Publ. RIMS Kyoto Univ. **54** (2018), 379–391
DOI 10.4171/PRIMS/54-2-5

A Lemma for Microlocal Sheaf Theory in the ∞-Categorical Setting

by

Marco Robalo and Pierre Schapira

Abstract

The microlocal sheaf theory of Kashiwara and Schapira (Sheaves on manifolds, Grundlehren der mathematischen Wissenschaften 292, Springer, Berlin, 1990) makes essential use of an extension lemma for sheaves due to Kashiwara, and this lemma is based on a criterion of the same author giving conditions in order that a functor defined in $\mathbb{R}^{\mathrm{op}}$ with values in the category Sets of sets be constant. In the first part of this paper, using classical tools, we show how to generalize the extension lemma to the case of the unbounded derived category. In the second part, we extend Kashiwara's result on constant functors by replacing the category Sets with the ∞-category of spaces and apply it to generalize the extension lemma to ∞-sheaves, the ∞-categorical version of sheaves. Finally, we define the micro-support of sheaves with values in a stable $(\infty, 1)$-category.

2010 Mathematics Subject Classification: 35A27, 13D09, 55U35, 55P42.
Keywords: Microlocal sheaf theory, higher categories.

§1. Introduction

Microlocal sheaf theory appeared in [KS82] and was developed in [KS85, KS90]. However, this theory is constructed in the framework of the bounded (or bounded from below) derived category of sheaves $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ on a real manifold M, for a commutative unital ring $\mathbf{k}$, and it appears necessary in various problems to extend the theory to the unbounded derived category of sheaves $\mathrm{D}(\mathbf{k}_M)$. See in particular [Tam08, Tam15].

A crucial result in this theory is the "non-characteristic deformation lemma" [KS90, Prop. 2.7.2]. This lemma, which first appeared in [Kas75, Kas83]), asserts that if one has an increasing family of open subsets $\{U_s\}_{s\in\mathbb{R}}$ of a topological

Communicated by T. Mochizuki. Received September 4, 2017. Revised September 29, 2017.

M. Robalo: Sorbonne Universités, UPMC Univ. Paris 6, Institut de Mathématiques de Jussieu;
e-mail: marco.robalo@imj-prg.fr
P. Schapira: Sorbonne Universités, UPMC Univ. Paris 6, Institut de Mathématiques de Jussieu;
e-mail: pierre.schapira@imj-prg.fr

Hausdorff space M and an object F of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ such that the cohomology of F on U_s extends through the boundary of U_s for all s, then $\mathrm{R}\Gamma(U_s;F)$ is constant with respect to s. A basic tool for proving this result is the "constant functor criterion", again due to Kashiwara, a result which gives a condition in order that a functor $X\colon \mathbb{R}^{\mathrm{op}} \to \mathrm{Sets}$ is constant, where Sets is the category of sets in a given universe.

In Section 2 we generalize the extension lemma to the unbounded setting, that is, to objects of $\mathrm{D}(\mathbf{k}_M)$. Our proof is rather elementary and is based on the tools of [KS90]. This generalization being achieved, readers can persuade themselves that most of the results of [KS90], such as the functorial behavior of the micro-support, extend to the unbounded case.

Next, we consider a higher categorical generalization of this result. In Section 3 we generalize the constant functor criterion to the case where the 1-category Sets is replaced with the ∞-category $\mathcal{S}$ of spaces. Using this new tool, in Section 4.1, we generalize the extension lemma for ∞-sheaves with values in any stable compactly generated ∞-category $\mathscr{D}$. When $\mathscr{D}$ is the ∞-category $\mathrm{Mod}^{\infty}(\mathbf{k}_M)$ of ∞-sheaves of unbounded complexes of $\mathbf{k}$-modules we recover the results of Section 2.

Finally, in Section 4.2 we define the micro-support of any ∞-sheaf F with general stable higher coefficient.

Remark 1.1. After this paper was written, David Treumann informed us of the result by Dmitri Pavlov [Pav16] which generalizes Kashiwara's "constant functor criterion" to the case where the functor takes values in the ∞-category of spectra. Note that Theorem 3.2 below implies Pavlov's result on spectra.

§2. Unbounded derived category of sheaves

Let Sets denote the category of sets, in a given universe $\mathscr{U}$. In the sequel, we consider $\mathbb{R}$ as a category with the morphisms being given by the natural order $\leq$.

We first recall a result due to Kashiwara (see [KS90, §1.12]).

