

# Hodge–Tate Conditions for Landau–Ginzburg Models

by

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## Abstract

We give a sufficient condition for a class of tame compactified Landau–Ginzburg models in the sense of Katzarkov–Kontsevich–Pantev to satisfy some versions of their conjectures. We also give examples that satisfy the condition. The relations to the quantum  $\mathcal{D}$ -modules of Fano manifolds and the original conjectures are explained in the appendices.

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## §1. Introduction

Let  $X$  be a smooth projective variety over  $\mathbb{C}$  with a Zariski open subset  $Y$ . We assume that  $D := X \setminus Y$  is a simple normal crossing hypersurface. Let  $f : X \rightarrow \mathbb{P}^1$  be a flat projective morphism such that the restriction  $w := f|_Y$  is a regular function. In general, the meromorphic flat connection  $(\mathcal{O}_X(*D), d + df)$  has irregular singularities along  $D$ . Let  $H_{\mathrm{dR}}^\bullet(Y, w)$  denote the de Rham cohomology group of  $(\mathcal{O}_X(*D), d + df)$ . It has been studied from the viewpoint of generalized Hodge theories. (See twistor  $\mathcal{D}$ -modules [30], [29]; irregular Hodge structures [10], [18],

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[37], [38]; non-commutative Hodge structures [25], [26]; TERP-structures [23] and so on.)

In some cases,  $(Y, \mathbf{w})$  can be considered a “mirror dual” of a smooth projective Fano variety  $F$  called a sigma model. In that case,  $(Y, \mathbf{w})$  is called a Landau–Ginzburg model, and it is predicted that some categories associated to  $(Y, \mathbf{w})$  are equivalent to the corresponding categories associated to  $F$ . This prediction is called a homological mirror symmetry conjecture (HMS). Some parts of HMS are proved in some cases ([1], [2], [43]).

From this point of view, Katzarkov–Kontsevich–Pantev [26] proposed some conjectures as conjectural consequences of HMS. As emphasized in [26], some of their conjectures can be seen as “purely algebro-geometric” conjectures on the generalized Hodge theory of  $H_{\mathrm{dR}}^{\bullet}(Y, \mathbf{w})$ . Such conjectures are the main subjects of this paper.

As an introduction, we survey some versions of the conjectures in Sections 1.1 and 1.2. (The relations to the original ones are explained in Appendix B.) Then we explain our main result in Section 1.3. In this paper, we always assume that the pole divisor  $(f)_{\infty}$  of  $f$  is reduced and the support  $|(f)_{\infty}|$  is equal to  $D$ , although this assumption is more restrictive than that of [26].

### §1.1. Hodge numbers

The cohomology group  $H_{\mathrm{dR}}^{\bullet}(Y, \mathbf{w})$  is given by taking the hypercohomology of the complex  $(\Omega_X^{\bullet}(*D), d + df \wedge)$ . There are  $\mathcal{O}_X$ -coherent subsheaves  $\Omega_f^k$  of  $\Omega_X^k(*D)$  which give a subcomplex  $(\Omega_f^{\bullet}, d + df \wedge)$  (see Section 3.1.1). It is known that the inclusion  $(\Omega_f^{\bullet}, d + df \wedge) \hookrightarrow (\Omega_X^{\bullet}(*D), d + df \wedge)$  is a quasi-isomorphism (see [18, Cor. 1.4.3]). The Hodge number  $f^{p,q}(Y, \mathbf{w})$  is defined by

$$f^{p,q}(Y, \mathbf{w}) := \dim H^q(X, \Omega_f^p).$$

It is proved by Esnault–Sabbah–Yu, Kontsevich and M. Saito [18] that we have  $\dim H^k(Y, \mathbf{w}) = \sum_{p+q=k} f^{p,q}(Y, \mathbf{w})$ , which can be considered a consequence of the  $E_1$ -degeneration property of the “Hodge filtration”.

Take a sufficiently small holomorphic disk  $\Delta$  in  $\mathbb{P}^1$  centered at infinity so that  $Y_b := f^{-1}(b)$  is smooth for any  $b \in \Delta \setminus \{\infty\}$ . It is proved in [26] (see also [11]) that we have the equality

$$\dim H_{\mathrm{dR}}^k(Y, \mathbf{w}) = \dim H^k(Y, Y_b),$$

where  $b \in \Delta \setminus \{\infty\}$ , and  $H^k(Y, Y_b)$  denotes the relative cohomology with  $\mathbb{C}$ -coefficient. In our situation, the monodromy  $T_k$  at infinity is known to be unipotent ([27, Thm. I']). Let  ${}^k W$  be the monodromy weight filtration of  $N_k := \log T_k$  on

$H^k(Y, Y_b)$  centered at  $k$  (see (2.2), (2.3)). The number  $h^{p,q}(Y, \mathbf{w})$  is defined by

$$h^{p,q}(Y, \mathbf{w}) := \dim \operatorname{Gr}_{2^p}^{k^W} H^k(Y, Y_b) \quad (k = p + q).$$

By an HMS consideration, Katzarkov–Kontsevich–Pantev [26] conjectured

$$(1.1) \quad f^{p,q}(Y, \mathbf{w}) = h^{p,q}(Y, \mathbf{w}).$$

It is easy to observe that conjecture (1.1) does not hold if the fiber  $D$  at infinity is smooth and  $f^{p,q}(Y, \mathbf{w})$  are not zero for two different pairs  $(p, q)$  and  $(p', q')$  with  $p + q = p' + q'$ . Actually, such an example is given in [28]. However, also in [28], there are examples of  $(X, f)$  that satisfy (1.1). There remains a question when equality (1.1) holds. The counterexample suggests that we need to impose some conditions on the degeneration property of  $Y_b$  as  $b \rightarrow \infty$ .

### §1.2. Speciality

Let  $(\lambda, \tau)$  be a pair of complex numbers. The dimension of the hypercohomology  $\mathbb{H}^\bullet(X; (\Omega_f^\bullet, \lambda d + \tau df \wedge))$  is known to be independent of the choice of  $(\lambda, \tau)$  ([18], [30]). Let  $\mathbb{C}_\lambda, \mathbb{C}_\tau$  be complex planes with coordinate  $\lambda$  and  $\tau$  respectively. Put  $\mathbb{P}_\lambda^1 := \mathbb{C}_\lambda \cup \{\infty\}$  and  $S := \mathbb{P}_\lambda^1 \times \mathbb{C}_\tau$ . It follows that we have a locally free  $\mathbb{Z}/2\mathbb{Z}$ -graded  $\mathcal{O}_S(*(\lambda)_\infty)$ -module  ${}^bH$  whose fiber at  $(\lambda, \tau)$  is  $\mathbb{H}^\bullet(X; (\Omega_f^\bullet, \lambda d + \tau df \wedge))$ . The  $\mathcal{O}_S(*(\lambda)_\infty)$ -module  ${}^bH$  is equipped with a grade-preserving meromorphic flat connection

$${}^b\nabla : {}^bH \rightarrow {}^bH \otimes_{\mathcal{O}_S} \Omega_S^1(\log \lambda \tau)((\lambda)_0),$$

where  $\Omega_S^1(\log \lambda \tau)((\lambda)_0)$  denotes the  $\mathcal{O}_S$ -module locally generated by  $\lambda^{-1}\tau^{-1} d\tau$  and  $\lambda^{-2} d\lambda$ .

For a smooth projective Fano variety  $F$ , the quantum  $\mathcal{D}$ -module for the quantum parameters  $c_1(F) \log \tau \in H^2(F)$  gives a similar pair  $({}^aH, {}^a\nabla)$ . These pairs are considered as one parameter variation of non-commutative Hodge structures  $({}^AH, {}^A\nabla) := ({}^aH, {}^a\nabla)|_{\tau=1}$ , and  $({}^BH, {}^B\nabla) := ({}^bH, {}^b\nabla)|_{\tau=1}$ . It is conjectured [26, Conj. 3.11] that homological mirror correspondences for a pair  $F \mid (Y, \mathbf{w})$  should induce an isomorphism  $({}^aH, {}^a\nabla) \simeq ({}^bH, {}^b\nabla)$  (more precisely, we need to fix more data to determine the mirror pair).

On the one hand,  $({}^AH, {}^A\nabla)$  has a trivial logarithmic extension to  $\lambda = \infty$ . On the other hand, it is a non-trivial problem to construct a logarithmic extension of  $({}^BH, {}^B\nabla)$  such that the induced vector bundle on  $\mathbb{P}_\lambda^1$  is trivial. The problem is called the Birkhoff problem (see [36] for example), and the solution to the problem for  $({}^BH, {}^B\nabla)$  plays a key role in the construction of primitive forms ([12], [35]).

Katzarkov–Kontsevich–Pantev observed that the trivial solution of the Birkhoff problem for the connection  $({}^AH, {}^A\nabla)$  can be described in terms of the Deligne

canonical extension and the weight filtration for the nilpotent part of the residue endomorphism along  $\{\lambda = \infty\}$ . An extension given in a similar way is called a skewed canonical extension in [26]. The skewed canonical extension can be defined for more general objects including  $({}^B H, {}^B \nabla)$ . The property that the skewed canonical extension gives a solution to the Birkhoff problem is called “speciality” (see [26, Def. 3.21] or Definition 2.13 for details).

From the point of view of the conjecture  $({}^a H, {}^a \nabla) \simeq ({}^b H, {}^b \nabla)$ , they conjectured that  $({}^B H, {}^B \nabla)$  is special ([26, Conj. 3.22(a)]). Combining it with their unobstructedness result on the versal deformation of  $(Y, w)$ , they also conjectured the existence of a version of a primitive form under the assumption that  $\Omega_X^{\dim X}(D)$  is trivial ([26, Conj. 3.22(b)]).

**§1.3. Rescaling structures and Hodge–Tate conditions**

To treat the conjectures in Sections 1.1 and 1.2 simultaneously, we introduce the notion of rescaling structure (see Section 2 for details). Let  $\sigma : \mathbb{C}_\theta^* \times S \rightarrow S$  be the action of  $\mathbb{C}_\theta^*$  defined by  $(\theta, \lambda, \tau) \mapsto (\theta\lambda, \theta\tau)$ . Let  $p_2 : \mathbb{C}_\theta^* \times S \rightarrow S$  denote the projection. A rescaling structure is a triple  $(\mathcal{H}, \nabla, \chi)$  of  $\mathbb{Z}$ -graded locally free  $\mathcal{O}_S(*(\lambda)_\infty)$ -module  $\mathcal{H}$ , a grade-preserving meromorphic flat connection

$$\nabla : \mathcal{H} \rightarrow \mathcal{H} \otimes \Omega_S^1(\log \lambda\tau)((\lambda)_0),$$

and an isomorphism  $\chi : p_2^* \mathcal{H} \xrightarrow{\sim} \sigma^* \mathcal{H}$  with some conditions (see Definition 2.5).

For a rescaling structure  $(\mathcal{H}, \nabla, \chi)$ , take a fiber  $V$  of  $\mathcal{H}$  at  $(\lambda, \tau) = (1, 0)$ . Under an assumption, we associate two filtrations  $F$  and  $W$  on  $V$ , where  $F$  is called a Hodge filtration and  $W$  is called a weight filtration of  $\mathcal{H}$  (Section 2.3). We also define an abstract version of Hodge numbers  $f^{p,q}(\mathcal{H})$  and  $h^{p,q}(\mathcal{H})$ .

The rescaling structure is said to satisfy the Hodge–Tate condition if these two filtration behave like a Hodge filtration and a weight filtration of a mixed Hodge structure of Hodge–Tate type in the sense of Deligne [9] (see Definition 2.11 for details). If  $(\mathcal{H}, \nabla, \chi)$  satisfies the Hodge–Tate condition, we have  $f^{p,q}(\mathcal{H}) = h^{p,q}(\mathcal{H})$ , and we also have that  $\mathcal{H}|_{\tau=1}$  is special.

In Appendix A, we show that a “Tate twisted” version  $\mathcal{H}_F$  of  ${}^a H$  comes equipped with a rescaling structure for any smooth projective Fano variety  $F$ . The rescaling structure  $\mathcal{H}_F$  satisfies the Hodge–Tate condition, and we have

$$f^{p,q}(\mathcal{H}_F) = h^{p,q}(\mathcal{H}_F) = \dim H^q(F, \Omega_F^{n-p}).$$

For the pair  $(X, f)$ , we also have a version  $\mathcal{H}_f$  of  ${}^b H$ , which comes equipped with a rescaling structure (see Section 3; the relation between  $\mathcal{H}_f$  and  ${}^b H$  is given in Appendix B). The main result of this paper is the following:

**Theorem 1.1** (Theorem 3.30). *Let  $\mathcal{H}_f$  be the rescaling structure for  $(X, f)$ .*

- (1) *If  $\mathcal{H}_f$  satisfies the Hodge–Tate condition, then equation (1.1) holds and  $\mathcal{H}_f|_{\tau=1}$  is special.*
- (2) *The rescaling structure  $\mathcal{H}_f$  satisfies the Hodge–Tate condition if and only if the mixed Hodge structure  $(H^k(Y, Y_\infty; \mathbb{Q}), F, W)$  is Hodge–Tate for every  $k \in \mathbb{Z}$ .*

The definition of the mixed Hodge structure  $(H^k(Y, Y_\infty; \mathbb{Q}), F, W)$  is given in Section 3.4.3. In Section 4, we also give some examples such that  $\mathcal{H}_f$  satisfies the Hodge–Tate condition in the case where the dimension of  $X$  is 2 or 3.

## §2. Rescaling structures

### §2.1. Holomorphic extensions and filtrations

Let  $\mathbb{C}$  denote a complex plane. Set  $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ . Let  $H$  be a finitely generated locally free  $\mathcal{O}_{\mathbb{C}}(*\{0\})$ -module. Let  $V$  denote the fiber of  $H$  at  $1 \in \mathbb{C}$ . Assume that we are given an increasing filtration  $G_\bullet V = (G_m V \mid m \in \mathbb{Z})$  on  $V$  such that

$$(2.1) \quad G_m V := \begin{cases} 0 & (m \ll 0), \\ V & (m \gg 0). \end{cases}$$

We shall recall some methods to construct an extension of  $H$  to an  $\mathcal{O}_{\mathbb{C}}$ -module by using  $G_\bullet V$ . Here, by an extension of  $H$ , we mean a locally free  $\mathcal{O}_{\mathbb{C}}$ -submodule  $L$  of  $H$  such that  $L \otimes_{\mathcal{O}_{\mathbb{C}}}(*\{0\}) = H$ .

**2.1.1. Construction using  $\mathbb{C}^*$ -actions.** Let  $m : \mathbb{C}^* \times \mathbb{C}^* \rightarrow \mathbb{C}^*$  and  $\sigma : \mathbb{C}^* \times \mathbb{C} \rightarrow \mathbb{C}$  denote the multiplications. Let  $p_2 : \mathbb{C}^* \times \mathbb{C} \rightarrow \mathbb{C}$  be the projection. Assume that  $H$  is  $\mathbb{C}^*$ -equivariant with respect to  $\sigma$ . Namely, we have an isomorphism  $\chi : p_2^* H \xrightarrow{\sim} \sigma^* H$  with the cocycle condition:

$$(m \times \text{id}_{\mathbb{C}})^* \chi = (\text{id}_{\mathbb{C}^*} \times \sigma)^* \chi \circ p_{23}^* \chi,$$

where  $p_{23} : \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C} \rightarrow \mathbb{C}^* \times \mathbb{C}$  is given by  $p_{23}(t_1, t_2, z) := (t_2, z)$ . This case is considered in [41, Lem. 19] for example. For any vector  $v \in V$ , there is a unique invariant section  $\phi_v \in \Gamma(\mathbb{C}, H)$  with  $\phi_v(1) = v$ . There exists a unique extension  $L_1$  such that  $v \in G_m V$  if and only if  $\phi_v \in L_1(m\{0\})$ . The extension  $L_1$  is isomorphic to the extension  $\sum_m G_m V \otimes \mathcal{O}_{\mathbb{C}}(-m\{0\})$  of  $V \otimes \mathcal{O}(*\{0\})$ . This construction gives a one-to-one correspondence between the sets of increasing filtrations on  $V$  with (2.1) and  $\mathbb{C}^*$ -equivariant holomorphic extensions of  $H$ .

**Example 2.1.** Let  $V$  be a finite-dimensional  $\mathbb{C}$ -vector space with a decomposition  $V = \bigoplus_{p \in \mathbb{Z}} V_p$ . Put  $H := \mathcal{O}_{\mathbb{C}}(*\{0\}) \otimes_{\mathbb{C}} V$ . Note that  $p_2^* H \simeq \mathcal{O}_{\mathbb{C}^* \times \mathbb{C}}(*\mathbb{C}^* \times \{0\}) \otimes V \simeq$

$\sigma^*H$ . Define  $\chi : p_2^*H \xrightarrow{\sim} \sigma^*H$  by  $\chi|_{\mathcal{O}_{\mathbb{C}^* \times \mathbb{C}}(*\mathbb{C}^* \times \{0\}) \otimes V_p}(t, z) := t^p \otimes \text{id}_{V_p}$ . Consider  $V$  as the fiber of  $H$  at  $1 \in \mathbb{C}$ . Then the trivial extension  $L_1 := \mathcal{O}_{\mathbb{C}} \otimes V$  corresponds to the filtration

$$G_m V = \bigoplus_{-p \leq m} V_p.$$

Indeed, for  $v \in V_p$ , the invariant section  $\phi_v$  is given by  $\phi_v(z) = z^p v \in L_1(-p\{0\})$ .

**2.1.2. Double complex.** Let  $(C^{\bullet, \bullet}, \delta_1, \delta_2)$  be a double complex of  $\mathbb{C}$ -vector spaces where  $\delta_1 : C^{p,q} \rightarrow C^{p+1,q}$  and  $\delta_2 : C^{p,q} \rightarrow C^{p,q+1}$  are the differentials. We assume that  $C^{p,q} = 0$  if  $p < 0$  or  $q < 0$ , and that the total complex  $(C^{\bullet}, \delta)$  has finite-dimensional cohomology. Here, we put  $C^\ell := \bigoplus_{p+q=\ell} C^{p,q}$  and  $\delta := \delta_1 + \delta_2$ . Let  $F$  be the filtration on  $(C^{\bullet}, \delta)$  given by  $F_m C^\ell := \bigoplus_{p+q=\ell, -p \leq m} C^{p,q}$ . We also assume that the morphisms  $H^k(F_m(C^{\bullet}, \delta)) \rightarrow H^k(C^{\bullet}, \delta)$  are injective for all  $k$  and  $m$ .

Put  $\mathcal{C}^{p,q} := \mathcal{O}_{\mathbb{C}} \otimes C^{p,q}$  and  $\mathcal{C}^\ell := \bigoplus_{p+q=\ell} \mathcal{C}^{p,q}$ . We have a complex  $(\mathcal{C}^{\bullet}, z\delta_1 + \delta_2)$ . Let  $L_1$  be the  $k$ th cohomology group of this complex. By the assumption,  $L_1$  is a finitely generated locally free  $\mathcal{O}_{\mathbb{C}}$ -module. Put  $H := L_1 \otimes \mathcal{O}_{\mathbb{C}}(*\{0\})$  and consider  $L_1$  as an extension of  $H$ . Define  $\chi_p : p_2^* \mathcal{C}^{p,q} \xrightarrow{\sim} \sigma^* \mathcal{C}^{p,q}$  by  $\chi_p(t, z) := t^p \otimes \text{id}$ . This induces an isomorphism  $\chi : p_2^* H \xrightarrow{\sim} \sigma^* H$  with the cocycle condition.

**Lemma 2.2.** *Consider the  $k$ th cohomology  $H^k(C^{\bullet}, \delta)$  as the fiber of  $H$  at  $1 \in \mathbb{C}$ . Then the extension  $L_1$  corresponds to the filtration*

$$G_m H^k(C^{\bullet}, \delta) := \text{Im}(H^k(F_m(C^{\bullet}, \delta)) \rightarrow H^k(C^{\bullet}, \delta)).$$

*Proof.* Put  $F_m \mathcal{C}^k := \bigoplus_{p+q=k, p \geq -m} \mathcal{C}^{p,q}$ . It induces a filtration on the complex  $(\mathcal{C}^{\bullet}, z\delta_1 + \delta_2)$ , which is also denoted by  $F$ . The induced filtration on  $L_1$  is also denoted by  $F$ . By the assumption, we have  $\text{Gr}_\ell^F L_1 \simeq H^k(\text{Gr}_\ell^F(C^{\bullet}))$ . Hence it reduces to the case where there exists a  $p_0 \in \mathbb{Z}$  such that  $\mathcal{C}^{p,q} = 0$  for  $p \neq p_0$ . In this case, we have  $L_1 \simeq H^{k-p_0}(C^{p_0, \bullet}, \delta_2) \otimes \mathcal{O}_{\mathbb{C}}$ , and we obtain the conclusion by Example 2.1. □

**2.1.3. Construction using flat connections with regular singularities.**

Assume that  $H$  is equipped with a flat connection  $\nabla$  with a regular singularity at  $\{0\}$ . We also assume that each  $G_k V$  is invariant with respect to the monodromy of  $\nabla$ . This case is considered in [25], [26], [36] for example. We have the flat subbundles  $G_\bullet H$  on  $H$  such that the fiber of  $G_k H$  at 1 is  $G_k V$ . For any  $t \in \mathbb{C}^*$ , let  $V_t$  be the fiber of  $H$  at  $t$ . Let  $G_\bullet V_t$  denote the induced filtration on  $V_t$ . Set  $I_t := \{st \mid 0 < s \leq 1\}$ . For any vector  $v \in V_t$ , we have the flat section  $\psi_{v,t} \in \Gamma(I_t, H)$  with  $\psi_{v,t}(t) = v$ . There exists a unique logarithmic lattice  $L_2$  with the following property: Fix a frame of  $L_2$  near 0, and let  $\|*\|_{L_2}$  be the Hermitian

metric on  $L_2$  near 0 so that the frame is a orthogonal with respect to  $\|*\|_{L_2}$ . A vector  $v \in V_t$  is contained in  $G_m V_t$  if and only if  $\psi_{v,t}$  satisfies

$$\|\psi_{v,t}(r \cdot t)\|_{L_2} \leq C|r|^{-m}(-\log r)^N \quad (0 < r \ll 1)$$

for some positive constants  $C$  and  $N$ . This construction also gives a one-to-one correspondence between the logarithmic extension of  $H$  and monodromy-invariant filtrations on  $V$  with (2.1).

**2.1.4. Characterization by using the Deligne lattice.** The extension  $L_2$  can be characterized by using the Deligne lattice of  $(H, \nabla)$ . Let  $L'$  be the Deligne lattice of  $(H, \nabla)$ , which means that  $L'$  is the logarithmic at 0 and the residue with eigenvalues whose real parts are contained in  $(-1, 0]$ . The flat subbundles  $G_m H$  extend to  $\{0\}$  and give subbundles of  $L'$ . Let  $G_m L'$  denote the subbundles of  $L'$ .

**Lemma 2.3** ([26, Sect. 3.3.1]). *The extension  $L_2$  is given by*

$$L_2 = \sum_{m \in \mathbb{Z}} G_m L'(-m\{0\})$$

as a submodule of  $L'(*\{0\})$ .

