\hat{\phi}_2\}=\hat{c}\,\hat{\theta}+c_0\hat{\theta}\hat{\phi}_1+c_1\hat{\phi}_1^2+c_2\hat{\phi}_2$*.*

Lemma 2.5. *Assume that* $\{\xi_0,\hat{\theta}\}=0$*,* $\{\{\xi_0-\phi_1,\phi_2\},\phi_2\}=0$ *on* S*. Then we have* $\{\xi_0-\phi_1,\hat{\theta}\}=c_0\hat{\theta}+c_1\hat{\phi}_1^2+c_2\hat{\phi}_2$*.*

Proof. Note that $\{\xi_0-\phi_1,\hat{\theta}\}=\alpha\hat{\theta}+\beta\hat{\phi}_1+\gamma\hat{\phi}_2$. On the other hand we see

$$\{\hat{\theta},\{\xi_0-\phi_1,\phi_2\}\}=O(|\hat{\phi}|),\quad \{\xi_0-\phi_1,\{\hat{\theta},\phi_2\}\}=O(|(\hat{\theta},\hat{\phi})|).$$

Then from the Jacobi identity it follows that $\beta=O(|(\hat{\theta},\hat{\phi})|)$ and hence we have $\{\xi_0-\phi_1,\hat{\theta}\}=\alpha\hat{\theta}+c_0\hat{\theta}\hat{\phi}_1+c_1\hat{\phi}_1^2+c_2\hat{\phi}_2$ which proves the assertion. □

Corollary 2.6. *We have* $\{\xi_0,\hat{\theta}\}=c_0\hat{\theta}+c_1\hat{\phi}_1^2+c_2\hat{\phi}_2$*.*

§3. Cutoff and weight symbols

We use the same notation as in [N4]. We first make a dilation of the coordinate x_0: $x_0\to\mu x_0$ with small $\mu>0$ so that

$$\begin{aligned}P(x,\xi,\mu)&=\mu^2P(\mu x_0,x',\mu^{-1}\xi_0,\xi')\\&=p(\mu x_0,x',\xi_0,\mu\xi')+\mu P_1(\mu x_0,x',\xi_0,\xi')+\mu^2P_0(\mu x_0,x')\\&=:p(x,\xi,\mu)+P_1(x,\xi,\mu)+P_0(x,\mu).\end{aligned}$$

In what follows we often express such symbols dropping μ. It is easy to see that $a(\mu x_0,x',\mu\xi')=a(x,\xi',\mu)\in S(\langle\mu\xi'\rangle^m,g_0)$ if $a(x,\xi')\in S^m_{1,0}$ where

$$g_0=|dx|^2+\langle\xi'\rangle_\mu^{-2}|d\xi'|^2,\quad \langle\xi'\rangle_\mu=(\mu^{-2}+|\xi'|^2)^{1/2}=\mu^{-1}\langle\mu\xi'\rangle.$$

To prove the well-posedness of the Cauchy problem, applying [N5, Thm. 1.1], it suffices to derive energy estimates for $P_{\xi'}$, which coincides with the original P in a conic neighborhood of $(0,0,\xi')$, $|\xi'|=1$. Thus we can assume that the following conditions are satisfied globally:

$$
(3.1)\qquad
\begin{cases}
p(x,\xi) = -\xi_0^2 + \phi_1(x,\xi')^2 + \phi_2(x,\xi')^2, & \phi_j \in S(\langle \mu\xi'\rangle, g_0),\\
\{\xi_0, \phi_1\} = d_1\phi_1 + d_2\phi_2, & d_j \in \mu S(1, g_0),\\
\{\xi_0 - \phi_1, \hat{\phi}_2\} = \mu\hat{c}\,\hat{\theta} + c_0\hat{\theta}\hat{\phi}_1 + c_1\hat{\phi}_1^2 + c_2\hat{\phi}_2, & \hat{c} > 0,\ \hat{c} \in S(1, g_0),\\
\{\xi_0, \hat{\theta}\} = c_0'\hat{\theta} + c_1'\hat{\phi}_1^2 + c_2'\hat{\phi}_2, & \\
\{\phi_1, \hat{\phi}_2\} \geq c\mu, & c > 0,
\end{cases}
$$

where c_j, $c_j' \in \mu S(1, g_0)$ and $\hat{\theta} \in S(1, g_0)$ verifies

$$
(3.2)\qquad \{\hat{\theta}, \phi_j\} = c_j\hat{\phi}_2, \quad c_j \in \mu S(1, g_0)
$$

and $\sup|\hat{\theta}|$, $\sup|\hat{\phi}_j|$ can be assumed to be sufficiently small, shrinking a conic neighborhood of $\bar{\rho}'$ where we are working.

Let us put $P_{\mathrm{sub}} = P_1^s + iP_2^s$ with real $P_i^s \in \mu S(\langle\mu\xi'\rangle, g_0)$. Then from (1.4) and (2.4) the following conditions can be assumed to be satisfied globally:

$$
(3.3)\qquad
\begin{cases}
\mu^{1/2}\sqrt{\hat{c}\,\{\phi_1, \hat{\phi}_2\}|\hat{\theta}|}|e| + P_1^s \geq c\mu\langle\mu\xi'\rangle & \text{in } \hat{\theta} < 0,\\
P_1^s \geq c\mu\langle\mu\xi'\rangle & \text{in } \hat{\theta} > 0,\\
P_2^s = \mu c_0\hat{\theta}\langle\mu\xi'\rangle + c_{11}\phi_1 + c_{12}\phi_2 & (c_0 = 0 \text{ for } \hat{\theta} < 0),
\end{cases}
$$

with a constant $c > 0$ and $c_0 \in S(1, g_0)$ and $c_{ij} \in \mu S(1, g_0)$. Recall from [N4] that

$$
\begin{cases}
\phi = \langle\xi'\rangle_\mu^{1/2}\big(\hat{\phi}_2 + w\big),\\
\Phi = \pi + i\big\{\log(\hat{\phi}_2 + i\omega) - \log(\hat{\phi}_2 - i\omega)\big\} = \pi - 2\arg(\hat{\phi}_2 + i\omega),\\
w = (\hat{\phi}_2^2 + \langle\xi'\rangle_\mu^{-1})^{1/2}, \quad \omega = (\hat{\phi}_1^4 + \langle\xi'\rangle_\mu^{-1})^{1/2},\\
\rho^2 = \hat{\phi}_2^2 + \omega^2 = \hat{\phi}_2^2 + \hat{\phi}_1^4 + \langle\xi'\rangle_\mu^{-1} \geq (w^2 + \omega^2)/2,
\end{cases}
$$

where ϕ plays a major role in our arguments and Φ is introduced in order to manage the energy estimates in the region $C\hat{\phi}_1^2 \geq w$. Note that

$$
(3.4)\qquad \{F, \Phi\} = 2(\omega\{F, \hat{\phi}_2\} - \hat{\phi}_2\{F, \omega\})/\rho^2.
$$

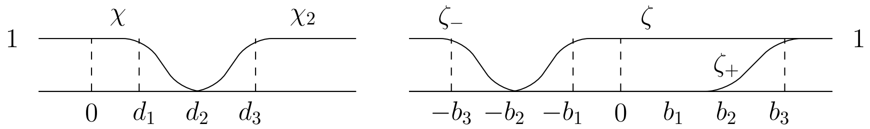
We use the following metrics:

$$
\begin{cases}
g = w^{-2}|dx|^2 + w^{-1}\langle\xi'\rangle_\mu^{-2}|d\xi'|^2,\\
g_1 = (\rho^{-1} + \omega^{-1/2})^2|dx|^2 + \omega^{-1}\langle\xi'\rangle_\mu^{-2}|d\xi'|^2,\\
\tilde{g} = (w^{-1} + \omega^{-1/2})^2|dx|^2 + \langle\xi'\rangle_\mu^{-3/2}|d\xi'|^2,\\
\bar{g} = \langle\xi'\rangle_\mu^{-1}|dx|^2 + \langle\xi'\rangle_\mu^{-3/2}|d\xi'|^2.
\end{cases}
$$

Note that $g,\, g_1 \le \tilde{g} \le 4\bar{g}$ and $\bar{g}$ is the metric defining the class $S_{3/4,1/2}$ for any fixed $\mu > 0$. As checked in [N4] we have $\omega \in S(\omega, g_1)$, $\rho \in S(\rho, g_1)$ and $\Phi \in S(1, g_1)$. With a cutoff symbol $\zeta(x, \xi') = \zeta(\hat{\theta} w^{-1})$ we define the weight

$$T = \exp\left(n\zeta^2(\chi^2 \log \phi + \Phi)\right), \tag{3.5}$$

where $\chi = \chi(\hat{\phi}_1^2 w^{-1})$ and $\zeta(s) = 1$ in $s \ge -b_1$ and $\zeta(s) = 0$ in $s \le -b_2$ with $\zeta'(s) \ge 0$ and n is a positive parameter.



Let $\zeta_{\pm}(x, \xi') = \zeta_{\pm}(\hat{\theta} w^{-1})$ and $\chi_2(x, \xi') = \chi_2(\hat{\phi}_1^2 w^{-1})$ where $\zeta_{\pm}(s) = 1$ in $\pm s \ge b_3$ and 0 in $\pm s \le b_2$ so that $\zeta\zeta_+ = \zeta_+$ and $\zeta\zeta_- = 0$. We simply write χ, χ_2 for $\chi(x, \xi')$ and $\chi_2(x, \xi')$ and ζ, $\zeta_{\pm}$ for $\zeta(x, \xi')$ and $\zeta_{\pm}(x, \xi')$ if there is no confusion. It is easy to check $\chi,\, \chi_2 \in S(1, g)$. As for cutoff symbols ζ, $\zeta_{\pm}$ we have the following lemma.

Lemma 3.1. *Let* $G = w^{-2}|dx|^2 + \langle \xi' \rangle_{\mu}^{-2} |d\xi'|^2 \ (\le g)$, *then* $w \in S(w, G)$ *and* $\phi \in S(\phi, G)$. *We have also* $\zeta, \zeta_{\pm} \in S(1, G)$. *Let* $s \in \mathbb{R}$, *then* $\zeta_+ \hat{\theta}^s \in S(|\hat{\theta}|^s, G)$. *Moreover if* $0 < s \le 1$ *and* $|\alpha| \neq 0$ *we have* $|(\zeta_+ \hat{\theta}^s)^{(\alpha)}_{(\beta)}| \le C_{\alpha\beta} w^s \langle \xi' \rangle_{\mu}^{-|\alpha|} w^{-|\beta|}$.

Proof. To prove $\phi \in S(\phi, G)$, with $\tilde{\phi} = \hat{\phi}_2 + w$, it is enough to show $\tilde{\phi} \in S(\tilde{\phi}, G)$. Note that one can write

$$\partial_x^{\beta} \partial_{\xi'}^{\alpha} \tilde{\phi} = \frac{\partial_x^{\beta} \partial_{\xi'}^{\alpha} \hat{\phi}_2(x)}{w} \tilde{\phi} + \frac{\partial_x^{\beta} \partial_{\xi'}^{\alpha} \langle \xi' \rangle_{\mu}^{-1}}{2w} = b_{\alpha\beta} \tilde{\phi} + a_{\alpha\beta}$$

with $b_{\alpha\beta} \in S(w^{-|\beta|} \langle \xi' \rangle_{\mu}^{-|\alpha|}, G)$ and $a_{\alpha\beta} \in S((w^{-1} \langle \xi' \rangle_{\mu}^{-1}) w^{-|\beta|} \langle \xi' \rangle_{\mu}^{-|\alpha|}, G)$ for $|\alpha + \beta| = 1$. By induction on $|\alpha + \beta|$ we see easily $\partial_x^{\beta} \partial_{\xi'}^{\alpha} \tilde{\phi} = b_{\alpha\beta} \tilde{\phi} + a_{\alpha\beta}$ with $b_{\alpha\beta} \in S(w^{-|\beta|} \langle \xi' \rangle_{\mu}^{-|\alpha|}, G)$ and $a_{\alpha\beta} \in S((w^{-1} \langle \xi' \rangle_{\mu}^{-1}) w^{-|\beta|} \langle \xi' \rangle_{\mu}^{-|\alpha|}, G)$ for any α, β. Since $w^{-1} \langle \xi' \rangle_{\mu}^{-1} \le 2\tilde{\phi}$ we get the assertion. To prove $\zeta \in S(1, G)$ it suffices to show

$$|\zeta' \partial_x^{\beta} \partial_{\xi'}^{\alpha} (\hat{\theta} w^{-1})| \le C_{\alpha\beta} w^{-|\beta|} \langle \xi' \rangle_{\mu}^{-|\alpha|}. \tag{3.6}$$

By Lemma 2.1 without restrictions we may assume $\hat{\theta}(x, \xi') = \psi(x') + f(x, \xi') \hat{\phi}_2(x)$ from which it follows $|\partial_{\xi'}^{\alpha} \hat{\theta}| \le C_{\alpha} \langle \xi' \rangle_{\mu}^{-|\alpha|} w$ for $|\alpha| \ge 1$. Noting $|\zeta' \hat{\theta} w^{-1}| \le C$ we get (3.6). On the support of ζ_+ the estimate

$$|(\hat{\theta}^s)^{(\alpha)}_{(\beta)}| \le \sum C_{\alpha_1, \ldots, \beta_k} \hat{\theta}^s |\hat{\theta}^{(\alpha_1)}_{(\beta_1)}| \hat{\theta}^{-1} \cdots |\hat{\theta}^{(\alpha_k)}_{(\beta_k)}| \hat{\theta}^{-1}$$

holds where $|\alpha_i + \beta_i| \ge 1$ and $\alpha_1 + \cdots + \alpha_k = \alpha$, $\beta_1 + \cdots + \beta_k = \beta$. On the other hand Lemma 2.1 shows that $|\hat{\theta}^{(\alpha_i)}_{(\beta_i)}| \le C_{\alpha_i \beta_i} \langle \xi' \rangle_{\mu}^{-|\alpha_i|} w^{1-|\beta_i|}$ if $|\alpha_i| \neq 0$ and bounded by C_{β_i} if $|\alpha_i|=0$. Since $\hat{\theta}^{-1}w$ is bounded on the support of ζ_+ the third assertion is clear. If $|\alpha_i|\neq 0$ then noting $\hat{\theta}^s|\hat{\theta}^{(\alpha_i)}_{(\beta_i)}\hat{\theta}^{-1}|\leq C_{\alpha_i\beta_i}w^s\langle\xi'\rangle_\mu^{-|\alpha_i|}w^{-|\beta_i|}$ on the support of ζ_+ one gets the last assertion. □

Remark 3.2. If $n>2$ the $\psi(x')$ in Lemma 2.1 would depend on ξ' also and hence ζ, $\zeta_\pm$ do not belong to $S(1,g)$ in general.

To factorize p let us define

$$\psi=(-h\zeta_-^2+\nu\zeta_+^2)\hat{\theta}\phi_1+\chi_2\phi_1^3\langle\mu\xi'\rangle^{-2}=\tilde{\zeta}\hat{\theta}\phi_1+\chi_2\phi_1^3\langle\mu\xi'\rangle^{-2}, \tag{3.7}$$

with a positive parameter $0<\nu\ll 1$ which will be determined later, where $\tilde{\zeta}=-h\zeta_-^2+\nu\zeta_+^2$ with $h=\mu\hat{c}\{\phi_1,\hat{\phi}_2\}^{-1}>0$. Using ψ we rewrite p as

$$\begin{aligned}p&=-(\xi_0+\phi_1-\psi)(\xi_0-\phi_1+\psi)+2\psi\phi_1-\psi^2+\phi_2^2\\&=-(\xi_0+\phi_1-\psi)(\xi_0-\phi_1+\psi)+q,\end{aligned} \tag{3.8}$$

where

$$\begin{cases}q=\phi_2^2+2a^2\tilde{\zeta}\hat{\theta}\phi_1^2+2a^2\chi_2\phi_1^4\langle\mu\xi'\rangle^{-2},\\ a=(1-\tilde{\zeta}\hat{\theta}/2-\chi_2\phi_1^2\langle\mu\xi'\rangle^{-2}/2)^{1/2}.\end{cases}$$

The main part of $\{\xi_0-\phi_1+\psi,q\}$ will be $\{\xi_0-\phi_1+\psi,\phi_2^2\}$ which is required to be $O(q)$ in $\theta<0$ as explained above. Indeed, by our choice of ψ, we have

$$\begin{aligned}\{\xi_0-\phi_1+\psi,\hat{\phi}_2\}&=\mu(1-\zeta_-^2)\hat{c}\hat{\theta}+\mu\nu\hat{h}^{-1}\hat{c}\zeta_+^2\hat{\theta}\\&\quad+c_1\hat{\phi}_1^2+c_2\hat{\theta}\hat{\phi}_1+c_3\hat{\phi}_2,\end{aligned} \tag{3.9}$$

where $1-\zeta_-^2=0$ in $\hat{\theta}\leq -b_3w$ so that $|(1-\zeta_-^2)\hat{\theta}|\leq Cw$ in $\hat{\theta}\leq 0$.

