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Regularity of a D-Module Along a Submanifold

by

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Abstract

We study how regularity along a submanifold of a differential or microdifferential system can propagate from a family of submanifolds to another. The first result is that a microdifferential system regular along a Lagrangian foliation is regular. However, when restricted to a fixed submanifold the corresponding result is true only under a condition on the characteristic variety.

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§1. Introduction

The initial idea of this paper is a question from M. Granger relative to a paper with F. Castro-Jimenez [1]: If a holonomic $\mathcal{D}_X$-module is regular along a family of hyperplanes crossing on a linear subvariety, is it regular along the intersection?

As we will see in Section 6, from the microlocal point of view, this problem is equivalent to the following one: If a holonomic $\mathcal{D}_X$-module is regular along the points of a submanifold, is it regular along the submanifold itself?

In the second formulation, the problem is similar to the well-known theorem of Kashiwara–Kawaï [5, Thm. 6.4.1] which shows that a $\mathcal{D}_X$-module regular along the points of an open set Ω is regular on Ω. However, we show here that the answer to our question is not yes in general, but only under a condition on the characteristic variety of the $\mathcal{D}_X$-module. To solve the problem we first prove a microlocal version of Kashiwara–Kawaï's result. Consider a conic Lagrangian foliation of an open set of the cotangent bundle to a complex manifold. Let $\mathcal{M}$ be a holonomic microdifferential module defined on the open set. If $\mathcal{M}$ is regular along each leaf of the foliation then the module $\mathcal{M}$ is regular.

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Then we may give an answer to the initial problem. A natural framework to state it is a "maximally degenerate" involutive submanifold of the cotangent bundle. Such a variety carries a canonical conic Lagrangian foliation and contains also a canonical conic Lagrangian submanifold, its degeneracy locus. Then the result is that if a holonomic microdifferential module is regular along the leaves of the foliation, it is regular along the degeneracy locus under the condition that its characteristic variety is contained in the maximally degenerate involutive submanifold.

This applies to our initial problem. If a holonomic differential or microdifferential module has its characteristic variety contained in the set $\langle x, \xi\rangle = 0$ of $T^*\mathbb{C}^n$ and is regular along the hyperplanes containing the origin, then it is regular along the origin (Example 6.6). In the same way, a holonomic module which is regular along the points of a submanifold Y of a complex variety X is regular along Y under the condition that, in a neighborhood of the conormal to Y, the characteristic variety is contained in the inverse image of Y by the projection $T^*X \to X$ (Example 6.5).

When the condition on the characteristic variety is not satisfied, the result may not be true. In Example 6.5, it may be untrue if the singular support of the module has irreducible components tangent to the variety Y. We show this by constructing a counterexample.

In Sections 2 and 3, we recall the definitions of regularity and give some classical results that we will use later.

In Section 4, we prove a complex microlocal Cauchy theorem that we use in Section 5 to prove our main result. In Section 6, we show how this applies to maximally degenerate involutive manifolds and give examples.

Section 7 is devoted to the complete calculation of a counterexample when the condition on the characteristic variety is not fulfilled.

§2. Regularity

In dimension 1, regularity of $\mathcal{D}_X$-modules is equivalent to the notion of differential equations with regular singularity. In higher dimensions, there are two different kinds of regularity. The first one is global, i.e., it concerns a $\mathcal{D}_X$-module on an open set, while the second is relative to a submanifold. In both cases, there is an equivalence between growth conditions on the solutions and algebraic conditions on the module itself. There are a lot of works on the subject: here we refer mostly to Kashiwara–Kawaï [4], [5] concerning the global theory and to our works [8], [9], [11] for the regularity relative to a submanifold.

Let us first recall the definition of the V-filtration of Kashiwara which is the main tool in the definition of regularity.

Let X be a complex manifold and Y a submanifold of X. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X, and $\mathcal{D}_X$ the sheaf of differential operators on X with coefficients in $\mathcal{O}_X$.

The sheaf $\mathcal{D}_X$ is provided with the usual filtration by the order of operators; this filtration will be denoted by $(\mathcal{D}_{X,m})_{m\geq 0}$. Kashiwara defined in [3] another filtration, the V-filtration:

$$V_k\mathcal{D}_X = \left\{P \in \mathcal{D}_X \mid \forall\, \ell \in \mathbb{Z},\ P\mathcal{I}_Y^{\ell} \subset \mathcal{I}_Y^{\ell-k}\right\} \tag{2.1}$$

where $\mathcal{I}_Y$ is the defining ideal of Y and $\mathcal{I}_Y^{\ell} = \mathcal{O}_X$ if $\ell \leq 0$.

If Y is given in local coordinates by $Y = \{(x_1, \dots, x_p, t_1, \dots, t_q) \mid t = 0\}$, then the function x_i and the derivations $D_{x_i} = \frac{\partial}{\partial x_i}$ are of order 0 for the V-filtration while t_j is of order -1 and D_{t_j} of order 1.

Let $\tau : T_Y X \to Y$ be the normal bundle to Y in X and $\mathcal{O}_{[T_Y X]}$ the sheaf of holomorphic functions on $T_Y X$ which are polynomial in the fibers of τ. Let $\mathcal{O}_{[T_Y X]}[k]$ be the subsheaf of $\mathcal{O}_{[T_Y X]}$ of homogeneous functions of degree k in the fibers of τ. There are canonical isomorphisms between $\mathcal{I}_Y^k/\mathcal{I}_Y^{k-1}$ and $\tau_*\mathcal{O}_{[T_Y X]}[k]$, between $\bigoplus \mathcal{I}_Y^k/\mathcal{I}_Y^{k-1}$ and $\tau_*\mathcal{O}_{[T_Y X]}$. Hence the graded ring $\operatorname{gr}^V \mathcal{D}_X$ associated to the V-filtration on $\mathcal{D}_X$ acts naturally on $\mathcal{O}_{[T_Y X]}$. An easy calculation [14, Chap. III, Lem. 1.4.1] shows that as a subring of $\mathcal{E}\mathrm{nd}(\tau_*\mathcal{O}_{[T_Y X]})$ it is identified to $\tau_*\mathcal{D}_{[T_Y X]}$ the sheaf of differential operators on $T_Y X$ with coefficients in $\tau_*\mathcal{O}_{[T_Y X]}$.