*Proof.* It is enough to show that  $L_2 = L'$  if  $G_\bullet V$  is given by  $G_{-1}V = 0$  and  $G_0V = V$ . Let  $\text{rk } L'$  be the rank of  $L'$ . We have an isomorphism of logarithmic connections  $(L', \nabla) \simeq (\mathcal{O}_{\mathbb{C}}^{\oplus \text{rk } L'}, \nabla')$ , where  $\nabla' = d - \mathcal{U}t^{-1} dt$  for a matrix  $\mathcal{U} \in \text{End}(\mathbb{C}^{\oplus \text{rk } L'})$  with eigenvalues whose real parts are contained in  $[0, 1)$ . Take the standard frame  $v_1, \dots, v_{\text{rk } L'}$  of  $\mathcal{O}_{\mathbb{C}}^{\oplus \text{rk } L'}$ . It induces a Hermitian metric  $\|*\|_{L'}$ . For fixed  $t \in \mathbb{C}^*$ , take  $\alpha \in \mathbb{C}$  with  $\exp \alpha = t$ . We have the flat section  $\psi_i(r \cdot t) := \exp(\alpha \log r \mathcal{U})v_i(rt)$  on  $I_t$  for all  $i = 1, \dots, \text{rk } L'$ . Since the flat sections on  $I_t$  are  $\mathbb{C}$ -linear combinations of  $\psi_i$ , we obtain the conclusion.  $\square$

**2.1.5. Relation between two constructions.** Assume that  $H$  is  $\mathbb{C}^*$ -equivariant and equipped with a flat connection  $\nabla$ . We also assume the compatibility of the action and flat connection. In other words, for all  $t \in \mathbb{C}^*$ , the action of  $t$  on  $H$  is assumed to be equal to the parallel transport of  $\nabla$ . Then we have the following:

**Lemma 2.4.** *The connection  $\nabla$  is regular singular at  $\{0\}$ . The extensions  $L_1$  and  $L_2$  constructed in Section 2.1.1 and Section 2.1.3 coincide.*

*Proof.* By the compatibility of the action and the connection, the invariant section  $\phi_v$  for  $v \in V$  is  $\nabla$ -flat. Since  $L_2$  is generated by  $t^m \phi_v$  ( $v \in G_m V$ ,  $t$  is a coordinate on  $\mathbb{C}$ ), it gives a logarithmic extension of  $H$ . This shows that the connection  $\nabla$  is regular singular at  $\{0\}$ . Fix a trivialization of  $L_1$  around  $\{0\}$  and let  $\|*\|_{L_1}$  be the

induced Hermitian metric on  $L_1$  around  $\{0\}$ . For  $v \in G_m V$ ,  $\phi_v$  is in  $L_1(m\{0\})$ , which implies

$$\|\phi_v(t)\|_{L_1} \leq C|t|^{-m} \quad (t \in \mathbb{C}^*)$$

for some positive constant  $C$ . This shows the conclusion:  $L_1 = L_2$ . □

**§2.2. Definition of rescaling structure**

Let  $\mathbb{C}_\lambda, \mathbb{C}_\tau$  be complex planes with coordinate  $\lambda$  and  $\tau$  respectively. Put  $\mathbb{P}_\lambda^1 := \mathbb{C}_\lambda \cup \{\infty\}$  and  $S := \mathbb{P}_\lambda^1 \times \mathbb{C}_\tau$ . Let  $\sigma : \mathbb{C}_\theta^* \times S \rightarrow S$  be the action of  $\mathbb{C}_\theta^*$  defined by  $\sigma(\theta, \lambda, \tau) := (\theta\lambda, \theta\tau)$ . For a meromorphic function  $h$  on a variety,  $(h)_0$  and  $(h)_\infty$  denote the zero divisor of  $h$  and the pole divisor of  $h$ , respectively. The supports of these divisors are denoted by  $|(h)_0|$  and  $|(h)_\infty|$ , respectively. Let  $p_2 : \mathbb{C}_\theta^* \times S \rightarrow S$  be the projection. We define the notion of rescaling structure as follows.

**Definition 2.5.** A rescaling structure is a triple  $(\mathcal{H}, \nabla, \chi)$  of a  $\mathbb{Z}$ -graded locally free  $\mathcal{O}_S(*(\lambda)_\infty)$ -module  $\mathcal{H}$ , a grade-preserving meromorphic flat connection

$$\nabla : \mathcal{H} \rightarrow \mathcal{H} \otimes \Omega_S^1(*(|(\lambda)_\infty| \cup |(\lambda\tau)_0|))$$

and a grade-preserving isomorphism  $\chi : p_2^* \mathcal{H} \xrightarrow{\sim} \sigma^* \mathcal{H}$  with the following properties:

- (1) We have  $\nabla_{\lambda\tau\partial_\tau}(\mathcal{H}) \subset \mathcal{H}$  and  $\nabla_{\lambda^2\partial_\lambda} \mathcal{H} \subset \mathcal{H}$ .
- (2) On  $\mathbb{C}_\lambda^* \times \mathbb{C}_\tau^*$ ,  $\chi$  is flat with respect to  $p_2^* \nabla$  and  $\sigma^* \nabla$ .
- (3) The isomorphism  $\chi$  satisfies the cocycle condition. In other words, we have

$$(m \times \text{id}_S)^* \chi = (\text{id}_{\mathbb{C}_\theta^*} \times \sigma)^* \chi \circ p_{23}^* \chi,$$

where  $m : \mathbb{C}_\theta^* \times \mathbb{C}_\theta^* \rightarrow \mathbb{C}_\theta^*$  denotes the multiplication and  $p_{23} : \mathbb{C}_\theta^* \times \mathbb{C}_\theta^* \times S \rightarrow \mathbb{C}_\theta^* \times S$  denotes the projection given by  $p_{23}(\theta_1, \theta_2, (\lambda, \tau)) = (\theta_2, (\lambda, \tau))$ .

We often omit  $\nabla$  and  $\chi$  if there is no confusion. The  $k$ th graded piece of  $\mathcal{H}$  is denoted by  $\mathcal{H}^k$ . We assume  $\sum_k \text{rank } \mathcal{H}^k < \infty$  in this paper.

We note that we introduce the notion of rescaling structure only for convenience for the later use. Similar structures have been studied in [23], [26], [30], [37], [38] for example. Operations acting on  $\mathcal{H}$  are often assumed to preserve the grading without mention. If  $\mathcal{H}$  and  $\mathcal{H}'$  are rescaling structures, we can naturally define the tensor product  $\mathcal{H} \otimes \mathcal{H}'$  which is also a rescaling structure. The dual  $\mathcal{H}^\vee$  can also be defined canonically.

**Example 2.6.** Set  $\mathbb{T} := \mathcal{O}_S(*(\lambda)_\infty)v$  where  $v$  is a global section, and  $\text{deg } v = 2$ . The connection  $\nabla$  is defined by  $\nabla v := -v\lambda^{-1} d\lambda$ . The isomorphism  $\chi : p_2^* \mathbb{T} \xrightarrow{\sim} \sigma^* \mathbb{T}$



is given by  $\chi(p_2^*v) := \theta\sigma^*v$ . Then the tuple  $\mathbb{T}(-1) := (\mathbb{T}, \nabla, \chi)$  is a rescaling structure. We define

$$\mathbb{T}(-k) := \begin{cases} \mathbb{T}(-1)^{\otimes k} & \text{if } k \in \mathbb{Z}_{\geq 0}, \\ (\mathbb{T}(-1)^\vee)^{\otimes -k} & \text{if } k \in \mathbb{Z}_{< 0}. \end{cases}$$

For a rescaling structure  $\mathcal{H}$ , we define  $\mathcal{H}(k) := \mathcal{H} \otimes \mathbb{T}(k)$ .

**§2.3. Hodge numbers and Hodge–Tate condition for rescaling structures**

**2.3.1. Hodge filtrations for rescaling structures.** Let us consider the restriction  $\mathcal{H}|_{\tau=0} := \mathcal{H}/\tau\mathcal{H}$ . It admits a  $\mathbb{C}_\theta^*$  action, and hence we can apply the correspondence of Section 2.1.1 to get the filtration  $F_\bullet V$  on  $V := \mathcal{H}|_{\lambda=1, \tau=0}$  corresponding to the lattice at  $\lambda = 0$ .

**Definition 2.7.** Let  $(\mathcal{H}, \nabla, \chi)$  be a rescaling structure. Then we define

$$f^{p,q}(\mathcal{H}) := \dim \text{Gr}_{-p}^F V^{p+q},$$

where  $V^k$  is the  $k$ th graded part of  $V$ .

**2.3.2. Weight filtrations for nilpotent rescaling structures.** We consider the following condition on rescaling structures:

**Definition 2.8.** A rescaling structure  $(\mathcal{H}, \nabla, \chi)$  is called nilpotent if the residue endomorphism  $\text{Res}_{\{\tau=0\}} \nabla$  on  $\mathcal{H}|_{\tau=0}$  is nilpotent.

By definition, we have the following:

**Lemma 2.9.** *We have that  $\mathcal{H}(*(\lambda)_0)$  is the Deligne lattice of the meromorphic connection  $\mathcal{H}(*(\lambda\tau)_0)$  along the divisor  $|(\tau)_0|$ .*

We have a nilpotent endomorphism  $N := (\text{Res}_{\{\tau=0\}} \nabla)|_{\lambda=1}$  on  $V$ , where  $V$  is the fiber of  $\mathcal{H}$  at  $(\lambda, \tau) = (1, 0)$ . Let  $V^k$  be the fiber of  $\mathcal{H}^k$  at  $(\lambda, \tau) = (1, 0)$ . The graded piece of  $N$  on  $V^k$  is denoted by  $N_k$ . Let  ${}^k W$  denote the weight filtration of  $N_k$  centered at  $k$ , i.e.,  ${}^k W$  is the unique filtration on  $V^k$  with the following properties:

$$(2.2) \quad N_k({}^k W_i) \subset {}^k W_{i-2} \quad \text{for all } i \in \mathbb{Z},$$

$$(2.3) \quad N_k^j : \text{Gr}_{k+j}^{{}^k W} V^k \xrightarrow{\sim} \text{Gr}_{k-j}^{{}^k W} V^k \quad \text{for all } j \in \mathbb{Z}.$$

The induced filtration on  $V$  is simply denoted by  $W$ .

**Definition 2.10.** Let  $(\mathcal{H}, \nabla, \chi)$  be a nilpotent rescaling structure. We define

$$h^{p,q}(\mathcal{H}) := \dim \text{Gr}_{2p}^W V^{p+q}.$$

**2.3.3. Hodge–Tate condition.** In [9], a mixed ( $\mathbb{Q}$ -)Hodge structure  $(V_{\mathbb{Q}}, F, W)$  is called Hodge–Tate if the Hodge filtration  $F$  on  $V := V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$  and the weight filtration  $W$  satisfy

$$(2.4) \quad W_{2i+1} = W_{2i} \quad \text{for all } i \in \mathbb{Z},$$

$$(2.5) \quad F_{-j} \oplus W_{2j+2} \xrightarrow{\sim} V \quad \text{for all } j \in \mathbb{Z}.$$

We use the same notation in this paper. Imitating this notion, we define the following:

**Definition 2.11.** Let  $(\mathcal{H}, \nabla, \chi)$  be a nilpotent rescaling structure. Let  $F$  and  $W$  be the filtrations on  $V := \mathcal{H}|_{(\lambda, \tau)=(1,0)}$  defined in Sections 2.3.1 and 2.3.2. Then  $(\mathcal{H}, \nabla, \chi)$  is said to satisfy the Hodge–Tate condition if  $(V, F, W)$  satisfies (2.4) and (2.5). A rescaling structure is said to be of Hodge–Tate type if it satisfies the Hodge–Tate condition.

The following is trivial by definition:

**Lemma 2.12.** *If a rescaling structure  $(\mathcal{H}, \nabla, \chi)$  satisfies the Hodge–Tate condition, then  $f^{p,q}(\mathcal{H}) = h^{p,q}(\mathcal{H})$  for all  $p, q$ .*

**§2.4. Hodge–Tate condition implies the speciality**

Let  $H = \bigoplus_k H^k$  be a  $\mathbb{Z}$ -graded finitely generated locally free  $\mathcal{O}_{\mathbb{P}^1_\lambda}(*\infty)$  module with a grade-preserving meromorphic flat connection  $\nabla$ . We assume that  $\nabla$  has singularity at most at  $\{\lambda = 0\}$  in  $\mathbb{C}_\lambda$  and  $\nabla_{\lambda^2 \partial_\lambda}(H) \subset H$ . We also assume that  $\nabla$  is regular singular at infinity. Take the Deligne lattice  $U_0 H$  at  $\lambda = \infty$ . Let  $N$  be the nilpotent part of  $\text{Res}_{\{\lambda=\infty\}} \nabla$ . Define  ${}^k W_\bullet(U_0 H|_{\lambda=\infty})$  as the weight filtration of  $N$  centered at  $k$ . It induces a filtration  $W_\bullet(U_0 H|_{\lambda \neq 0})$  of  $\mathbb{Z}$ -graded logarithmic subbundles of  $U_0 H|_{\lambda \neq 0}$ .

**Definition 2.13** ([26, Def. 3.21]). Let  $H, \nabla, U_0 H$  and  $W_\bullet(U_0 H|_{\lambda \neq 0})$  be as above. We define a vector bundle  $\hat{H}$  on  $\mathbb{P}^1_\lambda$  by

$$\hat{H}|_{\lambda \neq 0} := \text{Im} \left\{ \bigoplus_{\ell} W_{2\ell}(U_0 H) \otimes \mathcal{O}_{\mathbb{P}^1_\lambda}(-\ell \cdot \infty) \rightarrow U_0 H(*\infty) \right\},$$

and  $\hat{H}|_{\lambda \neq \infty} := H$ . We call  $\hat{H}$  a skewed canonical extension of  $H$ . The  $\mathbb{Z}$ -graded flat bundle  $(H, \nabla)$  is called special if  $\hat{H}$  is isomorphic to a trivial bundle over  $\mathbb{P}^1_\lambda$ .

*Remark 2.14.* Our definition of speciality is slightly different to that of [26]. This construction of  $\hat{H}$  is the same as in Section 2.1.3 if we take the filtration  $G_\bullet V$  to be  $G_\ell := W_{2\ell}$ .

**Proposition 2.15.** *Let  $(\mathcal{H}, \nabla, \chi)$  be a rescaling structure of Hodge–Tate type. Then  $H_1 := \mathcal{H}|_{\tau=1}$  is special.*

The rest of this section is devoted to proving this proposition.

**2.4.1. Regular singularity along  $|(\lambda)_\infty|$ .** Let  $(\mathcal{H}, \nabla, \chi)$  be a rescaling structure. Put  $S^* := \mathbb{C}_\lambda^* \times \mathbb{C}_\tau^* \subset S$ . Let  $\iota : S^* \hookrightarrow \mathbb{C}_\theta^* \times S$  be the embedding given by  $\iota(\lambda, \tau) := (\lambda^{-1}\tau^{-1}, \lambda, \tau)$ . We observe that  $\iota_\sigma := \sigma \circ \iota$  gives  $\iota_\sigma(\lambda, \tau) = (\tau^{-1}, \lambda^{-1})$  and  $\iota_p := p_2 \circ \iota$  is the inclusion  $S^* \hookrightarrow S$ . Hence we have the isomorphism

$$(2.6) \quad \iota^* \chi : \iota_p^* \mathcal{H} = \mathcal{H}|_{S^*} \xrightarrow{\sim} \iota_\sigma^* \mathcal{H}.$$

We also remark that  $\iota_\sigma$  extends to the map  $S \setminus |(\lambda)_0| \rightarrow S$  given by  $(\lambda, \tau) \mapsto (\tau^{-1}, \lambda^{-1})$ , which is denoted by  $\bar{\iota}_\sigma$ .

**Lemma 2.16.** *The meromorphic connection  $(\mathcal{H}(*(\lambda\tau)_0), \nabla)$  is regular singular along  $|(\lambda)_\infty|$ .*

*Proof.* The isomorphism (2.6) gives a logarithmic extension  $\tilde{\mathcal{H}}$  of  $\mathcal{H}|_{\tau \neq 0}$  along  $|(\lambda)_\infty|$ . The pull-back  $\bar{\iota}_\sigma^* \tilde{\mathcal{H}}$  is isomorphic to  $\mathcal{H}|_{S \setminus |(\lambda\tau)_0|}$ .  $\square$

**2.4.2. Deligne lattice.** Since  $\mathcal{H}(*(\lambda\tau)_0)$  is regular singular along  $|(\lambda)_\infty| \cup |(\tau)_0| \subset S$ , we have the Deligne lattice  $U_0 \mathcal{H}$  of  $\mathcal{H}(*(\lambda\tau)_0)$  along  $|(\lambda)_\infty| \cup |(\tau)_0|$ . Assume that  $\mathcal{H}$  is nilpotent. Then  $U_0 \mathcal{H}|_{\tau=0}$  is equal to  $\mathcal{H}(*(\lambda)_0)|_{\tau=0}$  by Lemma 2.9. In particular, we have  $U_0 \mathcal{H}|_{(\lambda, \tau)=(1,0)} = V$ . By (2.6), we have that the residue endomorphism  $N := \text{Res}_{\lambda=\infty} \nabla$  on  $U_0 \mathcal{H}|_{\lambda=\infty}$  is nilpotent. We have the weight filtration  ${}^k W$  on degree  $k$  part of  $U_0 \mathcal{H}|_{\lambda=\infty}$  with respect to  $N$  centered at  $k$ . Let  $W$  be the resulting filtration on  $U_0 \mathcal{H}|_{\lambda=\infty}$ . Then we have logarithmic  $\mathcal{O}_S(*(\lambda)_0)$ -submodules  $W_\bullet(U_0 \mathcal{H})$  of  $\mathcal{H}(*(\lambda)_0)$  which coincide with  $W_\bullet U_0 \mathcal{H}|_{\lambda=\infty}$  on  $\lambda = \infty$ . Define  $\hat{\mathcal{H}}$  by

$$(2.7) \quad \hat{\mathcal{H}}|_{\lambda \neq 0} := \text{Im} \left\{ \bigoplus_{\ell} W_{2\ell}(U_0 \mathcal{H}) \otimes \mathcal{O}_S(-\ell(\lambda)_\infty) \rightarrow \mathcal{H}(*(\lambda)_\infty)|_{\lambda \neq 0} \right\},$$

and  $\hat{\mathcal{H}}|_{\lambda \neq \infty} = \mathcal{H}$ . It is easy to see that  $\hat{\mathcal{H}}|_{\tau=1}$  is  $\hat{H}_1$ .

**Lemma 2.17.** *The filtration on  $V$  induced by  $W_\bullet U_0 \mathcal{H}$  is equal to the weight filtration given in Section 2.3.2.*

*Proof.* Let  $T_1$  be the monodromy around  $\{\lambda = \infty\}$  acting on  $V' := \mathcal{H}|_{(\lambda, \tau)=(1,1)}$ . Let  $T_2$  be the monodromy around  $\{\tau = 0\}$  acting on  $V'$ . By the  $\mathbb{C}^*$ -equivariance of  $\mathcal{H}$  (or by (2.6)),  $N^{(i)} := \log T_i$  ( $i = 1, 2$ ) coincide with each other (both of them are nilpotent). We have a trivialization  $(U_0 \mathcal{H}, \nabla) \simeq (V' \otimes \mathcal{O}_S(*(\lambda)_0), \nabla')$ , where  $\nabla' = d - N^{(1)} \lambda^{-1} d\lambda + N^{(2)} \tau^{-1} d\tau$ . Identify  $V$  and  $V'$  via this isomorphism. Then

the filtration induced by  $W_\bullet U_0 \mathcal{H}$  corresponds to the filtration induced by  $N^{(1)}$ , and the filtration given in Section 2.3.2 corresponds to the filtration induced by  $N^{(2)}$ . Since  $N^{(1)} = N^{(2)}$ , these filtrations are equal.  $\square$

**2.4.3. Proof of the Proposition 2.15.** Put  $\hat{H}_0 := \hat{\mathcal{H}}|_{\tau=0}$ . By Lemmas 2.17 and 2.4,  $\hat{H}_{0|\lambda \neq 0}$  is given by construction in Section 2.1.1 taking  $G_\ell = W_{2\ell}$  ( $\ell \in \mathbb{Z}$ ). Then the Hodge–Tate condition implies the triviality of  $\hat{H}_0$ . By the rigidity of triviality of vector bundles on  $\mathbb{P}^1$ , there is an open neighborhood  $U$  in  $\mathbb{C}_\tau$  such that the restriction  $\hat{\mathcal{H}}|_{\mathbb{P}^1_\lambda \times U}$  is trivial along  $\mathbb{P}^1_\lambda$ . Using the  $\mathbb{C}_\theta^*$ -action, we can show that  $\hat{\mathcal{H}}$  itself is trivial along  $\mathbb{P}^1_\lambda$ . In particular,  $\hat{H}_1$  is trivial.  $\square$

**2.4.4. Relation to M. Saito’s criterion.** The referee of this paper indicated the relation between Proposition 2.15 and M. Saito’s criterion for Birkhoff’s problem. To see this, we recall M. Saito’s criterion in a special case. Let  $H, \nabla$  and  $U_0 H$  be as in Section 2.4. For simplicity, we assume that  $\text{Res}_{\lambda=\infty} \nabla$  is nilpotent. Let  $V_\infty$  denote the fiber of  $U_0 H$  at  $\lambda = \infty$ . Note that the residue  $N$  acts on  $V_\infty$ . We define a filtration  $F$  on  $V_\infty$  as

$$F_k V_\infty := \text{Im} \left( \Gamma(\mathbb{P}^1_\lambda, U_0 H \otimes \mathcal{O}_{\mathbb{P}^1_\lambda}(k\{0\})) \rightarrow V_\infty \right)$$

where the map is the restriction.

**Theorem 2.18** ([40, Lem. 2.8], [36, IV 5.b]). *Assume that we have an increasing filtration  $G_\bullet V_\infty$  such that*

- (1)  $G_k V_\infty$  is invariant under the morphism  $N$  for each  $k \in \mathbb{Z}$ ;  $N(G_k V_\infty) \subset G_k V_\infty$  and
- (2)  $G_\bullet V_\infty$  is opposed to  $F_\bullet V_\infty$ ;  $F_{-p} \oplus G_{p+1} = V_\infty$  for each  $p \in \mathbb{Z}$ .

*Then the extension defined by replacing  $W_{2\ell}$  by  $G_\ell$  in Definition 2.13 is logarithmic at infinity and isomorphic to trivial  $\mathcal{O}_{\mathbb{P}^1_\lambda}$ -module.*

Let us consider the case  $H = H_1 = \mathcal{H}|_{\tau=1}$  where  $\mathcal{H}$  is a nilpotent rescaling structure. We first observe that  $\mathcal{H}$  is reconstructed from its restriction  $H_1$  as follows. Let  $\varpi : \mathbb{P}^1_\lambda \times \mathbb{C}_\tau^* \rightarrow \mathbb{P}^1_\lambda$  be the map defined by  $\varpi(\lambda, \tau) := (\lambda/\tau)$ . Let  $\iota_\tau : \mathbb{P}^1_\lambda \times \mathbb{C}_\tau^* \rightarrow \mathbb{C}_\theta^* \times S$  be the map defined by  $\iota_\tau(\lambda, \tau) = (\tau^{-1}, \lambda, \tau)$ . Then by taking the pull-back of the morphism  $\chi$  by  $\iota_\tau$ , similarly to (2.6), we can identify  $\mathcal{H}_{\mathbb{P}^1_\lambda \times \mathbb{C}_\tau^*}$  with  $\varpi^*(\mathcal{H}|_{\tau=1})$ . Hence, by Lemma 2.9, we obtain  $\mathcal{H}$  by taking the Deligne lattice of  $\varpi^*(\mathcal{H}|_{\tau=1})$  along  $\{\tau = 0\}$ .

