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On Enriques Surfaces with Four Cusps

by

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Abstract

We study Enriques surfaces with four disjoint A_2-configurations. In particular, we construct open Enriques surfaces with fundamental groups $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$, completing the picture of the A_2-case from Keum and Zhang (*Fundamental groups of open K3 surfaces, Enriques surfaces and Fano 3-folds*, J. Pure Appl. Algebra **170** (2002), 67–91; Zbl 1060.14057). We also construct an explicit Gorenstein $\mathbb{Q}$-homology projective plane of singularity type $A_3 + 3A_2$, supporting an open case from Hwang, Keum and Ohashi (*Gorenstein $\mathbb{Q}$-homology projective planes*, Science China Mathematics **58** (2015), 501–512; Zbl 1314.14072).

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§1. Introduction

The main aim of this article is to study Enriques surfaces with four disjoint A_2-configurations, the maximum number possible (because an Enriques surface has Picard number 10). We shall make heavy use of elliptic fibrations to study the moduli of such Enriques surfaces:

Theorem 1.1. *Enriques surfaces with four disjoint A_2-configurations come in exactly two irreducible two-dimensional families $\mathcal{F}_{3,3,3,3}$, $\mathcal{F}_{4,3,1}$.*

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This result, which relies on the understanding of Picard–Lefschetz reflections on the Enriques surface and its K3-cover following [10], enables us to determine the fundamental groups of the open Enriques surfaces obtained by removing the A_2-configurations (often also referred to as the cusps).

Our paper draws on the classification of possible fundamental groups of open Enriques surfaces (i.e., complements of configurations of smooth rational curves) initiated in [10]. Keum and Zhang state a list of 26 possible groups and give 24 examples. Here we supplement and correct their results by adding one example and one group supported by another example. Our second main result is as follows.

Theorem 1.2. *Let* $G \in \{S_3 \times \mathbb{Z}/3\mathbb{Z}, (\mathbb{Z}/3\mathbb{Z})^{\oplus 2} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}\}$. *Then there is a complex Enriques surface* S *with a set* $\mathcal{A}$ *of four disjoint* A_2*-configurations such that*

$$\pi_1(S \setminus \mathcal{A}) \cong G.$$

For a more concise statement, the reader is referred to Theorem 4.3. This completes the picture for the A_2-case.

Another key point of our paper is the clarification that there are indeed Enriques surfaces admitting different sets of four disjoint A_2-configurations which lead to each alternative of the fundamental group in Theorem 1.2. This issue will be discussed in detail in Section 5 and also supported by an explicit example; see Section 5.6.

While some of the constructions involved in our methods are analytic in nature, notably the notion of logarithmic transformations of elliptic surfaces, we will crucially facilitate Enriques involutions of base change type, as studied systematically in [6], since this algebro-geometric technique grants us good control of the curves on the surfaces and their moduli. We review this, among all the prerequisites and basics necessary for the understanding of this paper, in Section 2. Section 3 introduces the two families $\mathcal{F}_{3,3,3,3}$ and $\mathcal{F}_{4,3,1}$ and proves Theorem 1.1. The proof of Theorem 1.2 is given in Section 4. As a by-product, our examples produce an explicit Gorenstein $\mathbb{Q}$-homology projective plane of singularity type $A_3 + 3A_2$; by Proposition 3.14 we settle an open case from [7]. The paper concludes with further considerations concerning the moduli of Enriques surfaces with four disjoint A_2-configurations.

Convention. In this note the base field is always $\mathbb{C}$. Root lattices A_n, D_k, E_l are taken to be negative definite.

§2. Preliminaries and basics

§2.1. A_2-configurations

Let S be an Enriques surface that contains four disjoint A_2-configurations, i.e., eight smooth rational curves $F_1', F_1'', \dots, F_4', F_4''$ such that

$$F_j'.F_j'' = 1 \quad \text{for } j = 1, \dots, 4,$$

and the rational curves in question are mutually disjoint otherwise. We say that a collection of disjoint A_2-configurations $F_1', F_1'', \dots, F_l', F_l''$ is 3-*divisible* if one can label the rational curves in each A_2-configuration such that the divisor

$$\sum_{j=1}^{l}(F_j' - F_j'') \tag{2.1}$$

is divisible by 3 in $\mathrm{Pic}(S)$. Equivalently, since

$$\mathrm{Pic}(S) = \mathrm{H}^2(S, \mathbb{Z}) = \mathbb{Z}^{10} \oplus \mathbb{Z}/2\mathbb{Z},$$

the class given by (2.1) is 3-divisible in $\mathrm{Num}(S)$, the lattice given by divisors up to numerical equivalence. Recall that $\mathrm{Num}(S)$ is a unimodular even hyperbolic lattice; in fact

$$\mathrm{Num}(S) = U + E_8$$

where U denotes the unimodular hyperbolic plane. The following result (which is similar to [1, Lem. 1] and [3, Lem. 1.1]) will be instrumental for some of our investigations.

Lemma 2.1. *Let* $\sum_{j=1}^{l}(F_j' - F_j'') = 3D$ *in* $\mathrm{Num}(S)$. *Then* $l = 3$, *but* D *is neither effective nor anti-effective.*

Proof. The fact $l = 3$ is easily derived using $(F_j' - F_j'')^2 = -6$ and the integrality and rank of the lattice $\mathrm{Num}(S)$. Assume that D is effective. Then since $F_j'.D = -1$ for each $j = 1, 2, 3$, each of these curves F_j' is contained in the support of D. Hence $D' = D - (F_1' + F_2' + F_3')$ is still effective. On the other hand, we obviously have $3D' \leq 0$, hence $3D \sim 0$, but this is not compatible with $D^2 = (D')^2 = -2$, a contradiction.

If $D \leq 0$, then an analogous argument applies to the F_j'', thus completing the proof of the lemma. □

We follow the approach of [10, Sect. 3] and let M (resp. $\overline{M}$) denote the lattice spanned by $F_1', \dots, F_4''$ in $\mathrm{Num}(S)$ (resp. its primitive closure).

Lemma 2.2. *The index of* M *inside* $\overline{M}$ *satisfies* $[\overline{M} : M] \in \{3, 3^2\}$.

Proof. The lattice M has discriminant $d(M) = 3^4$, so $[\overline{M} : M] \in \{1, 3, 3^2\}$. We claim that the first case is impossible. Indeed, suppose that $M = \overline{M}$. Then $M \hookrightarrow \mathrm{Num}(S)$ is a primitive embedding, so

$$M^\vee/M \cong (M^\perp)^\vee/M^\perp.$$

By assumption the left-hand side is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^4$ while the right-hand side comes from the rank-2 lattice $M^\perp$, and thus has length at most 2, a contradiction. □

Corollary 2.3. *The four A_2-configurations $F_1', \dots, F_4''$ contain either one or four 3-divisible sets.*

In particular, we can infer that

(2.2) $F_1', \dots, F_4''$ contain four 3-divisible sets if and only if $\overline{M}$ is unimodular.

In other words, in this case each triplet of the A_2-configurations in question is 3-divisible up to relabeling the rational curves.

§2.2. Elliptic fibrations

We start by recalling some basic concepts and relations. Any complex Enriques surface S admits an elliptic fibration

$$\varphi: \quad S \to \mathbb{P}^1. \tag{2.3}$$

There are two fibers of multiplicity 2; their supports are usually called half-pencils. The difference of the two half-pencils gives the canonical divisor which represents the 2-torsion in $\mathrm{H}^2(S, \mathbb{Z})$. This already shows that the fibration cannot have a section, but by [2, Prop. VIII.17.6] there always is a bisection R of square $R^2 = 0$ or -2, i.e., an irreducible curve R such that $R.F = 2$ for any fiber F of (2.3).

The moduli of Enriques surfaces can be studied through the universal cover

$$\pi : Y \to S \tag{2.4}$$

which is a K3 surface. By construction, this induces an elliptic fibration

$$\tilde{\varphi}: \quad Y \to \mathbb{P}^1, \tag{2.5}$$

which fits into the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{2:1} & S \\ {\scriptstyle\tilde{\varphi}}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathbb{P}^1 & \xrightarrow{2:1} & \mathbb{P}^1. \end{array} \tag{2.6}$$

The bottom row degree-2 morphism

$$\mathbb{P}^1 \xrightarrow{2:1} \mathbb{P}^1 \tag{2.7}$$

ramifies exactly in the points below the multiple fibers. Moreover, the universal covering induces a primitive embedding

$$U(2) + E_8(2) \cong \pi^* \operatorname{Num}(S) \hookrightarrow \operatorname{Pic}(Y),$$

which lends itself to a study of K3 surfaces with the above lattice polarization. Abstractly, a complex K3 surface Y admits an Enriques involution if and only if there is a primitive embedding of $U(2) + E_8(2)$ into $\operatorname{Pic}(Y)$ without perpendicular roots (i.e., classes of smooth rational curves) by [8]. In view of this, it is evident that a bisection R of square $R^2 = 0$ occurs generically, since on the contrary any (-2)-curve on S necessarily splits into two disjoint smooth rational curves on the K3 cover Y; these give sections of (2.5), causing the Picard number to go up to 11 at least. The same generic behavior will occur on our families $\mathcal{F}_{3,3,3,3}$, $\mathcal{F}_{4,3,1}$ in Section 3.

On the other hand, we can consider the Jacobian fibration of (2.3). This will be a rational elliptic surface

$$X \to \mathbb{P}^1 \tag{2.8}$$

with section and is thus governed by means of explicit classifications, e.g., using the theory of Mordell–Weil lattices in [16]. Naturally S and X share the same singular fibers, except that on S, smooth or semi-stable fibers (Kodaira type I_n, $n \geq 0$) may come with multiplicity 2. The Enriques surface S can be recovered from X through a logarithmic transformation which depends on the choice of non-trivial 2-torsion points in two distinct smooth or semi-stable fibers of (2.8) (see, e.g., [4, Sect. 1.6]). Intrinsically this leads to another K3 surface in terms of the Jacobian elliptic fibration arising from (2.8) through the quadratic base change (2.7) ramified in the two distinct fibers where the logarithmic transformation changed the multiplicities of fibers. It is clear from the construction that at the same time this K3 surface features as the Jacobian of (2.5). That is, we get another commutative diagram

$$\begin{array}{ccc} \operatorname{Jac}(Y) & \xrightarrow{2:1} & X \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{2:1} & \mathbb{P}^1. \end{array} \tag{2.9}$$

Recall that the depicted elliptic fibrations on Y and $\operatorname{Jac}(Y)$ share the same configurations of singular fibers and the same Picard numbers.

For some purposes, the above construction has the drawback of being analytical in nature. This can be circumvented in the special situation where the elliptic fibration (2.5) is already Jacobian, i.e., admits a section. For instance, this occurs in the presence of a bisection R of (2.3) with square $R^2 = -2$ as indicated above. A more general framework for this to occur was introduced in terms of involutions of base change type in [6]. Here one considers the quadratic twist X' of X which acquires I_n^* fibers $(n \geq 0)$ at the two ramification points of (2.7). In consequence, the quadratic base change (2.7) applied to either X or X' gives the same K3 surface Y.

For any section on X', the pull-back to Y is anti-invariant with respect to the involution $\imath$ on Y induced by the deck transformation of (2.7) (s.t. $Y/\imath = X$). It follows that $\imath$ composed with translation by the anti-invariant section defines another non-symplectic involution on Y. This has fixed points, necessarily in the ramified fibers, if and only if the section meets the identity components of the two twisted fibers on X'. Otherwise, for instance if the section is 2-torsion, we obtain an Enriques involution on Y which we will refer to as an involution of base change type.