The Euler vector field θ of $T_Y X$ is the vector field which acts on $\mathcal{O}_{[T_Y X]}[k]$ by multiplication by k. Let ϑ be any differential operator in $V_0\mathcal{D}_X$ whose image in $\operatorname{gr}_0^V \mathcal{D}_X$ is θ.

Definition 2.1. The holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is *regular along* Y if any section u of $\mathcal{M}$ is annihilated by a differential operator of the form $\vartheta^N + P + Q$ where P is in $\mathcal{D}_{X,N-1} \cap V_0\mathcal{D}_X$ and Q is in $\mathcal{D}_{X,N} \cap V_{-1}\mathcal{D}_X$.

Definition 2.2. A polynomial b is *a regular b-function* for u along Y if there exists a differential operator Q in $V_{-1}\mathcal{D}_X \cap \mathcal{D}_{X,m}$ such that $(b(\vartheta) + Q)u = 0$. Here m is the degree of b.

It is proved in [10, Thm. 3.4] that $\mathcal{M}$ is regular along Y if and only if all sections of u admit a regular b-function.

We denote by $\mathcal{O}_{\widehat{X|Y}}$ the formal completion of $\mathcal{O}_X$ along Y, that is,

$$\mathcal{O}_{\widehat{X|Y}} = \varprojlim_k \mathcal{O}_X/\mathcal{I}_Y^k.$$

If d is the codimension of Y, as usual we denote by $\mathcal{B}_{Y|X}^{\infty} = \mathcal{H}_Y^d(\mathcal{O}_X)$ the cohomology of $\mathcal{O}_X$ with support in Y and by $\mathcal{B}_{Y|X} = \mathcal{H}_{[Y]}^d(\mathcal{O}_X)$ the corresponding algebraic cohomology.

From [9, Thm. 3.2.11] applied with $r_0 = s = 1$ and $r = +\infty$, we get that if $\mathcal{M}$ is regular along Y then

$$\forall j \geq 0, \quad \mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{\widehat{X|Y}}) = \mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y, \tag{2.2}$$

$$\forall j \geq 0, \quad \mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{Y|X}) = \mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}^\infty_{Y|X}). \tag{2.3}$$

We proved in [11, Thm. 2.4.2] that the converse is true if Y is a hypersurface: if equality (2.2) or equality (2.3) is true, then $\mathcal{M}$ is regular along Y.

If Y is not a hypersurface, this is no longer true. A microlocalization of the definition is necessary to extend the results to any submanifold, as will be seen in the next section.

§3. Microlocal regularity

We denote by $\mathcal{E}_X$ the sheaf of microdifferential operators of [13]. It is filtered by the order, $\mathcal{E}_X = \bigcup \mathcal{E}_{X,k}$ and we call it the usual filtration.

Let Λ be a Lagrangian conic submanifold of the cotangent bundle T^*X. A section of a coherent $\mathcal{E}_X$-module supported by Λ is nondegenerate if the ideal of the principal symbols of the microdifferential operators annihilating it is the defining ideal of Λ ([13, Chap. II, Def. 4.1.1]. An $\mathcal{E}_X$-module $\mathcal{M}$ is a simple holonomic module supported by Λ if it is generated by a nondegenerate section. This always exists locally for a given Λ.

If Λ is the conormal $T^*_Y X$ to a submanifold Y of X, a canonical simple holonomic module supported by Λ is the sheaf $\mathcal{C}_{Y|X}$ of holomorphic microfunctions. It was defined in [13, Chap. II, Prop. 1.5.8], where it was denoted by $\mathcal{C}^f_{Y|X}$. The corresponding sheaf of microfunctions of infinite order is $\mathcal{C}^\infty_{Y|X}$ which was denoted by $\mathcal{C}_{Y|X}$ in [13]. We refer to [14] for these definitions.

In [8], we extended the definitions of V-filtrations and b-functions to microdifferential equations and Lagrangian subvarieties of the cotangent bundle.

Let $\mathcal{M}_\Lambda$ be a simple holonomic $\mathcal{E}_X$-module supported by Λ generated by a nondegenerate section u_Λ. Let $\mathcal{M}_{\Lambda,k} = \mathcal{E}_{X,k} u_\Lambda$. Then the V-filtration on $\mathcal{E}_X$ along Λ is defined in [8, Sect. 2.1]:

$$V_k \mathcal{E}_X = \big\{ P \in \mathcal{E}_X \mid \forall \ell \in \mathbb{Z},\ P\mathcal{M}_{\Lambda,\ell} \subset \mathcal{M}_{\Lambda,\ell+k} \big\}. \tag{3.1}$$

This filtration is independent of the choices of $\mathcal{M}_\Lambda$ and u_Λ, so it is globally defined.

Let $\mathcal{O}_\Lambda[k]$ be the sheaf of holomorphic functions on Λ homogeneous of degree k in the fibers of $\Lambda \to X$ and $\mathcal{O}_{(\Lambda)} = \bigoplus_{k\in\mathbb{Z}} \mathcal{O}_\Lambda[k]$. Then there is an isomorphism between $\mathcal{M}_{\Lambda,k}/\mathcal{M}_{\Lambda,k-1}$ and $\mathcal{O}_{\Lambda,k}$. By this isomorphism the graded ring $\operatorname{gr}^V \mathcal{E}_X$ acts on $\mathcal{O}_{(\Lambda)}$ and may be identified to the sheaf $\mathcal{D}_{(\Lambda)}$ of differential operators on Λ with coefficients in $\mathcal{O}_{(\Lambda)}$.

