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Grothendieck Duality and $\mathbb{Q}$-Gorenstein Morphisms

by

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Abstract

The notions of a $\mathbb{Q}$-Gorenstein scheme and a $\mathbb{Q}$-Gorenstein morphism are introduced for locally Noetherian schemes by dualizing complexes and (relative) canonical sheaves. These cover all the previously known notions of a $\mathbb{Q}$-Gorenstein algebraic variety and a $\mathbb{Q}$-Gorenstein deformation satisfying the Kollár condition, over a field. By studying the relative $\mathbf{S}_2$-condition and base change properties, valuable results are proved for $\mathbb{Q}$-Gorenstein morphisms, which include the infinitesimal criteria, the valuative criterion, and $\mathbb{Q}$-Gorenstein refinements.

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Contents

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§1. Introduction

The notion of a $\mathbb{Q}$-Gorenstein variety is important for the minimal model theory of algebraic varieties in characteristic zero: A normal algebraic variety X defined over a field of any characteristic is said to be $\mathbb{Q}$-Gorenstein if rK_X is Cartier for some positive integer r, where K_X stands for the canonical divisor of X. In some papers, X is additionally required to be Cohen–Macaulay. Reid used this notion essentially to define the canonical singularity in [49, Def. (1.1)], and he named the notion "$\mathbb{Q}$-Gorenstein" in [50, (0.12.e)], where the Cohen–Macaulay condition is required. The notion without the Cohen–Macaulay condition appears in [24] for example. In the minimal model theory of algebraic varieties of dimension more than two, we must deal with varieties with mild singularities such as terminal, canonical, log-terminal, and log-canonical (cf. [24, §0–2] for the definition). The notion of $\mathbb{Q}$-Gorenstein is hence frequently used in studying the higher dimensional birational geometry.

The notion of a $\mathbb{Q}$-Gorenstein deformation is also popular in the study of degenerations of normal algebraic varieties in characteristic zero related to the minimal model theory and the moduli theory since the paper [31] by Kollár and Shepherd-Barron. Roughly speaking, a $\mathbb{Q}$-Gorenstein deformation $\mathcal{X} \to C$ of a $\mathbb{Q}$-Gorenstein normal algebraic variety X is considered as a flat family of algebraic varieties over a smooth curve C with a closed fiber being isomorphic to X such that $rK_{\mathcal{X}/C}$ is Cartier and $rK_{\mathcal{X}/C}|_X \sim rK_X$ for some $r > 0$, where $K_{\mathcal{X}/C}$ stands for the relative canonical divisor. We call such a deformation "naively $\mathbb{Q}$-Gorenstein" (cf. Definition 7.1 below). This is said to be "weakly $\mathbb{Q}$-Gorenstein" in [15, §3], or satisfying *Viehweg's condition* (cf. [21, §2, Property $\mathbf{V}^{[N]}$]). We say that $\mathcal{X} \to C$ is a $\mathbb{Q}$-Gorenstein deformation if

$$\mathcal{O}_{\mathcal{X}}(mK_{\mathcal{X}/C}) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_X \simeq \mathcal{O}_X(mK_X)$$

for any integer m. This additional condition seems to be considered first by Kollár [27, 2.1.2], and it is called the *Kollár condition*; a similar condition is named Property $\mathbf{K}$ in [21, §2] for example. A typical example of a $\mathbb{Q}$-Gorenstein deformation appears as a deformation of the weighted projective plane $\mathbb{P}(1, 1, 4)$: Its versal deformation space has two irreducible components, in which the one-dimensional component corresponds to the $\mathbb{Q}$-Gorenstein deformation and its general fibers are $\mathbb{P}^2$ (cf. [46, §8]). The $\mathbb{Q}$-Gorenstein deformation is also used in constructing some simply connected surfaces of general type over the complex number field $\mathbb{C}$ in [34]. The authors have succeeded in generalizing the construction to the positive characteristic case in [33], where a special case of $\mathbb{Q}$-Gorenstein deformation over a mixed characteristic base scheme is considered.

During the preparation of the joint paper [33], the authors began generalizing the notion of a $\mathbb{Q}$-Gorenstein morphism to the case of morphisms between locally Noetherian schemes. The purpose of this article is to give good definitions of $\mathbb{Q}$-*Gorenstein scheme* and $\mathbb{Q}$-*Gorenstein morphism*: We define the notion of "$\mathbb{Q}$-Gorenstein" for locally Noetherian schemes admitting dualizing complexes (cf. Definition 6.1 below) and define the notion of "$\mathbb{Q}$-Gorenstein" for flat morphisms locally of finite type between locally Noetherian schemes (cf. Definition 7.1 below). So, we try to define the notion of "$\mathbb{Q}$-Gorenstein" as generally as possible. We do not require the Cohen–Macaulay condition for fibers, which is assumed in most articles on $\mathbb{Q}$-Gorenstein deformations, and we allow all locally Noetherian schemes as the base scheme of a $\mathbb{Q}$-Gorenstein morphism.

$\mathbb{Q}$-Gorenstein schemes and $\mathbb{Q}$-Gorenstein morphisms

The definition of a $\mathbb{Q}$-Gorenstein scheme in Definition 6.1 below is interpreted as follows (cf. Lemma 6.4(3)): A locally Noetherian scheme X is said to be $\mathbb{Q}$-Gorenstein if and only if

- it satisfies Serre's condition $\mathbf{S}_2$,
- it is Gorenstein in codimension one,
- there exists a dualizing sheaf $\mathcal{L}$ locally on X and the double dual of $\mathcal{L}^{\otimes r}$ is invertible for some integer $r > 0$ locally on X.

Here we consider a dualizing sheaf (cf. Definition 4.13) as the 0th cohomology $\mathcal{H}^0(\mathcal{R}^\bullet)$ of an ordinary dualizing complex $\mathcal{R}^\bullet$, which is a dualizing complex of special type and exists for any *locally equi-dimensional* (cf. Definition 2.2(3)) and locally Noetherian schemes admitting dualizing complexes (cf. Lemma 4.14). The dualizing sheaf of the Gorenstein locus $U = \mathrm{Gor}(X)$ is isomorphic to $\mathcal{L}|_U$ up to tensor product with invertible sheaves, and $\mathcal{L}$ is a reflexive $\mathcal{O}_X$-module with an isomorphism $\mathcal{L} \simeq j_*(\mathcal{L}|_U)$ for the open immersion $j\colon U \hookrightarrow X$. This definition generalizes the usual definition of $\mathbb{Q}$-Gorenstein normal algebraic varieties over a field (cf. Example 6.6).

On the other hand, in order to define $\mathbb{Q}$-Gorenstein morphisms, we need to discuss the *relative canonical sheaf* of an $\mathbf{S}_2$-morphism. An $\mathbf{S}_2$-morphism is defined as a flat morphism of locally Noetherian schemes which is locally of finite type and every fiber satisfies Serre's condition $\mathbf{S}_2$ (cf. Definition 2.30). By Conrad [7, Sect. 3.5] and Sastry [51], we have a good notion of the relative canonical sheaf for Cohen–Macaulay morphisms. For an $\mathbf{S}_2$-morphism $f\colon Y \to T$ of locally Noetherian schemes, the relative Cohen–Macaulay locus $Y^\flat = \mathrm{CM}(Y/T)$ is an open subset (cf. Definition 2.28 and Fact 2.29), and we define the relative canonical sheaf $\omega_{Y/T}$ as the direct image by the open immersion $Y^\flat \subset Y$ of the relative canonical sheaf $\omega_{Y^\flat/T}$ of the Cohen–Macaulay morphism $Y^\flat \to T$ (cf. Definition 5.3). The sheaf $\omega_{Y/T}$ is coherent (cf. Proposition 5.5), and it is reflexive when every fiber is Gorenstein in codimension one (cf. Proposition 5.6). We set $\omega_{Y/T}^{[m]}$ to be the double dual of $\omega_{Y/T}^{\otimes m}$ for $m \in \mathbb{Z}$, and we define $\mathbb{Q}$-Gorenstein morphisms as follows: A flat morphism $f: Y \to T$ locally of finite type between locally Noetherian schemes is called a $\mathbb{Q}$-Gorenstein morphism (cf. Definition 7.1) if and only if

- every fiber is a $\mathbb{Q}$-Gorenstein scheme and
- $\omega_{Y/T}^{[m]}$ satisfies relative $\mathbf{S}_2$ over T for any $m \in \mathbb{Z}$.

Note that f is an $\mathbf{S}_2$-morphism and every fiber is Gorenstein in codimension one, by the first condition. The second condition corresponds to the Kollár condition. A weaker notion, a naively $\mathbb{Q}$-Gorenstein morphism, is defined by replacing the second condition with

- $\omega_{Y/T}^{[m]}$ is invertible for some m locally on Y.

This condition corresponds to Viehweg's condition.

By our definition above, we can consider $\mathbb{Q}$-*Gorenstein deformations* $f: Y \to T$ of a $\mathbb{Q}$-Gorenstein scheme X defined over a field $\Bbbk$. Here f is a $\mathbb{Q}$-Gorenstein morphism, T contains a point o with residue field $\Bbbk$, and the fiber $Y_o = f^{-1}(o)$ is isomorphic to X over $\Bbbk$. The scheme X is not necessarily assumed to be reduced nor normal, and f is not necessarily a morphism of $\Bbbk$-schemes. The $\mathbb{Q}$-Gorenstein deformations of non-normal schemes have been treated in articles in some special cases: Hacking [15] and Tziolas [56] consider $\mathbb{Q}$-Gorenstein deformations of slc surfaces, which are not normal in general, over $\mathbb{C}$. The work of Abramovich–Hassett [1] covers also non-normal reduced Cohen–Macaulay algebraic schemes over a fixed field. By further study of $\mathbb{Q}$-Gorenstein morphisms, we may have a well-defined theory of infinitesimal $\mathbb{Q}$-Gorenstein deformations, which is now in progress in the authors' joint work.

Notable results on $\mathbb{Q}$-Gorenstein morphisms

Some expected properties on $\mathbb{Q}$-Gorenstein morphisms can be verified by standard methods. For example, we prove that $\mathbb{Q}$-Gorenstein morphisms are stable under base change and composition (cf. Propositions 7.22(5) and 7.23(3)). Besides such elementary properties, we have notable results in the topics below, which show that our definition of $\mathbb{Q}$-Gorenstein morphism is reasonable and widely applicable:

(1) A sufficient condition for a virtually $\mathbb{Q}$-Gorenstein morphism to be $\mathbb{Q}$-Gorenstein
(2) Infinitesimal and valuative criteria for a morphism to be $\mathbb{Q}$-Gorenstein

(3) Some conditions on fibers related to Serre's $\mathbf{S}_3$-condition which are sufficient for a morphism to be $\mathbb{Q}$-Gorenstein
(4) The existence of $\mathbb{Q}$-Gorenstein refinement

We shall explain results on these topics briefly.

(1): The virtually $\mathbb{Q}$-Gorenstein morphism is introduced in Section 7.2 as a weak form of a $\mathbb{Q}$-Gorenstein morphism (cf. Definition 7.13). This is inspired by the definition [15, Def. 3.1] by Hacking on a $\mathbb{Q}$-Gorenstein deformation of an slc surface in characteristic zero: His definition is generalized to the notion of a Kollár family of $\mathbb{Q}$-line bundles in [1]. Hacking defines the $\mathbb{Q}$-Gorenstein deformation by the property that it locally lifts to an equivariant deformation of an index-one cover. This definition essentially coincides with our definition of a virtually $\mathbb{Q}$-Gorenstein morphism (cf. Lemma 7.16 and Remark 7.17). A $\mathbb{Q}$-Gorenstein morphism is always a virtually $\mathbb{Q}$-Gorenstein morphism. The converse holds if every fiber satisfies $\mathbf{S}_3$; it is proved as a part of Theorem 7.18. This theorem is derived from Theorems 3.16 and 5.10 on criteria for certain sheaves to be invertible, and from a study of the relative canonical dualizing complex in Section 5.1. By Theorem 7.18, we can study infinitesimal $\mathbb{Q}$-Gorenstein deformations of a $\mathbb{Q}$-Gorenstein algebraic scheme over a field satisfying $\mathbf{S}_3$ via the equivariant deformations of the index-one cover.

(2): The infinitesimal criterion says that, for a given flat morphism $f\colon Y \to T$ locally of finite type between locally Noetherian schemes, it is a $\mathbb{Q}$-Gorenstein morphism if and only if the base change $f_A\colon Y_A = Y \times_T \operatorname{Spec} A \to \operatorname{Spec} A$ is a $\mathbb{Q}$-Gorenstein morphism for any morphism $\operatorname{Spec} A \to T$ for any Artinian local ring A. The valuative criterion is similar but T is assumed to be reduced and A is replaced with any discrete valuation ring. These criteria and some variants are proved in Theorems 7.25 and 7.29 and Corollaries 7.26 and 7.27. The proofs of these criteria use Proposition 3.19 on infinitesimal and valuative criteria for a reflexive sheaf on Y to satisfy relative $\mathbf{S}_2$ over T.

(3): Theorem 7.30 proves that, for a morphism $f\colon Y \to T$ in (2) above, it is $\mathbb{Q}$-Gorenstein along a fiber $Y_t = f^{-1}(t)$ if Y_t is $\mathbb{Q}$-Gorenstein and Gorenstein in codimension two and if

$$\omega^{[m]}_{Y_t/\Bbbk(t)} = \omega^{[m]}_{Y_t/\operatorname{Spec}\Bbbk(t)}$$

satisfies $\mathbf{S}_3$ for any $m \in \mathbb{Z}$. Here $\Bbbk(t)$ denotes the residue field of $\mathcal{O}_{T,t}$.

(4): For an $\mathbf{S}_2$-morphism $f\colon Y \to T$ of locally Noetherian schemes such that every fiber is $\mathbb{Q}$-Gorenstein, the $\mathbb{Q}$-*Gorenstein refinement* is defined as a monomorphism $S \to T$ satisfying the following universal property (cf. Definition 7.31): For a morphism $T' \to T$ from another locally Noetherian scheme T', the base change

$Y \times_T T' \to T'$ is a $\mathbb{Q}$-Gorenstein morphism if and only if $T' \to T$ factors through $S \to T$. Theorem 7.32 proves the existence of a $\mathbb{Q}$-Gorenstein refinement, for example in the case where f is proper and $\omega^{[m]}_{Y_t/\Bbbk(t)}$ is invertible for a constant $m > 0$ for any fiber Y_t. In this case, $S \to T$ is shown to be separated and locally of finite type. Similar results are given as Theorem 7.34 for a local version and as Theorem 7.35 for naively $\mathbb{Q}$-Gorenstein morphisms. Kollár's result [29, Cor. 25] is stronger than Theorem 7.32 when f is a projective morphism.

The role of our key proposition

The deep results above on topics (1)–(4) and some basic properties of $\mathbf{S}_2$-morphisms and $\mathbb{Q}$-Gorenstein morphisms are obtained by applying our key proposition (= Proposition 3.7). It proves that, for a flat morphism $Y \to T$ of locally Noetherian schemes and for an exact sequence

$$0 \to \mathcal{F} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{G} \to 0$$

of coherent $\mathcal{O}_Y$-modules satisfying suitable conditions, the relative $\mathbf{S}_2$-condition for $\mathcal{F}$ over T is equivalent to the flatness of $\mathcal{G}$ over T. For example, if f is an $\mathbf{S}_2$-morphism, then a reflexive sheaf $\mathcal{F}$ on Y admits such an exact sequence locally on Y when $\mathcal{F}$ is locally free in codimension one on each fiber (cf. Lemma 3.14). Therefore, the relative $\mathbf{S}_2$-condition for $\mathcal{F}$ over T can be studied by the flatness of another sheaf $\mathcal{G}$ defined locally on Y. This is useful, since the sheaves $\omega^{[m]}_{Y/T}$ are reflexive. For the relative canonical sheaf $\omega_{Y/T}$ of an $\mathbf{S}_2$-morphism $Y \to T$, we can show in the proof of Proposition 5.5 that Y is locally embedded into an affine smooth T-scheme P as a closed subscheme and $\omega_{Y/T}$ admits such an exact sequence as $\mathcal{F}$ on P.