Lemma 3.3. *We have $(\tilde{\zeta}\hat{\theta})^{(\alpha)}_{(\beta)}$, $(\chi_2\hat{\phi}_1^2)^{(\alpha)}_{(\beta)}\in S(\langle\xi'\rangle_\mu^{-|\alpha|},g)$ for $|\alpha+\beta|=1$. Hence the same holds for $a^{(\alpha)}_{(\beta)}$. In particular $|(\tilde{\zeta}\hat{\theta})^{(\alpha)}_{(\beta)}|$, $|(\chi_2\hat{\phi}_1^2)^{(\alpha)}_{(\beta)}|$ and $|a^{(\alpha)}_{(\beta)}|$ are bounded by $C_{\alpha\beta}w^{1/2}\langle\xi'\rangle_\mu^{-|\alpha|}w^{-|\alpha|/2-|\beta|}$ for $|\alpha+\beta|\geq 1$ and bounded by $C_{\alpha\beta}w\langle\xi'\rangle_\mu^{-|\alpha|}w^{-|\alpha|/2-|\beta|}$ for $|\alpha+\beta|\geq 2$.*

In this paper $\mathrm{Op}(\phi)$ denotes the Weyl quantized pseudodifferential operator with symbol ϕ and we denote $\mathrm{Op}(\phi)\,\mathrm{Op}(\psi)=\mathrm{Op}(\phi\#\psi)$. We often use the same letter to denote a symbol and the operator with such a symbol if there is no confusion. Thus we denote

$$\mathrm{Op}(\phi\psi)u=\phi\psi u,\quad \mathrm{Op}(\phi)\,\mathrm{Op}(\psi)u=\phi(\psi u).$$

We make some additional preparation (see [Iv2]). Let $c=id_1+ic_{11}$ with d_1, c_{11} in (3.2), (3.3) and we set $M=\xi_0+\phi_1-\psi+c$, $\Lambda=\xi_0-\phi_1+\psi-c$ and write

$$p+P_1^s+iP_2^s=-M\#\Lambda+Q=-M\#\Lambda+q+T_1+iT_2.$$

Note that $-(\xi_0+\phi_1-\psi)(\xi_0-\phi_1+\psi)=-M\Lambda-c\phi_1-2c\psi-c^2$. In view of Lemma 3.3 it is not difficult to check

$$M\#\Lambda = M\Lambda + i\{\xi_0, \phi_1-\psi+c_{11}\} + c_1 w^{1/2}\phi_1 + c_2\hat{\phi}_1^2\langle\mu\xi'\rangle + R$$

with $c_i \in \mu S(1,\bar{g})$ and $R \in \mu^2 S(w^{-1},\bar{g})$. Therefore we see from (3.1) that T_1 satisfies

$$\begin{cases} \mu^{1/2}\sqrt{\hat{c}\,\{\phi_1,\hat{\phi}_2\}|\hat{\theta}|}|e(x,\xi')| + T_1 \geq 2\bar{\kappa}\mu\langle\mu\xi'\rangle \text{ in } \hat{\theta}<0, \\ T_1 \geq 2\bar{\kappa}\mu\langle\mu\xi'\rangle \text{ in } \hat{\theta}>0, \end{cases} \tag{3.10}$$

with some $\bar{\kappa}>0$, and T_2 can be written

$$T_2 = \mu c_0\hat{\theta}\langle\mu\xi'\rangle + b_0\hat{\theta}\phi_1 + b_1\hat{\phi}_1^2\langle\mu\xi'\rangle + b_2\phi_2 + b_3 w^{1/2}\phi_1 \tag{3.11}$$

with $b_i \in \mu S(1,\bar{g})$. Thus $T_2\langle\mu\xi'\rangle^{-1} = O(|(\hat{\theta},\hat{\phi}_1^2,\hat{\phi}_2,w^{1/2}\hat{\phi}_1)|)$ so that we can get rid of the term $O(\hat{\phi}_1)$ in the expression of T_2. We transform P by T so that

$$PT = T\tilde{P}, \quad \tilde{P} = -\tilde{M}\tilde{\Lambda} + \tilde{Q}.$$

To simplify notation we set $\Psi = \zeta^2(\chi^2\log\phi + \Phi)$. Then we have the following lemma.

Lemma 3.4. *We have $T = e^{n\Psi} \in S(e^{n\Psi}, (\log^2\langle\xi'\rangle_\mu)\bar{g})$.*

Proof. Note that $\partial_x^\beta\partial_{\xi'}^\alpha \log\phi = \phi^{-1}\partial_x^\beta\partial_{\xi'}^\alpha\phi$ for $|\alpha+\beta|=1$ and $\phi^{-1}\in S(\phi^{-1},g)$. Since $|\log\phi| \leq C\log\langle\xi'\rangle_\mu$ and g, $g_1 \leq 4\bar{g}$ the assertion is clear. □

Let us write $\tilde{M} = D_0 - \tilde{m}(x,D')$, $\tilde{\Lambda} = D_0 - \tilde{\lambda}(x,D')$ and fix any small $\varepsilon>0$.

Proposition 3.5 ([N3, N4]). *Let $\tilde{P} = -(\tilde{M}-i\gamma\lambda_\mu^{2\epsilon})(\tilde{\Lambda}-i\gamma\lambda_\mu^{2\epsilon})+\tilde{Q}$, then we have*

$$\begin{aligned} 2\,\mathsf{Im}(\tilde{P}u,\tilde{\Lambda}u) \geq{}& \frac{d}{dx_0}(\|\tilde{\Lambda}u\|^2 + ((\mathsf{Re}\,\tilde{Q})u,u) + \gamma^2\|\langle D'\rangle_\mu^{2\varepsilon}u\|^2) \\ &+ \gamma\|\lambda_\mu^\varepsilon(\tilde{\Lambda}u)\|^2 + 2\gamma\,\mathsf{Re}(\lambda_\mu^{2\varepsilon}(\tilde{Q}u),u) + 2((\mathsf{Im}\,\tilde{m})\tilde{\Lambda}u,\tilde{\Lambda}u) \\ &+ 2\,\mathsf{Re}(\tilde{\Lambda}u,(\mathsf{Im}\,\tilde{Q})u) + \mathsf{Im}([D_0-\mathsf{Re}\,\tilde{\lambda},\mathsf{Re}\,\tilde{Q}]u,u) \\ &+ 2\,\mathsf{Re}((\mathsf{Re}\,\tilde{Q})u,(\mathsf{Im}\,\tilde{\lambda})u) + (\gamma^3/2)\|\lambda_\mu^{3\varepsilon}u\|^2 \\ &+ 2\gamma^2(\lambda_\mu^{4\varepsilon}(\mathsf{Im}\,\tilde{\lambda})u,u). \end{aligned} \tag{3.12}$$

In this paper the positive large parameters n, γ and the positive small parameter μ are assumed to satisfy $n\mu^{1/4} \ll 1$ and $\gamma\mu^4 \gg 1$.

Remark 3.6. The weight $\langle\mu D'\rangle^{2\varepsilon}$ is introduced to control error terms $\log^N\langle D'\rangle$, caused by the metric $(\log^2\langle\xi'\rangle_\mu)\bar{g}$, and hence we can choose $\varepsilon>0$ as small as we please, which determines the well-posed Gevrey class $\gamma^{(1/2\varepsilon)}$. Actually the Cauchy problem is well posed in the space consisting of all C_0^∞ functions of whose Fourier transform is bounded by $\exp\left(-C\log^N\langle\xi'\rangle\right)$ with some $C>0$, $N>0$.

Definition 3.7. We set $\lambda=\langle\mu\xi'\rangle$, $\lambda_\mu=\langle\xi'\rangle_\mu$ and we write $a\in S(\lambda_\mu^{s+0},g)$ (resp. $a\in S(\lambda^{s+0},g)$) if $a\in S(\langle\xi'\rangle_\mu^{s+\varepsilon},g)$ (resp. $a\in S(\langle\mu\xi'\rangle^s\langle\xi'\rangle_\mu^\varepsilon,g)$) for any $\varepsilon>0$. We also write

$$\|Au\|\le C\|\lambda_\mu^{s+0}u\|\quad(\text{resp. }\|\lambda^{s+0}u\|)$$

if $\|Au\|\le C_\varepsilon\|\langle D'\rangle_\mu^{s+\varepsilon}u\|$ (resp. $\|Au\|\le C_\varepsilon\|\langle\mu D'\rangle^s\langle D'\rangle_\mu^\varepsilon u\|$) for any $\varepsilon>0$ with $C_\varepsilon>0$ independent of $\mu>0$.

§4. Transformed symbols $\tilde\lambda$, $\tilde m$

We first list several properties of cutoff symbols.

Lemma 4.1. *We have*

$$\begin{aligned}&\chi\chi_2=0,\quad \zeta\zeta_-=0,\quad \zeta\zeta_+=\zeta_+,\quad \tilde\zeta\zeta=\nu\zeta_+^2,\\ &\hat\phi_2,\quad \chi\hat\phi_1^2,\quad \zeta'\hat\theta,\quad \zeta'_\pm\hat\theta\in S(w,g),\quad \chi\hat\phi_1\in S(w^{1/2},g),\\ &(1-\zeta_-^2-\zeta_+^2)\hat\theta,\quad \zeta(1-\zeta_+^2)\hat\theta\in S(w,g),\end{aligned}\tag{4.1}$$

where $\zeta'=\zeta'(\hat\theta w^{-1})$. We also have $\{\chi,\lambda_\mu^s\}$, $\{\zeta,\lambda_\mu^s\}\in S(w^{-1}\lambda_\mu^{s-1},g)$.

Denote $W_\beta^\alpha=T^{-1}\partial_x^\beta\partial_{\xi'}^\alpha T$ and note that we have for $a\in S(\lambda_\mu^{s+0}w^t,g)$ or $a\in S(\lambda^s,g_0)$,

$$\begin{aligned}a\#T&=T\#a-inT\{a,\Psi\}\\&\quad+\frac{i}{8}\,T\sum_{|\alpha+\beta|=3}\frac{(-1)^{|\beta|}}{\alpha!\beta!}\big(a_{(\beta)}^{(\alpha)}W_\alpha^\beta-W_\beta^\alpha a_{(\alpha)}^{(\beta)}\big)+T\#R\end{aligned}$$

with some $R\in S(w^t\lambda_\mu^{s-5/4+0},\bar g)$ or $R\in S(\lambda^s\lambda_\mu^{-5/2+0},\bar g)$ respectively. From Lemma 3.3 it follows that $\psi_{(\beta)}^{(\alpha)}W_\alpha^\beta\in S(1,\bar g)$ for $|\alpha+\beta|=3$ then the main parts of $\mathsf{Im}\,\tilde m$ and $\mathsf{Im}\,\tilde\lambda$ are, up to the parameter n,

$$\begin{aligned}\{\xi_0\pm\phi_1\mp\psi,\Psi\}&=\zeta^2\{\xi_0\pm\phi_1\mp\psi,\chi^2\log\phi+\Phi\}\\&\quad+\{\xi_0\pm\phi_1\mp\psi,\zeta^2\}(\chi^2\log\phi+\Phi).\end{aligned}$$

To estimate $\{\xi_0\pm\phi_1\mp\psi,\chi^2\log\phi+\Phi\}$ it suffices to repeat similar arguments as in [N4] to get

$$\{\xi_0\pm\phi_1\mp\psi,\chi^2\log\phi+\Phi\}=\{\xi_0\pm\phi_1\mp\psi,\hat\phi_2\}(r+2\omega\rho^{-2})+R\tag{4.2}$$

with $R \in \mu S(\lambda_\mu^{+0}, \bar{g})$ where

$$0 \le r = \chi^2 w^{-1} + \delta \in S(w^{-1}\lambda_\mu^{+0}, g),$$
$$0 \le \delta = -2\chi\chi' \hat{\phi}_1^2 w^{-3} \hat{\phi}_2 \log\phi \in S(w^{-1}\lambda_\mu^{+0}, g)$$

and the fact $\delta \ge 0$ follows from [N4, Lem. 3.6] which was a key point in treating (4.2). We examine how the term $\{\xi_0 \pm \phi_1 \mp \psi, \zeta^2\}(\chi^2 \log\phi + \Phi)$ can be managed. It is not difficult to see

$$\begin{aligned}\{\xi_0 \pm \phi_1 \mp \psi, \zeta^2\}&(\chi^2 \log\phi + \Phi) \\ &= -2\zeta\zeta' \hat{\theta} w^{-3} \hat{\phi}_2 (\chi^2 \log\phi + \Phi)\{\xi_0 \pm \phi_1 \mp \psi, \hat{\phi}_2\} + R,\end{aligned} \tag{4.3}$$

with $R \in \mu S(\lambda_\mu^{+0}, \bar{g})$. Here we note the following result.

Lemma 4.2. *We have*

$$0 \le \Delta = -2\zeta\zeta' \hat{\theta} w^{-3} \hat{\phi}_2 (\chi^2 \log\phi + \Phi) \in S(w^{-1}\lambda_\mu^{+0}, \bar{g}).$$

Proof. Since $\hat{\phi}_2 \log\phi \ge 0$ by [N5, Lem. 3.6] it is clear $0 \le -2\chi^2\zeta\zeta'\hat{\theta} w^{-3}\hat{\phi}_2 \log\phi \in S(w^{-1}\lambda_\mu^{+0}, g)$ because $\zeta'(\hat{\theta} w^{-1})\hat{\theta} \le 0$. Noting that $0 \le \Phi = \pi - 2\arg(\hat{\phi}_2 + i\omega) \le \pi$ if $\hat{\phi}_2 \ge 0$ and $-\pi \le \Phi = \pi - 2\arg(\hat{\phi}_2 + i\omega) \le 0$ for $\hat{\phi}_2 \le 0$ it is also clear $\hat{\phi}_2\Phi \ge 0$ and hence $0 \le -2\zeta\zeta'\hat{\theta} w^{-3}\hat{\phi}_2\Phi \in S(w^{-1}, \bar{g})$. Thus we get the assertion. □

To simplify notation we set $\Gamma = r + 2\omega\rho^{-2}$. From (4.2) and (4.3) it suffices to consider $n(\Delta + \zeta^2\Gamma)\{\xi_0 \pm \phi_1 \mp \psi, \hat{\phi}_2\}$. As in [N4] we set

$$\begin{cases} e_1 = \mu\hat{c} + \nu\{\phi_1, \hat{\phi}_2\}, \quad e_3 = \{\xi_0 + \phi_1, \hat{\phi}_2\}, \\ e_2 = \{\xi_0 + \phi_1, \hat{\phi}_2\} - \nu\hat{\theta}\{\phi_1, \hat{\phi}_2\}\zeta_+^2. \end{cases}$$

Noting Lemma 4.1 it is easy to see

$$\begin{aligned}\{\xi_0 - \phi_1 + \psi, \hat{\phi}_2\} &= \mu\hat{c}\hat{\theta} + \tilde{\zeta}\{\phi_1, \hat{\phi}_2\}\hat{\theta} + c_0\hat{\theta}\hat{\phi}_1 + 3\chi_2\hat{\phi}_1^2\{\phi_1, \hat{\phi}_2\}, \\ \{\xi_0 + \phi_1 - \psi, \hat{\phi}_2\} &= \{\xi_0 + \phi_1, \hat{\phi}_2\} - \tilde{\zeta}\{\phi_1, \hat{\phi}_2\}\hat{\theta} - 3\chi_2\hat{\phi}_1^2\{\phi_1, \hat{\phi}_2\}\end{aligned} \tag{4.4}$$

modulo $S(w, \bar{g})$. Noting $\zeta = \zeta_+^2 + \zeta(1 - \zeta_+^2)$, $\zeta(1 - \zeta_+^2)\hat{\theta} \in S(w, \bar{g})$ we have

$$\begin{cases} \zeta^2\{\xi_0 - \phi_1 + \psi, \hat{\phi}_2\} = (e_1 + a_1\hat{\phi}_1)\zeta_+^2\hat{\theta} + a_2\zeta\hat{\phi}_1^2, \\ \zeta^2\{\xi_0 + \phi_1 - \psi, \hat{\phi}_2\} = e_2\zeta^2 + a_3\zeta\hat{\phi}_1^2, \end{cases}$$

with $a_i \in \mu S(1, \bar{g})$ modulo $S(w, \bar{g})$. Since $\Delta\hat{\theta}$, $\Gamma\hat{\phi}_1^2 \in S(\lambda_\mu^{+0}, \bar{g})$ by Lemma 4.1 we see $\mathsf{Im}\,\tilde{\lambda} = n\zeta_+^2(e_1 + a\hat{\phi}_1)\Gamma\hat{\theta} + R$ with $R \in S(\lambda_\mu^{+0}, \bar{g})$ and $a \in \mu S(\lambda_\mu^{+0}, \bar{g})$. Similarly we have $\mathsf{Im}\,\tilde{m} = n(e_3 + a'\hat{\phi}_1^2)\Delta + ne_2\zeta^2\Gamma + R$ with $R \in S(\lambda_\mu^{+0}, \bar{g})$. Noting that the main part of $\mathsf{Re}\,\tilde{\lambda}$ comes from $\{\{\xi_0 - \phi + \psi, \Psi\}, \Psi\}$ we summarize the following lemma.

Lemma 4.3. *We have*

$$\begin{cases} \mathsf{Im}\,\tilde{\lambda} = n(e_1 + b_1\hat{\phi}_1)\Gamma\zeta_+^2\hat{\theta} + R_1, \\ \mathsf{Re}\,\tilde{\lambda} = \phi_1 - \psi + n(b_2\hat{\theta} + b_3\hat{\phi}_1^2)w^{-1/2} + R_2, \\ \mathsf{Im}\,\tilde{m} = ne_2\zeta^2\Gamma + n(e_3 + b_4\hat{\phi}_1^2)\Delta + R_3, \end{cases}$$

where $b_i \in \mu S(\lambda_\mu^{+0}, \bar{g})$ *and* $R_i \in S(\lambda_\mu^{+0}, \bar{g})$.