All of these definitions are invariant under quantized canonical transformations [8, Sect. 2.1].

Definition 3.1. The holonomic $\mathcal{E}_X$-module $\mathcal{M}$ is *regular along* Λ (on an open set of T^*X) if any section u of $\mathcal{M}$ is annihilated by a microdifferential operator of the form $\vartheta^N + P + Q$ where P is in $\mathcal{E}_{X,N-1} \cap V_0\mathcal{E}_X$ and Q is in $\mathcal{E}_{X,N} \cap V_{-1}\mathcal{E}_X$.

We have a fundamental result:

Theorem 3.2 ([11, Thm. 2.4.2]). *The holonomic $\mathcal{E}_X$-module $\mathcal{M}$ is regular along the conormal T^*_YX to a submanifold Y of X on the open set Ω of T^*X if and only if*

$$\forall\, j \geq 0, \quad \mathcal{E}\mathrm{xt}^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{Y|X})|_\Omega = \mathcal{E}\mathrm{xt}^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^\infty_{Y|X})|_\Omega. \tag{3.2}$$

As the restriction to the zero section of T^*X of the sheaf $\mathcal{C}_{Y|X}$ is $\mathcal{B}_{Y|X}$ while the restriction of $\mathcal{C}^\infty_{Y|X}$ is $\mathcal{B}^\infty_{Y|X}$, equation (2.3) is a special case of (3.2).

If Λ is not the conormal bundle to a submanifold Y, the result is still true if $\mathcal{C}_{Y|X}$ is replaced by a simple holonomic module $\mathcal{M}_\Lambda$:

$$\forall\, j \geq 0, \quad \mathcal{E}\mathrm{xt}^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{M}_\Lambda) = \mathcal{E}\mathrm{xt}^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}^\infty_X \otimes_{\mathcal{E}_X} \mathcal{M}_\Lambda). \tag{3.3}$$

The regularity property is also equivalent to other properties like the existence of a regular b-function or conditions on the microcharacteristic varieties (see [10]).

Definition 3.1 is equivalent to what Kashiwara–Kawaï call having "regular singularities along Λ" in [5, Def. 1.1.11] when Λ is a smooth part of an irreducible component of the characteristic variety of $\mathcal{M}$ (this is proved in [7, Thm. 3.1.7]).

So [5, Def. 1.1.16] may be reformulated as follows.

Definition 3.3. The holonomic $\mathcal{E}_X$-module $\mathcal{M}$ has *regular singularities* (or is *regular*) if for each irreducible component Λ of its characteristic variety, $\mathcal{M}$ is regular along a Zariski open subset of the regular part of Λ.

A holonomic $\mathcal{D}_X$-module is regular if $\mathcal{E}_X \otimes_{\mathcal{D}_X} \mathcal{M}$ is regular.

Let us now recall two important results of Kashiwara–Kawaï:

Theorem 3.4 ([4, Thm. 4.1.1]). *If $\mathcal{M}$ is a regular holonomic $\mathcal{E}_X$-module then it is regular along any Lagrangian submanifold of T^*X.*

Theorem 3.5 ([5, Thm. 6.4.1]). *A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is regular on X if and only if at each point $x \in X$,*

$$\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_x \simeq \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \widehat{\mathcal{O}}_x).$$

Here $\mathcal{O}_X$ is the sheaf of holomorphic functions while $\widehat{\mathcal{O}}_x$ is the set of formal power series at x.

§4. A complex Cauchy problem for $\mathcal{E}_X$-modules

Let X be a complex analytic manifold and Y be a submanifold of X. Let $\mathcal{O}_X$ be (resp. $\mathcal{O}_Y$) the sheaf of holomorphic functions on X (resp. Y) and Ω_X (resp. Ω_Y) the sheaf of holomorphic differential forms of maximum degree on X (resp. Y).

Let $\varpi : T^*X \times_X Y \to T^*X$, $\varrho : T^*X \times_X Y \to T^*Y$ and $\pi : T^*X \times_X Y \to Y$ be the canonical maps.

The sheaf $\mathcal{E}_{Y\to X}$ is defined by

$$\mathcal{E}_{Y\to X} = \pi^{-1}\mathcal{O}_Y \otimes_{\pi^{-1}\mathcal{O}_X} \varpi^{-1}\mathcal{E}_X.$$

If t is an equation of Y, $\mathcal{E}_{Y\to X}$ is isomorphic to $\mathcal{E}_X/t\mathcal{E}_X$ as a $(\varrho^{-1}\mathcal{E}_Y, \varpi^{-1}\mathcal{E}_X)$-bimodule.

The inverse image of a coherent left $\mathcal{E}_X$-module $\mathcal{M}$ by $i : Y \hookrightarrow X$ is defined in [13] as

$$i^*\mathcal{M} = \mathcal{M}_Y = \varrho_*\left(\mathcal{E}_{Y\to X} \otimes_{\varpi^{-1}\mathcal{E}_X} \varpi^{-1}\mathcal{M}\right).$$

Here i^{-1} denotes the inverse image of sheaves and i^* the inverse image of $\mathcal{E}_X$-modules.

