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Eigenfunction of the Laplacian as a Degenerate Case of a Function with its Fourier Transform Supported in an Annulus

by

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Abstract

The aim of this paper is to reach a characterization of the eigenfunction of the Laplace–Beltrami operator as a degenerate case of the inverse Paley–Wiener theorem (for functions whose Fourier transform is supported on a compact annulus) on the rank-1 Riemannian symmetric spaces of noncompact type. The most distinguished prototypes of these spaces are the hyperbolic spaces.

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§1. Introduction

Let X be a rank-1 Riemannian symmetric space of noncompact type of dimension d, Δ be the Laplace–Beltrami operator on X induced by its Riemannian structure and B be its maximal distinguished boundary which is diffeomorphic to $\mathbb{S}^{d-1}$. For a suitable function f on X, let $\widetilde{f}$ be its Helgason Fourier transform (henceforth called a Fourier transform) defined on $\mathbb{R}^+ \times B$. Let $L^{2,\infty}(X)$ be the Lorentz space of weak L^2-functions on X. A weak L^2-function f is an L^2-tempered distribution and we can consider its Fourier transform $\widetilde{f}$ in the sense of a tempered distribution. For a function F on B we denote its Poisson transform at $\alpha \in \mathbb{C}$ by $\mathcal{P}_\alpha F$. Let $-\rho^2$ be the bottom of the L^2-spectrum of Δ. In the latter part of this section and in Section 2 we shall elaborate on this. Using this notation we first state a result that characterizes a weak L^2-functions with its Fourier transform supported on a sphere as a Poisson transform of an L^2-function on the boundary B.

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Theorem 1.1. *The following two statements are equivalent for any* $f \in L^{2,\infty}(X)$*:*

(i) $\widetilde{f}$ *is supported on the sphere of radius* $\alpha > 0$ *in* $\mathbb{R}^+ \times B$*;*

(ii) $f = \mathcal{P}_\alpha F$ *for some* $F \in L^2(B)$ *and in particular* f *is an eigenfunction of* Δ*.*

This result serves as an intermediate step for the theorem below, which is the main result of this paper. It is a detailed account of the characterization of weak L^2-functions with compactly supported Fourier transform.

Theorem 1.2. *Suppose that for a nonzero function* $f \in L^{2,\infty}(X)$*, there are constants* $c_1 \geq \rho^2, 0 < c_2 \leq 1/\rho^2$ *such that*

$$\lim_{n\to\infty} \|\Delta^n f\|_{2,\infty}^{1/n} = c_1, \qquad \lim_{n\to\infty} \|\Delta^{-n} f\|_{2,\infty}^{1/n} = c_2.$$

Let $\alpha = \sqrt{c_1 - \rho^2}$ *and* $\beta = \sqrt{1/c_2 - \rho^2}$*. Then we have the following conclusions:*

(a) $c_1 c_2 \geq 1$*.*

(b) If $c_1 c_2 > 1$ *then* $\widetilde{f}$ *is supported in the annulus* $\mathbb{A}_\beta^\alpha = [\beta, \alpha] \times B$ *around the origin, but not inside any smaller annulus* $\mathbb{A}_{\beta'}^{\alpha'}$ *with* $\beta < \beta'$ *or* $\alpha' < \alpha$*.*

(c) If $c_1 c_2 = 1$ *then* $f = \mathcal{P}_\alpha F$ *for some* $F \in L^2(B)$*. Hence, in particular,* f *is an eigenfunction of* Δ *with eigenvalue* $-c_1$*.*

(d) The annulus containing the support of $\widetilde{f}$ *may reduce to a ball* $\mathbb{A}_0^\alpha = [0, \alpha] \times B$*, but cannot collapse to the origin, namely* $\alpha > 0$*.*

In the final section we provide some generalization of these results (Proposition 5.1 and Theorem 5.3). The Poisson transform $\mathcal{P}_\alpha$ is an analogue of the operator P_λ on $\mathbb{R}^n$ given by $P_\lambda F(x) = \int_{\mathbb{S}^{n-1}} F(y) e^{i\lambda x\cdot y}\, dy$. While P_λ maps a suitable function F on the *boundary* $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$ to a function on $\mathbb{R}^n$, $\mathcal{P}_\alpha$ maps a function F defined on B to a function on X. Indeed $\mathcal{P}_\alpha F$ are all the eigenfunctions of Δ where F is a function or more general object on B. In the hypothesis of the theorem above, $\Delta^n f$ is used in the sense of a distribution while $\Delta^{-n} f$ is in the sense of a multiplier. This is possible as the spectrum of Δ on X is $(-\infty, -\rho^2]$ where ρ, the half-sum of positive roots, is realized as a positive number. We keep away from these interpretational worries, as we shall discuss them in detail in Section 3.

The choice of the weak L^2-norm (i.e., $L^{2,\infty}$-norm) in the hypotheses of these theorems are not at all arbitrary. Indeed, among all the Lorentz norms, which include all L^p-norms, the $L^{2,\infty}$-norm is the unique option for the statements to be true. In particular the second theorem can accommodate the two possibilities (b) and (c) only when the $L^{2,\infty}$-norm is used. We shall elaborate on this in Section 3 and cite some other "close to L^2"-norms that can be used in place of the weak L^2-norm.

Similar theorems can be proved for $\mathbb{R}^n$ replacing the $L^{2,\infty}$-norm by the L^∞-norm or by the L^p-norm with $p > 2n/(n-1)$ for $n > 1$, to ensure the possibility of accommodating the eigenfunctions of the Laplacian. One also obtains an analogue for the Heisenberg groups $\mathbb{H}_n$ with the L^∞-norm in the hypothesis, and using the "Fourier transform" as defined in [42]. See [16, 23, 42], where some parts of these results for $\mathbb{R}^n$ and $\mathbb{H}_n$ are implicit. The situation in the Riemannian symmetric spaces of noncompact type appears to be more intriguing, as indicated in [42] by constructing a counterexample of the Euclidean result for a complex hyperbolic space.

To orient the readers we shall add some perspective for this study. We are motivated by two different sets of papers. The first one deals with the inverse Paley–Wiener theorem. An inverse Paley–Wiener theorem gives a criterion on a function (with some integrability or regularity) which is necessary and sufficient for its Fourier transform to be compactly supported in a ball around the origin, through the holomorphic extension of the function along with a growth condition on it. For Euclidean spaces it is the same as the usual Paley–Wiener theorem. But for other spaces (e.g., a semisimple or nilpotent Lie group or a symmetric space) where it is plausible to talk about Fourier transforms, the usual and the inverse Paley–Wiener theorems are distinguished by the fact that the domain of a function and its Fourier transform may be quite different and it is not at all clear where the complex analytic extension of the function has to be considered. For non-Euclidean spaces, such inverse Paley–Wiener theorems are rather recent (see, e.g., [14, 27, 32]). Very roughly, they state again that a suitable function with its Fourier transform compactly supported on its domain can be characterized from the holomorphic extension (in an appropriate domain) and growth of the function.

Unlike these results, a *real* inverse Paley–Wiener theorem (see [1, 2, 3, 8, 9, 10, 33, 44] and the references therein) does not consider the holomorphic extension of the inverse Fourier transform, but gives a criterion involving norm estimates on the integral powers of the Laplacian acting on the function. While most of these papers deal with Euclidean spaces, in [1, 33] the authors consider the setup of L^2-functions on Riemannian symmetric spaces. See also [12, 15, 25] and the references therein for related results on band-limited functions and sampling results.

A second set of papers, starting with Roe [36] and followed by many including [16, 22, 23, 29, 35, 42], try to characterize the eigenfunctions of differential operators, in particular of the Laplacian, from a normed estimate of a double sequence of functions $\{f_k\}$ related by $\Delta f_k = f_{k+1}$, where Δ is the Laplacian of the space in question. Most of these papers deal with Euclidean spaces. The first notable exception is [42] where Strichartz establishes the failure of the Euclidean result for hyperbolic spaces, as mentioned above. But through [29] and [35] the result is restored by the author and his collaborators for all Riemannian symmetric spaces of noncompact type (which includes hyperbolic spaces) and is also generalized to harmonic NA groups. A common thread between these two sets of results is the use of estimates of integral powers of the Laplacian applied on the function. Our aim is to exploit this connection to consolidate these two sets of results.

We pause briefly to point out the distinguishing features of our study. As mentioned above, in [1] Andersen considered real inverse Paley–Wiener theorem characterizing functions in $L^2(X)$ whose Fourier transform is supported in a ball around the origin in $\mathbb{R}^+ \times B$. This is a representative result of the first set mentioned above. Part (b) of Theorem 1.2 is an extension of this, where we characterize the functions whose Fourier transforms are supported in a compact annulus around the origin, under a weaker norm condition. As [1] deals with L^2-functions, the Plancherel theorem has a crucial presence in the proof. But the use of the L^2-norm precludes the possibility of the support degenerating to a sphere vis-à-vis the possibility of capturing an eigenfunction, as there is no L^p-eigenfunction of Δ for $p \leq 2$. However, as we go out of the L^2-setup and consider $L^{2,\infty}$-functions, we face new difficulties. But finally we are rewarded with a concrete realization of the function as the Poisson transform of an L^2-function F on the boundary, which is the same as a matrix coefficient of the class-1 principal series representation π_α. The aim of the second set, on the other hand, is to obtain a characterization of the eigenfunction of Δ. The hypotheses of these theorems cannot accommodate functions other than the eigenfunctions of Δ.

We note in passing that "the compactly supported Fourier transform" binds the real and the usual inverse Paley–Wiener theorems together, vindicating a relation between the estimates of $\Delta^n f$ and the regularity (e.g., complex or real analyticity) of f. Indeed, the use of estimates of iterated action of the Laplacian or more general operators on a function to retrieve regularity properties of the function is classical. We may cite for example the works of Stein, Nelson, Kotake and Narasimhan [26, 31, 40].

The organization of the paper is as follows. Section 2 explains the notation and contains the preliminaries. Section 3 prepares the reader further by supplying more results (some of which are not easy to locate) and by explaining the *sharpness* of the main results. Section 4 contains the proofs of Theorems 1.1 and 1.2. In Section 5 we provide some generalizations of the main results.

§2. Preliminaries

In this section we shall establish notation and collect all the ingredients to explain the statements and proofs of the main results.

§2.1. Generalities

For any $p \in [1, \infty)$, let $p' = p/(p-1)$. The letters $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$, $\mathbb{C}$ denote respectively the set of natural numbers, the ring of integers and the fields of real and complex numbers. We denote the sets of the nonzero real numbers, nonnegative real numbers and nonnegative integers respectively by $\mathbb{R}^\times$, $\mathbb{R}^+$ and $\mathbb{Z}^+$. For $z \in \mathbb{C}$, $\Re z$, $\Im z$ and $\bar{z}$ denote respectively the real and imaginary parts of z and the complex conjugate of z. For a set S in a topological space, $\overline{S}$ denotes its closure and for a set S in a measure space, $|S|$ denotes its measure. We shall follow the standard practice of using the letters C, C_1, C_2, C' etc. for positive constants, whose value may change from one line to another. The constants may be suffixed to show their dependencies on important parameters. The notation $\langle f_1, f_2 \rangle$ for two functions or distributions f_1, f_2 is frequently used in this article. It may mean $\int f_1 f_2$ when it makes sense. It may also mean that the distribution f_1 is acting on f_2. Depending on the functions/distributions f_1, f_2 involved, the space could be X or its Fourier dual $\mathbb{R}^+ \times B$, or $\mathbb{R}$ with the canonical measures on them. As this notation is widely used in the literature, we hope this will not create any confusion. For two positive expressions f_1 and f_2, by $f_1 \asymp f_2$ we mean that there are constants C_1, $C_2 > 0$ such that $C_1 f_1 \le f_2 \le C_2 f_1$.

§2.2. Lorentz spaces

We shall briefly introduce Lorentz spaces (see [18, 34, 41] for details). Let (M, m) be a σ-finite measure space, $f : M \longrightarrow \mathbb{C}$ be a measurable function and $p \in [1, \infty)$, $q \in [1, \infty]$. We define

$$\|f\|^*_{p,q} = \begin{cases} \left(\frac{q}{p} \int_0^\infty [f^*(t) t^{1/p}]^q \, \frac{dt}{t} \right)^{1/q} & \text{if } q < \infty, \\ \sup_{t>0} t d_f(t)^{1/p} = \sup_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty, \end{cases}$$

where for $\alpha > 0$, $d_f(\alpha) = |\{x \mid f(x) > \alpha\}|$, the distribution function of f, and $f^*(t) = \inf\{s \mid d_f(s) \le t\}$, the *decreasing rearrangement* of f. Let $L^{p,q}(M)$ be the set of all measurable $f : M \longrightarrow \mathbb{C}$ such that $\|f\|^*_{p,q} < \infty$. We note the following:

(i) The space $L^{p,\infty}(M)$ is known as the weak L^p-space.