§2.3. Picard–Lefschetz reflections

Recall that by Kodaira's work [11], the irreducible components of a singular fiber of an elliptic fibration correspond to an extended Dynkin diagram; a Dynkin diagram, or equivalently root lattice, can be obtained from the singular fiber by omitting any simple component. Given A_2-configurations, it is thus natural to ask whether these correspond to rational curves supported on the fibers of an elliptic fibration on S. While this may not be true in general, we can weaken the limitations by considering the question up to isometries of $H^2(S, \mathbb{Z})$. This will allow us to reduce the problem of 3-divisible sets of A_2-configurations to the study of certain elliptic fibrations on Enriques surfaces. To this end, recall that each (-2)-class E in $H^2(S, \mathbb{Z})$ defines a *Picard–Lefschetz reflection*

$$s_E : H^2(S, \mathbb{Z}) \ni D \mapsto D + (D.E)E \in H^2(S, \mathbb{Z}).$$

In general, such a reflection does not act effectively on divisors, but the situation changes drastically when restricted to smooth rational curves. Namely, if E and E' are both represented by a smooth rational curve, then

$$s_E(E') \text{ is either effective (if } E \neq E'\text{), or anti-effective (if } E = E'\text{).} \tag{2.10}$$

In the sequel we will use the following corrected version of [10, Claim 3.5.1] (which included neither the configurations (2.12) nor the degenerate case of (2.13); cf. also Remark 2.10).

Lemma 2.4. *There exists a half-pencil H on S and smooth rational curves $E_1,\dots,E_k \subset S$ such that the image of each curve F_j', F_j'', where $j=1,\dots,4$, under the isometry*

$$\mathfrak{p}_S := (s_{E_k} \circ \cdots \circ s_{E_1}) \tag{2.11}$$

is, up to some multiple of H, the class of a smooth rational curve which is an irreducible component of a member of the pencil $|2H|$. Moreover, the elliptic fibration given by $|2H|$ is either of the type

$$I_3^4,\quad I_3^3 \oplus 2I_3,\quad I_3^2 \oplus (2I_3)^2 \tag{2.12}$$

or of the type

$$IV^* \oplus I_3 \oplus I_1,\quad IV^* \oplus 2I_3 \oplus I_1,\quad IV^* \oplus I_3 \oplus 2I_1,\quad IV^* \oplus 2I_3 \oplus 2I_1,\quad IV^* \oplus IV. \tag{2.13}$$

Proof. We argue with the lattices $M, M^\perp \subset \mathrm{Num}(S)$ from Section 2.1. Since M has discriminant 81 and $M^\perp$ is hyperbolic of rank 2, $M^\perp$ represents zero. Thus there is an isotropic class $H_0 \in M^\perp$ which we may assume to be primitive in $\mathrm{H}^2(S,\mathbb{Z})$. By Riemann–Roch, either H_0 or $-H_0$ is effective, so let us assume the former. Following [2, Lem. VIII.17.4], it remains to subtract the base locus of $|2H_0|$ to derive an elliptic fibration. This precisely amounts to a composition

$$\mathfrak{p}_0 = (s_{E_l} \circ \cdots \circ s_{E_1}) \tag{2.14}$$

of reflections in smooth rational curves $E_1,\dots,E_l$ (each meeting the image of H_0 under the previous reflections negatively). By construction, we obtain the half-pencil $H := \mathfrak{p}_0(H_0)$ such that $|2H|$ induces an elliptic fibration on S.

To study the impact of the isometry $\mathfrak{p}_0$ on the (-2)-classes F_j', F_j'', the following generalization of (2.10) enters crucially:

Claim 2.5. For $j=1,\dots,4$, the class $\mathfrak{p}_0(F_j')$ (and also the class $\mathfrak{p}_0(F_j''))$ is an effective or anti-effective divisor supported on components of a singular fiber of the elliptic pencil $|2H|$.

In order to simplify the exposition of the proof of the lemma, the proof of Claim 2.5 will be given, also for later use, in Section 2.4.

Continuing the proof of Lemma 2.4, we infer from Claim 2.5 that singular fibers of the elliptic pencil $|2H|$ contain four disjoint A_2-configurations (given by effective or anti-effective (-2)-divisors, but not necessarily (yet) by irreducible curves). As explained in Section 2.2, the Jacobian fibration of $|2H|$ is a rational elliptic surface X. As it shares the four disjoint A_2-divisor configurations in the fibers, X is automatically extremal by the Shioda–Tate formula [20, Cor. 6.13], i.e., X has finite Mordell–Weil group. Going through the classification in [13], one finds that X may have the following configurations:

$$I_3^4,\ IV^* \oplus I_3 \oplus I_1,\ IV^* \oplus IV,\ II^* \oplus I_1^2,\ II^* \oplus II. \tag{2.15}$$

Note that the configurations (2.12) and (2.13) from Lemma 2.4 correspond to the first three entries in (2.15), with fiber multiplicities. In order to complete the proof of Lemma 2.4, we shall now prove all claims for the first three configurations above before ruling out the last two configurations from (2.15).

Before going into the details, recall that for any irreducible root lattice R, any two roots are equivalent under reflections. Naturally this extends to the extended Dynkin diagrams $\tilde{R}$:

Tool 2.6. Any two roots in $\tilde{R}$ are equivalent under reflections.

(Note that this holds true even though $\tilde{R}$ contains infinitely many roots — but only finitely many modulo the primitive isotropic vector.)

Consider the configurations (2.12) and (2.13) from Lemma 2.4. To complete the proof, we have to show that there are reflections in fiber components of $|2H|$ such that the composition of all reflections takes each curve F_j', F_j'' to a single smooth rational curve, up to a multiple of H. To see this, fix $D = \mathfrak{p}_0(F_j')$ or $\mathfrak{p}_0(F_j'')$ for some $j = 1, \dots, 4$. We claim that there is an integer $n \in \mathbb{Z}$ and a divisor $\tilde{D}$ such that

$$D = nH + \tilde{D} \quad \text{and} \quad 0 < \tilde{D} < 2H. \tag{2.16}$$

This can be seen without difficulty because fiber components generate a semi-negative-definite lattice. Indeed, if there were some $n \in \mathbb{Z}$ such that

$$D = nH + \tilde{D} - \hat{D}$$

with $\tilde{D}$, $\hat{D}$ both effective and supported on distinct fiber components, then by construction,

$$-2 = D^2 = \tilde{D}^2 - 2\tilde{D}.\hat{D} + \hat{D}^2.$$

Since all entries on the right-hand side are non-positive, even integers, we deduce that either $\tilde{D}$ or $\hat{D}$ has square zero, and hence equals some fiber multiple. Upon subtracting or adding the fiber class $2H$, we thus obtain the representation (2.16) of D. Note that in particular $\tilde{D}$ is supported on a single fiber (and naturally $\tilde{D}$ and n can be chosen such that $0 < \tilde{D} < H$ if $\tilde{D}$ is supported on a multiple fiber), so we can now complete the proof of Lemma 2.4 fiber by fiber. In particular, we only have to distinguish two cases.

If the fiber has Kodaira type I_3 or IV, with components Θ_0, Θ_1, Θ_2 meeting each other transversally, then up to permutations of components, the only possibilities for a configuration given by effective (-2)-divisors $\tilde{D}_1, \tilde{D}_2 < 2H$ (or $< H$ if the fiber has multiplicity 2) such that $\tilde{D}_1.\tilde{D}_2 = 1$ are easily determined as

$$\tilde{D}_1 = \Theta_1, \tilde{D}_2 = \Theta_2 \quad \text{and} \quad \tilde{D}_1 = \Theta_0 + \Theta_1, \;\; \tilde{D}_2 = \Theta_0 + \Theta_2.$$

Since the second configuration is obtained from the first by reflection in Θ_0, the remaining statement of Lemma 2.4 holds on fibers of type I_3 and IV.

Suppose the fiber in question has Kodaira type IV^*, thus supporting three disjoint A_2-type configurations given by the divisors $\tilde{D}$ obtained from $\mathfrak{p}_0(F_1'), \dots, \mathfrak{p}_0(F_3'')$ as in (2.16). Let us write the type IV^* fiber as

$$\Theta_1 + 2\Theta_2 + \Theta_3 + 2\Theta_4 + \Theta_5 + 2\Theta_6 + 3\Theta_0 \tag{2.17}$$

where Θ_{2i} meets exactly Θ_{2i-1} and Θ_0 $(i = 1, 2, 3)$. Recall that an additive fiber on an Enriques surface cannot be multiple.

Start with the root $D_1 = \mathfrak{p}_0(F_1')$. By Tool 2.6, there is a composition of reflections $\mathfrak{p}_1'$ such that $\mathfrak{p}_1'(D_1) = \Theta_1$ as claimed.

Let $\mathfrak{p}_1 = \mathfrak{p}_1' \circ \mathfrak{p}_0$ and $\tilde{D}_2$ denote the effective divisor given by the decomposition of $\mathfrak{p}_1(F_1'')$ defined in (2.16). Since $\tilde{D}_2.\Theta_1 = 1$ and $0 \leq \tilde{D}_2 < 2H$, we infer that $\Theta_1 \not\subseteq \operatorname{supp}(\tilde{D}_2)$ while Θ_2 appears with multiplicity 1 in $\tilde{D}_2$. We claim that there are reflections in $\Theta_3, \dots, \Theta_6, \Theta_0$ exclusively, taking $\tilde{D}_2$ to Θ_2. To see this, we refer to the following more general property which will be useful in the sequel, too.

Tool 2.7. Let v be a root in a root lattice R which contains the vertex e with multiplicity 1. Then there is a composition $\mathfrak{p}$ of reflections in the other vertices of R such that $\mathfrak{p}(v) = e$.

Proof. Denote the vertices of the Dynkin diagram of R by $e_1, \dots, e_n$ and write

$$v = \sum_{j=1}^{n} a_j e_j \quad (a_j \in \mathbb{Z}).$$

Here $e = e_i$, say, and $a_i = 1$ by assumption. Since the roots in R are always effective or anti-effective, we infer $v \geq 0$ from a_i. Since $v^2 = -2$, there is some j such that $v.e_j < 0$. If $j \neq i$, then the reflection s_{e_j} reduces the complexity of the root (measured in terms of $\sum_j a_j \geq 0$), so we may continue with the root $s_{e_j}(v) \geq 0$ instead of v.

Assume that at some point during this process, we have

$$v.e_j \geq 0 \quad \forall j \neq i.$$

We claim that this implies $v = e_i$. To see this, we compute

$$-2 = v^2 = v.\sum_{j=1}^{n} a_j e_j = -2 + \sum_{j \neq i} \underbrace{a_j v.e_j}_{\geq 0}.$$

More precisely, the summands on the right-hand side are all zero if and only if $a_j = 0$ for all j with e_j adjacent to e_i. But then, since roots always have connected support, we infer that all a_j for $j \neq i$ are zero as claimed.

In summary, we can apply reflections away from $e = e_i$ (reducing the complexity and preserving effectivity) until v is mapped to e as stated. □

Remark 2.8. If e has coefficient -1 in the root $v \in R$, then one can show analogously that reflections away from e map v to $-e$.

Applied to $\tilde{D}_2$, we deduce that there is a reflection $\mathfrak{p}_2'$ such that

$$\mathfrak{p}_2'(\tilde{D}_2) = \Theta_2, \quad \mathfrak{p}_2'(\Theta_1) = \Theta_1.$$

That is, $\mathfrak{p}_2 = \mathfrak{p}_2' \circ \mathfrak{p}_1$ maps F_1', F_1'' to Θ_1, Θ_2 up to multiples of H.

The same kind of reasoning applies to $F_2', \dots, F_3''$ to show that a composition of reflections (in $\Theta_3, \dots, \Theta_6$ only!) maps their image under $\mathfrak{p}_2$, up to multiples of H, to the fiber components $\Theta_3, \dots, \Theta_6$. The details are omitted for shortness. This proves Lemma 2.4 for the first three fiber configurations from (2.15).