Concerning right $\mathcal{E}_X$-modules, the definition is

$$\mathcal{E}_{X\leftarrow Y} = \varpi^{-1}\mathcal{E}_X \otimes_{\pi^{-1}\Omega_X} \pi^{-1}\Omega_Y$$

and the inverse image of a right $\mathcal{E}_X$-module $\mathcal{N}$ is given by

$$i^*\mathcal{N} = \varrho_*\left(\varpi^{-1}\mathcal{N} \otimes_{\varpi^{-1}\mathcal{E}_X} \mathcal{E}_{X\leftarrow Y}\right).$$

The characteristic variety of an $\mathcal{E}_X$-module is, by definition, its support. A submanifold Y of X is noncharacteristic on an open set V of T^*Y for an $\mathcal{E}_X$-module $\mathcal{M}$ if the map $\varrho : \varpi^{-1}Ch(\mathcal{M}) \to T^*Y$ is finite and proper on $\varrho^{-1}V$. In that case, the module $i^*\mathcal{M}$ is $\mathcal{E}_Y$-coherent and by [13, Thm. 3.5.6], duality and inverse image by i of $\mathcal{E}_X$-modules commute, that is,

$$i^*\,\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_X)[\dim X] = \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{E}_Y)[\dim Y]. \tag{4.1}$$

Let Z be a submanifold of Y. The conormal bundle $T^*_Z X$ to Z in X is a sub-bundle of $T^*X \times_X Y$ and the restriction to $T^*_Z X$ of the map ϖ is the identity. The restriction to $T^*_Z X$ of the map $\varrho : T^*X \times_X Y \to T^*Y$ is the map $T^*_Z X \to T^*_Z Y$ which we still denote by ϱ.

Proposition 4.1. *Let $\mathcal{M}$ be holonomic $\mathcal{E}_X$-module which is defined in a neighborhood Ω of $T^*_Z X$. We assume that Y is noncharacteristic for $\mathcal{M}$ and denote by*

d the codimension of Y. Then we have

$$\varrho_* \, \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{Z|X})[d] \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{C}_{Z|Y}),$$
$$\varrho_* \, \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^\infty_{Z|X})[d] \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{C}^\infty_{Z|Y}).$$

Proof. This result has been proved for $(\mathcal{C}^{\mathbb{R}}_{Z|X}, \mathcal{C}^{\mathbb{R}}_{Z|Y})$ in [6, Lem. 6.1]. Our proof is similar.

Applying the tensor product of $\mathcal{C}_{Z|Y}$ to both sides of equation (4.1) we get

$$\begin{aligned}\mathbb{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{C}_{Z|Y}) &= \varrho_* \left(\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_X) \otimes^{\mathbb{L}}_{\mathcal{E}_X} \mathcal{E}_{X\leftarrow Y}\right) \otimes^{\mathbb{L}}_{\mathcal{E}_Y} \mathcal{C}_{Z|Y}[d] \\ &= \varrho_* \left(\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_X) \otimes^{\mathbb{L}}_{\mathcal{E}_X} \mathcal{E}_{X\leftarrow Y} \otimes^{\mathbb{L}}_{\varrho^{-1}\mathcal{E}_Y} \varrho^{-1}\mathcal{C}_{Z|Y}\right)[d].\end{aligned}$$

From the proof of [13, Lem. 3.5.7], we have

$$\mathcal{C}_{Z|X} = \mathcal{E}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{B}_{Z|Y} = \mathcal{E}_{X\leftarrow Y} \otimes_{\varrho^{-1}\mathcal{E}_Y} \varrho^{-1}\mathcal{C}_{Z|Y}$$

and thus

$$\begin{aligned}\mathbb{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{C}_{Z|Y}) &= \varrho_* \left(\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_X) \otimes^{\mathbb{L}}_{\mathcal{E}_X} \mathcal{C}_{Z|X}\right)[d] \\ &= \varrho_* \, \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{Z|X})[d].\end{aligned}$$

The proof of the formula for $(\mathcal{C}^\infty_{Z|Y}, \mathcal{C}^\infty_{Z|X})$ is the same. □

Corollary 4.2. *Let $Z \subset Y \subset X$ be three manifolds and assume that Y is non-characteristic for a coherent $\mathcal{E}_X$-module $\mathcal{M}$. If $\mathcal{M}$ is regular along Z on an open subset Ω of T^*X, then $\mathcal{M}_Y$ is regular along Z on $\varrho(\Omega)$.*

§5. Regularity and Lagrangian foliations

The aim of this section is to prove a microlocal version of Theorem 3.5.

Let M be a conic symplectic manifold. Here we assume that M is a complex manifold but Proposition 5.1 is true as well in the real differentiable case.

We recall that a structure of conic symplectic manifold on a complex manifold M is given by a 1-form ω_M whose differential $\sigma_M = d\omega_M$ is a symplectic 2-form on M. The same structure may also be given by a symplectic 2-form σ_M and a compatible action of $\mathbb{C}^*$.

Let θ_M be the Euler vector field of the $\mathbb{C}^*$-action. The correspondence between ω_M and (σ_M, θ_M) is given by the interior product $\omega_M = \sigma_M \rfloor \theta_M$ [2].

A *conic Lagrangian foliation* of M is a foliation by conic Lagrangian submanifolds.

Proposition 5.1. *Let M be conic symplectic manifold with a conic Lagrangian foliation. There is locally a homogeneous symplectic map from M to the cotangent* *bundle T^*X of a complex manifold X which transforms the leaves into the fibers of $\pi : T^*X \to X$.*

This result has been proved in the nonhomogeneous case when M is a Banach space by Weinstein [15, Cor. 7.2].

Proof. A Lagrangian variety is involutive, hence if two functions vanish on a Lagrangian variety Λ, their Poisson bracket vanishes on Λ. But the Poisson bracket depends only on the derivative of the functions, hence if two functions are constant on Λ, their Poisson bracket vanishes on Λ.

Let $\mathcal{O}$ be the sheaf of rings of holomorphic functions on M which are constant on the leaves of the foliation. The Poisson bracket of two functions of $\mathcal{O}$ vanishes everywhere. Let $2n$ be the dimension of M. As the foliation has codimension n, we can find locally n functions $u_1, \dots, u_n$ in $\mathcal{O}$ whose differentials are linearly independent at each point. The Poisson bracket of two of them is always 0, hence by the proof of Darboux theorem as it is given in [2, Thm. 3.5.6], there are n functions $v_1, \dots, v_n$ on M such that $(u_1, \dots, u_n, v_1, \dots, v_n)$ is a canonical symplectic (nonhomogeneous) system of coordinates for M. By definition, the canonical symplectic 2-form σ_M of M is equal to $\sum dv_i \wedge du_i$.