(ii) $L^{p,p}(M) = L^p(M)$ and $\| \cdot \|^*_{p,p} = \| \cdot \|_p$.

(iii) For $1 < p, q < \infty$, the dual space of $L^{p,q}(M)$ is $L^{p',q'}(M)$ and the dual of $L^{p,1}(M)$ is $L^{p',\infty}(M)$.

(iv) If $q_1 \le q_2 \le \infty$ then $L^{p,q_1}(M) \subset L^{p,q_2}(S)$ and $\|f\|^*_{p,q_2} \le \|f\|^*_{p,q_1}$.

The Lorentz "norm" $\| \cdot \|^*_{p,q}$ is actually a quasi-norm and $L^{p,q}(M)$ is a quasi Banach space (see [18, p. 50]). For $1 < p \le \infty$, there is an equivalent norm $\| \cdot \|_{p,q}$

which makes it a Banach space (see [41, Chap. V, Thms. 3.21, 3.22]). We shall slur over this difference and use the notation $\|\cdot\|_{p,q}$.

§2.3. Symmetric space

We shall mostly use standard notation for objects related to semisimple Lie groups and the associated Riemannian symmetric spaces of noncompact type. Along with the required preliminaries this can be found for example in [17, 19]. To make this article self-contained, we shall gather them without elaboration. We recall that a rank-1 Riemannian symmetric space of noncompact type (which we denote by X throughout this article) can be realized as a quotient space G/K, where G is a connected noncompact semisimple Lie group with finite center and of real rank 1 and K is a maximal compact subgroup of G. Thus $\boldsymbol{o} = \{K\}$ is the origin of X and a function on X can be identified with a function on G which is invariant under the right K-action. The group G acts naturally on $X = G/K$ by left translations $\ell_g : xK \to gxK$ for $g \in G$. The Killing form on the Lie algebra $\mathfrak{g}$ of G induces a G-invariant Riemannian structure and a G-invariant measure on X. Let Δ be the corresponding Laplace–Beltrami operator. For an element $x \in X$, let $|x| = d(x, \boldsymbol{o})$, where d is the distance associated to the Riemannian structure on X. Let $\mathfrak{k}$ be the Lie algebra of K, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Then $\dim \mathfrak{a} = 1$ as G is of real rank 1. We denote the real dual of $\mathfrak{a}$ by $\mathfrak{a}^*$. Let $\Sigma \subset \mathfrak{a}^*$ be the subset of nonzero roots of the pair $(\mathfrak{g}, \mathfrak{a})$. We recall that either $\Sigma = \{-\gamma, \gamma\}$ or $\{-2\gamma, -\gamma, \gamma, 2\gamma\}$ where γ is a positive root and the Weyl group W associated to Σ is $\{\mathrm{Id}, -\mathrm{Id}\}$ where Id is the identity operator. Let $m_\gamma = \dim \mathfrak{g}_\gamma$ and $m_{2\gamma} = \dim \mathfrak{g}_{2\gamma}$ where $\mathfrak{g}_\gamma$ and $\mathfrak{g}_{2\gamma}$ are the root spaces corresponding to γ and 2γ. Then $\rho = \frac{1}{2}(m_\gamma + 2m_{2\gamma})\gamma$ denotes the half-sum of the positive roots. Let H_0 be the unique element in $\mathfrak{a}$ such that $\gamma(H_0) = 1$ and through this we identify $\mathfrak{a}$ with $\mathbb{R}$ as $t \mapsto tH_0$. Then $\mathfrak{a}_+ = \{H \in \mathfrak{a} \mid \gamma(H) > 0\}$ is identified with the set of positive real numbers. We identify $\mathfrak{a}^*$ and its complexification $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{R}$ and $\mathbb{C}$ by $t \mapsto t\gamma$ respectively $z \mapsto z\gamma$, $t \in \mathbb{R}$, $z \in \mathbb{C}$. By abuse of notation we will denote $\rho(H_0) = \frac{1}{2}(m_\gamma + 2m_{2\gamma})$ by ρ. Let $\mathfrak{n} = \mathfrak{g}_\gamma + \mathfrak{g}_{2\gamma}$, $N = \exp \mathfrak{n}$, $A = \exp \mathfrak{a}$, $A^+ = \exp \mathfrak{a}_+$ and $\overline{A^+} = \exp \overline{\mathfrak{a}_+}$. Then N is a nilpotent Lie group and A is a one-dimensional vector subgroup identified with $\mathbb{R}$. Precisely, A is parametrized by $a_s = \exp(sH_0)$. The Lebesgue measure on $\mathbb{R}$ induces a Haar measure on A by $da_s = ds$. Let M be the centralizer of A in K. The groups M and A normalize N.

The group G has the Iwasawa decomposition $G = KAN$ and the polar decomposition $G = K\overline{A^+}K$. Through polar decomposition X is realized as $\overline{A^+} \times B = \mathbb{R}^+ \times B$, quotiented by the relation $(0, k_1) \sim (0, k_2)$ for all $k_1, k_2 \in B$ where $B = K/M$ is the compact boundary of X. Using the Iwasawa decomposition

$G = KAN$, we write an element $x \in G$ uniquely as $k(x) \exp H(x) n(x)$ where $k(x) \in K$, $n(x) \in N$ and $H(x) \in \mathfrak{a}$. For $x \in X = G/K$, $b \in B = K/M$ let $A(x, b) = A(gK, kM) = -H(x^{-1}k)$. Let dg, dk and dm be the Haar measures of G, K and M respectively with $\int_K dk = 1$ and $\int_M dm = 1$. Let db be the normalized measure on $K/M = B$ induced by dk on K. Using them we write the integral formulae corresponding to the Iwasawa decompositions $G = KAN$ and the polar decomposition, which hold for any integrable function:

$$\int_G f(g)\, dg = C_1 \int_K \int_{\mathbb{R}} \int_N f(ka_t n) e^{2\rho t}\, dn\, dt\, dk \tag{2.1}$$

and

$$\int_G f(g)\, dg = C_2 \int_K \int_0^\infty \int_K f(k_1 a_t k_2)(\sinh t)^{m_\gamma} (\sinh 2t)^{m_{2\gamma}}\, dk_1\, dt\, dk_2. \tag{2.2}$$

The constants C_1, C_2 depend on the normalization of the Haar measures involved. Since $\sinh t \approx te^t/(1 + t)$, $t \geq 0$ it follows from (2.2) that

$$\begin{aligned} \int_G |f(g)|\, dg \asymp C_3 \int_K \int_0^1 \int_K |f(k_1 a_t k_2)| t^{d-1}\, dk_1\, dt\, dk_2 \\ + C_4 \int_K \int_1^\infty \int_K |f(k_1 a_t k_2)| e^{2\rho t}\, dk_1\, dt\, dk_2, \end{aligned} \tag{2.3}$$

where $d = m_\gamma + m_{2\gamma} + 1$ is the dimension of the symmetric space. For an integrable function f on X, $\int_G f(g)\, dg = \int_X f(x)\, dx$ where in the left-hand side f is considered as a right K-invariant function on G and dg is the Haar measure on G, while on the right-hand side dx is the G-invariant measure on X.

2.3.1. Poisson transform. For $\lambda \in \mathbb{C}$, the complex power of the Poisson kernel $x \mapsto e^{-(i\lambda+\rho)H(x^{-1})}$ is an eigenfunction of the Laplace–Beltrami operator Δ with eigenvalue $-(\lambda^2 + \rho^2)$. For any $\lambda \in \mathbb{C}$ and $F \in L^1(B)$, we define the Poisson transform $\mathcal{P}_\lambda$ of F by (see [19, p. 279])

$$\mathcal{P}_\lambda F(x) = \int_B F(b) e^{(i\lambda+\rho)A(x,b)}\, db \quad \text{for } x \in X.$$

Then

$$\Delta \mathcal{P}_\lambda F = -(\lambda^2 + \rho^2) \mathcal{P}_\lambda F.$$

Readers who are oriented more towards representation theory will look at X as a quotient space G/K and will recognize that $\mathcal{P}_\alpha F = \langle \pi_\alpha(x) 1_B, F \rangle$, a matrix coefficient of the class-1 principal series representation π_α of G, realized in the compact picture on $L^2(B)$. Here by 1_B we mean the constant function 1 on $B = K/M$. By Schur's orthogonality relation it is clear that these are the only matrix coefficients which are relevant for functions on G/K.

A function f on X is said to be left K-invariant or *radial* if $f(kx) = f(x)$ for all $k \in K$ and $x \in X$. Note that a left K-invariant function on X can be identified with a K-biinvariant function on G. We shall use both the terms radial and K-biinvariant for such functions. For any function space $\mathcal{L}(X)$, by $\mathcal{L}(G//K)$ we denote its subset of K-biinvariant functions. For a suitable function f on X we define its *radialization* Rf by $Rf(x) = \int_K f(kx)\,dk$. It is clear that Rf is a radial function and if f is radial then $Rf = f$. We also note that for $\phi, \psi \in C_c^\infty(X)$, we have (i) $\langle R\phi, \psi\rangle = \langle \phi, R\psi\rangle$ and (ii) $R(\Delta\phi) = \Delta(R\phi)$. From (i) it follows that $\int_X f(x)\,dx = \int_X Rf(x)\,dx$ and hence $\|Rf\|_1 \le \|f\|_1$. We have also the trivial L^∞-boundedness of the operator R: $\|Rf\|_\infty \le \|f\|_\infty$. Then a standard interpolation argument (see, e.g., [41, p. 197]) yields that

$$\|Rf\|_{p,q} \le \|f\|_{p,q} \quad \text{for } 1 < p < \infty,\ 1 \le q \le \infty.$$

For any $\lambda \in \mathbb{C}$, the elementary spherical function ϕ_λ is given by

$$\phi_\lambda(x) = \mathcal{P}_\lambda 1_B(x) = \int_K e^{-(i\lambda+\rho)H(xk)}\,dk = \int_K e^{(i\lambda-\rho)H(xk)}\,dk \quad \text{for all } x \in G.$$

Hence $\Delta\phi_\lambda = -(\lambda^2+\rho^2)\phi_\lambda$ for $\lambda \in \mathbb{C}$. It follows that for $\lambda \in \mathbb{C}$, ϕ_λ is radial, $\phi_\lambda = \phi_{-\lambda}$ and it satisfies the following estimates (see [6], [17, (4.6.5)]):

$$|\phi_{\alpha+i\gamma_p\rho}(a_t)| \asymp e^{-(2\rho/p')|t|} \quad \text{as } |t| \to \infty,\ \alpha \in \mathbb{R},\ 0 < p < 2,\ \gamma_p = 2/p - 1; \tag{2.4}$$

$$|\phi_0(a_t)| \le Ce^{\rho t}(1+|t|) \quad \text{for } t > 0 \tag{2.5}$$

and

$$\left|\frac{d^n}{d\lambda^n}\phi_\lambda(x)\right| \le C(1+|x|)^n \phi_{\Im\lambda}(x) \quad \text{for } \lambda \in \mathbb{C}. \tag{2.6}$$

2.3.2. Spherical Fourier transform. For a measurable function f of X, we define its *spherical Fourier transform* $\widehat{f}$ and its inverse as (see [19, p. 425, p. 454])

$$\widehat{f}(\lambda) = \int_X f(x)\phi_{-\lambda}(x)\,dx, \quad \lambda \in \mathfrak{a}^*, \quad f(x) = C\int_{\mathfrak{a}^*} \widehat{f}(\lambda)\,\phi_\lambda(x)\,|\mathrm{c}(\lambda)|^{-2}\,d\lambda,$$

whenever the integrals make sense. Here $\mathrm{c}(\lambda)$ is the Harish-Chandra c-function, $d\lambda$ is the Lebesgue measure on $\mathfrak{a}^* \equiv \mathbb{R}$ and $|\mathrm{c}(\lambda)|^{-2}\,d\lambda$ is the spherical Plancherel measure on $\mathfrak{a}^*$ and C is a normalizing constant. Since $\phi_\lambda = \phi_{-\lambda}$ we have $\widehat{f}(\lambda) = \widehat{f}(-\lambda)$, hence we can consider $\widehat{f}$ as a function on $\mathbb{R}^+$.