Lemma 2.1 (Constant functor criterion). *Consider a functor* $X\colon \mathbb{R}^{\mathrm{op}} \to \mathrm{Sets}$. *Assume that for each* $s \in \mathbb{R}$,

$$\varinjlim_{t>s} X_t \xrightarrow{\sim} X_s \xrightarrow{\sim} \varprojlim_{r<s} X_r. \tag{2.1}$$

Then the functor X *is constant.*

Let $\mathbf{k}$ denote a unital ring and denote by $\mathrm{Mod}(\mathbf{k})$ the abelian Grothendieck category of $\mathbf{k}$-modules. For brevity, we set

$$\begin{aligned} \mathrm{C}(\mathbf{k}) &:= \mathrm{C}(\mathrm{Mod}(\mathbf{k})), \quad \text{the category of chain complexes of } \mathrm{Mod}(\mathbf{k}),\\ \mathrm{D}(\mathbf{k}) &:= \mathrm{D}(\mathrm{Mod}(\mathbf{k})), \quad \text{the (unbounded) derived category of } \mathrm{Mod}(\mathbf{k}). \end{aligned}$$

We look at the ordered set $(\mathbb{R}, \leq)$ as a category and consider a functor $X\colon \mathbb{R}^{\mathrm{op}} \to \mathrm{C}(\mathbf{k})$. For brevity, we write $X_s = X(s)$.

The next result is a variant on Lemma 2.1 and the results of [KS90, §1.12].

Lemma 2.2. *Assume that*

(2.2) *for any $k \in \mathbb{Z}$, any $r \leq s$ in $\mathbb{R}$, the map $X^k_s \to X^k_r$ is surjective;*

(2.3) *for any $k \in \mathbb{Z}$, any $s \in \mathbb{R}$, $X^k_s \xrightarrow{\sim} \varprojlim_{r<s} X^k_r$;*

(2.4) *for any $j \in \mathbb{Z}$, any $s \in \mathbb{R}$, $\varinjlim_{t>s} H^j(X_t) \xrightarrow{\sim} H^j(X_s)$.*

Then for any $j \in \mathbb{Z}$, $r, s \in \mathbb{R}$ with $r \leq s$, one has the isomorphism $H^j(X_t) \xrightarrow{\sim} H^j(X_s)$. In other words, for all $j \in \mathbb{Z}$, the functor $H^j(X)$ is constant.

Proof. Consider the assertions for all $j \in \mathbb{Z}$, all $r, s \in \mathbb{R}$ with $r \leq s$:

(2.5) for any $j \in \mathbb{Z}$, $s \in \mathbb{R}$, the map $H^j(X_s) \to \varprojlim_{r<s} H^j(X_r)$ is surjective;

(2.6) for any $j \in \mathbb{Z}$, $r \leq s$, the map $H^j(X_s) \to H^j(X_r)$ is surjective;

(2.7) for any $j \in \mathbb{Z}$, $s \in \mathbb{R}$, the map $H^j(X_s) \to \varprojlim_{r<s} H^j(X_r)$ is bijective.

Assertion (2.5) follows from hypotheses (2.2) and (2.3) by applying [KS90, Prop. 1.12.4(a)].

Assertion (2.6) follows from (2.5) and hypothesis (2.4) in view of [KS90, Prop. 1.12.6].

It follows from (2.6) that for any $j \in \mathbb{Z}$ and $s \in \mathbb{R}$, the projective system $\{H^j(X_r)\}_{r<s}$ satisfies the Mittag-Leffler condition. We get (2.7) by using [KS90, Prop. 1.12.4(b)].

To conclude, apply [KS90, Prop. 1.12.6], using (2.4) and (2.7). □

Theorem 2.3 (Non-characteristic deformation lemma). *Let[1] M be a Hausdorff space and let $F \in \mathrm{D}(\mathbf{k}_M)$. Let $\{U_s\}_{s\in\mathbb{R}}$ be a family of open subsets of M. We assume*

(a) for all $t \in \mathbb{R}$, $U_t = \bigcup_{s<t} U_s$;

(b) for all pairs (s,t) with $s \leq t$, the set $\overline{U_t \setminus U_s} \cap \operatorname{supp} F$ is compact;

(c) setting $Z_s = \bigcap_{t>s} \overline{(U_t \setminus U_s)}$, we have for all pairs (s,t) with $s \leq t$ and all $x \in Z_s$, $(\mathrm{R}\Gamma_{X\setminus U_t} F)_x \simeq 0$.

[1] Stéphane Guillermou informed us that, some time ago, Claude Viterbo obtained a similar result (unpublished).

Then for all $t \in \mathbb{R}$, *we have the isomorphism in* $\mathrm{D}(\mathbf{k})$,

$$\mathrm{R}\Gamma\Big(\bigcup_s U_s; F\Big) \xrightarrow{\sim} \mathrm{R}\Gamma(U_t; F).$$

We shall adapt the proof of [KS90, Prop. 2.7.2], using Lemma 2.2.