The functions $u_1, \dots, u_n$, are constant on the leaves of the foliation which are conic varieties, hence they are constant on the fibers of the $\mathbb{C}^*$-action. Let θ_M be the Euler vector field associated to this action and for $i = 1, \dots, n$ let $w_i = \theta_M(v_i)$. As $\theta_M(u_i) = 0$, θ_M is equal to $\sum_{i=1}^n w_i \frac{\partial}{\partial v_i}$ and thus $\omega_M = \sigma_M \rfloor \theta_M = \sum w_i \, du_i$.

Hence, the functions $(u_1, \dots, u_n, w_1, \dots, w_n)$ define a canonical symplectic homogeneous system of coordinates for M. In the coordinates (u, w), the leaves are given by $u = \text{constant}$, which shows the proposition. □

Theorem 5.2. *Let Ω be a conic open subset of T^*X and (Λ_α) be a conic Lagrangian foliation of Ω.*

Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-module defined on Ω. If $\mathcal{M}$ is regular along each leaf Λ_α, then $\mathcal{M}$ is a regular holonomic module on Ω.

Proof. Let $\dot{T}^*X$ be the complement of the zero section in T^*X. As Definition 3.3 is local and concerns only the intersection of the characteristic variety with $\dot{T}^*X$, we consider a neighborhood of a point of $\dot{T}^*X$.

We will prove the theorem by induction on the dimension of X. If the dimension of X is 1, the characteristic variety of $\mathcal{M}$ is the union of the conormal bundles to isolated points of X. On the other hand, a conic Lagrangian foliation is necessarily given by the conormal bundles to the points of X. Hence by the hypothesis, $\mathcal{M}$ is regular along each component of its characteristic variety, hence $\mathcal{M}$ is regular by [5, Def. 1.1.16].

Assume now that the dimension of X is >1. The problem being local on $\dot{T}^*X$, we may use Proposition 5.1 and transform the foliation into the union of the conormal bundles to the points of an open subset U of X. So Ω is a conic open subset of $\pi^{-1}(U)$ and $\mathcal{M}$ is regular along $T^*_{\{x\}}X$ on Ω for $x \in U$.

According to Definition 3.3, we will prove that $\mathcal{M}$ is regular by proving that it is regular along a Zariski open subset of each irreducible component of its characteristic variety. Such an irreducible component is a conic Lagrangian subvariety of T^*X, hence generically it is the conormal to a smooth subvariety Z of X. So we consider a neighborhood of a point x^* where an irreducible component Λ is the conormal to a submanifold Z of X. If Z is a point of X, $\mathcal{M}$ is regular along Λ by the hypothesis.

If not, there are local coordinates $(x_1 \dots, x_p, t_1, \dots, t_q)$ of X such that x^* has coordinates $x{=}0$, $t{=}0$, $\xi{=}0$, $\tau{=}(1,0,\dots,0)$ and $\Lambda{=}\{(x,t,\xi,\tau) \in T^*X \mid x{=}0, \tau{=}0\}$.

Let $Y_a = \{(x,t) \in X \mid t_1 = a\}$. The conormal to the hypersurface Y_a does not meet Λ, hence Y_a is noncharacteristic for $\mathcal{M}$. As $\mathcal{M}$ is regular along each point of X on Ω, by Corollary 4.2, $\mathcal{M}_{Y_a}$ is regular along each point of Y_a if $a \ll 1$.

By the induction hypothesis, $\mathcal{M}_{Y_a}$ is then regular if $a \ll 1$ which implies by [5, Thm. 6.4.5] that $\mathcal{M}$ is regular near x^* and we are done. □

§6. Applications and examples

Let Σ be a submanifold of T^*X with a conic Lagrangian foliation. This implies that Σ is conic involutive [2, Thm. 3.6.2]. We assume that this foliation is, locally on Σ, the restriction of a conic Lagrangian foliation of T^*X.

Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-module whose characteristic variety is contained in Σ and which is regular along each leaf of this foliation. Then by Theorem 5.2, $\mathcal{M}$ is regular, hence regular along any Lagrangian submanifold of T^*X.

A typical example of conic involutive manifolds which have a canonical Lagrangian foliation restriction of a foliation of X is given by "maximally degenerate" involutive manifolds, as we will explain now.

The canonical projection $T^*X \to X$ defines a map $T^*X \times_X T^*X \to T^*(T^*X)$ which, composed with the diagonal map $T^*X \to T^*X \times_X T^*X$, defines the canonical 1-form of T^*X, that is, $\omega_X : T^*X \to T^*(T^*X)$. We now restrict our attention to the complement $\dot{T}^*X$ of the zero section in T^*X. The set of points of $\Sigma \cap \dot{T}^*X$ where $\omega_X|_\Sigma$ vanishes is isotropic, hence of dimension less than or equal to the dimension of X. It is called the *degeneracy locus* of Σ.

Definition 6.1 ([12]). The involutive submanifold Σ of $\dot{T}^*X$ is said to be *maximally degenerate* if the degeneracy locus is of maximal dimension, that is, the dimension of X.

Then the degeneracy locus is a Lagrangian subset Λ_0 of $\dot{T}^*X$. In fact it is a Lagrangian submanifold of $\dot{T}^*X$ by [12]. The manifold Σ is involutive, hence has a canonical foliation by bicharacteristic leaves.

Lemma 6.2. *Let Σ be a maximally degenerate involutive submanifold of $\dot{T}^*X$ and Λ_0 its degeneracy locus. After restricting Σ to a neighborhood of Λ_0 we have*

(1) Λ_0 is the union of the **conic** *bicharacteristic leaves of Σ;*

(2) for each conic leaf S of Σ, there is one and only one Lagrangian conic submanifold of Σ whose intersection with Λ_0 is exactly S;

(3) these Lagrangian submanifolds define a foliation of Σ which we will call the "Lagrangian foliation";

(4) locally on Σ, the Lagrangian foliation is the restriction of a Lagrangian foliation of T^*X.