2.3.3. Helgason Fourier transform. For a function f on X, its Helgason Fourier transform (which we shall call the Fourier transform) is defined by

$$\widetilde{f}(\xi, b) = \int_X f(x)e^{(-i\xi+\rho)(A(x,b))}\,dx$$

for $\xi \in \overline{\mathfrak{a}_+^*} \equiv \mathbb{R}^+$, $b \in B$ for which the integral exists. (See [20, pp. 199–203] for details.) The Fourier transform $f(x) \to \widetilde{f}(\xi, b)$ extends to an isometry of $L^2(X)$ onto $L^2(\mathbb{R}^+ \times B, |\mathrm{c}(\xi)|^{-2}\, d\xi\, db)$ and we have

$$\int_X f_1(x)\overline{f_2(x)}\, dx = C \int_{\mathbb{R}^+ \times B} \widetilde{f}_1(\xi, b)\overline{\widetilde{f}_2(\xi, b)}|\mathrm{c}(\xi)|^{-2}\, d\xi\, db.$$

For functions f, g on X with g radial, $\widetilde{g}(\xi, k) = \widehat{g}(\xi)$ and $\widetilde{f * g}(\xi, b) = \widetilde{f}(\xi, b)\widehat{g}(\xi)$ for $\xi \in \mathbb{C}$ and $b \in B$ whenever the quantities $f * g$, $\widetilde{f * g}$, $\widetilde{f}$ and $\widehat{g}$ make sense.

2.3.4. Schwartz spaces, tempered distributions. For $1 \le p \le 2$, the L^p-Schwartz space $\mathcal{C}^p(X)$ is defined (see [5]) as the set of C^∞-functions on X such that

$$\gamma_{r,D}(f) = \sup_{x \in S} |Df(x)|\phi_0^{-2/p}(1 + |x|)^r < \infty,$$

for all nonnegative integers r and left-invariant differential operators D on X. We topologize $\mathcal{C}^p(X)$ by the seminorms $\gamma_{r,D}$. In what follows, we shall often abbreviate "continuous seminorm" as "seminorm". Then $\mathcal{C}^p(X)$ is a dense subset of $L^p(X)$. Let $\mathcal{C}^p(G//K)$ be the set of radial functions in $\mathcal{C}^p(X)$. We shall primarily use $\mathcal{C}^2(X)$, the L^2-Schwartz space. Let $\mathcal{C}^2(\widehat{X})$ (respectively $\mathcal{C}^2(\widehat{G//K})$) be the image of $\mathcal{C}^2(X)$ (respectively of $\mathcal{C}^2(G//K)$) under $f \mapsto \widetilde{f}$ (respectively $f \mapsto \widehat{f}$). Then (see [5]) $f \mapsto \widehat{f}$ is a topological isomorphism from $\mathcal{C}^2(G//K)$ to $\mathcal{C}^2(\widehat{G//K}) = S(\mathbb{R})_{\mathrm{even}}$ where $S(\mathbb{R})$ is the set of Schwartz class functions on $\mathbb{R}$, and $S(\mathbb{R})_{\mathrm{even}}$ denotes the subspace of even functions in $S(\mathbb{R})$. We do not need an explicit description of $\mathcal{C}^2(\widehat{X})$. For this and for a proof of the isomorphism $f \mapsto \widetilde{f}$ from $\mathcal{C}^2(X)$ to $\mathcal{C}^2(\widehat{X})$, we refer to [13, Thm. 4.8.1].

We denote the dual space of $\mathcal{C}^p(G//K)$ (respectively $\mathcal{C}^p(X)$) by $\mathcal{C}^p(G//K)'$ (respectively $\mathcal{C}^p(X)'$). Elements of $\mathcal{C}^p(G//K)'$ and $\mathcal{C}^p(X)'$ are called respectively the K-biinvariant L^p-tempered distributions and L^p-tempered distributions on X. It is clear that $L^{p'}(G//K) \subset \mathcal{C}^p(G//K)'$ and $L^{p'}(X) \subset \mathcal{C}^p(X)'$ for $1 \le p \le 2$. For an L^2-tempered distribution f, $\widetilde{f}$ is defined as a continuous linear functional on $\mathcal{C}^2(\widehat{X})$: for $\phi \in \mathcal{C}^2(X)$, $\langle \widetilde{f}, \widetilde{\phi} \rangle = \langle f, \phi \rangle$.

For a function $\phi \in \mathcal{C}^2(X)$, we define support of $\widetilde{\phi}$ as a subset of $\mathbb{R}^+ \times B$ by

$$\mathrm{Suppt}\, \widetilde{\phi} = \overline{\{(\lambda, b) \in \mathbb{R}^+ \times B \mid \widetilde{\phi}(\lambda, b) \ne 0\}}.$$

If ϕ is also K-biinvariant then $\widetilde{\phi}(\lambda, b) = \widehat{\phi}(\lambda)$ for all $b \in B$ and hence $\mathrm{Suppt}\, \widehat{\phi} = \overline{\{\lambda \in \mathbb{R}^+ \mid \widehat{\phi}(\lambda) \ne 0\}} \times B$. When ϕ is K-biinvariant, by abuse of terminology, the set $\overline{\{\lambda \in \mathbb{R}^+ \mid \widehat{\phi}(\lambda) \ne 0\}}$ will also be called the support of ϕ. We recall that $L^{2,\infty}(X) \subset \mathcal{C}^2(X)'$ (see Proposition 3.2(ii) below). For a function $f \in L^{2,\infty}(X)$, the distributional support of $\widetilde{f}$ is the complement of the largest open set $U \subset \mathbb{R}^+ \times B$ (with respect to the relative topology) such that for any $\phi \in \mathcal{C}^2(X)$ with $\operatorname{Suppt}\widetilde{\phi}$ contained in U, $\langle f, \phi\rangle = 0$.

If for a function $f \in L^{2,\infty}(X)$, $\operatorname{Suppt}\widetilde{f}$ is an empty set then $f \equiv 0$. Indeed, $\operatorname{Suppt}\widetilde{f}$ is empty implies that f annihilates all functions in $\mathcal{C}^2(X)$ and hence it is zero as an L^2-tempered distribution.

2.3.5. Abel transform. For a radial function f on X its Abel transform $\mathcal{A}f$ is defined by

$$\mathcal{A}f(a) = e^{\rho(\log a)} \int_N f(an)\, dn \quad \text{for } a \in A,$$

whenever the integral makes sense. Through the identification of A with $\mathbb{R}$ we can write it as

$$\mathcal{A}f(t) = e^{\rho t} \int_N f(a_t n)\, dn \quad \text{for } t \in \mathbb{R}.$$

For $f \in S(\mathbb{R})$, let $\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x}\, dx$ be its Euclidean Fourier transform at $\xi \in \mathbb{R}$.

We recall two fundamental properties of the Abel transform (see, e.g., [5]): (a) (*slice projection* theorem) for any $f \in \mathcal{C}^2(G//K)$, $\lambda \in \mathbb{R}$, $\mathcal{F}(\mathcal{A}f)(\lambda) = \widehat{f}(\lambda)$, (b) $\mathcal{A} : \mathcal{C}^2(G//K) \to S(\mathbb{R})_{\text{even}}$ is a topological isomorphism. Moreover, by duality, (b) implies that the adjoint of the Abel transform $\mathcal{A}^* : S(\mathbb{R})'_{\text{even}} \to \mathcal{C}^2(G//K)'$ is an isomorphism (see [21, p. 541]).

§3. Some preparatory discussion

In this section we shall explain the statements of the main results, highlight some of their features and gather some facts which will be used in the next section.

(1) As mentioned in the introduction, the weak L^2-norm in the hypothesis is the only possible Lorentz norm for the formulation. We shall elaborate on this.

As the statements of Theorems 1.1 and 1.2 involve Fourier transforms, tempered distributions are natural and perhaps the most general class of objects to work with. An $L^{2,\infty}$-function on X is an L^2-tempered distribution and the space $L^{2,\infty}(X)$ is *close to* $L^2(X)$, where usually the inverse Paley–Wiener theorems are stated. We recall that for $1 \le q < \infty$, $L^{2,q}(X) \subset \mathcal{C}^2(X)'$ (see Proposition 3.2(ii) below), i.e., an $L^{2,q}$-function is also an L^2-tempered distribution. But the $L^{2,q}$-norm with $q < \infty$ (which in particular includes the $L^2 = L^{2,2}$-norm) discards the possibility of f being an eigenfunction (see Proposition 3.2(vi) below).

Suppose that we take $f \in L^{p,q}(X)$ with $1 \le p < 2$, $1 \le q \le \infty$ and use the $L^{p,q}$-norm in the hypothesis instead of the $L^{2,\infty}$-norm. Then again f is an L^2-tempered distribution. Indeed $\mathcal{C}^2(X) \subset L^2(X) \cap L^\infty(X)$ and hence by interpolation, $\mathcal{C}^2(X) \subset L^{p',q}(X)$ for p, q in the range above. This implies by duality that $L^{p,q'}(X) \subset \mathcal{C}^2(X)'$. But the Fourier transform $\widetilde{f}(\lambda, b)$ of such a function f exists pointwise and has a complex-analytic extension in λ in a strip, for almost every $b \in B$ (see [30, 34]). Therefore $\widetilde{f}$ cannot be compactly supported. Thus Theorem 1.1 with this norm is not meaningful. If we formulate Theorem 1.2 with this norm, then the only possibilities are $c_1 = \infty$ and $c_2 = \rho^{-2}$, i.e., the annulus $\mathbb{A}_\beta^\alpha = \mathbb{R}^+ \times B$.

Lastly if $f \in L^{p,q}(X)$ with $p > 2$, $1 \leq q \leq \infty$, then f is an $L^{p'}$-tempered distribution where $p' < 2$ (and in general not an L^2-tempered distribution). See [29, Sect. 6]. It is clear that the usual definition of distributional support of the Fourier transform is not meaningful for such a function, since there is no nonzero function in $\mathcal{C}^{p'}(X)$ whose Fourier transform is compactly supported. Thus Theorem 1.1 cannot be formulated using the $L^{p,q}$-norm with p, q in this range. On the other hand there are functions $f \in L^{p,q}(X)$ satisfying

$$\lim_{n\to\infty} \|\Delta^n f\|_{p,q}^{1/n} = c_1, \qquad \lim_{n\to\infty} \|\Delta^{-n} f\|_{p,q}^{1/n} = 1/c_1$$

which are not eigenfunctions, not even generalized eigenfunctions of Δ. We recall that (see, e.g., [23, p. 205]) $f \neq 0$ is a generalized eigenfunction of Δ with eigenvalue λ if $(\Delta - \lambda)^N f = 0$ for some $N \in \mathbb{N}$, $N > 1$. For example $\frac{\partial}{\partial\lambda}\phi_\lambda|_{\lambda=\alpha}$ for any $\alpha \in \mathbb{R}$ is a generalized eigenfunction of Δ (see Lemma 5.2). Coming back to the main discussion, we take two points $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\lambda_1^2 \neq \lambda_2^2$, $|\Im\lambda_i| < |2/p - 1|\rho$ for $i = 1, 2$ and $|\lambda_1^2 + \rho^2| = |\lambda_2^2 + \rho^2| = \delta$ for some fixed $\delta > (4\rho^2)/(pp')$. Indeed, uncountably many $\lambda \in \mathbb{C}$ satisfy this for any such fixed δ. Then it is easy to verify that if $f = \phi_{\lambda_1} + \phi_{\lambda_2}$ then f is not a generalized eigenfunction but satisfies the hypothesis of Theorem 1.2 with the substitution of the $L^{2,\infty}$-norm by the $L^{p,q}$-norm for p, q as above.