Applying the local criterion of flatness (cf. Proposition A.1) and the valuative criterion of flatness (cf. [12, IV, Thm. (11.8.1)]) for $\mathcal{G}$, we have infinitesimal and valuative criteria for $\mathcal{F}$ to satisfy relative $\mathbf{S}_2$ over T in Proposition 3.19. This is applied to the reflexive sheaf $\mathcal{F} = \omega^{[m]}_{Y/T}$ in topic (2) above. The flattening stratification theorem by Mumford in [39, Lect. 8] and the representability theorem of unramified functors by Murre [41] applied to $\mathcal{G}$ yield Theorem 3.26 on the *relative* $\mathbf{S}_2$ *refinement* for $\mathcal{F}$ (cf. Definition 3.20). This is defined as a monomorphism $S \to T$ satisfying the following universal property: For a morphism $T' \to T$ from a locally Noetherian scheme T' and for the induced morphisms $p \colon Y' \to Y$ and $Y' \to T'$ from the fiber product $Y' = Y \times_T T'$, the double dual of $p^*\mathcal{F}$ satisfies relative $\mathbf{S}_2$ over T' if and only if $T' \to T$ factors through $S \to T$. When $Y \to T$ is projective, Theorem 3.26 is similar to Kollár's result [29, Thm. 2] on "hulls and husks". We also have a "local version" of the relative $\mathbf{S}_2$ refinement for $\mathcal{F}$ as Theorem 3.28 by applying to $\mathcal{G}$ theorems on a local universal flattening functor by Raynaud–Gruson [48, Part 1, Thm. (4.1.2)] or Raynaud [47, Chap. 3, Thm. 1]. Applying the results on relative $\mathbf{S}_2$ refinement for $\mathcal{F} = \omega_{Y/T}^{[m]}$, we have theorems on $\mathbb{Q}$-Gorenstein refinement mentioned in topic (4) above.

The implication (b′) ⇒ (b) in Proposition 3.7 is important. It is essential in the proof of Corollary 3.10, and it is used in the proofs of Theorem 3.16 on a criterion for a certain sheaf to be invertible, mentioned in the explanation of (1) above and of Theorem 7.30 in (3) above. Corollary 3.10 is similar to a special case of [28, Thm. 12], but to which we have found a counterexample (cf. Example 3.12).

Various other results and remarks

Most parts of Sections 2 and 4 are surveys: Section 2 discusses basic properties of $\mathbf{S}_k$-conditions and relative $\mathbf{S}_k$-conditions, and Section 4 discusses the dualizing complex and relative dualizing complex in a little detail. Even in the surveys, we present the following results and remarks, which seem to be new or not well known.

- Some general properties on reflexive modules $\mathcal{F}$ over a locally Noetherian scheme X are presented in Lemmas 2.21, 2.33, 2.34, and Corollary 2.22. These are related to the $\mathbf{S}_2$-conditions and the relative $\mathbf{S}_2$-conditions for $\mathcal{F}$ and for X. Similar properties can be found in [21, §3]. Note that reflexive modules are well understood over an integral domain (cf. [6, VII, §4]).
- *Every* $\mathbf{S}_2$-morphism (cf. Definition 2.30) *of locally Noetherian schemes locally has pure relative dimension* (cf. Lemma 2.38(1)).
- *For a locally Noetherian scheme X admitting a dualizing complex and for a coherent sheaf $\mathcal{F}$ on it, the $\mathbf{S}_k$-locus $\mathbf{S}_k(\mathcal{F})$, the Cohen–Macaulay locus* $\mathrm{CM}(\mathcal{F})$*, and the Gorenstein locus* $\mathrm{Gor}(X)$ *are open* (cf. Proposition 4.11).
- Let X be a locally equi-dimensional and locally Noetherian scheme admitting a dualizing complex. In this case, X has an *ordinary* dualizing complex $\mathcal{R}^\bullet$ and the dualizing sheaf $\mathcal{L} = \mathcal{H}^0(\mathcal{R}^\bullet)$ in the sense of Definition 4.13 (cf. Lemma 4.14). *Then $\mathcal{L}$ satisfies $\mathbf{S}_2$, and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})$ is an $\mathcal{O}_X$-algebra defining the $\mathbf{S}_2$-ification of X* (cf. Proposition 4.21 and its remark).
- (Cf. Corollary 4.38.) *For a flat separated morphism $Y \to T$ of finite type of Noetherian schemes and for the canonical inclusion morphism $\psi_t\colon Y_t \to Y$ from the fiber $Y_t = f^{-1}(t)$ over a point $t \in T$, there is a quasi-isomorphism*

  $$\mathbf{L}\psi_t^*(f^!\mathcal{O}_T) \simeq_{\mathrm{qis}} \omega^\bullet_{Y_t/\Bbbk(t)}$$

  of complexes of $\mathcal{O}_{Y_t}$-modules, where $f^!\mathcal{O}_T$ is the twisted inverse image *of $\mathcal{O}_T$ by f* (cf. Example 4.33), *and $\omega^\bullet_{Y_t/\Bbbk(t)}$ is the* canonical dualizing complex *of the algebraic scheme Y_t over the residue field $\Bbbk(t)$ of $\mathcal{O}_{T,t}$* (cf. Definition 4.28). When $\mathcal{O}_{T,t}$ is regular, the assertion has been proved by [45, Prop. 3.3(1)].

In the other Sections 3, 5, 6, and 7 can be found the following interesting results and remarks which are not listed above as notable ones:

- A counterexample of Kollár's theorem [28, Thm. 12] is given in Example 3.12. The ideal sheaf $\mathcal{J}$ in Example 3.12 gives a counterexample to assertion $(*)$ in Remark 3.13 on formal completions and direct images by open immersions, which seems to be used implicitly in many articles. The referee informed us that a modified version of [28, Thm. 12] is given in [30, Thm. 10.70].

- *For an $\mathbf{S}_2$-morphism $f\colon Y \to T$ of locally Noetherian schemes and for the open immersion $j\colon Y^\flat \hookrightarrow T$ from the relative Cohen–Macaulay locus $Y^\flat = \mathrm{CM}(Y/T)$, the pushforward $j_*\omega_{Y^\flat/T}$ is coherent, and it is isomorphic to $\mathcal{H}^{-d}(f^!\mathcal{O}_T)$ locally on Y for the relative dimension d and the twisted inverse image $f^!\mathcal{O}_T$* (cf. Proposition 5.5). *Moreover, this sheaf is reflexive if every fiber is Gorenstein in codimension one* (cf. Proposition 5.6).

- In Section 6.2 we discuss affine cones X of connected polarized projective schemes $(S, \mathcal{A})$ over a field $\Bbbk$. Here the projective scheme S is not assumed to be reduced nor irreducible. In the sequel, we give useful conditions for $\omega^{[r]}_{X/\Bbbk}$ to satisfy $\mathbf{S}_k$ and for X to be $\mathbb{Q}$-Gorenstein, in several situations of $(S, \mathcal{A})$ (cf. Proposition 6.13, Corollaries 6.14 and 6.15).

- By Lemma 7.8 and Example 7.9 we present a new example of naively $\mathbb{Q}$-Gorenstein morphisms which is not $\mathbb{Q}$-Gorenstein in the case of morphisms of algebraic varieties over an algebraically closed field of characteristic zero. For known examples, see Fact 7.7.

- For a naively $\mathbb{Q}$-Gorenstein morphism $f\colon Y \to T$ and a point $t \in T$, the relative Gorenstein index of f along the fiber $Y_t = f^{-1}(t)$ coincides with the Gorenstein index of Y_t under suitable conditions (cf. Proposition 7.11). This covers [31, Lem. 3.16], but the proof has problems as explained in Remark 7.12.

- In Remark 7.28, applying Corollary 7.27, we verify the unboundedness of $\{r_n\}$ for Kollár's example of naively $\mathbb{Q}$-Gorenstein morphisms over the spectra of Artinian rings, explained in [16, 14.7] and [32, Exam. 7.6]. There, the proof is left to the reader, but we are afraid that the expected proof might have a problem similar to the second problem in Remark 7.12.

- For a famous example (Example 7.4) of deformations of the weighted projective plane $\mathbb{P}(1, 1, 4)$ which is not $\mathbb{Q}$-Gorenstein, its $\mathbb{Q}$-Gorenstein refinement is determined in Example 7.33 by using Lemma 7.5.

Organization of this article

In Section 2 we recall some basic notions and properties related to Serre's $\mathbf{S}_k$-condition. Section 2.1 recalls basic properties on dimension, depth, and the $\mathbf{S}_k$-condition. The relative $\mathbf{S}_k$-condition is explained in Section 2.2. In Section 3 we proceed with the study of the relative $\mathbf{S}_2$-condition and give some criteria for this condition. Section 3.1 is devoted to proving the key proposition (Proposition 3.7) and related properties. Some applications of Proposition 3.7 are given in Section 3.2: Theorem 3.16 on a criterion for a certain sheaf to be invertible, Proposition 3.19 on infinitesimal and valuative criteria, Theorem 3.26 on the relative $\mathbf{S}_2$ refinement, and its local version: Theorem 3.28.

The theory of Grothendieck duality is surveyed briefly in Section 4 with a few original results. In Sections 4.1 and 4.2 we recall some well-known properties on the dualizing complex based on arguments in [17] and [7]. The twisted inverse image functor is explained in Section 4.3 with the famous Grothendieck duality theorem for proper morphisms (cf. Theorem 4.30). The base change theorem for the relative dualizing sheaf for a Cohen–Macaulay morphism is mentioned in Section 4.4. In Section 5 we give some technical base change results for the relative canonical sheaf of an $\mathbf{S}_2$-morphism. Section 5.2 is devoted to proving Theorem 5.10 on a criterion for a certain sheaf related to the relative canonical sheaf to be invertible. This technical theorem also gives sufficient conditions for the base change homomorphism of the relative canonical sheaf to the fiber to be an isomorphism, and it is applied to the proof of Theorem 7.18 on virtually $\mathbb{Q}$-Gorenstein morphism (cf. topic (1) above).

In Section 6 we study $\mathbb{Q}$-Gorenstein schemes. The definition and its basic properties are given in Section 6.1. As an example of $\mathbb{Q}$-Gorenstein schemes, in Section 6.2 we consider the case of affine cones over polarized projective schemes over a field. In Section 7 we study $\mathbb{Q}$-Gorenstein morphisms, and two variants: naively $\mathbb{Q}$-Gorenstein morphisms and virtually $\mathbb{Q}$-Gorenstein morphisms. The $\mathbb{Q}$-Gorenstein morphism and the naively $\mathbb{Q}$-Gorenstein morphism are defined in Section 7.1, and their basic properties are discussed. The virtually $\mathbb{Q}$-Gorenstein morphism is defined in Section 7.2, and we prove Theorem 7.18 on a criterion for a virtually $\mathbb{Q}$-Gorenstein morphism to be $\mathbb{Q}$-Gorenstein (cf. topic (1) above). In Section 7.3 several basic properties including base change of $\mathbb{Q}$-Gorenstein morphisms and of their variants are discussed. Theorems mentioned in topics (2)–(4) above are proved in Section 7.4.

Some elementary facts on the local criterion of flatness and base change isomorphisms are explained in Appendix A for the readers' convenience. In this article, we try to cite references as much as possible for the readers' convenience and for the authors' assurance. We also try to refer to the original article if possible.

Notation and conventions

(1) For a complex $K^\bullet = [\cdots \to K^i \xrightarrow{d^i} K^{i+1} \to \cdots]$ in an abelian category and for an integer q, we denote by $\tau^{\leq q}(K^\bullet)$ (resp. $\tau^{\geq q}(K^\bullet)$) the "truncation" of $K^\bullet$, which is defined as the complex

$$[\cdots \to K^{q-2} \xrightarrow{d^{q-2}} K^{q-1} \to \mathrm{Ker}(d^q) \to 0 \to \cdots]$$

$$(\text{resp. } [\cdots \to 0 \to \mathrm{Coker}(d^{q-1}) \to K^{q+1} \xrightarrow{d^{q+1}} K^{q+2} \to \cdots])$$

(cf. [10, Déf. 1.1.13]). The complex $K^\bullet[m]$ shifted by an integer m is defined as the complex $L^\bullet = [\cdots \to L^i \xrightarrow{d^i_L} L^{i+1} \to \cdots]$ such that $L^i = K^{i+m}$ and $d^i_L = (-1)^m d^{i+m}$ for any $i \in \mathbb{Z}$. It is known that the mapping cone of the natural morphism $\tau^{\leq q}(K^\bullet) \to K^\bullet$ is quasi-isomorphic to $\tau^{\geq q+1}(K^\bullet)$ for any $q \in \mathbb{Z}$.

(2) For a complex $K^\bullet$ in an abelian category (resp. for an object $K^\bullet$ of the derived category), the i-th cohomology of $K^\bullet$ is denoted usually by $\mathrm{H}^i(K^\bullet)$. For a complex $\mathcal{K}^\bullet$ of sheaves on a scheme, the i-th cohomology is a sheaf and is denoted by $\mathcal{H}^i(\mathcal{K}^\bullet)$.

(3) The derived category of an abelian category A is denoted by $\mathbf{D}(\mathsf{A})$. Moreover, we write $\mathbf{D}^+(\mathsf{A})$ (resp. $\mathbf{D}^-(\mathsf{A})$, resp. $\mathbf{D}^b(\mathsf{A})$) for the full subcategory consisting of bounded below (resp. bounded above, resp. bounded) complexes.

(4) An *algebraic scheme* over a field $\Bbbk$ means a $\Bbbk$-scheme of finite type. An *algebraic variety* over $\Bbbk$ is an integral separated algebraic scheme over $\Bbbk$.

(5) For a scheme X, a sheaf of $\mathcal{O}_X$-modules is called an $\mathcal{O}_X$*-module* for simplicity. A coherent (resp. quasi-coherent) sheaf on X means a coherent (resp. quasi-coherent) $\mathcal{O}_X$-module. The (abelian) category of $\mathcal{O}_X$-modules (resp. quasi-coherent $\mathcal{O}_X$-modules) is denoted by $\mathsf{Mod}(\mathcal{O}_X)$ (resp. $\mathsf{QCoh}(\mathcal{O}_X)$).

(6) For a scheme X and a point $x \in X$, the maximal ideal (resp. the residue field) of the local ring $\mathcal{O}_{X,x}$ is denoted by $\mathfrak{m}_{X,x}$ (resp. $\Bbbk(x)$). The stalk of a sheaf $\mathcal{F}$ on X at x is denoted by $\mathcal{F}_x$.

(7) For a morphism $f\colon Y \to T$ of schemes and for a point $t \in T$, the fiber $f^{-1}(t)$ over t is defined as $Y \times_T \operatorname{Spec} \Bbbk(t)$ and is denoted by Y_t. For an $\mathcal{O}_Y$-module $\mathcal{F}$, the restriction $\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t}$ to the fiber Y_t is denoted by $\mathcal{F}_{(t)}$ (cf. Notation 2.24).

(8) The derived category of a scheme X is defined as the derived category of $\mathsf{Mod}(\mathcal{O}_X)$ and is denoted by $\mathbf{D}(X)$. The full subcategory consisting of complexes with quasi-coherent (resp. coherent) cohomology is denoted by $\mathbf{D}_{\mathrm{qcoh}}(X)$

(resp. $\mathbf{D}_{\mathrm{coh}}(X)$). For $* = +, -, b$ and for $\dagger = \mathrm{qcoh}, \mathrm{coh}$, we set

$$\mathbf{D}^*(X) = \mathbf{D}^*(\mathsf{Mod}(\mathcal{O}_X)) \quad \text{and} \quad \mathbf{D}^*_\dagger(X) = \mathbf{D}^*(X) \cap \mathbf{D}_\dagger(X).$$

(9) For a sheaf $\mathcal{F}$ on a scheme X and for a closed subset Z, the i-th local cohomology sheaf of $\mathcal{F}$ with support in Z is denoted by $\mathcal{H}^i_Z(\mathcal{F})$ (cf. [18]).