We now turn to the last two fiber configurations from (2.15). Here the configuration of four A_2's is supported on a single fiber of Kodaira type II^*. We shall seek to establish a contradiction to Lemma 2.1, using Tools 2.6, 2.7.

By Lemma 2.2, there is a configuration of three A_2's involving a 3-divisible class, say

$$\sum_{j=1}^{3} (F_j' - F_j'') = 3D.$$

We start by embedding the remaining A_2-summand into the II^* fiber whose components we label as in Figure 1.

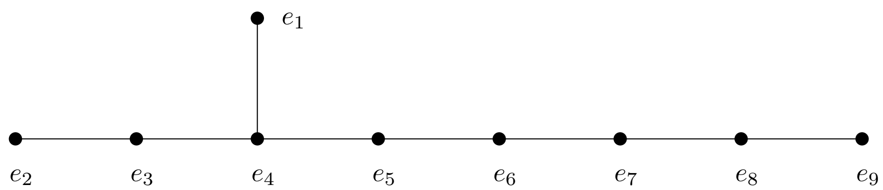

Figure 1. Components of a fiber of Kodaira type II^*.

We let $D' := \mathfrak{p}_0(F_4')$ and $D'' := \mathfrak{p}_0(F_4'')$ denote the two (-2)-divisors of the remaining A_2, both supported on the singular fiber (effective or anti-effective by Claim 2.5). By Tool 2.6, there is a composition $\mathfrak{p}_1'$ of reflections mapping D' to e_9. Decomposing

$$\mathfrak{p}_1'(D'') = \tilde{D} + mH \quad (0 \leq \tilde{D} < 2H)$$

as before, we infer from the intersection number with $\mathfrak{p}_1'(D') = e_9$ that $e_9 \notin \mathrm{supp}(\tilde{D})$ while e_8 has multiplicity 1 in $\tilde{D}$; i.e., $\tilde{D} \in E_8 = \tilde{E}_8 \setminus \{e_9\}$, and by Tool 2.7, there is a composition $\mathfrak{p}_2'$ of reflections in $e_1, \ldots, e_7$ mapping $\tilde{D}$ to e_8.

It follows that $\mathfrak{p}_2 = \mathfrak{p}_2' \circ \mathfrak{p}_1' \circ \mathfrak{p}_0$ maps $F_1', \ldots, F_3''$ to effective or anti-effective (-2)-divisors in the orthogonal complement of $\langle e_8, e_9 \rangle$ inside the extended Dynkin diagram $\tilde{E}_8$ which is given by

$$\langle e_8, e_9 \rangle^{\perp} = \langle e_1, \ldots, e_6, 2H \rangle \cong \tilde{E}_6.$$

In consequence, the above analysis of the IV^* case applies verbatim to show that, after a suitable composition $\mathfrak{p}$ of reflections (in $e_1, \ldots, e_6$) and up to fiber multiples, the three A_2's in question can be realized as

$$\langle e_2, e_3 \rangle \oplus \langle e_5, e_6 \rangle \oplus \langle e_1, -(2e_1 + e_2 + 2e_3 + 3e_4 + 2e_5 + e_6) \rangle. \tag{2.18}$$

Indeed, after mapping $\langle F_2', F_2'' \rangle$, $\langle F_3', F_3'' \rangle$ to $\langle e_2, e_3 \rangle$, $\langle e_5, e_6 \rangle$ and F_1' to e_1, we solve the system of equations given by the fact that $\mathfrak{p}(F_1'')$ is a (-2)-divisor with prescribed intersection pattern. This gives exactly the above solution modulo $2H$.

Apparently (2.18) does not give the signs of the six classes $D_1', \ldots, D_3''$ in the 3-divisible divisor $\mathfrak{p}(3D)$. Yet, since the intersection numbers $\mathfrak{p}(3D).e_j$ are multiples of 3 for $j = 1, \ldots, 9$, one obtains only one possibility (up to sign and a multiple of $(2H)$):

$$(e_3 - e_2) + (e_5 - e_6) + (e_1 + (2e_1 + e_2 + 2e_3 + 3e_4 + 2e_5 + e_6)) = \pm 3\mathfrak{p}(D)$$

which yields

$$\mathfrak{p}(D) = \pm(e_1 + e_3 + e_4 + e_5) + 2mH$$

for some $m \in \mathbb{Z}$. In particular, $\mathfrak{p}(D)$ is either effective or anti-effective, and applying $\mathfrak{p}^{-1}$, we infer the same for D from Observation 2.11 (to be derived in Section 2.4). This contradicts Lemma 2.1 and thus concludes the proof of Lemma 2.4. □

Remark 2.9. On the extremal rational elliptic surfaces, the orthogonal A_2-configurations gives rise to 3-torsion sections by way of 3-divisibility. Essentially, this holds because $\mathrm{H}^2(X, \mathbb{Z})$ is unimodular. Since the same applies to $\mathrm{Num}(S)$, we will be able to establish the same results on S, even though there is no section; see Lemma 3.6.

Remark 2.10. The proof of [10, Claim 3.5.1] states that one can let go the fiber multiple in Lemma 2.4, i.e., there is a composition of reflections $\mathfrak{p}_S$ such that the image of each smooth rational curve F'_j, F''_j under $\mathfrak{p}_S$ is again represented by a smooth rational curve (without possibly adding a multiple of H). Since we were not able to find a reference for this statement, we decided to follow the advice of a referee and give a detailed proof of the weaker statement recorded in Lemma 2.4 (which fortunately will be sufficient for our purposes).

§2.4. Picard–Lefschetz reflections on the K3-cover

In the last part of this section we study Picard–Lefschetz reflections on the K3-cover Y of S.

Let $\pi : Y \to S$ be the K3-cover with induced elliptic fibration (2.5) and let $\psi \curvearrowright Y$ be the Enriques involution. Given a smooth rational curve E in S, the preimage $\pi^{-1}(E)$ consists of two disjoint smooth rational curves E^+, E^-.

With this preparation we proceed to the proof of Claim 2.5.

Proof of Claim 2.5. We maintain the notation of the proof of Lemma 2.4 (see (2.14)), and put

$$\mathfrak{p}_{0,Y} := (s_{E_l^+} \circ s_{E_l^-} \circ \cdots \circ s_{E_1^+} \circ s_{E_1^-}).$$

This map is independent of the order of the elements of the pairs E_i^+, E_i^- as we shall exploit below. Let $D \in \mathrm{Pic}(S)$. Observe that $(D.E_1) = (\pi^* D.E_1^+) = (\pi^* D.E_1^-)$. In particular, we have

$$\begin{aligned}(s_{E_1^+} \circ s_{E_1^-})(\pi^* D) &= \pi^* D + (\pi^* D.E_1^+)E_1^+ + (\pi^* D.E_1^-)E_1^- \\ &= \pi^*(D + (D.E_1)E_1) = \pi^*(s_{E_1}(D)).\end{aligned}$$

This yields the equality

$$\mathfrak{p}_{0,Y} \circ \pi^* = \pi^* \circ \mathfrak{p}_0.$$

Similarly, one can show that

$$\mathfrak{p}_{0,Y} \circ \psi^* = \psi^* \circ \mathfrak{p}_{0,Y}.$$

Moreover, one has the equality

$$\pi_*(\mathfrak{p}_{0,Y})(E^+) = \pi_*(\mathfrak{p}_{0,Y}(E^-)) = \mathfrak{p}_0(E). \tag{2.19}$$

Since $\mathfrak{p}_{0,Y}$ is an isometry, we have $\mathfrak{p}_{0,Y}(F_j'^{\pm})^2 = -2$. Therefore Riemann–Roch implies that either $|\mathfrak{p}_{0,Y}(F_j'^{\pm})| \neq \emptyset$ or $|-\mathfrak{p}_{0,Y}(F_j'^{\pm})| \neq \emptyset$. Suppose that $D' \in |\pm \mathfrak{p}_{0,Y}(F_j'^{\pm})|$. From $(\mathfrak{p}_{0,Y}(F_j'^{\pm}).\pi^* H) = 0$, we infer that all components of $\mathrm{supp}(D')$ are components of fibers of the fibration $|\pi^* H|$. Claim 2.5 now follows directly from (2.19). □

Essentially, the above proof shows the following observation:

Observation 2.11. If D is an effective (-2)-divisor on an Enriques surface, $\mathrm{supp}(D)$ consists of (-2)-curves, and $\mathfrak{p}$ is a composition of reflections in some (-2)-curves, then $\mathfrak{p}(D)$ is either effective or anti-effective.

Since the proof of Lemma 2.4 is now complete, we know that the map $\mathfrak{p}_S$ (see (2.11)) exists, and we define

$$\mathfrak{p}_Y := (s_{E_k^+} \circ s_{E_k^-} \circ \cdots \circ s_{E_1^+} \circ s_{E_1^-}). \tag{2.20}$$

Obviously we have the equalities $\mathfrak{p}_Y \circ \pi^* = \pi^* \circ \mathfrak{p}_S$, and $\mathfrak{p}_Y \circ \psi^* = \psi^* \circ \mathfrak{p}_Y$. Moreover, it is immediate that

$$\pi_*(\mathfrak{p}_Y(E^+)) = \pi_*(\mathfrak{p}_Y(E^-)) = \mathfrak{p}_S(E).$$

The latter implies, using Zariski's lemma, that for $j = 1, \ldots, 4$, the divisor $\mathfrak{p}_Y(F_j'^{\pm})$ (and also the divisor $\mathfrak{p}_Y(F_j''^{\pm})$) is represented, up to sign, by a sum of smooth rational curves contained in a singular fiber of the elliptic fibration (2.5) induced by $|\pi^*H|$ plus possibly a multiple of the general fiber of $|\pi^*H|$.

In particular, Y inherits eight orthogonal A_2-configurations from S. For later use, we label the curves $F_j'^{\pm}$, $F_j''^{\pm}$, in such a way that

$$\pi_*\mathfrak{p}_Y(F_j'^{\pm}) = \mathfrak{p}_S F_j' \quad \text{and} \quad \pi_*\mathfrak{p}_Y(F_j''^{\pm}) = \mathfrak{p}_S F_j'', \quad \text{for } j = 1, \ldots, 4. \tag{2.21}$$

In the sequel we will need the following simple observation. Suppose that for $j = 1, \ldots, 4$ the equalities

$$\mathfrak{p}_S F_j' = \Theta_{2j-1} + n_{2j-1} H \quad \text{and} \quad \mathfrak{p}_S F_j'' = \Theta_{2j} + n_{2j} H \tag{2.22}$$

hold, where Θ_{2j-1}, Θ_{2j} are components of singular fibers of $|2H|$ and $n_{2j-1}, n_{2j} \in \mathbb{Z}$. Then, up to a relabeling of the rational curves $\Theta_{2j-1}^{\pm}$, $\Theta_{2j}^{\pm}$, we have

$$\mathfrak{p}_Y F_j'^{\pm} = \Theta_{2j-1}^{\pm} + n_{2j-1}\pi^* H \quad \text{and} \quad \mathfrak{p}_Y F_j''^{\pm} = \Theta_{2j}^{\pm} + n_{2j}\pi^* H. \tag{2.23}$$

§3. Two families of Enriques surfaces with four cusps

In this section we construct families of Enriques surfaces with four disjoint A_2-configurations supported on the fibers of an elliptic fibration (following Lemma 2.4) and study 3-divisible sets on them.