This identification also gives an isomorphism  $\chi_V : V \xrightarrow{\sim} V_\infty$  and we have the following:

**Lemma 2.19.** *The isomorphism  $\chi_V : (V, F) \rightarrow (V_\infty, F)$  is a filtered isomorphism.*

*Proof.* For  $\bar{v} \in F_k V_\infty$ , take a lift  $v \in \Gamma(\mathbb{P}^1_\lambda, U_0 H_1(k\{0\}))$ . Then we have a unique  $\mathbb{C}_\theta^*$ -equivariant section  $\tilde{v} \in \Gamma(S, \mathcal{H}(k(\lambda)_0))$  whose restriction to  $\tau = 1$  is  $v$ . We have that the restriction of  $\tilde{v}$  to  $\tau = 0$  is the  $\mathbb{C}^*$ -invariant section with  $(\tilde{v}|_{\tau=0})(1) = \tilde{v}(1, 0) = \chi_V^{-1}(\bar{v})$ , and  $\tilde{v}|_{\tau=0} \in \mathcal{H}|_{\tau=0}(k\{0\})$ . This proves the lemma.  $\square$

By this lemma, Proposition 2.15 can be seen as a corollary of M. Saito’s criterion (Theorem 2.18). We also remark that the relation to the Hodge–Tate condition is mentioned in [34, Exa. 3.4.3] for classical Hodge structures.

*Remark 2.20.* From these observations, it seems that the parameter  $\tau$  plays a minor role. However, this parameter naturally appears in some examples [19], [26], [30], [37]. In particular, as we will see in Appendix A, the parameter  $\tau$  appears as a quantum parameter for Tate twisted quantum  $\mathcal{D}$ -modules. In that case, the nilpotent-ness of the rescaling structure is deduced from the fact that the quantum cup product converges to the classical cup product as the quantum parameter goes to zero.

### §3. Landau–Ginzburg models

In this section we consider the following pair  $(X, f)$ , referred as a Landau–Ginzburg model:

- a smooth projective variety  $X$  of dimension  $n$  over  $\mathbb{C}$ ;
- a flat projective morphism  $f : X \rightarrow \mathbb{P}^1$  of varieties.

We also consider  $f$  as a meromorphic function on  $X$ . We assume that the pole divisor  $(f)_\infty$  of  $f$  is reduced. The support  $|(f)_\infty|$  is denoted by  $D$ . We also assume that  $D$  is a simple normal crossing. Put  $Y := X \setminus D$ . The restriction of  $f$  to  $Y$  is denoted by  $w$ .

*Remark 3.1.* The terminology “Landau–Ginzburg model” might be inappropriate for general  $(X, f)$ . We need to impose the condition that there is an isomorphism  $\mathcal{O} \xrightarrow{\sim} \Omega_X^n(D); 1 \mapsto \text{vol}_X$  in order to regard the tuple  $((X, f), D, \text{vol}_X)$  as a tame compactified Landau–Ginzburg model in [26] (see Appendix B). In this paper, we do not use this condition. However, since the main examples we have in mind are (tame compactified) Landau–Ginzburg models, we call the pair  $(X, f)$  a Landau–Ginzburg model for the sake of convenience.

#### §3.1. Rescaling structure for Landau–Ginzburg models

**3.1.1. The Kontsevich complex.** Let  $df : \Omega_X^k(\log D) \rightarrow \Omega_X^{k+1}(\log D)(D)$  be a morphism induced by the multiplication of  $df$ . The inverse image of  $\Omega_X^{k+1}(\log D) \subset$

$\Omega_X^{k+1}(\log D)(D)$  is denoted by  $\Omega_f^k$ . The multiplication  $df$  induces a morphism  $df : \Omega_f^k \rightarrow \Omega_f^{k+1}$ . The exterior derivative  $d$  induces a morphism  $d : \Omega_f^k \rightarrow \Omega_f^{k+1}$ .

Let  $\pi_S : S \times X \rightarrow X$  be the projection. Recall that  $S = \mathbb{P}_\lambda^1 \times \mathbb{C}_\tau$ . Put  $\Omega_{f,\lambda,\tau}^k := \pi_S^{-1} \Omega_f^k \otimes \lambda^{-k} \mathcal{O}_{S \times X}(*(\lambda)_\infty)$ . We have morphisms of sheaves  $d + \lambda^{-1} \tau df : \Omega_{f,\lambda,\tau}^k \rightarrow \Omega_{f,\lambda,\tau}^{k+1}$  where  $d$  is the relative exterior derivative, i.e.,  $d = d_{S \times X/S}$ . Since  $(d + \lambda^{-1} \tau df)^2 = 0$ , we have a complex  $(\Omega_{f,\lambda,\tau}^\bullet, d + \lambda^{-1} \tau df)$ .

**Definition 3.2.** Let  $p_S : S \times X \rightarrow S$  denote the projection. For each  $k \in \mathbb{Z}$ , we put

$$(3.1) \quad \mathcal{H}_f^k := \mathbb{R}^k p_{S*} (\Omega_{f,\lambda,\tau}^\bullet, d + \lambda^{-1} \tau df).$$

We define a  $\mathbb{Z}$ -graded  $\mathcal{O}_S(*(\lambda)_\infty)$ -module by  $\mathcal{H}_f := \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_f^k$ .

**3.1.2. The rescaling structure.** Let  $\sigma : \mathbb{C}_\theta^* \times S \rightarrow S$  denote the action of  $\mathbb{C}_\theta^*$  given in Section 2.2. Let  $\tilde{\sigma} : \mathbb{C}_\theta^* \times S \times X \rightarrow S \times X$  be the action induced by  $\sigma$  and trivial  $\mathbb{C}_\theta^*$ -action on  $X$ . Let  $\tilde{p}_2 : \mathbb{C}_\theta^* \times S \times X \rightarrow S \times X$  denote the projection. We have the natural isomorphism  $\tilde{\chi}_f : \tilde{p}_2^*(\Omega_{f,\lambda,\tau}^\bullet, d + \lambda^{-1} \tau df) \xrightarrow{\sim} \tilde{\sigma}^*(\Omega_{f,\lambda,\tau}^\bullet, d + \lambda^{-1} \tau df)$ . It induces an isomorphism  $\chi_f : p_2^* \mathcal{H}_f \xrightarrow{\sim} \sigma^* \mathcal{H}_f$  with the cocycle condition (Definition 2.5(3)).

**Proposition 3.3.** *The pair  $(\mathcal{H}_f, \chi_f)$  comes equipped with a rescaling structure.*

*Proof.* By the theorem of Esnault–Sabbah–Yu, M. Saito and Kontsevich [18] (see also [26], [30]),  $\mathcal{H}_f$  is locally free over  $\mathcal{O}_S(*(\lambda)_\infty)$ . Moreover, [30, Thm. 3.5] (see also its consequences in [30, Sect. 3.1.8]) implies that we have a connection  $\nabla$  on each  $\mathcal{H}_f^k$  with the properties in Definition 2.5.  $\square$

**3.1.3. Hodge filtration.** Since  $\mathcal{H}_f$  is a rescaling structure,  $V_f := \mathcal{H}_f|_{(\lambda,\tau)=(1,0)}$  is equipped with a filtration  $F_\bullet V_f$  (see Section 2.3.1). Note that  $V_f \simeq \mathbb{H}^\bullet(X, (\Omega_f^\bullet, d))$ .

**Lemma 3.4** ([18],[30]). *Let  $F_\bullet(\Omega_f^\bullet, d)$  be the stupid filtration on  $(\Omega_f^\bullet, d)$ , i.e., we put  $F_{-p} \Omega_f^k = 0$  for  $p > k$  and  $F_{-p} \Omega_f^k = \Omega_f^k$  for  $p \leq k$ . Then we have*

$$(3.2) \quad F_{-p} V_f^k \simeq \text{Im} (\mathbb{H}^k(X, F_{-p}(\Omega_f^\bullet, d)) \rightarrow \mathbb{H}^k(X, (\Omega_f^\bullet, d))).$$

*Proof.* Let  $\pi_\lambda : \mathbb{C}_\lambda \times X \rightarrow X$  be the projection. Define  $\Omega_{f,\lambda}^k := \pi_\lambda^* \Omega_f^k$ . Let  $p_\lambda : \mathbb{C}_\lambda \times X \rightarrow \mathbb{C}_\lambda$  denote the projection. By the local freeness, we have an  $\mathbb{C}_\theta^*$ -equivariant isomorphism

$$\mathcal{H}_{f|\tau=0}^k \simeq \mathbb{R}^k p_{\lambda*} (\Omega_{f,\lambda}^\bullet, \lambda d).$$

The isomorphism  $\chi$  on  $\mathbb{R}^k p_{\lambda*} (\Omega_{f,\lambda}^\bullet, \lambda d)$  is induced by  $\theta^p \tilde{\chi}_{f|\tau=0} : (\tilde{p}_2^* \Omega_{f,\lambda,\tau}^p)|_{\tau=0} \xrightarrow{\sim} (\tilde{\sigma}^* \Omega_{f,\lambda,\tau}^p)|_{\tau=0}$ .

Let  $\mathcal{A}_X^{p,q}$  denote the sheaf of  $(p, q)$ -forms on  $X$ . Let  $\partial : \mathcal{A}_X^{p,q} \rightarrow \mathcal{A}_X^{p+1,q}$  and  $\bar{\partial} : \mathcal{A}_X^{p,q} \rightarrow \mathcal{A}_X^{p,q+1}$  be the Dolbeault operators. Set  $\mathcal{A}_f^{p,q} := \Omega_f^p \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,q}$ . Put  $\mathcal{A}_{f,\lambda}^{p,q} := \mathcal{O}_{\mathbb{C}_\lambda \times X} \otimes_{\pi_\lambda^{-1} \mathcal{O}_X} \pi_\lambda^{-1} \mathcal{A}_f^{p,q}$ . The operators on  $\mathcal{A}_{f,\lambda}^{p,q}$  induced by  $\partial$  and  $\bar{\partial}$  are denoted by the same notation. Then we obtain the double complex  $(\mathcal{A}_{f,\lambda}^{\bullet,\bullet}, \lambda\partial, \bar{\partial})$ . Let  $(\mathcal{A}_{f,\lambda}^\bullet, \lambda\partial + \bar{\partial})$  be the total complex. Note that  $\mathcal{A}_{f,\lambda}^k = \bigoplus_{p+q=k} \mathcal{A}_{f,\lambda}^{p,q}$ . We obtain a  $\mathbb{C}_\theta^*$ -equivariant quasi-isomorphism

$$(\Omega_{f,\lambda}^\bullet, \lambda d) \xrightarrow{\sim} (\mathcal{A}_{f,\lambda}^\bullet, \lambda\partial + \bar{\partial}),$$

where the isomorphism on  $\mathcal{A}_{f,\lambda}^{p,q}$  is induced by  $\theta^p \tilde{\chi}_f|_{\tau=0}$ . Hence, we have a  $\mathbb{C}_\theta^*$ -equivariant isomorphism:

$$(3.3) \quad \mathbb{R}^k p_{\lambda*}(\Omega_{f,\lambda}^\bullet, \lambda d) \simeq \mathcal{H}^k p_{\lambda*}(\mathcal{A}_{f,\lambda}^\bullet, \lambda\partial + \bar{\partial}).$$

Applying Lemma 2.2 for  $C^{p,q} := \Gamma(X, \mathcal{A}_f^{p,q})$ ,  $\delta_1 := \partial$  and  $\delta_2 := \bar{\partial}$ , the fiber of the cohomology sheaf  $\mathcal{H}^k p_{\lambda*}(\mathcal{A}_{f,\lambda}^\bullet, \lambda\partial + \bar{\partial})$  at  $\lambda = 1$  has the filtration  $G_\bullet$  as in Lemma 2.2 (the fact that we can apply the lemma is due to [18, Thm. 1.3.2]). Since the restriction of (3.3) to  $\lambda = 1$  gives a filtered isomorphism  $(V_f^k, F) \simeq (H^k(C^\bullet, \delta), G)$ , we obtain the conclusion.  $\square$

By this lemma, we have  $\text{Gr}_{-p}^F V_f^k = H^{k-p}(X, \Omega_f^p)$ . Define  $f^{p,q}(Y, \mathbf{w}) := \dim H^q(X, \Omega_f^p)$ . Then we have  $f^{p,q}(Y, \mathbf{w}) = f^{p,q}(\mathcal{H}_f)$ . In the rest of Section 3, we investigate  $h^{p,q}(\mathcal{H}_f)$ , or the weight filtration of the rescaling structure.

### §3.2. Meromorphic connections for Landau–Ginzburg models

We set  $X^{(1)} := \mathbb{C}_\tau \times X$ . We also set  $D^{(1)} := \mathbb{C}_\tau \times D$ . Let  $p_\tau : X^{(1)} \rightarrow \mathbb{C}_\tau$  and  $\pi_\tau : X^{(1)} \rightarrow X$  denote the projections. We shall review some results on a meromorphic flat bundle  $\mathcal{M} := \mathcal{O}(*D^{(1)})v$  with  $\nabla v = d(\tau f)v$  in [30], where  $v$  denotes a global frame. We have

$$\mathcal{M} \simeq \left( \mathcal{O}_{X^{(1)}}(*D^{(1)}), d + d(\tau f) \right); \quad v \mapsto 1.$$

Note that, in our case, some of the results in [30] are simplified since we assume that  $(f)_\infty$  is reduced and the horizontal divisor (denoted by  $H$  in [30]) is empty.

**3.2.1. V-filtration along  $\tau$ .** Regard  $\pi_\tau^* \mathcal{D}_X$  as a sheaf of subalgebra in  $\mathcal{D}_{X^{(1)}}$ . Let  $\tau V_0 \mathcal{D}_{X^{(1)}}$  denote the sheaf of subalgebra generated by  $\pi_\tau^* \mathcal{D}_X$  and  $\tau \partial_\tau$ . For  $\alpha = 0, 1$ , we set

$$U_\alpha \mathcal{M} := \pi_\tau^* \mathcal{D}_X \cdot \mathcal{O}_{X^{(1)}} \left( (\alpha + 1)D^{(1)} \right) v \subset \mathcal{M}.$$

For  $\alpha \in \mathbb{Z}_{<0}$ , we set  $U_\alpha \mathcal{M} := \tau^{-\alpha} U_0 \mathcal{M}$ . For  $\alpha \in \mathbb{Z}_{>0}$ , we set  $U_\alpha \mathcal{M} := \sum_{p+q \leq \alpha} \partial_\tau^p U_q \mathcal{M}$ . Then we have the following:

**Proposition 3.5** ([30, Prop. 2.3]). *We have that  $U_\bullet \mathcal{M}$  is a  $V$ -filtration on  $\mathcal{M}$  along  $\tau$  indexed by integers with the standard order (up to shift of degree by 1). More precisely, we have the following:*

- $U_\alpha \mathcal{M}$  are coherent  $V_0 \mathcal{D}_{X^{(1)}}$ -modules such that  $\bigcup_\alpha U_\alpha \mathcal{M} = \mathcal{M}$ .
- We have  $\tau U_\alpha \mathcal{M} \subset U_{\alpha-1} \mathcal{M}$  and  $\partial_\tau U_\alpha \mathcal{M} \subset U_{\alpha+1} \mathcal{M}$ .
- Define  $\text{Gr}_\alpha^U \mathcal{M} := U_\alpha \mathcal{M} / U_{\alpha-1} \mathcal{M}$ . Then  $\tau \partial_\tau + \alpha$  is nilpotent on  $\text{Gr}_\alpha^U \mathcal{M}$ .

**3.2.2. Relative de Rham complexes.** We set  $\Omega_{f,\tau}^k := \pi_\tau^* \Omega_f^k$ . We obtain a complex  $(\Omega_{f,\tau}^\bullet, d + \tau df)$  where  $d = d_{X^{(1)}/\mathbb{C}_\tau}$  is the relative exterior derivative. We have the following:

**Proposition 3.6** ([30]). *We have a quasi-isomorphism of complexes*

$$(3.4) \quad (\Omega_{f,\tau}^\bullet, d + \tau df) \xrightarrow{\sim} U_0 \mathcal{M} \otimes \Omega_{X^{(1)}/\mathbb{C}_\tau}^\bullet.$$

*Proof.* Combine [30, Prop. 2.21] and [30, Prop. 2.22] in the case  $\alpha = 0$ . □

As a consequence, we have the following (see also the proof of [30, Cor. 2.23]):

**Corollary 3.7.** *We have the following isomorphism of logarithmic connections:*

$$\mathcal{H}_{f|\lambda=1}^k \xrightarrow{\sim} \mathbb{R}^k p_{\tau*} \left( U_0 \mathcal{M} \otimes \Omega_{X^{(1)}/\mathbb{C}_\tau}^\bullet \right).$$

We also have a quasi-isomorphism of complexes:

$$(\Omega_f^\bullet, d) \Big|_{\tau=0} \xrightarrow{\sim} (\Omega_{f,\tau}^\bullet, d + \tau df) \Big|_{\tau=0} \xrightarrow{\sim} \text{Gr}_0^U \mathcal{M} \otimes \Omega_X^\bullet,$$

which induces  $V_f^k \xrightarrow{\sim} \mathbb{H}^k \left( X, (\text{Gr}_0^U \mathcal{M} \otimes \Omega_X^\bullet) \right)$ . The residue endomorphism on  $V_f^k$  is identified with the nilpotent endomorphism on  $\mathbb{H}^k \left( X, (\text{Gr}_0^U \mathcal{M} \otimes \Omega_X^\bullet) \right)$  associated with  $\varphi_0$  on  $\text{Gr}_0^U \mathcal{M} \otimes \Omega_X^\bullet$ , where  $\varphi_0$  denotes the endomorphism induced by  $\tau \partial_\tau$ .

**3.2.3. Residue endomorphisms.** We shall give an alternative description of  $\varphi_0$  to the one in Corollary 3.7. Consider  $\Omega_{X^{(1)}}^k(\log \tau)_0 := \mathcal{O}_{\{0\} \times X} \otimes \Omega_{X^{(1)}}^k(\log \tau)$  as an  $\mathcal{O}_X$ -module. It naturally decomposes to the module

$$\Omega_X^k \oplus [\tau^{-1} d\tau] \cdot \Omega_X^{k-1},$$

where  $[\tau^{-1} d\tau]$  denotes the section induced by  $\tau^{-1} d\tau$ .

Since  $U_\alpha \mathcal{M}$  is a  $V_0 \mathcal{D}_{X^{(1)}}$ -module, we have  $\nabla : U_\alpha \mathcal{M} \otimes \Omega_{X^{(1)}}^k(\log \tau) \rightarrow U_\alpha \mathcal{M} \otimes \Omega_{X^{(1)}}^{k+1}(\log \tau)$ . This induces

$$\nabla' : \text{Gr}_0^U \mathcal{M} \otimes \Omega_{X^{(1)}}^k(\log \tau)_0 \rightarrow \text{Gr}_0^U \mathcal{M} \otimes \Omega_{X^{(1)}}^{k+1}(\log \tau)_0.$$



The morphisms

$$\begin{aligned} \nabla'_0 &: \mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_X^k \rightarrow \mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_X^{k+1}, \text{ and} \\ \nabla'_0 &: \mathrm{Gr}_0^U \mathcal{M} \otimes [\tau^{-1}d\tau] \cdot \Omega_X^{k-1} \rightarrow \mathrm{Gr}_0^U \mathcal{M} \otimes [\tau^{-1}d\tau] \cdot \Omega_X^k \end{aligned}$$

induced by  $\nabla'$  are the same as the flat connection  $\nabla_0$  given by the  $\mathcal{D}_X$ -module structure of  $\mathrm{Gr}_0^U \mathcal{M}$ . The morphism  $\mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_X^k \rightarrow \mathrm{Gr}_0^U \mathcal{M} \otimes [\tau^{-1}d\tau] \cdot \Omega_X^k$  induced by  $\nabla'$  is given by  $m \mapsto [\tau^{-1}d\tau]\varphi_0(m)$ .

We have the following exact sequence of complexes:

$$(3.5) \quad \begin{aligned} 0 \longrightarrow \mathrm{Gr}_0^U \mathcal{M} \otimes ([\tau^{-1}d\tau] \cdot \Omega_X^\bullet[-1]) &\longrightarrow \mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_{X^{(1)}}^\bullet(\log \tau)_0 \\ &\xrightarrow{h} \mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_X^\bullet \longrightarrow 0. \end{aligned}$$

From this exact sequence, we obtain a morphism

$$\varphi_1 : \mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_X^\bullet \rightarrow \mathrm{Gr}_0^U \mathcal{M} \otimes ([\tau^{-1}d\tau] \cdot \Omega_X^\bullet) \simeq \mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_X^\bullet$$

in the derived category  $D^b(\mathbb{C}_X)$  of  $\mathbb{C}_X$ -modules.

**Lemma 3.8.**  $\varphi_0 = \varphi_1$ .

*Proof.* Let  $C^\bullet(h)$  be the mapping cone of  $h$  in (3.5), i.e.,

$$C^k(h) := \mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_{X^{(1)}}^{k+1}(\log \tau)_0 \oplus \mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_X^k, \quad d_{C^\bullet(h)}(a, b) := (-\nabla' a, ha + \nabla_0 b),$$

where  $a \in \mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_{X^{(1)}}^{k+1}(\log \tau)_0$  and  $b \in \mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_X^k$ . Then the morphism

$$\mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_X^k \ni \omega \mapsto [\tau^{-1}d\tau] \cdot \omega \in \mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_{X^{(1)}}^{k+1}(\log \tau)_0$$

induces a quasi-isomorphism  $\iota_0 : \mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_X^\bullet \rightarrow C^\bullet(h)$ . The morphism  $\varphi_1$  is induced by a natural morphism  $\iota_1 : \mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_X^\bullet \rightarrow C^\bullet(h)$ .

Using the identification  $\Omega_{X^{(1)}}^k(\log \tau)_0 = \Omega_X^k \oplus [\tau^{-1}d\tau] \cdot \Omega_X^{k-1}$ , we obtain a morphism  $\Omega_X^k \rightarrow \Omega_{X^{(1)}}^k(\log \tau)_0$  of  $\mathcal{O}_X$ -modules. This morphism induces  $\Psi : \mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_X^k \rightarrow C^{k-1}(h)$ . For a section  $\omega \in \mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_X^k$ , we have  $\Psi \circ d_{\mathrm{Gr}_0^U \mathcal{M} \otimes \Omega_X^\bullet}(\omega) = \Psi(\nabla_0(\omega)) = \nabla'_0(\Psi(\omega))$ . We also have

$$\begin{aligned} d_{C^\bullet(h)} \circ \Psi(\omega) &= (-\nabla'(\Psi(\omega)), h \circ \Psi(\omega)) \\ &= -\nabla'_0(\Psi(\omega)) - [\tau^{-1}d\tau] \cdot \varphi_0(\omega) + \iota_1(\omega) \\ &= -\nabla'_0(\Psi(\omega)) - \iota_0\varphi_0(\omega) + \iota_1(\omega). \end{aligned}$$

Hence we obtain  $(d \circ \Psi + \Psi \circ d)(\omega) = \iota_1(\omega) - \iota_0\varphi_0(\omega)$ , which implies  $\varphi_1 = \varphi_0$ .  $\square$

**§3.3. Relative cohomology groups for Landau–Ginzburg models**

Let  $t$  denote a coordinate on the target space of  $w : Y \rightarrow \mathbb{C}$ . Put  $s := 1/t$  and let  $\mathbb{C}_s \subset \mathbb{P}^1$  be the complex plane with coordinate  $s$ . Take a sufficiently small holomorphic disk  $\Delta_s \subset \mathbb{C}_s$  centered at infinity so that no critical values of  $f$  are contained in  $\Delta_s^\times := \Delta_s \setminus \{\infty\}$ .