Lemma 4.4. *There exists* $c > 0$ *which is independent of* $\mu > 0$ *such that we have*

$$\begin{aligned} C(\mathsf{Im}\,\tilde{\lambda}\,u, u) &\geq c\mu n(\Gamma\zeta_+^2\hat{\theta}u, u) - C_1\|\lambda_\mu^{+0}u\|^2 \\ &\geq c\mu n(\Gamma(\zeta_+\hat{\theta}^{1/2})u, (\zeta_+\hat{\theta}^{1/2})u) - C_2\|\lambda_\mu^{+0}u\|^2, \\ C(\mathsf{Im}\,\tilde{m}\,u, u) &\geq c\mu n((\zeta^2\Gamma + \Delta)u, u) - C_3\|\lambda_\mu^{+0}u\|^2 \\ &\geq c\mu n(\Gamma(\zeta u), \zeta u) + c\mu n(\Delta u, u) - C_4\|\lambda_\mu^{+0}u\|^2. \end{aligned}$$

We have also

$$\begin{aligned} C(\mathsf{Im}\,\tilde{\lambda}\,u, u) &\geq c\mu n(\|\chi\zeta_+\hat{\theta}^{1/2}w^{-1/2}u\|^2 + \|\zeta_+\hat{\theta}^{1/2}\rho^{-1/2}u\|^2), \\ C(\mathsf{Im}\,\tilde{m}\,u, u) &\geq c\mu n(\|\zeta\chi w^{-1/2}u\|^2 + \|\zeta\rho^{-1/2}u\|^2) \end{aligned}$$

modulo $C'\|\lambda_\mu^{+0}u\|^2$ *with some* C, $c > 0$ *independent of* μ.

Proof. Since $\hat{\phi}_1(0, \mathbf{e}_2) = 0$ we may assume $\tilde{e}_1 = e_1 + b_1\hat{\phi}_1 \geq \mu c_1$ with $c_1 > 0$. Take $M > 0$ so that $M\tilde{e}_1 \geq \mu$. Since $0 \leq (M\tilde{e}_1 - \mu)\Gamma\zeta_+^2\hat{\theta} \in \mu S(w^{-1}\lambda_\mu^{+0}, \bar{g}) \subset \mu S^{1/2+0}_{3/4,1/2}$ then from the Fefferman–Phong inequality (see [H2, Thm. 18.6.8]) it follows that

$$M(\tilde{e}_1\Gamma\zeta_+^2\hat{\theta}u, u) \geq \mu(\Gamma\zeta_+^2\hat{\theta}u, u) - C_1\|\lambda_\mu^{+0}u\|^2.$$

Here note that $\Gamma\zeta_+^2\hat{\theta} = (\zeta_+\hat{\theta}^{1/2})\#\Gamma\#(\zeta_+\hat{\theta}^{1/2}) + R$ with $R \in S(\lambda_\mu^{+0}, \bar{g})$. Since $|(Ru, u)| \leq C'\|\lambda_\mu^{+0}u\|^2$ the first assertion follows. To show the second assertion it suffices to repeat the same arguments proving the first assertion. To prove the third assertion we first note that

$$(\delta\zeta_+^2\hat{\theta}u, u),\ (\zeta^2\delta u, u),\ (\Delta u, u) \geq -C\|\lambda_\mu^{+0}u\|^2,$$

which follows from the Fefferman–Phong inequality since δ, $\Delta \in S(w^{-1}\lambda_\mu^{+0}, \bar{g})$ are non-negative. We then write $\chi^2\zeta_+^2\hat{\theta}w^{-1} = \chi\zeta_+\hat{\theta}^{1/2}w^{-1/2}\#\chi\zeta_+\hat{\theta}^{1/2}w^{-1/2} + R$ with $R \in S(\lambda_\mu^{+0}, \bar{g})$ because $\zeta_+\hat{\theta}^{1/2} \in S(1, G) \subset S(1, \bar{g})$ by Lemma 3.1, which gives the first term on the right-hand side. To get the second term on the right-hand side we note that $w^{-1} + \omega\rho^{-2} \geq \rho^{-1}/2$; on the other hand if $\chi < 1$ we have $C\omega \geq \rho$ with some $C > 0$ and hence $C\omega\rho^{-2} \geq \rho^{-1}$. Therefore it follows $C(\chi^2\zeta_+^2\hat{\theta}w^{-1} + \zeta_+^2\hat{\theta}\omega\rho^{-2}) \geq \zeta_+^2\rho^{-1}\hat{\theta}$ and the Fefferman–Phong inequality proves

$$C(\chi^2\zeta_+^2\hat{\theta}w^{-1}u, u) + C(\zeta_+^2\hat{\theta}\omega\rho^{-2}u, u) \geq \|\zeta_+\rho^{-1/2}\hat{\theta}u\|^2 - C\|\lambda_\mu^{+0}u\|^2,$$

which gives the second term. The proof of the last assertion is similar. □

Applying Lemma 4.4 one can show the following proposition.

Proposition 4.5. *We have*

$$2((\mathrm{Im}\,\tilde m)\tilde\Lambda u,\tilde\Lambda u)\geq c\mu n((\Gamma+\Delta)(\zeta\tilde\Lambda u),(\zeta\tilde\Lambda u))+c\mu n\|\chi\zeta w^{-1/2}\tilde\Lambda u\|^2$$
$$+c\mu n\|\zeta\omega^{1/2}\rho^{-1}\tilde\Lambda u\|^2+c\mu n\|\zeta\rho^{-1/2}\tilde\Lambda u\|^2-C\|\lambda_\mu^{+0}\tilde\Lambda u\|^2,$$

with some $c>0$ independent of $\mu>0$ and some $C>0$.

§5. Estimate $\|\tilde\Lambda u\|$

From now on we often disregard error terms which are bounded by $\gamma^2\|\lambda_\mu^{+0}u\|^2$ without special mention because we have $\gamma^3\|\lambda_\mu^{3\varepsilon}u\|^2$ in (3.12). We first note the following lemma which is easily checked using (3.1) and (3.2).

Lemma 5.1. *Let $\hat\zeta$, $\hat\chi\in C^\infty(\mathbb{R})$ such that $\hat\zeta'$, $\hat\chi'\in C_0^\infty(\mathbb{R})$. Set $\hat\zeta=\hat\zeta(\hat\theta w^{-1})$ and $\hat\chi=\hat\chi(\hat\phi_1^2w^{-1})$. Then we have*

$$\{\hat\phi_1,\hat\zeta\}\in S(w^{-1}\lambda_\mu^{-1},\bar g),\quad \{\hat\phi_1,\hat\chi\}\in S(w^{-1}\lambda_\mu^{-1},\bar g),$$
$$\{\hat\phi_1,w^{-1}\}\in S(w^{-2}\lambda_\mu^{-1},\bar g),\quad \{\hat\phi_1,\omega\rho^{-2}\}\in S(\rho^{-2}\lambda_\mu^{-1},\bar g),$$
$$\{\hat\phi_2,\hat\chi\}\in S(w^{-1/2}\lambda_\mu^{-1},\bar g),\quad \{\hat\phi_2,\hat\zeta\}\in S(\lambda_\mu^{-1},\bar g),$$
$$\{\hat\phi_2,\omega\rho^{-2}\}\in S(\rho^{-3/2}\lambda_\mu^{-1},\bar g),\quad \{\hat\phi_2,w^{-1}\}\in S(w^{-1}\lambda_\mu^{-1},\bar g),$$
$$\{\hat\zeta,\hat\theta\}\in S(\lambda_\mu^{-1},\bar g),\quad \{\hat\zeta,w^{-1}\}\in S(w^{-2}\lambda_\mu^{-1},\bar g),\quad \{\hat\zeta,\omega\rho^{-2}\}\in S(\rho^{-3/2}w^{-1}\lambda_\mu^{-1},\bar g),$$
$$\{\hat\chi,\hat\theta\}\in S(\lambda_\mu^{-1},\bar g),\quad \{\hat\chi,w^{-1}\}\in S(w^{-1/2},\bar g),\quad \{\hat\chi,\omega\rho^{-2}\}\in S(\rho^{-5/2}\lambda_\mu^{-1},\bar g),$$
$$\{\hat\zeta,\hat\chi\}\in S(w^{-3/2}\lambda_\mu^{-1},\bar g),\quad \{w^{-1},\omega\rho^{-2}\}\in S(w^{-2}\rho^{-3/2}\lambda_\mu^{-1},\bar g),$$
$$\{\hat\theta,w^{-1}\}\in S(w^{-1}\lambda_\mu^{-1},\bar g),\quad \{\hat\theta,\omega\rho^{-2}\}\in S(\rho^{-1}\lambda_\mu^{-1},\bar g),\quad \{w^{-1},\lambda_\mu^{-1}\}\in S(\lambda_\mu^{-1},\bar g).$$

From Lemma 4.4 it follows that

$$-2\,\mathrm{Im}(\tilde\Lambda v,v)\geq\frac{d}{dx_0}\|v\|^2+\frac{3}{2}\gamma\|\lambda_\mu^{\varepsilon}v\|^2+c\mu n(\chi^2\zeta_+^2\hat\theta w^{-1}v,v)$$
$$+c\mu n\|\chi\zeta_+\hat\theta^{1/2}w^{-1/2}v\|^2-C\|\lambda_\mu^{+0}v\|^2,\tag{5.1}$$

with some $c>0$. Let $\zeta_0(s)$, $\chi_0(s)\in C^\infty(\mathbb{R})$ be such that $\operatorname{supp}\zeta_0$ is contained in $\{\zeta_+=1\}$ and $\chi_0=1$ for $s\leq c$ with some $c>0$ and $\operatorname{supp}\chi_0\subset\{\chi=1\}$. Set $\zeta_0=\zeta_0(\hat\theta w^{-1})$ and $\chi_0=\chi_0(\hat\phi_1^2w^{-1})$. Replacing u by $w^{-1}\eta\hat\theta^{1/2}u$, $\eta=\chi_0\zeta_0$ in (5.1) it follows that

$$
\begin{aligned}
&-2\,\mathsf{Im}(\tilde{\Lambda}(w^{-1}\eta\hat{\theta}^{1/2}u), w^{-1}\eta\hat{\theta}^{1/2}u)\\
&\quad\geq \frac{d}{dx_0}\|w^{-1}\eta\hat{\theta}^{1/2}u\|^2 + c\mu n(w^{-1}\chi^2\zeta_+^2\hat{\theta}(w^{-1}\eta\hat{\theta}^{1/2}u), w^{-1}\eta\hat{\theta}^{1/2}u).
\end{aligned}
\tag{5.2}
$$

We first examine $[\tilde{\Lambda}, w^{-1}\eta\hat{\theta}^{1/2}]u$. Note $\{\xi_0-\phi_1, w^{-1}\} = -2\hat{\phi}_2 w^{-3}\{\xi_0-\phi_1,\hat{\phi}_2\}$ modulo $S(w^{-1},\bar{g})$. From (3.1) we see $\eta\hat{\theta}^{1/2}\{\xi_0-\phi_1, w^{-1}\} - c\eta w^{-2}\hat{\theta}^{3/2} \in S(w^{-1},\bar{g})$ with some $c\in S(1,\bar{g})$. Since $\eta\hat{\theta}^{1/2}\{\psi, w^{-1}\} = \eta\hat{\theta}^{1/2}\{\tilde{\zeta}\hat{\theta}\phi_1, w^{-1}\}$ for $\chi\chi_2 = 0$ then noting Lemma 5.1 we have $\eta\hat{\theta}^{1/2}\{\xi_0-\phi_1+\psi, w^{-1}\} - b\eta w^{-2}\hat{\theta}^{3/2} \in S(w^{-1},\bar{g})$. We next examine $w^{-1}\{\xi_0-\phi_1+\psi, \eta\hat{\theta}^{1/2}\}$. Since $\hat{\theta}^{-1/2}\zeta_0 \in S(w^{-1/2},\bar{g})$ from (3.1) we have $w^{-1}\{\xi_0-\phi_1, \eta\hat{\theta}^{1/2}\} - c\eta\hat{\theta}w^{-3/2} \in \mu S(w^{-1},\bar{g})$ by similar arguments. Noting $\{\tilde{\zeta}\hat{\theta}\phi_1, \eta\hat{\theta}^{1/2}\} \in \mu S(w^{-1},\bar{g})$ we get

$$
\{\tilde{\Lambda}, w^{-1}\eta\hat{\theta}^{1/2}\} - c\eta w^{-2}\hat{\theta}^{3/2} - c'\eta\hat{\theta}w^{-3/2} \in \mu S(w^{-1},\bar{g})
\tag{5.3}
$$

with some c, $c'\in \mu S(1,\bar{g})$. From (5.3) one has

$$
\begin{aligned}
|\,\mathsf{Im}([\tilde{\Lambda}, w^{-1}\eta\hat{\theta}^{1/2}]u, w^{-1}\eta\hat{\theta}^{1/2}u)| &\leq \mathsf{Re}(c\eta w^{-2}\hat{\theta}^{3/2}u, w^{-1}\eta\hat{\theta}^{1/2}u)\\
&\quad + \mathsf{Re}(c'\eta\hat{\theta}w^{-3/2}u, w^{-1}\eta\hat{\theta}^{1/2}u) + C\|w^{-1}u\|^2.
\end{aligned}
$$

Writing $\mathsf{Re}(w^{-1}\eta\hat{\theta}^{1/2}\#c\eta w^{-2}\hat{\theta}^{3/2}) = \mathsf{Re}(w^{-3/2}\eta\hat{\theta}\#c\#\eta w^{-3/2}\hat{\theta})$ modulo $S(w^{-2},\bar{g})$ and $\tilde{\Lambda}(w^{-1}\eta\hat{\theta}^{1/2}u) = w^{-1}\eta\hat{\theta}^{1/2}(\tilde{\Lambda}u) + [\tilde{\Lambda}, w^{-1}\eta\hat{\theta}^{1/2}]u$ we get

$$
\begin{aligned}
&\mathsf{Im}(\tilde{\Lambda}w^{-1}(\eta\hat{\theta}^{1/2}u), w^{-1}\eta\hat{\theta}^{1/2}u)\\
&\quad\leq \mathsf{Im}(w^{-1}\eta\hat{\theta}^{1/2}(\tilde{\Lambda}u), w^{-1}\eta\hat{\theta}^{1/2}u)\\
&\qquad + C\mu\|\eta w^{-3/2}\hat{\theta}u\|^2 + C\|\eta w^{-1}\hat{\theta}^{1/2}u\|^2 + C\|w^{-1}u\|^2.
\end{aligned}
\tag{5.4}
$$

We now estimate $\mathsf{Im}(w^{-1}\eta\hat{\theta}^{1/2}(\tilde{\Lambda}u), w^{-1}\eta\hat{\theta}^{1/2}u)$. Thanks to Lemma 3.3 one can write

$$
w^{-1}\eta\hat{\theta}^{1/2}\#w^{-1}\eta\hat{\theta}^{1/2} = \eta w^{-1/2}\#\eta w^{-3/2}\hat{\theta} + b\zeta_0 w^{-3/2}\hat{\theta} + R
$$

with $b\in S(1,\bar{g})$ where $R\in S(w^{-1},\bar{g})$ and therefore we have

$$
\begin{aligned}
&\mathsf{Im}(w^{-1}\eta\hat{\theta}^{1/2}(\tilde{\Lambda}u), w^{-1}\eta\hat{\theta}^{1/2}u)\\
&\quad\leq \mathsf{Im}(w^{-1/2}\eta(\tilde{\Lambda}u), \eta w^{-3/2}\hat{\theta}u)\\
&\qquad + C\gamma^{-1/2}(\|\zeta_0 w^{-3/2}\hat{\theta}u\|^2 + \|w^{-1}u\|^2) + C\gamma^{1/2}(\|\tilde{\Lambda}u\|^2 + \|u\|^2)\\
&\quad\leq (\epsilon_1\mu n)^{-1}\|\eta w^{-1/2}\tilde{\Lambda}u\|^2 + (\epsilon_1\mu n + C\gamma^{-1/2})\|\zeta_0 w^{-3/2}\hat{\theta}u\|^2\\
&\qquad + C\gamma^{1/2}\|\tilde{\Lambda}u\| + C(\|w^{-1}u\|^2 + \|u\|^2),
\end{aligned}
\tag{5.5}
$$

where $b \in S(1, \bar{g})$ and $\epsilon_1 > 0$ will be chosen later. Combining (5.2), (5.4) and (5.5) one obtains

$$
\begin{aligned}
&\frac{d}{dx_0}\|w^{-1}\eta\hat{\theta}^{1/2}u\|^2 + c\mu n(w^{-1}\chi^2\zeta_+^2\hat{\theta}(w^{-1}\eta\hat{\theta}^{1/2}u), w^{-1}\eta\hat{\theta}^{1/2}u)\\
&\quad\le (\epsilon_1\mu n)^{-1}\|\eta w^{-1/2}\tilde{\Lambda}u\|^2 + (\epsilon_1\mu n + C\gamma^{-1/2} + C\mu)\|\zeta_0 w^{-3/2}\hat{\theta}u\|^2\\
&\qquad + C\big(\|\eta w^{-1}\hat{\theta}^{1/2}u\|^2 + \|w^{-1}u\|^2 + \gamma^{1/2}\|\tilde{\Lambda}u\|^2\big).
\end{aligned} \tag{5.6}
$$

We now estimate $(w^{-1}\chi^2\zeta_+^2\hat{\theta}(w^{-1}\eta\hat{\theta}^{1/2}u), w^{-1}\eta\hat{\theta}^{1/2}u)$ from below. Note that

$$w^{-1}\eta\hat{\theta}^{1/2}\# w^{-1}\chi^2\zeta_+^2\hat{\theta}\# w^{-1}\eta\hat{\theta}^{1/2} = w^{-3}\eta^2\chi^2\zeta_+^2\hat{\theta}^2 + R$$

with $R \in S(w^{-2}, \bar{g})$ and hence we have

$$(w^{-1}\chi^2\zeta_+^2\hat{\theta}(w^{-1}\eta\hat{\theta}^{1/2}u), w^{-1}\eta\hat{\theta}^{1/2}u) \ge (w^{-3}\eta^2\hat{\theta}^2u, u) - C\|w^{-1}u\|^2$$

for $\eta^2\chi^2\zeta_+^2 = \eta^2$. We have $\eta^2 w^{-3}\hat{\theta}^2 = \eta w^{-3/2}\hat{\theta}\#\eta w^{-3/2}\hat{\theta} + R$ with $R \in S(w^{-2}, \bar{g})$ which proves that

$$
\begin{aligned}
2\,\mathsf{Re}(w^{-1}\chi^2\zeta_+^2\hat{\theta}(\eta w^{-1}\hat{\theta}^{1/2}u), \eta w^{-1}\hat{\theta}^{1/2}u) &\ge (\eta^2 w^{-3}\hat{\theta}^2u, u)\\
&\quad + \|\eta w^{-3/2}\hat{\theta}u\|^2 - C\|w^{-1}u\|^2.
\end{aligned} \tag{5.7}
$$