(10) For a morphism $X \to Y$ of schemes, $\Omega^1_{X/Y}$ denotes the sheaf of relative one-forms. When $X \to Y$ is smooth, $\Omega^p_{X/Y}$ denotes the p-th exterior power $\bigwedge^p \Omega^1_{X/Y}$ for integers $p \geq 0$.

§2. Serre's $\mathbf{S}_k$-condition

We shall recall several fundamental properties on locally Noetherian schemes, which are indispensable for understanding the explanation of dualizing complex and Grothendieck duality in Section 4, as well as the discussion of relative canonical sheaves and $\mathbb{Q}$-Gorenstein morphisms in Sections 5 and 6, respectively. In Section 2.1 we recall basic properties on dimension, depth, Serre's $\mathbf{S}_k$-condition especially for $k = 1$ and 2, and reflexive sheaves. The relative $\mathbf{S}_k$-condition is discussed in Section 2.2.

§2.1. Basics on Serre's condition

The $\mathbf{S}_k$-condition is defined by "depth" and "dimension". We begin by recalling some elementary properties on dimension, codimension, and on depth.

Property 2.1 (Dimension, codimension). Let X be a scheme and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module *of finite type* (cf. [12, 0_{I}, (5.2.1)]), i.e., $\mathcal{F}$ is quasi-coherent and locally finitely generated as an $\mathcal{O}_X$-module. Then $\operatorname{Supp} \mathcal{F}$ is a closed subset (cf. [12, 0_{I}, (5.2.2)]).

(1) If Y is a closed subscheme of X such that $Y = \operatorname{Supp} \mathcal{F}$ as a set, then

$$\dim \mathcal{F}_y = \dim \mathcal{O}_{Y,y} = \operatorname{codim}(\overline{\{y\}}, Y)$$

for any point $y \in Y$, where $\dim \mathcal{F}_y$ is considered as the dimension of the closed subset $\operatorname{Supp} \mathcal{F}_y$ of $\operatorname{Spec} \mathcal{O}_{X,y}$ (cf. [12, IV, (5.1.2), (5.1.12)]).

(2) The dimension of $\mathcal{F}$, denoted by $\dim \mathcal{F}$, is defined as $\dim \operatorname{Supp} \mathcal{F}$ (cf. [12, IV, (5.1.12)]). Then

$$\dim \mathcal{F} = \sup\{\dim \mathcal{F}_x \mid x \in X\}$$

(cf. [12, IV, (5.1.12.3)]). If X is locally Noetherian, then

$$\dim \mathcal{F} = \sup\{\dim \mathcal{F}_x \mid x \text{ is a closed point of } X\}$$

by [12, IV, (5.1.4.2), (5.1.12.1), Cor. (5.1.11)]. Note that the local dimension of $\mathcal{F}$ at a point x, denoted by $\dim_x \mathcal{F}$, is just the infimum of $\dim \mathcal{F}|_U$ for all the open neighborhoods U of x.

(3) For a closed subset $Z \subset X$, the equality

$$\operatorname{codim}(Z, X) = \inf\{\dim \mathcal{O}_{X,z} \mid z \in Z\}$$

holds, and moreover, if X is locally Noetherian, then

$$\operatorname{codim}_x(Z, X) = \inf\{\dim \mathcal{O}_{X,z} \mid z \in Z,\, x \in \overline{\{z\}}\}$$

for any point $x \in X$ (cf. [12, IV, Cor. (5.1.3)]). Note that $\operatorname{codim}(\emptyset, X) = +\infty$ and that $\operatorname{codim}_x(Z, X) = +\infty$ if $x \notin Z$. Furthermore, if Z is locally Noetherian, then the function $x \mapsto \operatorname{codim}_x(Z, X)$ is lower semi-continuous on X (cf. [12, 0_{IV}, Cor. (14.2.6)(ii)]).

Definition 2.2 (Equi-dimensional). Let X be a scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module of finite type. Let A be a ring and M a finitely generated A-module.

(1) We call X (resp. $\mathcal{F}$) *equi-dimensional* if all the irreducible components of X (resp. $\operatorname{Supp} \mathcal{F}$) have the same dimension.

(2) We call A (resp. M) *equi-dimensional* if all the irreducible components of $\operatorname{Spec} A$ (resp. $\operatorname{Supp} M$) have the same dimension, where $\operatorname{Supp} M$ is the closed subset of $\operatorname{Spec} A$ defined by the annihilator ideal $\operatorname{Ann}(M)$. Note that $\operatorname{Supp} M$ equals $\operatorname{Supp} M^\sim$ for the associated quasi-coherent sheaf $M^\sim$ on $\operatorname{Spec} A$.

(3) We call X (resp. $\mathcal{F}$) *locally equi-dimensional* if the local ring $\mathcal{O}_{X,x}$ (resp. the stalk $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module) is equi-dimensional for any point $x \in X$.

Property 2.3 (Catenary). A scheme X is said to be *catenary* if

$$\operatorname{codim}(Y, Z) + \operatorname{codim}(Z, T) = \operatorname{codim}(Y, T)$$

for any irreducible closed subsets $Y \subset Z \subset T$ of X (cf. [12, 0_{IV}, Prop. (14.3.2)]). A ring A is said to be catenary if $\operatorname{Spec} A$ is so. Then, for a scheme X, it is catenary if and only if every local ring $\mathcal{O}_{X,x}$ is catenary (cf. [12, IV, Cor. (5.1.5)]). If X is a locally Noetherian scheme and if $\mathcal{O}_{X,x}$ is catenary for a point $x \in X$, then

$$\operatorname{codim}_x(Y, X) = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,x}$$

for any closed subscheme Y of X containing x (cf. [12, IV, Prop. (5.1.9)]).

Property 2.4 (Depth). Let A be a Noetherian ring, I an ideal of A, and let M be a finitely generated A-module. The I-*depth* of M, denoted by $\operatorname{depth}_I M$, is defined as the length of any maximal M-regular sequence contained in I when $M \neq IM$, and as $+\infty$ when $M = IM$. Here an element $a \in I$ is said to be M-*regular* if a is not a zero divisor of M, i.e., the multiplication map $x \mapsto ax$ induces an injection $M \to M$, and a sequence $a_1, a_2, \ldots, a_n$ of elements of I is said to be M-*regular* if a_i is M_i-regular for any i, where $M_i = M/(a_1, \ldots, a_{i-1})M$. The following equality is well known (cf. [18, Prop. 3.3], [14, III, Prop. 2.4], [37, Thms. 16.6, 16.7]):

$$\operatorname{depth}_I M = \inf\{i \in \mathbb{Z}_{\geq 0} \mid \operatorname{Ext}^i_A(A/I, M) \neq 0\}.$$

If A is a local ring and if I is the maximal ideal $\mathfrak{m}_A$, then $\operatorname{depth}_I M$ is denoted simply by $\operatorname{depth} M$; in this case, we have $\operatorname{depth} M \leq \dim M$ when $M \neq 0$ (cf. [12, 0_{IV}, (16.4.5.1)], [37, Exer. 16.1, Thm. 17.2]).

Definition 2.5 (Z-depth). Let X be a locally Noetherian scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. For a closed subset Z of X, the Z-*depth* of $\mathcal{F}$ is defined as

$$\operatorname{depth}_Z \mathcal{F} = \inf\{\operatorname{depth} \mathcal{F}_z \mid z \in Z\}$$

(cf. [18, p. 43, Def.], [12, IV, (5.10.1.1)], [2, III, Def. (3.12)]), where the stalk $\mathcal{F}_z$ of $\mathcal{F}$ at z is regarded as an $\mathcal{O}_{X,z}$-module. Note that $\operatorname{depth}_Z 0 = +\infty$.

Property 2.6 (Cf. [18, Thm. 3.8]). In the situation above, for a given integer $k \geq 1$, one has the equivalence

$$\operatorname{depth}_Z \mathcal{F} \geq k \iff \mathcal{H}^i_Z(\mathcal{F}) = 0 \quad \text{for any } i < k.$$

Here $\mathcal{H}^i_Z(\mathcal{F})$ stands for the i-th local cohomology sheaf of $\mathcal{F}$ with support in Z (cf. [18], [14]). In particular, the condition $\operatorname{depth}_Z \mathcal{F} \geq 1$ (resp. ≥ 2) is equivalent to the condition that the restriction homomorphism $\mathcal{F} \to j_*(\mathcal{F}|_{X \setminus Z})$ is an injection (resp. isomorphism) for the open immersion $j\colon X \setminus Z \hookrightarrow X$. Furthermore, the condition $\operatorname{depth}_Z \mathcal{F} \geq 3$ is equivalent to $\mathcal{F} \simeq j_*(\mathcal{F}|_{X \setminus Z})$ and $R^1 j_*(\mathcal{F}|_{X \setminus Z}) = 0$.

Remark (Cf. [18, Cor. 3.6], [2, III, Cor. 3.14]). Let A be a Noetherian ring with an ideal I and let M be a finitely generated A-module. Then

$$\operatorname{depth}_I M = \operatorname{depth}_Z M^{\sim}$$

for the closed subscheme $Z = \operatorname{Spec} A/I$ of $X = \operatorname{Spec} A$ and for the coherent $\mathcal{O}_X$-module $M^{\sim}$ associated with M.

Remark 2.7 (Associated prime). Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module on a locally Noetherian scheme X. A point $x \in X$ is called an *associated point* of $\mathcal{F}$ if the maximal ideal $\mathfrak{m}_x$ is an associated prime of the stalk $\mathcal{F}_x$ (cf. [12, IV, Déf. (3.1.1)]). This condition is equivalent to $\operatorname{depth} \mathcal{F}_x = 0$. We denote by $\operatorname{Ass}(\mathcal{F})$ the set of associated points. This is a discrete subset of $\operatorname{Supp} \mathcal{F}$. If an associated point x of

$\mathcal{F}$ is not a generic point of $\mathcal{F}$, i.e., $\operatorname{depth} \mathcal{F}_x = 0$ and $\dim \mathcal{F}_x > 0$, then x is called the *embedded point* of $\mathcal{F}$. If $X = \operatorname{Spec} A$ and $\mathcal{F} = M^\sim$ for a Noetherian ring A and for a finitely generated A-module M, then $\operatorname{Ass}(\mathcal{F})$ is just the set of associated primes of M, and the embedded points of $\mathcal{F}$ are the embedded primes of M.

Remark 2.8. Let $\phi\colon \mathcal{F} \to j_*(\mathcal{F}|_{X\setminus Z})$ be the homomorphism in Property 2.6 and set $U = X \setminus Z$. Then ϕ is an injection (resp. isomorphism) at a point $x \in Z$, i.e., the homomorphism

$$\phi_x \colon \mathcal{F}_x \to (j_*(\mathcal{F}|_U))_x$$

of stalks is an injection (resp. isomorphism), if and only if

$$\operatorname{depth} \mathcal{F}_{x'} \geq 1 \quad (\text{resp. } \geq 2)$$

for any $x' \in Z$ such that $x \in \overline{\{x'\}}$. In fact, ϕ_x is identical to the inverse image $p_x^*(\phi)$ by a canonical morphism $p_x\colon \operatorname{Spec} \mathcal{O}_{X,x} \to X$, and it is regarded as the restriction homomorphism of $p_x^*(\mathcal{F})$ to the open subset $U_x = p_x^{-1}(U)$ via the base change isomorphism

$$p_x^*(j_*(\mathcal{F}|_U)) \simeq j_{x*}((p_x^*\mathcal{F})|_{U_x})$$

(cf. Lemma A.9 below), where j_x stands for the open immersion $U_x \hookrightarrow \operatorname{Spec} \mathcal{O}_{X,x}$. For the complement $Z_x = p_x^{-1}(Z)$ of U_x in $\operatorname{Spec} \mathcal{O}_{X,x}$, by Property 2.6 we know that $p_x^*(\phi)$ is an injection (resp. isomorphism) if and only if

$$\operatorname{depth}_{Z_x} p_x^*(\mathcal{F}) \geq 1 \quad (\text{resp. } \geq 2).$$

This implies the assertion, since Z_x is identical to the set of points $x' \in Z$ such that $x \in \overline{\{x'\}}$.

We recall Serre's condition $\mathbf{S}_k$ (cf. [12, IV, Déf. (5.7.2)], [2, VII, Def. (2.1)], [37, p. 183]).

Definition 2.9. Let X be a locally Noetherian scheme, $\mathcal{F}$ a coherent $\mathcal{O}_X$-module and k a positive integer. We say that $\mathcal{F}$ satisfies $\mathbf{S}_k$ if the inequality

$$\operatorname{depth} \mathcal{F}_x \geq \inf\{k, \dim \mathcal{F}_x\}$$

holds for any point $x \in X$, where the stalk $\mathcal{F}_x$ at x is considered as an $\mathcal{O}_{X,x}$-module. We say that $\mathcal{F}$ *satisfies* $\mathbf{S}_k$ *at a point* $x \in X$ if

$$\operatorname{depth} \mathcal{F}_y \geq \inf\{k, \dim \mathcal{F}_y\}$$

for any point $y \in X$ such that $x \in \overline{\{y\}}$. We say that X satisfies $\mathbf{S}_k$ if $\mathcal{O}_X$ does so.

Remark. In the situation of Definition 2.9, assume that $\mathcal{F} = i_*(\mathcal{F}')$ for a closed immersion $i\colon X' \hookrightarrow X$ and for a coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$. Then $\mathcal{F}$ satisfies $\mathbf{S}_k$ if and only if $\mathcal{F}'$ does so. In fact,

$$\operatorname{depth} \mathcal{F}_x = +\infty \quad \text{and} \quad \dim \mathcal{F}_x = -\infty$$

for any $x \notin X'$, and

$$\operatorname{depth} \mathcal{F}_x = \operatorname{depth} \mathcal{F}'_x \quad \text{and} \quad \dim \mathcal{F}_x = \dim \mathcal{F}'_x$$

for any $x \in X'$ (cf. [12, 0_{IV}, Prop. (16.4.8)]).

Remark 2.10. Let A be a Noetherian ring and M a finitely generated A-module. For a positive integer k, we say that M *satisfies* $\mathbf{S}_k$ if the associated coherent sheaf $M^\sim$ on $\operatorname{Spec} A$ satisfies $\mathbf{S}_k$. Then, for X, $\mathcal{F}$ and x in Definition 2.9, $\mathcal{F}$ satisfies $\mathbf{S}_k$ at x if and only if the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ satisfies $\mathbf{S}_k$. In fact, by considering $\operatorname{Supp} \mathcal{F}_x$ as a closed subset of $\operatorname{Spec} \mathcal{O}_{X,x}$ and by the canonical morphism $\operatorname{Spec} \mathcal{O}_{X,x} \to X$, we can identify $\operatorname{Supp} \mathcal{F}_x$ with the set of points $y \in \operatorname{Supp} \mathcal{F}$ such that $x \in \overline{\{y\}}$.

Definition 2.11 (Cohen–Macaulay). Let A be a Noetherian local ring and M a finitely generated A-module. Then M is said to be *Cohen–Macaulay* if $\operatorname{depth} M = \dim M$ unless $M = 0$ (cf. [12, 0_{IV}, Déf. (16.5.1)], [37, §17]). In particular, if $\dim A = \operatorname{depth} A$, then A is called a Cohen–Macaulay local ring. Let X be a locally Noetherian scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. If the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ is Cohen–Macaulay for any $x \in X$, then $\mathcal{F}$ is said to be Cohen–Macaulay (cf. [12, IV, Déf. (5.7.1)]. If $\mathcal{O}_X$ is Cohen–Macaulay, then X is called a Cohen–Macaulay scheme.

Remark 2.12. For A and M above, it is known that if M is Cohen–Macaulay, then the localization $M_{\mathfrak{p}}$ is also Cohen–Macaulay for any prime ideal $\mathfrak{p}$ of A (cf. [12, 0_{IV}, Cor. (16.5.10)], [37, Thm. 17.3]). Hence, M is Cohen–Macaulay if and only if M satisfies $\mathbf{S}_k$ for any $k \geq 1$.