Set  $\mathfrak{X} := X \times \Delta_s$ ,  $\mathfrak{D} := D \times \Delta_s$ . Let  $\pi_s : \mathfrak{X} \rightarrow X$  and  $p_s : \mathfrak{X} \rightarrow \Delta_s$  be the projections. Put  $g := 1/f$ . Set  $\Gamma := \{(x, s) \in \mathfrak{X} \mid g(x) = s\}$ . The inclusion  $\Gamma \hookrightarrow \mathfrak{X}$  is denoted by  $i_\Gamma$ . The divisor  $\mathfrak{D} \cup \Gamma$  is a normal crossing. The intersection  $\mathfrak{D} \cap \Gamma$  is denoted by  $\mathfrak{D}_\Gamma$ .

**3.3.1. De Rham complexes.** For  $k \in \mathbb{Z}_{\geq 0}$ , we have a natural morphism

$$\phi^k : \Omega_{\mathfrak{X}}^k(\log \mathfrak{D} \cup \{s = 0\}) \longrightarrow i_{\Gamma*} \Omega_\Gamma^k(\log \mathfrak{D}_\Gamma).$$

Let  $E^k$  be the kernel of  $\phi^k$ . This gives a subcomplex  $E^\bullet$  of  $\Omega_{\mathfrak{X}}^\bullet(\log \mathfrak{D} \cup \{s = 0\})$ .

**Lemma 3.9.** *For each  $k$ , we have*

$$(3.6) \quad E^k = \left( \frac{ds}{s} - \frac{dg}{g} \right) \cdot \pi_s^* \Omega_X^{k-1}(\log D) \oplus (s - g) \cdot \pi_s^* \Omega_X^k(\log D).$$

*In particular,  $E^k$  is a locally free  $\mathcal{O}_{\mathfrak{X}}$ -module.*

*Proof.* It is trivial that the right-hand side of (3.6) is included in  $E^k$ . Let  $s^{-1}ds \cdot \omega_1 + \omega_2$  be a section of  $E^k$ , where  $\omega_1 \in \pi_s^* \Omega_X^{k-1}(\log D)$  and  $\omega_2 \in \pi_s^* \Omega_X^k(\log D)$ . Since  $(s^{-1}ds - g^{-1}dg) \cdot \omega_1$  is a (local) section of  $E^k$ ,  $g^{-1}dg \cdot \omega_1 + \omega_2$  is a section of  $E^k$ . We observe that  $g^{-1}dg \cdot \omega_1 + \omega_2$  is also a section of  $\pi_s^* \Omega_X^k(\log D)$ . Since  $E^k \cap \pi_s^* \Omega_X^k(\log D) = (g - s)\pi_s^* \Omega_X^k(\log D)$ , we obtain that  $g^{-1}dg \cdot \omega_1 + \omega_2$  is a section of  $(s - g)\pi_s^* \Omega_X^k(\log D)$ . This implies that  $s^{-1}ds \cdot \omega_1 + \omega_2$  is a section of the right-hand side of (3.6). □

**3.3.2. Relative de Rham complex.** For  $k \in \mathbb{Z}_{\geq 0}$ , we have a canonical morphism

$$\phi^k : \pi_s^* \Omega_X^k(\log D) \longrightarrow i_{\Gamma*} \Omega_{\Gamma/\Delta_s}^k(\log \mathfrak{D}_\Gamma).$$

Note that  $\pi_s^* \Omega_X^k(\log D)$  is given by

$$\Omega_{\Gamma/\Delta_s}^k(\log \mathfrak{D}_\Gamma) := \frac{\Omega_\Gamma^k(\log \mathfrak{D}_\Gamma)}{\Omega_\Gamma^{k-1}(\log \mathfrak{D}_\Gamma) \wedge p_{\Gamma*} \Omega_{\Delta_s}^1(\log s)},$$

where  $p_\Gamma$  denotes the composition of  $i_\Gamma$  and  $p_s$ .

**Definition 3.10** ([26]). The kernel of the morphism  $\phi^k$  is denoted by  $E^k$ . The induced subcomplex of  $\pi_s^* \Omega_X^\bullet(\log D)$  is denoted by  $E^\bullet$ .

By definition,  $E^k_{|\mathfrak{X} \setminus \mathfrak{D}} \simeq ((s - g)\pi_s^* \Omega_X^k(\log D))_{|\mathfrak{X} \setminus \mathfrak{D}}$ .

**Lemma 3.11** ([26]). *Let  $Q$  be a point in  $D$ . If we take a sufficiently small neighborhood  $U$  of  $Q$ , we have*

$$(3.7) \quad E^k_{|\mathfrak{U}} = \frac{dg}{g} \cdot (\pi_s^* \Omega_X^{k-1}(\log D))_{|\mathfrak{U}} + ((s - g)\pi_s^* \Omega_X^{k-1}(\log D))_{|\mathfrak{U}},$$

where  $\mathfrak{U} := U \times \Delta_s$ . Moreover,  $E^k$  is a locally free  $\mathcal{O}_{\mathfrak{X}}$ -module.

*Proof.* Since the complex  $(\Omega_X^\bullet(\log D), g^{-1}dg)$  is acyclic near  $Q \in D$ , we have a decomposition

$$\pi_s^* \Omega_X^\ell(\log D)_{|\mathfrak{U}(=U \times \Delta_s)} = \mathcal{F}^\ell \oplus \mathcal{G}^\ell$$

such that  $g^{-1}dg : \mathcal{F}^{\ell-1} \xrightarrow{\sim} \mathcal{G}^\ell$  ( $\ell \in \mathbb{Z}_{\geq 0}$ ) for a sufficiently small neighborhood  $U$  of  $Q$  (see the proof of [30, Lem. 2.29]). We have  $E^k_{|\mathfrak{U}} \supset \mathcal{G}^k$ , and  $E^k_{|\mathfrak{U}} \cap \mathcal{F}^k = (s - g)\mathcal{F}^k$ . The local freeness of  $E^k$  and equation (3.7) are obvious by this description.  $\square$

By this lemma, the restriction of  $E^\bullet$  to  $s = 0$  is identified with  $(\Omega_f, d)$ . Here, we remark that we have

$$\Omega_{f|U}^k = g \cdot \Omega_X^k(\log D)_{|U} + \frac{df}{f} \wedge \Omega_X^{k-1}(\log D)_{|U}$$

for sufficiently small  $U$  (see [26, (2.3.1)], [30, Lem. 2.29] for example).

**3.3.3. Gauss–Manin connection.** We have a canonical epimorphism  $\varphi : E^k \rightarrow E^k$ .

**Lemma 3.12.**  $\text{Ker } \varphi = s^{-1}ds \cdot E^{k-1}$ .

*Proof.* It is trivial that  $\text{Ker } \varphi \supset s^{-1}ds \cdot E^{k-1}$ . Let  $(s^{-1}ds - g^{-1}dg)\omega_1 + (s - g)\omega_2$  be a (local) section of  $\text{Ker } \varphi$ , where  $\omega_1 \in \pi_s^* \Omega_X^{k-1}(\log D)$  and  $\omega_2 \in \pi_s^* \Omega_X^k(\log D)$ . We have

$$-g^{-1}dg\omega_1 + (s - g)\omega_2 = 0.$$

Hence, we have

$$\omega_1 = (s - g)\tau_1 + g^{-1}dg\tau_2, \quad \omega_2 = g^{-1}dg\tau_1$$

for some  $\tau_1 \in \Omega_X^{k-1}(\log D)$  and  $\tau_2 \in \Omega_X^{k-2}(\log D)$ . We obtain

$$\begin{aligned} & (s^{-1}ds - g^{-1}dg)\omega_1 + (s - g)\omega_2 \\ &= (s^{-1}ds - g^{-1}dg)((s - g)\tau_1 + g^{-1}dg\tau_2) + (s - g)g^{-1}dg\tau_1 \\ &= s^{-1}ds(s - g)\tau_1 + (s^{-1}ds - g^{-1}dg)g^{-1}dg\tau_2 \\ &= s^{-1}ds((s - g)\tau_1 + g^{-1}dg\tau_2). \end{aligned}$$

This implies  $\text{Ker } \varphi \subset s^{-1}ds \cdot E^{k-1}$ . □

By this lemma, we have the following diagram, whose rows and columns are exact:

(3.8)

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & s^{-1}ds \cdot E^\bullet[-1] & \longrightarrow & E^\bullet & \xrightarrow{\varphi} & E^\bullet \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & s^{-1}ds \cdot \pi_s^* \Omega_X^\bullet(\log D)[-1] & \longrightarrow & \Omega_X^\bullet(\log(\mathfrak{D} \cup \{s=0\})) & \longrightarrow & \pi_s^* \Omega_X^\bullet(\log D) \longrightarrow 0 \\
 & & \downarrow \phi^\bullet[-1] & & \downarrow \phi^\bullet & & \downarrow \phi^\bullet \\
 0 & \longrightarrow & s^{-1}ds \cdot i_{\Gamma^*} \Omega_{\Gamma/\Delta_s}^\bullet(\log \mathfrak{D}_\Gamma)[-1] & \longrightarrow & i_{\Gamma^*} \Omega_\Gamma^\bullet(\log \mathfrak{D}_\Gamma) & \longrightarrow & i_{\Gamma^*} \Omega_{\Gamma/\Delta_s}^\bullet(\log \mathfrak{D}_\Gamma) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0.
 \end{array}$$

From this exact sequence, we obtain a morphism

$$E^\bullet \longrightarrow s^{-1}ds \cdot E^\bullet$$

in the derived category  $D^b(\mathbb{C}_X)$ . This gives a logarithmic connection

$$(3.9) \quad \nabla^{\text{GM}} : \mathbb{R}^k p_{s*} E^\bullet \longrightarrow \mathbb{R}^k p_{s*} E^\bullet \otimes \Omega_{\Delta_s}^1(\log s).$$

On  $\Delta_s^\times$ , the kernel of  $\nabla^{\text{GM}}$  is the local system of the relative cohomology  $H^k(Y, Y_b)$  ( $b \in \Delta_s^\times$ ) ([26]). Hence (3.9) gives a logarithmic extension of the flat connection associated with the local system of the relative cohomology  $H^k(Y, Y_b)$  ( $b \in \Delta_s^\times$ ).

**3.3.4. Residue endomorphisms.** Put  $E_0^\bullet := E^\bullet \otimes \mathcal{O}_{X \times \{0\}}$ . The complex  $E_0^\bullet$  can naturally be considered as a complex on  $X$ . The complex  $E_0^\bullet$  is a subcomplex of the complex  $\Omega_X^\bullet(\log D) \oplus s^{-1}ds \otimes \Omega_X^{k-1}(\log D)$ . On  $\pi_s(\Gamma)$ , we have

$$E_0^k = g \cdot \Omega_X^k(\log D) \oplus \left( \frac{ds}{s} - \frac{dg}{g} \right) \otimes \Omega_X^{k-1}(\log D).$$

On  $X \setminus \pi_s(\Gamma)$ , we have

$$E_0^k = \Omega_X^k(\log D) \oplus \frac{ds}{s} \otimes \Omega_X^{k-1}(\log D).$$

From the exact sequence (3.8), we have the exact sequence

$$(3.10) \quad 0 \longrightarrow \frac{ds}{s} \otimes (\Omega_f^\bullet, d)[-1] \longrightarrow E_0^\bullet \longrightarrow (\Omega_f^\bullet, d) \longrightarrow 0.$$

From this exact sequence, we obtain a morphism

$$\varphi_2 : (\Omega_f^\bullet, d) \longrightarrow \frac{ds}{s} \otimes (\Omega_f^\bullet, d) \simeq (\Omega_f^\bullet, d).$$

This induces a residue endomorphism

$$\text{Res}_{\{s=0\}}(\nabla^{\text{GM}}) : \mathbb{H}^k(X, (\Omega_f, d)) \longrightarrow \mathbb{H}^k(X, (\Omega_f, d))$$

of  $\nabla^{\text{GM}}$  along  $\{s = 0\}$ .

**§3.4. Hodge-Tate conditions for Landau-Ginzburg models**

**3.4.1. Comparison of the residue endomorphisms.** We shall compare the residue endomorphisms given in Sections 3.2.2 (see also Section 3.2.3) and 3.3.4. Put  $\Omega_{\mathfrak{X}}^\bullet(\log s)_0 := \Omega_{\mathfrak{X}}^\bullet(\log s) \otimes \mathcal{O}_{X \times \{0\}}$ . Let  $[s^{-1}ds]$  denote the section of  $\Omega_{\mathfrak{X}}^\bullet(\log s)_0$  induced by  $s^{-1}ds$ . The correspondence  $[s^{-1}ds] \leftrightarrow [\tau^{-1}d\tau]$  gives an isomorphism  $\Omega_{\mathfrak{X}}^\bullet(\log s)_0 \simeq \Omega_{X^{(1)}}^\bullet(\log \tau)_0$ . Via this isomorphism, we identify  $\Omega_{\mathfrak{X}}^\bullet(\log s)_0$  with  $\Omega_{X^{(1)}}^\bullet(\log \tau)_0$ . Similarly, we identify

$$\Omega_{\mathfrak{X}}^\bullet(\log(\mathfrak{D} \cup \{s = 0\}))_0 := \Omega_{\mathfrak{X}}^\bullet(\log(\mathfrak{D} \cup \{s = 0\})) \otimes \mathcal{O}_{X \times \{0\}}$$

with

$$\Omega_{X^{(1)}}^\bullet(\log(D^{(1)} \cup \{\tau = 0\}))_0 := \Omega_{X^{(1)}}^\bullet(\log(D^{(1)} \cup \{\tau = 0\})) \otimes \mathcal{O}_{\{0\} \times X}.$$

By the construction of  $U_0\mathcal{M}$ , we have an inclusion

$$\Omega_{X^{(1)}}^k(\log \tau) \otimes \mathcal{O}_{X^{(1)}}(D^{(1)}) \cdot v \hookrightarrow \Omega_{X^{(1)}}^k(\log \tau) \otimes \mathcal{M}.$$

We also have another inclusion

$$\Omega_{X^{(1)}}^k(\log(D^{(1)} \cup \{\tau = 0\})) \cdot v \hookrightarrow \Omega_{X^{(1)}}^k(\log \tau) \otimes \mathcal{O}_{X^{(1)}}(D^{(1)}) \cdot v.$$

Hence we obtain a morphism

$$\Omega_{\mathfrak{X}}^k(\log(\mathfrak{D} \cup \{s = 0\})) \longrightarrow U_0\mathcal{M} \otimes \Omega_{X^{(1)}}^k(\log \tau).$$

Since the filtration  $U_\bullet\mathcal{M}$  is indexed by  $\mathbb{Z}$ , we have a morphism

$$\Omega_{\mathfrak{X}}^k(\log(\mathfrak{D} \cup \{s = 0\}))_0 \rightarrow \text{Gr}_0^U \mathcal{M} \otimes \Omega_{X^{(1)}}^k(\log \tau)_0$$

given by  $\eta \mapsto v \otimes \eta$ , where  $v$  denotes the section of  $\text{Gr}_0^U \mathcal{M}$  induced by the global section  $v$  of  $\mathcal{M}$ . By restricting this morphism to  $\mathbf{E}_0^k$ , we obtain a morphism

$$\Phi : \mathbf{E}_0^k \longrightarrow \text{Gr}_0^U \mathcal{M} \otimes \Omega_{X^{(1)}}^k(\log \tau)_0.$$

**Lemma 3.13.** *We have that  $\Phi$  defines a morphism of complexes.*

*Proof.* First we verify the lemma on  $\pi_s(\Gamma)$ . Since  $f = 1/g$ , we have

$$\nabla'(v) = v\tau dg^{-1} + v\tau g^{-1}[\tau^{-1}d\tau] = v\tau g^{-1}([\tau^{-1}d\tau] - g^{-1}dg)$$

in  $\text{Gr}_0^U \otimes \Omega_{X^{(1)}}^1(\log \tau)_0$ . Hence, we have

$$\nabla'(v) \cdot (-g^{-1}dg + [\tau^{-1}d\tau]) = 0$$

in  $\text{Gr}_0^U \otimes \Omega_{X^{(1)}}^2(\log \tau)_0$ . Since  $vg^{-1}dg$  and  $v\tau^{-1}d\tau$  are sections of  $U_0\mathcal{M} \otimes \Omega_{X^{(1)}}^1(\log \tau)$ , we have

$$g\nabla'(v) = v\tau(-g^{-1}dg + [\tau^{-1}d\tau]) = 0$$

in  $\text{Gr}_0^U \otimes \Omega_{X^{(1)}}^1(\log \tau)_0$ .

Let  $(g^{-1}dg - [\tau^{-1}d\tau])\omega_1 + g\omega_2$  be a section of  $E_0^k$ , where  $\omega_1 \in \Omega_X^{k-1}(\log D)$  and  $\omega_2 \in \Omega_X^k(\log D)$  (see Section 3.3.4). We then obtain

$$\begin{aligned} &\nabla'(v \cdot (g^{-1}dg - [\tau^{-1}d\tau])\omega_1) \\ &= \nabla'(v) \cdot (g^{-1}dg - [\tau^{-1}d\tau])\omega_1 + v \cdot d((g^{-1}dg - [\tau^{-1}d\tau])\omega_1) \\ &= v \cdot d((g^{-1}dg - [\tau^{-1}d\tau])\omega_1) \end{aligned}$$

and

$$\nabla'(v \cdot g\omega_2) = \nabla'(v) \cdot g \cdot \omega_2 + vd(g \cdot \omega_2) = vd(g \cdot \omega_2).$$

Hence we have  $\nabla' \circ \Phi = \Phi \circ d$  on  $\pi_s(\Gamma)$ .

On  $X \setminus \pi_s(\Gamma)$ ,  $f = 1/g$  is a holomorphic function. Hence,  $\nabla'(v) = v\tau df + v\tau f \cdot \tau^{-1}d\tau$  is a section of  $\tau U_0\mathcal{M} \otimes \Omega_{X^{(1)}}^1(\log \tau)$ . This implies  $\nabla'(v) = 0$  on  $\text{Gr}_0^U \mathcal{M} \otimes \Omega_{X^{(1)}}^1$ . Then we can prove  $\nabla' \circ \Phi = \Phi \circ d$  on  $X \setminus \pi_s(\Gamma)$  similarly.  $\square$

We then obtain the following:

**Theorem 3.14.** *The nilpotent endomorphism  $(\text{Res}_{\{\tau=0\}} \nabla)|_{\lambda=1}$  on  $V_f^k$  coincides with the residue endomorphism of the Gauss–Manin connection  $\nabla^{\text{GM}}$  for the relative cohomology group.*

*Proof.* By Lemma 3.13, we obtain the following commutative diagram in the abelian category of complexes on  $X$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \frac{ds}{s} \cdot (\Omega_f^\bullet, d)[-1] & \longrightarrow & E_0^\bullet & \longrightarrow & (\Omega_f^\bullet, d) \longrightarrow 0 \\ & & \downarrow \text{qis} & & \downarrow \Phi & & \downarrow \text{qis} \\ 0 & \longrightarrow & \text{Gr}_0^U \mathcal{M} \otimes ([\tau^{-1}d\tau] \cdot \Omega_X^\bullet[-1]) & \longrightarrow & \text{Gr}_0^U \mathcal{M} \otimes \Omega_{X^{(1)}}^\bullet(\log \tau)_0 & \longrightarrow & \text{Gr}_0^U \mathcal{M} \otimes \Omega_X^\bullet \longrightarrow 0. \end{array}$$

The rows of this diagram are the exact sequences (3.10) and (3.5). Left and right columns are the quasi-isomorphisms given in Corollary 3.7. This diagram shows  $\varphi_1 = \varphi_2$  in the derived category, which implies the theorem (see Lemma 3.8).  $\square$

**3.4.2. Koszul complex.** Let  $W_\bullet \Omega_X^\ell(\log D)$  be the weight filtration given by

$$W_m \Omega_X^\ell(\log D) := \begin{cases} \Omega_X^\ell(\log D) & (m \geq \ell), \\ \Omega_X^{\ell-m} \wedge \Omega_X^m(\log D) & (0 \leq m < \ell), \\ 0 & (m < 0). \end{cases}$$

Take the irreducible decomposition  $D = \bigcup_{i \in \Lambda} D_i$ . Fix an order of  $\Lambda$ . Note that each  $D_i$  is a smooth hypersurface in  $X$  by the assumption. Put  $D(0) := X$ , and  $D(m) := \bigsqcup_{I \subset \Lambda, |I|=m} (\bigcap_{i \in I} D_i)$  for  $m \in \mathbb{Z}_{>0}$ . We have the isomorphism of complexes  $\text{Rés}_m : \text{Gr}_m^W \Omega_X^\bullet(\log D) \xrightarrow{\sim} a_{m*} \Omega_{D(m)}^\bullet[-m]$ , where  $a_m : D(m) \rightarrow X$  denotes the morphism induced by inclusions ([8], [21], [31]).