Proof. (i) Following [KS90, Prop. 2.7.2], we shall first prove the isomorphism

$$(a)^s\colon \varinjlim_{t>s} H^j(U_t; F) \xrightarrow{\sim} H^j(U_s; F) \quad \text{for all } j. \tag{2.8}$$

Replacing M with $\operatorname{supp} F$, we may assume from the beginning that $\overline{U_t \setminus U_s}$ is compact. For $s \le t$, consider the distinguished triangle

$$(\mathrm{R}\Gamma_{M\setminus U_t} F)|_{Z_s} \to (\mathrm{R}\Gamma_{M\setminus U_s} F)|_{Z_s} \to (\mathrm{R}\Gamma_{U_t\setminus U_s} F)|_{Z_s} \xrightarrow{+1}.$$

The first two terms are 0 by hypothesis (c). Therefore $(\mathrm{R}\Gamma_{U_t\setminus U_s} F)|_{Z_s} \simeq 0$ and we get

$$0 \simeq H^j(Z_s; \mathrm{R}\Gamma_{U_t\setminus U_s} F) \simeq \varinjlim_{U \supset Z_s} H^j(U \cap U_t; \mathrm{R}\Gamma_{M\setminus U_s} F) \quad \text{for all } j,$$

where U ranges over the family of open neighborhoods of Z_s.

For any such U there exists t' with $s < t' \le t$ such that $U \cap U_t \supset U_{t'} \setminus U_s$. Therefore,

$$\varinjlim_{t, t>s} H^j(U_t; \mathrm{R}\Gamma_{M\setminus U_s} F) \simeq 0 \quad \text{for all } j.$$

By using the distinguished triangle $\mathrm{R}\Gamma_{M\setminus U_s} F \to F \to \mathrm{R}\Gamma_{U_s} F \xrightarrow{+1}$, we get (2.8).

(ii) We shall follow [KS06, Prop. 14.1.6, Thm. 14.1.7] and recall that if $\mathscr{C}$ is a Grothendieck category, then any object of $\mathrm{C}(\mathscr{C})$ is qis (quasi-isomorphic) to a homotopically injective object whose components are injective. Hence, given $F \in \mathrm{D}(\mathbf{k}_M)$, we may represent it by a homotopically injective object $F^\bullet \in \mathrm{C}(\mathbf{k}_M)$ whose components F^k are injective. Then $\mathrm{R}\Gamma(U_s; F)$ is represented by $\Gamma(U_s; F^\bullet) \in \mathrm{C}(\mathbf{k})$. Set

$$X_s^k = \Gamma(U_s; F^k), \quad X_s = \Gamma(U_s; F^\bullet).$$

Then (2.2) is satisfied since F^k is flabby, (2.3) is satisfied since F^k is a sheaf and (2.4) is nothing but (2.8).

(iii) To conclude, apply Lemma 2.2. □

§3. The constant functor criterion for $\mathcal{S}$

§3.1. On ∞-categories

The aim of this subsection is essentially notational and references are made to [Lur09, Lur17]. We use Joyal's quasi-categories to model $(\infty, 1)$-categories. If not necessary we will simply use the terminology ∞-categories.

Denote by Cat_∞ the $(\infty, 1)$-category of all $(\infty, 1)$-categories in a given universe $\mathscr{U}$ and by Cat the 1-category of all 1-categories in $\mathscr{U}$.

To $\mathscr{C} \in \mathrm{Cat}$, one associates its nerve, $N(\mathscr{C}) \in \mathrm{Cat}_\infty$. Denoting by $N(\mathrm{Cat})$ the image of Cat by N, the embedding $\iota\colon N(\mathrm{Cat}) \hookrightarrow \mathrm{Cat}_\infty$ admits a left adjoint h, namely the functor which to an $(\infty, 1)$-category $\mathscr{C}$ associates its homotopy category $\mathrm{h}\mathscr{C}$. We get the functors

$$\mathrm{h}\colon\ \mathrm{Cat}_\infty \rightleftarrows N(\mathrm{Cat})\ :\iota$$

Hence, $\mathrm{h}\circ\iota \simeq \mathrm{id}_1$ and there exists a natural morphism of ∞-functors $\mathrm{id}_\infty \to \iota\circ\mathrm{h}$, where id_1 and id_∞ denote the identity functors of the categories Cat and Cat_∞, respectively.

Looking at Cat_∞ as a simplicial set, its degree-0 elements are the $(\infty, 1)$-categories, its degree-1 elements are the ∞-functors, etc. Hence the functor h sends a $(\infty, 1)$-category to the usual category, an ∞-functor to the usual functor, etc. It sends a stable $(\infty, 1)$-category to a triangulated category where the distinguished triangles are induced by the cofiber–fiber sequences. Moreover, it sends an ∞-functor to a triangulated functor, etc. See [Lur17, 1.1.2.15].