Write $\zeta_0^2 w^{-3}\hat{\theta}^2 = \eta^2 w^{-3}\hat{\theta}^2 + (1-\chi_0^2)\zeta_0^2 w^{-3}\hat{\theta}^2$ and consider

$$\mu^{-2}M\zeta_0^2\hat{\theta}\phi_1^2 - (1-\chi_0^2)\zeta_0^2 w^{-3}\hat{\theta}^2 = (\mu^{-1}H\zeta_0\hat{\theta}^{1/2}\phi_1)^2,$$

where $H = (M-(1-\chi_0^2)w^{-3}\lambda_\mu^{-2}\hat{\phi}_1^{-2}\hat{\theta})^{1/2} \in S(1, \bar{g})$ for large $M > 0$, which follows from $(1-\chi_0^2)\hat{\phi}_1^{-2} \in S(w^{-1}, \bar{g})$. Then it is not difficult to see

$$
\begin{aligned}
(\mu^{-1}H\zeta_0\hat{\theta}^{1/2}\phi_1)\#(\mu^{-1}H\zeta_0\hat{\theta}^{1/2}\phi_1) &= (\mu^{-1}H\zeta_0\hat{\theta}^{1/2}\phi_1)^2\\
&\quad + b_1 w^{-1/2}\phi_1 + b_2 w^{-1}\hat{\phi}_1^2\lambda + R
\end{aligned}
$$

with $b_j \in \mu^{-1}S(1, \bar{g})$ and $R \in S(w^{-2}, \bar{g})$. Noting that $b_1 w^{-1/2}\phi_1 = w^{-1}\# b_1 w^{1/2}\phi_1 + R_1$ and $b_2 w^{-1}\hat{\phi}_1^2\lambda = w^{-1}\# b_2\hat{\phi}_1^2\lambda + R_2$ with $R_i \in S(w^{-2}, \bar{g})$ we conclude that $\mu^{-2}M(\zeta_0^2\hat{\theta}\phi_1^2u, u) \ge ((1-\chi_0^2)\zeta_0^2 w^{-3}\hat{\theta}^2u, u)$ modulo a term $C\mu^{-2}(\|b_1 w^{1/2}\phi_1 u\|^2 + \|b_2\hat{\phi}_1^2\lambda u\|^2 + \mu^2\|w^{-1}u\|^2)$ which proves together with (5.7) that $\mu^3 n^3|(\zeta_0^2 w^{-3}\hat{\theta}^2u, u)|$ is bounded by

$$
\begin{aligned}
&C\mu n^3(a^2\tilde{\zeta}\hat{\theta}\phi_1^2u, u) + 2(w^{-1}\chi^2\zeta_+^2\hat{\theta}(\eta w^{-1}\hat{\theta}^{1/2})u, \eta w^{-1}\hat{\theta}^{1/2}u)\\
&\qquad + C\mu n^3\big(\|w^{1/2}\phi_1 u\|^2 + \|\hat{\phi}_1^2\lambda u\|^2 + \mu^2\|w^{-1}u\|^2\big)
\end{aligned} \tag{5.8}
$$

with some $C > 0$. To simplify notation we introduce the following definition.

Definition 5.2. We denote by $O(E)$ a symbol or the set of symbols of the form

$$a_1\mu w^{-1}+a_2\mu\omega^{-1}+a_3\mu^{1/2}\lambda^{1/2}+a_4w^{1/2}\phi_1 \\ +a_5\omega^{1/2}\phi_1+a_6\phi_2+a_7\hat{\phi}_1^2\lambda+a_8w\lambda+a_9\omega\lambda$$

with $a_i\in S(1,\bar{g})$. We denote by $S(\lambda^{t_1}\lambda_\mu^{t_2}w^s,\bar{g})O(E)$ a symbol or the set of symbols which is a linear combination of $\mu^{-1}w^{-1}$, $\mu^{-1}\omega^{-1}$, $\mu^{1/2}\lambda^{1/2}$, $w^{1/2}\phi_1$, $\omega^{1/2}\phi_1$, ϕ_2, $\lambda\hat{\phi}_1^2$, $w\lambda$ and $\omega\lambda$ with coefficients in $S(\lambda^{t_1}\lambda_\mu^{t_2}w^s,\bar{g})$. We also denote

$$\|O(E)u\|^2=\mu^2(\|w^{-1}u\|^2+\|\omega^{-1}u\|^2)+\mu\|\lambda^{1/2}u\|^2+\|w^{1/2}\phi_1u\|^2 \\ +\|\omega^{1/2}\phi_1u\|^2+\|\phi_2u\|^2+\|\hat{\phi}_1^2\lambda u\|^2+\|w\lambda u\|^2+\|\omega\lambda u\|^2.$$

We choose $\epsilon_1>0$ small so that we have a positive contribution $(\zeta_0w^{-3}\hat{\theta}u,u)$ when adding (5.6) and (5.8). Then we obtain the following proposition.

Proposition 5.3. *Let χ_0, ζ_0 be as above. Then there exist $n_0>0$, $\mu_0>0$ and $\gamma_0>0$ such that we have*

$$C\mu n\|\chi_0\zeta_0w^{-1/2}\tilde{\Lambda}u\|^2+C\gamma\|\tilde{\Lambda}u\|^2\geq c\mu^2n^2\frac{d}{dx_0}\|\chi_0\zeta_0w^{-1}\hat{\theta}^{1/2}u\|^2 \\ +c\mu^3n^3(\|\zeta_0w^{-3/2}\hat{\theta}u\|^2+(\zeta_0^2w^{-3}\hat{\theta}^2u,u)) \\ -C\mu(\tilde{\zeta}a^2\hat{\theta}\phi_1^2u,u)-C\mu\|O(E)u\|^2$$

with some $c>0$ for $n\geq n_0$, $0<\mu\leq\mu_0$ and $\gamma\geq\gamma_0$.

§6. Transformed symbol $\tilde{Q}$

We start with the following lemma.

Lemma 6.1. *One can write $O(E)=T\#(O(E)+R)$ with $R\in S(\lambda_\mu^{+0},\bar{g})$.*

Proof. Let $A\in O(E)$; then it is easy to check $T^{-1}T_{(\beta)}^{(\alpha)}A_{(\alpha)}^{(\beta)}\in S(\lambda_\mu^{-1/4+0},\bar{g})O(E)$ for $|\alpha+\beta|=1$. Then we have $TA-T\#A=TA_1$ with $A_1\in S(\lambda_\mu^{-1/4+0},\bar{g})O(E)$. Repeating the same arguments we get $TA=T\#(A+A_1+\cdots+A_4)+K$ where $K\in S(\lambda_\mu^{-1},\bar{g})O(E)\subset S(\lambda_\mu^{+0},\bar{g})$. Since $T\#T^{-1}=1-r$ with $r\in\mu^{1/4}S(1,\bar{g})$ and hence the inverse of $1-r$ exists in $\mathcal{L}(L^2,L^2)$ which is given by $\mathrm{Op}(b)$ with $b\in S(1,\bar{g})$ (see [Be]) and hence $T\#\tilde{T}=1$ with $\tilde{T}=T^{-1}\#b\in S(\lambda_\mu^{+0},\bar{g})$. Then writing $K=T\#(\tilde{T}\#K)$ we get the assertion. □

Recall $W^\alpha_\beta = T^{-1}\partial_x^\beta \partial_{\xi'}^\alpha T \in S(\lambda_\mu^{-3|\alpha|/4+|\beta|/2+0}, \bar{g})$. Since $q^{(\alpha)}_{(\beta)} \in S(\lambda^2\lambda_\mu^{-|\alpha|}, \bar{g})$ for $|\alpha+\beta| = 1$ by Lemma 3.3 we see

$$q\#T = T\#q - inT\{q,\Psi\} + \frac{i}{8}T \sum_{|\alpha+\beta|=3} \frac{(-1)^{|\beta|}}{\alpha!\beta!}\big(q^{(\alpha)}_{(\beta)}W^\beta_\alpha - W^\alpha_\beta q^{(\beta)}_{(\alpha)}\big) \tag{6.1}$$

modulo $R \in \mu^{3/2}S(\lambda^{1/2+0}, \bar{g})$. We first check the following lemma.

Lemma 6.2. *We have*

$$\sum_{|\alpha+\beta|=3} (-1)^{|\beta|}(q^{(\alpha)}_{(\beta)}W^\beta_\alpha - W^\alpha_\beta q^{(\beta)}_{(\alpha)})/\alpha!\beta! \in \mu S(\lambda_\mu^{+0}, \bar{g})O(E).$$

Proof. Write $q = \phi_2^2 + f\phi_1^2$ with $f = 2a^2\tilde{\zeta}\hat{\theta} + 2a^2\chi_2\hat{\phi}_1^2$ and recall $f \in S(1, g)$ and $f^{(\alpha)}_{(\beta)} \in S(\lambda_\mu^{-|\alpha|}, g)$ for $|\alpha+\beta| = 1$ by Lemma 3.3. Applying Lemma 3.3 again one can check that $(f\phi_1^2)^{(\alpha)}_{(\beta)}W^\beta_\alpha$ with $|\alpha+\beta| = 3$ is a linear combination of $\lambda^{1/2}$, $w^{1/2}\phi_1$ and $\hat{\phi}_1^2\lambda$ with coefficients in $\mu S(\lambda_\mu^{+0}, \bar{g})$, which proves the assertion. □

We make a more detailed study of $\{q,\Psi\}$ and $\{\Psi, \{q,\Psi\}\}$.

Lemma 6.3. *We have*

$$\{q,\Psi\} = \nu\zeta_+^2 a^2\hat{\theta}\phi_1(\Gamma+\Delta)\{\phi_1, \hat{\phi}_2\} + a_1O(E) + a_2O(E) + a_3O(E),$$

where $a_1 = \zeta\chi a_1'$, $a_2 = \zeta a_2'$ *with* $a_1' \in \mu S(w^{-1/2}\lambda_\mu^{+0}, \bar{g})$, $a_2' \in \mu S(\rho^{-1/2}, \bar{g})$ *and* $a_3 \in \mu S(\lambda_\mu^{+0}, \bar{g})$.

Proof. Denote $\Psi_1 = \zeta^2\chi^2\log\phi \in S(\lambda_\mu^{+0}, g)$ and $\Psi_2 = \zeta^2\Phi$, $\Phi \in S(1, g_1)$ so that $\Psi = \Psi_1 + \Psi_2$. Thanks to (3.4) and Lemma 5.1 we can see $\{\phi_2^2 + a^2\chi_2\hat{\phi}_1^4\lambda^2, \Psi_2\} = a_2O(E)$ where $a_2 = \zeta a_2'$ with $a_2 \in \mu S(\rho^{-1/2}, \bar{g})$. Similarly, from Lemma 5.1 and

$$\{F, \log\phi\} = \{F, \hat{\phi}_2\}/w + \lambda_\mu^{-1/2}\{F, \lambda_\mu^{1/2}\} + \lambda_\mu^{1/2}\{F, \lambda_\mu^{-1}\}/2w\phi, \tag{6.2}$$

we obtain $\{\phi_2^2 + a^2\chi_2\hat{\phi}_1^4\lambda^2, \Psi_1\} = a_1O(E)$ with $a_1 \in \mu S(w^{-1/2}\lambda_\mu^{+0}, \bar{g})$ where clearly $a_1 = \zeta\chi a_1'$. We turn to $\{a^2\tilde{\zeta}\hat{\theta}\phi_1^2, \Psi_j\}$. Repeating similar arguments one can check that $\{a^2\tilde{\zeta}\hat{\theta}\phi_1^2, \Psi_j\} = a^2\tilde{\zeta}\hat{\theta}\{\phi_1^2, \Psi_j\} + a_jO(E)$ where a_j verifies the same properties as above. From the same arguments proving (4.2) and (4.3) one can show

$$\{\phi_1, \Psi\} = (\zeta^2\Gamma + \Delta)\{\phi_1, \hat{\phi}_2\} + R \tag{6.3}$$

with $R \in S(\lambda_\mu^{+0}, \bar{g})$. Since $\tilde{\zeta}\zeta = \nu\zeta_+^2$ and $\tilde{\zeta}\Delta = \nu\zeta_+^2\Delta$ we get the assertion. □

Lemma 6.4. *We have*

$$\{\Psi, \{q,\Psi\}\} = -A_1 + \zeta_+(a_2w^{-1/2}\hat{\theta}\phi_1 + a_3\phi_1) + S(\lambda_\mu^{+0}, \bar{g})O(E)$$

where $a_j \in \mu S(\lambda_\mu^{+0}, \bar{g})$, $j = 2, 3$ *and*

$$0 \le A_1 = \nu\zeta_+^2 a^2 \hat{\theta}\Gamma(\zeta^2\Gamma + \Delta)\{\phi_1, \hat{\phi}_2\}^2 \in \mu^2 S(w^{-2}\lambda_\mu^{+0}, \bar{g}).$$

Proof. By obvious abbreviated notation we see $\Psi^{(\alpha)}_{(\beta)}(a_j O(E))^{(\beta)}_{(\alpha)} \in O(E)$ for $|\alpha + \beta| = 1$ and hence $\{\Psi, \sum a_j O(E)\} \in O(E)$. By Lemma 6.3 it remains to check that $\{\Psi, \zeta_+^2 a^2 \hat{\theta}\phi_1(\Gamma + \Delta)\{\phi_1, \hat{\phi}_2\}\}$. Since $\hat{\theta}\Delta \in S(\lambda_\mu^{+0}, \bar{g})$ and $b = \zeta_+^2 a^2 \hat{\theta}\Delta\{\phi_1, \hat{\phi}_2\} \in \mu S(\lambda_\mu^{+0}, \bar{g})$ it is easy to check that $\{\Psi, b\phi_1\}$ is a linear combination of $w^{1/2}\phi_1$ and $w\lambda$ with coefficients $\mu S(\lambda_\mu^{+0}, \bar{g})$ because $\zeta^2\Phi \in S(1, \tilde{g})$. Therefore $\{\Psi, b\phi_1\} \in \mu S(\lambda_\mu^{+0}, \bar{g})O(E)$. Let us consider $\{\Psi, B\}$ with $B = \zeta_+^2 a^2 \hat{\theta}\phi_1\Gamma\{\phi_1, \hat{\phi}_2\}$. Thanks to (6.3) taking Lemma 5.1 into account we can prove that

$$\{\Psi, B\} = -\zeta_+^2 a^2 \hat{\theta}\Gamma(\zeta^2\Gamma + \Delta)\{\phi_1, \hat{\phi}_2\}^2 + \zeta_+(a_2 w^{-1/2}\hat{\theta}\phi_1 + a_3\phi_1)$$

modulo $S(\lambda_\mu^{+0}, \bar{g})O(E)$ where $a_j \in \mu S(\lambda_\mu^{+0}, \bar{g})$. This proves the assertion. □

Proposition 6.5. *We have*

$$\begin{aligned} q\#T &= T\#\big(q - in\{q, \Psi\} + n^2\{\Psi, \{\Psi, q\}\}\big) \\ &\quad + i(a_1\mu w^{-3/2}\hat{\theta} + a_2 w^{1/2}\hat{\theta}\lambda + a_3\hat{\theta}\phi_1) + \mu S(\lambda_\mu^{+0}, \bar{g})O(E), \end{aligned}$$

where $a_j \in \mu S(\lambda_\mu^{+0}, \bar{g})$ *are real valued and* $\operatorname{supp} a_j \subset \operatorname{supp} \zeta_+$.