We recall that the morphism  $\text{Rés}_m$  is locally described as follows. Let  $(U; (z_1, \dots, z_n))$  be a local coordinate system such that  $U \cap D = \bigcup_{1 \leq j \leq k} \{z_j = 0\}$ . Assume that we have  $\{i_1 < i_2 < \dots < i_k\} \subset \Lambda$  such that  $D_{i_j} \cap U = \{z_j = 0\}$ . For  $J = (j_1, \dots, j_m)$  with  $1 \leq j_1 < j_2 < \dots < j_m \leq k$ , put  $D_J := \{z_{j_1} = \dots = z_{j_m} = 0\}$  and  $(z^{-1} dz)_J := z_{j_1}^{-1} dz_{j_1} \wedge \dots \wedge z_{j_m}^{-1} dz_{j_m}$ . For  $\omega \in W_m \Omega_X^\ell(\log D)$ , we have a unique expression

$$\omega = (z^{-1} dz)_J \wedge \alpha + \beta,$$

where  $\alpha \in \Omega_X^{\ell-m}$ ,  $\beta \in \Omega_X^k(\log D)$  such that  $\beta$  does not have the component  $(z^{-1} dz)_J$ . The residue  $\text{Rés}_{J, \omega}$  is defined by  $\text{Rés}_{J, \omega} := \alpha|_{D_J}$ , and  $\text{Rés}_m$  is defined by

$$\text{Rés}_m(\omega) := \sum_{J \subset \{1, \dots, k\}, |J|=m} \text{Rés}_J(\omega).$$

Let  $\mathcal{M}_{X,D}^{\text{gp}}$  be the sheaf of invertible sections of  $\mathcal{O}_X(*D)$ . We have the morphism  $\mathcal{O}_X \rightarrow \mathcal{M}_{X,D}^{\text{gp}}$  given by  $h \mapsto \exp(2\pi i h)$ , where  $i := \sqrt{-1}$ . We have the exact sequence of  $\mathbb{Z}_X$ -modules

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \xrightarrow{\exp(2\pi i -)} \mathcal{M}_{X,D}^{\text{gp}} \xrightarrow{v_{D(1)}} a_{1*} \mathbb{Z}_{D(1)} \longrightarrow 0,$$

where the  $\mathbb{Z}_X$ -module structure of  $\mathcal{M}_{X,D}^{\text{gp}}$  is given by the multiplication and  $v_{D(1)}$  denotes taking the valuation along the divisors. The induced morphism  $\mathcal{O}_X \rightarrow \mathcal{M}_{X,D}^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$  is denoted by  $\mathbf{e}$ . We also have the exact sequence

$$(3.11) \quad 0 \longrightarrow \mathbb{Q}_X \longrightarrow \mathcal{O}_X \xrightarrow{\mathbf{e}} \mathcal{M}_{X,D}^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow a_{1*} \mathbb{Q}_{D(1)} \longrightarrow 0.$$

We shall consider the ‘‘Koszul complex’’ of  $\mathbf{e}$  ([24], [31]),

$$K_m^\ell := \text{Sym}_{\mathbb{Q}}^{m-\ell}(\mathcal{O}_X) \otimes_{\mathbb{Q}} \bigwedge_{\mathbb{Q}}^{\ell} (\mathcal{M}_{X,D}^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}).$$

We have the natural inclusion  $K_m^\ell \hookrightarrow K_{m+1}^\ell$  by  $h_1 \cdots h_{m-\ell} \otimes y \mapsto 1 \cdot h_1 \cdots h_{m-\ell} \otimes y$  and the differential  $d : K_m^\ell \rightarrow K_m^{\ell+1}$  by

$$d(h_1 \cdots h_{m-\ell} \otimes y) := \sum_{i=1}^{m-\ell} h_1 \cdots h_{i-1} \cdot h_{i+1} \cdots h_{m-\ell} \otimes \mathbf{e}(h_i) \wedge y.$$

**Lemma 3.15** ([24, Prop. 4.3.1.6], [31, Thm. 4.15]).

$$\mathcal{H}^q(K_p^\bullet) \simeq \begin{cases} a_{q*} \mathbb{Q}_{D(q)} & \text{for } q \leq p, \\ 0 & \text{for } q > p. \end{cases}$$

By this lemma, the natural inclusion  $K_p^\bullet \hookrightarrow K_{p+1}^\bullet$  is a quasi-isomorphism for  $p \geq n = \dim X$ . We put  $K_\infty^\bullet := K_n^\bullet$  and let  $W_m K_\infty^k$  be the image of  $K_m^k$  to  $K_n^k$  for  $m < n$  and  $W_m K_\infty^k := K_\infty^k$  for  $m \geq n$ . We obtain a filtered complex  $(K_\infty^\bullet, W)$ .

**Theorem 3.16** ([31, Thm. 4.15, Cor. 4.16]). *The morphism  $K_m^\ell \rightarrow W_m \Omega_X^\ell(\log D)$  given by*

$$(3.12) \quad h_1 \cdots h_{m-\ell} \otimes y_1 \wedge \cdots \wedge y_\ell \mapsto \frac{1}{(2\pi\mathbf{i})^\ell} \left( \prod_{i=1}^{m-\ell} h_i \right) \cdot \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_\ell}{y_\ell}$$

*induces a filtered quasi-isomorphism  $\alpha : (K_\infty^\bullet, W) \otimes \mathbb{C} \rightarrow (\Omega_X^\bullet(\log D), W)$ , or an isomorphism in the derived category of filtered complexes  $D^b(\mathrm{FC}_X)$  [8, Sect. 7.1].*

**Corollary 3.17** ([31, Prop.-Def. 4.11, Cor. 4.17]). *Let  $F$  be the stupid filtration on  $\Omega_X^\bullet(\log D)$ . Then the tuple*

$$\mathrm{Hdg}(X \log D) := ((K_\infty^\bullet, W), (\Omega_X^\bullet(\log D), F, W), \alpha)$$

*is isomorphic to the cohomological mixed  $\mathbb{Q}$ -Hodge complex  $((\mathbb{R}j_* \mathbb{Q}_Y, \tau_{\leq}), (\Omega_X^\bullet(\log D), F, W), \alpha')$  on  $X$  in [8, (8.1.8)], [7]. Here,  $j : Y \hookrightarrow X$  is the inclusion,  $\tau_{\leq}$  denotes the filtration by truncation functor and  $\alpha' : (\mathbb{R}j_* \mathbb{C}_Y, \tau_{\leq}) \rightarrow (\Omega_X^\bullet(\log D), W)$  is an isomorphism in  $D^+(\mathrm{FC}_X)$ .*

*Proof.* By Theorem 3.16, we have the following commutative diagram:

$$(3.13) \quad \begin{array}{ccccccc} (\mathbb{R}j_* \mathbb{C}_Y, \tau_{\leq}) & \xrightarrow{\sim} & (\mathbb{R}j_* \Omega_Y^\bullet, \tau_{\leq}) & \xleftarrow{\sim} & (\Omega_X^\bullet(\log D), \tau_{\leq}) & \longrightarrow & (\Omega_X^\bullet(\log D), W) \\ \parallel & & \mathbb{R}j_* j^{-1} \alpha \uparrow \simeq & & \alpha \uparrow \simeq & & \alpha \uparrow \simeq \\ (\mathbb{R}j_* \mathbb{C}_Y, \tau_{\leq}) & \xrightarrow{\sim} & \mathbb{R}j_* j^{-1} (K_\infty^\bullet, \tau_{\leq}) \otimes \mathbb{C} & \xleftarrow{\sim} & (K_\infty^\bullet, \tau_{\leq}) \otimes \mathbb{C} & \longrightarrow & (K_\infty^\bullet, W) \otimes \mathbb{C}. \end{array}$$

Here, the arrows  $\xrightarrow{\sim}$  and  $\uparrow \simeq$  denote filtered quasi-isomorphisms. Since the natural morphism

$$(\Omega_X^\bullet(\log D), \tau_{\leq}) \longrightarrow (\Omega_X^\bullet(\log D), W)$$



is a filtered quasi-isomorphism, the morphism

$$(K_\infty^\bullet, \tau_\leq) \otimes \mathbb{C} \longrightarrow (K_\infty^\bullet, W) \otimes \mathbb{C}$$

is also a filtered quasi-isomorphism. Note that  $\alpha'$  is defined by the first row of (3.13), and the second row comes from the sequence

$$(3.14) \quad (\mathbb{R}j_*\mathbb{Q}_Y, \tau_\leq) \xrightarrow{\sim} \mathbb{R}j_*j^{-1}(K_\infty^\bullet, \tau_\leq) \xleftarrow{\sim} (K_\infty^\bullet, \tau_\leq) \xrightarrow{\sim} (K_\infty^\bullet, W).$$

It follows that (3.14) defines the isomorphism of cohomological mixed Hodge complexes.  $\square$

The cohomological mixed Hodge complex  $\mathcal{H}dg(X \log D)$  gives a mixed  $\mathbb{Q}$ -Hodge structure on the cohomology groups  $H^k(Y, \mathbb{Q})$ ,  $k \in \mathbb{Z}_{\geq 0}$ , which is denoted by  $H^k(Y) := (H^k(Y, \mathbb{Q}), F, W)$ .

**3.4.3. Cohomological mixed Hodge complex.** Put  $\tilde{A}^{p,q} := \Omega_X^{p+q}(\log D)/W_{q-1}\Omega_X^{p+q}(\log D)$  and  $\tilde{C}^{p,q} := (K_\infty^{p+q}/W_{q-1}K_\infty^{p+q})(q)$ , where  $p, q \in \mathbb{Z}_{\geq 0}$  and  $(q)$  denotes the Tate twist. We have the differentials

$$\begin{aligned} \delta' : \tilde{A}^{p,q} &\rightarrow \tilde{A}^{p+1,q}; [\eta \bmod W_{q-1}] \mapsto [d\eta \bmod W_{q-1}], \\ \delta'' : \tilde{A}^{p,q} &\rightarrow \tilde{A}^{p,q+1}; [\eta \bmod W_{q-1}] \mapsto [g^{-1}dg \wedge \eta \bmod W_q], \\ \delta' : \tilde{C}^{p,q} &\rightarrow \tilde{C}^{p+1,q}; [x \otimes y \bmod W_{q-1}] \otimes (2\pi i)^q \mapsto [d(x \otimes y) \bmod W_{q-1}] \otimes (2\pi i)^q, \\ \delta'' : \tilde{C}^{p,q} &\rightarrow \tilde{C}^{p,q+1}; [x \otimes y \bmod W_{q-1}] \otimes (2\pi i)^q \mapsto [x \otimes g \wedge y \bmod W_q] \otimes (2\pi i)^{q+1}, \end{aligned}$$

where  $\eta \in \Omega_X^{p+q}(\log D)$ ,  $x \in \text{Sym}_{\mathbb{Q}}^k(\mathcal{O}_X)$  for  $k \geq 0$  and  $y \in \bigwedge_{\mathbb{Q}}^{p+q}(\mathcal{M}_{X,D}^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q})$ . The total complexes of these double complexes are denoted by  $s(\tilde{A}^{\bullet,\bullet})$  and  $s(\tilde{C}^{\bullet,\bullet})$ , i.e.,  $s(\tilde{A}^{\bullet,\bullet})^k := \bigoplus_{p+q=k} \tilde{A}^{p,q}$  and  $\delta := \delta' + \delta'' : s(\tilde{A}^{\bullet,\bullet})^k \rightarrow s(\tilde{A}^{\bullet,\bullet})^{k+1}$  is the differential;  $s(\tilde{C}^{\bullet,\bullet})$  is defined similarly. We also have the filtrations

$$\begin{aligned} W_r \tilde{A}^{p,q} &:= W_{r+2q}\Omega_X^{p+q}(\log D)/W_{q-1}\Omega_X^{p+q}(\log D) \subset \tilde{A}^{p,q}, \\ W_r \tilde{C}^{p,q} &:= (W_{r+2q}K_\infty^{p+q}/W_{q-1}K_\infty^{p+q})(q) \subset \tilde{C}^{p,q}, \end{aligned}$$

which induce the filtrations  $W_r s(\tilde{A}^{\bullet,\bullet})^k := \bigoplus_{p+q=k} W_r \tilde{A}^{p,q}$  and  $W_r s(\tilde{C}^{\bullet,\bullet})^k := \bigoplus_{p+q=k} W_r \tilde{C}^{p,q}$  on  $s(\tilde{A}^{\bullet,\bullet})$  and  $s(\tilde{C}^{\bullet,\bullet})$  respectively. We define the filtration  $F$  by  $F_\ell s(\tilde{A}^{\bullet,\bullet})^k := \bigoplus_{p+q=k} \bigoplus_{p \geq -\ell} \tilde{A}^{p,q}$ .

Since  $\delta'' W_r \subset W_{r-1}$  for the filtrations  $W$  on  $s(\tilde{A}^{\bullet,\bullet})$  and  $s(\tilde{C}^{\bullet,\bullet})$ , we obtain the isomorphisms  $\text{Gr}_j^W s(\tilde{A}^{\bullet,\bullet}) \simeq \bigoplus_{k \geq 0, -j} \text{Gr}_{j+2k}^W \Omega_X^{\bullet}(\log D)$  and  $\text{Gr}_j^W s(\tilde{C}^{\bullet,\bullet}) \simeq \bigoplus_{k \geq 0, -j} \text{Gr}_{j+2k}^W K_\infty^{\bullet}(k)$  of complexes. Then the following lemma is trivial by Theorem 3.16:

**Lemma 3.18.** *The morphisms  $K_\infty^{p+q}(q) \rightarrow \Omega_X^{p+q}(\log D)$  given by*

$$(h_1 \cdots h_{n-p-q} \otimes y_1 \wedge \cdots \wedge y_{p+q}) \otimes (2\pi i)^q \mapsto \frac{1}{(2\pi i)^p} \left( \prod_{i=1}^{n-p-q} h_i \right) \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{p+q}}{y_{p+q}}$$

*induce a filtered quasi-isomorphism  $\alpha_1 : (s(\tilde{C}^{\bullet,\bullet}), W) \otimes \mathbb{C} \rightarrow (s(\tilde{A}^{\bullet,\bullet}), W)$ .*

Put  $A^{p,q} := \tilde{A}^{p,q+1}$ ,  $C^{p,q} := \tilde{C}^{p,q+1}$  for  $p, q \in \mathbb{Z}_{\geq 0}$ . The total complexes are denoted by  $s(A^{\bullet,\bullet})$  and  $s(C^{\bullet,\bullet})$ . We note that  $s(A^{\bullet,\bullet})$  and  $s(C^{\bullet,\bullet})$  are supported on  $D$ . Let  $W_r A^{p,q} := W_{r-1} \tilde{A}^{p,q+1}$  and  $W_r C^{p,q} := W_{r-1} \tilde{C}^{p,q+1}$ . These filtrations induce filtrations on  $s(A^{\bullet,\bullet})$  and  $s(C^{\bullet,\bullet})$ . We have a quasi-isomorphism  $\alpha_0 : (s(C^{\bullet,\bullet}), W) \otimes \mathbb{C} \rightarrow (s(A^{\bullet,\bullet}), W)$  by restricting  $\alpha_1$ . We also have the filtration  $F$  on  $s(A^{\bullet,\bullet})$  by  $F_\ell s(A^{\bullet,\bullet})^k := \bigoplus_{p+q=k} \bigoplus_{p \geq -\ell} A^{p,q}$ .

**Theorem 3.19** ([31, Thm. 11.22]). *The tuple*

$$\psi_g^{\text{Hdg}} := ((s(C^{\bullet,\bullet}), W), (s(A^{\bullet,\bullet}), F, W), \alpha_0)$$

*is a cohomological mixed  $\mathbb{Q}$ -Hodge complex on  $X$ , which defines a mixed Hodge structure on the hypercohomology  $\mathbb{H}^\bullet(X, \psi_g(\mathbb{Q}_X))$  of the nearby cycle  $\psi_g(\mathbb{Q}_X)$ .*

The mixed  $\mathbb{Q}$ -Hodge structure on the hypercohomology group  $\mathbb{H}^k(X, \psi_g \mathbb{Q}_X)$  is denoted by  $H^k(Y_\infty)$ . Define  $\vartheta_{\mathbb{C}} : \Omega_X^p(\log D) \rightarrow A^{p,0}$  by  $\vartheta_{\mathbb{C}}(\eta) := (-1)^p [g^{-1} dg \wedge \eta \bmod W_0]$ . It induces a morphism of complexes  $\vartheta_{\mathbb{C}} : \Omega_X^\bullet(\log D) \rightarrow s(A^{\bullet,\bullet})$ . Define  $\vartheta_{\mathbb{Q}} : K_\infty^p \rightarrow C^{p,0}$  by  $\vartheta_{\mathbb{Q}}(x \otimes y) := (-1)^p [x \otimes g \wedge y]$ . It induces a morphism of complexes  $\vartheta_{\mathbb{Q}} : K_\infty^\bullet \rightarrow s(C^{\bullet,\bullet})$ . By the construction, we have  $\alpha_0 \circ \vartheta_{\mathbb{Q}} = \vartheta_{\mathbb{C}} \circ \alpha$ . Hence, we obtain a morphism of cohomological mixed  $\mathbb{Q}$ -Hodge complexes  $\vartheta : \mathcal{H}dg(X \log D) \rightarrow \psi_g^{\text{Hdg}}$  (see [17, Sect. 3.3.4.2] for the definition of a morphism of a cohomological mixed Hodge complex). We have the mixed cone complex  $C(\vartheta) = ((C(\vartheta_{\mathbb{Q}}), W), (C(\vartheta_{\mathbb{C}}), W, F), \alpha_\vartheta)$  ([17, Sect. 3.3.4.2]). We also have the notion of shift ([17, Sect. 3.3.3.1]) for cohomological mixed Hodge complexes.

**Proposition 3.20.** *The tuple  $\Xi_g^{\text{Hdg}} := ((s(\tilde{C}^{\bullet,\bullet}), W), (s(\tilde{A}^{\bullet,\bullet}), W, F), \alpha_1)$  constitutes a cohomological mixed  $\mathbb{Q}$ -Hodge complex on  $X$ , which is isomorphic to  $C(\vartheta)[-1]$ .*

*Proof.* The shifted cone  $C(\vartheta_{\mathbb{Q}})[-1]$  of  $\vartheta_{\mathbb{Q}}$  is given by

$$\begin{aligned} (C(\vartheta_{\mathbb{Q}})[-1])^k &= K_\infty^k \oplus s(C^{\bullet,\bullet})^{k-1} \\ &= \tilde{C}^{k,0} \oplus \bigoplus_{p+q=k-1} C^{p,q} = \bigoplus_{p+q=k} \tilde{C}^{p,q}. \end{aligned}$$

The differential  $d : \bigoplus_{p+q=k} \tilde{C}^{p,q} \rightarrow \bigoplus_{p+q=k+1} \tilde{C}^{p,q}$  of  $C(\vartheta_{\mathbb{Q}})[-1]$  is given by  $d|_{\tilde{A}^{p,q}} = -\delta$  for  $q > 0$ , and  $d|_{\tilde{A}^{p,0}} = \delta' + (-1)^{p+1}\delta''$ . The isomorphism  $h_{\mathbb{Q}} : s(\tilde{C}^{\bullet,\bullet}) \rightarrow C^{\bullet}(\vartheta_{\mathbb{Q}})[-1]$  is given by  $h_{\mathbb{Q}}|_{\tilde{C}^{p,0}} := \text{id}_{\tilde{C}^{p,0}}$  and  $h_{\mathbb{Q}}|_{\tilde{C}^{p,q+1}} = (-1)^{p+q}\text{id}_{\tilde{C}^{p,q+1}}$ . The weight filtration on  $C(\vartheta_{\mathbb{Q}})[-1]$  is given by

$$\begin{aligned} W_{\ell}(C^k(\vartheta_{\mathbb{Q}})[-1])^k &= W_{\ell}K_{\infty}^k \oplus W_{\ell+1}s(C^{\bullet,\bullet})^{k-1} \\ &= W_{\ell}s(\tilde{C}^{\bullet,\bullet})^k. \end{aligned}$$

This shows the compatibility of the weight filtrations. A similar argument can be applied to  $C(\vartheta_{\mathbb{C}})$ . The compatibility of Hodge filtration  $F$  can easily be checked. Let  $h_{\mathbb{C}} : s(\tilde{A}^{\bullet,\bullet}) \rightarrow C(\vartheta_{\mathbb{C}})$  be the isomorphism defined in the same way as  $h_{\mathbb{Q}}$ . It can also be checked that

$$(\alpha_{\vartheta}[-1]) \circ (h_{\mathbb{Q}} \otimes \text{id}_{\mathbb{C}}) = h_{\mathbb{C}} \circ \alpha_1.$$

Note that  $\alpha_{\vartheta} : C(\vartheta_{\mathbb{Q}}) \otimes \mathbb{C} \rightarrow C(\vartheta_{\mathbb{C}})$  is defined by  $\alpha_{\vartheta}(x, y) := (\alpha x, \alpha_0 y)$  for  $x \in K_{\infty}^{k+1} \otimes \mathbb{C}$ ,  $y \in s(C^{\bullet,\bullet})^k \otimes \mathbb{C}$ . This proves the proposition.  $\square$

The mixed Hodge complex  $\Xi_g^{\text{Hdg}}$  defines a mixed Hodge structure on  $\mathbb{H}^k(X, s(\tilde{C}^{\bullet,\bullet}))$ , which we denote by  $H^k(Y, Y_{\infty}) = (H^k(Y, Y_{\infty}; \mathbb{Q}), F, W)$ .

**Corollary 3.21** ([31]). *We have the following long exact sequence of mixed Hodge structures:*

$$(3.15) \quad \dots \longrightarrow H^{k-1}(Y_{\infty}) \longrightarrow H^k(Y, Y_{\infty}) \longrightarrow H^k(Y) \longrightarrow H^k(Y_{\infty}) \longrightarrow \dots$$

*Proof.* Apply [31, Thm. 3.22(2)] to the cone  $C(\vartheta)$ .  $\square$

*Remark 3.22.* Although we postpone clarification of the precise relation, the notation  $\Xi_g^{\text{Hdg}}$  comes from the notation for Beilinson’s maximal extension (see [18, Thm. E.3]).

**3.4.4. Monodromy weight filtration.** Let  $\nu : \tilde{A}^{p,q} \rightarrow \tilde{A}^{p-1,q+1}$  be the morphism given by  $\nu([\eta \bmod W_{q-1}]) := [\eta \bmod W_q]$ . It induces a nilpotent endomorphism on  $s(\tilde{A}^{\bullet,\bullet})$ , which is also denoted by  $\nu$ . It can easily be observed that  $\nu(W_r) \subset W_{r-2}$ , and  $\nu(F_i) \subset F_{i+1}$ . We also define  $\nu : \tilde{C}^{p,q} \rightarrow \tilde{C}^{p-1,q+1}(-1)$  similarly:  $[x \otimes y \bmod W_{q-1}] \otimes (2\pi i)^{q-1} \mapsto [x \otimes y \bmod W_q] \otimes (2\pi i)^{q-1}$ . Hence we have a morphism  $\nu : H^k(Y, Y_{\infty}) \rightarrow H^k(Y, Y_{\infty})(-1)$  of mixed Hodge structures for each  $k$ . The following theorem is proved in Section 3.4.6:

**Theorem 3.23.** *The map  $\nu$  induces isomorphisms*

$$\nu^r : \text{Gr}_{k+r}^W H^k(Y, Y_{\infty}) \xrightarrow{\sim} \text{Gr}_{k-r}^W H^k(Y, Y_{\infty})(-r),$$

i.e., the weight filtration  $W$  on  $H^k(Y, Y_\infty)$  is the monodromy weight filtration of  $\nu$  centered at  $k$ .

The way to prove this theorem is essentially the same as in [21, Thm. 5.2]. We remark that  $n$  in [21] corresponds to  $n-1$  in this paper.