Let $\mathcal{S}$ (resp. $\mathcal{S}_*$) denote the $(\infty, 1)$-category of spaces (resp. pointed spaces) [Lur09, 1.2.16.1]. Informally, one can think of $\mathcal{S}$ as a simplicial set whose vertices are CW-complexes, 1-cells are continuous maps, 2-cells are homotopies between continuous maps, etc. Recall that $\mathcal{S}$ admits small limits and colimits in the sense of [Lur09, 1.2.13]. Moreover, by Whitehead's theorem, a map $f : X \to Y$ in $\mathcal{S}$ is an equivalence if and only if the induced map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is an isomorphism of sets and for every base point $x \in X$, the induced maps $\pi_n(X, x) \to \pi_n(Y, f(x))$ are isomorphisms for all $n \geq 1$.

It is also convenient to recall the existence of a Grothendieck construction for $(\infty, 1)$-categories. Namely, for any $(\infty, 1)$-category $\mathscr{C}$, the *straightening* and *unstraightening* constructions establish an equivalence of $(\infty, 1)$-categories

$$\mathrm{St} : (\mathrm{Cat}_\infty/\mathscr{C})^{\mathrm{cart}} \simeq \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Cat}_\infty) : \mathrm{Un} \tag{3.1}$$

where on the left-hand side we have the $(\infty, 1)$-category of ∞-functors $\mathscr{D} \to \mathscr{C}$ that are Cartesian fibrations and functors that preserve Cartesian morphisms (see [Lur09, Def. 2.4.1.1]), and on the right-hand side we have the $(\infty, 1)$-category of

∞-functors from $\mathscr{C}^{\mathrm{op}}$ to Cat_∞. See [Lur09, 3.2.0.1]. The same holds for diagrams in $\mathcal{S}$, where we find

$$\mathrm{St} : (\mathrm{Cat}_\infty/\mathscr{C})^{\text{Right-fib}} \simeq \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathcal{S}) : \mathrm{Un} \tag{3.2}$$

where this time on the left-hand side we have the $(\infty, 1)$-category of ∞-functors $\mathscr{D} \to \mathscr{C}$ that are right fibrations. See [Lur09, 2.2.1.2]. The equivalence (3.2) will be useful for the following reason: for any diagram $X : \mathscr{C}^{\mathrm{op}} \to \mathcal{S}$, its limit in $\mathcal{S}$ can be identified with the space of sections of $\mathrm{Un}(X)$ [Lur09, 3.3.3.4]:

$$\lim X \simeq \mathrm{Map}_{\mathscr{C}}(\mathscr{C}, \mathrm{Un}(X)). \tag{3.3}$$

§3.2. A criterion for a functor to be constant

In this subsection, we generalize [KS90, Prop. 1.12.6] to the case of an ∞-functor. Let $X\colon \mathbb{R}^{\mathrm{op}} \to \mathcal{S}$ be an ∞-functor. We set

$$X_s = X(s), \quad \rho_{s,t}\colon X_t \to X_s \quad (s \le t). \tag{3.4}$$

Lemma 3.1 (Constant ∞-functor criterion for spaces). *Let $X\colon \mathbb{R}^{\mathrm{op}} \to \mathcal{S}$ be an ∞-functor. Assume that for each $s \in \mathbb{R}$, the natural morphisms in $\mathcal{S}$,*

$$\operatorname*{colim}_{s<t} X_t \to X_s \to \lim_{r<s} X_r, \tag{3.5}$$

are both equivalences. Then for every $t \ge s$, the morphism $X_t \to X_s$ in $\mathcal{S}$ is an equivalence.

The proof adapts and uses that of [KS90, Prop. 1.12.6] to the case of $\mathcal{S}$.

Proof. **Step I.** It is enough to prove that for each $c \in \mathbb{R}$, the restriction of X to $\mathbb{R}_{<c}$ is constant.

Step II: Choosing base points. Let $c \in \mathbb{R}$ and again let X denote the restriction of X to $\mathbb{R}_{<c}$. The hypothesis

$$\lim_{s<c} X_s \simeq X_c$$

ensures that the choice of a base point in X_c determines a compatible system of base points up to homotopy at every X_s with $s < c$, i.e., the choice of a 2-simplex $\sigma : \Delta^2 \to \mathrm{Cat}_\infty$:

$$\begin{array}{ccc} & & \mathcal{S}_* \\ & \overset{\overline{X}}{\nearrow} \;\; \sigma & \downarrow \\ \mathrm{N}(\mathbb{R}^{\mathrm{op}}_{<c}) & \xrightarrow[X]{} & \mathcal{S}. \end{array} \tag{3.6}$$

For the reader's convenience we explain how to construct the 2-simplex σ. Thanks to (3.3), the limit $\lim_{[s<c]} X_s$ can be identified with the category of sections of the right fibration $\mathrm{Un(X)} \to \mathrm{N}(\mathbb{R}_{<c})$. Therefore, the hypothesis of the lemma ensures that the choice of a base point in X_c provides a section of $\mathrm{Un}(X)$.[2] Its image via the functor St of (3.2) provides the lifting (3.6).