Proof. From Lemma 6.3 it is clear that $T^{-1}T^{(\alpha)}_{(\beta)}\{q, \Psi\}^{(\beta)}_{(\alpha)}$ is $c_1\hat{\theta}\phi_1 + c_2 w^{1/2}\hat{\theta}\lambda$ modulo $\mu^{1/2}S(\lambda_\mu^{+0}, \bar{g})O(E)$ because $\lambda_\mu^{-1/4} \in S(w^{1/2}, \bar{g})$ for $|\alpha + \beta| = 2$. Therefore we get

$$\begin{aligned} T\#\{q, \Psi\} &= T\{q, \Psi\} + nT\{\Psi, \{q, \Psi\}\}/2i + c_1\hat{\theta}\phi_1 \\ &\quad + c_2 w^{1/2}\hat{\theta}\lambda + \mu S(\lambda_\mu^{+0}, \bar{g})O(E), \end{aligned}$$

where $c_i \in \mu S(\lambda_\mu^{+0}, \bar{g})$ is real. It is clear that $\operatorname{supp} c_j \subset \operatorname{supp} \zeta_+$. Thus we have $T\{q, \Psi\} = T\#\big(\{q, \Psi\} + n\{\Psi, \{\Psi, q\}\}/2i - c_1\hat{\theta}\phi_1 - c_2\mu w^{1/2}\hat{\theta}\lambda\big) + \mu S(\lambda_\mu^{+0}, \bar{g})O(E)$. From Lemma 6.4 it can be seen that $\Psi^{(\alpha)}_{(\beta)}\{\Psi, \{q, \Psi\}\}^{(\beta)}_{(\alpha)}$ for $|\alpha + \beta| = 1$ are written as $a_1\mu w^{-3/2}\hat{\theta} + a_2\omega^{1/2}\hat{\theta}\lambda + a_3\hat{\theta}\phi_1$ modulo $\mu S(\lambda_\mu^{+0}, \bar{g})O(E)$. This proves

$$\begin{aligned} T\{q, \Psi\} &= T\#(\{q, \Psi\} - n\{\Psi, \{\Psi, q\}\}/2i + a_1\mu w^{-3/2}\hat{\theta} + a_2 w^{1/2}\hat{\theta}\lambda + a_3\hat{\theta}\phi_1) \\ &\quad + \mu S(\lambda_\mu^{+0}, \bar{g})O(E) \end{aligned}$$

where $a_j \in \mu S(\lambda_\mu^{+0}, \bar{g})$ with $\operatorname{supp} a_j \subset \operatorname{supp} \zeta_+$. This proves the assertion. □

Corollary 6.6. *We have*

$$\operatorname{Im}\tilde{Q} = T_2 - \nu n\zeta_+^2 a^2\hat{\theta}\phi_1\Gamma\{\phi_1,\hat{\phi}_2\} + a_1\mu w^{-3/2}\hat{\theta} + a_2 w^{1/2}\hat{\theta}\lambda$$
$$+ a_3\hat{\theta}\phi_1 + c_1 O(E) + c_2 O(E) + \mu S(\lambda_\mu^{+0},\bar{g})O(E),$$

where $a_j \in \mu S(\lambda_\mu^{+0},\bar{g})$ *are real valued with supports are contained in* $\operatorname{supp}\zeta_+$ *and* $c_1 = \zeta\chi c_1'$ *with* $c_1' \in \mu S(w^{-1/2}\lambda_\mu^{+0},\bar{g})$ *and* $c_2 = \zeta c_2'$ *with* $c_2' \in \mu S(\rho^{-1/2},\bar{g})$.

Corollary 6.7. *We have*

$$\operatorname{Re}\tilde{Q} = q + T_1 + \zeta_+(a_1\mu w^{-2}\hat{\theta} + a_2 w^{-1/2}\hat{\theta}\phi_1 + a_3\phi_1) + S(\lambda_\mu^{+0},\bar{g})O(E),$$

where $a_j \in \mu S(\lambda_\mu^{+0},\bar{g})$ *and* $a_1 \geq 0$.

§7. Estimate $((\operatorname{Re}\tilde{Q} - T_1 + \bar{\kappa}\mu\lambda)u, u)$

We write $\operatorname{Re}\tilde{Q} = \operatorname{Re}\tilde{Q} - T_1 + \bar{\kappa}\mu\lambda + (T_1 - \bar{\kappa}\mu\lambda)$ with $\bar{\kappa} > 0$ in (3.10) and study $((\operatorname{Re}\tilde{Q} - T_1 + \bar{\kappa}\mu\lambda)u, u)$.

Proposition 7.1. *Let* $c_\pm \in S(1,\bar{g})$ *be real. Then we have*

$$\begin{aligned} C\big((q+\bar{\kappa}\mu\lambda)u,u\big) \geq & \sum(\|c_\pm\zeta_\pm|\hat{\theta}|^{1/2}\phi_1 u\|^2 + |(c_\pm\zeta_\pm^2|\hat{\theta}|\phi_1^2 u,u)|) \\ & + |(\phi_2^2 u,u)| + |(w\phi_1^2 u,u)| + |(\omega\phi_1^2 u,u)| \\ & + |(w^2\lambda^2 u,u)| + |(\omega^2\lambda^2 u,u)| + \|O(E)u\|^2. \end{aligned} \tag{7.1}$$

Proof. In the proof we may assume $\bar{\kappa} = 1$ without restrictions. Write

$$Ma^2\tilde{\zeta}\hat{\theta}\phi_1^2 - (c_+\zeta_+^2\hat{\theta} + c_-\zeta_-^2|\hat{\theta}|)\phi_1^2 = H_+^2\zeta_+^2\hat{\theta}\phi_1^2 + H_-^2\zeta_-^2|\hat{\theta}|\phi_1^2$$

with $H_+ = (Ma^2\nu - c_+)^{1/2}$ and $H_- = (Ma^2\hat{h} - c_-)^{1/2}$ where $M > 0$ is chosen so that $Ma^2\nu - c_+ \geq c$ and $Ma^2\hat{h} - c_- \geq c > 0$. Since $\zeta_\pm|\hat{\theta}|^{1/2} \in S(|\hat{\theta}|^{1/2},\bar{g})$ by Lemma 3.1 noting $H_\pm \in S(1,\bar{g})$ we can write

$$\begin{aligned} &\zeta_\pm|\hat{\theta}|^{1/2}\phi_1 H_\pm \# \zeta_\pm|\hat{\theta}|^{1/2}\phi_1 H_\pm - \zeta_\pm^2|\hat{\theta}|\phi_1^2 H_\pm^2 \\ &\quad = \sum_{|\alpha+\beta|=2} C_{\alpha\beta}(\zeta_\pm|\hat{\theta}|^{1/2}\phi_1 H_\pm)^{(\alpha)}_{(\beta)}(\zeta_\pm|\hat{\theta}|^{1/2}\phi_1 H_\pm)^{(\beta)}_{(\alpha)} \\ &\quad = b_1 w^{-3}\hat{\phi}_1^2 + b_2 w^{-5/2}\hat{\phi}_1 \end{aligned}$$

modulo $\mu^2 S(w^{-2},\bar{g})$, where $b_i \in \mu^2 S(1,\bar{g})$. Writing $b_1 w^{-3}\hat{\phi}_1^2 = c_1 w^{1/2}\phi_1 \# w^{1/2}\phi_1 + R_1$ and $b_2 w^{-5/2}\hat{\phi}_1 = \mu c_2 w^{-1} \# w^{1/2}\phi_1 + R_2$ with $c_i \in S(1,\bar{g})$ and $R_i \in \mu S(w^{-2},\bar{g})$ we conclude

$$\sum|(c_\pm\zeta_\pm^2|\hat{\theta}|\phi_1^2 u,u)| \leq M(a^2\tilde{\zeta}\hat{\theta}\phi_1^2 u,u) + C\|O(E)u\|^2. \tag{7.2}$$

Similarly $c_\pm^2\zeta_\pm^2|\hat\theta|\phi_1^2$ can be written

$$c_\pm\zeta_\pm|\hat\theta|^{1/2}\phi_1\#c_\pm\zeta_\pm|\hat\theta|^{1/2}\phi_1+b_\pm\mu^2w^{-5/2}\hat\phi_1+b'_\pm\mu^2w^{-3}\hat\phi_1^2+R$$

with $R\in S(w^{-2},\bar g)$. Thus $\|c_\pm\zeta_\pm|\hat\theta|^{1/2}\phi_1u\|^2$ is estimated also by the right-hand side of (7.2).

We next study $\tilde q=\phi_2^2+\chi_2a^2\phi_1^4\lambda^{-2}+\mu\lambda$. If $\chi_2\neq 1$ so that $\hat\phi_1^2\leq d_3w$ it is clear $\hat\phi_1^4\leq C(\hat\phi_2^2+\lambda_\mu^{-1})$ and then $\hat\phi_1^4\lambda^2\leq C(\phi_2^2+\mu\lambda)$ so we have $\tilde q\geq c\hat\phi_1^4\lambda^2$ with some $c>0$. If $\chi_2=1$ this inequality is obvious. Since $\phi_1^4\lambda^{-2}+\mu\lambda=\lambda^2\omega^2$ and $\phi_2^2+\mu\lambda=w^2\lambda^2$ it is obvious $\tilde q\geq c(w^2+\omega^2)\lambda^2$ with some $c>0$. Let us set $\tilde q-c\omega^2\lambda^2=F^2$ with $F=\lambda(\tilde q\lambda^{-2}-c\omega^2)^{1/2}\in S(\lambda,\bar g)$. If we note $\chi_2a^2\hat\phi_1^4\in S(w^2,g)$ and $\omega\in S(\omega,G_1)$ with $G_1=\omega^{-1/2}(|dx|^2+\langle\xi'\rangle_\mu^{-2}|d\xi'|^2)$ it is not difficult to see that $F^2=F\#F+R$ with $R\in\mu^2S(w^{-1}+\omega^{-1},\bar g)$. Thus we conclude that

$$(\tilde qu,u)\geq c(\omega^2\lambda^2u,u)-C\mu^4(\|\omega^{-1}u\|^2+\|w^{-1}u\|^2)-C\|u\|^2.$$

Repeating a similar argument we get $(\tilde qu,u)\geq c\|\hat\phi_1^2\lambda u\|^2-C\mu^4\|w^{-1}u\|^2-C\|u\|^2$. Since $\omega^2\lambda^2=\omega\lambda\#\omega\lambda+R$ with $R\in\mu^2S(\omega^{-1},\bar g)$, hence $(\omega^2\lambda^2u,u)\geq\|\omega\lambda u\|^2-C(\mu^4\|\omega^{-1}u\|^2+\|u\|^2)$. Recalling $\phi_2^2+\mu\lambda=w^2\lambda^2$ similar arguments show

$$((\phi_2^2+\mu\lambda)u,u)\geq c(w^2\lambda^2u,u)+\|w\lambda u\|^2-C(\mu^4\|w^{-1}u\|^2+\|u\|^2).$$

Noting $\mu\lambda\leq w^2\lambda^2\in S(\lambda^2,g_0)$ we see $(w^2\lambda^2u,u)\geq\mu\|\lambda^{1/2}u\|^2-C\|u\|^2$. On the other hand, since one can write $w^{-1}=(w^{-1}\lambda^{-1/2})\#\lambda^{1/2}+R$ with $R\in S(1,\bar g)$, noting that $w^{-1}\lambda^{-1/2}\in\mu^{-1/2}S(1,\bar g)$ we have $\|w^{-1}u\|^2\leq C\mu^{-1}\|\lambda^{1/2}u\|^2+C\|u\|^2$. Similarly we have $\|\omega^{-1}u\|^2\leq C\mu^{-1}\|\lambda^{1/2}u\|^2+C\|u\|^2$. Thus we get

$$\begin{aligned}&\mu^2(\|w^{-1}u\|^2+\|\omega^{-1}u\|^2)+\mu\|\lambda^{1/2}u\|^2+\|w\lambda u\|^2+\|\omega\lambda u\|^2\\&\qquad+|(w^2\lambda^2u,u)|+|(\omega^2\lambda u,u)|+\|\hat\phi_1^2\lambda u\|^2\\&\quad\leq C(\tilde qu,u)+C\|u\|^2.\end{aligned}\tag{7.3}$$

Noting $w^{1/2}\phi_1\#w^{1/2}\phi_1=w\phi_1^2+R$ with $R\in\mu^2S(w^{-2},\bar g)$ and $w\phi_1^2=\mathsf{Re}(\lambda\hat\phi_1^2\#w\lambda)+R$ with $R\in\mu^2S(w^{-2},\bar g)$, we have

$$\|w^{1/2}\phi_1u\|^2+|(w\phi_1^2u,u)|\leq C\|O(E)u\|^2.$$

We get $\|\omega^{1/2}\phi_1u\|^2+|(\omega\phi_1^2u,u)|\leq C(\|\lambda\hat\phi_1^2u\|^2+\|\omega\lambda u\|^2+\mu^2\|\omega^{-1}u\|^2)$ by a repetition of similar arguments. It is easy to see $\|\phi_2u\|^2+|(\phi_2^2u,u)|\leq C((\phi_2^2+\mu\lambda)u,u)+\|u\|^2)$; then we conclude the assertion by (7.3). □

Corollary 7.2. *We have* $\|\hat\theta\phi_1u\|^2+|(\hat\theta\phi_1^2u,u)|\leq C((q+\mu\lambda)u,u)+C\|u\|^2$.

Proof. Take $\eta(s) \in C_0^\infty(\mathbb{R})$ so that $\zeta_- + \zeta_+ + \eta = 1$. Thanks to Proposition 7.1 it suffices to prove $|(\eta\hat{\theta}\phi_1^2 u, u)| \le C((q+\mu\lambda)u, u) + C\|u\|^2$. Note that one can write $\eta\hat{\theta}\phi_1^2 = cw\phi_1^2$; then the assertion follows immediately. □

Lemma 7.3. *Let $\chi_0 = \chi_0(\hat{\phi}_1^2 w^{-1})$ with $\chi_0(s) \in C_0^\infty(\mathbb{R})$ which is 1 near $s = 0$. Then we have*

$$((1-\chi_0)\zeta_\pm^2|\hat{\theta}|w\lambda^2 u, u) \le C((q+\mu\lambda)u, u) + C\|u\|^2.$$

Proof. Note that $Ma^2\tilde{\zeta}\hat{\theta}\phi_1^2 - (1-\chi_0)(\zeta_+^2\hat{\theta} + \zeta_-^2|\hat{\theta}|)w\lambda^2 = H_+^2\zeta_+^2\hat{\theta}\phi_1^2 + H_-^2\zeta_-^2|\hat{\theta}|\phi_1^2$ where $H_+ = (Ma^2\nu - (1-\chi_0)w\hat{\phi}_1^{-2})^{1/2}$ and $H_- = (Ma^2\hat{h} - (1-\chi_0)w\hat{\phi}_1^{-2})^{1/2}$ which are in $S(1,\bar{g})$ taking $M > 0$ large. The rest of the proof is just a repetition of the proof of Proposition 7.1. □

Since $0 \le \zeta_+ a_1\hat{\theta}w^{-2} \in \mu S(w^{-2}\lambda_\mu^{+0}, \bar{g})$ then $(\zeta_+ a_1 w^{-2}\hat{\theta}u, u)$ is bounded from below by $-C\|\lambda_\mu^{+0}u\|^2$. On the other hand, noting $\|w^{-1/2}u\|^2 \le C\gamma^{-1}\|w^{-1}u\|^2 + C\gamma\|u\|^2$ it is easy to check

$$|((a_2 w^{-1/2}\hat{\theta}\phi_1 + a_3\phi_1)u, u)| \le C\gamma^{-1/2}\|O(E)u\|^2 + C\gamma^{3/2}\|\lambda_\mu^{+0}u\|^2. \tag{7.4}$$

From Proposition 7.1 and Corollary 7.2 together with (7.4) we obtain the following proposition.

Proposition 7.4. *There exist $\gamma_0 > 0$, $\mu_0 > 0$, $n_0 > 0$ such that we have*

$$C((\mathsf{Re}\,\tilde{Q} - T_1 + \bar{\kappa}\mu\lambda)u, u) \ge |(\hat{\theta}\phi_1^2 u, u)| + \|\hat{\theta}\phi_1 u\|^2 + \|O(E)u\|^2$$

modulo $C\gamma^{3/2}\|\lambda_\mu^{+0}u\|^2$ for $\gamma \ge \gamma_0$, $0 < \mu < \mu_0$ and $n \ge n_0$. We have also

$$\begin{aligned} &C(\lambda_\mu^{2\varepsilon}(\mathsf{Re}\,\tilde{Q} - T_1 + \bar{\kappa}\mu\lambda)u, u) + C\gamma^2\|\lambda_\mu^{3\varepsilon}u\|^2 \\ &\qquad \ge |(\lambda_\mu^{2\varepsilon}\hat{\theta}\phi_1^2 u, u)| + \|\lambda_\mu^\varepsilon\hat{\theta}\phi_1 u\|^2 + \|\lambda_\mu^\varepsilon O(E)u\|^2. \end{aligned}$$

§8. Estimate $\mathsf{Re}((\mathsf{Re}\,\tilde{Q} - T_1 + \bar{\kappa}\mu\lambda)u, (\mathsf{Im}\,\tilde{\lambda})u)$

Recall Lemma 4.3 which gives $\mathsf{Im}\,\tilde{\lambda} = n\tilde{e}_1\Gamma\zeta_+^2\hat{\theta} + R_1$ with $R_1 \in S(\lambda_\mu^{+0}, \bar{g})$. Denote $\tilde{q} = \phi_2^2 + \chi_2 a^2\phi_1^4\lambda^{-2} + \mu\lambda$ again. Note $\mathsf{Re}(\tilde{e}_1\hat{\theta}\zeta_+^2\Gamma\#\tilde{q}) = \tilde{e}_1\zeta_+^2\hat{\theta}\tilde{q}\Gamma + R$ with $R \in \mu S(\lambda^{1+0}, \bar{g})$ since $\Gamma \in S(w^{-1}\lambda_\mu^{+0}, \tilde{g})$ and $\phi_2^2 + \chi_2 a^2\phi_1^4\lambda^{-2} \in S(w^2\lambda^2, \tilde{g})$. Thus noting $|(Ru, u)| \le C\mu\|\lambda^{1/2+0}u\|^2$ we get

$$\mathsf{Re}(\tilde{q}u, \tilde{e}_1\hat{\theta}\zeta_+^2\Gamma u) \ge (\tilde{e}_1\zeta_+^2\hat{\theta}\tilde{q}\Gamma u, u) - C\|\lambda_\mu^{+0}O(E)u\|^2.$$

Write $M\tilde{e}_1\zeta_+^2\hat{\theta}\tilde{q}\Gamma - \mu\zeta_+^2\hat{\theta}w^2\lambda^2\Gamma = H\#(M\tilde{e}_1\tilde{q}w^{-2}\lambda^{-2} - \mu)\Gamma\#H + R$ with $H = \zeta_+\hat{\theta}^{1/2}w\lambda$ and $R \in \mu S(\lambda^{1+0}, \bar{g})$. Since $0 \le (M\tilde{e}_1\tilde{q}w^{-2}\lambda^{-2} - \mu)\Gamma \in \mu S(w^{-1}\lambda_\mu^{+0}, \bar{g}) \subset$

$\mu S^{1/2+0}_{3/4,1/2}$ then from the Fefferman–Phong inequality it follows that

$$M(\tilde{e}_1\zeta_+^2\hat{\theta}\tilde{q}\Gamma u,u)-\mu(\zeta_+^2\hat{\theta}w^2\lambda^2\Gamma u,u)\geq -C\|\lambda_\mu^{+0}O(E)u\|^2.$$

Since $\zeta_+^2\hat{\theta}w^2\lambda^2\Gamma=H\#\Gamma\#H+R$ with $R\in\mu S(\lambda^{1+0},\bar{g})$, taking $(\zeta_+^2-\zeta^2)\hat{\theta}\in S(w,g)$ into account we conclude

$$\begin{aligned}\mu(\Gamma(\zeta\hat{\theta}^{1/2}w\lambda u),\zeta\hat{\theta}^{1/2}w\lambda u)+\mu|(\zeta^2\hat{\theta}w^2\lambda^2\Gamma u,u)|\\ \leq M\,\mathsf{Re}(\tilde{q}u,\tilde{e}_1\hat{\theta}\zeta_+^2\Gamma u)+C\|\lambda_\mu^{+0}O(E)u\|^2.\end{aligned}\tag{8.1}$$

Noting $\Gamma=r+2\omega\rho^{-2}$ and $\omega^s r\in S(w^{s-1}\lambda_\mu^{+0},\tilde{g})$ for $s\geq 0$, a repetition of a similar argument for ω instead of w shows (8.1) where w is replaced by ω. Note that $(w^2+\omega^2)\Gamma\geq\chi^2 w+\omega^3\rho^{-2}$. It is easy to check that $w+\omega^3\rho^{-2}\geq c\rho$ with some $c>0$ and on the support of $1-\chi^2$ we have $C\omega\geq\rho\geq\omega$. Therefore we obtain

$$C\rho\geq(w^2+\omega^2)\Gamma\geq c\rho$$

with some $c>0$. Then applying the Fefferman–Phong inequality one obtains $(\zeta^2\hat{\theta}(w^2+\omega^2)\lambda^2\Gamma u,u)\geq c\|\zeta\hat{\theta}^{1/2}\rho^{1/2}\lambda u\|^2-C\|\lambda_\mu^{+0}O(E)u\|^2$. Thus we have the following lemma.