**3.4.5. Monodromy weight spectral sequence.** By Proposition 3.20, we have the following:

**Corollary 3.24.** *The spectral sequence for  $(\mathbb{R}\Gamma(X, s(\tilde{C}^{\bullet,\bullet})), W)$  whose  $E_1$ -term is given by*

$$E_1^{-r,q+r} = \mathbb{H}^q(X, \text{Gr}_r^W s(\tilde{C}^{\bullet,\bullet}))$$

degenerates at the  $E_2$ -term. In other words,  $\text{Gr}_{q+r}^W \mathbb{H}^q(X, s(\tilde{C}^{\bullet,\bullet}))$  is the cohomology of the complex

$$E_1^{-r-1,q+r} \xrightarrow{d_1} E_1^{-r,q+r} \xrightarrow{d_1} E_1^{-r+1,q+r}.$$

*Proof.* Apply ([8, (8.1.9)]) to the cohomological mixed  $\mathbb{Q}$ -Hodge complex  $\Xi_g^{\text{Hdg}}$  on  $X$ . □

By Theorem 3.16 we have a quasi-isomorphism  $\text{Gr}_m^W K_\infty^\bullet \simeq a_{m*} \mathbb{Q}_{D(m)}[-m](-m)$ . Recall that  $\text{Gr}_j^W s(\tilde{C}^{\bullet,\bullet}) \simeq \bigoplus_{k \geq 0, -j} \text{Gr}_{j+2k}^W K_\infty^\bullet(k)$ . Hence,

$$\begin{aligned} E_1^{-r,q+r} &= \mathbb{H}^q(X, \text{Gr}_r^W s(\tilde{C}^{\bullet,\bullet})) \simeq \bigoplus_{k \geq 0, -r} \mathbb{H}^q(X, \text{Gr}_{r+2k}^W K_\infty^\bullet(k)) \\ &\simeq \bigoplus_{k \geq 0, -r} H^{q-r-2k}(D(2k+r); \mathbb{Q})(-r-k). \end{aligned}$$

Following [21], we put  $K_{\mathbb{Q}}^{i,j,k} := H^{i+j-2k+n}(D(2k-i); \mathbb{Q})(i-k)$  for  $k \geq 0, i$ , and  $K_{\mathbb{Q}}^{i,j,k} = 0$  otherwise. Then we have  $E_1^{-r,q+r} \simeq \bigoplus_{k \in \mathbb{Z}} K_{\mathbb{Q}}^{-r,q-n,k}$ . We also put  $E_{1,\mathbb{R}}^{-r,q+r} := E_1^{-r,q+r} \otimes \mathbb{R}$ ,  $K^{i,j,k} := K_{\mathbb{Q}}^{i,j,k} \otimes \mathbb{R}$ , and  $K^{i,j} := \bigoplus_k K^{i,j,k}$ . The induced morphism  $d_1 \otimes \text{id}_{\mathbb{R}}$  is also denoted by  $d_1$ .

**Proposition 3.25** (cf. [21, Lem. (2.7), Prop. (2.9)]). *The restriction of  $d_1$  to  $K^{i,j,k}$  decomposes to  $d'_1 : K^{i,j,k} \rightarrow K^{i+1,j+1,k}$  and  $d''_1 : K^{i,j,k} \rightarrow K^{i+1,j+1,k+1}$ . Moreover,  $d'_1$  is the alternating sum of the Gysin map  $\gamma^{(2k-i)}$  in [21, (1.3)] times  $(-1)$ , and  $d''_1$  is the alternating sum of restriction map  $\rho^{(2k-i)}$  in [21, (1.3)].*

*Proof.* By the definition,  $d_1 : E_1^{-r,q+r} \rightarrow E_1^{-r+1,q+r}$  is induced by the short exact sequence

$$0 \rightarrow \text{Gr}_{r-1}^W s(\tilde{C}^{\bullet,\bullet}) \rightarrow W_r s(\tilde{C}^{\bullet,\bullet}) / W_{r-2} s(\tilde{C}^{\bullet,\bullet}) \rightarrow \text{Gr}_r^W s(\tilde{C}^{\bullet,\bullet}) \rightarrow 0.$$

We shall compute the complex version of  $d_1$  using Dolbeault resolution, and then observe the compatibility with the rational structure.

Let  $\mathcal{A}_X^{p,q}$  be the sheaf of  $(p, q)$ -forms on  $X$ . Put  $\mathcal{A}_{X,D}^{p,q} := \Omega_X^p(\log D) \otimes_{\mathcal{O}_X} \mathcal{A}_X^{0,q}$ ,  $\mathcal{A}_X^k := \bigoplus_{p+q=k} \mathcal{A}_X^{p,q}$ ,  $\mathcal{A}_{X,D}^k := \bigoplus_{p+q=k} \mathcal{A}_{X,D}^{p,q}$  and  $W_m \mathcal{A}_{X,D}^k := \mathcal{A}_X^{k-m} \wedge \mathcal{A}_{X,D}^m$ . Let  $d := \partial + \bar{\partial} : \mathcal{A}_\star^k \rightarrow \mathcal{A}_\star^{k+1}$  be the differential ( $\star = X$ , or  $X, D$ ). We have a resolution  $(\Omega_X^\bullet(\log D), d) \simeq (\mathcal{A}_{X,D}^\bullet, d)$  compatible with the filtrations. Put  $\widetilde{\mathcal{A}}^{p,q} := \mathcal{A}_{X,D}^{p+q} / W_{q-1} \mathcal{A}_{X,D}^{p+q}$  and define  $\delta' : \widetilde{\mathcal{A}}^{p,q} \rightarrow \widetilde{\mathcal{A}}^{p+1,q}$ ,  $\delta'' : \widetilde{\mathcal{A}}^{p,q} \rightarrow \widetilde{\mathcal{A}}^{p,q+1}$  by  $\delta'([\eta \bmod W_{q-1}]) = [d\eta \bmod W_{q-1}]$  and  $\delta''([\eta \bmod W_{q-1}]) = [g^{-1}dg \wedge \eta \bmod W_q]$ . Denote by  $s(\widetilde{\mathcal{A}}^{\bullet,\bullet})$  the associated single complex. We also define the filtration on  $s(\widetilde{\mathcal{A}}^{\bullet,\bullet})$  by  $W_r \widetilde{\mathcal{A}}^{p,q} = W_{r+2q} \mathcal{A}_{X,D}^{p+q} / W_{q-1} \mathcal{A}_{X,D}^{p+q}$ . We have the quasi-isomorphism  $s(\widetilde{\mathcal{A}}^{\bullet,\bullet}) \simeq s(\widetilde{\mathcal{A}}^{\bullet,\bullet})$  compatible with the filtrations.

For  $k \geq 0, -r$ , take a class

$$[x] \in \mathbb{H}^q(X, \text{Gr}_{r+2k}^W \Omega_X^\bullet(\log D)) \subset \mathbb{H}^q(X, \text{Gr}_r^W s(\widetilde{\mathcal{A}}^{\bullet,\bullet})).$$

Since we have the isomorphism  $\mathbb{H}^q(X, \text{Gr}_{r+2k}^W \Omega_X^\bullet(\log D)) \simeq H^q(\Gamma(X, \text{Gr}_{r+2k}^W \mathcal{A}_{X,D}^\bullet))$ , then we can take a representative  $x \in \Gamma(X, \text{Gr}_{r+2k}^W \mathcal{A}_{X,D}^q)$  with  $0 = dx \in \Gamma(X, \text{Gr}_{r+2k}^W \mathcal{A}_{X,D}^{q+1})$ . Take a lift  $\tilde{x} \in \Gamma(X, W_{r+2k} \mathcal{A}_{X,D}^q / W_{k-1} \mathcal{A}_{X,D}^q) = \Gamma(X, W_r \widetilde{\mathcal{A}}^{q-k,k})$ . We have  $\delta'' \tilde{x} \in \Gamma(X, W_{r-1} \widetilde{\mathcal{A}}^{q-k,k+1})$ . Since  $dx = 0$ , we have  $\delta' \tilde{x} \in \Gamma(X, W_{r-1} \widetilde{\mathcal{A}}^{q-k+1,k})$ . We obtain

$$\begin{aligned} d_1[x] &= [\delta' \tilde{x}] + [\delta'' x] \\ &\in \mathbb{H}^{q+1}(X, \text{Gr}_{r+2k-1}^W \Omega_X^\bullet(\log D)) \oplus \mathbb{H}^{q+1}(X, \text{Gr}_{r+2k+1}^W \Omega_X^\bullet(\log D)). \end{aligned}$$

Defining  $d'_1[x] := [\delta' \tilde{x}]$ , and  $d''_1[x] := [\delta'' \tilde{x}]$ , we have the decomposition  $d_1 = d'_1 + d''_1$ .

Then, by the construction,  $d'_1 : \mathbb{H}^q(X, \text{Gr}_{r+2k}^W \Omega_X^\bullet(\log D)) \rightarrow \mathbb{H}^{q+1}(X, \text{Gr}_{r+2k-1}^W \Omega_X^\bullet(\log D))$  is induced by the short exact sequence

$$0 \longrightarrow \text{Gr}_{r+2k-1}^W \Omega_X^\bullet(\log D) \longrightarrow \frac{W_{r+2k} \Omega_X^\bullet(\log D)}{W_{r+2k-2} \Omega_X^\bullet(\log D)} \longrightarrow \text{Gr}_{r+2k}^W \Omega_X^\bullet(\log D) \longrightarrow 0.$$

The differential  $d''_1 : \mathbb{H}^q(X, \text{Gr}_{r+2k}^W \Omega_X^\bullet(\log D)) \rightarrow \mathbb{H}^{q+1}(X, \text{Gr}_{r+2k+1}^W \Omega_X^\bullet(\log D))$  is induced by

$$g^{-1}dg : \text{Gr}_m^W \Omega_X^p(\log D) \rightarrow \text{Gr}_{m+1}^W \Omega_X^{p+1}(\log D) \quad (m \geq 0).$$

In [21], it is shown that  $\text{R\acute{e}s}_{r+2k-1} \circ d'_1 = (-\gamma^{(r+2k)}) \circ \text{R\acute{e}s}_{r+2k}$  and  $\text{R\acute{e}s}_{r+2k+1} \circ d''_1 = \rho^{(r+2k)} \circ \text{R\acute{e}s}_{r+2k}$  holds, where  $\gamma^{(m)} : \mathbb{H}^{k-m}(D(m); \mathbb{C}) \rightarrow \mathbb{H}^{k-m+2}(D(m-1); \mathbb{C})$  denotes the (alternating sum of) Gysin map and  $\rho^{(m)} : \mathbb{H}^k(D(m); \mathbb{C}) \rightarrow \mathbb{H}^k(D(m+1); \mathbb{C})$  denotes (the alternating sum of) restriction [21, (1.3)]. It is also shown that similar commutativity holds for rational cohomology ([21, (1.8),(2.9)]). Hence, we obtain the conclusion.  $\square$

The morphism  $\nu : s(\tilde{C}^{\bullet,\bullet}) \rightarrow s(\tilde{C}^{\bullet,\bullet})(-1)$  induces morphisms  $\nu : K^{i,j,k} \rightarrow K^{i+2,j,k+1}(-1)$ , which is the identity whenever  $k \geq 0, i$ . Hence, we obtain the following lemma:

**Lemma 3.26** ([21, Lem. (2.7), Prop. (2.9)], [31, Prop. 11.34]).

- (1) For all  $i \geq 0$ ,  $\nu$  induces an isomorphism  $\nu^i : K^{-i,j} \xrightarrow{\sim} K^{i,j}(-i)$ .
- (2)  $\text{Ker}(\nu^{i+1}) \cap K^{-i,j} = K^{-i,j,0}$ .

**3.4.6. Polarized Hodge–Lefschetz modules.** We shall use Guillén–Navarro Aznar’s formulation [21, Sect. 4] of the result of Saito [39] and Deligne on the Hodge–Lefschetz modules. Let  $L^{\bullet,\bullet} = \bigoplus_{i,j \in \mathbb{Z}} L^{i,j}$  be a bi-graded finite-dimensional  $\mathbb{R}$ -vector space. Let  $\ell_1, \ell_2$  be endomorphisms on  $L$  such that  $\ell_1(L^{i,j}) \subset L^{i+2,j}$ ,  $\ell_2(L^{i,j}) \subset L^{i,j+2}$  and  $[\ell_1, \ell_2] = 0$ . The tuple  $(L^{\bullet,\bullet}, \ell_1, \ell_2)$  is called a Lefschetz module if  $\ell_1^i : L^{-i,j} \rightarrow L^{i,j}$  are isomorphisms for all  $i > 0$  and  $\ell_2^j : L^{i,-j} \rightarrow L^{i,j}$  are isomorphisms for all  $j > 0$ . A Lefschetz module  $(L^{\bullet,\bullet}, \ell_1, \ell_2)$  is called a Hodge–Lefschetz module if every  $L^{i,j}$  has real Hodge structure and  $\ell_1, \ell_2$  are morphisms of real Hodge structures of some types ([20, (1.2)] or [44, Def. 7.22]).

A polarization  $\psi$  of a Hodge–Lefschetz module  $(L^{\bullet,\bullet}, \ell_1, \ell_2)$  is a morphism of real Hodge structures  $\psi : L^{\bullet,\bullet} \otimes L^{\bullet,\bullet} \rightarrow \mathbb{R}$  of certain type with the following properties:

- (P1)  $\psi(\ell_i x, y) + \psi(x, \ell_i y) = 0$  for  $i = 1, 2$  and
- (P2)  $\psi(-, \ell_1^i \ell_2^j C -)$  is symmetric positive definite on  $L_0^{-i,-j} := L^{-i,-j} \cap \text{Ker}(\ell_1^{i+1}) \cap \text{Ker}(\ell_2^{j+1})$ .

Here  $C$  denotes the Weil operator. The tuple  $(L^{\bullet,\bullet}, \ell_1, \ell_2, \psi)$  of a Hodge–Lefschetz module and its polarization is called a polarized Hodge–Lefschetz module.

A differential  $d$  on a polarized Hodge–Lefschetz module  $(L^{\bullet,\bullet}, \ell_1, \ell_2, \psi)$  is a morphism of real Hodge structures  $d : L^{\bullet,\bullet} \rightarrow L^{\bullet,\bullet}$  of certain type such that

- (D1)  $d(L^{i,j}) \subset L^{i+1,j+1}$  for  $i, j \in \mathbb{Z}$ ;
- (D2)  $d^2 = 0$ ;
- (D3)  $[d, \ell_i] = 0$  for  $i = 1, 2$ ; and
- (D4)  $\psi(dx, y) = \psi(x, dy)$ .

The tuple  $(L^{\bullet,\bullet}, \ell_1, \ell_2, \psi, d)$  is called a differential polarized Hodge–Lefschetz module. By definition  $\ell_i$  defines an endomorphism on the cohomology group  $H^*(L^{\bullet,\bullet}, d)$  for  $i = 1, 2$ , which is denoted by the same notation. We also have a bilinear map on  $H^*(L^{\bullet,\bullet}, d)$ , which is also denoted by  $\psi$ .

**Theorem 3.27.** [21, Thm. (4.5)] *Let  $(L^{\bullet,\bullet}, \ell_1, \ell_2, \psi, d)$  be a differential polarized Hodge-Lefschetz module. Then  $(H^*(L^{\bullet,\bullet}, d), \ell_1, \ell_2, \psi)$  is a polarized Hodge-Lefschetz module.*

Fix a Kähler form  $\omega_{\text{Kähler}}$  on  $X$ . Let  $[\omega_{\text{Kähler}}] \in H^2(X; \mathbb{R})$  be its cohomology class. The cup product with the restriction of the class  $[\omega_{\text{Kähler}}]$  to  $H^2(D(2k+i); \mathbb{R})$  defines mappings  $L : K^{i,j,k} \rightarrow K^{i,j+2,k}$  for all  $k \geq 0, i$ . Define the linear mapping  $\psi : K^{\bullet,\bullet} \otimes K^{\bullet,\bullet} \rightarrow \mathbb{R}$  by

$$\psi(x, y) := \begin{cases} \varepsilon(i+j-n) (2\pi i)^{2k+i} \int_{D(2k+i)} x \wedge y & \text{if } x \in K^{-i,-j,k}, y \in K^{i,j,k+i}, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\varepsilon(a) := (-1)^{a(a-1)/2}$ .

**Theorem 3.28** (cf. [21, Thm. (5.1)]). *The tuple  $(K^{\bullet,\bullet}, (2\pi i)\nu, L, \psi, d_1)$  is a differential polarized Hodge-Lefschetz module.*

*Proof.* By Lemma 3.26,  $(2\pi i\nu)^i : K^{-i,j} \xrightarrow{\sim} K^{i,j}$  for  $i > 0$ . By the hard Lefschetz theorem, we also have  $L^j : K^{i,-j} \xrightarrow{\sim} K^{i,j}$  for  $j > 0$ . Hence,  $(K^{\bullet,\bullet}, (2\pi i)\nu, L)$  is a Hodge-Lefschetz module. Since the trace map and the cup product are the morphisms of Hodge structures,  $\psi$  is a morphism of real Hodge structures. By some direct computation as in [21, Prop. 3.5], we have  $\psi(x, y) = (-1)^n \psi(y, x)$ ,  $\psi((2\pi i)\nu x, y) + \psi(x, (2\pi i)\nu y) = 0$ ,  $\psi(Lx, y) + \psi(x, Ly) = 0$ . This proves (P2). By Lemma 3.26, and the last formula in [21, (1.3)], we also have  $\psi(d_1^i x, y) = \psi(x, d_1^i y)$ . It follows that  $\psi(d_1 x, y) = \psi(x, d_1 y)$ . This proves (D4). Statements (D1), (D2) are trivial by definition, and (D3) follows from Proposition 3.25.

It remains to prove (P1). Put  $K_0^{-i,-j} := K^{-i,-j} \cap \text{Ker}(\nu^{i+1}) \cap \text{Ker}(L^{j+1})$ . By the hard Lefschetz theorem and Lemma 3.26,  $K_0^{-i,-j}$  is the primitive part of  $H^{n-i-j}(D(i); \mathbb{R})(-i)$ . If we put  $Q(x, y) := \psi(x, ((2\pi i)\nu)^i L^j C y)$  for  $x, y \in K_0^{-i,-j}$ , we have

$$Q(x, y) = \varepsilon(i+j-n) \int_{D(i)} ((2\pi i)^i x) \wedge L^j C (2\pi i)^i y.$$

Note that  $\xi := (2\pi i)^i x$  and  $\eta := (2\pi i)^i y$  are the elements of the primitive part of  $H^{n-i-j}(D(i); \mathbb{R})$ . Since  $L$  is the Lefschetz operator on  $D(i)$ , the map  $(\xi, \eta) \mapsto \varepsilon(i+j-n) \int_{D(i)} \xi \wedge L^j C \eta$  is positive definite by the classical Hodge-Riemann bilinear relations. This implies (P1).  $\square$

*Proof of Theorem 3.23.* By Theorem 3.27 and Theorem 3.28, the tuple

$$(H^*(K^{\bullet,\bullet}, d_1), (2\pi i)\nu, L, \psi)$$

is a polarized Hodge–Lefschetz module. In particular,  $(2\pi i\nu)^i : H^*(K^{\bullet,\bullet}, d_1)^{-i,j} \rightarrow H^*(K^{\bullet,\bullet}, d_1)^{i,j}$  are isomorphisms for  $i > 0$ . By Corollary 3.24, this implies the theorem.  $\square$

**3.4.7. Main theorem.** We first compare the nilpotent endomorphisms in Section 3.4.1 with  $\nu$  in Section 3.4.3. Recall that the stupid filtration on  $(\Omega_f^\bullet, d)$  was denoted by  $F$  in Lemma 3.4.

**Proposition 3.29.** *We have a filtered quasi-isomorphism  $\rho : ((\Omega_f^\bullet, d), F) \xrightarrow{\sim} (s(\tilde{A}^{\bullet,\bullet}), F)$ , which is compatible with the nilpotent endomorphisms  $\varphi_2$  and  $\nu$ . In other words,  $\nu \circ \rho = \rho \circ \varphi_2$  in the derived category.*

*Proof.* The morphism  $\rho$  is given by the natural inclusion  $\Omega_f^p \hookrightarrow \Omega_X^p(\log D) = \tilde{A}^{p,0}$ . It is trivial that  $\rho$  is strictly compatible with  $F$ . By (3.8), we have a short exact sequence

$$0 \longrightarrow \Omega_f^p \longrightarrow \Omega_X^p(\log D) \longrightarrow \Omega_X^p(\log D) \otimes \mathcal{O}_D \longrightarrow 0.$$

By [42], we have an exact sequence

$$0 \longrightarrow \Omega_X^p(\log D) \otimes \mathcal{O}_D \xrightarrow{\theta_p} A^{p,0} \xrightarrow{\delta''} \dots$$

where  $\theta_p(\eta) := (-1)^p [g^{-1}dg \wedge \eta \bmod W_0]$ . Hence, we obtain an exact sequence

$$0 \longrightarrow \Omega_f^p \xrightarrow{\rho} \tilde{A}^{p,0} \xrightarrow{\delta''} \tilde{A}^{p,1} \xrightarrow{\delta''} \dots$$

This implies that  $\rho$  is a filtered quasi-isomorphism.

Take the shifted cone  $B^\bullet := C^\bullet(\nu)[-1]$  of  $\nu$ . Define  $\varrho : \mathbf{E}_0^k \rightarrow B^k = s(\tilde{A}^{\bullet,\bullet})^k \oplus s(\tilde{A}^{\bullet,\bullet})^{k-1}$  as the restriction of the morphism

$$\begin{aligned} \Omega_{\mathfrak{X}}^k(\log(\mathfrak{D} \cup \{s = 0\}))_0 &= \Omega_X^k(\log D) \oplus s^{-1}ds\Omega_X^{k-1}(\log D) \ni \\ \omega_1 + s^{-1}ds\omega_2 &\mapsto \omega_1 \oplus \omega_2 \\ &\in A^{k,0} \oplus A^{k-1,0} \subset s(\tilde{A}^{\bullet,\bullet})^k \oplus s(\tilde{A}^{\bullet,\bullet})^{k-1}. \end{aligned}$$

Then  $\varrho$  gives a morphism of complexes. Indeed, it is trivial on  $X \setminus \pi_s(\Gamma)$ . On  $\pi_s(\Gamma)$ , take a section  $g\omega_1 + (s^{-1}ds - g^{-1}dg)\omega_2$  of  $\mathbf{E}_0^k$ . Note that  $[g\omega_1 \bmod W_0] = 0$ , and  $[dg \wedge \omega_1 \bmod W_0] = [g(g^{-1}dg \wedge \omega_1) \bmod W_0] = 0$ . Then we have

$$\begin{aligned} d\varrho(g\omega_1) &= (dg \wedge \omega_1 + gd\omega_1) \oplus 0 \\ &= \varrho(d(g\omega_1)), \\ d\varrho((s^{-1}ds - g^{-1}dg) \cdot \omega_2) &= d((-g^{-1}dg\omega_2) \oplus \omega_2) \\ &= (g^{-1}dg d\omega_2) \\ &\quad \oplus (-d\omega_2, [-g^{-1}dg \wedge \omega_2 + g^{-1}dg \wedge \omega_2 \bmod W_0]) \end{aligned}$$



$$\begin{aligned} &= (g^{-1}dg d\omega_2) \oplus -d\omega_2 \\ &= \varrho \circ d((s^{-1}ds - g^{-1}dg)\omega_2). \end{aligned}$$

We obtain the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & s^{-1}ds \cdot (\Omega_f^\bullet, d)[-1] & \longrightarrow & E_0 & \longrightarrow & (\Omega_f^\bullet, d) \longrightarrow 0 \\ & & \downarrow \rho & & \downarrow e & & \downarrow \rho \\ 0 & \longrightarrow & s(\tilde{A}^{\bullet, \bullet})[-1] & \longrightarrow & B^\bullet & \longrightarrow & s(\tilde{A}^{\bullet, \bullet}) \longrightarrow 0. \end{array}$$

Compatibility with  $\varphi_2$  and  $\nu$  follows from this diagram. □

Combining Theorems 3.14 and 3.23 and Proposition 3.29 we attain the following main theorem of this paper:

**Theorem 3.30.** *The filtrations  $F$  and  $W$  on  $V_f^k$  are identified with the Hodge filtration and the weight filtration on  $H^k(Y, Y_\infty; \mathbb{C})$ . In particular, the rescaling structure  $\mathcal{H}_f$  is of Hodge-Tate type if and only if the mixed Hodge structures  $(H^k(Y, Y_\infty; \mathbb{Q}), F, W)$  are Hodge-Tate for all  $k$ .*

We also have the equation

$$(3.16) \quad h^{p,q}(\mathcal{H}_f) = \dim \text{Gr}_{2p}^W H^{p+q}(Y, Y_\infty).$$

The right-hand side of (3.16) is denoted by  $h^{p,q}(Y, \mathbf{w})$  in Section 1. By Lemma 2.12 and Proposition 2.15, we obtain Theorem 1.1(1). Theorem 1.1(2) follows from Theorem 3.30 immediately.