Recall that the forgetful functor $\mathcal{S}_* \to \mathcal{S}$ preserves filtrant colimits and all small limits and is conservative. Therefore the hypotheses are also valid for $\overline{X}$.

Step III: Working with a fixed choice of base points. Given an ∞-functor $Y : \mathbb{N}^{\mathrm{op}} \to \mathcal{S}_*$ we have for each $n \in \mathbb{N}$, a short exact sequence (of groups when $n \geq 1$ and pointed sets when $n = 0$) called the Milnor exact sequence (see for instance [MP12, Prop. 2.2.9] or [GJ09, Prop. 2.15 Chap. VI]):

$$0 \to R^1 \lim_{i \in \mathbb{N}^{\mathrm{op}}} \pi_{n+1}(Y_i) \to \pi_n(\lim_{i \in \mathbb{N}^{\mathrm{op}}} Y_i) \to \lim_{i \in \mathbb{N}^{\mathrm{op}}} \pi_n(Y_i) \to 0. \tag{3.7}$$

As the inclusion $\mathbb{N}^{\mathrm{op}} \subseteq \mathbb{R}^{\mathrm{op}}$ is cofinal, the Milnor sequence is also valid for $\mathbb{R}^{\mathrm{op}}$-towers. In what follows, we fix for each $s \in \mathbb{R}$ an isomorphism of posets $\alpha_s : \mathbb{R} \simeq \mathbb{R}_{<s}$ and use α_s to transfer the cofinality argument to the poset $\mathbb{R}_{<s}$. Clearly the arguments below are independent of the chosen α_s.

Choose any lifting $\overline{X}$ of X. The preceding discussion applied to $Y = \overline{X}$ yields for each $n \in \mathbb{N}$, $s \in \mathbb{R}_{<c}$, a short exact sequence

$$0 \to R^1 \lim_{r<s} \pi_{n+1}(\overline{X}_r) \to \pi_n(\lim_{r<s} \overline{X}_r) \to \lim_{r<s} \pi_n(\overline{X}_r) \to 0. \tag{3.8}$$

Under the hypothesis of the lemma, we get short exact sequences

$$0 \to R^1 \lim_{r<s} \pi_{n+1}(\overline{X}_r) \to \pi_n(\overline{X}_s) \to \lim_{r<s} \pi_n(\overline{X}_r) \to 0. \tag{3.9}$$

For each $n \geq 0$, each $s, t \in \mathbb{R}_{\leq x}$ with $t \geq s$, we shall prove

$$\text{the map } \operatorname*{colim}_{c>t>s} \pi_n(\overline{X}_t) \to \pi_n(\overline{X}_s) \text{ is bijective;} \tag{3.10}$$

$$\text{the map } \pi_n(\overline{X}_s) \to \lim_{r<s} \pi_n(\overline{X}_r) \text{ is surjective;} \tag{3.11}$$

$$\text{the map } \pi_n(\overline{X}_t) \to \pi_n(\overline{X}_s) \text{ is surjective;} \tag{3.12}$$

$$\text{the map } \pi_n(\overline{X}_s) \to \lim_{r<s} \pi_n(\overline{X}_r) \text{ is bijective.} \tag{3.13}$$

Assertion (3.10) follows from the hypothesis, the fact that the system $\{t : c > t > s\}$ is cofinal in $\{t : t > s\}$ and the fact that π_n commutes with filtrant colimits for $n \geq 0$.

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[2] Thanks to [Lur09, 2.4.2.4] such a section is automatically Cartesian.

Assertion (3.11) follows from (3.9).

Let us prove (3.12). By the surjectivity result in [KS90, Prop. 1.12.6], it is enough to prove the surjectivity of $\mathrm{colim}_{[c>t>s]}\,\pi_n(\overline{X}_t) \to \pi_n(\overline{X}_s)$ and $\pi_n(\overline{X}_s) \to \lim_{[r<s]}\pi_n(\overline{X}_r)$ for all $s \in \mathbb{R}_{<c}$, which follows from (3.10) and (3.11).

By (3.12), we know that the projective systems $\{\pi_n(\overline{X}_r)\}_{r<s}$ satisfy the Mittag-Leffler condition for all $n \geq 0$, $s < c$. Therefore, $R^1 \lim_{[r<s]}\pi_{n+1}(\overline{X}_r) \simeq 0$ for all n, all $s \in \mathbb{R}_{<c}$ and (3.13) follows from (3.9). Therefore, we have isomorphisms for every $n \geq 0$,

$$\operatorname*{colim}_{s<t<c}\pi_n(\overline{X}_t) \simeq \pi_n(\overline{X}_s) \simeq \lim_{r<s}\pi_n(\overline{X}_r).$$

Applying [KS90, Prop. 1.12.6], we get that the diagram of sets $s \mapsto \pi_n(\overline{X}_s)$ is constant for every n.