Lemma 8.1. *We have*

$$\mu|(\zeta^2\hat{\theta}\rho\lambda^2u,u)|+\mu\|\zeta\hat{\theta}^{1/2}\rho^{1/2}\lambda u\|^2\leq C\,\mathsf{Re}(\tilde{q}u,\tilde{e}_1\hat{\theta}\zeta_+^2\Gamma u)+C\|\lambda_\mu^{+0}0(E)u\|^2.$$

We turn to $\mathsf{Re}(a^2\tilde{\zeta}\hat{\theta}\phi_1^2u,\tilde{e}_1\zeta_+^2\hat{\theta}\Gamma u)$. Since $\Gamma=r+2\omega\rho^{-2}$ and $r\hat{\theta}^2\in S(w,\bar{g})$ and $\omega\rho^{-2}\hat{\phi}_1^2\in S(1,\bar{g})$ we see that $\mathsf{Re}(\tilde{e}_1\zeta_+^2\hat{\theta}\Gamma\#a^2\tilde{\zeta}\hat{\theta}\phi_1^2)$ can be written

$$\nu\tilde{e}_1\zeta_+^4\hat{\theta}^2a^2\phi_1^2\Gamma+\sum_{|\alpha+\beta|=2}\frac{(-1)^{|\beta|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!}(\tilde{e}_1\zeta_+^2\hat{\theta}\Gamma)^{(\alpha)}_{(\beta)}(a^2\tilde{\zeta}\hat{\theta}\phi_1^2)^{(\beta)}_{(\alpha)}+R$$

with $R\in\mu S(\lambda,\bar{g})$. Consider $(\tilde{e}_1\zeta_+^2\hat{\theta})^{(\alpha_2)}_{(\beta_2)}\Gamma^{(\alpha_1)}_{(\beta_1)}(a^2\tilde{\zeta}\hat{\theta})^{(\beta'')}_{(\alpha'')}(\phi_1^2)^{(\beta')}_{(\alpha')}$ for $|\alpha+\beta|=2$. By Lemma 3.3 it is not difficult to see that we can write such a term as

$$\begin{aligned}\nu c\zeta_+^2w\lambda^2\hat{\theta}^2+\zeta_+(c_{21}w^{-1/2}\lambda\hat{\theta}+c_{22}w^{-1}\phi_1\hat{\theta}+c_{23}w^{-3/2}\hat{\theta}\hat{\phi}_1^2\lambda)\\ +(c_{31}w^{-1/2}\phi_1+c_{32}w^{-1}\hat{\phi}_1^2\lambda)\end{aligned}\tag{8.2}$$

with $c\in\mu S(1,\bar{g})$ and $c_{ij}\in\mu^2S(1,\bar{g})$. One can estimate the last term, the linear combination with c_{3j} coefficients, applying Proposition 7.1. The second term, the linear combination with c_{2j} coefficients, can be estimated thanks to Propositions 5.3 and 7.1. Indeed, writing $c_{23}\zeta_+\hat{\theta}\hat{\phi}_1^2\lambda=\mathsf{Re}(c_{23}\zeta_+w^{-3/2}\hat{\theta}\#\hat{\phi}_1^2\lambda)+R$ with $R\in S(w^{1/2}\lambda,\bar{g})$ we have

$$|\,\mathsf{Re}(c_{23}\zeta_+w^{-3/2}\hat{\theta}\hat{\phi}_1^2\lambda u,u)|\leq C\mu^2\gamma^{-1/2}\|\zeta_+w^{-3/2}\hat{\theta}u\|^2+C\gamma^{1/2}\|O(E)u\|^2.$$

Other terms can be estimated similarly. To estimate the first term in (8.2), choosing $\nu > 0$ small we write $\zeta_+^2\hat{\theta}w\lambda^2 - \nu c\zeta_+^2 w\hat{\theta}^2\lambda^2 = H\#H + R$ with $H = \zeta_+\hat{\theta}^{1/2}w^{1/2}\lambda(1-\nu c\hat{\theta})^{1/2}$ and $R \in S(w^2\lambda^{2+0}, \bar{g})$ and apply Lemma 8.1. We now prove the following result.

Lemma 8.2. *There are $c > 0$ and $\nu_0 > 0$ such that we have*

$$\begin{aligned} \mathsf{Re}(a^2\tilde{\zeta}\hat{\theta}\phi_1^2 u, \tilde{e}_1\zeta_+^2\hat{\theta}\Gamma u) \geq\; & c\nu\mu(\Gamma(\zeta_+\hat{\theta}\phi_1)u, \zeta_+\hat{\theta}\phi_1 u) \\ & - C(\mu^3 n + \gamma^{-1/2})\|\zeta w^{-3/2}\hat{\theta}u\|^2 \\ & - C(\mu n^{-1} + \gamma^{-1/2})\|\zeta w^{1/2}\hat{\theta}\lambda u\|^2 \\ & - C\|\hat{\theta}\phi_1 u\|^2 - C\gamma^{1/2}\|O(E)u\|^2 \end{aligned} \tag{8.3}$$

for $0 < \nu \leq \nu_0$.

Proof. It remains to estimate $\nu\,\mathsf{Re}(\tilde{e}_1\zeta_+^4\hat{\theta}^2 a^2\phi_1^2\Gamma u, u)$ from below. Since $(\zeta_+^4 - \zeta_+^2)\hat{\theta}^2\phi_1^2\Gamma \in S(w\phi_1^2\lambda_\mu^{+0}, \bar{g})$ it suffices to study $\nu\,\mathsf{Re}(\tilde{e}_1\zeta_+^2\hat{\theta}^2 a^2\phi_1^2\Gamma u, u)$. Note that

$$\begin{aligned} \mathsf{Re}(\zeta_+\hat{\theta}\phi_1\#\tilde{e}_1 a^2\Gamma\#\zeta_+\hat{\theta}\phi_1) = \tilde{e}_1\zeta_+^2\hat{\theta}^2 a^2\phi_1^2\Gamma - \sum \frac{(-1)^{|\beta_1+\beta_2+\beta_3|}}{4\alpha_1!\beta_1!\cdots\beta_3!} \\ \times (\zeta_+\hat{\theta}\phi_1)^{(\alpha_1+\alpha_2)}_{(\beta_1+\beta_2)}(\tilde{e}_1 a^2\Gamma)^{(\beta_1+\alpha_3)}_{(\alpha_1+\beta_3)}(\zeta_+\hat{\theta}\phi_1)^{(\beta_2+\beta_3)}_{(\alpha_2+\alpha_3)} + R, \end{aligned}$$

where the sum is taken over $|\alpha_1 + \beta_1 + \cdots + \beta_3| = 2$ and $R \in \mu^2 S(\lambda^{1+0}, \bar{g})$ which follows from Lemma 3.3. Here it can be checked that the second term is written

$$c_1\zeta^2 w^{-1}\lambda\hat{\theta}^2 + c_2\zeta w^{-1}\hat{\theta}\phi_1 + c_3 w^{-1}\hat{\phi}_1^2\lambda + c_4 w^{-1/2}\phi_1 + c_5 w^{-1/2}\zeta\hat{\theta}\lambda$$

with $c_i \in \mu^2 S(\lambda_\mu^{+0}, \bar{g})$ modulo $\mu^2 S(w^{-1}\lambda_\mu^{+0}, \bar{g})$. To estimate the first term let us write $c_1\zeta^2 w^{-1}\lambda\hat{\theta}^2 = \mathsf{Re}(c_1\zeta w^{-3/2}\hat{\theta}\#\zeta w^{1/2}\lambda\hat{\theta}) + R$ with $R \in \mu^2 S(\lambda^{1+0}, \bar{g})$. Then one can estimate $|\,\mathsf{Re}(c_1\zeta^2 w^{-1}\hat{\theta}^2 u, u)|$ by

$$C\mu^3 n\|\zeta w^{-3/2}\hat{\theta}u\|^2 + C\mu n^{-1}\|\zeta w^{1/2}\lambda\hat{\theta}u\|^2 + C\|\lambda_\mu^{+0}O(E)u\|^2.$$

It is easy to see that $|((c_2\zeta w^{-1}\hat{\theta}\phi_1 + c_3 w^{-1}\hat{\phi}_1^2\lambda + c_4 w^{-1/2}\phi_1 + c_5 w^{-1/2}\zeta\hat{\theta}\lambda)u, u)|$ is bounded by $C\gamma^{-1/2}(\|\zeta w^{-3/2}\hat{\theta}u\|^2 + \|\zeta w^{1/2}\hat{\theta}\lambda u\|^2) + C\gamma^{1/2}\|O(E)u\|^2$. To end the proof it suffices to apply the Fefferman–Phong inequality to obtain

$$\mathsf{Re}(\tilde{e}_1 a^2\Gamma(\zeta_+\hat{\theta}\phi_1 u), \zeta_+\hat{\theta}\phi_1 u) \geq c\mu\,\mathsf{Re}(\Gamma(\zeta_+\hat{\theta}\phi_1 u), \zeta_+\hat{\theta}\phi_1 u) - C\|\hat{\theta}\phi_1 u\|^2$$

because $\tilde{e}_1 a^2 - c\mu \geq 0$ with some $c > 0$. □

Similar arguments proving Lemma 8.2 show the estimate

$$\begin{aligned} \mathsf{Re}(a^2\chi_2\hat{\phi}_1^4\lambda^2 u, \tilde{e}_1\zeta_+^2\hat{\theta}\Gamma u) \geq & -C\gamma^{-1/2}(\|\zeta_+ w^{-3/2}\hat{\theta}u\|^2 + \|\zeta_+ w^{1/2}\hat{\theta}\lambda u\|^2) \\ & - C\gamma^{1/2}\|\lambda_\mu^{+0}O(E)u\|^2. \end{aligned}$$

We turn to consider

$$((a_1\mu w^{-2}\hat{\theta}+a_2 w^{-1/2}\hat{\theta}\phi_1+a_3\phi_1)u,(\mathsf{Im}\,\tilde{\lambda})u). \tag{8.4}$$

To handle (8.4) we prepare a lemma.

Lemma 8.3. *We have*

$$\mathsf{Re}(\Gamma u,v)\le(\Gamma v,v)+(\Gamma w,w)+C(\|\lambda_\mu^{+0}v\|^2+\|\lambda_\mu^{+0}w\|^2).$$

Proof. Since $0\le\Gamma\in S(\lambda_\mu^{1/2+0},\bar{g})$ it follows from the Fefferman–Phong inequality that $(\Gamma u,u)\ge -C\|\lambda_\mu^{+0}u\|^2$ with some $C>0$. Thus with $L=\Gamma+C\lambda_\mu^{+0}$ we have $(Lu,u)\ge 0$ so that $|\,\mathsf{Re}(Lu,v)|\le(Lu,u)+(Lv,v)$ which proves the assertion. □

Write $\mathsf{Re}(\tilde{e}_1\zeta_+^2\hat{\theta}\Gamma\#a_1\mu w^{-2}\hat{\theta})=\mu\,\mathsf{Re}(\Gamma\zeta_+\hat{\theta}w\lambda\#a\zeta_+w^{-1}\hat{\theta})+R$ where $a=\tilde{e}_1a_1w^{-2}\lambda^{-1}\in\mu S(\lambda_\mu^{+0},\bar{g})$ and $R\in\mu^2S(w^{-2},\bar{g})$ and apply Lemma 8.3 to get

$$\begin{aligned}|\,\mathsf{Re}(\tilde{e}_1\Gamma\zeta_+^2\hat{\theta}u,a_1\mu w^{-2}\hat{\theta}u)|\le{}& C\mu n^{-1}\,\mathsf{Re}(\Gamma\zeta_+\hat{\theta}w\lambda u,\zeta_+\hat{\theta}w\lambda u)\\&+C\mu n\,\mathsf{Re}(\Gamma a\zeta_+w^{-1}\hat{\theta}u,a\zeta_+w^{-1}\hat{\theta}u)\\&+C\|\lambda_\mu^{+0}O(E)u\|^2.\end{aligned}$$

Since $|\,\mathsf{Re}(\Gamma a\zeta_+w^{-1}\hat{\theta}u,a\zeta_+w^{-1}\hat{\theta}u)|\le C\mu^2(\|\zeta_+w^{-3/2}\hat{\theta}u\|^2+\|w^{-1}\lambda_\mu^{+0}u\|^2)$ we conclude

$$\begin{aligned}|\,\mathsf{Re}(\tilde{e}_1\Gamma\zeta_+^2\hat{\theta}u,a_1\mu w^{-2}\hat{\theta}u)|\le{}& C\mu n^{-1}\,\mathsf{Re}(\Gamma\zeta_+\hat{\theta}w\lambda u,\zeta_+\hat{\theta}w\lambda u)\\&+C\mu^3n\|\zeta_+w^{-3/2}\hat{\theta}u\|^2+C\|\lambda_\mu^{+0}O(E)u\|^2).\end{aligned}$$

Similar arguments show

$$|\,\mathsf{Re}(\tilde{e}_1\Gamma\zeta_+^2\hat{\theta}u,a_3\phi_1u)|\le\gamma^{-1/2}\,\mathsf{Re}(\Gamma\zeta_+\hat{\theta}w\lambda u,\zeta_+\hat{\theta}w\lambda u)+C\gamma^{1/2}\|\lambda_\mu^{+0}O(E)u\|^2.$$

Repeating similar arguments we conclude that (8.4) is bounded by

$$\begin{aligned}&C(\mu n^{-1}+\gamma^{-1/2})\,\mathsf{Re}(\Gamma\zeta_+\hat{\theta}w\lambda u,\zeta_+\hat{\theta}w\lambda u)\\&\quad+C\mu^3n\|\zeta_+w^{-3/2}\hat{\theta}u\|^2+C\gamma^{1/2}\|O(E)u\|^2.\end{aligned}$$

We finally consider the term (qu,bu) with $b\in S(\lambda_\mu^{+0},\bar{g})$. Noting $\tilde{\zeta}'\hat{\theta}^{1/2}\in S(w^{1/2},\bar{g})$ one sees

$$\mathsf{Re}(b\#a^2\tilde{\zeta}^2\hat{\theta}\phi_1^2)=\mathsf{Re}(ba\tilde{\zeta}\hat{\theta}^{1/2}\phi_1\#a\tilde{\zeta}\hat{\theta}^{1/2}\phi_1)+O(E)\cdot O(E)+O(E)$$

and hence one obtains $|(qu,bu)|\le C\|O(E)u\|^2$. We summarize in the following proposition.

Proposition 8.4. *There exist* $c > 0$ *and* $\gamma_0 > 0$, $\mu_0 > 0$, $n_0 > 0$, $\nu_0 > 0$ *such that we have*

$$C\{\gamma((q+\mu\lambda)u,u)+\gamma^3\|u\|^2+\mathsf{Re}((\mathsf{Re}\,\tilde{Q}-T_1+\bar{\kappa}\mu\lambda)u,\mathsf{Im}\,\tilde{\lambda}\,u)\\ +\mu n\|\chi\zeta w^{-1/2}\tilde{\Lambda}u\|^2\}\\ \geq cn\nu\mu(\Gamma(\zeta_+\hat{\theta}\phi_1)u,\zeta_+\hat{\theta}\phi_1 u)+cn\mu|(\zeta^2\hat{\theta}\rho\lambda^2u,u)|+cn\mu\|\zeta\hat{\theta}^{1/2}\rho^{1/2}\lambda u\|^2$$

for $\gamma \geq \gamma_0$, $0 < \mu < \mu_0$, $n \geq n_0$ *and* $0 < \nu \leq \nu_0$.

