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Erratum to Hodge Theory of the Middle Convolution

by

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Abstract

We give a correction to the statement of Theorem 3.2.3 of [2].

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Theorem 3.2.3 of [2] is incorrectly stated. The correct statement is as follows. Given $\lambda \in S^1$, we set $\lambda = \exp(-2\pi \mathrm{i}\alpha')$ with $\alpha' \in (0,1]$ (not $[0,1)$). With this in mind, we have the following theorem.

Theorem 3.2.3 ([10, Thm. 5.4]).

$$\mathrm{gr}_F^p \phi_{s,\lambda}(M_1 \boxtimes M_2) = \bigoplus_{\substack{(\lambda_1,\lambda_2)\\ \lambda_1\lambda_2=\lambda}} \begin{cases} \bigoplus_{j+k=p-1} \mathrm{gr}_F^j \phi_{t_1,\lambda_1} M_1 \otimes \mathrm{gr}_F^k \phi_{t_2,\lambda_2} M_2 & \textit{if } \alpha'_1+\alpha'_2 \in (0,1], \\ \bigoplus_{j+k=p} \mathrm{gr}_F^j \phi_{t_1,\lambda_1} M_1 \otimes \mathrm{gr}_F^k \phi_{t_2,\lambda_2} M_2 & \textit{if } \alpha'_1+\alpha'_2 \in (1,2]. \end{cases}$$

The statement of Theorem 3.1.2 is unchanged. Note that 3.1.2(2) would be more symmetric by setting $\lambda = \exp(-2\pi i\alpha')$ with $\alpha' \in (0,1]$:

3.1.2(2)′
$$\mu^p_{x_i,\lambda,\ell}(\mathrm{MC}_\chi(M)) = \begin{cases} \mu^{p-1}_{x_i,\lambda/\lambda_o,\ell}(M) & \text{if } \alpha' \in (\alpha_o, 1], \\ \mu^{p}_{x_i,\lambda/\lambda_o,\ell}(M) & \text{if } \alpha' \in (0, \alpha_o]. \end{cases}$$

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We make clear below the side-changing relations to relate our setting to that of [10]. Assume $(M, F^\bullet M)$ is a polarizable complex Hodge module on the disc Δ as defined in [2, §3.2], and that M is a minimal extension at the origin. Let $V^\bullet M$ be its V-filtration (cf. the notation in [2, §2.2]).

Since Δ has a global coordinate, we can identify the associated right $\mathscr{D}_\Delta$-module with M on which $\mathscr{D}_\Delta$ acts in a transposed way. We denote it by M^r. The V-filtration and the F-filtration are now denoted increasingly. We have the following relations:

$$F_p M^r = F^{-p-1} M, \quad V_\gamma M^r = V^{-\gamma-1} M.$$

By the definition in [9], we have, for $\lambda \in S^1$ and $\lambda = \exp(2\pi i\gamma)$ with $\gamma \in [-1, 0)$,

$$(*) \qquad \begin{aligned} F_p\psi_\lambda M^r &:= F_{p-1}\operatorname{gr}^V_\gamma M^r = F^{-p}\operatorname{gr}^\beta_V M \quad (\beta = -\gamma - 1),\\ F_p\phi_1 M^r &:= F_p \operatorname{gr}^V_0 M^r = F^{-p-1}\operatorname{gr}^{-1}_V M. \end{aligned}$$

Due to our previous definition of $F^q\psi_\lambda M$ and $F^q\phi_1 M$ (given before Theorem 2.2.4 and Proposition 2.2.5), we find that

$$F_p\psi_\lambda M^r = F^{-p}\psi_\lambda M, \quad F_p\phi_1 M^r = F^{-p}\phi_1 M.$$