Definition 2.13 ($\mathbf{S}_k(\mathcal{F})$, $\mathrm{CM}(\mathcal{F})$). Let X be a locally Noetherian scheme and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. For an integer $k \geq 1$, the $\mathbf{S}_k$*-locus* $\mathbf{S}_k(\mathcal{F})$ of $\mathcal{F}$ is defined to be the set of points $x \in X$ at which $\mathcal{F}$ satisfies $\mathbf{S}_k$ (cf. Definition 2.9). The *Cohen–Macaulay locus* $\mathrm{CM}(\mathcal{F})$ of $\mathcal{F}$ is defined to be the set of points $x \in \mathcal{F}$ such that $\mathcal{F}_x$ is a Cohen–Macaulay $\mathcal{O}_{X,x}$-module. By definition and by Remark 2.12, one has $\mathrm{CM}(\mathcal{F}) = \bigcap_{k \geq 1} \mathbf{S}_k(\mathcal{F})$. We define $\mathbf{S}_k(X) := \mathbf{S}_k(\mathcal{O}_X)$ and $\mathrm{CM}(X) = \mathrm{CM}(\mathcal{O}_X)$, and call them the $\mathbf{S}_k$-locus and the Cohen–Macaulay locus of X, respectively.

Remark. It is known that $\mathbf{S}_k(\mathcal{F})$ and $\mathrm{CM}(\mathcal{F})$ are open subsets when X is locally a subscheme of a regular scheme (cf. [12, IV, Prop. (6.11.2)(ii)]). In Proposition 4.11 below, we shall prove the openness when X admits a dualizing complex.

Remark. For a locally Noetherian scheme X, every generic point of X is contained in the Cohen–Macaulay locus $\mathrm{CM}(X)$ because $\dim A = \operatorname{depth} A = 0$ for any Artinian local ring A.

Lemmas 2.14 and 2.15 below are basic properties on the condition $\mathbf{S}_k$.

Lemma 2.14. *Let X be a locally Noetherian scheme and let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module. For a positive integer k, the following conditions are equivalent to each other:*

(i) *The sheaf $\mathcal{G}$ satisfies $\mathbf{S}_k$.*

(ii) *The inequality*

$$\operatorname{depth}_Z \mathcal{G} \geq \inf\{k, \operatorname{codim}(Z, \operatorname{Supp} \mathcal{G})\}$$

holds for any closed (resp. irreducible and closed) subset $Z \subset \operatorname{Supp} \mathcal{G}$.

(iii) *The sheaf $\mathcal{G}$ satisfies $\mathbf{S}_{k-1}$ when $k \geq 2$, and $\operatorname{depth}_Z \mathcal{G} \geq k$ for any closed (resp. irreducible and closed) subset $Z \subset \operatorname{Supp} \mathcal{G}$ such that $\operatorname{codim}(Z, \operatorname{Supp} \mathcal{G}) \geq k$.*

(iv) *There is a closed subset $Z \subset \operatorname{Supp} \mathcal{G}$ such that $\operatorname{depth}_Z \mathcal{G} \geq k$ and $\mathcal{G}|_{X \setminus Z}$ satisfies $\mathbf{S}_k$.*

Proof. We may assume that $\mathcal{G}$ is not zero. The equivalence (i) $\Leftrightarrow$ (ii) follows from Definitions 2.5 and 2.9 and from the equality $\dim \mathcal{G}_x = \operatorname{codim}(\overline{\{x\}}, \operatorname{Supp} \mathcal{G})$ for $x \in \operatorname{Supp} \mathcal{G}$ in Property 2.1(1). The equivalence (i) $\Leftrightarrow$ (ii) implies the equivalence (ii) $\Leftrightarrow$ (iii). We have (i) $\Rightarrow$ (iv) by taking a closed subset Z with $\operatorname{codim}(Z, \operatorname{Supp} \mathcal{G}) \geq k$ using the inequality in (ii). It is enough to show (iv) $\Rightarrow$ (i). More precisely, it is enough to prove that, in the situation of (iv), the inequality

$$\operatorname{depth} \mathcal{G}_x \geq \inf\{k, \dim \mathcal{G}_x\}$$

holds for any point $x \in X$. If $x \notin Z$, then this holds, since $\mathcal{G}|_{X \setminus Z}$ satisfies $\mathbf{S}_k$. If $x \in Z$, then $\dim \mathcal{G}_x \geq \operatorname{depth} \mathcal{G}_x \geq \operatorname{depth}_Z \mathcal{G} \geq k$ (cf. Property 2.4 and Definition 2.5), and it induces the inequality above. Thus, we are done. □

Lemma 2.15. *Let X be a locally Noetherian scheme and $\mathcal{G}$ a coherent $\mathcal{O}_X$-module. Then, for any closed subset Z of X, the following hold:*

(1) *One has the inequality*

$$\operatorname{depth}_Z \mathcal{G} \leq \operatorname{codim}(Z \cap \operatorname{Supp} \mathcal{G}, \operatorname{Supp} \mathcal{G}).$$

(2) *For an integer* $k > 0$*, if* $\mathcal{G}$ *satisfies* $\mathbf{S}_k$ *and if* $\operatorname{codim}(Z \cap \operatorname{Supp} \mathcal{G}, \operatorname{Supp} \mathcal{G}) \geq k$*, then* $\operatorname{depth}_Z \mathcal{G} \geq k$*.*

Proof. The inequality in (1) follows from the inequality $\operatorname{depth} \mathcal{G}_x \leq \dim \mathcal{G}_x$ for any $x \in \operatorname{Supp} \mathcal{G}$, since

$$\operatorname{codim}(Z \cap \operatorname{Supp} \mathcal{G}, \operatorname{Supp} \mathcal{G}) = \inf\{\dim \mathcal{G}_x \mid x \in Z \cap \operatorname{Supp} \mathcal{G}\} \quad \text{and}$$
$$\operatorname{depth}_Z \mathcal{G} = \inf\{\operatorname{depth} \mathcal{G}_x \mid x \in Z \cap \operatorname{Supp} \mathcal{G}\}$$

when $Z \cap \operatorname{Supp} \mathcal{G} \neq \emptyset$, by Property 2.1 and Definition 2.5. Assertion (2) is derived from the equivalence (i) $\Leftrightarrow$ (ii) of Lemma 2.14. □

For the conditions $\mathbf{S}_1$ and $\mathbf{S}_2$, we have immediately the following corollary of Lemma 2.14 by considering Property 2.6.

Corollary 2.16. *Let* X *be a locally Noetherian scheme and let* $\mathcal{G}$ *be a coherent* $\mathcal{O}_X$*-module. The following three conditions are equivalent to each other, where* j *denotes the open immersion* $X \setminus Z \hookrightarrow X$*:*

(i) *The sheaf* $\mathcal{G}$ *satisfies* $\mathbf{S}_1$ *(resp.* $\mathbf{S}_2$*).*

(ii) *For any closed subset* $Z \subset \operatorname{Supp} \mathcal{G}$ *with* $\operatorname{codim}(Z, \operatorname{Supp} \mathcal{G}) \geq 1$ *(resp.* ≥ 2*), the restriction homomorphism* $\mathcal{G} \to j_*(\mathcal{G}|_{X \setminus Z})$ *is injective (resp. an isomorphism, and* $\mathcal{G}$ *satisfies* $\mathbf{S}_1$*).*

(iii) *There is a closed subset* $Z \subset \operatorname{Supp} \mathcal{G}$ *such that* $\mathcal{G}|_{X \setminus Z}$ *satisfies* $\mathbf{S}_1$ *(resp.* $\mathbf{S}_2$*) and the restriction homomorphism* $\mathcal{G} \to j_*(\mathcal{G}|_{X \setminus Z})$ *is injective (resp. an isomorphism).*

Remark. Let X be a locally Noetherian scheme and $\mathcal{G}$ a coherent $\mathcal{O}_X$-module. Then, by definition, $\mathcal{G}$ satisfies $\mathbf{S}_1$ if and only if $\mathcal{G}$ has no embedded points (cf. Remark 2.7). In particular, the following hold when $\mathcal{G}$ satisfies $\mathbf{S}_1$:

(1) Every coherent $\mathcal{O}_X$-submodule of $\mathcal{G}$ satisfies $\mathbf{S}_1$ (cf. Lemma 2.17(2) below).

(2) The sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ satisfies $\mathbf{S}_1$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$.

(3) Let T be the closed subscheme defined by the annihilator of $\mathcal{G}$, i.e., $\mathcal{O}_T$ is the image of the natural homomorphism $\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{G})$. Then T also satisfies $\mathbf{S}_1$.

Lemma 2.17. *Let* X *be a locally Noetherian scheme and let* $\mathcal{G}$ *be the kernel of a homomorphism* $\mathcal{E}^0 \to \mathcal{E}^1$ *of coherent* $\mathcal{O}_X$*-modules.*

(1) *Let* Z *be a closed subset of* X*. If* $\operatorname{depth}_Z \mathcal{E}^0 \geq 1$*, then* $\operatorname{depth}_Z \mathcal{G} \geq 1$*. If* $\operatorname{depth}_Z \mathcal{E}^0 \geq 2$ *and* $\operatorname{depth}_Z \mathcal{E}^1 \geq 1$*, then* $\operatorname{depth}_Z \mathcal{G} \geq 2$*.*

(2) *If* $\mathcal{E}^0$ *satisfies* $\mathbf{S}_1$*, then* $\mathcal{G}$ *satisfies* $\mathbf{S}_1$*.*

(3) *Assume that* $\operatorname{Supp}\mathcal{G} \subset \operatorname{Supp}\mathcal{E}^1$. *If* $\mathcal{E}^1$ *satisfies* $\mathbf{S}_1$ *and* $\mathcal{E}^0$ *satisfies* $\mathbf{S}_2$, *then* $\mathcal{G}$ *satisfies* $\mathbf{S}_2$.

Proof. Let $\mathcal{B}$ be the image of $\mathcal{E}^0 \to \mathcal{E}^1$. Then we have an exact sequence

$$0 \to \mathcal{H}^0_Z(\mathcal{G}) \to \mathcal{H}^0_Z(\mathcal{E}^0) \to \mathcal{H}^0_Z(\mathcal{B}) \to \mathcal{H}^1_Z(\mathcal{G}) \to \mathcal{H}^1_Z(\mathcal{E}^0)$$

and an injection $\mathcal{H}^0_Z(\mathcal{B}) \to \mathcal{H}^0_Z(\mathcal{E}^1)$ of local cohomology sheaves with support in Z (cf. [18, Prop. 1.1]). Thus, (1) is derived from Property 2.6. The remaining assertions (2) and (3) are consequences of (1) above and the equivalence (i) $\Leftrightarrow$ (ii) in Lemma 2.14. □

Lemma 2.18. *Let P be the n-dimensional projective space $\mathbb{P}^n_{\Bbbk}$ over a field $\Bbbk$ and let $\mathcal{G}$ be a coherent $\mathcal{O}_P$-module such that $\mathcal{G}$ satisfies $\mathbf{S}_1$ and that every irreducible component of* $\operatorname{Supp}\mathcal{G}$ *has positive dimension. Then* $\mathrm{H}^0(P, \mathcal{G}(m)) = 0$ *for any* $m \ll 0$, *where we write* $\mathcal{G}(m) = \mathcal{G} \otimes_{\mathcal{O}_P} \mathcal{O}_P(m)$.

Proof. We shall prove by contradiction. Assume that $\mathrm{H}^0(P, \mathcal{G}(-m)) \neq 0$ for infinitely many $m > 0$. There is a member D of $|\mathcal{O}_P(k)|$ for some $k > 0$ such that $D \cap \operatorname{Ass}(\mathcal{G}) = \emptyset$ (cf. Remark 2.7). Thus, the inclusion $\mathcal{O}_P(-D) \subset \mathcal{O}_P$ induces an injection $\mathcal{G}(-D) := \mathcal{G} \otimes_{\mathcal{O}_P} \mathcal{O}_P(-D) \to \mathcal{G}$. Hence, we have an injection $\mathcal{G}(-k) \simeq \mathcal{G}(-D) \to \mathcal{G}$, and we may assume that $\mathrm{H}^0(P, \mathcal{G}(-m)) = \mathrm{H}^0(P, \mathcal{G}) \neq 0$ for any $m > 0$ by replacing $\mathcal{G}$ with $\mathcal{G}(-l)$ for some $l > 0$. Let ξ be a non-zero element of $\mathrm{H}^0(P, \mathcal{G})$, which corresponds to a non-zero homomorphism $\mathcal{O}_P \to \mathcal{G}$. Let T be the closed subscheme of P such that $\mathcal{O}_T$ is the image of $\mathcal{O}_P \to \mathcal{G}$. Then T is non-empty and is contained in the affine open subset $P \setminus D$, since $\xi \in \mathrm{H}^0(P, \mathcal{G}(-D))$. Therefore, T is a finite set, and $T \subset \operatorname{Ass}(\mathcal{G})$. Since $\mathcal{G}$ satisfies $\mathbf{S}_1$, every point of T is an irreducible component of $\operatorname{Supp}\mathcal{G}$. This contradicts the assumption. □

Remark. The referee pointed out that Lemma 2.18 is proved by a similar argument in the proof of [19, III, Thm. 7.6(b)].

Definition 2.19 (Reflexive sheaf). For a scheme X and an $\mathcal{O}_X$-module $\mathcal{F}$, we write $\mathcal{F}^\vee$ for the dual $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. The double-dual $\mathcal{F}^{\vee\vee}$ of $\mathcal{F}$ is defined as $(\mathcal{F}^\vee)^\vee$. The natural composition homomorphism $\mathcal{F} \otimes \mathcal{F}^\vee \to \mathcal{O}_X$ defines a canonical homomorphism $c_{\mathcal{F}} \colon \mathcal{F} \to \mathcal{F}^{\vee\vee}$. Note that $c_{\mathcal{F}^\vee}$ is always an isomorphism. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite type and if $c_{\mathcal{F}}$ is an isomorphism, then $\mathcal{F}$ is said to be reflexive.

Remark 2.20. Let $\pi \colon Y \to X$ be a flat morphism of locally Noetherian schemes. Then the dual operation $^\vee$ commutes with π^*, i.e., there is a canonical isomorphism

$$\pi^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \simeq \mathcal{H}om_{\mathcal{O}_Y}(\pi^*\mathcal{F}, \mathcal{O}_Y)$$

for any coherent $\mathcal{O}_X$-module $\mathcal{F}$. In particular, if $\mathcal{F}$ is reflexive, then so is $\pi^*\mathcal{F}$. This isomorphism is derived from [12, $0_{\rm I}$, (6.7.6)], since every coherent $\mathcal{O}_X$-module has a finite presentation locally on X.