Proof. Let $\pi : \dot{T}^*X \to X$ be the canonical projection. By [12, Thm. 2.15], there is locally on Λ_0, a homogeneous symplectic transformation of $\dot{T}^*X$ which transforms Σ into $\pi^{-1}(Z)$ for some submanifold Z of X.

Taking coordinates $(x_1, \ldots, x_{n-p}, t_1, \ldots, t_p)$ of X such that $Z = \{(x,t) \in X \mid t = 0\}$ and $\Sigma = \{(x,t,\xi,\tau) \in \dot{T}^*X \mid t = 0\}$, we have $\Lambda_0 = \{(x,t,\xi,\tau) \in \dot{T}^*X \mid t = 0,\ \xi = 0\}$ and the bicharacteristic leaves of Σ are the sets

$$S_{x_0,\xi_0} = \big\{(x,t,\xi,\tau) \in \dot{T}^*X \mid t = 0,\ x = x_0,\ \xi = \xi_0\big\}.$$

So, there is a map $\varrho : \Sigma \to T^*Z$ such that

- the bicharacteristics of Σ are the inverse images by ϱ of the points of T^*Z;
- Λ_0 is the inverse image of the zero section of T^*Z;
- the conic Lagrangian submanifolds of Σ are the inverse images by ϱ of the conic Lagrangian submanifolds of T^*Z [2, Thm. 3.6.2].

So, a Lagrangian conic submanifold of Σ whose intersection with Λ_0 is exactly a bicharacteristic S is the inverse image of a Lagrangian conic submanifold of T^*Z whose intersection with the zero section is exactly a point x_0. So it is the inverse image of the conormal $T^*_{\{x_0\}}Z$.

The Lagrangian foliation of Σ is given by the conormals $T^*_{\{x\}}X$ to the points of Z.

The result is now proved locally on Λ_0. From the proof of [2, Thm. 3.6.2], we know that the bicharacteristic leaves and the Lagrangian leaves are uniquely determined so they are globally defined in a neighborhood Ω of Λ_0. As Ω is a union of characteristic leaves, Ω itself is a maximally degenerated involutive manifold with degeneracy locus Λ_0. So we replace Σ by Ω to get the result. □

We may now apply Theorem 5.2 to these involutive manifolds:

Corollary 6.3. *Let Σ be a maximally degenerate involutive submanifold of $\dot{T}^*X$ with degeneracy locus Λ_0 and Lagrangian foliation (L_α).*

Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-module whose characteristic variety is contained in Σ and which is regular along each leaf L_α. Then $\mathcal{M}$ is regular holonomic hence regular along Λ_0.

Applied to $\mathcal{D}_X$-modules, Corollary 6.3 gives the following result:

Corollary 6.4. *Let Σ be a maximally degenerate involutive submanifold of $\dot{T}^*X$ with degeneracy locus Λ_0 and Lagrangian foliation (L_α).*

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. We assume that the characteristic variety of $\mathcal{M}$ is contained in Σ in a conic neighborhood Ω of Λ_0. If $\mathcal{M}$ is regular along each leaf L_α in Ω, then $\mathcal{M}$ is regular along Λ_0.

Note that in this corollary, the $\mathcal{D}_X$-module $\mathcal{M}$ is microlocally regular in a neighborhood of $T^*_Y X \cap \dot{T}^*X$ but may not be regular as a $\mathcal{D}_X$-module.

These corollaries may be applied in various situations according to the choice of the manifold Σ.

Example 6.5. The first example was studied in the proof of Lemma 6.2. Let Z be a submanifold of X and $\Sigma = \pi^{-1}(Z)$ where $\pi : \dot{T}^*X \to X$ is the canonical projection. Then the degeneracy locus of Σ is $T^*_Z X$, the conormal bundle to Z. The Lagrangian foliation is given by the conormal bundles to the points of Z.

Example 6.6. Let $X = \mathbb{C}^n$ and $\Sigma = \{(x, \xi) \in T^*X \mid \langle x, \xi \rangle = 0,\ \xi \neq 0\}$. Then the degeneracy locus of Σ is the conormal bundle to the origin of $\mathbb{C}^n$ and the Lagrangian foliation is given by the conormal bundles to the hyperplanes of X which contain the origin.

Example 6.7. More generally, we may consider a linear subvariety Z of X and Σ the union of the conormal bundles to the hyperplanes which contain Z. The Lagrangian foliation is given by these conormals and the degeneracy locus is the conormal bundle to Z.

Example 6.8. A hypersurface S of X with normal crossings is a hypersurface which may be locally defined by an equation $x_1 x_2 \dots x_p = 0$ in coordinates $(x_1, \dots, x_n)$. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module whose characteristic variety is contained in the union of all the conormal bundles to the irreducible components of S and to the intersections of these components.

Let Y be an irreducible component of S (e.g. $\{x_1 = 0\}$). Then if $\mathcal{M}$ is regular along each point of Y it is regular along Y. Indeed, any irreducible component of the characteristic variety of $\mathcal{M}$ is contained in $\pi^{-1}Y$ or does not meet the conormal $T^*_Y X$.

§7. A counterexample

In this section, we give an example showing that the fact that the characteristic variety is contained in the maximally degenerated involutive manifold is a necessary condition in Corollary 6.3. In particular, if the singular support of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ (which is an analytic subset of X) has an irreducible component tangent to the manifold Y, then $\mathcal{M}$ may be regular along each point of Y but not regular along Y.