(2) Outside the set of Lorentz norms and L^p-norms there are some prominent size estimates which are used in the literature to characterize eigenfunctions of the Laplacian as Poisson transforms. We shall mention only two of them. Let $B(\mathbf{o}, r) = \{x \in X \mid |x| < r\}$ be the geodesic ball of radius r centered at the origin $\mathbf{o}$ of X. For $1 < p < \infty$, $1 \leq q < \infty$ and a function f on X we define

$$M_p(f) = \left(\limsup_{r\to\infty} \frac{1}{r} \int_{B(o,r)} |f(x)|^p \, dx \right)^{1/p}, \tag{3.1}$$

$$\mathcal{K}_{p,q}(f) = \|\mathcal{K}_q(f)\|_{p,\infty} \quad \text{where } \mathcal{K}_q(f)(x) = \left(\int_K |f(kx)|^q \, dk \right)^{1/q}. \tag{3.2}$$

Any function f on X satisfying $M_2(f) < \infty$ or $\mathcal{K}_{2,q}(f) < \infty$ is an L^2-tempered distribution. (See the last sentence of this section.) Since the argument in the proof of Theorem 1.2 works under the assumption that f is an L^2-tempered distribution, we can substitute the $L^{2,\infty}$-norm by the M_2-norm or by the $\mathcal{K}_{2,q}$-norm. See [29] for the background relevant to these norms.

(3) Negative powers of Δ used in the statement of Theorem 1.2 can be interpreted in terms of radial multipliers. Precisely, Δ^{-1} is an L^p-multiplier for $1 < p < 2$ (see [4]) and hence an $L^{p'}$-multiplier. Hence by interpolation (see, e.g., [41, p. 197]) Δ^{-1} defines a bounded operator from $L^{2,\infty}(X)$ to itself. This is a benefit of the fact that in X (and in NA groups) the spectrum of Δ does not contain 0 (see [43]). However, keeping in mind the spaces (e.g., $\mathbb{R}^n$) where this interpretation is not valid, we can have an alternative formulation of Theorem 1.2 following [16, 23, 42], which in our case is only a change of notation.

Theorem 3.1. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a doubly infinite sequence of nonzero functions in $L^{2,\infty}(X)$ with $\Delta f_k = f_{k+1}$ for all $k \in \mathbb{Z}$. Suppose for constants $c_1 \geq \rho^2, c_2 \leq 1/\rho^2$,*

$$\lim_{k\to\infty} \|f_k\|_{2,\infty}^{1/k} = c_1, \qquad \lim_{k\to\infty} \|f_{-k}\|_{2,\infty}^{1/k} = c_2.$$

Then we have conclusions (a)–(d) of Theorem 1.2 for $f = f_0$.

Indeed the substitution $f = f_0$ and $f_k = \Delta^k f_0 = \Delta^k f$ for $k \in \mathbb{Z}$ reduces the hypothesis of this theorem to that of Theorem 1.2.

(4) We recall that Δ^n for $n \in \mathbb{N}$ commutes with translations, precisely $\Delta^n \ell_x f = \ell_x \Delta^n f$ for any $x \in G$ and a locally integrable function f on X. It is also not difficult to see that $\Delta^{-n}\ell_x f = \ell_x \Delta^{-n} f$ for any $n \in \mathbb{N}$. Similarly it can be verified that Δ^n for $n \in \mathbb{Z}$ commutes with the radialization operator R, i.e., $\Delta^n(R(f)) = R(\Delta^n f)$.

(5) We conclude this section collecting a few not-so-well-known results, some of which are used in the discussion above and some will be required for the main argument.

Proposition 3.2. *(i) $\mathcal{C}^2(X)$ is a dense subset of $L^{2,1}(X)$ and there exists a seminorm ν of $\mathcal{C}^2(X)$ such that for all $\phi \in \mathcal{C}^2(X)$, $\|\phi\|_{2,1} \leq C\nu(\phi)$.*

(ii) For $f \in L^{2,\infty}(X)$, there exists a seminorm ν of $\mathcal{C}^2(X)$ such that for all $\phi \in \mathcal{C}^2(X)$, $|\langle f, \phi\rangle| \leq C\|f\|_{2,\infty}\nu(\phi)$. That is, $f \in L^{2,\infty}(X)$ is an L^2-tempered distribution. Since for any $q < \infty$, $L^{2,q}(X) \subset L^{2,\infty}(X)$ and $\|f\|_{2,\infty} \leq \|f\|_{2,q}$, any $f \in L^{2,q}(X)$ is also an L^2-tempered distribution.

(iii) Let $1 \leq q \leq \infty$ be fixed. If for a nonnegative radial measure μ on X, $\widehat{\mu}(0) < \infty$, then $T_\mu : f \to f * \mu$ defines a bounded operator from $L^{2,q}(X)$ to itself and the operator norm satisfies $\|T_\mu\|_{L^{2,q}\to L^{2,q}} \leq \widehat{\mu}(0)$.

(iv) For $f \in L^{2,\infty}(X)$ and $\psi \in \mathcal{C}^2(G//K)$, $\|f * \psi\|_{2,\infty} \leq \|f\|_{2,\infty}\nu(\psi)$ for some seminorm ν of $\mathcal{C}^2(X)$.

(v) *If a nonzero function f on X satisfies $\Delta f = -\rho^2 f$, then $f \notin L^{2,\infty}(X)$. In particular $\phi_0 \notin L^{2,\infty}(X)$.*

(vi) *If a nonzero function f on X satisfies $\Delta f = -(\lambda^2 + \rho^2) f$, for some $\lambda \in \mathbb{R}^\times$, then $f \notin L^{2,q}(X)$ for any $q < \infty$.*

(vii) *For any $\lambda \in \mathbb{R}^\times$, $\phi_\lambda \in L^{2,\infty}(X)$.*

(viii) *Suppose that a function f on X satisfies $\Delta f = -(\lambda^2 + \rho^2) f$ with $\lambda \in \mathbb{R}^\times$. Then $f = \mathcal{P}_\lambda u$ for some $u \in L^2(B)$ if and only if $f \in L^{2,\infty}(X)$ and in that case $\|\mathcal{P}_\lambda u\|_{2,\infty} \le C_\lambda \|u\|_{L^2(B)}$.*

Proof. (i) follows from the definition of $\mathcal{C}^2(X)$ and the fact that for an appropriately large M, the function $\phi_0(x)(1 + |x|)^{-M} \in L^{2,1}(X)$. See [29, Lem. 6.1.1]. Denseness of $\mathcal{C}^2(X)$ is a consequence of denseness of $C_c^\infty(X)$ in $L^{2,1}(X)$. (ii) is immediate from (i) and Hölder's inequality. See also [29, Lem. 6.1.1]. (iii) is a particular case of a more general result proved in [38, Lem. 3.2.1] and [6]. For (iv) we have

$$\begin{aligned}\widehat{|\psi|}(0) &= \int_X |\psi(x)| \phi_0(x)\, dx \\ &\le \sup_{x \in X} \left[|\psi(x)| \phi_0^{-1}(x) (1 + |x|)^M \right] \int_X \phi_0^2(x) (1 + |x|)^{-M}\, dx.\end{aligned}$$

It follows from the estimate of ϕ_0 and the measure on X (see Section 2) that $C = \int_X \phi_0^2(x)(1 + |x|)^{-M}\, dx < \infty$ for suitably large M. We define

$$\nu(\psi) = \sup_{x \in X} \left[|\psi(x)| \phi_0^{-1}(x) C (1 + |x|)^M \right]$$

to get $\widehat{|\psi|}(0) \le \nu(\psi)$. Thus by (iii),

$$\|f * \psi\|_{2,\infty} \le \|\, |f| * |\psi|\, \|_{2,\infty} = \|T_{|\psi|}(|f|)\|_{2,\infty} \le \|f\|_{2,\infty} \widehat{|\psi|}(0) \le \|f\|_{2,\infty} \nu(\psi).$$

For (v), (vi), (vii) and (viii) we refer to [29, Prop. 3.1.1, (2.2.6) and Thm. 4.3.5] and [28]. ((vii) is also a particular case of (viii).) □

For the corresponding results, in particular those of (i), (ii) and (viii) above for the M_2-norm and the $\mathcal{K}_{2,q}$-norm, we refer to [29, Lem. 6.1.1] and [11, 24].

§4. Proof of the main results

This section is devoted to the proofs of Theorems 1.1 and 1.2. We begin with a few results which relate the support of the Fourier transform of a function on X to the support of the Fourier transform of its translation and radialization.

Proposition 4.1. *Let $g \in \mathcal{C}^2(X)$ and $\lambda \in \mathbb{R}^+$. Then $(\lambda, b) \in \operatorname{Suppt} \widetilde{g}$ for some $b \in B$ if and only if $\lambda \in \operatorname{Suppt} \widehat{R(\ell_x g)}$ for some $x \in G$.*

Proof. Note that for $\lambda \in \mathbb{R}$ (see [20, p. 200]),

$$\widehat{R(\ell_x g)}(\lambda) = g * \phi_\lambda(x^{-1}) = \int_B \widetilde{g}(\lambda, b)\, e^{(i\lambda+\rho)(A(x^{-1},b))}\, db = \mathcal{P}_\lambda\, \widetilde{g}(\lambda, \cdot)(x^{-1}),$$

where in the last equality above we have considered $\widetilde{g}(\lambda, \cdot)$ as a function on B. If $(\lambda, b) \notin \operatorname{Suppt} \widetilde{g}$ for all $b \in B$ then clearly $\lambda \notin \operatorname{Suppt} \widehat{R(\ell_x g)}$ for all $x \in G$. Conversely, if $\lambda \notin \operatorname{Suppt} \widehat{R(\ell_x g)}$ for all $x \in G$, then $\mathcal{P}_\lambda \widetilde{g}(\lambda, \cdot) \equiv 0$. Using the simplicity criterion ([20, pp. 152, 165]) this implies that $\widetilde{g}(\lambda, \cdot) \equiv 0$. □

Proposition 4.2. *Let $g \in \mathcal{C}^2(X)$. If the support of $\widetilde{g}$ intersects the sphere $\{\gamma\} \times B$ for some $\gamma \geq 0$, then for any $y \in G$, the support of $\widetilde{\ell_y g}$ also intersects $\{\gamma\} \times B$.*

Proof. We have

$$\widetilde{\ell_y g}(\xi, kM) = \int_X g(y^{-1}x) e^{(i\xi-\rho)H(x^{-1}k)} dx.$$

With the substitution $y^{-1}x = z$ and using the identity $H(z^{-1}y^{-1}k) = H(y^{-1}k) + H(z^{-1}k(y^{-1}k))$ ([20, p. 200]) we get from above

$$\begin{aligned}\widetilde{\ell_y g}(\xi, kM) &= [e^{(i\xi-\rho)H(y^{-1}k)}] \int_X g(z) e^{(i\xi-\rho)H(z^{-1}k(y^{-1}k))}\, dz \\ &= [e^{(i\xi-\rho)H(y^{-1}k)}]\ \ \widetilde{g}(\xi, k(y^{-1}k)).\end{aligned}$$

Suppose that $\widetilde{g}(\gamma, b) \neq 0$ for $b = k_1 M$. Let $k(yk_1) = k$. Then $k(y^{-1}k) = k_1$ and hence $\widetilde{\ell_y g}(\gamma, kM) \neq 0$, which proves the assertion. □

We note that for Theorem 1.2, it is required to find only the outer and inner radii of the support of $\widetilde{f}$. Precisely, the outer and inner radii of the support of $\widetilde{f}$ are α and β respectively if the support of $\widetilde{f}$ is contained in the annulus $[\beta, \alpha] \times B$ but not contained in $[\beta', \alpha'] \times B$ when $\beta < \beta'$ or $\alpha' < \alpha$. We need two more results in this vein, relating the support of the Fourier transform of a function with the support of the Fourier transform of its translation and radialization.