Lemma 2.21. *Let X be a locally Noetherian scheme, Z a closed subset and $\mathcal{G}$ a coherent $\mathcal{O}_X$-module.*

(1) *For an integer $k = 1$ or 2, assume that $\operatorname{depth}_Z \mathcal{O}_X \geq k$ and that $\mathcal{G}$ is reflexive. Then $\operatorname{depth}_Z \mathcal{G} \geq k$.*
(2) *For an integer $k = 1$ or 2, assume that X satisfies $\mathbf{S}_k$ and that $\mathcal{G}$ is reflexive. Then $\mathcal{G}$ satisfies $\mathbf{S}_k$.*
(3) *Assume that $\operatorname{depth}_Z \mathcal{O}_X \geq 1$ and that $\mathcal{G}|_{X\setminus Z}$ is reflexive. If $\operatorname{depth}_Z \mathcal{G} \geq 2$, then $\mathcal{G}$ is reflexive.*

Proof. For the proof of (1), by localizing X, we may assume that there is an exact sequence $\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{G}^\vee \to 0$ for some free $\mathcal{O}_X$-modules $\mathcal{E}_0$ and $\mathcal{E}_1$ of finite rank. Taking the dual, we have an exact sequence $0 \to \mathcal{G} \simeq \mathcal{G}^{\vee\vee} \to \mathcal{E}_0^\vee \to \mathcal{E}_1^\vee$ (cf. the proof of [20, Prop. 1.1]). The condition $\operatorname{depth}_Z \mathcal{O}_X \geq k$ implies that $\operatorname{depth}_Z \mathcal{E}_i^\vee \geq k$ for $i = 0, 1$. Thus, $\operatorname{depth}_Z \mathcal{G} \geq k$ by Lemma 2.17(1). This proves (1). Assertion (2) is a consequence of (1) (cf. Definition 2.9). We shall show (3). Let $j\colon X \setminus Z \hookrightarrow X$ be the open immersion. Then $\mathcal{G} \simeq j_*(\mathcal{G}_{X\setminus Z})$ by Property 2.6, since $\operatorname{depth}_Z \mathcal{G} \geq 2$ by assumption. Hence we have a splitting of the canonical homomorphism $\mathcal{G} \to \mathcal{G}^{\vee\vee}$ into the double-dual by the commutative diagram

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{G}^{\vee\vee} \\ \simeq\big\downarrow & & \big\downarrow \\ j_*(\mathcal{G}|_{X\setminus Z}) & \xrightarrow{\ \simeq\ } & j_*(\mathcal{G}^{\vee\vee}|_{X\setminus Z}). \end{array}$$

Thus, we have an injection $\mathcal{C} \hookrightarrow \mathcal{G}^{\vee\vee}$ from $\mathcal{C} := \mathcal{G}^{\vee\vee}/\mathcal{G}$, where $\operatorname{Supp} \mathcal{C} \subset Z$. The injection corresponds to a homomorphism $\mathcal{C} \otimes \mathcal{G}^\vee \to \mathcal{O}_X$, but this is zero, since $\operatorname{depth}_Z \mathcal{O}_X \geq 1$. Therefore, $\mathcal{C} = 0$ and $\mathcal{G}$ is reflexive. This proves (3), and we are done. □

Remark. The proof of (1) is essentially the same as the proof of [21, Prop. 3.6].

Corollary 2.22. *Let X be a locally Noetherian scheme, Z a closed subset, and $\mathcal{G}$ a coherent $\mathcal{O}_X$-module. Assume that $\mathcal{G}|_{X\setminus Z}$ is reflexive and $\operatorname{codim}(Z, X) \geq 1$. Let us consider the following three conditions:*

(i) *$\mathcal{G}$ satisfies $\mathbf{S}_2$ and $\operatorname{codim}(Z \cap \operatorname{Supp} \mathcal{G}, \operatorname{Supp} \mathcal{G}) \geq 2$;*
(ii) *$\operatorname{depth}_Z \mathcal{G} \geq 2$;*

(iii) $\mathcal{G}$ *is reflexive.*

Then (i) $\Rightarrow$ (ii) *holds always. If* $\operatorname{depth}_Z \mathcal{O}_X \geq 1$, *then* (ii) $\Rightarrow$ (iii) *holds, and if* $\operatorname{depth}_Z \mathcal{O}_X \geq 2$, *then* (ii) $\Leftrightarrow$ (iii) *holds. If* X *satisfies* $\mathbf{S}_2$ *and* $\operatorname{codim}(Z, X) \geq 2$, *then these three conditions are equivalent to each other.*

Proof. The implication (i) $\Rightarrow$ (ii) is shown in Lemma 2.15(2). The next implication (ii) $\Rightarrow$ (iii) in the case $\operatorname{depth}_Z \mathcal{O}_X \geq 1$ follows from Lemma 2.21(3), and the converse implication (iii) $\Rightarrow$ (ii) in the case $\operatorname{depth}_Z \mathcal{O}_X \geq 2$ follows from Lemma 2.21(1). Assume that X satisfies $\mathbf{S}_2$ and $\operatorname{codim}(Z, X) \geq 2$. Then $\operatorname{depth}_Z \mathcal{O}_X \geq 2$ by Lemma 2.15(2), and we have (ii) $\Leftrightarrow$ (iii) in this case. It remains to prove (ii) $\Rightarrow$ (i). Assume that $\operatorname{depth}_Z \mathcal{G} \geq 2$. Then $\operatorname{codim}(Z \cap \operatorname{Supp} \mathcal{G}, \operatorname{Supp} \mathcal{G}) \geq 2$ by Lemma 2.15(1). On the other hand, the reflexive sheaf $\mathcal{G}|_{X \setminus Z}$ satisfies $\mathbf{S}_2$ by Lemma 2.21(2), since $X \setminus Z$ satisfies $\mathbf{S}_2$. Thus, $\mathcal{G}$ satisfies $\mathbf{S}_2$ by the equivalence (i) $\Leftrightarrow$ (iv) of Lemma 2.14, and we are done. □

Remark. If X is a locally Noetherian scheme satisfying $\mathbf{S}_1$, then the support of a reflexive $\mathcal{O}_X$-module is a union of irreducible components of X. In fact, if $\mathcal{G}$ is reflexive, then $\operatorname{depth}_Z \mathcal{G} \geq 1$ for any closed subset Z with $\operatorname{codim}(Z, X) \geq 1$, by Lemma 2.21(1), and we have $\operatorname{codim}(Z \cap \operatorname{Supp} \mathcal{G}, \operatorname{Supp} \mathcal{G}) \geq 1$ by Lemma 2.15(1). This means that $\operatorname{Supp} \mathcal{G}$ is a union of irreducible components of X. In particular, if X is irreducible and satisfies $\mathbf{S}_1$, and if $\mathcal{G} \neq 0$, then $\operatorname{Supp} \mathcal{G} = X$. However, $\operatorname{Supp} \mathcal{G} \neq X$ in general when X is reducible. For example, let R be a Noetherian ring with two R-regular elements u and v, and set $X := \operatorname{Spec} R/uvR$ and $\mathcal{G} := (R/uR)^{\sim}$. Then we have an isomorphism $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_X) \simeq \mathcal{G}$ by the natural exact sequence

$$0 \to R/uR \to R/uvR \xrightarrow{u\times} R/uvR \to R/uR \to 0.$$

Thus, $\mathcal{G}$ is a reflexive $\mathcal{O}_X$-module, but $\operatorname{Supp} \mathcal{G} \neq X$ when $u \notin \sqrt{vR}$.

We have discussed properties $\mathbf{S}_1$ and $\mathbf{S}_2$ for general coherent sheaves. Finally in Section 2.1, we note the following well-known facts on locally Noetherian schemes satisfying $\mathbf{S}_2$.

Fact 2.23. Let X be a locally Noetherian scheme satisfying $\mathbf{S}_2$.

(1) If X is catenary (cf. Property 2.3), then X is locally equi-dimensional (cf. Definition 2.2(3)) (cf. [12, IV, Cor. (5.1.5), (5.10.9)]).
(2) For any open subset X° with $\operatorname{codim}(X \setminus X^\circ, X) \geq 2$ and for any connected component X_α of X, the intersection $X_\alpha \cap X^\circ$ is connected. This is a consequence of a result of Hartshorne (cf. [12, IV, Thm. (5.10.7)], [14, III, Thm. 3.6]).

§2.2. Relative $\mathbf{S}_k$-conditions

Here we shall consider the relative $\mathbf{S}_k$-condition for morphisms of locally Noetherian schemes.

Notation 2.24. Let $f: Y \to T$ be a morphism of schemes. For a point $t \in T$, the fiber $f^{-1}(t)$ of f over t is defined as $Y \times_T \operatorname{Spec} \Bbbk(t)$, and it is denoted by Y_t. For an $\mathcal{O}_Y$-module $\mathcal{F}$, the restriction $\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq \mathcal{F} \otimes_{\mathcal{O}_T} \Bbbk(t)$ to the fiber Y_t is denoted by $\mathcal{F}_{(t)}$.

Remark. The restriction $\mathcal{F}_{(t)}$ is identified with the inverse image $p_t^*(\mathcal{F})$ for the projection $p_t: Y_t \to Y$, and $\operatorname{Supp} \mathcal{F}_{(t)}$ is identified with $Y_t \cap \operatorname{Supp} \mathcal{F} = p_t^{-1}(\operatorname{Supp} \mathcal{F})$. If f is the identity morphism $Y \to Y$, then $\mathcal{F}_{(y)}$ is a sheaf on $\operatorname{Spec} \Bbbk(y)$ corresponding to the vector space $\mathcal{F}_y \otimes \Bbbk(y)$ for $y \in Y$.

Definition 2.25. For a morphism $f: Y \to T$ of schemes and for an $\mathcal{O}_Y$-module $\mathcal{F}$, let $\operatorname{Fl}(\mathcal{F}/T)$ be the set of points $y \in Y$ such that $\mathcal{F}_y$ is a flat $\mathcal{O}_{T,f(y)}$-module. If $Y = \operatorname{Fl}(\mathcal{F}/T)$, then $\mathcal{F}$ is said to be *flat over* T, or f*-flat*. If S is a subset of $\operatorname{Fl}(\mathcal{F}/T)$, then $\mathcal{F}$ is said to be *flat over* T *along* S, or f*-flat along* S.

Fact 2.26. Let $f: Y \to T$ be a morphism of locally Noetherian schemes and k a positive integer. For a coherent $\mathcal{O}_Y$-module $\mathcal{F}$ and a coherent $\mathcal{O}_T$-module $\mathcal{G}$, the following results are known, where in (2), (3), and (4), we fix an arbitrary point $y \in Y$, and set $t = f(y)$:

(1) If f is locally of finite type, then $\operatorname{Fl}(\mathcal{F}/T)$ is open.

(2) If $\mathcal{F}_y$ is flat over $\mathcal{O}_{T,t}$ and if $(\mathcal{F}_{(t)})_y$ is a free $\mathcal{O}_{Y_t,y}$-module, then $\mathcal{F}_y$ is a free $\mathcal{O}_{Y,y}$-module. In particular, if $\mathcal{F}$ is flat over T and if $\mathcal{F}_{(t)}$ is locally free for any $t \in T$, then $\mathcal{F}$ is locally free.

(3) If $\mathcal{F}_y$ is non-zero and flat over $\mathcal{O}_{T,t}$, then the following equalities hold:

$$\dim(\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{G})_y = \dim(\mathcal{F}_{(t)})_y + \dim \mathcal{G}_t, \tag{II-1}$$

$$\operatorname{depth}(\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{G})_y = \operatorname{depth}(\mathcal{F}_{(t)})_y + \operatorname{depth} \mathcal{G}_t. \tag{II-2}$$

(4) If $\mathcal{F}_y$ is non-zero and flat over $\mathcal{O}_{T,t}$ and if $\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{G}$ satisfies $\mathbf{S}_k$ at y, then $\mathcal{G}$ satisfies $\mathbf{S}_k$ at t.

(5) Assume that $\mathcal{F}$ is flat over T along the fiber Y_t over a point $t \in f(\operatorname{Supp} \mathcal{F})$. If $\mathcal{F}_{(t)}$ satisfies $\mathbf{S}_k$ and if $\mathcal{G}$ satisfies $\mathbf{S}_k$ at t, then $\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{G}$ also satisfies $\mathbf{S}_k$ at any point of Y_t.

(6) Assume that f is flat and that every fiber Y_t satisfies $\mathbf{S}_k$. Then $f^*\mathcal{G}$ satisfies $\mathbf{S}_k$ at y if and only if $\mathcal{G}$ satisfies $\mathbf{S}_k$ at $f(y)$.

Assertion (1) is just [12, IV, Thm. (11.1.1)]. Assertion (2) is a consequence of Proposition A.1 and Lemma A.5, since $\mathcal{O}_{Y_t,y} = \mathcal{O}_{Y,y}/I$ for the ideal $I = \mathfrak{m}_{T,t}\mathcal{O}_{Y,y}$ and we have

$$\mathrm{Tor}_1^{\mathcal{O}_{Y,y}}(\mathcal{F}_y, \mathcal{O}_{Y,y}/I) = 0 \quad \text{and} \quad (\mathcal{F}_{(t)})_y \simeq \mathcal{F}_y/I\mathcal{F}_y$$

under the assumption of (2). The two equalities (II-1) and (II-2) in (3) follow from [12, IV, Cor. (6.1.2), Prop. (6.3.1)], since

$$(\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{G})_y \simeq \mathcal{F}_y \otimes_{\mathcal{O}_{T,t}} \mathcal{G}_t \quad \text{and} \quad (\mathcal{F}_{(t)})_y \simeq \mathcal{F}_y \otimes_{\mathcal{O}_{T,t}} \Bbbk(t).$$

Assertions (4) and (5) are shown in [12, IV, Prop. (6.4.1)] by equalities (II-1) and (II-2), and assertion (6) is a consequence of (4) and (5) (cf. [12, IV, Cor. (6.4.2)]).

Corollary 2.27. *Let $f\colon Y \to T$ be a flat morphism of locally Noetherian schemes. Let W be a closed subset of T contained in $f(Y)$. Then*

$$\mathrm{codim}(f^{-1}(W), Y) = \mathrm{codim}(W, T) \quad \textit{and} \quad \mathrm{depth}_{f^{-1}(W)} f^*\mathcal{G} = \mathrm{depth}_W \mathcal{G}$$

for any coherent $\mathcal{O}_T$-module $\mathcal{G}$.

Proof. We may assume that $\mathcal{G} \neq 0$. Then

$$\mathrm{codim}(f^{-1}(W), Y) = \inf\{\dim \mathcal{O}_{Y,y} \mid y \in f^{-1}(W)\},$$
$$\mathrm{depth}_{f^{-1}(W)} f^*\mathcal{G} = \inf\{\mathrm{depth}(f^*\mathcal{G})_y \mid y \in f^{-1}(W)\},$$

by Property 2.1 and Definition 2.5. Thus, we can prove the assertion by applying (II-1) to $(\mathcal{F}, \mathcal{G}) = (\mathcal{O}_Y, \mathcal{O}_T)$ and (II-2) to $(\mathcal{F}, \mathcal{G}) = (\mathcal{O}_Y, \mathcal{G})$, since

$$\dim \mathcal{O}_{T,t} = \mathrm{codim}(W, T) \quad \text{and} \quad \dim \mathcal{O}_{Y_t,y} = 0$$

for a certain generic point t of W and a generic point y of Y_t, and since

$$\mathrm{depth}\, \mathcal{G}_t = \mathrm{depth}_W \mathcal{G} \quad \text{and} \quad \mathrm{depth}(((f^*\mathcal{G})_{(t)})_y = \mathrm{depth}\, \mathcal{O}_{Y_t,y} = 0$$

for a certain point $t \in W \cap \mathrm{Supp}\, \mathcal{G}$ and for a generic point y of Y_t. ☐

Definition 2.28. Let $f\colon Y \to T$ be a morphism of locally Noetherian schemes and $\mathcal{F}$ a coherent $\mathcal{O}_Y$-module. As a relative version of Definition 2.13, for a positive integer k, we define

$$\mathbf{S}_k(\mathcal{F}/T) := \mathrm{Fl}(\mathcal{F}/T) \cap \bigcup\nolimits_{t\in T} \mathbf{S}_k(\mathcal{F}_{(t)}) \quad \text{and}$$
$$\mathrm{CM}(\mathcal{F}/T) := \mathrm{Fl}(\mathcal{F}/T) \cap \bigcup\nolimits_{t\in T} \mathrm{CM}(\mathcal{F}_{(t)}),$$

and call them the *relative* $\mathbf{S}_k$*-locus* and the *relative Cohen–Macaulay locus* of $\mathcal{F}$ *over* T, respectively. We also write

$$\mathbf{S}_k(Y/T) = \mathbf{S}_k(\mathcal{O}_Y/T) \quad \text{and} \quad \mathrm{CM}(Y/T) = \mathrm{CM}(\mathcal{O}_Y/T),$$

and call them the *relative* $\mathbf{S}_k$*-locus* and the *relative Cohen–Macaulay locus for* f, respectively. The relative $\mathbf{S}_k$-condition and the relative Cohen–Macaulay condition are defined as follows:

- For a point $y \in Y$ (resp. a subset $S \subset Y$), we say that $\mathcal{F}$ satisfies *relative* $\mathbf{S}_k$ *over* T *at* y (resp. *along* S) if $y \in \mathbf{S}_k(\mathcal{F}/T)$ (resp. $S \subset \mathbf{S}_k(\mathcal{F}/T)$). We also say that $\mathcal{F}$ is *relatively Cohen–Macaulay over* T *at* y (resp. *along* S) if $y \in \mathrm{CM}(\mathcal{F}/T)$ (resp. $S \subset \mathrm{CM}(\mathcal{F}/T)$).
- We say that $\mathcal{F}$ satisfies *relative* $\mathbf{S}_k$ *over* T if $Y = \mathbf{S}_k(\mathcal{F}/T)$. We also say that $\mathcal{F}$ is *relatively Cohen–Macaulay over* T if $Y = \mathrm{CM}(\mathcal{F}/T)$.