Let $X = \mathbb{C}^2$ with coordinates (x, t); we denote by $D_x = \frac{\partial}{\partial x}$ and $D_t = \frac{\partial}{\partial t}$ the corresponding derivatives. We consider two differential operators $P = x^2 D_x + 1$ and $Q = t$. Let $\mathcal{I}$ be the ideal of $\mathcal{D}_X$ generated by P and Q and $\mathcal{M}$ be the holonomic $\mathcal{D}_X$-module $\mathcal{D}_X/\mathcal{I}$.

We denote by Y_0 the hypersurface of equation $\{t = 0\}$ and by Y_φ the hypersurface of equation $\{t - \varphi(x) = 0\}$ where φ is a holomorphic function of one variable defined in a neighborhood of 0 in $\mathbb{C}$.

Proposition 7.1.

(1) The module $\mathcal{M}$ is regular along Y_0.

(2) If $\varphi(0) = 0$ and $\varphi(x) \neq 0$ when $x \neq 0$, the module $\mathcal{M}$ is not regular along Y_φ.

(3) The module $\mathcal{M}$ is not regular along $\{0\}$. In fact, it is not regular along $T^*_{\{0\}}X$ at any point of $T^*_{\{0\}}X$.

Proof. As the $\mathcal{D}_X$-module is supported by Y_0 it is trivially regular along Y_0.

To prove that $\mathcal{M}$ is not regular along Y_φ, we will prove that

$$\mathcal{E}\mathrm{xt}^1_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}^\infty_{Y_\varphi|X}/\mathcal{B}_{Y_\varphi|X})_0 \neq 0.$$

Let $j : X \setminus Y_\varphi \hookrightarrow X$ be the canonical injection. Let $j_* j^{-1}\mathcal{O}_X$ be the sheaf of holomorphic functions with singularities on Y_φ and $\mathcal{O}_X[*Y_\varphi]$ be the sheaf of meromorphic functions with poles on Y_φ. By definition $\mathcal{B}^\infty_{Y_\varphi|X} = j_* j^{-1}\mathcal{O}_X/\mathcal{O}_X$ and $\mathcal{B}_{Y_\varphi|X} = \mathcal{O}_X[*Y_\varphi]/\mathcal{O}_X$; hence we have to calculate $\mathcal{E}\mathrm{xt}^1$ with values in $\mathcal{F} = j_* j^{-1}\mathcal{O}_X/\mathcal{O}_X[*Y_\varphi]$. As P and Q commute, the module $\mathcal{M}$ admits a free resolution

$$0 \longrightarrow \mathcal{D}_X \xrightarrow{\binom{P}{Q}} (\mathcal{D}_X)^2 \xrightarrow{(P,Q)} \mathcal{D}_X \longrightarrow \mathcal{M} \longrightarrow 0.$$

Hence $\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M},\mathcal{F})$ vanishes if and only if the system

$$\begin{cases} Pf &= g, \\ Qf &= h, \end{cases} \qquad \text{with } Ph = Qg, \tag{7.1}$$

has a solution f in $\mathcal{F}$ for any data (g,h) in $\mathcal{F}\times\mathcal{F}$.

Lemma 7.2. *There exists a function $h(x,t)$ in $j_*j^{-1}\mathcal{O}_X$ such that $h(x,0) = \exp(1/x)$.*

Proof. There is some integer $n > 0$ and a function ψ such that $\varphi(x) = x^n\psi(x)$ and $\psi(0) \neq 0$. Let $\lambda_0(x) = 1$ and for $j \geq 1$, let $\lambda_j(x) = \sum_{k=0}^{n-1} \frac{x^k}{(nj-k)!}$. We have $\lambda_0(x) + \sum_{j\geq 1} \lambda_j(x)\frac{1}{x^{nj}} = \exp(1/x)$. The function

$$h(x,t) = \sum_{j\geq 0} \psi(x)^j \lambda_j(x) \frac{1}{(\varphi(x)-t)^j}$$

is a solution to the lemma. □

Proof of proposition continued: Let $h(x,t)$ be the function given by Lemma 7.2. We have

$$(P(x,D_x)h(x,t))\,|_{t=0} = P(x,D_x)h(x,0) = P(x,D_x)\exp(1/x) = 0;$$

hence there exists some function g in $j_*j^{-1}\mathcal{O}_X$ such that $Qg(x,t)= P(x,D_x)h(x,t)$.

As $h(x,0)$ is a function on $\mathbb{C}\setminus\{0\}$ which is not meromorphic, the equation $tf = h$ has no solution in $\mathcal{F} = j_*j^{-1}\mathcal{O}_X/\mathcal{O}_X[*Y_\varphi]$. Hence equation (7.1) has no solution and $\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M},\mathcal{F})$ does not vanish. This shows point (2) of the proposition.

To prove point (3) we consider the function $f(x,t) = (1/t)\exp(1/x)$ which defines the element $u = \sum_{j\geq 0} \frac{(-1)^j}{j!(j+1)!}\delta^{(j)}(x)\delta(t)$ in $\mathcal{B}^\infty_{\{0\}|X}$. As u is a solution of $Pu = Qu = 0$ which does not belong to $\mathcal{B}_{\{0\}|X}$, the module $\mathcal{M}$ is not regular on $\{0\}$.