Proposition 4.3. *Let $f \in L^{2,\infty}(X)$. Then for any $x \in G$, the outer and inner radii of support of $\widetilde{\ell_x f}$ are the same as those of the support of $\widetilde{f}$.*

Proof. Suppose that the outer and inner radii of support of $\widetilde{f}$ are α, β respectively. We take a function $g \in \mathcal{C}^2(G/K)$, such that $\operatorname{Suppt} \widetilde{g}$ is contained in $\{(\lambda, b) \in \mathbb{R}^+ \times B \mid \lambda > \alpha\}$. Then by Proposition 4.2, $\operatorname{Suppt} \widetilde{\ell_{x^{-1}} g}$ for any $x \in G$ is also contained in $\{(\lambda, b) \in \mathbb{R}^+ \times B \mid \lambda > \alpha\}$. Hence $\langle f, \ell_{x^{-1}} g\rangle = 0$. Therefore $\langle \widetilde{\ell_x f}, g\rangle = \langle f, \ell_{x^{-1}} g\rangle = 0$. This shows that the outer radius of Suppt $\widetilde{\ell_x f} \le \alpha$ for all $x \in G$. Since f is a translation of $\ell_x f$, the outer radius of the support of $\widetilde{\ell_x f}$ is the same as the outer radius of the support of $\widetilde{f}$. Similarly we can show that the inner radii of $\widetilde{f}$ and of $\widetilde{\ell_x f}$ are the same. □

Proposition 4.4. *Let $f \in L^{2,\infty}(X)$. Suppose that* Suppt $\widetilde{f} \subset \{\alpha\} \times B$. *Then* Suppt $\widehat{R(\ell_x f)} \subset \{\alpha\}$ *for any $x \in G$.*

Proof. If for some $x \in G$, $R(\ell_x f) = 0$ we have nothing to show. Therefore we assume $R(\ell_x f) \neq 0$. We take a function $g \in \mathcal{C}^2(X)$ with Suppt $\widetilde{g} \subset \{(\lambda, b) \in \mathbb{R}^+ \times B \mid \lambda \neq \alpha\}$. By Proposition 4.1, Suppt $\widehat{R(g)} \subset \{\lambda \in \mathbb{R}^+ \mid \lambda \neq \alpha\}$. Therefore by Proposition 4.3, $\langle \ell_x f, Rg\rangle = 0$ and hence $\langle R(\ell_x f), g\rangle = \langle \ell_x f, Rg\rangle = 0$. □

This makes us ready to present the proofs of the main results. First we shall take up Theorem 1.1 which will be proved through a series of lemmas.

We shall write ∂_λ, ∂_λ^n respectively for $\frac{\partial}{\partial \lambda}$ and $\frac{\partial^n}{\partial \lambda^n}$.

Lemma 4.5. *For any $\lambda_0 > 0$, $\partial_\lambda \phi_\lambda|_{\lambda=\lambda_0} \notin L^{2,\infty}(X)$.*

Proof. In view of the polar decomposition, the corresponding integral formula (2.3), and the identification of A with $\mathbb{R}$, it suffices to show that $\partial_\lambda \phi_\lambda|_{\lambda=\lambda_0}$ restricted to $[1, \infty)$ does not belong to $L^{2,\infty}([1,\infty), e^{2\rho t}\, dt)$. We shall use the facts that $e^{-\rho t} \in L^{2,\infty}([1,\infty), e^{2\rho t}\, dt)$ and $te^{-\rho t} \notin L^{2,\infty}([1,\infty), e^{2\rho t}\, dt)$, which are easily verifiable through straightforward computation. We recall that for $\lambda \in \mathbb{R}^\times$, ϕ_λ has the expansion (see [24, 39])

$$\phi_\lambda(t) = e^{-\rho t}[\mathrm{c}(\lambda) e^{i\lambda t} + \mathrm{c}(-\lambda) e^{-i\lambda t} + E(\lambda, t)],$$

where

$$E(\lambda, t) = \mathrm{c}(\lambda) e^{i\lambda t} \sum_{k=1}^{\infty} \Gamma_k(\lambda) e^{-2kt} + \mathrm{c}(-\lambda) e^{-i\lambda t} \sum_{k=1}^{\infty} \Gamma_k(-\lambda) e^{-2kt}$$

and the Γ_k are recursively defined by $\Gamma_0(\lambda) = 1$ and

$$(k+1)(k+1-i\lambda)\Gamma_{k+1} = (\rho+k)(\rho+k-i\lambda)\Gamma_k + m_{2\gamma} \sum_{j=0}^{k} (-1)^{k+j+1}(\rho+2j-i\lambda)\Gamma_j.$$

For $t \ge 1$ the series defining $E(\lambda, t)$ and its λ-derivative at $\lambda = \lambda_0$ are uniformly convergent. Term-by-term differentiation shows that $|E(\lambda, t)| \le C_\lambda$ for some constant C_λ for $t \ge 1$. Thus $e^{-\rho t} E(\lambda, t) \in L^{2,\infty}([1,\infty), e^{2\rho t}\, dt)$. Therefore we need to show that

$$e^{-\rho t} \partial_\lambda [\mathrm{c}(\lambda) e^{i\lambda t} + \mathrm{c}(-\lambda) e^{-i\lambda t}]|_{\lambda=\lambda_0} \notin L^{2,\infty}([1,\infty), e^{2\rho t}\, dt).$$

Noting that $\overline{\mathrm{c}(\lambda)} = \mathrm{c}(-\lambda)$ and writing $\mathrm{c}(\lambda) = a(\lambda) + i\,b(\lambda)$ where $a(\lambda), b(\lambda)$ are real functions, we have

$$\begin{aligned} e^{-\rho t}\partial_\lambda[\mathrm{c}(\lambda)e^{i\lambda t} + \mathrm{c}(-\lambda)e^{-i\lambda t}] &= 2e^{-\rho t}\partial_\lambda(\Re(\mathrm{c}(\lambda)e^{i\lambda t})) \\ &= 2e^{-\rho t}\partial_\lambda(a(\lambda)\cos\lambda t - b(\lambda)\sin\lambda t) \\ &= -2te^{-\rho t}(a(\lambda)\sin\lambda t + b(\lambda)\cos\lambda t) \\ &\quad + 2e^{-\rho t}(\partial_\lambda(a(\lambda))\cos\lambda t - \partial_\lambda(b(\lambda))\sin\lambda t). \end{aligned}$$

Since at $\lambda{=}\lambda_0$ the last term in the equality above is in $L^{2,\infty}([1,\infty), e^{2\rho t}\,dt)$, we need only to show that $g(t) = te^{-\rho t}(a(\lambda_0)\sin\lambda_0 t + b(\lambda_0)\cos\lambda_0 t) \notin L^{2,\infty}([1,\infty), e^{2\rho t}\,dt)$. For the sake of deriving a contradiction, we assume that $g \in L^{2,\infty}([1,\infty), e^{2\rho t}\,dt)$. Then its translation by $\pi/2\lambda_0$ is $g(t+\pi/2\lambda_0) = C(t+\pi/2\lambda_0)e^{-\rho t}(a(\lambda_0)\cos\lambda_0 t - b(\lambda_0)\sin\lambda_0 t)$, which is also in $L^{2,\infty}([1,\infty), e^{2\rho t}\,dt)$. This follows from interpolation of the facts that for $1 < p < 2 < q$, translation by a fixed element in $\mathbb{R}$ is a bounded operator from L^p to L^p and from L^q to L^q in the measure space $([1,\infty), e^{2\rho t}\,dt)$.

We note that the part $C(\pi/2\lambda_0)e^{-\rho t}(a(\lambda_0)\cos\lambda_0 t - b(\lambda_0)\sin\lambda_0 t)$ of $g(t + \pi/2\lambda_0)$ is in $L^{2,\infty}([1,\infty), e^{2\rho t}\,dt)$. Therefore the other part of $g(t+\pi/2\lambda_0)$ given by $h(t) = te^{-\rho t}(-b(\lambda_0)\sin\lambda_0 t + a(\lambda_0)\cos\lambda_0 t) \in L^{2,\infty}([1,\infty), e^{2\rho t}\,dt)$. Since $g(t)$ and $h(t)$ are in $L^{2,\infty}([1,\infty), e^{2\rho t}\,dt)$ we have

$$\begin{aligned} b(\lambda_0)g(t) + a(\lambda_0)h(t) &= te^{-\rho t}(a(\lambda_0)^2 + b(\lambda_0)^2)\cos\lambda_0 t \in L^{2,\infty}([1,\infty), e^{2\rho t}\,dt), \\ a(\lambda_0)g(t) - b(\lambda_0)h(t) &= te^{-\rho t}(a(\lambda_0)^2 + b(\lambda_0)^2)\sin\lambda_0 t \in L^{2,\infty}([1,\infty), e^{2\rho t}\,dt). \end{aligned}$$

Hence $(a(\lambda_0)^2 + b(\lambda_0)^2)e^{i\lambda_0 t}te^{-\rho t} \in L^{2,\infty}([1,\infty), e^{2\rho t}\,dt)$ which amounts to saying that $te^{-\rho t} \in L^{2,\infty}([1,\infty), e^{2\rho t}\,dt)$, a contradiction. □

Lemma 4.6. *For any nonconstant polynomial P and $\lambda_0 \geq 0$, if $P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} \not\equiv 0$, then $P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} \notin L^{2,\infty}(X)$.*

Proof. First we shall take up the case $\lambda_0 > 0$. Let P be a polynomial of degree n given by $P(y) = a_0 y^n + a_1 y^{n-1} + \cdots + a_n$ with $a_0 \neq 0$. We shall show that if $P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} \in L^{2,\infty}(X)$, then $\partial_\lambda\phi_\lambda|_{\lambda=\lambda_0} \in L^{2,\infty}(X)$. Use of Lemma 4.5 then completes the proof.

So, we assume that $P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} \in L^{2,\infty}(X)$. If $n = 1$, then $P(\partial_\lambda)\phi_\lambda = a_0\partial_\lambda\phi_\lambda + a_1\phi_\lambda$. Since $P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} \in L^{2,\infty}(X)$ and $a_1\phi_{\lambda_0} \in L^{2,\infty}(X)$ (see Proposition 3.2(vii)) we have $a_0\partial_\lambda\phi_\lambda|_{\lambda=\lambda_0} \in L^{2,\infty}(X)$. If $n \geq 2$, we take a function $\psi \in \mathcal{C}^2(G//K)$ such that $\widehat{\psi}$ and its derivatives of orders up to $(n-2)$ are zero at λ_0 and $\partial_\lambda^{n-1}(\widehat{\psi}(\lambda))|_{\lambda=\lambda_0} \neq 0$ (e.g., $\psi \in \mathcal{C}^2(G//K)$ given by $\widehat{\psi}(\lambda) = (\lambda^2 - \lambda_0^2)^{n-1}e^{-\lambda^2}, \lambda \in \mathbb{R}$). Then $\partial_\lambda^{n-r}(\widehat{\psi}(\lambda)\phi_\lambda)|_{\lambda=\lambda_0} = 0$ for all $2 \leq r \leq n$.

We note that $P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} * \psi = P(\partial_\lambda)(\phi_\lambda * \psi)|_{\lambda=\lambda_0}$ where the convolution can be justified from the estimate of $P(\partial_\lambda)\phi_\lambda$ (see (2.4), (2.6)). Hence,

$$P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} * \psi = \{a_0\partial_\lambda^n(\widehat{\psi}(\lambda)\phi_\lambda) + a_1\partial_\lambda^{n-1}(\widehat{\psi}(\lambda)\phi_\lambda)\}|_{\lambda=\lambda_0}.$$

Expanding the derivatives in the right-hand side by the Leibniz rule and using that $\widehat{\psi}$ and its derivatives of order $1, 2, \ldots, n-2$ vanish at λ_0 we get

$$\begin{aligned} &P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} * \psi \\ &\quad = \left[a_0\{\phi_\lambda\partial_\lambda^n(\widehat{\psi}(\lambda)) + n(\partial_\lambda\phi_\lambda)\partial_\lambda^{n-1}(\widehat{\psi}(\lambda))\} + a_1\phi_\lambda\partial_\lambda^{n-1}(\widehat{\psi}(\lambda))\right]_{\lambda=\lambda_0}. \end{aligned}$$

The assumption $P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} \in L^{2,\infty}(X)$ implies that $P(\partial_\lambda)\phi_\lambda|_{\lambda=\lambda_0} * \psi \in L^{2,\infty}(X)$ (Proposition 3.2(iv)). Since $\phi_{\lambda_0} \in L^{2,\infty}(X)$ (Proposition 3.2(vii)), we get from above that $\partial_\lambda\phi_\lambda|_{\lambda=\lambda_0} \in L^{2,\infty}(X)$.

Next we assume that $\lambda_0 = 0$. Note that for any fixed $x \in G$, $\phi_\lambda(x)$ is an even function of λ. Hence for any odd $m \in \mathbb{N}$, $\partial_\lambda^m\phi_\lambda(x)$ is an odd function of λ and consequently $\partial_\lambda^m\phi_\lambda|_{\lambda=0} \equiv 0$. As above we take a polynomial $P(y) = a_0y^n + a_1y^{n-1} + \cdots + a_n$ with $a_0 \neq 0$ and for the sake of deriving a contradiction we assume that $P(\partial_\lambda)\phi_\lambda|_{\lambda=0} \not\equiv 0$ and $P(\partial_\lambda)\phi_\lambda|_{\lambda=0} \in L^{2,\infty}(X)$. Since $\partial_\lambda^n\phi_\lambda|_{\lambda=0} \equiv 0$ if n is odd, we take the degree of P, i.e., n, to be even without loss of generality.