Step IV: End of the proof. The conclusion of Step III holds for any lifting $\overline{X}$ of the restriction of X to $\mathbb{R}_{<c}$. As the result holds for $n = 0$, the diagram $s < c \mapsto \pi_0(X_s)$ is also constant, seen as a diagram of sets rather than pointed sets.

To conclude one must show that for any $n \in \mathbb{N}$, $t \geq s \in \mathbb{R}_{<c}$ and for every choice of a base point y in X_t, the induce maps

$$\rho^n_{s,t}\colon \pi_n(X_t, y) \to \pi_n(X_s, \rho^n_{s,t}(y)) \tag{3.14}$$

are bijective. Since, for $\alpha < c$, $\alpha \mapsto \pi_0(X_\alpha) \in \mathrm{Sets}$ is constant, choosing $l \in \mathbb{R}$ with $t < l < c$, y determines a unique element $\bar{y}$ in $\pi_0(X_l)$. Again, using the hypothesis $X_l \simeq \lim_{r<l} X_r$ with the argument of Step II, the choice of a representative for $\bar{y}$ determines a homotopy compatible system of base points at every X_r for $r < l$ and therefore a new lifting $\overline{X}$ of the restriction of X to $\mathbb{R}_{<l}$. The associated base point of $\overline{X}$ at X_t is a representative of y and the composition with π_n provides the maps (3.14). By (3.10), (3.11), (3.12), (3.13) and [KS90, Prop. 1.12.6] the maps (3.14) are isomorphisms. This conclusion holds for any $c \in \mathbb{R}$ and thus for any $t \geq s$ in $\mathbb{R}^{\mathrm{op}}$. □

We refer to [Lur09, 5.5.7.1] for the notion of a presentable compactly generated $(\infty, 1)$-category.

Theorem 3.2 (Presentable constant ∞-functor criterion). *Let $\mathscr{C}$ be a presentable compactly generated $(\infty, 1)$-category and let $X\colon \mathbb{R}^{\mathrm{op}} \to \mathscr{C}$ be an ∞-functor. Assume that for each $s \in \mathbb{R}$, the natural morphisms*

$$\operatorname*{colim}_{s<t} X_t \to X_s \to \lim_{r<s} X_r$$

are both equivalences. Then for any $t \geq s$ the induced map $X_t \to X_s$ is an equivalence.

Proof. Apply Lemma 3.1 to all mapping spaces $\mathrm{Map}(Z, X_t)$ for each compact object Z. □

Remark 3.3. This result does not apply to $\mathscr{C} = \mathbb{R}^{\mathrm{op}}$ and X the identity functor. Indeed, $\mathbb{R}^{\mathrm{op}}$ is not compactly generated in the sense of [Lur09, 5.5.7.1].

Remark 3.4. The conclusion of Theorem 3.2 is that the ∞-functor $X\colon \mathbb{R}^{\mathrm{op}} \to \mathscr{C}$ sends all morphisms to equivalences in $\mathscr{C}$. As indicated to us by M. Porta, this implies that $X\colon \mathbb{R}^{\mathrm{op}} \to \mathscr{C}$ factors through the ∞-categorical localization of $N(\mathbb{R}^{\mathrm{op}})$ along the class W consisting of all arrows, $N(\mathbb{R}^{\mathrm{op}})[W^{-1}]$.[3] The category $\mathbb{R}^{\mathrm{op}}$ is contractible so that $N(\mathbb{R}^{\mathrm{op}})[W^{-1}] \simeq *$ and X is equivalent to the constant diagram.

§4. Micro-support

§4.1. The non-characteristic deformation lemma with stable coefficients

In this subsection, we generalize [KS90, Prop. 2.7.2] and Theorem 2.3 to more general coefficients. Let $\mathscr{D}$ be a presentable compactly generated stable $(\infty,1)$-category. Given a topological space M, we denote by Op_M the category of its open subsets. One defines a higher categorical version of sheaves on M as follows. Let $\mathrm{Psh}(M, \mathscr{D})$ denote the $(\infty,1)$-category of ∞-functors

$$\mathrm{N}(\mathrm{Op}_M^{\mathrm{op}}) \to \mathscr{D}.$$

See [Lur09, 1.2.7.2, 1.2.7.3]. The category Op_M is equipped with a Grothendieck topology whose covering of U is the families $\{U_i\}_i$ such that $U_i \subseteq U$ and $\bigcup_i U_i = U$. We let $\mathrm{Sh}(M, \mathscr{D})^\wedge$ denote the full subcategory of $\mathrm{Psh}(M, \mathscr{D})$ spanned by those functors that satisfy the sheaf condition and are hypercomplete. See [Lur09, 6.2.2] and [Lur11a, Sect. 1.1] for the theory of ∞-sheaves and [Lur09, 6.5.2, 6.5.3, 6.5.4] for the notion of hypercomplete. The $(\infty,1)$-category $\mathrm{Sh}(M, \mathscr{D})^\wedge$ is again a stable compactly generated $(\infty,1)$-category and when $M = \mathrm{pt}$ one recovers $\mathrm{Sh}(M, \mathscr{D})^\wedge \simeq \mathscr{D}$.