Lastly, the theorem of Saito (for filtered right $\mathscr{D}_\Delta$-modules) gives, setting $\lambda = \exp(-2\pi i\beta)$ with $\beta \in (-1, 0]$ (since we are now interested in vanishing cycles),

$$\operatorname{gr}^F_p \phi_{s,\lambda}(M_1^r \boxtimes M_2^r) = \bigoplus_{\substack{(\lambda_1,\lambda_2)\\ \lambda_1\lambda_2=\lambda}} \begin{cases} \displaystyle\bigoplus_{j+k=p+1} \operatorname{gr}^F_j \phi_{t_1,\lambda_1} M_1^r \otimes \operatorname{gr}^F_k \phi_{t_2,\lambda_2} M_2^r & \text{if } \beta_1+\beta_2 \in (-2,-1],\\ \displaystyle\bigoplus_{j+k=p} \operatorname{gr}^F_j \phi_{t_1,\lambda_1} M_1^r \otimes \operatorname{gr}^F_k \phi_{t_2,\lambda_2} M_2^r & \text{if } \beta_1+\beta_2 \in (-1,0]. \end{cases}$$

We now replace β, β_1, β_2 by α', α'_1, $\alpha'_2 \in (0, 1]$ (by adding 1 to each number). The previous formula is immediately translated to the above statement by replacing M^r with M and increasing F-filtrations with decreasing ones.

In the setting of Theorem 3.1.2(2), we have $\alpha'_2 = \alpha_o \in (0, 1)$, and $\operatorname{gr}^k_F \phi_{t_2,\lambda_o} M_o = 0$ unless $k = 0$. For $\alpha', \alpha'_1 \in (0, 1]$, we have

$$\alpha' = \alpha'_1 + \alpha_o \iff \alpha' \in (0, 1] \cap (\alpha_o, \alpha_o + 1] = (\alpha_o, 1].$$

If $\alpha'_1 + \alpha_o \in (1, 2]$, we must set $\alpha' = \alpha'_1 + \alpha_o - 1$, and similarly $\alpha' \in (0, \alpha_o]$. We thus find the above expression for $\mu^p_{x_i,\lambda,\ell}(\mathrm{MC}_\chi(M))$ depending on the position of α'. Going back to $\alpha \in [0, 1)$, the condition becomes as stated in [2].

Remark 1 (Suggested by the referee). The formula of Theorem 3.2.3 is essentially the same as that given in [11]. The referee emphasizes that the results of

[10], [11] involve $\mathbb{Q}$-mixed Hodge modules, while Theorem 3.2.3 concerns polarizable complex Hodge modules as defined in [2, §3.2]. Fortunately, the last version of [4] proves a Thom–Sebastiani-type theorem for filtered $\mathscr{D}$-modules in a sufficiently general case including our case, where the V-filtration is indexed by $\mathbb{R}$.

In [2, §2], we have used the (still unpublished) results of Schmid in the context of polarizable variations of real or complex Hodge structures of some weight, according to [13] (cf. also [1, §1.11]) in order to ensure that, by taking their intermediate extensions, we obtain a polarizable complex Hodge module as defined in [2, §3.2]. Recall that another proof is given in [7, §3.a–3.g] relying on the theory of tame harmonic bundles on curves [12].

Remark 2. Since we are interested only in proving Theorem 3.1.2 of [2], we will indicate precisely a direct proof of 3.1.2(2)$'$ via twistor D-modules, avoiding Thom–Sebastiani in its local form, and using instead the stationary phase formula proved in [8, (A.11) & (A.12)].

To a filtered $\mathbb{C}[t]\langle\partial_t\rangle$-module $(M, F^\bullet M)$ we associate the Rees module $R_F M := \bigoplus_p F^p M z^{-p}$, where z is a new variable. It is endowed with the action of $z^2\partial_z$ such that, for $m \in F^p M$, we have $z^2\partial_z(mz^{-p}) = -pmz^{-(p-1)}$. To a variation of polarized complex Hodge structure $(V, \nabla, F^\bullet V)$ of weight 0 on $\mathbb{A}^1 \smallsetminus \boldsymbol{x}$ is associated a polarized pure twistor $\mathscr{D}_{\mathbb{P}^1}$-module $\mathscr{T}$ of weight 0 whose restriction to $\mathbb{A}^1 \smallsetminus \boldsymbol{x}$ is $(R_F V, R_F V, R_F k)$, where the sesquilinear $R_F k$ is obtained by the Rees procedure from the flat sesquilinear pairing k inducing the polarization (cf. [7, §3]). Then $\mathscr{T}$ is also endowed with a compatible action of $z^2\partial_z$: one says that it is integrable.