Moreover, we may consider the microfunction v in $\mathcal{C}^\infty_{\{0\}|X}$, the symbol of which is $\sum_{j\geq 0} \frac{(-1)^j}{j!(j+1)!}\xi^j$, that is the microfunction associated to u. It is a solution of $Pv = Qv = 0$ which does not belong to $\mathcal{C}_{\{0\}|X}$ at any point of $T^*_{\{0\}}X$. This shows point (3) of the proposition. □

Let $Z = \mathbb{C}^2$ with coordinates (y,s). The coordinates will be (x,t,ξ,τ) on T^*X and (y,s,η,σ) on T^*Z. The Legendre transform is defined from T^*X to T^*Z when $\tau \neq 0$ and $\sigma \neq 0$ by the equations

$$y = \xi\tau^{-1}, \quad s = t + x\xi\tau^{-1}, \quad \eta = -x\tau, \quad \sigma = \tau.$$

According to [13, §3.3, Chap. II], a quantized canonical transform associated to it is given by

$$\begin{cases} x = -D_y D_s^{-1}, \\ t = (D_s s + D_y y) D_s^{-1} = s + y D_y D_s^{-1} + 2 D_s^{-1}, \\ D_x = y D_s, \\ D_t = D_s. \end{cases}$$

So, if we apply this transformation to the $\mathcal{E}_X$-module $\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$ with $\pi : T^*X \to X$, we get the coherent $\mathcal{E}_Z$-module $\widetilde{\mathcal{N}} = \mathcal{E}_Z/(\mathcal{E}_Z \widetilde{P} + \mathcal{E}_Z \widetilde{Q})$ with

$$\begin{cases} \widetilde{P}(y, s, D_y, D_s) = D_y^2 D_s^{-2} y D_s + 1 = y D_y^2 D_s^{-1} + 2 D_y D_s^{-1} + 1, \\ \widetilde{Q}(y, s, D_y, D_s) = s + y D_y D_s^{-1} + 2 D_s^{-1}. \end{cases}$$

For $(\alpha, \lambda) \in \mathbb{C}^* \times \mathbb{C}^*$, let us consider the following Lagrangian manifolds:

$$\begin{aligned}
&\Lambda_0 = T^*_{Y_0} X = \{(x, t, \xi, \tau) \in T^*X \mid t = 0,\ \xi = 0\}, \\
&\Lambda_\alpha = T^*_{Y_\varphi} X = \{(x, t, \xi, \tau) \in T^*X \mid t = -\alpha x,\ \xi = \alpha\tau\} \qquad \text{with } \varphi(x) = -\alpha x, \\
&\Lambda'_\lambda = T^*_{Y_\varphi} X = \{(x, t, \xi, \tau) \in T^*X \mid t = \tfrac{1}{4}\lambda x^2,\ \xi = -\tfrac{1}{2}\lambda x \tau\} \qquad \text{with } \varphi(x) = \tfrac{\lambda x^2}{4}, \\
&\Lambda'_0 = T^*_{\{0\}} X = \{(x, t, \xi, \tau) \in T^*X \mid t = 0,\ x = 0\},
\end{aligned}$$

$$\begin{aligned}
&\widetilde{\Lambda}_0 = T^*_{\{0\}} Z = \{(s, y, \eta, \sigma) \in T^*Z \mid s = 0,\ y = 0\}, \\
&\widetilde{\Lambda}_\alpha = T^*_{\{p_\alpha\}} Z = \{(s, y, \eta, \sigma) \in T^*Z \mid s = 0,\ y = \alpha\}, \\
&\widetilde{\Lambda}'_\lambda = T^*_{Y_\varphi} Z = \{(s, y, \eta, \sigma) \in T^*Y \mid s + y^2/\lambda = 0,\ \eta = 2(y/\lambda)\sigma\} \text{ with } \varphi(y) = \tfrac{-y^2}{\lambda}, \\
&\widetilde{\Lambda}'_0 = T^*_{Y_0} Z = \{(s, y, \eta, \sigma) \in T^*Z \mid s = 0,\ \eta = 0\}
\end{aligned}$$

By the Legendre transform, the manifolds Λ_0, Λ_α, Λ'_λ and Λ'_0 are respectively transformed into $\widetilde{\Lambda}_0$, $\widetilde{\Lambda}_\alpha$, $\widetilde{\Lambda}'_\lambda$ and $\widetilde{\Lambda}'_0$ outside of the set $\{\tau = 0\}$. As the regularity along a Lagrangian manifold is invariant under quantized canonical transformation, we get from Proposition 7.1 that $\widetilde{\mathcal{N}}$ is a holonomic $\mathcal{E}_Z$-module which is regular along $\widetilde{\Lambda}_0$ but not regular along $\widetilde{\Lambda}_\alpha$, $\widetilde{\Lambda}'_\lambda$ and $\widetilde{\Lambda}'_0$.

Again let be $Z = \mathbb{C}^2$ with coordinates (y, s) and let us define the following differential operators:

$$\begin{cases} P_1(y, s, D_y, D_s) = y D_y^2 + 2 D_y + D_s, \\ Q_1(y, s, D_y, D_s) = s D_s + y D_y + 3, \end{cases}$$

Let $\mathcal{J}$ be the ideal of $\mathcal{D}_Z$ generated by P_1 and Q_1 and $\mathcal{N}$ be the holonomic $\mathcal{D}_Z$-module $\mathcal{D}_Z/\mathcal{J}$.

Proposition 7.3. *The module $\mathcal{N}$ is regular along all points of $Y = \{(y,s) \mid s = y^2\}$ but is not regular along Y itself.*

Remark 7.4. We have also that $\mathcal{N}$ is irregular along each point of $Y_0 = \{(y,s) \mid s = 0\}$ except 0 and is irregular along Y_0.

Proof. The operator Q_1 is a b-function for $\mathcal{N}$ at 0 hence $\mathcal{N}$ is regular along this point (locally but also microlocally at each point of T_0^*Z). Moreover, the characteristic variety of $\mathcal{N}$ is contained in $\{(s,y,\eta,\sigma) \in T^*Z \mid s\sigma = 0,\ y\eta = 0\}$ hence $\mathcal{N}$ is elliptic at each point of Y except 0, so it is regular along these points.

The module $\mathcal{E}_Z \otimes_{\pi^{-1}\mathcal{D}_Z} \pi^{-1}\mathcal{N}$ is by definition isomorphic to $\widetilde{\mathcal{N}}$ outside of the set $\{\tau = 0\}$. Hence by Proposition 7.1, $\mathcal{N}$ is not regular along Y as well as Y_0 and its nonzero points. □

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