Let $\psi \in \mathcal{C}^2(G//K)$ be defined by $\widehat{\psi}(\lambda) = \lambda^n e^{-\lambda^2}$ for $\lambda \in \mathbb{R}$. Then $\partial_\lambda^m\widehat{\psi}|_{\lambda=0} = 0$ for $m = 0, 1, \ldots, n-1$ and $\partial_\lambda^n\widehat{\psi}|_{\lambda=0} \neq 0$.

As above, by Proposition 3.2(iv), $P(\partial_\lambda)\phi_\lambda|_{\lambda=0} * \psi \in L^{2,\infty}(X)$, hence

$$P(\partial_\lambda)(\widehat{\psi}(\lambda)\phi_\lambda)|_{\lambda=0} = P(\partial_\lambda)(\phi_\lambda * \psi)|_{\lambda=0} = P(\partial_\lambda)\phi_\lambda|_{\lambda=0} * \psi \in L^{2,\infty}(X).$$

We also have

$$\begin{aligned} P(\partial_\lambda)(\widehat{\psi}(\lambda)\phi_\lambda)|_{\lambda=0} &= a_0\left[\phi_\lambda\partial_\lambda^n(\widehat{\psi}(\lambda)) + a_0\partial_\lambda\phi_\lambda\partial_\lambda^{n-1}(\widehat{\psi}(\lambda))\right]|_{\lambda=0} \\ &= a_0\partial_\lambda^n(\widehat{\psi}(\lambda))|_{\lambda=0}\ \phi_0. \end{aligned}$$

Since $\partial_\lambda^n(\widehat{\psi}(\lambda))|_{\lambda=0} \neq 0$, we have $\phi_0 \in L^{2,\infty}(X)$, which contradicts Proposition 3.2(v). ☐

Lemma 4.7. *For any polynomial P in one variable and $\xi \in \mathbb{R}$, $\mathcal{A}^*(P(\partial_\xi)e^{-i\xi t}) = P(\partial_\xi)\phi_\xi$ as an L^2-tempered distribution on X; equivalently $(\mathcal{A}^*)^{-1}(P(\partial_\xi)\phi_\xi) = P(\partial_\xi)e^{-i\xi t}$ as a tempered distribution on $\mathbb{R}$.*

Proof. It is enough to show this for $P(\partial_\xi) = \partial_\xi$. Let $\psi \in \mathcal{C}^2(G//K)$. Then $\mathcal{A}\psi \in S(\mathbb{R})_{\text{even}}$. We have

$$\langle \mathcal{A}\psi, \partial_\xi e^{-i\xi t}\rangle = \langle \psi, \mathcal{A}^*(\partial_\xi e^{-i\xi t})\rangle.$$

On the other hand, using the slice-projection theorem (see Section 2.3.5) we have

$$\langle \mathcal{A}\psi, \partial_\xi e^{-i\xi t}\rangle = \partial_\xi \mathcal{F}(\mathcal{A}\psi)(\xi) = \partial_\xi \widehat{\psi}(\xi) = \partial_\xi \langle \psi, \phi_\xi\rangle = \langle \psi, \partial_\xi \phi_\xi\rangle.$$

Thus $\langle \psi, \mathcal{A}^*(\partial_\xi e^{-i\xi t})\rangle = \langle \psi, \partial_\xi \phi_\xi\rangle$, for all $\psi \in \mathcal{C}^2(G//K)$, implying $\mathcal{A}^*(\partial_\xi e^{-\xi t}) = \partial_\xi \phi_\xi$ as L^2-tempered distributions. As $\mathcal{A}^*$ is an isomorphism from $S(\mathbb{R})_{\mathrm{even}}$ to $\mathcal{C}^2(G//K)$, the equivalent statement follows. □

Lemma 4.8. *Let f_1, f_2 be two nonzero functions in $L^{2,\infty}(X)$. Then the following statements are true.*

(a) There exists $x \in G$ such that $R(\ell_x f_1) \neq 0$.

(b) If for some $x \in G$, $R(\ell_x f_1) \neq 0$, then $R(\Delta^n \ell_x f_1) \neq 0$ for all $n \in \mathbb{Z}$.

(c) If $R(\ell_x f_1) = R(\ell_x f_2)$ for all $x \in G$, then $f_1 = f_2$.

Proof. If $R(\ell_x f_1) = 0$ for all $x \in G$, then for any $h \in \mathcal{C}^2(G//K)$, $\langle \ell_x f_1, h\rangle = 0$. Let $h_t, t > 0$ be the heat kernel which is an element in $\mathcal{C}^2(G//K)$ defined through its spherical Fourier transform $\widehat{h_t}(\lambda) = e^{-t(\lambda^2+\rho^2)}$. Taking $h = h_t$ we thus get $\langle \ell_x f_1, h_t\rangle = 0$, i.e., $f_1 * h_t \equiv 0$ for all $t > 0$. But $f_1 * h_t \to f_1$ as $t \to 0$ in the sense of a distribution. Therefore $f_1 = 0$ which contradicts that f_1 is nonzero. This proves (a). Applying this on $f_1 - f_2$ we get (c).

For (b) it is enough to show that $R(\ell_x f_1) \neq 0$ implies that $\Delta^{-1} R(\ell_x f_1) \neq 0$ and $\Delta R(\ell_x f_1) \neq 0$. Indeed, $\Delta^{-1} R(\ell_x f_1) = 0$ implies $R(\ell_x f_1) = \Delta\Delta^{-1} R(\ell_x f_1) = 0$. On the other hand, if $\Delta R(\ell_x f_1) = 0$, then $\langle \Delta R(\ell_x f_1), \psi\rangle = 0$ and hence $\langle R(\ell_x f_1), \Delta\psi\rangle = 0$ for all $\psi \in \mathcal{C}^2(G//K)$. Since for any $\phi \in \mathcal{C}^2(G//K)$, $\widehat{\phi}(\lambda)(\lambda^2 + \rho^2)^{-1} \in \mathcal{C}^2(\widehat{G//K})$ (see Section 2.3.4), ϕ can be written as $\phi = \Delta\psi$ for some $\psi \in \mathcal{C}^2(G//K)$. Thus $\langle R(\ell_x f_1), \phi\rangle = 0$ for any $\phi \in \mathcal{C}^2(G//K)$, i.e., $R(\ell_x f_1) = 0$. □

Lemma 4.9. *Let f be a nonzero locally integrable radial function on X which defines an L^2-tempered distribution. If the support of the (distributional) spherical Fourier transform $\widehat{f}$ is $\{\alpha\}$ for some $\alpha \geq 0$, then $f = P(\partial_\lambda)\phi_\lambda|_{\lambda=\alpha}$ for some polynomial P. If moreover $f \in L^{2,\infty}(G//K)$ then $f = c\phi_\alpha$ for some constant c and $\alpha > 0$.*

Proof. Since f is an L^2-tempered distribution, $(\mathcal{A}^*)^{-1} f$ is an even tempered distribution on $\mathbb{R}$ (see Section 2.3.5). We recall that $\mathcal{C}^2(\widehat{G//K}) = S(\mathbb{R})_{\mathrm{even}}$. The Euclidean Fourier transform of $(\mathcal{A}^*)^{-1} f$ in the sense of a distribution, denoted by $\mathcal{F}((\mathcal{A}^*)^{-1} f)$, is the same as the spherical Fourier transform of f in the sense of an L^2-tempered distribution, denoted by $\widehat{f}$. Indeed, we take $\phi, \psi \in S(\mathbb{R})_{\mathrm{even}}$ such that $\mathcal{F}(\psi) = \phi$. As an Abel transform is an isomorphism between $\mathcal{C}^2(G//K)$ and $S(\mathbb{R})_{\mathrm{even}}$, there is $g \in \mathcal{C}^2(G//K)$ such that $\mathcal{A}g = \psi$; hence by the slice-projection theorem $\widehat{g} = \mathcal{F}(\psi)$. Then we have

$$\begin{aligned}\langle \mathcal{F}((\mathcal{A}^*)^{-1}f), \phi\rangle &= \langle (\mathcal{A}^*)^{-1}f, \psi\rangle = \langle (\mathcal{A}^*)^{-1}f, \mathcal{A}g\rangle \\ &= \langle \mathcal{A}^*(\mathcal{A}^*)^{-1}f, g\rangle = \langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle \\ &= \langle \widehat{f}, \mathcal{F}(\psi)\rangle = \langle \widehat{f}, \phi\rangle.\end{aligned}$$

Thus $\langle \mathcal{F}((\mathcal{A}^*)^{-1}f), \phi\rangle = \langle \widehat{f}, \phi\rangle$ where in the left-hand side ϕ is interpreted as a function of $S(\mathbb{R})_{\text{even}}$ and on the right-hand side ϕ is an element of $\mathcal{C}^2(\widehat{G//K})$. Therefore $\mathcal{F}((\mathcal{A}^*)^{-1}f)$ is supported on $\{\alpha\}$.

Therefore by [37, Thm. 6.25],

$$(\mathcal{A}^*)^{-1}f(t) = [P_1(\partial_\lambda)e^{i\lambda t} + P_2(\partial_\lambda)e^{-i\lambda t}]|_{\lambda=\alpha}$$

for two polynomials P_1 and P_2.

As $\phi_\lambda = \phi_{-\lambda}$ we have by Lemma 4.7, $f = P(\partial_\lambda)\phi_\lambda|_{\lambda=\alpha}$ for some polynomial P. If $f \in L^{2,\infty}(X)$, then by Proposition 3.2(ii), f is an L^2-tempered distribution, hence the conclusion above is valid for f. But by Lemma 4.6 the polynomial P is constant. Hence $f = c\phi_\alpha$ for some constant c. By Proposition 3.2(v), $\alpha > 0$. □

Lemma 4.10. *Let f be a nonzero function in $L^{2,\infty}(X)$ with* $\operatorname{Suppt}\widetilde{f} \subset \{\alpha\} \times B$. *Then $\alpha > 0$.*

Proof. For any $x \in G$, $R(\ell_x f)$ is in $L^{2,\infty}(G//K)$. By Proposition 4.4 for any $x \in G$, $\operatorname{Suppt}\widehat{R(\ell_x f)} \subset \{\alpha\}$. Therefore by Lemma 4.9, $R(\ell_x f) = c_x\phi_\alpha$, for some constant c_x which depends on x and if $\alpha = 0$ then $c_x = 0$ for all $x \in G$. That is, if $\alpha = 0$ then $\mathcal{R}(\ell_x f) = 0$ for all $x \in G$, which by Lemma 4.8(a) implies that $f \equiv 0$. □

We shall now complete the proof of Theorem 1.1.

Completion of the proof of Theorem 1.1. Suppose that $f = \mathcal{P}_\alpha F$ for $F \in L^2(B)$. We take a function $\phi \in \mathcal{C}^2(X)$ such that $\widetilde{\phi}(\alpha, b) = 0$ for all $b \in B$. We note that $\phi \in L^{2,1}(X)$ (Proposition 3.2(i)). Then by duality, $\langle \phi, \mathcal{P}_\alpha F\rangle = 0$. Indeed, using Fubini's theorem,

$$\begin{aligned}\int_G \phi(x)\overline{\mathcal{P}_\alpha F(x)}\,dx &= \int_G\int_K \phi(x)e^{(i\lambda-\rho)H(x^{-1}k)}F(kM)\,dk\,dx \\ &= \int_B \widetilde{\phi}(\alpha, b)\overline{F(b)}\,db = 0.\end{aligned}$$

From this it is easy to see that the distributional Fourier transform of f is supported on the sphere $\{\alpha\} \times B$.

For the converse we have by Proposition 4.4 that for any $x \in G$ either $R(\ell_x f)$ is zero or its spherical Fourier transform is supported on $\{\alpha\}$. We also note that since $f \in L^{2,\infty}(X)$, $R(\ell_x f) \in L^{2,\infty}(G//K)$. Therefore by Lemma 4.9, $\Delta R(\ell_x f) = -(\alpha^2+\rho^2)R(\ell_x f)$ for all $x \in G$. That is, $R(\ell_x \Delta f) = R(\ell_x[-(\alpha^2+\rho^2)f])$ for all $x \in G$. Hence by Lemma 4.8(c), $\Delta f = -(\alpha^2+\rho^2)f$. Since $f \in L^{2,\infty}(X)$, by Proposition 3.2(viii), we have $f = \mathcal{P}_\alpha u$ for some $u \in L^2(B)$. $\square$

Next we shall prove Theorem 1.2.