§3.1. First family of Enriques surfaces

Let $X_{3,3,3,3}$ be the extremal rational elliptic surface with four singular fibers of type I_3. Locating them at the third roots μ_3 of (-1) and at ∞, the surface is given by the Hesse pencil

$$X_{3,3,3,3}: \quad x^3 + y^3 + z^3 + 3\lambda xyz = 0.$$

Here the 3-torsion sections alluded to in Remark 2.9 enter as the base points of the cubic pencil. An Enriques surface is obtained from $X_{3,3,3,3}$ by applying logarithmic transformations of order 2 to the elliptic fibers over two distinct points $P_1, P_2 \in \mathbb{P}^1$. As explained in Section 2.2, this depends on the choice of 2-torsion points in the fibers of $X_{3,3,3,3}$ over P_1, P_2. However, this subtlety will not cause us any trouble:

Lemma 3.1. *The Enriques surfaces obtained by a logarithmic transformation of order* 2 *from* $X_{3,3,3,3}$ *as above form an irreducible two-dimensional family* $\mathcal{F}_{3,3,3,3}$.

Proof. The moduli space $\mathcal{F}_{3,3,3,3}$ is a degree 9 ramified covering of the configuration space

$$\mathrm{Sym}^2(\mathbb{P}^1) \setminus \text{diagonal}. \tag{3.1}$$

Thus $\mathcal{F}_{3,3,3,3}$ has dimension 2. Here the fiber above a non-ordered pair $\{P_1, P_2\}$ consists of all non-ordered pairs in

$$(E_{P_1}[2] \setminus \{O\}) \times (E_{P_2}[2] \setminus \{O\})$$

where E_P denotes the fiber of $X_{3,3,3,3}$ at $P \in \mathbb{P}^1$ with zero element O and 2-torsion subgroup $E_P[2]$. It follows that the covering (3.1) ramifies exactly at the singular fibers. Since the monodromy action of

$$\pi_1(\mathrm{Sym}^2(\mathbb{P}^1 \setminus (\mu_3 \cup \{\infty\})) \setminus \text{diagonal})$$

on a general fiber is transitive, the moduli space $\mathcal{F}_{3,3,3,3}$ is irreducible. □

This proves the first part of Theorem 1.1. In view of Lemma 3.1, we will allow ourselves to abuse notation and denote the resulting Enriques surface(s) simply by S_{P_1,P_2}.

Now, let $S = S_{P_1,P_2} \in \mathcal{F}_{3,3,3,3}$ be an Enriques surface with K3-cover Y and elliptic fibrations φ, $\tilde{\varphi}$ in the notation of Section 2.2. We continue by establishing some information about Y with the help of $\mathrm{Jac}(Y)$. As far as P_1, P_2 are different from ∞ and third roots of (-1) (i.e., outside the branch locus of (3.1)), we obtain eight fibers of type I_3 on Y and $\mathrm{Jac}(Y)$, so the Picard number $\rho(Y) = \rho(\mathrm{Jac}(Y))$ is at least 18 by the Shioda–Tate formula.

Lemma 3.2. *If* $\rho(Y) = 18$, *then* $\mathrm{NS}(Y)$ *has discriminant* $\mathrm{d}(\mathrm{NS}(Y)) = -324$.

Proof. By assumption, $\mathrm{Jac}(Y)$ has finite Mordell–Weil group. The configuration of singular fibers accommodates only 3-torsion, so we infer

$$\mathrm{MW}(\mathrm{Jac}(Y)) \cong (\mathbb{Z}/3\mathbb{Z})^2$$

by pull-back from $X_{3,3,3,3}$. Hence $\mathrm{d}(\mathrm{NS}(\mathrm{Jac}(Y))) = -81$. By the existence of a bisection on Y (induced from S; see Section 2.2), we infer from [9, Lem. 2.1] that

$$\text{either} \quad \mathrm{d}(\mathrm{NS}(Y)) = -81 \quad \text{or} \quad \mathrm{d}(\mathrm{NS}(Y)) = -324 \tag{3.2}$$

as soon as $\rho(Y) = 18$. (Here the former equality holds iff $\tilde{\varphi}$ admits a section, i.e., iff $Y = \mathrm{Jac}(Y)$.) Lemma 3.2 now results immediately from the following proposition. □

Proposition 3.3. *Let Y be the K3-cover of an Enriques surface. Then*

$$2^{20-\rho(Y)} \mid \mathrm{d}(\mathrm{NS}(Y)).$$

In particular, if $\mathrm{d}(\mathrm{NS}(Y))$ is odd, then $\rho(Y) = 20$.

Proof. We shall use the primitive embedding

$$L := U(2) + E_8(2) \cong \pi^* \operatorname{Num}(S) \hookrightarrow \mathrm{NS}(Y).$$

We follow the notation of [14, Sect. 5°] and denote the discriminant group of L by

$$\mathrm{A}_L := L^\vee/L;$$

likewise for other primitive sublattices of $\mathrm{NS}(Y)$ such as $L^\perp$. Define the finite abelian group

$$H := \mathrm{NS}(Y)/(L \oplus L^\perp).$$

Obviously we have the inclusion

$$H \subset \mathrm{A}_L \oplus \mathrm{A}_{L^\perp}.$$

Let p_L (resp. $p_{L^\perp}$) be the projection from $\mathrm{A}_L \oplus \mathrm{A}_{L^\perp}$ onto the first (resp. the second) summand. By [14, p. 111] either projection is an embedding. The first embedding implies

$$H \cong (\mathbb{Z}/2\mathbb{Z})^l,$$

while the second shows $l \leq \rho - 10$ since the length of $A_{L^\perp}$ is bounded by the rank of $L^\perp$. We obtain

$$\mathrm{d}(\mathrm{NS}(Y)) = \mathrm{d}(L \oplus L^\perp)/|H|^2 = 2^{10-l} \cdot (\mathrm{d}(L^\perp)/2^l).$$

Note that the rightmost term in brackets is an integer since $|H| = |p_{L^\perp}(H)|$ divides $|\mathrm{A}_{L^\perp}| = \mathrm{d}(L^\perp)$. Hence we infer that $2^{20-\rho} | \mathrm{d}(\mathrm{NS}(Y))$ as claimed. □

Remark 3.4. A detailed analysis using the 2-length of the groups involved allows one to strengthen the above line of arguments to prove that the K3 cover Y of an Enriques surface has $A_{\mathrm{NS}(Y)}$ of 2-length at least $20 - \rho(Y)$.

§3.2. 3-divisible sets

We shall now investigate the 3-divisible sets among the four A_2-configurations supported on fibers of an Enriques surface $S \in \mathcal{F}_{3,3,3,3}$. Our main results will be formulated in Lemmas 3.6 and 3.7.

Let G be a 2-section of the elliptic fibration φ and let F_j, F_j', F_j'', where $j = 1, \dots 4$, be the components of the I_3-fibers of φ. In order to streamline our notation we label the components of the singular fibers in the following way relative to G:

Notation 3.5. If G meets only one component of an I_3-fiber we denote this component by F_j. Otherwise, F_j', F_j'' stand for the components of the I_3-fiber that meet the 2-section G (i.e., we have $G.F_j = 0$ then).

In particular, if $(F_j + F_j' + F_j'')$ happens to be a half-pencil of the fibration in question, we assume that $G.F_j = 1$. After this preparation we can study 3-divisible sets in the fibers of the elliptic fibration φ on S and $\tilde{\varphi}$ on Y.

Lemma 3.6. *Let $S \in \mathcal{F}_{3,3,3,3}$. The A_2-configurations*

$$F_1', F_1'', \dots, F_4', F_4'' \tag{3.3}$$

contain four 3-divisible sets.

Proof. By (2.2) it suffices to prove that $\overline{M}$, the primitive closure of the lattice M spanned in $\mathrm{Num}(S)$ by the curves (3.3), is unimodular. Equivalently, the lattice $M^\perp = \overline{M}^\perp$ is unimodular. To see this, define an auxiliary divisor class

$$D := G + \sum_{\{j : G.F_j = 0\}} (F_j' + F_j'') \in M^\perp.$$

Let B denote a half-pencil of the fibration φ. By construction, $B \in M^\perp$, and B, D span the hyperbolic plane U since $D.B = G.B = 1$ and $B^2 = 0$. Thus $M^\perp$ and $\overline{M}$ are unimodular, and the proof of Lemma 3.6 is completed by (2.2). □

We shall now eliminate all but one 3-divisible class by considering a different configuration of four A_2's on $S \in \mathcal{F}_{3,3,3,3}$. Recall that F_j^+, F_j^- stand for the (-2)-curves on the K3-cover $\pi : Y \to S$ that lie over the smooth rational curve F_j, and likewise for F_j', F_j''. A discussion of properties of 3-divisible sets of A_2-configurations on K3 surfaces can be found in [1]. In particular, by [1, Lem. 1], a

3-divisible set of A_2-configurations on a K3 surface consists always of six or nine such configurations.

Lemma 3.7. *Let $S \in \mathcal{F}_{3,3,3,3}$. Then*

(a) the four A_2-configurations

$$F_1', F_1'', \ldots, F_3', F_3'', F_4', F_4 \tag{3.4}$$

on the Enriques surface S contain exactly one 3-divisible set;

(b) if the I_3-configuration $(F_4 + F_4' + F_4'')$ is not a half-pencil, then the eight A_2-configurations

$$F_1'^+, F_1''^+, F_1'^-, F_1''^-, \ldots, F_3'^-, F_3''^-, F_4'^+, F_4^+, F_4'^-, F_4^- \tag{3.5}$$

on the K3-cover Y contain exactly one 3-divisible set.

Proof. (a): By (2.3) we are to show that (3.4) does not contain four 3-divisible sets. Suppose to the contrary. Then each triplet of A_2-configurations in (3.4) is 3-divisible. In particular, we have

$$\sum_{j=2}^{3} (\lambda_j' F_j' + \lambda_j'' F_j'') + \lambda_4 F_4 + \lambda_4' F_4' = 3\mathcal{L}, \quad \text{where } \{\lambda_j', \lambda_j''\} = \{\lambda_4, \lambda_4'\} = \{1, -1\}.$$

Since $G.(\lambda_j' F_j' + \lambda_j'' F_j'') = 0$ for $j = 2, 3$, we obtain $G.(\lambda_4 F_4 + \lambda_4' F_4') \in 3\mathbb{Z}$.

If G meets only the curve F_4 in the fiber $(F_4 + F_4' + F_4'')$ (resp. $2(F_4 + F_4' + F_4'')$ iff we deal with a half-pencil) we have $G.F_4 \in \{2, 1\}$ and $G.F_4' = 0$, so $\lambda_4 \in 3\mathbb{Z}$, a contradiction.

Otherwise, G meets the fiber $(F_4 + F_4' + F_4'')$ in two different components, i.e., $G.F_4' = G.F_4'' = 1$ and $G.F_4 = 0$, which yields $\lambda_4' \in 3\mathbb{Z}$. Again we arrive at a contradiction, which implies by symmetry and Lemma 2.2 that

$$F_1', F_1'', F_2', F_2'', F_3', F_3'' \text{ form the unique 3-divisible set in (3.4).} \tag{3.6}$$

(b): Since the pull-back of a (non-trivial) 3-divisible divisor under π is (non-trivially) 3-divisible, (3.6) implies that the six A_2-configurations

$$F_1'^+, F_1''^+, F_1'^-, F_1''^-, \ldots, F_3'^+, F_3''^+, F_3'^-, F_3''^- \tag{3.7}$$

are 3-divisible on the K3-cover Y. To show that they form the unique 3-divisible configuration in (3.5), assume that the A_2-configuration F_4^+, $F_4'^+$ is contained in another non-trivial 3-divisible set on Y.

Suppose that neither the curve F_4^- nor the curve $F_4'^-$ is contained in the 3-divisible divisor in question. Since π is unramified, push-forward yields a non-trivial 3-divisible set of three A_2-configurations in (3.4) that contains F_4, F_4'. The latter is impossible by (3.6).