The usual pullback and push-forward functorialities can be lifted to the higher categorical setting and are given by exact functors. See for instance the discussion

[3] See [Lur17, 4.1.3.1].

in [PYY16, Sect. 2.4]. Let $j_U\colon U \hookrightarrow M$ be an open embedding and let $a_M\colon M \to \mathrm{pt}$ be the map from M to one point. We introduce the notation

$$\Gamma^\infty(U;\,\bullet\,) := a^\infty_{M\,*} \circ j^\infty_{U\,*} \circ j^{\infty\,-1}_U \colon \mathrm{Sh}(M,\mathscr{D})^\wedge \to \mathscr{D},$$

where a^∞_{M*}, $j^\infty_{U\,*}$, $j^{\infty\,-1}_U$ are the direct and inverse image functors for $(\infty,1)$-categories of sheaves. If Z is a closed subset of U, using the cofiber–fiber sequence associated to $\Gamma^\infty(U;\,\bullet\,) \to \Gamma^\infty(U \setminus Z;\,\bullet\,)$, we define

$$\Gamma^\infty_Z(U;\,\bullet\,)\colon \mathrm{Sh}(M,\mathscr{D})^\wedge \to \mathscr{D}.$$

The following result generalizes [KS90, Prop. 2.7.2] and Theorem 2.3 to any context of sheaves with stable coefficients.

Theorem 4.1 (Non-characteristic deformation lemma for stable coefficients). *Let M be a Hausdorff space and let $F \in \mathrm{Sh}(M,\mathscr{D})^\wedge$. Let $\{U_s\}_{s\in\mathbb{R}}$ be a family of open subsets of M. We assume*

(a) for all $t \in \mathbb{R}$, $U_t = \bigcup_{s<t} U_s$;

(b) for all pairs (s,t) with $s \le t$, the set $\overline{U_t \setminus U_s} \cap \operatorname{supp} F$ is compact;

(c) setting $Z_s = \bigcap_{t>s} \overline{(U_t \setminus U_s)}$, we have for all pairs (s,t) with $s \le t$ and all $x \in Z_s$, $(\Gamma^\infty_{X\setminus U_t} F)_x \simeq 0$.

Then we have the equivalences in $\mathscr{D}$, for all $s,t \in \mathbb{R}$,

$$\Gamma^\infty\Big(\bigcup_s U_s; F\Big) \xrightarrow{\ \sim\ } \Gamma^\infty(U_t;F).$$

We shall almost mimic the proof of [KS90, Prop. 2.7.2].

Proof. (i) We shall prove the equivalences

$$(a)^t\colon\ \lim_{s<t} \Gamma^\infty(U_s;F) \xleftarrow{\ \sim\ } \Gamma^\infty(U_t;F),$$
$$(b)^s\colon\ \operatorname*{colim}_{t>s} \Gamma^\infty(U_t;F) \xrightarrow{\ \sim\ } \Gamma^\infty(U_s;F).$$

(ii) Equivalence $(a)^t$ is always true by hypothesis (a). Indeed, one has $\mathbf{k}_{U_s} \simeq \varprojlim_{[r<s]} \mathbf{k}_{U_r}$, which implies $\lim_{[s<t]} \Gamma^\infty_{U_s} F \xleftarrow{\ \sim\ } \Gamma^\infty_{U_t} F$, and the result follows since the direct image functor commutes with lim (because it is a right adjoint).

(iii) The proof of the equivalence $(b)^s$ for all s is formally the same as the proof of (2.8) which itself mimics that of [KS90, Prop. 2.7.2] and we shall not repeat it.