Proof of Theorem 1.2. We shall prove (b) and (c) and then use them to prove (a) and (d).

(b) We take $\lambda_1, \lambda_2 \in \mathbb{R}^+$ such that $\alpha < \lambda_1 < \lambda_2$. Let $\phi \in \mathcal{C}^2(\widehat{G//K})$ be supported on $[\lambda_1, \lambda_2]$. We claim that $\langle \widetilde{f}, \phi\rangle = 0$.

Let $\epsilon = \frac{1}{2}(\lambda_1^2 - \alpha^2) > 0$ where $\alpha = \sqrt{c_1 - \rho^2}$. From the hypothesis we know that there exists $N \in \mathbb{N}$, such that for all $n \geq N$,

$$| \, \|\Delta^n f\|_{2,\infty}^{1/n} - c_1| < \epsilon \quad \text{and hence} \quad (c_1-\epsilon)^n < \|\Delta^n f\|_{2,\infty} < (c_1+\epsilon)^n. \tag{4.1}$$

As $\widetilde{\Delta^k f} = (-1)^k(\lambda^2+\rho^2)^k \widetilde{f}$ (where λ is a dummy variable),

$$\begin{aligned}
|\langle \widetilde{f}, \phi\rangle| &= \left| \left\langle \widetilde{\Delta^k f}, \frac{1}{(\lambda^2+\rho^2)^k}\phi \right\rangle \right| \\
&= |\langle \Delta^k f, \psi_k\rangle| \\
&\leq \|\Delta^k f\|_{2,\infty}\|\psi_k\|_{2,1} \\
&\leq \|\Delta^k f\|_{2,\infty}\nu(\psi_k) \\
&\leq \|\Delta^k f\|_{2,\infty}\mu(\widehat{\psi_k}),
\end{aligned}$$

where $\psi_k \in \mathcal{C}^2(G//K)$ is the inverse spherical transform of $(\lambda^2+\rho^2)^{-k}\phi \in \mathcal{C}^2(\widehat{G//K})$ and ν, μ are seminorms of $\mathcal{C}^2(X)$ and of $\mathcal{C}^2(\widehat{X})$ respectively. Above, we have used Hölder's inequality, that $\|\psi_k\|_{2,1} \leq \nu(\psi_k)$ (Proposition 3.2(i)) and the isomorphism between $\mathcal{C}^2(G//K)$ and $\mathcal{C}^2(\widehat{G//K})$ (see Section 2.3.4).

Thus for $k \geq N$, we have

$$|\langle \widetilde{f}, \phi\rangle| \leq (c_1+\epsilon)^k \mu(\widehat{\psi_k}) = \mu\left[\left(\frac{\alpha^2+\rho^2+\epsilon}{\lambda^2+\rho^2}\right)^k \phi \right]. \tag{4.2}$$

Recall that ϕ is supported on $[\lambda_1, \lambda_2]$. For $\lambda \in [\lambda_1, \lambda_2]$ and the ϵ chosen above,

$$\lambda^2+\rho^2 \geq \lambda_1^2+\rho^2 = \alpha^2+\rho^2+2\epsilon > \alpha^2+\rho^2+\epsilon.$$

Hence given any $\delta > 0$ we can find $N_1 \in \mathbb{N}$ with $N_1 \geq N$ such that for $k \geq N_1$, $\mu[\ldots] < \delta$ in (4.2) and hence $|\langle \widetilde{f}, \phi\rangle| < \delta$. This establishes the claim and proves that

f annihilates any function $\phi \in \mathcal{C}^2(G//K)$ such that $\widehat{\phi}$ is supported in a compact set of $\mathbb{R}^+$ outside $[0,\alpha]$.

A step-by-step adaptation of this argument will show that f also annihilates any function $\psi \in \mathcal{C}^2(G//K)$ such that $\widehat{\psi}$ is supported in a compact set of $\mathbb{R}^+$ outside $[\beta,\infty)$. We include a sketch of the proof: We take ξ_1, ξ_2 with $0 < \xi_1 < \xi_2 < \beta$. Let $\phi \in \mathcal{C}^2(\widehat{G//K})$ be supported on $[\xi_1,\xi_2]$. We need to show that $\langle \widetilde{f}, \phi\rangle = 0$. We take

$$\epsilon = \frac{\beta^2 - \xi_2^2}{2(\xi_2^2+\rho^2)(\beta^2+\rho^2)} > 0. \tag{4.3}$$

It follows from the hypothesis that there exists $N \in \mathbb{N}$, such that for all $n \geq N$,

$$| \, \|\Delta^{-n} f\|_{2,\infty}^{1/n} - c_2| < \epsilon \quad \text{and hence} \quad (c_2-\epsilon)^n < \|\Delta^{-n}f\|_{2,\infty} < (c_2+\epsilon)^n. \tag{4.4}$$

Following the steps of the previous part of the proof we get

$$|\langle \widetilde{f}, \phi\rangle| = |\langle \widetilde{\Delta^{-k} f}, (\lambda^2+\rho^2)^k \phi\rangle| \leq \|\Delta^{-k} f\|_{2,\infty} \mu(\widehat{\psi_k})$$

where $\psi_k \in \mathcal{C}^2(G//K)$ is the inverse image of $(\lambda^2+\rho^2)^k\phi \in \mathcal{C}^2(\widehat{G//K})$ and μ is a seminorm of $\mathcal{C}^2(\widehat{X})$. Taking $k \geq N$, we have

$$|\langle \widetilde{f}, \phi\rangle| \leq (c_2+\epsilon)^k \mu(\widehat{\psi_k}) = \mu\left[\left(\frac{1}{\beta^2+\rho^2}+\epsilon\right)^k (\lambda^2+\rho^2)^k \phi\right].$$

Since ϕ is supported on $[\xi_1, \xi_2]$, by (4.3) we have for $\lambda \in [\xi_1,\xi_2]$,

$$2\epsilon + \frac{1}{\beta^2+\rho^2} = \frac{1}{\xi_2^2+\rho^2} \leq \frac{1}{\lambda^2+\rho^2}.$$

The rest of the argument is the same as the first part.

We have shown that f annihilates any function $\psi \in \mathcal{C}^2(G//K)$ with $\widehat{\psi}$ compactly supported outside $[\beta,\alpha]$. We shall now remove the condition that ϕ is K-biinvariant. By Proposition 4.3, for any $x \in G$, $\ell_x f$ also annihilates all $\psi \in \mathcal{C}^2(G//K)$ for which $\widehat{\psi}$ is compactly supported outside $[\beta,\alpha]$. Since $\psi(x) = \psi(x^{-1})$, this implies that $f * \psi(x) = 0$ for all $x \in G$. Noting that $f*\psi \in L^{2,\infty}(X)$ (Proposition 3.2(iv)) we have for any $g \in \mathcal{C}^2(X)$, $\langle f*\psi, g\rangle = 0$ and hence by Fubini's theorem $\langle f, g*\psi\rangle = 0$.

We take $g \in \mathcal{C}^2(X)$ with Suppt $\widetilde{g}$ contained in an open set $U \subset \mathbb{R}^+ \times B$ such that $([\beta,\alpha]\times B) \cap U = \emptyset$. We find another open set $U_1 \subset \mathbb{R}^+ \times B$ satisfying $\overline{U} \subset U_1$, U_1 is B-invariant (i.e., if $(\lambda, b) \in U_1$ for some $b \in B$, then $\{\lambda\} \times B \subset U_1$) and $([\beta,\alpha] \times B) \cap U_1 = \emptyset$. We take a $\psi \in \mathcal{C}^2(G//K)$ such that $\widehat{\psi}$ is supported on U_1 and $\widehat{\psi} \equiv 1$ on U (hence on the set $\{\lambda \mid (\lambda, b) \in U \text{ for some } b \in B\} \times B$). Then

$g*\psi = g$ since $\widetilde{g*\psi}(\lambda,k) = \widetilde{g}(\lambda,k)\widehat{\psi}(\lambda) = \widetilde{g}(\lambda,k)$. Thus by the argument above, $\langle f, g\rangle = \langle f, g*\psi\rangle = 0$.

Thus it follows that $\widetilde{f}$ is supported on a subset of $[\beta,\alpha]\times B$. We shall now show that it is not supported in a smaller annulus. We define

$$R_f^+ = \sup\{\lambda^2+\rho^2 \mid (\lambda,b)\in \mathrm{Suppt}\,\widetilde{f}\}, \qquad R_f^- = \inf\{\lambda^2+\rho^2 \mid (\lambda,b)\in \mathrm{Suppt}\,\widetilde{f}\}.$$

Above we have proved that $c_1 \geq R_f^+$ and $1/c_2 \leq R_f^-$. Now we shall show that given any $\epsilon > 0$, $c_1 < R_f^+ + \epsilon$ and $1/c_2 > R_f^- - \epsilon$. For this we fix an $\epsilon > 0$. We take a $\psi \in \mathcal{C}^2(G//K)$ such that $\widehat{\psi}$ is compactly supported, $\widehat{\psi}\equiv 1$ on the support of $\widetilde{f}$ (hence $\mathrm{Suppt}\,\widetilde{f} \subseteq \mathrm{Suppt}\,\widehat{\psi}$) and $R_f^+ < R_\psi^+ < R_f^+ + \epsilon, R_f^- - \epsilon < R_\psi^- < R_f^-$. Then $\widehat{\psi}\widetilde{f} = \widetilde{f}$ and hence $f = f*\psi$. Thus by Proposition 3.2(iv) and an isomorphism of $\mathcal{C}^2(G//K)$ and $\mathcal{C}^2(\widehat{G//K})$, there exist seminorms ν of $\mathcal{C}^2(X)$ and μ of $\mathcal{C}^2(\widehat{X})$ such that

$$\begin{aligned}\|\Delta^n f\|_{2,\infty} &= \|\Delta^n f*\psi\|_{2,\infty} = \|f*\Delta^n\psi\|_{2,\infty}\\ &\leq \|f\|_{2,\infty}\,\nu(\Delta^n\psi) \leq \|f\|_{2,\infty}\mu(\widehat{\Delta^n\psi}).\end{aligned}$$

Thus,

$$\|\Delta^n f\|_{2,\infty} \leq \|f\|_{2,\infty}\mu((\lambda^2+\rho^2)^n\widehat{\psi}) \leq \|f\|_{2,\infty}(R_\psi^+)^n n^{N_\mu} C_{\psi,\mu}$$

for some positive integer N_μ depending on μ and some constant $C_{\psi,\mu} > 0$ which depends on ψ and μ. This implies

$$c_1 = \lim_{n\to\infty}\|\Delta^n f\|_{2,\infty}^{1/n} \leq R_\psi^+ < R_f^+ + \epsilon.$$

Replacing $\Delta^n f$ by $\Delta^{-n} f$ in the argument above, we get similarly,

$$\|\Delta^{-n} f\|_{2,\infty} \leq \|f\|_{2,\infty}(R_\psi^-)^{-n} n^{N'_\mu} C'_{\psi,\mu},$$

for some $N'_\mu \in \mathbb{N}$ and $C'_{\psi,\mu} > 0$, which implies $c_2 = \lim_{n\to\infty}\|\Delta^{-n}f\|_{2,\infty}^{1/n} \leq (R_\psi^-)^{-1}$, hence $1/c_2 \geq R_\psi^- > R_f^- - \epsilon$. This completes the proof of part (b)

(c) If $c_1c_2 = 1$ then $\alpha = \beta$, hence $\widetilde{f}$ is supported on the sphere $\{\alpha\}\times B$. This and Lemma 4.10 implies that $\alpha > 0$. Therefore (c) follows from Theorem 1.1.

(a) We have used the two conditions of the hypothesis independently to prove that $\widetilde{f}$ is supported in a subset of $[0,\alpha]$ and also in a subset of $[\beta,\infty)$ for $\alpha,\beta\in\mathbb{R}^+$. If $\alpha < \beta$ then the support of $\widetilde{f}$ is empty and hence $f = 0$, contradicting the hypothesis. Therefore $\alpha \geq \beta$, equivalently $c_1c_2 \geq 1$.