Thus we can assume that the curves F_4^-, $F_4'^-$, F_4^+, $F_4'^+$ are contained in the support of the 3-divisible divisor in question. From the properties of the push-forward π_* and (3.6), we infer the existence of $\lambda_j'^{\pm}$, $\lambda_j''^{\pm} \in \{0,1,-1\}$, such that one has

$$\sum_{j=1}^{3}\left(\lambda_j'^+F_j'^+ + \lambda_j'^-F_j'^- + \lambda_j''^+F_j''^+ + \lambda_j''^-F_j''^-\right) + (F_4'^- - F_4^-) - (F_4'^+ - F_4^+) = 3\tilde{\mathcal{L}}$$

for a divisor $\tilde{\mathcal{L}}$ on Y. By Lemma 3.6, each triplet of A_2-configurations in (3.3) is 3-divisible, so we can assume that for $j = 2,3,4$ there exist μ_j', μ_j'', such that

$$\sum_{j=2}^{4}(\mu_j'(F_j'^+ + F_j'^-) + \mu_j''(F_j''^+ + F_j''^-)) = 3\hat{\mathcal{L}} \quad \text{and} \quad \{\mu_j', \mu_j''\} = \{1,-1\}$$

for some $\hat{\mathcal{L}} \in \mathrm{Pic}(Y)$. After exchanging components, if necessary, we can assume that $(\mu_4', \mu_4'') = (1,-1)$. By adding the previous two equalities we arrive at a 3-divisible divisor

$$D + (2F_4'^+ + F_4''^+) + 3(F_4'^- + F_4^+) - (F_4^- + F_4'^- + F_4''^-) - 2(F_4^+ + F_4'^+ + F_4''^+)$$

with $\mathrm{supp}(D)$ contained in the union of the curves $F_j'^{\pm}$, $F_j''^{\pm}$ for $j = 1,2,3$. Since both triangles $(F_4^- + F_4'^- + F_4''^-)$, $(F_4^+ + F_4'^+ + F_4''^+)$ are fibers of the elliptic fibration $\tilde{\varphi}$, we derive a 3-divisible divisor

$$D - (F_4'^+ - F_4''^+) \tag{3.8}$$

with $\mathrm{supp}(D)$ satisfying the condition given above. We continue to establish a contradiction.

Recall (see, e.g., [17, Sect. 5]) that each non-trivial 3-divisible set on Y corresponds to a line $\mathbb{F}_3 v$, where v is a non-zero vector in the kernel of the $\mathbb{F}_3$-linear map

$$\mathbb{F}_3^8 \ni (\lambda_1^+, \ldots, \lambda_4^-) \mapsto \sum_1^4 \lambda_j^+(F_j'^+ - F_j''^+) + \sum_1^4 \lambda_j^-(F_j'^- - F_j''^-) \in \mathrm{Pic}(Y) \otimes \mathbb{F}_3.$$

Thus the kernel in question is a ternary $[8, d, \{6\}]$-code (i.e., a d-dimensional subspace of $\mathbb{F}_3^8$, such that all its non-zero vectors have exactly six non-zero coordinates). By the Griesmer bound (see, e.g., [25, Thm. (5.2.6)]), we have $d \leq$

2 and $F_j'^{\pm}$, $F_j''^{\pm}$, where $j = 1, \dots, 4$, contain at most four sets of 3-divisible A_2-configurations. On the other hand we obtain four non-trivial ψ^*-invariant 3-divisible sets by pulling-back the 3-divisible sets from S (see Lemma 3.6). Observe that the 3-divisible set given by (3.8) is not ψ^*-invariant, a contradiction. □

Remark 3.8. Since any elliptic fibration on an Enriques surface has exactly two multiple fibers, we can always ensure by exchanging fibers that the assumption in Lemma 3.7(b) holds.

§3.3. Second family of Enriques surfaces

In the following paragraphs, we work out Enriques surfaces with elliptic fibrations of the types (2.13). To this end, we consider another two extremal rational elliptic surfaces, given in Weierstrass form

$$X_{4,3,1} : y^2 + xy + ty = x^3,$$
$$X_{4,4} : y^2 + ty = x^3.$$

Each has a fiber of Kodaira type IV^* at ∞ and a 3-torsion section at $(0, 0)$. The fiber type at $t = 0$ is IV for $X_{4,4}$ and I_3 for $X_{4,3,1}$ (which thus has one further singular fiber, of type I_1). The surface $X_{4,4}$ gives the special case in (2.13) omitted in [10]. We point out that both elliptic surfaces live inside an isotrivial family

$$\mathcal{X} : y^2 + cxy + ty = x^3, \quad c \in \mathbb{C} \tag{3.9}$$

whose fibers $\mathcal{X}_c$ for $c \neq 0$ are all isomorphic to $X_{4,3,1}$ after rescaling while $\mathcal{X}_0 = X_{4,4}$. Implicitly, this connection will feature again shortly.

As in Section 3.1, we shall apply logarithmic transformations of order 2 to $X_{4,3,1}$ and $X_{4,4}$. On $X_{4,3,1}$, we can do so for any two distinct points $P_1, P_2 \in \mathbb{P}^1 \setminus \{\infty\}$; again, the logarithmic transformation depends on the choice of 2-torsion points in the fibers, but as in the proof of Lemma 3.1, we obtain an irreducible two-dimensional family $\mathcal{F}$ of Enriques surfaces, parametrized by a 9-fold covering of the configuration space $\mathrm{Sym}^2(\mathbb{P}^1 \setminus \{\infty\}) \setminus$ diagonal.

The situation is quite different for the logarithmic transformations of $X_{4,4}$: since $X_{4,4}$ has only two singular fibers (and the above Weierstrass form has only three terms), we can still rescale the base curve $\mathbb{P}^1$ without changing the surface at all! Hence the resulting Enriques surfaces do only come in a one-dimensional family.

Lemma 3.9. *The Enriques surfaces arising from $X_{4,4}$ via logarithmic transformation lie in the boundary of $\mathcal{F}$.*

We did not attempt to prove Lemma 3.9 directly from the above data (although the isotrivial family (3.9) certainly points in this direction). Instead, we pursue an alternative algebraic approach towards the Enriques surfaces with fibers of type IV^* and I_3 or IV in Section 5. Incidentally, this will provide an easy proof of Lemma 3.9; see Remark 5.7.

Definition 3.10. We denote the resulting irreducible family of Enriques surfaces by $\mathcal{F}_{4,3,1}$ (comprising surfaces arising from $X_{4,3,1}$ as well as from $X_{4,4}$).

On an Enriques surface $S' \in \mathcal{F}_{4,3,1}$, we put

$$3F_0 + \sum_{j=1}^{3}(F_j' + 2F_j'')$$

(resp. $F_4 + F_4' + F_4''$) to denote the IV^*-fiber (resp. the I_3 or IV-fibers) of the induced elliptic fibration φ. It is immediate that, up to the choice of the curve F_4, the rational curves

$$F_1', F_1'', \ldots, F_4', F_4''$$

form the only set of four disjoint A_2-configurations contained in the singular fibers of the fibration φ.

Let $\pi : Y' \to S'$ be the K3-cover and let $\tilde{\varphi}$ be the fibration induced by φ on Y'. The number of 3-divisible sets on S' (resp. Y') supported on the components of fibers of φ (resp. $\tilde{\varphi}$) can be found using [10, Lem. 3.5(1)].

Lemma 3.11. *Let $S' \in \mathcal{F}_{4,3,1}$. Then the four A_2-configurations*

$$F_1', F_1'', \ldots, F_4', F_4''$$

contain exactly one 3-divisible set, whereas the eight A_2-configurations

$$F_1'^{+}, F_1''^{+}, \ldots, F_4'^{-}, F_4''^{-}$$

on the K3-cover contain four 3-divisible sets.

Proof. By [10, Lem. 3.5(1)] the set $F_1', F_1'', \ldots, F_3', F_3''$ is not 3-divisible, whereas the six A_2-configurations $F_1'^{+}, F_1''^{+}, F_1'^{-}, F_1''^{-}, \ldots, F_3'^{-}, F_3''^{-}$ form a 3-divisible set on Y' (pushing down to a trivially 3-divisible divisor on S'). The former assertion rules out the second possibility of Corollary 2.3.

On the other hand, the curves $F_1', F_1'', \ldots, F_4', F_4''$ contain at least one 3-divisible set by Corollary 2.3. As the pullback under π we obtain another 3-divisible set on Y', so the K3-cover contains exactly four 3-divisible sets (see the proof of Lemma 3.7). □

§3.4. Proof of Theorem 1.1

Let S be an Enriques surface admitting four disjoint A_2-configurations. Then S admits an elliptic fibration

$$\pi : S \to \mathbb{P}^1$$

whose singular fibers are classified in Lemma 2.4. The Jacobian of π is either $X_{3,3,3,3}$, $X_{4,3,1}$ or $X_{4,4}$ (as exploited in the proof of Lemma 2.4). In particular, S arises from $\mathrm{Jac}(\pi)$ by a logarithmic transformation of degree 2. Thus $S \in \mathcal{F}_{3,3,3,3} \cup \mathcal{F}_{4,3,1}$ as shown in Sections 3.1, 3.3. □

§3.5. Explicit examples supporting each case

Example 3.12. Let S' be the Enriques surface with finite automorphism group $S_4 \times \mathbb{Z}/2\mathbb{Z}$ considered in [12, Exa. V] arising from the Kummer surface of E^2 for the elliptic curve with zero j-invariant. By [12, Table 2, p. 132] we have

$$S' \in \mathcal{F}_{4,3,1} \setminus \mathcal{F}_{3,3,3,3}.$$

In fact, by [12, Rem. (4.29)] no surface in $\mathcal{F}_{3,3,3,3}$ has finite automorphism group, so we can find no example of a surface from $\mathcal{F}_{3,3,3,3}$ in [12]. Therefore, we will use the construction of Enriques involution of base change type (as reviewed in Section 2.2) to obtain an explicit example of such an Enriques surface.

Example 3.13. Let Y denote the singular K3 surface with transcendental lattice

$$T(Y) = \begin{pmatrix} 6 & 3 \\ 3 & 12 \end{pmatrix}. \tag{3.10}$$

In what follows, we will sketch in a rather conceptual way that Y admits an Enriques involution of base change type whose quotient surface is in $\mathcal{F}_{3,3,3,3}$.

We start from elliptic curves E parametrized by the j-invariant. To E^2, we can associate the Kummer surface $\mathrm{Kum}(E^2)$, but also by way of what is called a Shioda–Inose structure nowadays (cf. [23]) a K3 surface which recovers the transcendental lattice of E^2. We thus obtain a one-dimensional family of K3 surfaces $\mathcal{Y}'$ with generic transcendental lattice

$$T(\mathcal{Y}') = U + \langle 2\rangle.$$

By Nishiyama's method ([15]) $\mathcal{Y}'$ comes with an elliptic fibration with a fiber of Kodaira type I_{18} and generically $\mathrm{MW}(\mathcal{Y}') \cong \mathbb{Z}/3\mathbb{Z}$. Quotienting out by translation by the 3-torsion sections, we obtain another family of K3 surfaces $\mathcal{Y}$, generically with one fiber of type I_6, six fibers of type I_3 and

$$\mathrm{MW}(\mathcal{Y}) \cong (\mathbb{Z}/3\mathbb{Z})^2.$$

It follows that $\mathcal{Y}$ arises from $X_{3,3,3,3}$ by the one-dimensional family of base changes (2.7) ramified at a given singular fiber, say $\lambda = \infty$. That is, there is an involution $\imath \curvearrowright \mathcal{Y}$ such that $\mathcal{Y}/\imath = X_{3,3,3,3}$. For discriminant reasons, the transcendental lattice is scaled by a factor of 3,

$$T(\mathcal{Y}) \cong T(\mathcal{Y}')(3), \tag{3.11}$$

where this equality not only holds generically, but also on the level of single members of the families (cf. [18, Lem. 8]).