§9. Estimates of error terms

We estimate $\mathsf{Re}(\tilde{\Lambda}u,(\mathsf{Im}\,\tilde{Q}-T_2)u)$. Recall

$$\mathsf{Im}\,\tilde{Q}-T_2=-\nu n\zeta_+^2a^2\hat{\theta}\phi_1\Gamma\{\phi_1,\hat{\phi}_2\}+a_1\mu w^{-3/2}\hat{\theta}+a_2w^{1/2}\hat{\theta}\lambda\\ +a_3\hat{\theta}\phi_1+c_1O(E)+c_2O(E)+\mu S(\lambda_\mu^{+0},\bar{g})O(E).$$

Thanks to Lemma 3.3 one can write

$$a^2\zeta_+^2\hat{\theta}\phi_1\{\phi_1,\hat{\phi}_2\}\Gamma=\mu\zeta_+\#\hat{a}\Gamma\#\zeta_+\hat{\theta}\phi_1+c_1\zeta_+\hat{\theta}\lambda+c_2\zeta_+\phi_1$$

with $\hat{a}=\mu^{-1}a^2\{\phi_1,\hat{\phi}_2\}$ modulo $\lambda_\mu^{+0}O(E)$ where $c_i\in\mu S(\lambda_\mu^{+0},\bar{g})$. Noting $\zeta\zeta_+=\zeta_+$ it follows from Lemma 8.3 that

$$\nu n\,\mathsf{Re}(a^2\zeta_+^2\hat{\theta}\phi_1\{\phi_1,\hat{\phi}_2\}\Gamma u,\tilde{\Lambda}u)\leq\epsilon^{-1}n\nu^2\mu(\hat{a}\Gamma(\zeta_+\hat{\theta}\phi_1)u,(\zeta_+\hat{\theta}\phi_1)u)\\ +\epsilon n\mu(\hat{a}\Gamma\zeta(\tilde{\Lambda}u),\zeta(\tilde{\Lambda}u))+cn\nu\mu\|\zeta_+\rho^{-1/2}\tilde{\Lambda}u\|^2\\ +cn\nu\mu\|\zeta_+\rho^{1/2}\hat{\theta}\lambda u\|^2\\ +C(\|\lambda_\mu^{+0}\hat{\theta}\phi_1u\|^2+\|\lambda_\mu^{+0}\tilde{\Lambda}u\|^2+\|\lambda_\mu^{+0}O(E)u\|^2)$$

where $\epsilon>0$ will be determined later. We turn to estimating

$$((a_1\mu w^{-3/2}\hat{\theta}+a_2w^{1/2}\hat{\theta}\lambda+a_3\hat{\theta}\phi_1)u,\tilde{\Lambda}u).$$

It is easy to see that this is bounded by

$$C\gamma^{-1/2}(\|\zeta w^{-3/2}\hat{\theta}u\|^2+\|\zeta w^{1/2}\hat{\theta}\lambda u\|^2)\\ +C\gamma^{1/2}(\|\tilde{\Lambda}u\|^2+\|\hat{\theta}\phi_1u\|^2+\|\lambda_\mu^{+0}O(E)u\|^2).$$

Finally we consider $|(c_1O(E)u+c_2O(E)u,\tilde{\Lambda}u)|$. Recalling Corollary 6.6 it is easily seen that this term is estimated by

$$C\gamma^{-1/2}(\|\zeta\chi w^{-1/2}\tilde{\Lambda}u\|^2+\|\zeta\rho^{-1/2}\tilde{\Lambda}u\|^2)+C\gamma^{1/2}\|\lambda_\mu^{+0}O(E)u\|^2.$$

Noting $\|w^{1/2}\phi_1u\|^2+\|\omega^{1/2}\phi_1u\|^2\geq\|\rho^{1/2}\phi_1u\|^2-C\|O(E)u\|^2$ we obtain the following proposition.

Proposition 9.1. *The term* $|\,\mathsf{Re}(\tilde{\Lambda}u, (\mathsf{Im}\,\tilde{Q} - T_2)u)|$ *is bounded by*

$$\begin{aligned} &c\epsilon^{-1}n\nu^2\mu(\Gamma(\zeta_+\hat{\theta}\phi_1)u, (\zeta_+\hat{\theta}\phi_1)u) + c\epsilon n\mu(\Gamma\zeta(\tilde{\Lambda}u), \zeta(\tilde{\Lambda}u)) \\ &\quad + (cn\nu\mu + C\gamma^{-1/2})(\|\zeta\rho^{-1/2}\tilde{\Lambda}u\|^2 + \|\zeta\rho^{1/2}\hat{\theta}\lambda u\|^2) \\ &\quad + C\gamma^{-1/2}(\|\zeta w^{-3/2}\hat{\theta}u\|^2 + \|\zeta\chi w^{-1/2}\tilde{\Lambda}u\|^2) \\ &\quad + C\gamma^{1/2}(\|\hat{\theta}\phi_1 u\|^2 + \|\tilde{\Lambda}u\|^2 + \|\lambda_\mu^{+0}O(E)u\|^2) \end{aligned}$$

where $c > 0$ *is independent of* ϵ, ν, μ *and* γ.

We turn to consider $([D_0 - \mathsf{Re}\,\tilde{\lambda}, \mathsf{Re}\,\tilde{Q} - T_1]u, u)$. Recall

$$\xi_0 - \mathsf{Re}\,\tilde{\lambda} = \xi_0 - \phi_1 + \psi + n(b_2\hat{\theta} + b_3\hat{\phi}_1^2)w^{-1/2} + R_1$$

where $\psi = \tilde{\zeta}\hat{\theta}\phi_1 + \chi_2\hat{\phi}_1^3\lambda$ and $R \in S(\lambda_\mu^{+0}, \bar{g})$ and

$$\mathsf{Re}\,\tilde{Q} - T_1 = q + \zeta_+(a_1\mu w^{-2}\hat{\theta} + a_2 w^{-1/2}\hat{\theta}\phi_1 + a_3\phi_1) + \mu S(\lambda_\mu^{+0}, \bar{g})O(E)$$

with $q = \phi_2^2 + 2\tilde{\zeta}a^2\hat{\theta}\phi_1^2 + 2\chi_2 a^2\hat{\phi}_1^4\lambda^2$. Let us study $(\phi_2\{\xi_0 - \phi_1 + \psi, \phi_2\}u, u)$. Taking (3.9) into account it suffices to estimate

$$\nu(c_1\zeta_+^2\hat{\theta}\phi_2\lambda u, u), \quad (c_2\phi_2\hat{\phi}_1^2\lambda u, u), \quad (c_3\hat{\theta}\phi_2\phi_1 u, u)$$

where $c_j \in \mu S(1, \bar{g})$. Write $\zeta_+^2\hat{\theta}\phi_2\lambda = (1 - \chi^2)\zeta_+^2\hat{\theta}\phi_2\lambda + \chi^2\zeta_+^2\hat{\theta}\phi_2\lambda$ and consider $M\zeta_+^2\hat{\theta}\phi_1^2 - (1 - \chi^2)\zeta_+^2\hat{\theta}\phi_2\lambda$ with a large $M > 0$. Note

$$M^2\zeta_+^2\hat{\theta}\phi_1^2 - (1 - \chi^2)\zeta_+^2\hat{\theta}\phi_2\lambda = (M\phi_1)^2 F$$

where $0 \le F = \zeta_+^2\hat{\theta}\big(1 - (1 - \chi^2)\hat{\phi}_2\hat{\phi}_1^{-2}/M^2\big) \in S(1, g)$. Writing $(M\phi_1)^2 F = \mathsf{Re}(M\phi_1 \# F \# M\phi_1) + R$ with $R \in S(w^{-2}, \bar{g})$ we obtain from the Fefferman–Phong inequality that

$$M^2(\zeta_+^2\hat{\theta}\phi_1^2 u, u) \ge ((1 - \chi^2)\zeta_+^2\hat{\theta}\phi_2\lambda u, u) - C\|O(E)u\|^2.$$

Consider now $2w\chi^2\zeta_+^2\hat{\theta}\lambda^2 - \chi^2\zeta^2\hat{\theta}\phi_2\lambda = (w^{1/2}\lambda)^2 F$ with $0 \le F = \chi^2\zeta_+^2\hat{\theta}(2 - \hat{\phi}_2 w^{-1}) \in S(1, \bar{g})$. Since $(w^{1/2}\lambda)^2 F = \mathsf{Re}(w^{1/2}\lambda \# F \# w^{1/2}\lambda) + R$ with $R \in \mu S(\lambda, \bar{g})$ from the Fefferman–Phong inequality again one has

$$2(w\chi^2\zeta_+^2\hat{\theta}\lambda^2 u, u) \ge (\chi^2\zeta_+^2\hat{\theta}\phi_2\lambda u, u) - C\|w^{1/2}\lambda^{1/2}u\|^2 - C\|O(E)u\|^2.$$

Here we note that $w^{1/2}\lambda^{1/2} \# w^{1/2}\lambda^{1/2} = w\lambda + R$ with $R \in S(1, \bar{g})$ and hence

$$2(w\chi^2\zeta_+^2\hat{\theta}\lambda^2 u, u) \ge (\chi^2\zeta_+^2\hat{\theta}\phi_2\lambda u, u) - C\|O(E)u\|^2.$$

It is easy to see $|(c_3\hat{\theta}\phi_1\phi_2 u, u)| \le C(\|\hat{\theta}\phi_1 u\|^2 + \|O(E)u\|^2)$; then we have the following lemma.

Lemma 9.2. *There exists $C>0$ such that*

$$|(\{\xi_0-\phi_1+\psi,\phi_2^2\}u,u)|\le C\nu\mu(\chi^2\zeta_+^2\hat\theta w\lambda^2u,u)\\ +C(\zeta_+^2\hat\theta\phi_1^2u,u)+C(\|\hat\theta\phi_1u\|^2+\|O(E)u\|^2).$$

We next consider $\{\xi_0-\phi_1+\psi,\tilde\zeta a^2\hat\theta\phi_1^2\}$ which is

$$\{\xi_0-\phi_1,\tilde\zeta a^2\hat\theta\phi_1^2\}+\{\tilde\zeta\hat\theta\phi_1,a^2\phi_1\}\tilde\zeta\hat\theta\phi_1+\{\chi_2\hat\phi_1^3\lambda,\tilde\zeta a^2\hat\theta\phi_1^2\}.$$

It follows that $\{\chi_2\hat\phi_1^3\lambda,\tilde\zeta a^2\hat\theta\phi_1^2\}=c_1\hat\phi_1^4\lambda^2$ and $\{\tilde\zeta\hat\theta\phi_1,a^2\phi_1\}\tilde\zeta\hat\theta\phi_1=c_2\hat\theta\phi_1^2$ from Lemma 3.3. Since $\{\xi_0-\phi_1,\tilde\zeta a^2\hat\theta\phi_1^2\}=c_1\hat\theta\phi_1^2+c_2\hat\phi_1^2\lambda\phi_2+c_3\hat\theta\phi_1\phi_2+c_4\hat\phi_1^4\lambda^2$ by (3.1), (3.2) and Lemma 5.1 we get

$$\{\xi_0-\phi_1+\psi,\tilde\zeta a^2\hat\theta\phi_1^2\}=c_1\hat\theta\phi_1^2+c_2\hat\phi_1^2\lambda\phi_2+c_4\hat\theta\phi_1\phi_2+c_5\hat\phi_1^4\lambda^2.$$

We then consider $\{\xi_0-\phi_1+\psi,\chi_2a^2\hat\phi_1^4\lambda^2\}$ which is

$$\{\xi_0-\phi_1,\chi_2a^2\hat\phi_1^4\lambda^2\}+\{\tilde\zeta\hat\theta\phi_1,\chi_2a^2\hat\phi_1^4\lambda^2\}+\{\chi_2\hat\phi_1^3\lambda,a^2\hat\phi_1\lambda\}\chi_2\hat\phi_1^3\lambda.$$

A repetition of similar arguments shows

$$\{\xi_0-\phi_1+\psi,\chi_2a^2\hat\phi_1^4\lambda^2\}=c_1\hat\theta\phi_1^2+c_2\hat\phi_1^4\lambda^2+\hat\phi_1^2\lambda\phi_2.$$

Therefore $|(\{\xi_0-\phi_1+\psi,a^2\tilde\zeta\hat\theta\phi_1^2+a^2\chi_2\hat\phi_1^4\lambda^2\}u,u)|$ is bounded by $C|(\hat\theta\phi_1^2u,u)|+C\|O(E)u\|^2$. Denoting ζ_+a_j by a_j we turn to checking $\{\xi_0-\phi_1+\psi,a_2w^{-1/2}\hat\theta\phi_1\}$ where $a_2\in S(\lambda_\mu^{+0},\bar g)$ with support contained in $\operatorname{supp}\zeta_+$. Noting Lemmas 3.3 and 4.1 it is easy to see that

$$\{\xi_0-\phi_1+\psi,a_2w^{-1/2}\hat\theta\phi_1\}=c_0\zeta^2w^{1/2}\lambda\hat\theta\phi_1+c_1\mu\zeta w^{-1/2}\hat\theta\lambda+c_2\mu\lambda$$

with $c_j\in\mu S(\lambda_\mu^{+0},\bar g)$. Writing $c_0\zeta^2w^{1/2}\lambda\hat\theta\phi_1=\mathsf{Re}(c_0\zeta\hat\theta^{1/2}\phi_1\#\zeta w^{1/2}\hat\theta^{1/2}\lambda)+R_1$ and $c_1\zeta w^{-1/2}\hat\theta\lambda=\mathsf{Re}(\zeta w^{-3/2}\hat\theta\#c_1w\lambda)+R_2$ with $R_i\in\mu S(\lambda^{1+0},\bar g)$ we obtain the estimate

$$|(c_0\zeta^2w^{1/2}\lambda\hat\theta\phi_1u,u)|\le C\gamma^{-1/2}(\|\zeta w^{1/2}\hat\theta^{1/2}\lambda u\|^2+\|\zeta w^{-3/2}\hat\theta u\|^2)\\ +C\gamma^{1/2}\|c_0\zeta\hat\theta^{1/2}\phi_1u\|^2+C\|O(E)u\|^2.$$

The term $|(c_1\zeta w^{-1/2}\hat\theta\lambda u,u)|$ can be estimated similarly. In order to estimate $\{\xi_0-\phi_1+\psi,a_1w^{-2}\hat\theta\}$ we need to look at a_1 more carefully. Since $(w\phi)^{-1}\in S(\lambda_\mu,g)$ the main part of $\{F,\log\phi\}$ is $w^{-1}\{F,\hat\phi_2\}$ by (6.2). Therefore noting (3.4), it is not difficult to see from the proof of Lemma 6.4 that a_1 has the form

$$f(\zeta_+)^{k_1}(\zeta_+')^{k_2}(\chi)^{k_3}(\chi')^{k_4}\hat\phi_1^{\ell_1}\hat\phi_2^{\ell_2}w^{s_1}\omega^{s_2}\rho^{s_3}(\log\phi)^{\epsilon}, \tag{9.1}$$

where $f \in S(1, g_0)$ and k_i, $\ell_i \in \mathbb{N}$ and $s_i \in \mathbb{R}$, $\epsilon = 0$ or 1 which verifies

$$s_1 + s_2 + s_3 + \ell_1/2 + \ell_2 \geq 0$$

so that this is in $S(\lambda_\mu^{+0}, \bar{g})$. Here we examine that $\xi_0 - \phi_1 + \psi$ commutes better against such terms of the form (9.1) than against a general symbol in $S(\lambda_\mu^{+0}, \bar{g})$.

Lemma 9.3. *Denote $\Lambda = \xi_0 - \phi_1 + \psi$; then $\{\Lambda, \hat{\phi}_1\}$, $\{\Lambda, \hat{\phi}_2\}$ and $\{\Lambda, \hat{\theta}\}$ are linear combinations of $\hat{\phi}_1$, $\hat{\phi}_2$ and $\hat{\theta}$ with $\mu S(1, \bar{g})$ coefficients. We denote these by $\{\Lambda, \hat{\phi}_1\} = \mu S(1, \bar{g}) O(\Sigma)$ and so on.*

Proof. It follows easily from (3.1) and (3.2) that $\{\xi_0 - \phi_1, \hat{\phi}_1\}$, $\{\xi_0 - \phi_1, \hat{\phi}_2\}$ and $\{\xi_0 - \phi_1, \hat{\theta}\}$ are $O(\Sigma)$. Write $\psi = (\tilde{\zeta}\hat{\theta} + \chi_2 \hat{\phi}_1^2)\phi_1$ and note Lemma 3.3; then the desired assertion for $\{\psi, \hat{\phi}_1\}$, $\{\psi, \hat{\phi}_2\}$ and $\{\psi, \hat{\theta}\}$ follows immediately. ☐

Corollary 9.4. *One can write $\{\Lambda, w^{-1}\} = \mu S(w^{-1}, \bar{g}) + \mu S(w^{-2}, \bar{g}) O(\Sigma)$ and $\{\Lambda, \omega^{-1}\} = \mu S(\omega^{-1}, \bar{g}) + \mu S(\omega^{-3/2}, \bar{g}) O(\Sigma)$ and that $\{\Lambda, \rho^{-1}\} = \mu S(w^{-1}, \bar{g}) + \mu S(w^{-2}, \bar{g}) O(\Sigma) + S(\omega^{-3/2}, \bar{g}) O(\Sigma)$. We have also $\{\Lambda, \zeta\} = c_1 w^{-1}\hat{\theta} + c_2 w^{-1/2}$ with $c_i \in \mu S(1, \bar{g})$ and the same holds for $\{\Lambda, \chi\}$.*

Let us consider $\{\Lambda, a_1\}$ where a_1 has the form (9.1) with $k_1 + k_2 \geq 1$. Since $(\chi)^{k_3}(\chi')^{k_4}\hat{\phi}_1 \in S(w^{1/2}, \bar{g})$ it follows from Lemma 9.3 and Corollary 9.4 that $\{\Lambda, a_1\}$ can be written as $c_0 w^{-1}\hat{\theta} + c_1 w^{-1/2} + c_2 \omega^{-1/2}$ with $c_i \in \mu S(\lambda_\mu^{+0}, \bar{g})$. Since $\omega^{-1/2} w^{-1/2} \in \mu^{-1/2} S(\lambda^{1/2}, \bar{g})$ then applying Lemma 9.3 and Corollary 9.4 again to $\{\Lambda, w^{-2}\hat{\theta}\}$ we conclude that

$$\mu\{\Lambda, a_1 w^{-2}\hat{\theta}\} = c_0 \mu^3 w^{-3}\hat{\theta}^2 + c_1 \mu^{5/2} w^{-3/2}\hat{\theta}\lambda^{1/2} + c_2 \mu^2 \phi_1 + O(E)$$

where $c_i \in S(\lambda_\mu^{+0}, \bar{g})$. Writing $c_0 w^{-3}\hat{\theta}^2 = \mathsf{Re}(c_0 w^{-3/2}\hat{\theta} \# w^{-3/2}\hat{\theta}) + R$ with $R \in S(w^{-2}, \bar{g})$ and recalling that the supports of c_i are contained in the support of ζ_+ we obtain the estimate

$$\begin{aligned} &|\mu(\{\xi_0 - \phi_1 + \psi, a_1 w^{-2}\hat{\theta}\}u, u)| \\ &\qquad \leq C(\mu^3 + \gamma^{-1/2})\|\zeta w^{-3/2}\hat{\theta}u\|^2 + C\gamma^{1/2}\|O(E)u\|^2. \end{aligned} \tag{9.2}$$

Since the estimate $|(\{\xi_0 - \phi_1 + \psi, \zeta_+ a_3 \phi_1\}u, u)| \leq C(\|\hat{\theta}\phi_1 u\|^2 + \|O(E)u\|^2)$ is easy we obtain the following proposition.