*Remark 3.31.* A similar relation between  $V_f$  and  $H^\bullet(Y, Y_\infty)$  is obtained in [33, Thms. (4.3), (5.3)] in terms of Hodge modules. However, it is not clear whether the weight filtrations are the same as ours.

By the strictness of the morphisms of mixed Hodge structures [7, Thm. (2.3.5)], we have the following well-known fact (see [31, Cor. 3.8] for example):

**Lemma 3.32.** *Let  $V^i = (V_{\mathbb{Q}}^i, F, W)$  ( $i = 1, 2, 3$ ) be mixed  $\mathbb{Q}$ -Hodge structures, where  $V_{\mathbb{Q}}^i$  is the  $\mathbb{Q}$ -vector space,  $F$  is the Hodge filtration on  $V_{\mathbb{C}}^i := V_{\mathbb{Q}}^i \otimes \mathbb{C}$  and  $W$  is the weight filtration for each  $i$ . Assume that we have that*

$$V^1 \longrightarrow V^2 \longrightarrow V^3$$

*is an exact sequence of mixed  $\mathbb{Q}$ -Hodge structures.*

*Then for all  $k, p \in \mathbb{Z}$ , the sequences*

$$\text{Gr}_{-p}^F \text{Gr}_k^W V_{\mathbb{C}}^1 \longrightarrow \text{Gr}_{-p}^F \text{Gr}_k^W V_{\mathbb{C}}^2 \longrightarrow \text{Gr}_{-p}^F \text{Gr}_k^W V_{\mathbb{C}}^3$$

*of complex vector spaces are exact.*

Note that a mixed  $\mathbb{Q}$ -Hodge structure  $V = (V_{\mathbb{Q}}, F, W)$  is Hodge–Tate if and only if

$$\mathrm{Gr}_{-p}^F \mathrm{Gr}_{p+q}^W V_{\mathbb{C}} = 0$$

for  $p \neq q$ . Then we immediately have the following:

**Corollary 3.33.** *Let  $V^i$  be as in Lemma 3.32. If  $V^1$  and  $V^3$  are Hodge–Tate, then so is  $V^2$ .*

By the long exact sequence (3.15) of mixed Hodge structures, we have the following:

**Corollary 3.34.** *If the mixed Hodge structures  $H^k(Y)$  and  $H^k(Y_{\infty})$  are of Hodge–Tate type for all  $k$ , then  $\mathcal{H}_f$  is of Hodge–Tate type.*

## §4. Examples

In this section, we shall give some examples of the Landau–Ginzburg models  $(X, f)$  in Section 3 such that the induced rescaling structures  $\mathcal{H}_f$  are of Hodge–Tate type. In Section 4.1, we consider the case  $\dim X = 2$ . In Section 4.2, we consider the case  $\dim X = 3$ .

### §4.1. Two-dimensional examples

We shall prove the following:

**Proposition 4.1.** *Let  $f : X \rightarrow \mathbb{P}^1$  be a rational elliptic surface such that  $(f)_{\infty}$  is a reduced normal crossing, and  $D = |(f)_{\infty}|$  is a wheel of  $d$  smooth rational curves for  $2 \leq d \leq 9$ . Then the rescaling structure  $\mathcal{H}_f$  of  $(X, f)$  is of Hodge–Tate type.*

*Proof.* Since  $X$  is a rational surface, we have  $h^{p,q}(X) = 0$  for  $p \neq q$ . Since  $D$  is a wheel of  $d$  rational curves, the (co)homology of  $D$  is of Hodge–Tate type (see [31, Exa. 5.34] for example). We have the exact sequence of mixed Hodge structures [8, (9.2.1.2)]

$$\cdots \longrightarrow H^k(X) \longrightarrow H^k(Y) \longrightarrow H^{k-1}(D)(-1) \longrightarrow \cdots .$$

By Corollary 3.33, it follows that  $H^k(Y)$  are Hodge–Tate for all  $k$ . By the Clemens–Schmid exact sequence [20, (10.14), Thm. (10.16)], we have the following exact sequence of mixed Hodge structures:

$$H^k(D) \longrightarrow H^k(Y_{\infty}) \xrightarrow{N} H^k(Y_{\infty})(-1) \longrightarrow H_{2-k}(D)(-2),$$

where  $0 \leq k \leq 2$  and  $N$  is the nilpotent endomorphism. Since  $H^k(D)$  and  $H_{2-k}(D)$  are Hodge–Tate, by Corollary 3.33, we have the exact sequence

$$(4.1) \quad 0 \longrightarrow A_1 \longrightarrow H^k(Y_\infty) \longrightarrow H^k(Y_\infty)(-1) \longrightarrow A_2 \longrightarrow 0,$$

where  $A_1$  and  $A_2$  are Hodge–Tate. Let  $p_k(x, y)$  be the Hodge number polynomial of  $H^k(Y_\infty)$  (see [31, (II-1), Lem. 2.8 & (III-2)] for example). The exact sequence (4.1) implies that  $(1 - xy)p_k(x, y) = \sum_p a_p x^p y^p$  for some  $a_p$ . Hence, we have  $p_k(x, y) = \sum_p b_p x^p y^p$  for some  $b_p$ . Namely, we have that  $H^k(Y_\infty)$  is of mixed Hodge–Tate type for each  $k$ . By Corollary 3.34, we have the conclusion.  $\square$

By Theorem 3.30, Lemma 2.12, Proposition 2.15, (3.2) and (3.16), we obtain the following:

**Corollary 4.2.** *Let  $(X, f)$  be as in Proposition 4.1. Then we have  $f^{p,q}(Y, \mathbf{w}) = h^{p,q}(Y, \mathbf{w})$ , and  $\mathcal{H}_{f|\tau=1}$  is special.*

*Remark 4.3.* This example was studied by Auroux–Katzarkov–Orlov [1] as homological mirrors of del Pezzo surfaces. The equality of Hodge numbers  $f^{p,q}(Y, \mathbf{w})$  and  $h^{p,q}(Y, \mathbf{w})$  was proved by Lunts–Przjalkowski [28] who directly computed both of the numbers (the number  $f^{p,q}(Y, \mathbf{w})$  was also computed in Harder’s thesis [22]). Here, we gave a more conceptual proof of the equality. To the best of the author’s knowledge, the speciality of  $\mathcal{H}_{f|\tau=1}$  was not known.

### §4.2. Three-dimensional examples

We consider the toric Landau–Ginzburg models considered in Harder’s thesis [22].

**4.2.1. Fano polytope.** Let  $M$  be a free Abelian group of rank 3. Put  $M_{\mathbb{R}} := M \otimes \mathbb{R}$ , and  $N := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ . We have the natural pairing  $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$ . Define  $N_{\mathbb{R}}$  similarly. We consider an integral polytope  $\mathbf{P}$  with the following properties:

- (a) There is a finite set  $\{u_{\mathbf{F}} \mid \mathbf{F} \text{ is a facet of } \mathbf{P}\}$  of primitive vectors in  $N$  indexed by all facets of  $\mathbf{P}$  such that

$$\begin{cases} \mathbf{P} &= \{m \in M_{\mathbb{R}} \mid \langle m, u_{\mathbf{F}} \rangle \geq -1 \text{ for all } \mathbf{F}\}, \\ \mathbf{F} &= \{m \in \mathbf{P} \mid \langle m, u_{\mathbf{F}} \rangle = -1\}. \end{cases}$$

In particular, the origin  $0 \in M$  is contained in the interior of  $\mathbf{P}$ .

- (b) For each facet  $\mathbf{F}$ , the set of vertexes of  $\mathbf{F}$  form a basis of  $M$ . In particular,  $\mathbf{F}$  is a triangle whose interior does not contain the point of  $M$ .

*Remark 4.4.* Condition (a) is called reflexivity. Condition (b) implies that the cone generated by  $\mathbf{F}$  is smooth. These cones generate a smooth fan, which defines a smooth Fano variety.

**4.2.2. Toric varieties.** For a face  $Q$  of  $P$ , let  $\sigma_Q$  be the cone generated by  $\{u_F \mid Q \subset F\}$ . We remark that  $\sigma_P = \{0\}$  since  $\{0\}$  is the cone generated by the empty set. Then we have a fan  $\Sigma_P := \{\sigma_Q \mid Q \text{ is a face of } P\}$  (see [6, Thm. 2.3.2] for example). Although this fan is not smooth in general, we have a smooth refinement  $\Sigma$  of  $\Sigma_P$ . Since the dimension of  $\Sigma_P$  is 3, the refinement is given by a triangulation of the convex hull of the set  $\{u_F \mid F \text{ is a facet of } P\}$ . In particular, together with condition (a), we may assume that for every primitive vector  $u_\rho$  of a ray  $\rho$  in  $\Sigma$ , we have  $\min_{m \in P} \langle m, u_\rho \rangle = -1$ . The toric variety corresponding to  $\Sigma$  is denoted by  $X_\Sigma$ . It contains the algebraic torus  $T_N = \text{Spec}(\mathbb{C}[M])$  as an open dense subset. Put  $D_\Sigma := X_\Sigma \setminus T_N$ .

**4.2.3. A non-degenerate Laurent polynomial.** We consider a Laurent polynomial

$$f_P(\chi) = \sum_{m \in M} c_m \chi^m \in \mathbb{C}[M],$$

where  $c_m$  are complex numbers and  $\chi^m$  is the monomial corresponding to  $m \in M$ . The polynomial  $f_P$  is considered an algebraic function on  $T_N$ . Since  $T_N$  is an open dense subvariety of  $X_\Sigma$ ,  $f_P$  is considered a meromorphic function on  $X_\Sigma$ , whose pole divisor is contained in  $D_\Sigma$ . We impose the following non-degenerate condition on  $f_P$ :

- (c) The convex hull of  $\{m \mid c_m \neq 0\}$  in  $M_\mathbb{R}$  is  $P$ .
- (d) For every face  $Q \subset P$ , put  $f_Q(\chi) := \sum_{m \in Q} c_m \chi^m$ . Then the intersection of  $(df_Q)^{-1}(0)$  and  $f_Q^{-1}(0)$  in  $T_N$  is empty for every  $Q$ .

The meaning of the non-degenerate condition considering  $f_P$  as a meromorphic function on  $X_\Sigma$  is explained later.

**4.2.4. Coordinate system with respect to a cone.** Fix an isomorphism  $M \xrightarrow{\sim} \mathbb{Z}^3$ ;  $m \mapsto (m_1, m_2, m_3)$ . Let  $(e_i)_{i=1}^3$  be a canonical base of  $M$  via  $M \xrightarrow{\sim} \mathbb{Z}^3$ . We have an isomorphism  $\mathbb{C}[M] \xrightarrow{\sim} \mathbb{C}[x_1^\pm, x_2^\pm, x_3^\pm]$  by  $\chi^m \mapsto x_1^{m_1} x_2^{m_2} x_3^{m_3}$ . For a maximal cone  $\sigma \in \Sigma(3)$ , take primitive vectors  $u_\rho$  for rays  $\rho$  of  $\sigma$ . Then the open subvariety  $U_\sigma = \text{Spec}(\mathbb{C}[\sigma^\vee \cap M])$  of  $X_\Sigma$  has coordinate  $(y_\rho)_{\rho \in \sigma(1)}$ . The relation between the two coordinates is given by  $x_i = \prod_{\rho} y_\rho^{\langle e_i, u_\rho \rangle}$ . The function  $f_P$  considered as a meromorphic function on  $U_\sigma$  is given by

$$(4.2) \quad f_P(y) = \sum_{m \in P} c_m \prod_{\rho \in \sigma(1)} y_\rho^{\langle m, u_\rho \rangle}.$$

**4.2.5. Pole orders along invariant divisors.** For each ray  $\rho \in \Sigma(1)$ , we have the divisor  $D_\rho$  invariant under the action of  $T_N$ . If  $\rho \in \sigma(1)$ , the intersection

$U_\sigma \cap D_\rho$  is given by  $\{y_\rho = 0\}$ . Let  $Q_\rho$  be a face defined by

$$Q_\rho := \left\{ m \in P \mid \langle m, u_\rho \rangle = \min_{m' \in P} \langle m', u_\rho \rangle = -1 \right\}.$$

Note that  $Q_\rho \neq \emptyset$ . Equation (4.2) is written

$$(4.3) \quad f_P(y) = y_\rho^{-1} \left( y_\rho f_{Q_\rho}(y) + y_\rho \sum_{m \in P, \langle m, u_\rho \rangle \geq 0} c_m \prod_{\rho' \in \sigma(1)} y_{\rho'}^{\langle m, u_{\rho'} \rangle} \right).$$

Note that  $y_\rho f_{Q_\rho}(y)$  does not depend on  $y_\rho$ . The pole order along  $D_\rho$  is 1.

**4.2.6. Non-degenerate condition.** For a  $\tau \in \sigma(2)$ , take  $\rho, \rho' \in \sigma(1)$  so that  $\tau = \rho + \rho'$ . Put  $Q_\tau := Q_\rho \cap Q_{\rho'}$ . We have

$$(4.4) \quad f_P(y) = y_\rho^{-1} y_{\rho'}^{-1} \left( y_\rho y_{\rho'} f_{Q_\tau}(y) + y_\rho y_{\rho'} \sum_{\substack{m \in P, \\ \langle m, u_\rho \rangle \geq 0 \text{ or } \langle m, u_{\rho'} \rangle \geq 0}} c_m \prod_{\rho'' \in \sigma(1)} y_{\rho''}^{\langle m, u_{\rho''} \rangle} \right).$$

Note that  $y_\rho y_{\rho'} f_{Q_\tau}(y)$  does not depend on  $y_\rho$  or  $y_{\rho'}$ . There is also a similar description of  $f_P$  for the vertex  $Q_\sigma = \bigcap_{\rho \in \sigma(1)} Q_\rho$ . From these descriptions, we have the following properties of the zero divisor  $(f_P)_0$  in  $X_\Sigma$ :

- The divisor  $(f_P)_0$  is a (reduced) smooth hypersurface of  $X_\Sigma$ .
- The fixed points of the action of  $T_N$  are not contained in  $(f_P)_0$ .
- The divisor  $D_\Sigma \cup (f_P)_0$  is a simple normal crossing.

**4.2.7. Base locus.** Put  $B_\rho := |(f_P)_0| \cap D_\rho$  for all rays  $\rho$  in  $\Sigma$ .

**Lemma 4.5.** *For every  $\rho$ ,  $B_\rho$  is isomorphic to a projective line.*

*Proof.* By the non-degenerateness of  $f_P$ , all  $B_\rho$  are smooth curves in  $X_\Sigma$ . Since  $D_\Sigma \cup (f_P)_0$  is a normal crossing, the intersections of  $B_\rho$  and the lower-dimensional  $T_N$ -orbits in  $D_\rho$  are zero-dimensional. Therefore, it is enough to show that the intersection of  $|(f_P)_0|$  and the two-dimensional orbit in  $D_\rho$  is rational.

Take a facet  $F \subset P$  which contains  $Q_\rho$ . By assumptions (a), (b) in Section 4.2.1,  $F$  is a triangle, whose vertexes  $e_1, e_2, e_3$  form a  $\mathbb{Z}$ -basis of  $M$ . Using this basis, we take an isomorphism  $M \simeq \mathbb{Z}^3$ . Let  $(x_1, x_2, x_3)$  be the corresponding coordinate as in Section 4.2.4. Put

$$I := \{i \in \{1, 2, 3\} \mid e_i \text{ is a vertex of } Q_\rho\}.$$

Note that  $I \neq \emptyset$ , and  $f_{Q_\rho} = \sum_{i \in I} c_i x_i \neq 0$ .

Take  $\sigma \in \Sigma(2)$  so that  $\rho \in \sigma(1)$ . Let  $\rho_1 := \rho, \rho_2, \rho_3$  be the three ray of  $\sigma$ . Put  $y_i := y_{\rho_i}$  for  $i = 1, 2, 3$ . Then  $g := y_1 f_{\mathbb{Q}_\rho}$  is a Laurent polynomial depending only on  $y_2, y_3$ . We need to show that  $\{(y_2, y_3) \in (\mathbb{C}^*)^2 \mid g(y_2, y_3) = 0\}$  is rational. This space is isomorphic to the quotient space of  $\{(y_1, y_2, y_3) \in (\mathbb{C}^*)^3 \mid f_{\mathbb{Q}_\rho}(y_1, y_2, y_3) = 0\}$  by the  $\mathbb{C}^*$ -action defined by  $t \cdot (y_1, y_2, y_3) := (ty_1, y_2, y_3)$ .

Using the coordinate  $(x_1, x_2, x_3)$ , the  $\mathbb{C}^*$ -action is given by  $t \cdot (x_1, x_2, x_3) = (t^{-1}x_1, t^{-1}x_2, t^{-1}x_3)$  since  $\langle e_i, u_{\rho_1} \rangle = -1$ . We are considering the quotient space of  $\{(x_1, x_2, x_3) \in (\mathbb{C}^*)^3 \mid \sum_{i \in I} c_i x_i = 0\}$ . Since the quotient of  $\{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid \sum_{i \in I} c_i x_i = 0\}$  by the action defined above is a line in  $\mathbb{P}^2$ , we obtain the rationality.  $\square$

**4.2.8. Blow-ups.** Take an ordering  $\Sigma(1) = \{\rho_1, \dots, \rho_\ell\}$  for the set of all rays in  $\Sigma$ . We consider the following sequence of blow-ups:

$$X = X^{(\ell)} \xrightarrow{p^{(\ell-1)}} \dots \rightarrow X^{(j+1)} \xrightarrow{p^{(j)}} X^{(j)} \rightarrow \dots \xrightarrow{p^{(0)}} X^{(0)} = X_\Sigma,$$

where  $p^{(j)} : X^{(j+1)} \rightarrow X^{(j)}$  is the blow-up along the strict transform of  $B_{\rho_{j+1}}$  in  $X^{(j)}$ . The composition  $X \rightarrow X_\Sigma$  is denoted by  $\pi_\Sigma$ . The strict transform of  $D_{\rho_j}$  is denoted by  $D_j$  ( $1 \leq j \leq \ell$ ).

**Lemma 4.6.** *We have the following:*

- (1) *The divisor  $D_j$  is given by the composition of blow-ups of  $D_{\rho_j}$  along reduced 0-schemes.*
- (2) *The union  $D := \bigcup_j D_j$  is a simple normal crossing.*
- (3) *The pole divisor of  $\pi_\Sigma^* f_{\mathbb{P}}$  is reduced and the support  $|(\pi_\Sigma^* f_{\mathbb{P}})_\infty|$  is  $D$ .*
- (4) *The pull-back of  $f_{\mathbb{P}}$  by  $\pi_\Sigma$  gives a well-defined morphism  $\pi_\Sigma^* f_{\mathbb{P}} : X \rightarrow \mathbb{P}^1$ .*

*Proof.* Let  $\pi^{(i)} : X^{(i)} \rightarrow X^{(0)}$  be the composition  $p^{(i-1)} \circ \dots \circ p^{(0)}$  for  $i = 1, 2, \dots, \ell$ . We put  $\pi^{(0)} := \text{id}_{X^{(0)}}$ . Let  $f^{(i)}$  be the pull-back of  $f_{\mathbb{P}}$  by  $\pi^{(i)}$  for  $i = 0, 1, \dots, \ell$ . Let  $D_j^{(i)}$  (resp.  $B_j^{(i)}$ ) denote the strict transform of  $D_{\rho_j}$  (resp.  $B_{\rho_j}$ ) in  $X^{(i)}$  for  $i, j = 1, 2, \dots, \ell$ . Put  $D_j^{(0)} := D_{\rho_j}$  and  $B_j^{(0)} := B_{\rho_j}$ . We define  $D^{(i)} := \bigcup_j D_j^{(i)}$ . We shall prove the following by the induction on  $i$ :

- (1) <sub>$i$</sub>  *The divisor  $D_j^{(i)}$  is given by the composition of blow-ups of  $D_j^{(0)}$  along reduced 0-schemes.*
- (2) <sub>$i$</sub>  *The zero divisor  $(f^{(i)})_0$  is a reduced smooth hypersurface of  $X^{(i)}$ , and the union  $(f^{(i)})_0 \cup D^{(i)}$  is a simple normal crossing.*
- (3) <sub>$i$</sub>  *The pole divisor  $(f^{(i)})_\infty$  is reduced and the support  $|(f^{(i)})_\infty|$  is  $D^{(i)}$ .*
- (4) <sub>$i$</sub>  *The intersection  $(f^{(i)})_0 \cap (f^{(i)})_\infty \cap \left(\bigcup_{j=1}^i D_j^{(i)}\right)$  is empty.*

Note that  $(1)_0$ , and  $(4)_0$  are trivial. We also remark that  $(2)_0$  and  $(3)_0$  are shown in Sections 4.2.5 and 4.2.6.