We continue by specializing to a member $Y \in \mathcal{Y}$ in order to endow Y with a section which combines with $\imath$ to an Enriques involution of base change type. To this end, choose E to be the elliptic curve with CM by $\mathbb{Z}[\omega]$ for $\omega = (1+\sqrt{-7})/2$ (j-invariant -15^3). By [24],

$$T(E^2) \cong \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$$

which exactly gives rise to (3.10) by (3.11). Inside the family $\mathcal{Y}$, this can only be accounted for by a section Q of height $7/6$. It is induced from a section Q' of height $7/12$ on the quadratic twist $X'_{3,3,3,3}$ of $X_{3,3,3,3}$. Here $X'_{3,3,3,3}$ has singular fibers of types I_3^*, 3 times I_3 and I_0^*; the given height can be attained only if Q' is perpendicular to O' and intersects non-trivially I_3^* (in a far simple component), one I_3 and I_0^*. In consequence, the pull-back Q on Y is disjoint from O and anti-invariant for $\imath$. Hence

$$\jmath := (\text{translation by } Q) \circ \imath$$

defines an Enriques involution on Y such that $Y/\jmath \in \mathcal{F}_{3,3,3,3}$ as claimed. All of this can be made explicit without much difficulty. For instance, spelling out the conditions for Q' on $X'_{3,3,3,3}$ to take the above shape, one finds that the second ramification point of the quadratic base change (2.7) is located at $\lambda = 5/4$. We leave the details to the reader.

§3.6. Gorenstein $\mathbb{Q}$-homology projective planes

In [7], a classification of Gorenstein $\mathbb{Q}$-homology projective planes is completed in terms of their singularity types. A key case originates from Enriques surfaces after contracting a set of nine (-2)-curves. Among the 31 singularity types which are a priori possible, only two are so far not supported by an example. Here we note that the Enriques surface $S = Y/\jmath$ from Example 3.13 remedies this for one type. (See also [19] for subsequent results in the same spirit.)

Proposition 3.14. *The Enriques surface S contains an $A_3 + 3A_2$-configuration of smooth rational curves.*

Proof. By construction, S is equipped with an elliptic fibration with four singular fibers of Kodaira type I_3, the one at ∞ actually with multiplicity 2. Consider the bisection R whose pre-image decomposes into O and Q on the covering K3 surface. Since O and Q are disjoint, R is a smooth rational curve, and $R^2 = -2$. By the setup in Example 3.13, R meets all but one singular fiber in a single fiber component. Therefore there are three A_2-configurations supported on the fibers that are perpendicular to R. The remaining I_3 fiber connects with R for a square with one diagonal added. Omitting one fiber component meeting R, we obtain an A_3-configuration. □

§4. The fundamental groups of open Enriques surfaces

Let S be an Enriques surface that contains four disjoint A_2-configurations F_1', F_1'', …, F_4', F_4'' and let $\pi : Y \to S$ be the K3-cover. As in the preceding sections, the (-2)-curves in $\pi^{-1}(F_j')$ are denoted by $F_j'^{+}$, $F_j'^{-}$. Moreover, we put

$$\mathcal{A} := \{F_1', F_1'', \ldots, F_4', F_4''\} \quad \text{and} \quad \mathcal{A}^{\pm} := \{F_1'^{+}, F_1''^{+}, F_1'^{-}, F_1''^{-}, \ldots, F_4'^{-}, F_4''^{-}\}.$$

Given the pair $(S, \mathcal{A})$, we follow [10] and define the *fundamental group of the open Enriques surface* $S^\circ = S \setminus \mathcal{A}$:

$$\pi_1(S, \mathcal{A}) := \pi_1(S^\circ).$$

To deal with Enriques surfaces with four A_2-configurations in more generality we introduce the following notation:

Notation 4.1. We say that the pair $(S, \mathcal{A})$ belongs to $\mathcal{F}_{4,3,1}$ (resp. $\mathcal{F}_{3,3,3,3}$) iff there exists a composition of Picard–Lefschetz reflections (2.11) and an elliptic pencil $|2H|$ on S such that the elliptic fibration given by $|2H|$ has singular fibers of the types (2.13) (resp. (2.12)) and, up to multiples of the half-pencil H, each class $\mathfrak{p}_S(F_j')$, $\mathfrak{p}_S(F_j'')$, where $j = 1, \ldots, 4$, is an irreducible component of a singular fiber of the elliptic fibration $|2H|$. To simplify our notation we write

$$(S, \mathcal{A}) \in \mathcal{F}_{4,3,1} \quad (\text{resp. } (S, \mathcal{A}) \in \mathcal{F}_{3,3,3,3})$$

when the above condition is satisfied. Then $\mathfrak{p}_S(\mathcal{A})$ stands for the set of the four A_2-configurations defined, up to multiples of H, by $\mathfrak{p}_S(F_1')$, …, $\mathfrak{p}_S(F_4'')$ for a fixed composition of reflections $\mathfrak{p}_S$.

Recall that $\mathfrak{p}_S$ induces the map $\mathfrak{p}_Y$ (see (2.20)). In the sequel we maintain the notation (2.21) and use $\mathfrak{p}_Y(\mathcal{A}^{\pm})$ to denote the set of the eight A_2-configurations on the K3-cover (again supported on the fibers of an elliptic fibration).

As we explained around Lemma 2.4, the authors of [10] claim that after applying an appropriate composition of Picard–Lefschetz reflections $\mathfrak{p}_S$, the four A_2-configurations on the Enriques surface S become components of singular fibers of the fibration of type (2.13). The latter implies the erroneous claim that $\mathcal{A}$ never contains four 3-divisible sets ([10, Lem. 3.5(2)]) and the fundamental group $\pi_1(S^0)$ of the open Enriques surface is either $\mathbb{Z}/6\mathbb{Z}$ or $S_3 \times \mathbb{Z}/3\mathbb{Z}$ (see [10, Lem. 3.6(3)]). Here we correct these claims.

Lemma 4.2. *Let S be an Enriques surface with four A_2-configurations $\mathcal{A}$. Then*

$$\pi_1(S^0) \in \{S_3 \times \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, (\mathbb{Z}/3\mathbb{Z})^{\oplus 2} \times \mathbb{Z}/2\mathbb{Z}\}. \tag{4.1}$$

Moreover, one has the following characterizations:

(a) Both $\mathcal{A}$ and $\mathcal{A}^{\pm}$ contain exactly one 3-divisible set iff $\pi_1(S^0) = \mathbb{Z}/6\mathbb{Z}$.

(b) The A_2-configuration $\mathcal{A}$ contains exactly one 3-divisible set and $\mathcal{A}^{\pm}$ contains four 3-divisible sets iff

$$\pi_1(S^0) = S_3 \times \mathbb{Z}/3\mathbb{Z}.$$

(c) The A_2-configuration $\mathcal{A}$ contains four 3-divisible sets iff $\pi_1(S^0) = (\mathbb{Z}/3\mathbb{Z})^{\oplus 2} \times \mathbb{Z}/2\mathbb{Z}$.

Proof. The proof follows almost verbatim the first part of the proof of [10, Lem. 3.6(3)], but there is one addition to be made: Lemma 3.6 shows that one cannot use [10, Lem. 3.5(2)] to rule out the existence of an Enriques surface S with $\pi_1(S^0) = (\mathbb{Z}/3\mathbb{Z})^{\oplus 2} \times \mathbb{Z}/2\mathbb{Z}$. □

With this preparation we can prove the following precise version of Theorem 1.2:

Theorem 4.3. *Let S be an Enriques surface with a set of four mutually disjoint A_2-configurations $\mathcal{A}$:*

(a) If $(S, \mathcal{A}) \in \mathcal{F}_{4,3,1}$, then $\pi_1(S, \mathcal{A}) = S_3 \times \mathbb{Z}/3\mathbb{Z}$.

(b) If $S \in \mathcal{F}_{3,3,3,3}$, then there exist $\mathcal{A}'$ and $\mathcal{A}''$ such that

$$\pi_1(S, \mathcal{A}') = (\mathbb{Z}/3\mathbb{Z})^{\oplus 2} \times \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \pi_1(S, \mathcal{A}'') = \mathbb{Z}/6\mathbb{Z}.$$

In particular, all groups given in Lemma 4.2 are realized by Enriques surfaces.

Proof. We start with (b) which is much easier to prove. Indeed, the existence of $\mathcal{A}'$ (resp. $\mathcal{A}''$) results immediately from Lemma 3.6 and Lemma 4.2(c) (resp. Lemma 3.7, Remark 3.8 and Lemma 4.2(a)).

In order to prove part (a) we assume that $(S, \mathcal{A}) \in \mathcal{F}_{4,3,1}$, i.e., that there exist the fibration $|2H|$ and the map $\mathfrak{p}_S$. By Lemma 4.2 it suffices to show that $\mathcal{A}^{\pm}$ contains four 3-divisible sets, but $\mathcal{A}$ contains only one 3-divisible set.

We label the components of the type-IV^* fiber of the fibration $|2H|$ as in (2.17). Moreover, we put $\Theta_7, \ldots, \Theta_9$ to denote the components of the I_3 fiber. We can assume that (2.22), (2.23) hold. Then, the divisor

$$\Theta_1^+ - \Theta_2^+ + \Theta_3^+ - \Theta_4^+ + \Theta_5^+ - \Theta_6^+ - \Theta_1^- + \Theta_2^- - \Theta_3^- + \Theta_4^- - \Theta_5^- + \Theta_6^- \tag{4.2}$$

is 3-divisible (see the proof of [10, Lem. 3.5.1]). From (2.23) we infer that $\mathcal{A}^{\pm}$ contains a 3-divisible set that cannot be obtained as the image of a 3-divisible set in $\mathcal{A}$ under the pull-back π^*. On the other hand, the eight A_2-configurations contain the pull-back π^* of a 3-divisible set contained in $\mathcal{A}$. The latter exists by Corollary 2.3. This proves the first claim.

The second claim is a little more subtle due to the multiples of the half-pencil H involved. Indeed, the four A_2-configurations have to contain a 3-divisible set by Lemma 3.11, but the three A_2-configurations $\Theta_1, \ldots, \Theta_6$ supported on the IV^* fiber are themselves *not* 3-divisible because the divisor

$$D_0 = \Theta_1 - \Theta_2 + \Theta_3 - \Theta_4 + \Theta_5 - \Theta_6 + H \tag{4.3}$$

visibly is (in the 3-divisible divisor (4.2) on the covering K3, the contributions from H even out). That is, without multiples of the half-pencil, the four A_2-configurations contain only one 3-divisible set (by Lemma 3.11) while modulo the half-pencil, there are four 3-divisible sets. Therefore we will have to conduct a careful analysis of the precise multiples of H involved.