Fact 2.29. For $f\colon Y \to T$ and $\mathcal{F}$ in Definition 2.28, assume that f is *locally of finite type* and $\mathcal{F}$ is flat over T. Then the following properties are known:

(1) The subset $\mathrm{CM}(\mathcal{F}/T)$ is open (cf. [12, IV, Thm. (12.1.1)(vi)]).

(2) If $\mathcal{F}_{(t)}$ is locally equi-dimensional (cf. Definition 2.2(3)) for any $t \in T$, then $\mathbf{S}_k(\mathcal{F}/T)$ is open for any $k \geq 1$ (cf. [12, IV, Thm. (12.1.1)(iv)]).

(3) If $Y \to T$ is flat, then $\mathbf{S}_k(Y/T)$ is open for any $k \geq 1$ (cf. [12, IV, Thm. (12.1.6)(i)]).

Definition 2.30 ($\mathbf{S}_k$-morphism and Cohen–Macaulay morphism). Let $f\colon Y \to T$ be a morphism of locally Noetherian schemes and k a positive integer. Then f is called an $\mathbf{S}_k$*-morphism* (resp. a *Cohen–Macaulay morphism*) if f is a flat morphism *locally of finite type* and $Y = \mathbf{S}_k(Y/T)$ (resp. $Y = \mathrm{CM}(Y/T)$). For a subset S of Y, f is called an $\mathbf{S}_k$*-morphism* (resp. a *Cohen–Macaulay morphism*) *along* S if $f|_V\colon V \to T$ is so for an open neighborhood V of S (cf. Fact 2.29(3)).

Remark. The $\mathbf{S}_k$-morphisms and the Cohen–Macaulay morphisms defined in [12, IV, Déf. (6.8.1)] are not necessarily locally of finite type. The definition of a Cohen–Macaulay morphism in [17, V, Ex. 9.7] coincides with ours. The notion of a "CM map" in [7, p. 7] is the same as that of a Cohen–Macaulay morphism in our sense for morphisms of locally Noetherian schemes.

Lemma 2.31. *Suppose that we are given a Cartesian diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{p} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{q} & T \end{array}$$

of schemes consisting of locally Noetherian schemes. Let $\mathcal{F}$ be a coherent $\mathcal{O}_Y$-module, Z a closed subset of Y, k a positive integer, and let $t' \in T'$ and $t \in T$ be points such that $t = q(t')$.

(1) *If f is flat, then, for the fibers $Y'_{t'} = f'^{-1}(t')$ and $Y_t = f^{-1}(t)$, one has*

$$\operatorname{codim}(p^{-1}(Z) \cap Y'_{t'}, Y'_{t'}) = \operatorname{codim}(Z \cap Y_t, Y_t) \quad \textit{and}$$
$$\operatorname{depth}_{p^{-1}(Z)\cap Y'_{t'}} \mathcal{O}_{Y'_{t'}} = \operatorname{depth}_{Z\cap Y_t} \mathcal{O}_{Y_t}.$$

(2) *If $\mathcal{F}$ is flat over T, then*

$$\operatorname{depth}_{p^{-1}(Z)\cap Y'_{t'}} (p^*\mathcal{F})_{(t')} = \operatorname{depth}_{Z\cap Y_t} \mathcal{F}_{(t)}.$$

(3) *If $\mathcal{F}$ is flat over T, then $\mathbf{S}_k(p^*\mathcal{F}/T') \subset p^{-1}\mathbf{S}_k(\mathcal{F}/T)$. If f is locally of finite type in addition, then $\mathbf{S}_k(p^*\mathcal{F}/T') = p^{-1}\mathbf{S}_k(\mathcal{F}/T)$.*

(4) *If f is locally of finite type and if $\mathcal{F}$ satisfies relative $\mathbf{S}_k$ over T, then $p^*\mathcal{F}$ does so over T'.*

(5) *If f is an $\mathbf{S}_k$-morphism (resp. Cohen–Macaulay morphism), then so is f'.*

Proof. Assertions (1) and (2) follow from Corollary 2.27 applied to the flat morphism $Y'_{t'} \to Y_t$ and to $\mathcal{G} = \mathcal{O}_{Y_t}$ or $\mathcal{G} = \mathcal{F}_{(t)}$. The first half of (3) follows from Definition 2.28 and Fact 2.26(4) applied to $Y'_{t'} \to Y_t$ and to $(\mathcal{F}, \mathcal{G}) = (\mathcal{O}_{Y'_{t'}}, \mathcal{F}_{(t)})$. The latter half of (3) follows from Fact 2.26(6), since the fiber $p^{-1}(y)$ over a point $y \in Y_t$ is isomorphic to $\operatorname{Spec} \Bbbk(y) \otimes_{\Bbbk(t)} \Bbbk(t')$ and since $\Bbbk(y) \otimes_{\Bbbk(t)} \Bbbk(t')$ is Cohen–Macaulay (cf. [12, IV, Lem. (6.7.1.1)]). Assertion (4) is a consequence of (3), and assertion (5) follows from (3) in the case $\mathcal{F} = \mathcal{O}_Y$, by Definition 2.30. □

Lemma 2.32. *Let $Y \to T$ be a morphism of locally Noetherian schemes and let Z be a closed subset of Y. Let $\mathcal{F}$ be a coherent $\mathcal{O}_Y$-module and k a positive integer.*

(1) *If $\mathcal{F}$ is flat over T, then*

$$\operatorname{depth}_Z \mathcal{F} \geq \inf\{\operatorname{depth}_{Z\cap Y_t} \mathcal{F}_{(t)} \mid t \in f(Z)\}.$$

(2) *If $\mathcal{F}$ satisfies relative $\mathbf{S}_k$ over T and if*

$$\operatorname{codim}(Z \cap \operatorname{Supp} \mathcal{F}_{(t)}, \operatorname{Supp} \mathcal{F}_{(t)}) \geq k$$

for any $t \in T$, then $\operatorname{depth}_Z \mathcal{F} \geq k$.

(3) *If $Y \to T$ is flat and if one of the two conditions below is satisfied, then $\operatorname{depth}_Z \mathcal{O}_Y \geq k$:*

  (a) *$\operatorname{depth}_{Y_t\cap Z} \mathcal{O}_{Y_t} \geq k$ for any $t \in T$;*

  (b) *Y_t satisfies $\mathbf{S}_k$ and $\operatorname{codim}(Y_t \cap Z, Y_t) \geq k$ for any $t \in T$.*

Proof. For the first assertion (1), we may assume that $Z \cap \operatorname{Supp} \mathcal{F} \neq \emptyset$. Then by Definition 2.5 we have the inequality in (1) from equality (II-2) in Fact 2.26(3) in the case where $\mathcal{G} = \mathcal{O}_T$, since $\operatorname{depth} \mathcal{O}_{T,t} \geq 0$ for any $t \in T$. Assertion (2) is a consequence of (1) and Lemma 2.15(2) applied to $(Y_t, Z \cap Y_t, \mathcal{F}_{(t)})$. The last assertion (3) is derived from (1) and (2) in the case where $\mathcal{F} = \mathcal{O}_Y$. □

The following result gives some relations between the reflexive modules and the relative $\mathbf{S}_2$-condition. Similar results can be found in [21, §3].

Lemma 2.33. *Let $f\colon Y \to T$ be a flat morphism of locally Noetherian schemes, $\mathcal{F}$ a coherent $\mathcal{O}_Y$-module, and Z a closed subset of Y. Assume that*

$$\operatorname{depth}_{Y_t \cap Z} \mathcal{O}_{Y_t} \geq 1$$

for any fiber $Y_t = f^{-1}(t)$. Then the following hold for the open immersion $j\colon Y \setminus Z \hookrightarrow Y$ and for the restriction homomorphism $\mathcal{F} \to j_*(\mathcal{F}|_{Y \setminus Z})$:

(1) *If $\mathcal{F}|_{Y \setminus Z}$ is reflexive and if $\mathcal{F} \simeq j_*(\mathcal{F}|_{Y \setminus Z})$, then $\mathcal{F}$ is reflexive.*
(2) *If $\mathcal{F}$ is reflexive and if $\operatorname{depth}_{Y_t \cap Z} \mathcal{O}_{Y_t} \geq 2$ for any $t \in T$, then $\mathcal{F} \simeq j_*(\mathcal{F}|_{Y \setminus Z})$.*
(3) *If $\mathcal{F}$ is flat over T and if $\operatorname{depth}_{Y_t \cap Z} \mathcal{F}_{(t)} \geq 2$ for any $t \in T$, then $\mathcal{F} \simeq j_*(\mathcal{F}|_{Y \setminus Z})$.*
(4) *If Y_t satisfies $\mathbf{S}_2$ and $\operatorname{codim}(Y_t \cap Z, Y_t) \geq 2$ for any $t \in T$, and if $\mathcal{F}$ is reflexive, then $\mathcal{F} \simeq j_*(\mathcal{F}|_{Y \setminus Z})$.*
(5) *If $\mathcal{F}$ satisfies relative $\mathbf{S}_2$ over T and if $\operatorname{codim}(Z \cap \operatorname{Supp} \mathcal{F}_{(t)}, \operatorname{Supp} \mathcal{F}_{(t)}) \geq 2$ for any $t \in T$, then $\mathcal{F} \simeq j_*(\mathcal{F}|_{Y \setminus Z})$.*
(6) *In the situation of (3) or (5), if $\mathcal{F}_{(t)}|_{Y_t \setminus Z}$ is reflexive, then $\mathcal{F}_{(t)}$ is reflexive; if $\mathcal{F}|_{Y \setminus Z}$ is reflexive, then $\mathcal{F}$ is reflexive.*

Proof. Note that $\mathcal{F} \simeq j_*(\mathcal{F}|_{Y \setminus Z})$ if and only if $\operatorname{depth}_Z \mathcal{F} \geq 2$ (cf. Property 2.6). We have $\operatorname{depth}_Z \mathcal{O}_Y \geq 1$ by Lemma 2.32(3). Hence, (1) is a consequence of Lemma 2.21(3). In case (2) we have $\operatorname{depth}_Z \mathcal{O}_Y \geq 2$ by Lemma 2.32(3), and (2) is a consequence of Lemma 2.21(1). Assertion (3) follows from Lemma 2.32(1) with $k = 2$. Assertions (4) and (5) are special cases of (2) and (3), respectively. The first assertion of (6) follows from Corollary 2.22. The second assertion of (6) is derived from (1) and (3). □

Remark. The assumption of Lemma 2.33 holds when Y_t satisfies $\mathbf{S}_1$ and $\operatorname{codim}(Y_t \cap Z, Y_t) \geq 1$ for any $t \in T$ (cf. Lemma 2.15(2)).

Lemma 2.34. *In the situation of Lemma 2.31, assume that f is flat, $\mathcal{F}|_{Y \setminus Z}$ is locally free, and*

$$\operatorname{depth}_{Y_t \cap Z} \mathcal{O}_{Y_t} \geq 2$$

for any $t \in T$*. Then* $\mathcal{F}^{\vee\vee} \simeq j_*(\mathcal{F}|_{Y \setminus Z})$ *for the open immersion* $j \colon Y \setminus Z \hookrightarrow Y$*, and*

$$(p^*\mathcal{F})^{\vee\vee} \simeq (p^*(\mathcal{F}^{\vee\vee}))^{\vee\vee}.$$

Moreover, $\mathcal{F}$ *and* $p^*\mathcal{F}$ *are reflexive if* $\mathcal{F}$ *is flat over* T *and*

$$\operatorname{depth}_{Y_t \cap Z} \mathcal{F}_{(t)} \geq 2$$

for any $t \in T$*.*

Proof. Now, $\operatorname{depth}_Z \mathcal{O}_Y \geq 2$ by Lemma 2.32(3). Hence, $\operatorname{depth}_Z \mathcal{F}^{\vee\vee} \geq 2$ by Lemma 2.21(1), and this implies the first isomorphism for $\mathcal{F}^{\vee\vee}$. We have

$$\operatorname{depth}_{Y'_{t'} \cap p^{-1}(Z)} \mathcal{O}_{Y'_{t'}} \geq 2$$

by Lemma 2.31(1). Hence, by the previous argument applied to $p^*\mathcal{F}$ and $p^*(\mathcal{F}^{\vee\vee})$, we have isomorphisms

$$(p^*\mathcal{F})^{\vee\vee} \simeq j'_*(p^*\mathcal{F}|_{Y' \setminus p^{-1}(Z)}) \simeq (p^*(\mathcal{F}^{\vee\vee}))^{\vee\vee}$$

for the open immersion $j' \colon Y' \setminus p^{-1}(Z) \hookrightarrow Y'$. It remains to prove the last assertion. In this case, $\mathcal{F}$ is reflexive by (1) and (3) of Lemma 2.33. Moreover, by Lemma 2.31(2), we have

$$\operatorname{depth}_{Y'_{t'} \cap p^{-1}(Z)} (p^*\mathcal{F})_{(t')} \geq 2$$

for any point $t' \in T'$. Thus, $p^*\mathcal{F}$ is reflexive by the same argument as above. □

Lemma 2.35. *Let* $f \colon Y \to T$ *be a morphism of locally Noetherian schemes, and let* Z *be a closed subset of* Y*. Assume that* f *is* quasi-flat (*cf.* [12, IV, (2.3.3)])*, i.e., there is a coherent* $\mathcal{O}_Y$*-module* $\mathcal{F}$ *such that* $\mathcal{F}$ *is flat over* T *and* $\operatorname{Supp} \mathcal{F} = Y$*. Then*

$$\operatorname{codim}_y(Z, Y) \geq \operatorname{codim}_y(Z \cap Y_{f(y)}, Y_{f(y)}) \tag{II-3}$$

for any point $y \in Z$*. If* $\operatorname{codim}(Z \cap Y_t, Y_t) \geq k$ *for a point* $t \in T$ *and for an integer* k*, then there is an open neighborhood* V *of* Y_t *in* Y *such that* $\operatorname{codim}(Z \cap V, V) \geq k$*.*

Proof. For the sheaf $\mathcal{F}$ above, we have $\operatorname{Supp} \mathcal{F}_{(t)} = Y_t$ for any $t \in T$. If $z \in Z \cap Y_t$, then

$$\dim \mathcal{F}_z = \dim \mathcal{O}_{Y,z} \quad \text{and} \quad \dim(\mathcal{F}_{(t)})_z = \dim \mathcal{O}_{Y_t,z} \tag{II-4}$$

by Property 2.1(1), and moreover,

$$\dim \mathcal{F}_z = \dim(\mathcal{F}_{(t)})_z + \dim \mathcal{O}_{T,t} \geq \dim(\mathcal{F}_{(t)})_z \tag{II-5}$$

by (II-1) since $\mathcal{F}$ is flat over T. Thus, we have (II-3) from (II-4) and (II-5) by Property 2.1(3). The last assertion follows from (II-3) and the lower semi-continuity of the function $x \mapsto \operatorname{codim}_x(Z, Y)$ (cf. [12, 0_{IV}, Cor. (14.2.6)]). In fact, the set of points $y \in Y$ with $\operatorname{codim}_y(Z, Y) \geq k$ is an open subset containing Y_t. □

We introduce the following notion (cf. [12, IV, Déf. (17.10.1)] and [7, p. 6]).