Proposition 9.5. *We have*

$$
\begin{aligned}
|([D_0 - \mathrm{Re}\,\tilde{\lambda}, \mathrm{Re}\,\tilde{Q}]u, u)| \le c\nu\mu(\chi^2\zeta_+^2\hat{\theta}w\lambda^2 u, u)\\
+ (c\mu^3 + C\gamma^{-1/2})\|\zeta w^{-3/2}\hat{\theta}u\|^2 + C(\zeta_+^2\hat{\theta}\phi_1^2 u, u)\\
+ C\gamma^{-1/2}(\|\zeta\rho^{1/2}\hat{\theta}^{1/2}\lambda u\|^2 + \|\zeta w^{-3/2}\hat{\theta}u\|^2)\\
+ C\gamma^{1/2}(\|\hat{\theta}\phi_1 u\|^2 + \|\zeta\hat{\theta}^{1/2}\phi_1 u\|^2 + \|O(E)u\|^2)
\end{aligned}
$$

where $c > 0$ *is independent of* ν, μ *and* γ.

§10. Lower-order terms

Finally we handle the lower-order terms. By (3.11) one can write

$$T_2 = \mu c_0\hat{\theta}\lambda + b_0\hat{\theta}\phi_1 + b_1\hat{\phi}_1^2\lambda + b_2\phi_2 + b_3 w^{1/2}\phi_1$$

with $b_j \in \mu S(1, g)$ where $c_0 = 0$ for $\hat{\theta} < 0$ by assumption. Write $c_0\hat{\theta}\lambda = c_0\zeta_+^2\hat{\theta}\lambda + (1-\zeta_+^2)c_0\hat{\theta}\lambda$ where it is clear that we can write $(1-\zeta_+^2)c_0\hat{\theta}\lambda = b_4 w\lambda$. One can write

$$
\begin{aligned}
(1-\chi^2)\zeta_+^2\hat{\theta}\lambda = \omega^{1/2}\rho^{-1}\zeta_+\#\rho\omega^{-1/2}(1-\chi^2)\zeta_+\hat{\theta}\lambda\\
+ \omega^{1/2}\rho^{-1}\zeta_+\# c\,\rho\omega^{-1/2}\hat{\theta}\lambda + R
\end{aligned}
$$

with $c \in S(\lambda_\mu^{-1/4}, \bar{g})$ and $R \in \mu^{1/2}S(\lambda^{1/2}, \bar{g})$. Moreover $\mathrm{supp}\, c \subset \mathrm{supp}(1-\chi^2)$. Indeed, since $\omega^{\pm 1/2}\rho^{\mp 1} \in S(\omega^{\pm 1/2}\rho^{\mp 1}, \tilde{g})$ then $\omega^{1/2}\rho^{-1}\zeta_+\#\rho\omega^{-1/2}(1-\chi^2)\zeta_+\hat{\theta}\lambda$ can be written as $c\zeta_+\hat{\theta}\lambda + R$ with $c \in S(\lambda_\mu^{-1/4}, \bar{g})$ and $R \in \mu^{1/2}S(\lambda^{1/2}, \bar{g})$. Writing $c\zeta_+\hat{\theta}\lambda = \omega^{1/2}\rho^{-1}\zeta_+\# c\rho\omega^{-1/2}\hat{\theta}\lambda + R$ again we get the desired assertion. This proves

$$
\begin{aligned}
|((1-\chi^2)\zeta_+^2\hat{\theta}\lambda u, \tilde{\Lambda}u)| \le C\gamma^{-1/2}\|\omega^{1/2}\rho^{-1}(\zeta_+\tilde{\Lambda}u)\|^2\\
+ C\gamma^{1/2}\|c\rho\omega^{-1/2}\hat{\theta}\lambda u\|^2 + C(\|\tilde{\Lambda}u\|^2 + \|O(E)u\|^2)
\end{aligned}
$$

with $c \in S(\lambda_\mu^{-1/4}, \bar{g})$ where $\mathrm{supp}\, c \subset \mathrm{supp}(1-\chi^2)$. Now consider $\|c\rho\omega^{-1/2}\hat{\theta}\lambda u\|^2$. Note that $c\rho\omega^{-1/2} \in S(\omega^{1/2}, \bar{g})$ because if $c \neq 0$ then we have $C\hat{\phi}_1^2 \ge w$ and hence $C^2\omega^2 \ge w^2$. Thus it is clear $\omega^2 \le \hat{\phi}_2^2 + \omega^2 = \rho^2 \le (C^2+1)\omega^2$ so that $\omega^{1/2} \le \rho\omega^{-1/2} \le (1+C')\omega^{1/2}$. Then it is easily seen that $c\rho\omega^{-1/2}\hat{\theta}\lambda\# c\rho\omega^{-1/2}\hat{\theta}\lambda = a\omega\lambda^{3/2} + R$ with $a \in S(1, \bar{g})$ and $R \in S(\lambda, \bar{g})$ so that $\|c\rho w^{-1/2}\hat{\theta}\lambda u\|^2 \le C\|O(E)u\|^2$. Summarizing we get

$$
\begin{aligned}
|((1-\chi^2)\zeta_+^2\hat{\theta}\lambda u, \tilde{\Lambda}u)| \le C\gamma^{-1/2}\|\rho^{-1}\omega^{1/2}\zeta_+(\tilde{\Lambda}u)\|^2\\
+ C\gamma^{1/2}(\|\tilde{\Lambda}u\|^2 + \|O(E)u\|^2).
\end{aligned}
$$

We turn to studying $(\chi^2\zeta_+^2\hat{\theta}\lambda u, \tilde{\Lambda}u)$. Let us write $\chi^2\zeta_+^2\hat{\theta}\lambda = \chi\zeta_+ w^{-1/2}\#\chi\zeta_+ w^{1/2}\hat{\theta}\lambda + c\zeta_+ w^{1/2}\hat{\theta}\lambda + R$ with $c \in S(1, \bar{g})$ and $R \in S(\lambda^{1/2}, \bar{g})$ and hence we have

$$|(\chi^2\zeta_+^2\hat{\theta}\lambda u, \tilde{\Lambda}u)| \le cn^{1/2}\|\chi\zeta_+ w^{-1/2}\tilde{\Lambda}u\|^2 + (cn^{-1/2} + C\gamma^{-1/2})\|\zeta_+ w^{1/2}\hat{\theta}\lambda u\|^2 + C\gamma^{1/2}(\|O(E)u\|^2 + \|\tilde{\Lambda}u\|^2).$$

Since it is clear that $|((b_0\hat{\theta}\phi_1 + b_1\hat{\phi}_1^2\lambda + b_2\phi_2 + b_3 w^{1/2}\phi_1)u, \tilde{\Lambda}u)|$ is bounded by $C(\|\tilde{\Lambda}u\|^2 + \|\hat{\theta}\phi_1 u\|^2 + \|O(E)u\|^2)$ we get the following proposition.

Proposition 10.1. *We have*

$$|(T_2 u, \tilde{\Lambda}u)| \le (c\mu n^{1/2} + C\gamma^{-1/2})\|\chi\zeta_+ w^{-1/2}\tilde{\Lambda}u\|^2 + (c\mu n^{-1/2} + C\gamma^{-1/2})\|\zeta_+ w^{1/2}\hat{\theta}\lambda u\|^2 + C\gamma^{-1/2}\|\rho^{-1}\omega^{1/2}\zeta\tilde{\Lambda}u\|^2 + C\gamma^{1/2}(\|\tilde{\Lambda}u\|^2 + \|\lambda_\mu^{+0}O(E)u\|^2)$$

with $c > 0$ independent of n, ν, μ and λ.

We turn to studying $((T_1 - \bar{\kappa}\mu\lambda)u, u)$. From Lemma 3.1 it follows that $\zeta_-^2 h|\hat{\theta}|\phi_1^2 = h^{1/2}\phi_1\zeta_-|\hat{\theta}|^{1/2}\#h^{1/2}\phi_1\zeta_-|\hat{\theta}|^{1/2} + R$ with $R \in \mu^2 S(w^{-2}, g)$. By Lemma 3.1 again we see

$$\phi_2\#h^{1/2}\phi_1\zeta_-|\hat{\theta}|^{1/2} - h^{1/2}\phi_1\zeta_-|\hat{\theta}|^{1/2}\#\phi_2 = \{\phi_2, h^{1/2}\phi_1\zeta_-|\hat{\theta}|^{1/2}\}/i + R$$

with $R \in \mu^2 S(w^{-2}, g)$. Here, since $h = \mu\hat{c}\{\phi_1, \hat{\phi}_2\}^{-1}$, we have

$$\{h^{1/2}\phi_1\zeta_-|\hat{\theta}|^{1/2}, \phi_2\} = \mu^{1/2}\zeta_-(\hat{c}\{\phi_1, \hat{\phi}_2\}|\hat{\theta}|)^{1/2}e + c\mu w^{1/2}\phi_1$$

with $c \in S(1, g)$ thanks to Lemma 3.1 because $\phi_2^{(\alpha)} \in \mu S(w, g)$ for $|\alpha| = 1$. Then the following estimate follows easily:

$$\mu^{1/2}\big(\zeta_-(\hat{c}\{\phi_1, \hat{\phi}_2\}|\hat{\theta}|)^{1/2}e\, u, u\big) \le (\phi_2^2 u, u) + (h\zeta_-^2|\hat{\theta}|\phi_1^2 u, u) + C\gamma^{-1/2}\|O(E)u\|^2$$

modulo $C\gamma^{3/4}\|u\|^2$ because $\|w^{-1}u\|^2 \le C(\gamma^{-1}\|w^{-1}u\|^2 + \gamma\|u\|^2)$. From (3.10) it follows that

$$\mu^{1/2}\zeta_-(\hat{c}\{\phi_1, \hat{\phi}_2\}|\hat{\theta}|)^{1/2}\, e + T_1 \ge 2\bar{\kappa}\mu\lambda - C\mu w^{1/2}\lambda \tag{10.1}$$

with some $C > 0$. In fact if $\hat{\theta} \le -b_3 w$ then $\zeta_- = 1$ and the assertion follows by (3.10). If $-b_3 w \le \hat{\theta} \le 0$ then we have $Cw^{1/2}\lambda \ge \mu^{-1/2}(\hat{c}\{\phi_1, \hat{\phi}_2\}|\hat{\theta}|)^{1/2}\, e$ and hence the assertion. Since $S(\lambda, G) \subset S^1_{1,1/2}$ the Fefferman–Phong inequality gives

$$\mu^{1/2}(\hat{c}\zeta_-(\{\phi_1, \hat{\phi}_2\}|\theta|)^{1/2}e\, u, u) + (T_1 u, u) \ge 2\bar{\kappa}\mu(\lambda u, u) - C\mu\|O(E)u\|^2.$$

We summarize what we have proved in the next proposition.

Proposition 10.2. *We have*

$$(\phi_2^2 u, u) + (h\zeta_-^2|\hat{\theta}|\phi_1^2 u, u) + ((T_1 - \bar{\kappa}\mu\lambda)u, u)$$
$$\geq \bar{\kappa}\mu(\lambda u, u) - C(\mu + \gamma^{-1/2})\|O(E)u\|^2 - C\gamma^{3/2}\|\lambda_\mu^{+0} u\|^2.$$

Similarly $(\lambda_\mu^{2\varepsilon}\phi_2^2 u, u) + (\lambda_\mu^{2\varepsilon} h\zeta_-^2|\hat{\theta}|\phi_1^2 u, u) + (\lambda_\mu^{2\varepsilon}(T_1 - \bar{\kappa}\mu\lambda)u, u)$ *is bounded from below by* $\bar{\kappa}\mu(\lambda_\mu^{2\varepsilon}\lambda u, u) - C(\mu + \gamma^{-1/2})\|\lambda_\mu^{\varepsilon} O(E)u\|^2 - C\gamma^{3/2}\|\lambda_\mu^{2\varepsilon} u\|^2$.

Finally we estimate $\mathsf{Re}((T_1 - \bar{\kappa}\mu\lambda)u, (\mathsf{Im}\,\tilde{\lambda})\, u)$. Since $\zeta_-\zeta_+ = 0$ then from (10.1) we see that $\mathsf{Re}((T_1 - \bar{\kappa}\mu\lambda)u, (\mathsf{Im}\,\tilde{\lambda})\, u)$ is bounded from below by

$$\mathsf{Re}((\bar{\kappa}\mu\lambda - C\mu w^{1/2}\lambda)u, \tilde{e}_1\Gamma\zeta_+\hat{\theta}u) - C\|\lambda_\mu^{+0} O(E)u\|^2.$$

Note that $\mathsf{Re}(\tilde{e}_1\Gamma\zeta_+\hat{\theta}\#(\bar{\kappa}\mu\lambda - C\mu w^{1/2}\lambda)) = \bar{\kappa}\mu\tilde{e}_1\Gamma\zeta_+\hat{\theta}\lambda + c\zeta_+ w^{-1/2}\hat{\theta}\lambda + R$ with $c \in S(\lambda_\mu^{+0}, \bar{g})$ and $R \in \mu S(w^{-1}, \bar{g})$. Since $0 \leq \tilde{e}_1\Gamma\zeta_+\hat{\theta}\lambda \in S(w^{-1}\lambda^{1+0}, \bar{g})$ and noting $c\zeta_+ w^{-1/2}\hat{\theta}\lambda = \mathsf{Re}(\zeta_+ w^{-3/2}\hat{\theta}\#cw\lambda) + R$ with $R \in S(\lambda, \bar{g})$ one can see that $\mathsf{Re}(\tilde{e}_1\Gamma\zeta_+\hat{\theta}\#(\bar{\kappa}\mu\lambda - C\mu w^{1/2}\lambda)$ has a bound from below $-C\gamma^{-1/2}\|\zeta_+ w^{-3/2}\hat{\theta}u\|^2 - C\gamma^{1/2}\|\lambda_\mu^{+0} O(E)u\|^2$. Therefore we obtain the following lemma.

Lemma 10.3. *We have*

$$\mathsf{Re}((T_1 - \bar{\kappa}\mu\lambda)u, (\mathsf{Im}\,\tilde{\lambda})\, u) \geq -C\gamma^{-1/2}\|\zeta_+ w^{-3/2}\hat{\theta}u\|^2 - C\gamma^{1/2}\|\lambda_\mu^{+0} O(E)u\|^2.$$

We first choose $\epsilon > 0$ small so that $c\epsilon n\mu(\Gamma\zeta(\tilde{\Lambda}u), \zeta(\tilde{\Lambda}u))$ in Proposition 9.1 can be controlled by the corresponding term in Proposition 4.5. We next choose $\nu > 0$ small so that $cn\nu\mu\|\zeta\rho^{-1/2}\tilde{\Lambda}u\|^2$ and

$$c\epsilon^{-1}n\nu^2\mu(\Gamma(\zeta_+\hat{\theta}\phi_1)u, (\zeta_+\hat{\theta}\phi_1)u) + cn\nu\mu\|\zeta\rho^{1/2}\hat{\theta}\lambda u\|^2$$

in Proposition 9.1 will be small against the corresponding terms in Propositions 4.5 and 8.4. We then choose n such that $\mu^3\|\zeta w^{-3/2}\hat{\theta}u\|^2$ in Proposition 9.5 can be controlled by Proposition 5.3 and $c\mu n^{1/2}\|\chi\zeta_+ w^{-1/2}\tilde{\Lambda}u\|^2 + c\mu n^{-1/2}\|\zeta_+ w^{1/2}\hat{\theta}\lambda u\|^2$ in Proposition 10.1 can be estimated by Propositions 4.5 and 8.4. Finally we choose $\mu > 0$ small enough and then $\gamma > 0$ large enough so that μn^4 is small and $\gamma\mu^4$ is large. Then combining Propositions 4.5, 5.3, 7.4, 8.4, 9.1, 9.5, 10.1 and 10.2 we obtain the desired weighted energy estimates. Once we obtain the energy estimates, in order to conclude the well-posedness of the Cauchy problem it suffices to apply [N5, Thm. 1.1].

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