To conclude, apply Theorem 3.2 to $\mathscr{D}$. ∎

Remark 4.2. Let $\mathbf{k}$ denote a commutative unital ring. Theorem 4.1 recovers the result of Theorem 2.3 in the particular case where $\mathscr{D}$ is the ∞-version of the derived category of $\mathbf{k}$, which we will denote by $\mathrm{Mod}^{\infty}(\mathbf{k})$. We define it as follows: let $\mathrm{C}(\mathbf{k})$ denote the 1-category of (unbounded) chain complexes over $\mathbf{k}$. One considers the nerve $\mathrm{N}(\mathrm{C}(\mathbf{k}))$ and settles $\mathrm{Mod}^{\infty}(\mathbf{k})$ as the localization $\mathrm{N}(\mathrm{C}(\mathbf{k}))[\mathcal{W}^{-1}]$ along the class of edges $\mathcal{W}$ given by quasi-isomorphisms of complexes. This localization is taken inside the theory of $(\infty, 1)$-categories as in Remark 3.4. The homotopy category $\mathrm{h}(\mathrm{Mod}^{\infty}(\mathbf{k}))$ is canonically equivalent to $\mathrm{D}(\mathbf{k})$ by the universal properties of higher and classical localizations. In this case we settle on the notation

$$\mathrm{Mod}^{\infty}(\mathbf{k}_M) := \mathrm{Sh}(M, \mathrm{Mod}^{\infty}(\mathbf{k}))^{\wedge}.$$

The homotopy category of $\mathrm{Mod}^{\infty}(\mathbf{k}_M)$ recovers the usual derived category of (unbounded) complexes of sheaves of $\mathbf{k}$-modules, $\mathrm{D}(\mathbf{k}_M)$. Indeed, [PYY16, Prop. 5.3(i)–(iii)] identifies the subcategory of $\mathrm{Sh}(M, \mathrm{Mod}^{\infty}(\mathbf{k}))$ spanned by hypercomplete objects, with the full subcategory spanned by the left-t-complete objects (i.e., an object that is equal to the limit of its tower of truncations). As explained in [Lur17, 1.2.1.18] we are reduced to showing that the ∞-category of left-bounded objects in $\mathrm{Sh}(M, \mathrm{Mod}^{\infty}(\mathbf{k}))$ is equivalent to the usual left-bounded derived category. This follows from the fully faithful embedding of [Lur11b, Prop. 2.1.8]. When $M = \mathrm{pt}$, one recovers $\mathrm{Mod}^{\infty}(\mathbf{k})$ and $\mathrm{D}(\mathbf{k})$, respectively.

Remark 4.3. If we assume that M is a topological manifold (therefore homotopy equivalent to a CW-complex), then $\mathrm{Sh}(M, \mathscr{D})^{\wedge}$ is equivalent to $\mathrm{Sh}(M, \mathscr{D})$. In particular, $\mathrm{Mod}^{\infty}(\mathbf{k}_M)$ is equivalent to the higher category $\mathrm{Sh}(M, \mathrm{Mod}^{\infty}(\mathbf{k}))$ of ∞-sheaves obtained without imposing hyperdescent. To see this we use the standard fact in general topology that M being a topological manifold, it is of finite Lebesgue covering dimension. This is the notion of covering dimension used in [Lur09, Def. 7.2.3.1]. The relation between the covering dimension of M and the homotopy dimension of its associated ∞-topos is then given by [Lur09, 7.2.3.6]. Hyperdescent follows from [Lur09, 7.2.1.12].

Remark 4.4. In [KS90, Prop. 2.7.2], Z_s is defined as $Z_s = \overline{\bigcap_{t>s} (U_t \setminus U_s)}$, which is a mistake. This mistake has already been corrected in the errata of `https://webusers.imj-prg.fr/~pierre.schapira/books/`.

§4.2. Micro-support

The definition [KS90, Def. 5.1.2] of the micro-support of sheaves immediately extends to ∞-sheaves with stable coefficients.

Let M be a real manifold of class C^1 and denote by T^*M its cotangent bundle.

Definition 4.5. Let $F \in \mathrm{Sh}(M, \mathscr{D})$. The micro-support of F, denoted $\mu\mathrm{supp}(F)$, is the closed $\mathbb{R}^+$-conic subset of T^*M defined as follows. For U open in T^*M, $U \cap \mu\mathrm{supp}(F) = \emptyset$ if for any $x_0 \in M$ and any real C^1-function φ on M defined in a neighborhood of x_0 satisfying $d\varphi(x_0) \in U$ and $\varphi(x_0) = 0$, one has $(\Gamma^{\infty}_{\{x;\varphi(x)\geq 0\}}(F))_{x_0} \simeq 0$.

When $\mathscr{D}$ is $\mathrm{Mod}^{\infty}(\mathbf{k})$, one recovers the classical definition of the micro-support.

Remark 4.6. As already mentioned in the introduction, Theorem 2.3 is the main tool to develop microlocal sheaf theory in the framework of classical derived categories. We hope that similarly Theorem 4.1 will be the main tool to develop microlocal sheaf theory in the new framework of sheaves with stable coefficients.

Remark 4.7. In [KS90], the micro-support of F was denoted $\mathrm{SS}(F)$, a shorthand for "singular support".

Acknowledgements

The research of P.S. is supported by the ANR-15-CE40-0007 "MICROLOCAL".

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