Take  $i \in \{1, 2, \dots, \ell\}$ . Assume that  $(1)_{i-1}, (2)_{i-1}, (3)_{i-1}, (4)_{i-1}$  hold. Let  $Q$  be an arbitrary point in  $B_i^{(i-1)}$ . By assumption  $(2)_{i-1}, (3)_{i-1}$ , we have a local coordinate system  $(U_Q; z_0, z_1, z_2)$  centered at  $Q$  with the following properties:

$$D^{(i-1)} \cap U_Q = \bigcup_{i=1}^k \{z_i=0\}, \quad D_i^{(i-1)} \cap U_Q = \{z_1=0\} \quad \text{and} \quad f_{|U_Q}^{(i-1)}(z) = z_0 \cdot \prod_{i=1}^k z_i^{-1},$$

where  $k = 1$  or  $2$ . We have  $B_i^{(i-1)} \cap U_Q = \{z_0 = z_1 = 0\}$ . Let  $V_Q$  be the inverse image of  $U_Q$  by  $p^{(i-1)}$ . Then we have

$$V_Q = \{((z_0, z_1, z_2), [w_0 : w_1]) \in U_Q \times \mathbb{P}^1 \mid z_0 w_1 - z_1 w_0 = 0\}.$$

If  $k = 2$  and  $\{z_2 = 0\} = D_j^{(i-1)}$  then  $j > i$  by assumption  $(4)_{i-1}$ , and  $D_j^{(i)} \cap V_Q$  is given by the blow-up of  $D_j^{(i-1)} \cap U_Q$  at the reduced point  $Q$ . On  $V_Q^+ := V_Q \cap \{w_0 \neq 0\}$ , we have a local coordinate  $(u_0, u_1, u_2)$  with  $z_0 = u_0, z_1 = u_0 u_1, z_2 = u_2$  and  $w_1/w_0 = u_1$ . We have  $f_{|V_Q^+}^{(i)}(u) = \prod_{i=1}^k u_i^{-1}$ . The strict transform  $D_i^{(i)} \cap V_Q^+$  is given by  $\{u_1 = 0\}$ . On  $V_Q^- := V_Q \cap \{w_1 \neq 0\}$ , we have a local coordinate  $(v_0, v_1, v_2)$  with  $z_0 = v_0 v_1, z_1 = v_1, z_2 = v_2$  and  $w_0/w_1 = v_0$ . We have  $f_{|V_Q^-}^{(i)}(v) = v_0$  if  $k = 1$ , and  $f_{|V_Q^-}^{(i)}(v) = v_0 v_2^{-1}$  if  $k = 2$ . The strict transform  $D_i^{(i)} \cap V_Q^-$  is given by  $\{v_1 = 0\}$ . By this description and the assumptions, we have  $(1)_i, (2)_i, (3)_i, (4)_i$ . Then, by the induction, we obtain  $(1)_\ell, (2)_\ell, (3)_\ell, (4)_\ell$ . It is easy to prove that  $(1)_\ell, (2)_\ell, (3)_\ell, (4)_\ell$  implies the lemma.  $\square$

**4.2.9. Hodge–Tate condition.** We obtain the following:

**Proposition 4.7.** *Let  $f : X \rightarrow \mathbb{P}^1$  be the pull-back of  $f_{\mathbb{P}}$  by  $\pi_{\Sigma}$ . Then the rescaling structure  $\mathcal{H}_f$  is of Hodge–Tate type.*

*Proof.* By Lemma 4.6, the pair  $(X, f)$  satisfies the condition in Section 3. Since  $X$  is given by blow-ups of a toric manifold along projective lines,  $h^{p,q}(X) = 0$  for  $p \neq q$  ([44, Thm. 7.31]). Since  $D_j$  is given by the composition of blow-ups of  $D_{\rho_j}$  along reduced 0-schemes (Lemma 4.6(1)), and each  $D_i \cap D_j$  is isomorphic to  $\mathbb{P}^1$ , the (co)homology of  $D$  is Hodge–Tate (see [31, Exa. 5.34] for example). Hence, by Lemma 3.33 and the exact sequence

$$\dots \longrightarrow H^k(X) \longrightarrow H^k(Y) \longrightarrow H^{k-1}(D)(-1) \longrightarrow \dots,$$

we have that the mixed Hodge structure on  $H^k(Y)$  is Hodge–Tate for each  $k$ . By Corollary 3.34, it remains to show that the limit mixed Hodge structure  $H^k(Y_{\infty})$

is of Hodge–Tate type. From the Clemens–Schmid exact sequence [20, (10.14), Thm. (10.16)], we obtain the exact sequence of mixed Hodge structures

$$H^k(D) \longrightarrow H^k(Y_\infty) \xrightarrow{N} H^k(Y_\infty)(-1) \longrightarrow H_{4-k}(D)(-3),$$

where  $0 \leq k \leq 4$ . By Corollary 3.33, we have the exact sequence

$$0 \longrightarrow A_1 \longrightarrow H^k(Y_\infty) \longrightarrow H^k(Y_\infty)(-1) \longrightarrow A_2 \longrightarrow 0,$$

where  $A_1$  and  $A_2$  are Hodge–Tate. Then, by a similar argument to the proof of Proposition 4.1,  $H^k(Y_\infty)$  is also Hodge–Tate for each  $k$ .  $\square$

Similarly to Corollary 4.2, we have the following:

**Corollary 4.8.** *Let  $(X, f)$  be as in Proposition 4.7. Then we have  $f^{p,q}(Y, \mathbf{w}) = h^{p,q}(Y, \mathbf{w})$ . We also have that  $\mathcal{H}_{f|\tau=1}$  is special.*

*Remark 4.9.* In [22], A. Harder computed the number  $f^{p,q}(Y, \mathbf{w})$  and compared it with the Hodge number of the smooth toric Fano manifold  $X_{\mathbf{P}}$  associated to  $\mathbf{P}$  [22, Thm. 2.3.7]. In [32], Reichelt–Sevenheck studied the hypergeometric  $\mathcal{D}$ -module associated to (a family of)  $f_{\mathbf{P}}$ , and solved a kind of Birkhoff problem. The result here is a priori different from theirs since the cohomology considered here is different from the one considered in [32]. We also remark that T. Mochizuki informed us that we can obtain similar but a priori different results from the viewpoint of twistor  $\mathcal{D}$ -modules.

## Appendix A. Rescaling structures for quantum $\mathcal{D}$ -modules of Fano manifolds

### Appendix A.1. Square roots of Tate twists

We use the notation in Section 2.2. Set  $\mathbb{T}^{1/2} := \mathcal{O}_S(*(\lambda)_\infty)w$  where  $w$  is a global section with  $\deg w = 1$ . We define a connection  $\nabla$  on  $\mathbb{T}^{1/2}$  by  $\nabla w := -(1/2)w\lambda^{-1}d\lambda$ . Since  $p_2^*(\mathbb{T}^{1/2}, \nabla)$  is not isomorphic to  $\sigma^*(\mathbb{T}^{1/2}, \nabla)$ ,  $(\mathbb{T}^{1/2}, \nabla)$  is not equipped with a rescaling structure. However, we have a flat isomorphism  $(\mathbb{T}^{1/2})^{\otimes 2} \xrightarrow{\sim} \mathbb{T}; w^{\otimes 2} \mapsto v$ . Hence we use the notation  $\mathbb{T}(-1/2) := (\mathbb{T}^{1/2}, \nabla)$ . For each  $k \in \mathbb{Z}$ , we define

$$(A.1) \quad \mathbb{T}(-k/2) := \begin{cases} \mathbb{T}(-1/2)^{\otimes k} & (k \geq 0), \\ (\mathbb{T}(-1/2)^\vee)^{\otimes -k} & (k < 0). \end{cases}$$

In the case where  $k \in 2\mathbb{Z}$ ,  $\mathbb{T}(k/2)$  is identified with the rescaling structure defined in Example 2.6. For a meromorphic connection  $(\mathcal{H}, \nabla)$  as in Definition 2.5, we also define  $\mathcal{H}(k/2) := \mathcal{H} \otimes \mathbb{T}(k/2)$ .



**Appendix A.2. Tate twisted quantum  $\mathcal{D}$ -modules**

Let  $F$  be a smooth projective Fano variety over  $\mathbb{C}$  of dimension  $n$ . Put  $\mathrm{HH}_a(F) := \bigoplus_{a=q-p} H^q(F, \Omega_F^p)$ . Set  $\mathrm{HH}_\bullet(F) := \bigoplus_a \mathrm{HH}_a(F)$  and identify it with  $H^\bullet(F; \mathbb{C})$  by the Hodge decomposition. Let  $\star_\tau$  be the quantum cup product of  $F$  with respect to the parameter  $c_1(F) \log \tau \in H^2(F; \mathbb{C})$ , where  $c_1(F)$  is the first Chern class of the tangent bundle of  $F$ . This is well defined for all  $\tau \in \mathbb{C}$ . Indeed, the right-hand side of

$$(A.2) \quad (\alpha \star_\tau \beta, \gamma)_F = \sum_{d \in H_2(F; \mathbb{Z})} \langle \alpha, \beta, \gamma \rangle_{0,3,d}^F \tau^{c_1(F) \cdot d}$$

is a finite sum since  $F$  is Fano, where  $\alpha, \beta, \gamma \in H^\bullet(F; \mathbb{C}) \simeq \mathrm{HH}_\bullet(F)$ ,  $(\cdot, \cdot)_F$  denotes the Poincaré pairing, and  $\langle \cdot, \cdot, \cdot \rangle_{0,3,d}^F$  denotes genus-zero 3-points Gromov–Witten invariant of degree  $d \in H_2(F; \mathbb{Z})$  (see [3], [4], [5], and references therein).

For any non-negative integer  $k$ , we take a finite rank free  $\mathcal{O}_S(*(\lambda)_\infty)$ -module  ${}^a H^k := \mathrm{HH}_{k-n}(F) \otimes \mathcal{O}_S(*(\lambda)_\infty)$ . The  $\mathbb{Z}$ -grading of  ${}^a H^k$  is defined to be 0. Define  $\mu_F \in \mathrm{End}(\mathrm{HH}_{k-n}(F))$  by  $\mu_F|_{H^q(F, \Omega_F^p)} := (p + q - n)/2 \cdot \mathrm{id}_{H^q(F, \Omega_F^p)}$ . We also have an endomorphism  $c_1(F) \star_\tau$  on  $\mathrm{HH}_{k-n}(F)$ . We have the Dubrovin connection  ${}^a \nabla$  on  ${}^a H^k$  as follows ([13], [14], [15]):

$${}^a \nabla := d + \frac{c_1(F) \star_\tau}{\lambda} \frac{d\tau}{\tau} + \mu_F \frac{d\lambda}{\lambda} - c_1(F) \star_\tau \frac{d\lambda}{\lambda^2}.$$

**Proposition A.1.** *We have that  $\mathcal{H}_F^k := {}^a H^k(-k/2)$  comes equipped with a rescaling structure.*

*Proof.* We have that  $\mathcal{H}_F^k$  is identified with the free  $\mathcal{O}_S(*(\lambda)_\infty)$ -module  $\mathrm{HH}_{k-n}(F) \otimes \mathcal{O}_S(*(\lambda)_\infty)$  with the connection

$$\nabla = d + \frac{c_1(F) \star_\tau}{\lambda} \frac{d\tau}{\tau} + \left( \mu_F - \frac{k}{2} \cdot \mathrm{id} \right) \frac{d\lambda}{\lambda} - c_1(F) \star_\tau \frac{d\lambda}{\lambda^2}.$$

Taking the pull-back by  $\sigma : \mathbb{C}_\theta^* \times S \rightarrow S; (\theta, \lambda, \tau) \mapsto (\theta\lambda, \theta\tau)$ , we have

$$\sigma^* \nabla = d + \frac{c_1(F) \star_{\theta\tau}}{\theta\lambda} \frac{d\tau}{\tau} + \left( \mu_F - \frac{k}{2} \cdot \mathrm{id} \right) \left( \frac{d\lambda}{\lambda} + \frac{d\theta}{\theta} \right) - \frac{c_1(F) \star_{\theta\tau}}{\theta} \frac{d\lambda}{\lambda^2}.$$

Put  $\mu_k := \mu_F - (k/2) \cdot \mathrm{id}$ . On  $H^q(F, \Omega_F^p)$  with  $q - p = k - n$ , we have  $\mu_k = (q - k) \cdot \mathrm{id} = (p - n) \cdot \mathrm{id}$ . Hence we have a morphism of  $\mathcal{O}_{\mathbb{C}_\theta^* \times S}(*(\lambda)_\infty)$ -modules:

$$\theta^{-\mu_k} : p_2^* \mathcal{H}_F^k \xrightarrow{\sim} \sigma^* \mathcal{H}_F^k.$$

By (A.2), we obtain

$$c_1 \star_\tau = \theta^{\mu_k} \left( \frac{c_1(F) \star_{\theta\tau}}{\theta} \right) \theta^{-\mu_k},$$

which implies that  $\theta^{-\mu_k}$  is flat with respect to the connections (see [19, Sect. 2.2] for example).  $\square$

**Definition A.2.** We define a rescaling structure  $\mathcal{H}_F$  by

$$\mathcal{H}_F := \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_F^k.$$

We call  $\mathcal{H}_F$  a Tate twisted quantum  $\mathcal{D}$ -module of  $F$ .

*Remark A.3.* The  $\mathbb{Z}/2\mathbb{Z}$ -graded flat meromorphic connection  ${}^a H$  in the introduction (or [26]) is given by  ${}^a H = \bigoplus_k {}^a H^k$ , where the  $\mathbb{Z}/2\mathbb{Z}$ -grading on  ${}^a H^k$  is given by  $(k \bmod 2)$ .

**Appendix A.3. Hodge–Tate condition and Hodge numbers**

The fiber of  $\mathcal{H}_F^k$  at  $(\lambda, \tau) = (1, 0)$  is naturally identified with  $\mathrm{HH}_{k-n}(F)$ . We shall describe the Hodge and weight filtrations on  $\mathrm{HH}_{k-n}(F)$  in the sense of Section 2.3.1.

As we have seen in the proof of Proposition A.1, the  $\mathbb{C}^*$ -action on  $\mathcal{H}_F^k|_{\tau=0}$  is given by  $\theta^{-(q-k)} = \theta^{-(p-n)}$  on  $H^q(F, \Omega_F^p) \otimes \mathcal{O}_{\mathbb{C}^\times}$  with  $q - p = k - n$ . Hence the Hodge filtration on  $\mathrm{HH}_{k-n}(F)$  is

$$(A.3) \quad F_i \mathrm{HH}_{k-n}(F) = \bigoplus_{\substack{p-n \leq i, \\ q-p=k-n}} H^q(F, \Omega_F^p).$$

We obtain  $f^{p,q}(\mathcal{H}_F) = \dim H^q(F, \Omega_F^{n-p}) = h^{n-p,q}(F)$ .

The residue endomorphism  $N_k := \mathrm{Res}_\tau \nabla$  on  $\mathrm{HH}_{k-n}(F)$  is identified with  $c_1(F) \cup$ . It follows that the monodromy weight filtration centered at  $k$  is

$$(A.4) \quad {}^k W_i \mathrm{HH}_{k-n}(F) = \bigoplus_{\substack{p \geq n-i/2, \\ q-p=k-n}} H^q(F, \Omega_F^p).$$

Hence, we have  $h^{p,q}(\mathcal{H}_F) = h^{n-p,q}(F)$ . By (A.3) and (A.4), we obtain the following:

**Proposition A.4.** *The Tate twisted quantum  $\mathcal{D}$ -module  $\mathcal{H}_F$  satisfies the Hodge–Tate condition for any smooth projective Fano variety  $F$ .*

**Appendix B. Relation to the work of Katzarkov–Kontsevich–Pantev**

**Appendix B.1. Tame compactified Landau–Ginzburg model**

In [26], Katzarkov–Kontsevich–Pantev considered the following:

**Definition B.1** ([26, Def. 2.4, (T)]; see also [28, Def. 3]). A tame compactified Landau–Ginzburg model is a tuple  $((X, f), D, \text{vol}_X)$ , where

- (1)  $X$  is a smooth projective variety and  $f : X \rightarrow \mathbb{P}^1$  is a flat projective morphism;
- (2)  $D = (\bigcup_i D_i^h) \cup (\bigcup_j D_j^v) \subset X$  is a reduced normal crossing divisor such that
  - (a)  $D^v = \bigcup_j D_j^v$  is a scheme-theoretic pole divisor of  $f$ , i.e.,  $(f)_\infty = D^v$ , and in particular, the pole order of  $f$  along  $D_j^v$  is 1;
  - (b) each component  $D_i^h$  of  $D^h := \bigcup_i D_i^h$  is smooth and horizontal for  $f$ , i.e.,  $f|_{D_i^h}$  is a flat morphism;
  - (c) the critical locus of  $f$  does not intersect  $D^h$ ;
- (3)  $\text{vol}_X$  is a nowhere vanishing meromorphic section of the canonical bundle  $K_X$  with poles of order exactly 1 along each component of  $D$ . In other words, we have an isomorphism  $\mathcal{O}_X \xrightarrow{\sim} K_X(D); 1 \mapsto \text{vol}_X$ .

In this paper (Section 3), the horizontal divisor  $D^h$  is assumed to be empty, and each component  $D_j^v$  is assumed to be smooth. Although we do not impose the existence of  $\text{vol}_X$  in Section 3, all examples in Section 4 have  $\text{vol}_X$ .

### Appendix B.2. Landau–Ginzburg Hodge numbers

The Hodge number  $f^{p,q}(Y, \mathbf{w})$  in this paper corresponds to  $f^{q,p}(Y, \mathbf{w})$  in [26, Def. 3.1]. The definition in this paper is suited to the convention in classical Hodge theory. The number  $h^{p,q}(Y, \mathbf{w})$  in [26] is  $\dim \text{Gr}_p^W H^{p+q}(Y, Y_\infty)$  in our notation. Our definition of  $h^{p,q}(Y, \mathbf{w})$  is  $\dim \text{Gr}_{2p}^W H^{p+q}(Y, Y_\infty)$ , which is different from their definition. As mentioned in [28], their definition seems not to be what they had in mind. The definition of  $h^{p,q}(Y, \mathbf{w})$  in this paper corresponds to  $h^{q,p}(Y, \mathbf{w})$  in [28, Def. 3]. In [28], they also give a counterexample for the part of equality with the numbers  $i^{p,q}(Y, \mathbf{w})$  in [26, Conj. 3.6].

### Appendix B.3. One-parameter families

Recall that  $S = \mathbb{P}_\lambda^1 \times \mathbb{C}_\tau$ . We also recall that  $\pi_S : S \times X \rightarrow X$  and  $p_S : S \times X \rightarrow S$  denote the projections. Put

$$\Omega_{X,S}^k(*D) := \mathcal{O}_{X \times S}(*(\lambda)_\infty) \otimes \pi_S^{-1} \Omega_X^k(*D).$$

Let  ${}^b H^k$  be the  $\mathcal{O}_S(*(\lambda)_\infty)$ -module defined by

$${}^b H^k := \mathbb{R}^k p_{S*}(\Omega_{X,S}^\bullet(*D), \lambda d + \tau df \wedge).$$

Let  $\nabla : \Omega_{X,S}^\bullet(*D) \rightarrow \Omega_{X,S}^\bullet(*D) \otimes p_S^* \Omega_S^1(*|(\lambda\tau)_0|)$  be the connection on  $\Omega_{X,S}^\bullet(*D) := \bigoplus_k \Omega_{X,S}^k(*D)$  defined by

$$\nabla = d_S + \frac{f}{\lambda} d\tau + G \frac{d\lambda}{\lambda} - \tau f \frac{d\lambda}{\lambda^2},$$

where  $G = -(k/2)\text{id}$  on  $\Omega_{X,S}^k(*D)$ . Then we have  $[\nabla_{\partial_\tau}, \lambda d + \tau df \wedge] = 0$ , and  $[\nabla_{\partial_\lambda}, \lambda d + \tau df \wedge] = (2\lambda)^{-1}(\lambda d + \tau df \wedge)$ . Let  $\mathcal{A}_X^{p,q}$  be the sheaf of  $(p, q)$ -forms on  $X$  and  $\partial$  and  $\bar{\partial}$  be the Dolbeault operators. Put  $\mathcal{A}_{X,S,D}^{p,q} := \Omega_{X,S}^p(*D) \otimes_{\pi_S^{-1}\mathcal{O}_X} \pi_S^{-1}\mathcal{A}_X^{0,q}$ . Let  $\partial : \mathcal{A}_{X,S,D}^{p,q} \rightarrow \mathcal{A}_{X,S,D}^{p+1,q}$ , and  $\bar{\partial} : \mathcal{A}_{X,S,D}^{p,q} \rightarrow \mathcal{A}_{X,S,D}^{p,q+1}$  be the induced operators. Put  $\mathcal{A}_{X,S,D}^\ell := \bigoplus_{p+q=\ell} \mathcal{A}_{X,S,D}^{p,q}$  and

$$d_{\text{tot}} := \lambda \partial + \bar{\partial} + \tau \partial f : \mathcal{A}_{X,S,D}^\ell \rightarrow \mathcal{A}_{X,S,D}^{\ell+1}.$$

We have a natural quasi-isomorphism

$$\iota_{\text{Dol}} : (\Omega_{X,S}^\bullet(*D), \lambda d + \tau df) \xrightarrow{\sim} (\mathcal{A}_{X,S,D}^\bullet, d_{\text{tot}}).$$

We also have the connection  $\nabla : \mathcal{A}_{X,S,D}^\bullet \rightarrow \mathcal{A}_{X,S,D}^\bullet \otimes \Omega_S^1(*|(\lambda\tau)_0|)$  by

$$\nabla := d_S + \frac{f}{\lambda} d\tau + \mu_f \frac{d\lambda}{\lambda} - \tau f \frac{d\lambda}{\lambda^2},$$

where  $\mu_f|_{\mathcal{A}_{X,S,D}^{p,q}} = 2^{-1}(q-p) \cdot \text{id}$ . Then  $\iota_{\text{Dol}} \circ \nabla = \nabla \circ \iota_{\text{Dol}}$  by definition. We have  $[\nabla_{\partial_\tau}, d_{\text{tot}}] = 0$ , and

$$\begin{aligned} [\nabla_{\partial_\lambda}, d_{\text{tot}}] &= [\partial_\lambda + \lambda^{-1}\mu_f - \lambda^{-2}\tau f, \lambda \partial + \bar{\partial} + \tau \partial f] \\ &= \partial - (1/2)\partial + (1/2)\lambda^{-1}\bar{\partial} - (1/2)\lambda^{-1}\tau \partial f + \lambda^{-1}\tau \partial f \\ &= (2\lambda)^{-1}(\lambda \partial + \bar{\partial} + \tau \partial f) = (2\lambda)^{-1}d_{\text{tot}}. \end{aligned}$$

Hence  $\nabla$  gives a connection  ${}^b\nabla^k$  on  ${}^bH^k \simeq \mathcal{H}^k p_{S*}(\mathcal{A}_{X,S,D}^\bullet, d_{\text{tot}})$ . We remark that similar discussions are given in [16] and [25].

**Lemma B.2.** *For each  $k \in \mathbb{Z}_{\geq 0}$ , we have  $({}^bH^k, {}^b\nabla^k)(-k/2) \simeq \mathcal{H}_f^k$ .*

*Proof.* We have a natural isomorphism  $({}^bH^k, {}^b\nabla^k)(-k/2) \simeq ({}^bH^k, {}^b\nabla^k - (k/2)\lambda^{-1}d\lambda)$ . Then the connection  ${}^b\nabla^k - (k/2)\lambda^{-1}d\lambda$  is induced from the following connection on  $\mathcal{A}_{X,S,D}^\bullet$ :

$$\nabla' := d_S + \frac{f}{\lambda} d\tau + P \frac{d\lambda}{\lambda} - \tau f \frac{d\lambda}{\lambda^2},$$

where  $P|_{\mathcal{A}_{X,S,D}^{p,q}} = 2^{-1}((q-p) - (p+q)) \cdot \text{id} = (-p) \cdot \text{id}$ . Note that  $[\nabla', d_{\text{tot}}] = 0$ . Moreover, it is induced from the following connection on  $\Omega_{X,S}^\bullet(*D)$ :

$$\nabla' = d_S + \frac{f}{\lambda} d\tau + P \frac{d\lambda}{\lambda} - \tau f \frac{d\lambda}{\lambda^2},$$

where  $P|_{\Omega_{X,S}^p(*D)} = (-p) \cdot \text{id}$ . We also remark that  $[\nabla', \lambda d + \tau df] = 0$ . Then the quasi-isomorphism

$$\text{iso} : (\Omega_{f,\lambda,\tau}^\bullet, d + \lambda^{-1}\tau df) \xrightarrow{\sim} (\Omega_{X,S}^\bullet(*D), \lambda d + \tau df)$$

on  $S^* \times X = (\mathbb{C}_\lambda^* \times \mathbb{C}_\tau^*) \times X$  defined by  $\text{iso}|_{\Omega_{f,\lambda,\tau}^p} = \lambda^p$  induces the conclusion naturally.  $\square$

*Remark B.3.* It seems that the connection on  ${}^bH$  which Katzarkov–Kontsevich–Pantev had in mind in [26, (3.2.2)] was the one where  $f$  is replaced by  $qf$ . The dual of it (or, the connection  $({}^bH, {}^b\nabla)$  defined first in [26, Sect. 3.2.2]) is isomorphic to  $\bigoplus_{k \in \mathbb{Z}} ({}^bH^k, {}^b\nabla^k)$ .

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