After rearranging the (-2)-curves, if necessary, we can assume that the divisor

$$D_1 = \Theta_1 - \Theta_2 - \Theta_3 + \Theta_4 + \Theta_7 - \Theta_8 \tag{4.4}$$

is 3-divisible. By symmetry, the same applies to a compatible pair of divisors

$$D_2 = \Theta_3 - \Theta_4 - \Theta_5 + \Theta_6 + \begin{cases} \Theta_8 - \Theta_9, & \text{case (i)}, \\ \Theta_9 - \Theta_7, & \text{case (ii)}, \end{cases}$$

$$D_3 = \Theta_5 - \Theta_6 - \Theta_1 + \Theta_2 + \begin{cases} \Theta_9 - \Theta_7, & \text{case (i)}, \\ \Theta_8 - \Theta_9, & \text{case (ii)}. \end{cases}$$

Which case actually persists depends on whether the I_3 fiber supported on Θ_7, Θ_8, Θ_9 is multiple or not, i.e., whether

$$\Theta_7 + \Theta_8 + \Theta_9 = H \text{ or } 2H.$$

One easily verifies that if the fiber is multiple, case (i) persists while the unramified fiber type leads to case (ii). It remains to check whether having four 3-divisible sets is compatible with fixed H-multiplicities $m_1, \dots, m_4$ attached to the four A_2-configurations. To this end, we may assume that

$$\mathfrak{p}(F_i' - F_i'') = \Theta_{2i-1} - \Theta_{2i} + m_i H, \quad i = 1, \dots, 4.$$

Since we care only about 3-divisibility, all equations in the remainder of the proof of Theorem 4.3 should be understood in $\mathbb{Z}/3\mathbb{Z}$. Then (4.3) leads to

$$m_1 + m_2 + m_3 = 1$$

while (4.4) gives

$$m_1 - m_2 + m_4 = 0.$$

The two cases for D_2, D_3 lead to the same equations although (or because) they depend on the multiplicity of the I_3 fiber:

$$\begin{aligned} m_2 - m_3 + m_4 &= 1, \\ -m_1 + m_3 + m_4 &= -1. \end{aligned}$$

Thus we obtain a system of four linear equations over $\mathbb{Z}/3\mathbb{Z}$. One immediately verifies that the system has no solution. Hence, regardless of the multiples of H involved, the four A_2-configurations cannot support four 3-divisible sets. By Lemma 4.2, this concludes the proof of Theorem 4.3. □

Finally, we use the Jacobian fibration to verify that surfaces in $\mathcal{F}_{4,3,1} \cap \mathcal{F}_{3,3,3,3}$ have some special properties; notably,. the families $\mathcal{F}_{4,3,1}$, $\mathcal{F}_{3,3,3,3}$ overlap only on proper subfamilies:

Proposition 4.4. *Let S be an Enriques surface and let Y be the K3-cover of S. If $S \in \mathcal{F}_{4,3,1} \cap \mathcal{F}_{3,3,3,3}$, then $\rho(Y) \geq 19$.*

Proof. We compare the discriminants of the K3-covers. For $S \in \mathcal{F}_{3,3,3,3}$ with K3-cover Y of Picard number $\rho(Y) = 18$, Lemma 3.2 gives $\mathrm{d}(\mathrm{NS}(Y)) = -324$.

A completely analogous argument applies to $S' \in \mathcal{F}_{4,3,1}$ with K3-cover Y' such that $\rho(Y') = 18$. We find that $\mathrm{d}(\mathrm{NS}(Y')) = -36$. This implies that $S' \notin \mathcal{F}_{3,3,3,3}$ and vice versa for S. □

In the next section, we shall take this result as a starting point to take a closer look at the moduli of our families $\mathcal{F}_{3,3,3,3}$ and $\mathcal{F}_{4,3,1}$.

§5. Moduli

§5.1. Algebro-geometric construction

We start by giving an algebro-geometric description of the family $\mathcal{F}_{4,3,1}$. As opposed to the analytic construction of logarithmic transformations, it will be based on Enriques involutions of base change type as outlined in Section 2.2. Our starting point is another extremal rational elliptic surface $X_{6,3,2,1}$, this time with $\mathrm{MW}(X_{6,3,2,1}) \cong \mathbb{Z}/6\mathbb{Z}$. As a cubic pencil, it can be given by

$$X_{6,3,2,1}: \quad (x+y)(y+z)(z+x) + txyz = 0. \tag{5.1}$$

More precisely, $X_{6,3,2,1}$ is the relatively minimal resolution of the above cubic pencil model in $\mathbb{P}^2 \times \mathbb{P}^1$, obtained by blowing up the three double base points at $[1,0,0]$, $[0,1,0]$, $[0,0,1]$. The blowup results in a fiber of Kodaira type I_6 at ∞; the other singular fibers are I_3 at $t=0$, I_2 at $t=1$ and I_1 at $t=-8$. The other three base points of the cubic pencil are actually points of inflection. Fixing one of them as zero O for the group law, say $[1,-1,0]$, we find that $P=[0,0,1]$ has order 2 inside $\mathrm{MW}(X_{6,3,2,1})$. Thus it lends itself to (the classical case of) the construction of an Enriques involution of base change type (as reviewed in Section 2.2). To this end, consider a quadratic base change (2.7) that does not ramify at the I_3 and I_1 fibers. Denote the pull-back surface by $Y_{6,3,2,1}$; this is an elliptic K3 surface, generically with all singular fibers of $X_{6,3,2,1}$ duplicated. The deck transformation $\imath$ enables us to define a fixed-point-free involution ψ on $Y_{6,3,2,1}$ by

$$\psi = (\text{fiberwise translation by } P) \circ \imath.$$

The quotient surface will be an Enriques surface $S = S_{6,3,2,1}$ with elliptic fibration

$$\pi_0 : S \to \mathbb{P}^1$$

with the same singular fibers as $X_{6,3,2,1}$ (generically not multiple); here O and P map to a smooth rational bisection R.

Lemma 5.1. *The Enriques surface S contains four perpendicular A_2-configurations.*

Proof. Consider the singular fibers of S together with the bisection R. Figure 2 depicts how they intersect and indicates the four A_2-configurations. □

By Theorem 1.1, we conclude that $S \in \mathcal{F}_{3,3,3,3}$ or $S \in \mathcal{F}_{4,3,1}$. For discriminant reasons (cf. the proof of Proposition 4.4), the second alternative should hold. Here we will give a purely geometric argument:

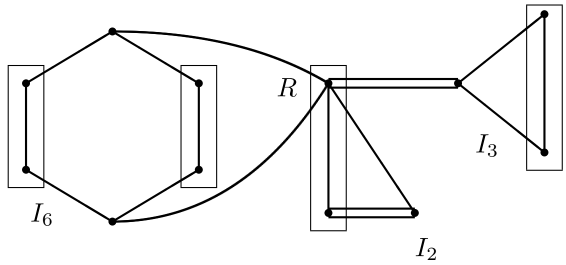

Figure 2. Four disjoint A_2-configurations on $S_{6,3,2,1}$.

Lemma 5.2. $S \in \mathcal{F}_{4,3,1}$.

Proof. It suffices to identify a divisor of Kodaira type IV^* on S with orthogonal A_2. Then its linear system will induce an elliptic fibration

$$\pi : S \to \mathbb{P}^1$$

with singular fibers of types IV^* and I_3 or IV; thus $S \in \mathcal{F}_{4,3,1}$. This is easily achieved: simply connect the three A_2's on the left in Figure 2 through one of the remaining components of the original I_6 fiber. □

Remark 5.3. A bisection for the fibration π from the proof of Lemma 5.2 can be given without difficulty (although Figure 2 displays only fiber components and 4-sections): take a half-pencil B of the fibration π_0. Out of the curves depicted in Figure 2, B meets the bisection R with only multiplicity 1. On the fibration π, B meets the IV^* fiber only in the double component R; since additive fibers cannot be multiple, B thus defines a bisection for π. The fiber of type I_3 or IV is met by B only in the component not displayed in Figure 2.

§5.2. K3-cover for $\mathcal{F}_{4,3,1}$

Overall, there are six configurations of how a bisection may intersect the two reducible fibers of a given Enriques surface $S \in \mathcal{F}_{4,3,1}$. For three of them, including the one sketched in Remark 5.3, we can conversely derive the configuration of rational curves on $S_{6,3,2,1}$ originating from the Enriques involution of base change type. Here we detail one example:

Example 5.4. For the configuration from Remark 5.3 comprising a bisection B (of arithmetic genus 0) meeting the fibers of type IV^* and I_3/IV on S, one finds that B automatically is a half-pencil inducing an elliptic fibration $\pi_{|2B|}$, since it is met by some (nodal) curve (a double component Θ of the IV^* fiber) with multiplicity 1 (so Θ gives a bisection for $\pi_{|2B|}$). This fibration has singular fibers accounting for the root lattices A_5+A_1 and A_2 obtained from the extended Dynkin diagrams $\tilde{E}_6$, $\tilde{A}_2$ by omitting the curves meeting B. With the nodal bisection Θ, we necessarily end up on a quotient of $X_{6,3,2,1}$ by an Enriques involution of base change type. Indeed, otherwise, there would be an additive fiber of type IV^*, III^* or II^*; but here it is easy to see that the root of the A_5-diagram met by Θ would correspond to a triple fiber component which, of course, cannot be met by a bisection.

For the other three possible configurations of bisection and singular fibers, the approach from Example 5.4 does not seem to work. However, the next proposition and its corollary show in a lattice-theoretic, hence non-explicit, way that also these Enriques surfaces are covered by $Y_{6,3,2,1}$. We expect that they arise from $Y_{6,3,2,1}$ by another kind of Enriques involution.

Proposition 5.5. *Let $S \in \mathcal{F}_{4,3,1}$ such that the K3 cover Y has $\rho(Y) = 18$. Then*

$$\mathrm{NS}(Y) \cong U(2) + A_2 + E_6 + E_8.$$

Proof. The elliptic fibration on Y induced from S comes automatically with a bisection R. Since $\rho(Y) = 18$, we can assume that $R^2 = 0$. The key step in proving the proposition is the observation that we can modify R to a divisor D by adding fiber components as correction terms such that D is perpendicular to two A_2- and two E_6-configurations on Y (in the fibers of π). For fibers of type I_3, IV, this has been exhibited in the proof of Lemma 3.6. For IV^* fibers, it is a similar exercise. For instance, if R meets a double component, then simply subtract from R the adjacent simple component Θ' (which is thus met by $R - \Theta'$ with multiplicity 2).

Crucially, we now use that the singular fibers come in pairs which are met by R in exactly the same way (there cannot be non-reduced singular fibers since $\rho(Y) = 18$). In consequence, the correction terms for D also come in pairs, so

$$D^2 \equiv R^2 \equiv 0 \mod 4.$$

Hence D and the general fiber F span the lattice $U(2)$, and we obtain a finite index sublattice

$$U(2) + A_2^2 + E_6^2 \subset \mathrm{NS}(Y).$$

To compute $\mathrm{NS}(Y)$, it remains to take the 3-divisible class in $A_2^2+E_6^2$ induced from the Enriques surface into account. From the lattice viewpoint, this behaves exactly like the 3-torsion section on $\mathrm{Jac}(Y)$; we obtain an integral index 3 overlattice M of $A_2^2 + E_6^2$ by adjoining a vector of square -4 obtained by adding up minimal vectors twice each of $A_2^\vee$ and $E_6^\vee$ (of square $-2/3$, resp. $-4/3$). Finally we verify that M and $A_2 + E_6$ have isomorphic discriminant forms. By [14, Cor. 1.13.3], it follows that $U(2) + M$ and $U(2) + A_2 + E_6 + E_8$ are isometric. □

Corollary 5.6. *A general Enriques surface $S \in \mathcal{F}_{4,3,1}$ is covered by a K3 surface $Y_{6,3,2,1}$.*

Proof. The Néron–Severi lattice of the covering K3 surface admits a unique embedding into the K3 lattice $U^3 + E_8^2$ up to isometries by [14, Thm. 1.14.4]. Hence the K3 surfaces with this lattice polarization form an irreducible two-dimensional family, and the corollary ends up being a consequence of Lemma 5.2 in the reverse direction (since the quadratic base changes of $X_{6,3,2,1}$ exactly form a two-dimensional family, the parameters being the non-ordered pairs comprising the two ramification points). □

Remark 5.7. We can use the above description to give a proof of Lemma 3.9. For this purpose, we check within our family where the singular fiber types degenerate from $(I_3 + I_1)$ to IV. For this, we normalize the base change (2.7) to take the shape

$$t \mapsto 1 - \gamma \frac{(t-1)(t-\lambda)}{t},$$

so that $Y_{6,3,2,1}$ has fibers of type I_6 at 0, ∞ and I_2 at 1, λ. Then we extract the elliptic fibration with two fibers of types IV^* and two perpendicular A_2 inducing generically a quadratic base change of $X_{4,3,1}$. This turns out to be isotrivial (with zero j-invariant and IV fibers instead of I_3 and I_1) exactly for $\lambda = 1 - 3/\gamma$.