Definition 2.36 (Pure relative dimension). Let $f \colon Y \to T$ be a morphism locally of finite type. The *relative dimension of* f at y is defined as $\dim_y Y_{f(y)}$, and it is denoted by $\dim_y f$. We say that f *has pure relative dimension* d if $d = \dim_y f$ for any $y \in Y$; this is equivalent to the condition that every non-empty fiber is equi-dimensional and has dimension equal to d.

Remark 2.37. If a flat morphism $f \colon Y \to T$ is locally of finite type and it has pure relative dimension, then it is an *equi-dimensional morphism* in the sense of [12, IV, Déf. (13.3.2), ($\mathrm{Err}_{\mathrm{IV}}$, 35)]. In fact, a generic point of Y is mapped a generic point of T by (II-1) applied to $\mathcal{F} = \mathcal{O}_Y$ and $\mathcal{G} = \mathcal{O}_T$, and condition (a$''$) of [12, IV, Prop. 13.3.1] is satisfied.

Lemma 2.38. *Let $f \colon Y \to T$ be a flat morphism locally of finite type between locally Noetherian schemes. For a point $y \in Y$ and its image $t = f(y)$, assume that the fiber Y_t satisfies $\mathbf{S}_k$ at y for some $k \geq 2$. Let Y° be an open subset of Y with $y \notin Y^\circ$. Then there exists an open neighborhood U of y in Y such that*

(1) *$f|_U \colon U \to T$ is an $\mathbf{S}_k$-morphism having pure relative dimension and*

(2) *the inequality*

$$\operatorname{codim}(U_{t'} \setminus Y^\circ, U_{t'}) \geq \operatorname{codim}_y(Y_t \setminus Y^\circ, Y_t)$$

holds for any $t' \in f(U)$, where $U_{t'} = U \cap Y_{t'}$.

Proof. By Fact 2.29(3), replacing Y with an open neighborhood of y, we may assume that f is an $\mathbf{S}_k$-morphism. For any point $y' \in Y$ and for the fiber $Y_{t'}$ over $t' = f(y')$, the local ring $\mathcal{O}_{Y_{t'}, y'}$ is equi-dimensional by Fact 2.23(1), since $Y_{t'}$ is catenary satisfying $\mathbf{S}_2$. Moreover, the local ring has no embedded primes by the condition $\mathbf{S}_1$. If an *associated prime cycle* Γ of $Y_{t'}$ (cf. [12, IV, Déf. (3.1.1)]) contains y', then Γ corresponds to an associated prime ideal $\mathfrak{p}$ of $\mathcal{O}_{Y', t'}$, and we have

$$\dim \Gamma = \dim_{y'} \Gamma = \dim \mathcal{O}_{Y_{t'}, y'}/\mathfrak{p} + \operatorname{tr.deg} \Bbbk(y')/\Bbbk(t')$$

by [12, IV, Prop. (5.2.1), Cor. (5.2.3)]. Since $\mathfrak{p}$ is minimal and $\mathcal{O}_{Y_{t'}, y'}$ is equi-dimensional, it follows that all the associated prime cycles of $Y_{t'}$ containing y' have the same dimension. Thus, by [12, IV, Thm. (12.1.1)(ii)], we may assume that f has pure relative dimension, by replacing Y with an open neighborhood of y. Consequently, $Y \to T$ is an equi-dimensional morphism (cf. Remark 2.37). Then the function

$$Y \ni y' \mapsto \operatorname{codim}_{y'}(Y_{f(y')} \setminus Y^{\circ}, Y_{f(y')})$$

is lower semi-continuous by [12, IV, Prop. (13.3.7)]. Hence, we can take an open neighborhood U of y satisfying the inequality in (2). Thus, we are done. □

Corollary 2.39. *Let $f\colon Y \to T$ be an $\mathbf{S}_2$-morphism of locally Noetherian schemes. Assume that every fiber Y_t is connected.*

(1) *If T are connected, then f has pure relative dimension. In particular, f is an equi-dimensional morphism.*
(2) *If f is proper, then the function $t \mapsto \operatorname{codim}(Y_t \cap Z, Y_t)$ is lower semi-continuous on T for any closed subset Z of Y.*

Proof. We may assume that T is connected. We know that every fiber Y_t is equi-dimensional by the proof of Lemma 2.38, since Y_t is connected. Moreover, $\dim Y_t$ is independent of the choice of $t \in T$ by Lemma 2.38(1), since T is connected. Hence, f has pure relative dimension, and (1) has been proved. In case (2), $f(Y) = T$, since $f(Y)$ is open and closed. Let us consider the set F_k of points $y \in Y$ such that

$$\operatorname{codim}_y(Z \cap Y_{f(y)}, Y_{f(y)}) \leq k$$

for an integer k. Then $f(F_k)$ is the set of points $t \in T$ with $\operatorname{codim}(Y_t \cap Z, Y_t) \leq k$. Now, F_k is closed by (1) and by [12, IV, Prop. (13.3.7)]. Since f is proper, $f(F_k)$ is closed. This proves (2), and we are done. □

§3. Relative $\mathbf{S}_2$-condition and flatness

We shall study *restriction homomorphisms* (cf. Definition 3.2 below) of coherent sheaves to open subsets by applying the local criterion of flatness (cf. Section A.1), and give several criteria for the restriction homomorphism on a fiber to be an isomorphism. In Section 3.1 we prove the key proposition (Proposition 3.7) and discuss related properties. Some applications of Proposition 3.7 are given in Section 3.2: Theorem 3.16 is a criterion for a sheaf to be invertible, which is used in the proof of Theorem 5.10 below. Proposition 3.19 gives infinitesimal and valuative criteria for a reflexive sheaf to satisfy the relative $\mathbf{S}_2$-condition. Theorem 3.26 on the relative $\mathbf{S}_2$ refinement is analogous to the flattening stratification theorem by Mumford in [39, Lect. 8] and to the representability theorem of unramified functors by Murre [41]. Its local version is given as Theorem 3.28.

§3.1. Restriction homomorphisms

In Section 3.1 we work under Situation 3.1 below unless otherwise stated.

Situation 3.1. We fix a morphism $f\colon Y \to T$ of locally Noetherian schemes, a closed subset Z of Y, and a coherent $\mathcal{O}_Y$-module $\mathcal{F}$. The complement of Z in Y is written as U, and $j\colon U \hookrightarrow Y$ stands for the open immersion.

Definition 3.2. The *restriction morphism* of $\mathcal{F}$ to U is defined as the canonical homomorphism

$$\phi = \phi_U(\mathcal{F})\colon \mathcal{F} \to j_*(\mathcal{F}|_U).$$

Similarly, for a point $t \in T$, the restriction homomorphism of $\mathcal{F}_{(t)}$ to U (or to $U \cap Y_t$) is defined as the canonical homomorphism

$$\phi_t = \phi_U(\mathcal{F}_{(t)})\colon \mathcal{F}_{(t)} \to j_*(\mathcal{F}_{(t)}|_{U\cap Y_t}).$$

Here $U \cap Y_t$ is identical to $U \times_Y Y_t$, and j stands also for the open immersion $U \cap Y_t \hookrightarrow Y_t$.

Remark. The homomorphism ϕ_t is an isomorphism along $U \cap Y_t$. In particular, ϕ_t is an isomorphism if $t \notin f(Z)$.

Remark. By Remark 2.8, we see that ϕ is an injection (resp. isomorphism) along Y_t if and only if

$$\operatorname{depth} \mathcal{F}_y \geq 1 \quad (\text{resp. } \geq 2)$$

for any point $y \in Z$ such that $Y_t \cap \overline{\{y\}} \neq \emptyset$.

We use the following notation only in Section 3.1.

Notation 3.3. For simplicity we write

$$\mathcal{F}_* := j_*(\mathcal{F}|_U) \quad \text{and} \quad \mathcal{F}_{(t)*} := j_*(\mathcal{F}_{(t)}|_{U\cap Y_t}).$$

When we fix a point t of $f(Z)$, we write A for the local ring $\mathcal{O}_{T,t}$ and $\mathfrak{m}$ for the maximal ideal $\mathfrak{m}_{T,t}$, and for an integer $n \geq 0$ we set

$$A_n := A/\mathfrak{m}^{n+1}, \quad T_n := \operatorname{Spec} A_n, \quad Y_n := Y \times_T T_n,$$
$$U_n = Y_n \cap U, \quad \mathcal{F}_n := \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_n}, \quad \mathcal{F}_{n*} := j_*(\mathcal{F}_n|_{U_n}).$$

In particular, $Y_t = Y_0$, $\mathcal{F}_{(t)} = \mathcal{F}_0$, $\mathcal{F}_{(t)*} = \mathcal{F}_{0*}$, and Y_n is a closed subscheme of Y_m for any $m \geq n$. Furthermore, the restriction homomorphisms of $\mathcal{F}_n$ and $(\mathcal{F}_{n*})_{(t)}$, respectively, are written by

$$\phi_n\colon \mathcal{F}_n \to \mathcal{F}_{n*} = j_*(\mathcal{F}_n|_{U_n}) \quad \text{and} \quad \varphi_n\colon (\mathcal{F}_{n*})_{(t)} = \mathcal{F}_{n*} \otimes_{\mathcal{O}_{Y_n}} \mathcal{O}_{Y_0} \to \mathcal{F}_{0*}.$$

Remark 3.4. The homomorphism ϕ_t in Definition 3.2 equals ϕ_0, and the diagram

$$\begin{array}{ccc} \mathcal{F}_n \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_0} & \xrightarrow{\phi_n \otimes \mathcal{O}_{Y_0}} & (\mathcal{F}_{n*}) \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_0} \\ \simeq \downarrow & & \downarrow \varphi_n \\ \mathcal{F}_0 & \xrightarrow{\phi_0} & \mathcal{F}_{0*} \end{array}$$

is commutative for any $n \geq 0$.

Lemma 3.5. *Assume that $\mathcal{F}|_U$ is flat over T.*

(1) *For a point $y \in Z$ and $t = f(y)$, if ϕ_t is injective at y, then $\mathcal{F}_y$ is flat over $\mathcal{O}_{T,t}$.*
(2) *For a point $y \in Z$ and $t = f(y)$, if ϕ_t is an isomorphism at y, then the restriction homomorphism $\phi_n \colon \mathcal{F}_n \to \mathcal{F}_{n*}$ is an isomorphism at y for any $n \geq 0$.*
(3) *If ϕ_t is an isomorphism for any $t \in f(Z)$, then ϕ is also an isomorphism.*

Proof. First we shall prove (3) assuming (1) and (2). Since ϕ_t is an isomorphism for any $t \in f(Z)$, $\mathcal{F}$ is flat over T by (1), and we have

$$\operatorname{depth}_{Y_t \cap Z} \mathcal{F}_{(t)} \geq 2$$

by (2) (cf. Property 2.6). Then $\operatorname{depth}_Z \mathcal{F} \geq 2$ by Lemma 2.32(1), and ϕ is an isomorphism (cf. Property 2.6).

Next, we shall prove (1) and (2). We may assume that $T = \operatorname{Spec} A$ for a local Noetherian ring A in which $t = f(y)$ corresponds to the maximal ideal $\mathfrak{m}$ of A and that $Y = \operatorname{Spec} \mathcal{O}_{Y,y}$ for the given point y (cf. Remark 2.8). We write $\Bbbk = A/\mathfrak{m} = \Bbbk(t)$ and use Notation 3.3. From the standard exact sequence

$$0 \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to A_n \to A_{n-1} \to 0$$

of A-modules, by taking tensor products with $\mathcal{F}$ over A, we have an exact sequence

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_{\Bbbk} \mathcal{F}_0 \xrightarrow{u_n} \mathcal{F}_n \to \mathcal{F}_{n-1} \to 0 \tag{III-1}$$

of $\mathcal{O}_Y$-modules. Here the left homomorphism u_n is injective at y for any $n \geq 0$ if and only if $\mathcal{F}_y$ is flat over $\mathcal{O}_{T,t}$ by the local criterion of flatness (cf. Proposition A.1). Now, u_n is injective on the open subset U_n, since $\mathcal{F}|_U$ is flat over T, and u_0 is the identity morphism. For each $n > 0$, there is a natural commutative diagram

$$\begin{array}{ccccccc} & & \mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_{\Bbbk} \mathcal{F}_0 & \xrightarrow{u_n} & \mathcal{F}_n & \longrightarrow & \mathcal{F}_{n-1} \\ & & \mathrm{id} \otimes \phi_0 \downarrow & & \phi_n \downarrow & & \phi_{n-1} \downarrow \\ 0 & \longrightarrow & \mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_{\Bbbk} j_*(\mathcal{F}_0|_{U_0}) & \xrightarrow{j_*(u_n|_{U_n})} & j_*(\mathcal{F}_n|_{U_n}) & \longrightarrow & j_*(\mathcal{F}_{n-1}|_{U_{n-1}}) \end{array}$$

of exact sequences. By assumption, $\phi_0 = \phi_t$ is an injection (resp. isomorphism) at y in case (1) (resp. (2)) (cf. Remark 2.8). Thus, u_n is injective at y for any n by the diagram. This shows (1). In case (2), by induction on n we see that ϕ_n is an isomorphism at y for any n, by the diagram. Thus, (2) also holds, and we are done. □

Applying Lemma 3.5 to $\mathcal{F} = \mathcal{O}_Y$, we have the following corollary.

Corollary 3.6. *Suppose that U is flat over T. If the restriction homomorphism $\phi_t(\mathcal{O}_Y)\colon \mathcal{O}_{Y_t} \to j_*(\mathcal{O}_{Y_t \cap U})$ is injective for a point $t \in f(Z)$, then f is flat along Y_t. If $\phi_t(\mathcal{O}_Y)$ is an isomorphism for any $t \in f(Z)$, then $\mathcal{O}_Y \simeq j_*(\mathcal{O}_U)$.*

Proposition 3.7 (Key proposition). *Suppose that there is an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{G} \to 0$$

of coherent $\mathcal{O}_Y$-modules such that

(i) *$\mathcal{E}^0$, $\mathcal{E}^1$, and $\mathcal{G}|_U$ are flat over T, and*

(ii) *the inequalities*

$$\operatorname{depth}_{Z \cap Y_t} \mathcal{E}^0_{(t)} \geq 2 \quad \text{and} \quad \operatorname{depth}_{Z \cap Y_t} \mathcal{E}^1_{(t)} \geq 1$$

hold for any $t \in f(Z)$.

Then the following hold:

(1) *The restriction homomorphism $\phi\colon \mathcal{F} \to \mathcal{F}_* = j_*(\mathcal{F}|_U)$ is an isomorphism.*

(2) *For a fixed point $t \in f(Z)$ and for any integer $n \geq 0$, $\mathcal{F}_{n*} = j_*(\mathcal{F}_n|_{U_n})$ (cf. Notation 3.3) is isomorphic to the kernel $\mathcal{F}'_n$ of the homomorphism*

$$\mathcal{E}^0_n = \mathcal{E}^0 \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_n} \to \mathcal{E}^1_n = \mathcal{E}^1 \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_n}$$

induced by $\mathcal{E}^0 \to \mathcal{E}^1$. In particular, $\mathcal{F}_{n*}$ is coherent for any $n \geq 0$, and $\mathcal{F}_{(t)*} = j_*(\mathcal{F}_{(t)}|_{U \cap Y_t})$ is coherent for any $t \in f(Z)$.

(3) *For any point $y \in Y$ and $t = f(y)$, the following conditions are equivalent to each other, where we use Notation 3.3 in (a′), (b′), and (b″):*

   (a) *$\phi_t\colon \mathcal{F}_{(t)} \to \mathcal{F}_{(t)*}$ is surjective at y;*

   (b) *ϕ_t is an isomorphism at y;*

   (c) *$\mathcal{G}_y$ is flat over $\mathcal{O}_{T,t}$;*

   (a′) *$\varphi_n\colon (\mathcal{F}_{n*})_{(t)} \to \mathcal{F}_{0*}$ is surjective at y for any $n \geq 0$;*

   (b′) *φ_n is an isomorphism at y for any $n \geq 0$;*

   (b″) *$\phi_n\colon \mathcal{F}_n \to \mathcal{F}_{n*}$ is an isomorphism at y for any $n \geq 0$.*

   Note that if (c) is satisfied, then $\mathcal{F}_y$ is also flat over $\mathcal{O}_{T,t}$.