Note that in [7, §3] the construction of $(\mathscr{T}, z^2\partial_z)$ uses the $\mathbb{R}$-variant of Schmid's results. In order to avoid this, we can use the property that the Hodge metric is a tame harmonic metric and then use the extension property of [12] (cf. also [6, Thm. 5.0.1], [5, Thm. 1.22], both in the simpler case of integrable objects).

The formulas (A.11) and (A.12) of [8] need to be modified in order to take care of the shift by 1 in the definition $(*)$ of $F^p\phi_1 M$, and of the shift of the filtration by the push-forward by a closed immersion, as explained in [3, (1.2.4)]. Here, the codimension-1 inclusion i_0 used in Lemma A.10 of [8] produces a shift by 1 in the formulas. With this slight change of convention, compatible with that of [9], (A.11) and (A.12) of [8] read, at $x_i = 0$ and with an adaptation of the notation,

$$(\mathrm{P}_\ell \phi_{t,\lambda}\mathscr{T}, z^2\partial_z) \simeq (\mathrm{P}_\ell \psi_{\tau',\lambda}{}^F\mathscr{T}, z^2\partial_z - \beta z)$$
$$\text{if } \lambda = \exp(-2\pi i\beta) \text{ and } \beta \in (-1, 0].$$

For $\chi = \lambda_o$, the meromorphic flat bundle L_χ defines a polarized pure twistor $\mathscr{D}$-module $\mathscr{T}_\chi$ of weight 0. We then have, setting $\beta_o = \alpha_o - 1 \in (-1, 0)$,

$$\begin{aligned}
&(\mathrm{P}_\ell\phi_{t,\lambda}(\mathrm{MC}_\chi\,\mathscr{T}), z^2\partial_z)\\
&\quad\simeq (\mathrm{P}_\ell\psi_{\tau',\lambda}{}^F(\mathrm{MC}_\chi\,\mathscr{T}), z^2\partial_z - \beta z) \quad (\beta\in(-1,0])\\
&\quad\simeq (\mathrm{P}_\ell\psi_{\tau',\lambda}({}^F\mathscr{T}\otimes{}^F\mathscr{T}_\chi), z^2\partial_z - \beta z)\\
&\quad\simeq (\mathrm{P}_\ell\psi_{\tau',\lambda/\lambda_o}({}^F\mathscr{T}), z^2\partial_z - (\beta-\beta_o)z)\otimes(\psi_{\tau',\lambda_o}{}^F\mathscr{T}_\chi, z^2\partial_z - \beta_o z)\\
&\quad\simeq (\mathrm{P}_\ell\phi_{t,\lambda/\lambda_o}\mathscr{T}, z^2\partial_z(-z))\otimes(\phi_{t,\lambda_o}\mathscr{T}_\chi, z^2\partial_z)\\
&\quad\simeq (\mathrm{P}_\ell\phi_{t,\lambda/\lambda_o}\mathscr{T}, z^2\partial_z(-z)),
\end{aligned}$$

where $(-z)$ means that we add $-z$ if $\beta\in(\beta_o, 0]$, that is, going back to the notation α', if $\alpha'\in(\alpha_o, 1]$. The $\mathbb{C}[z]$-module part of each side is $R_F\mathrm{P}_\ell\phi_{t,\lambda}(\mathrm{MC}_\chi\,M)$ (resp. $R_F\mathrm{P}_\ell\phi_{t,\lambda/\lambda_o}M$) and we recover $F^p\mathrm{P}_\ell\phi_{t,\lambda}(\mathrm{MC}_\chi\,M)$ (resp. $F^p\mathrm{P}_\ell\phi_{t,\lambda/\lambda_o}M$) by considering $\mathrm{Ker}(z^2\partial_z + pz)$. In such a way we obtain 3.1.2(2)$'$ at $x_i = 0$.

A similar formula applies at every singularity x_i of M after a twist by $e^{x_i/\tau' z}$ and gives 3.1.2(2)$'$ at any x_i.

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