§5.3. Comparison with $Y_{6,3,2,1}$

As a sanity check, we will compute $\mathrm{NS}(Y_{6,3,2,1})$ at a very general moduli point directly. Incidentally, this will allow us to draw interesting consequences; see Theorem 5.8.

Consider a K3 surface $Y_{6,3,2,1}$ with $\rho(Y_{6,3,2,1}) = 18$. By [20, (22)], $\mathrm{NS}(Y_{6,3,2,1})$ has discriminant -36. In order to compute $\mathrm{NS}(Y_{6,3,2,1})$ directly, we will identify two perpendicular divisors D_1, D_2 of Kodaira type II^* among the plentitude of (-2)-curves visible in the elliptic fibration π_0 as fiber components and torsion sections. To define D_1, connect the zero section O in three directions: by a component of either I_2 fiber, two components of an I_3 and a chain $\Theta_0, \dots, \Theta_4$ of five components of an I_6. Similarly, the divisor D_2 comprises the 6-torsion section disjoint from D_1 (i.e., meeting the remaining fiber component Θ_5 of the chosen I_6 fiber) and fiber components of the other I_2, I_3 and I_6 fibers.

This approach has several advantages. First it reveals that $Y_{6,3,2,1}$ admits an elliptic fibration $\pi_{|D_1|}$ with two fibers of type II^*. This comes with multisections of degree 6, given for instance by Θ_5. In consequence, the Jacobian has

$$\mathrm{NS}(\mathrm{Jac}(Y_{6,3,2,1}, \pi_{|D_1|})) \cong U + E_8^2.$$

With this Néron–Severi lattice, $\mathrm{Jac}(Y_{6,3,2,1}, \pi_{|D_1|})$ is sandwiched by the Kummer surface of two elliptic curves by [22]. All of this occurs in the framework of Shioda–Inose structures and shows the following result:

Theorem 5.8. *There are elliptic curves E, E' such that as transcendental $\mathbb{Q}$-Hodge structures,*

$$T(Y_{6,3,2,1}) \cong T(E \times E').$$

Remark 5.9. Theorem 5.8 provides a conceptual way to exhibit explicit K3 surfaces $Y_{6,3,2,1}$ with $\rho(Y_{6,3,2,1}) = 18$, parallelling [5, Sect. 4.7]. Using the involution of base change type from Section 5.1, we obtain explicit, very general members of the family $\mathcal{F}_{4,3,1}$, as opposed to the extraordinary Example 3.12.

As a second application, we return to the computation of $\mathrm{NS}(Y_{6,3,2,1})$. Consider the orthogonal projection inside $\mathrm{NS}(Y_{6,3,2,1})$ with respect to the sublattice E_8^2 specified above. The multisection Θ_5 is taken to a divisor D of square $D'^2 = -60$ orthogonal to E_8^2. It follows that D and a fiber of $\pi_{|D_1|}$ generate the lattice $U(6)$. Thus we obtain

$$\mathrm{NS}(Y_{6,3,2,1}) \cong U(6) + E_8^2. \tag{5.2}$$

(Here, a priori we have checked only the inclusion '$\supseteq$', but equality holds since the discriminants match.) One easily checks that the discriminant forms of the Néron–Severi lattices in Proposition 5.5 and in (5.2) agree. By [14, Cor. 1.13.3], this suffices to prove that the lattices are isometric as required.

Proposition 5.10. *If $Y_{6,3,2,1}$ has $\rho(Y_{6,3,2,1}) = 18$, then $T(Y_{6,3,2,1}) \cong U + U(6)$.*

Proof. This follows directly from (5.2) using [14, Prop. 1.6.1 & Cor. 1.13.3]. □

§5.4. K3-cover for $\mathcal{F}_{3,3,3,3}$

We can carry out similar calculations for the K3 cover Y' of an Enriques surface $S' \in \mathcal{F}_{3,3,3,3}$. Here we only sketch the results.

Generically, Y' comes equipped with an elliptic fibration with eight fibers of type I_3 and an irreducible bisection R' such that $R'^2 = 0$. Thus the argumentation from the proof of Lemma 3.6 applies to modify R' to a divisor D perpendicular to eight disjoint A_2-configurations (supported on the fibers). Generically, we obtain the finite index sublattice

$$U(2) + A_2^8 \hookrightarrow \mathrm{NS}(Y'),$$

which leads to the following analogue of Proposition 5.5.

Proposition 5.11. *Let $S' \in \mathcal{F}_{3,3,3,3}$ such that the K3 cover Y' has $\rho(Y') = 18$. Then*

$$\mathrm{NS}(Y') \cong U(2) + A_2^2 + E_6^2.$$

As before, it follows that the K3 covers of all Enriques surfaces in $\mathcal{F}_{3,3,3,3}$ form an irreducible two-dimensional family. Using the discriminant form, we can compute the transcendental lattice of a very general K3 cover:

Proposition 5.12. *Let $S' \in \mathcal{F}_{3,3,3,3}$ be an Enriques surface such that its K3-cover Y' has $\rho(Y') = 18$. Then*

$$T(Y') \cong U(3) + U(6).$$

Proof. The discriminant group A_{NS} of $\mathrm{NS}(Y')$ has 3-length 4 by Proposition 5.11. Since this length equals the rank of $T(Y')$, we deduce that $T(Y')$ is 3-divisible as an integral even lattice, i.e.,

$$T(Y') = M(3) \quad \text{for some even lattice } M.$$

By Lemma 3.2, M has determinant 4. From Proposition 5.11, we infer the equality of discriminant forms

$$q_M = -q_{U(2)} = q_{U(2)}.$$

Hence $M \cong U + U(2)$ by [14, Prop. 1.6.1 & Cor. 1.13.3]. □

§5.5. Overlap of $\mathcal{F}_{4,3,1}$ and $\mathcal{F}_{3,3,3,3}$

Recall from Proposition 4.4 that the two families of Enriques surfaces $\mathcal{F}_{4,3,1}$ and $\mathcal{F}_{3,3,3,3}$ intersect only on one-dimensional subfamilies. Here we shall give a lattice-theoretic characterization of two infinite series of subfamilies and work out the first case explicitly.

In essence, computing the one-dimensional subfamilies of overlap amounts to calculating even lattices T of signature $(2,1)$ admitting primitive embeddings into both generic transcendental lattices from Propositions 5.10 and 5.12. Then one can enhance the Néron–Severi lattices by a primitive vector perpendicular to T using the gluing data encoded in the discriminant form (see, e.g., [5, Sect. 3]). There are two obvious kinds of candidates for T with $N \in \mathbb{Z}_{>0}$:

$$U(3) + \langle 12N \rangle \hookrightarrow \begin{cases} U(3) + U(2) \cong U + U(6), \\ U(3) + U(6), \end{cases} \tag{5.3}$$

$$U(6) + \langle 6N \rangle \hookrightarrow \begin{cases} U(6) + U, \\ U(6) + U(3). \end{cases} \tag{5.4}$$

Remark 5.13. We point out that (5.3) includes families where the Jacobians of the K3 covers $Y_{3,3,3,3}$ and $Y_{4,3,1}$ overlap. In fact, this happens with transcendental lattices $U(3) + \langle 6M\rangle$, and one can show as in [6, Prop. 4.2] that a K3 surface with this transcendental lattice admits an Enriques involution if and only if M is even. Moreover, the involution turns out to be of base change type, so we can, at least in principle, give a very explicit description of these surfaces.

§5.6. Explicit component of $\mathcal{F}_{3,3,3,3} \cap \mathcal{F}_{4,3,1}$

We conclude this paper by working out the first case of (5.3) explicitly. That is, we aim for K3 surfaces with transcendental lattice

$$T = U(3) + \langle 12\rangle. \tag{5.5}$$

By Remark 5.13, this could be done purely on the level of Jacobians of $Y_{3,3,3,3}$ or $Y_{4,3,1}$, but here we shall rather continue to work with $Y_{6,3,2,1}$. The lattice enhancement raises the rank of the Néron–Severi lattice by 1 while the discriminant changes from -36 to 108. By the theory of Mordell–Weil lattices [21], this can be achieved only by adding a section Q of height 3. Up to adding a torsion section, we may assume Q to be induced by the quadratic twist X' of $X_{6,3,2,1}$ corresponding to the quadratic base change (2.7); i.e., Q comes from a section Q' of height $3/2$ on X'. Note that X' inherits the 2-torsion section from $X_{6,3,2,1}$. At the same time, this will ease the explicit computations and limit the possible configurations for Q'. Indeed, using the height formula from [21] it is easy to see that there are only two possible cases for Q' up to adding the 2-torsion section:

- either Q' meets exactly one I_0^* fiber (at a component not met by the 2-torsion section) and the I_6 fiber (at the component met by the 2-torsion section) non-trivially,
- or it intersects non-trivially exactly one I_0^* fiber (at a component not met by the 2-torsion section), the I_6 fiber (at a component adjacent to the zero component) and the I_3 fiber.

We can compute the Néron–Severi lattice and the transcendental lattice of the resulting covering K3 surfaces by the same means as in Section 5.3: simply compute the rank 3 orthogonal complement of E_8^2 inside NS. We obtain $T = U + \langle 108\rangle$ for the second case and the desired transcendental lattice from (5.5) for the first.

We continue to work out the first case in more detail. Let us assume that the quadratic base change (2.7) ramifies at $a, b \in \mathbb{P}^1$. For ease of computation, we shall use an extended Weierstrass form of X' which locates the 2-torsion section at $(0,0)$:

$$X': \; y^2 = x\big(x^2 + (t-a)(t-b)(t^2/4 + t - 2)x + (t-a)^2(t-b)^2(1-t)\big).$$

Then we can implement the section Q' to have the x-coordinate $c(t-a)$. Solving for this to give a square upon substituting into the extended Weierstrass form leads to

$$a = -\frac{1}{3}\frac{(b+2)^2}{b-4}, \quad c = (b+8)(b-1)^2/27.$$

Thus we obtain explicitly a one-dimensional family of K3 surfaces with transcendental lattice (5.5). Unless the base change degenerates or the ramification points hit the fibers of type I_1 or I_3, i.e., for $b \notin \{-8,-2,0,1,10\}$, the resulting K3 surface Y possesses the Enriques involution ψ of base change type constructed in Section 5.1. By Lemma 5.2, the quotient surface S lies in $\mathcal{F}_{4,3,1}$. We can also verify geometrically that $S \in \mathcal{F}_{3,3,3,3}$. To this end we use that the induced section Q of height 3 on Y meets only the two I_6 fibers non-trivially — in the same component as the 2-torsion section P, i.e., opposite the zero component — and it meets the zero section O in the ramified fiber above $t = b$. Hence Q, O and the identity component of either of the I_2 fibers form a triangle, i.e., they give a divisor D of Kodaira type I_3. Perpendicular to D, we find

- another I_3 formed by the sections P, $(P-Q)$ and the non-identity component of the other I_2 fiber;
- six A_2's contributed from the I_6 and I_3 fibers;
- four sections of the induced elliptic fibration $\pi_{|D|}$ given by the remaining components of the I_6 fibers.

We conclude that $\pi_{|D|}$ is a Jacobian elliptic fibration with eight fibers of type I_3. Hence it comes from $X_{3,3,3,3}$ by some quadratic base change. Finally, one directly verifies that the above rational curves on Y are interchanged by the Enriques involution ψ. Therefore, $\pi_{|D|}$ induces an elliptic fibration with four fibers of type I_3 on $S = Y/\psi$. That is, $S \in \mathcal{F}_{3,3,3,3}$ as claimed.

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