(d) When $\beta = 0$, equivalently $c_2 = 1/\rho^2$, then the annulus $\mathbb{A}_\beta^\alpha$ obviously reduces to a ball around the origin of radius α. But if the support of $\widetilde{f}$ collapses to the origin, i.e., if $\mathrm{Suppt}\,\widetilde{f} \subset \{0\}\times B$, then by Lemma 4.10, $f \notin L^{2,\infty}(X)$. $\square$

Before we close this section let us emphasize certain points that are implicit in the proof above. For the discussion below we take a nonzero function $f \in L^{2,\infty}(X)$ such that $\Delta^n f \in L^{2,\infty}(X)$ for all $n \in \mathbb{Z}$. Let $A_n = \|\Delta^n f\|_{2,\infty}^{1/n}$ and $A'_n = \|\Delta^{-n} f\|_{2,\infty}^{1/n}$. The notation R_f^+ and R_f^- is as defined in the proof of Theorem 1.2(b).

We consider the following assertions.

(i) $\{A_n\} \to \infty$ if and only if $\widetilde{f}$ is not compactly supported.

(ii) $\{A_n\} \to \alpha^2 + \rho^2$ for some $0 < \alpha < \infty$ if and only if the outer radius of $\operatorname{Suppt} \widetilde{f}$ is α. In particular, if $\widetilde{f}$ is compactly supported then $\{A_n\}$ converges to a finite limit.

(iii) $\{A'_n\} \to (\beta^2 + \rho^2)^{-1}$ for some $0 \le \beta < \infty$ if and only if the inner radius of $\operatorname{Suppt} \widetilde{f}$ is β. Thus, in particular, for any f as above $\{A'_n\}$ converges to a finite limit. It does not depend on the compactness of the support.

The forward sides of (ii) and (iii) are proved explicitly in Theorem 1.2(b) and the forward side of (i) is a weakening of the converse side of (ii). So it is enough to prove the converse sides of the three assertions. This again mainly involves revisiting the proof of Theorem 1.2(b). We include a sketch of the proof:

First we observe that if a subsequence of $\{A_n\}$ converges to a finite limit L, then $L > \rho^2$. Let $L = \alpha_L^2 + \rho^2$ for $\alpha_L > 0$. Then $\alpha_L \ge \alpha$ where α is the outer radius of $\operatorname{Suppt} \widetilde{f}$, i.e., (in particular) if $\{A_n\}$ has a convergent subsequence then $\widetilde{f}$ is compactly supported.

For (i) we assume that $\widetilde{f}$ is not compactly supported, but $\{A_n\}$ does not diverge to ∞. Then $\{A_n\}$ has a subsequence converging to a finite limit, say L. But then as observed above, $\widetilde{f}$ is compactly supported which is a contradiction.

For (ii), we take f as above with $\widetilde{f}$ compactly supported and the outer radius of support equal to α. Then $R_f^+ = \alpha^2 + \rho^2$. This implies that $\limsup_{n\to\infty} A_n < R_f^+ + \epsilon$ for any $\epsilon > 0$ (see the second part of the proof of Theorem 1.2(b)). Suppose that a subsequence of $\{A_n\}$ converges to a limit L. Then by our observation above $L = \alpha_L^2 + \rho^2$ with $\alpha_L \ge \alpha$, α being the outer radius of $\operatorname{Suppt} \widetilde{f}$. But $\alpha_L \ge \alpha$ implies $L \ge R_f^+$. Hence $L = R_f^+$, i.e., $A_n \to R_f^+ = \alpha^2 + \rho^2$.

For (iii) we note that the inner radius $\beta < \infty$ for any nonzero $f \in L^{2,\infty}(X)$, because $\beta = \infty$ implies $\operatorname{Suppt} \widetilde{f} = \emptyset$, hence $f = 0$. We have $R_f^- = \beta^2 + \rho^2$ and Theorem 1.2(b) shows that $\limsup_{n\to\infty} A'_n < 1/(R_f^- - \epsilon)$ for $0 < \epsilon < R_f^-$. Suppose that $\{A'_n\}$ has a subsequence converging to a limit L. Then $L \le 1/\rho^2$, since if we assume the contrary, then $L > 1/\rho^2 \ge 1/(\beta^2 + \rho^2) = 1/R_f^-$ which is a contradiction. We write $L = 1/(\beta_L^2 + \rho^2)$. Then by the first part of Theorem 1.2(b), $\beta_L \le \beta$, because any $\psi \in \mathcal{C}^2(G//K)$ with $\operatorname{Suppt} \widehat{\psi} \subset [\xi_1, \xi_2]$, $\xi_1 < \xi_2 < \beta_L$ is annihilated by f. But $\beta_L \le \beta$ implies $L \ge 1/R_f^-$. Thus $L = 1/R_f^-$, which proves the assertion.

§5. Concluding remarks

(1) Theorem 1.1 can be generalized in the following way. See [23, pp. 205], [42, Lem. 2.2] for Euclidean results of this genre.

Proposition 5.1. *Suppose that a locally integrable function f on X satisfies $f(x)(1+|x|)^{-M} \in L^{2,\infty}(X)$ for some fixed nonnegative integer M and $\widetilde{f}$ is supported on the sphere $\{\alpha\}\times B$ of radius $\alpha > 0$ in $\mathbb{R}^+ \times B$. Then $(\Delta+\alpha^2+\rho^2)^{M+1} f = 0$, i.e., f is a generalized eigenfunction of Δ with eigenvalue $-(\alpha^2+\rho^2)$. In particular, if $M = 0$ then f is an eigenfunction.*

We need the following lemma.

Lemma 5.2. *Let $e_\lambda, \lambda \in \mathbb{R}$ be a family of eigenfunctions of Δ with eigenvalues $A(\lambda)$ such that the function $(x,\lambda) \mapsto e_\lambda(x)$ is in $C^\infty(X\times\mathbb{R})$ and the function $\lambda \mapsto A(\lambda)$ is in $C^\infty(\mathbb{R})$. Then for any polynomial P in one variable of degree $m \in \mathbb{N}$, $(\Delta - A(\lambda))^{m+1} P(\partial_\lambda) e_\lambda = 0$, i.e., $P(\partial_\lambda)e_\lambda$ is a generalized eigenfunction of Δ with eigenvalue $A(\lambda)$.*

Proof. It suffices to show that $(\Delta - A(\lambda))^{m+1}\partial_\lambda^m e_\lambda = 0$, which can be verified by straightforward computation for $m = 1, 2$. Then we use induction. Suppose the result is true for $m = 1, 2, \ldots, n-1$. We have

$$(\Delta - A(\lambda))^{n+1}\partial_\lambda^n e_\lambda = (\Delta - A(\lambda))^n[\partial_\lambda^n(A(\lambda)e_\lambda) - A(\lambda)\partial_\lambda^n e_\lambda].$$

Expanding the part in square brackets $[\ldots]$ in the right-hand side above by the Leibniz rule we see that each term in it is of the form $C\partial_\lambda^r A(\lambda)\partial_\lambda^{n-r}e_\lambda$ for $r = 1, 2, \ldots, n$. From induction hypothesis it follows that $(\Delta - A(\lambda))^n\partial_\lambda^{n-r}e_\lambda = 0$. This completes the proof. □

We need this result only when $A(\lambda)$ is a polynomial, more specifically for $A(\lambda) = \lambda^2 + \rho^2$.

Proof of Proposition 5.1. We have $\ell_x(f(y)/(1+|y|)^M) = \ell_x f(y)/(1+|x^{-1}y|)^M \in L^{2,\infty}(X)$. Since $(1+|xy|) \le (1+|x|)(1+|y|)$ ([17, Prop. 4.6.11]), $\ell_x f(y)/(1+|y|)^M \in L^{2,\infty}(X)$. Now as $R((\ell_x f)(y)/(1+|y|)^M) = R(\ell_x f)(y)/(1+|y|)^M$, we have $R(\ell_x f)(y)/(1+|y|)^M \in L^{2,\infty}(G//K)$. Therefore by Proposition 3.2(ii) $R(\ell_x f)$ is an L^2-tempered distribution. By Proposition 4.4, if for some $x \in G$, $R(\ell_x f) \neq 0$ then $\widehat{R(\ell_x f)}$ is supported on $\{\alpha\}$. We fix $x \in G$, such that $R(\ell_x f) \neq 0$. Proceeding as in the proof of Lemma 4.9 we conclude that $R(\ell_x f) = P_x(\partial_\lambda)\phi_\lambda|_{\lambda=\alpha}$ where the polynomial P_x depends on $x \in G$. Hence by Lemma 5.2, $(\Delta + \alpha^2 + \rho^2)^{\deg P_x + 1} R(\ell_x f) = 0$. However, the condition $R(\ell_x f)/(1+|\cdot|)^M \in L^{2,\infty}(G//K)$ gives an upper bound for the degree of the polynomial, precisely $\deg P_x \le M$ as can be proved going through steps similar to Lemma 4.5. Thus for all $x \in G$, $(\Delta + \alpha^2 + \rho^2)^{M+1} R(\ell_x f) = 0$. That is, $R(\ell_x(\Delta + \alpha^2 + \rho^2)^{M+1} f) = 0$ and hence by Lemma 4.8, $(\Delta + \alpha^2 + \rho^2)^{M+1} f = 0$. □

Proposition 5.1 vindicates a generalization of Theorem 1.2. For a fixed $M > 0$ we define a weighted norm $\|\cdot\|_M$ in the following way. For a measurable function f on X, let $g(x) = f(x)(1+|x|)^{-M}$. Then $\|f\|_M = \|g\|_{2,\infty}$.

Theorem 5.3. *Let f be a nonzero measurable function on X with $\|f\|_M < \infty$. Suppose for constants $c_1 \geq \rho^2, c_2 \leq 1/\rho^2$,*

$$\lim_{n\to\infty} \|\Delta^n f\|_M^{1/n} = c_1, \qquad \lim_{n\to\infty} \|\Delta^{-n} f\|_M^{1/n} = c_2.$$

Let $\beta = \sqrt{1/c_2 - \rho^2}$ and $\alpha = \sqrt{c_1 - \rho^2}$. Then we have conclusions (a) and (b) of Theorem 1.2*, while (c) and (d) of that theorem are replaced by*

(c) if $c_1 c_2 = 1$ then f is a generalized eigenfunction with eigenvalue $-c_1$,

(d) the annulus $\mathbb{A}_\beta^\alpha$ may reduce to a ball around the origin and may also collapse to the origin.

The proof of Theorem 1.2 can be easily adapted to prove this, but we omit it for brevity. We note only that under the norm condition here which is more relaxed than that of Theorem 1.2, this theorem allows collapsing of the annulus to the origin (see (d) above). This corresponds to the case $c_1 = 1/c_2 = \rho^2$, hence $c_1 c_2 = 1$ and thus is a subcase of (c). Precisely, in this case f is a generalized eigenfunction of Δ with eigenvalue $-\rho^2$, a particular case of which is ϕ_0.

(2) We recall that through the Iwasawa decomposition $G = NAK$, $X = G/K$ can be identified with the solvable Lie group $N \rtimes A$. Thus a rank-1 Riemannian symmetric space X of noncompact type is also a Damek–Ricci space (which are also known as NA groups). We shall denote them by S and use both of these names. Rank-1 symmetric spaces are the most distinguished prototypes of the NA groups, though they account for a very thin subcollection in the set of all NA groups (see [6]). In general, a Damek–Ricci space is a Riemannian manifold and a solvable Lie group but not a symmetric space. The absence of semisimple machinery (which enters the analysis on symmetric spaces through the natural G-action on $X = G/K$) in a general Damek-Ricci space S offers many fresh challenges. For instance, we cannot decompose a function on S in its K-types, a very useful tool for symmetric spaces. In particular the sense of *radiality* in S is not connected with any group action. Keeping these in mind we have completely avoided such well-known techniques for symmetric spaces. Most of the basic ingredients of the proofs are also

available for NA groups. Thus the proof given here should be readily extendable to NA groups. We refer to [6, 7, 34] for a detailed account on harmonic analysis on NA groups. However, we have to make a compromise, as the characterization of the $L^{2,\infty}$-eigenfunction as a Poisson transform (see Proposition 3.2(viii)) is still unavailable for NA groups. This unavailability is perhaps rooted in a missing analogue (for NA groups) of a result for the symmetric spaces due to Helgason and *six authors* (see [20, Chap. V, Thm. 6.6]). Precisely, "f is a Poisson transform" has to be substituted by a weaker statement "f is an eigenfunction of Δ with eigenvalue $-(\alpha^2+\rho^2)$ (respectively $-c_1$)" in Theorem 1.1 (respectively in Theorem 1.2(c)).

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