Proof. By (i), (ii) and by Lemma 2.32(1), we have $\operatorname{depth}_Z \mathcal{E}^0 \geq 2$ and $\operatorname{depth}_Z \mathcal{E}^1 \geq 1$. Thus, $\operatorname{depth}_Z \mathcal{F} \geq 2$ by Lemma 2.17(1), and we have (1) (cf. Property 2.6). For each $n \geq 0$, the exact sequence

$$0 \to \mathcal{F}'_n \to \mathcal{E}^0_n \to \mathcal{E}^1_n \to \mathcal{G}_n \to 0$$

on Y_n satisfies conditions (i) and (ii) for the induced morphism $Y_n \to T_n$, where $\mathcal{G}_n = \mathcal{G} \otimes \mathcal{O}_{Y_n}$. Thus, by (1), the restriction homomorphism

$$\phi(\mathcal{F}'_n) \colon \mathcal{F}'_n \to (\mathcal{F}'_n)_* = j_*(\mathcal{F}'_n|_{U_n})$$

of $\mathcal{F}'_n$ is an isomorphism. On the other hand, there is a canonical homomorphism $\psi_n \colon \mathcal{F}_n = \mathcal{F} \otimes \mathcal{O}_{Y_n} \to \mathcal{F}'_n$. Note that ψ_n is an isomorphism at a point $y \in Y_t = Y_0$ if $\mathcal{G}_y$ is flat over $\mathcal{O}_{T,t}$. In particular, ψ_n is an isomorphism on U_n by condition (i). Hence, $(\mathcal{F}'_n)_* \simeq \mathcal{F}_{n*}$, and we have an isomorphism $\mathcal{F}'_n \simeq \mathcal{F}_{n*}$, by which ϕ_n is isomorphic to ψ_n. This proves (2). For the proof of (3), we may assume that $y \in Z$. We shall show that there is an exact sequence

$$\text{(III-2)}\quad \operatorname{Tor}_2^A(\mathcal{G}_y, \Bbbk) \to (\mathcal{F}_{(t)})_y = \mathcal{F}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y_t,y} \xrightarrow{(\psi_0)_y} (\mathcal{F}'_0)_y \to \operatorname{Tor}_1^A(\mathcal{G}_y, \Bbbk) \to 0$$

of $\mathcal{O}_{Y,y}$-modules, where $A = \mathcal{O}_{T,t}$ and $\Bbbk = \Bbbk(t)$. For the image $\mathcal{B}$ of $\mathcal{E}^0 \to \mathcal{E}^1$, we have two short exact sequences $0 \to \mathcal{F} \to \mathcal{E}^0 \to \mathcal{B} \to 0$ and $0 \to \mathcal{B} \to \mathcal{E}^1 \to \mathcal{G} \to 0$ on Y. Then the kernel of $\mathcal{B}_0 = \mathcal{B} \otimes \mathcal{O}_{Y_0} \to \mathcal{E}^1_0 = \mathcal{E}^1 \otimes \mathcal{O}_{Y_0}$ is isomorphic to $\mathcal{T}or_1^{\mathcal{O}_T}(\mathcal{G}, \Bbbk)$, and the kernel of $\mathcal{F}_0 \to \mathcal{E}^0_0$ is isomorphic to $\mathcal{T}or_1^{\mathcal{O}_T}(\mathcal{B}, \Bbbk) \simeq \mathcal{T}or_2^{\mathcal{O}_T}(\mathcal{G}, \Bbbk)$. Then we have the exact sequence (III-2) by applying the snake lemma to the commutative diagram

$$\begin{array}{ccccccc} & & \mathcal{F}_0 & \longrightarrow & \mathcal{E}^0_0 & \longrightarrow & \mathcal{B}_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle =} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}'_0 & \longrightarrow & \mathcal{E}^0_0 & \longrightarrow & \mathcal{E}^1_0 & & \end{array}$$

of exact sequences. Note that $\psi_0 \simeq \phi_0$ by the argument above. We shall prove (3) using (III-2). If (a) holds, then $\operatorname{Tor}_1^A(\mathcal{G}_y, \Bbbk) = 0$ by (III-2), and it implies (c) by the local criterion of flatness (cf. Proposition A.1), since $\mathcal{G}_y \otimes \mathcal{O}_{Y_t,y} \simeq \mathcal{G}_y \otimes \Bbbk$ is flat over $\Bbbk$. If (c) holds, then $\operatorname{Tor}_j^A(\mathcal{G}_y, \Bbbk) = 0$ for $j = 1$ and 2, and it implies (b) by (III-2). Thus, we have shown the equivalence of the three conditions (a), (b), and (c). By applying the equivalence of the three conditions to $\mathcal{F}'_n \simeq \mathcal{F}_{n*}$ and $Y_n \to T_n$ instead of $\mathcal{F}$ and $Y \to T$, we see that (a′) and (b′) are both equivalent to the condition that $(\mathcal{G}_n)_y$ is flat over $\mathcal{O}_{T_n,t}$ for any $n \geq 0$. This is also equivalent to (c) by the local criterion of flatness (cf. (i) ⇔ (iv) in Proposition A.1). If (c) holds, then $\psi_n \colon \mathcal{F}_n \to \mathcal{F}'_n$ is an isomorphism as we have noted before, and the isomorphism $\mathcal{F}'_n \simeq \mathcal{F}_{n*}$ in (2) implies (b″). Conversely, if (b″) holds, then φ_n is isomorphic to the canonical isomorphism $(\mathcal{F}_n)_{(t)} \simeq \mathcal{F}_{(t)}$ for any n (cf. Remark 3.4), and it implies (b′). Thus, we are done. □

Remark. The exact sequence (III-2) is obtained as the "edge sequence" of the spectral sequence

$$E_2^{p,q} = \mathcal{T}or_{-p}^{\mathcal{O}_T}(\mathcal{H}^q(\mathcal{E}^\bullet), \Bbbk(t)) \Rightarrow E^{p+q} = \mathcal{H}^{p+q}(\mathcal{E}^\bullet_{(t)})$$

of $\mathcal{O}_{Y_t}$-modules (cf. [12, III, (6.3.2.2)]) arising from the quasi-isomorphism

$$\mathcal{E}^\bullet_{(t)} \simeq_{\mathrm{qis}} \mathcal{E}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_T} \Bbbk(t),$$

where $\mathcal{E}^\bullet$ and $\mathcal{E}^\bullet_{(t)}$ denote the complexes $[0 \to \mathcal{E}^0 \to \mathcal{E}^1 \to 0]$ and $[0 \to \mathcal{E}^0_{(t)} \to \mathcal{E}^1_{(t)} \to 0]$, respectively.

Remark 3.8. In the situation of Proposition 3.7(2), the canonical homomorphism

$$\phi_\infty = \varprojlim\nolimits_n \phi_n \colon \varprojlim\nolimits_n \mathcal{F}_n \to \varprojlim\nolimits_n \mathcal{F}_{n*}$$

is an isomorphism, where the projective limit $\varprojlim_n$ is taken in the category of $\mathcal{O}_Y$-modules. This is shown as follows. Since $\mathcal{F}'_n \simeq \mathcal{F}_{n*}$, it is enough to show that the homomorphism

$$\psi_\infty(V) := \varprojlim\nolimits_n \mathrm{H}^0(V, \psi_n) \colon \varprojlim\nolimits_n \mathrm{H}^0(V, \mathcal{F}_n) \to \varprojlim\nolimits_n \mathrm{H}^0(V, \mathcal{F}'_n)$$

is an isomorphism for any open affine subset V of Y, where we note that the global section functor $\mathrm{H}^0(V, \bullet)$ commutes with $\varprojlim$. For $R = \mathrm{H}^0(V, \mathcal{O}_V)$ and $R_n = R/\mathfrak{m}^{n+1}R \simeq \mathrm{H}^0(V, \mathcal{O}_{Y_n})$, we have two exact sequences:

$$0 \to \mathrm{H}^0(V, \mathcal{F}) \to \mathrm{H}^0(V, \mathcal{E}^0) \to \mathrm{H}^0(V, \mathcal{E}^1),$$
$$0 \to \mathrm{H}^0(V, \mathcal{F}'_n) \to \mathrm{H}^0(V, \mathcal{E}^0) \otimes_R R_n \to \mathrm{H}^0(V, \mathcal{E}^1) \otimes_R R_n.$$

Since the $\mathfrak{m}R$-adic completion $\widehat{R} = \varprojlim R_n$ is flat over R and since $\varprojlim$ is left exact, we have an isomorphism

$$\mathrm{H}^0(V, \mathcal{F}) \otimes_R \widehat{R} \simeq \mathrm{Ker}(\mathrm{H}^0(V, \mathcal{E}^0) \otimes_R \widehat{R} \to \mathrm{H}^0(V, \mathcal{E}^1) \otimes_R \widehat{R}) \simeq \varprojlim\nolimits_n \mathrm{H}^0(V, \mathcal{F}'_n).$$

Then $\psi_\infty(V)$ is an isomorphism, since

$$\varprojlim\nolimits_n \mathrm{H}^0(V, \mathcal{F}_n) \simeq \varprojlim\nolimits_n (\mathrm{H}^0(V, \mathcal{F}) \otimes_R R_n) \simeq \mathrm{H}^0(V, \mathcal{F}) \otimes_R \widehat{R}.$$

Corollary 3.9. *In the situation of Proposition 3.7, assume that f is locally of finite type. Then condition* (c) *of Proposition* 3.7(3) *for a point $y \in Y$ is equivalent to*

(d) *there is an open neighborhood V of y in Y such that $\mathcal{F}|_V$ is flat over T, and ϕ_t is an isomorphism on $V \cap Y_t$ for any $t \in f(V)$.*

Furthermore, if $\mathcal{F}_{(t)}|_{U\cap Y_t}$ satisfies $\mathbf{S}_2$ for the point $t = f(y)$ and if $\mathcal{F}_{(t')}$ is equi-dimensional and

$$\text{(III-3)} \qquad \operatorname{codim}(Z \cap Y_{t'} \cap \operatorname{Supp}\mathcal{F}, Y_{t'} \cap \operatorname{Supp}\mathcal{F}) \geq 2$$

for any $t' \in T$, then (d) is equivalent to

(e) *there is an open neighborhood V of y in Y such that $\mathcal{F}|_V$ satisfies relative $\mathbf{S}_2$ over T, i.e., $V = \mathbf{S}_2(\mathcal{F}|_V/T)$.*

Proof. For the first assertion, by Proposition 3.7(3), it is enough to show (c) $\Rightarrow$ (d) assuming that f is locally of finite type and $y \in Z$. When (c) holds, $\mathcal{G}|_V$ is flat over T for an open neighborhood V of y in Y, by Fact 2.26(1). Thus, $\mathcal{F}|_V$ is flat over T by Proposition 3.7(i), and moreover, by Proposition 3.7(3) applied to any point in V, we see that ϕ_t is an isomorphism on $Y_t \cap V$ for any $t \in f(V \cap Z)$. Since ϕ_t is an isomorphism for any $t \notin f(Z)$, we have proved (c) $\Rightarrow$ (d).

We shall show (d) $\Leftrightarrow$ (e) in the situation of the second assertion. In this case, if ϕ_t is an isomorphism, then $\mathcal{F}_{(t)}$ satisfies $\mathbf{S}_2$ by Corollary 2.16. Hence, we have (d) $\Rightarrow$ (e) by Fact 2.29(2). Conversely, if (e) holds with $V = Y$, then

$$\operatorname{depth}_{Y_{t'}\cap Z} \mathcal{F}_{(t')} \geq 2$$

for any $t' \in f(Z)$ by Lemma 2.15(2), since $\mathcal{F}_{(t')}$ satisfies $\mathbf{S}_2$ and inequality (III-3) holds. Hence, $\phi_{t'}$ is an isomorphism for any $t' \in f(Z)$, and (d) holds. Thus, we are done. □

Corollary 3.10. *In the situation of Proposition 3.7, for a point $t \in f(Z)$, assume that the coherent $\mathcal{O}_{Y_t}$-module $\mathcal{F}_{(t)*} = j_*(\mathcal{F}_{(t)}|_{Y_t\cap U})$ satisfies*

$$\text{(III-4)} \qquad \operatorname{depth}_{Y_t\cap Z} \mathcal{F}_{(t)*} \geq 3.$$

Then the sheaves $\mathcal{F}$ and $\mathcal{G}$ are flat over T along Y_t, and the restriction homomorphism $\phi_t \colon \mathcal{F}_{(t)} \to \mathcal{F}_{(t)*}$ is an isomorphism.

Proof. By Proposition 3.7(3), it is enough to prove that ϕ_t is an isomorphism. By (III-4), we have

$$R^1 j_*(\mathcal{F}_{(t)}|_{U\cap Y_t}) = R^1 j_*(\mathcal{F}_0|_{U_0}) = 0$$

(cf. Property 2.6). Hence, the exact sequence (III-1) in the proof of Lemma 3.5 induces an exact sequence

$$0 \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_{\Bbbk} j_*(\mathcal{F}_0|_{U_0}) \to j_*(\mathcal{F}_n|_{U_n}) \to j_*(\mathcal{F}_{n-1}|_{U_{n-1}}) \to 0.$$

Since $\mathcal{F}_{n*} = j_*(\mathcal{F}_n|_{U_n})$, the homomorphism φ_n is surjective for any $n \geq 0$. Therefore, ϕ_t is an isomorphism by (b′) ⇒ (b) of Proposition 3.7(3). □

Remark 3.11. Corollary 3.10 is similar to a special case of [28, Thm. 12], where the sheaf corresponding to $\mathcal{F}$ above may not have an exact sequence of Proposition 3.7. However, Example 3.12 below provides a counterexample to [28, Thm. 12]. The referee informed us that a modified version of [28, Thm. 12] is given in [30, Thm. 10.70].

Example 3.12. Let Y be an affine space $\mathbb{A}^8_{\Bbbk}$ of dimension 8 over a field $\Bbbk$ with a coordinate system $(\mathrm{y}_1, \mathrm{y}_2, \ldots, \mathrm{y}_8)$. Let T be a three-dimensional affine space $\mathbb{A}^3_{\Bbbk}$ and let $f: Y \to T$ be the projection defined by $(\mathrm{y}_1, \ldots, \mathrm{y}_8) \mapsto (\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3)$. The fiber $Y_0 = f^{-1}(0)$ over the origin $0 = (0,0,0)$ of T is of dimension 5. We define closed subschemes Z and V of Y by

$$Z := \{\mathrm{y}_4 = \mathrm{y}_5 = \mathrm{y}_6 = 0\} \quad \text{and}$$
$$V := \{\mathrm{y}_1 + \mathrm{y}_2\mathrm{y}_7 + \mathrm{y}_3\mathrm{y}_8 = \mathrm{y}_4 - \mathrm{y}_1 = \mathrm{y}_5 - \mathrm{y}_2 = \mathrm{y}_6 - \mathrm{y}_3 = 0\}.$$

Then we can show the following properties:

(1) $V \simeq \mathbb{A}^4_{\Bbbk}$, and $V \cap Y_0 = V \cap Z = Y_0 \cap Z \simeq \mathbb{A}^2$;

(2) $\operatorname{codim}(Z, Y) = \operatorname{codim}(Z \cap Y_0, Y_0) = 3$ and $\operatorname{codim}(Z \cap V, V) = 2$;

(3) $V \setminus Y_0 \to T$ is a smooth morphism of relative dimension one, but the fiber $V \cap Y_0$ of $V \to T$ over 0 is two-dimensional.

Let $j: U \hookrightarrow Y$ be the open immersion from the complement $U := Y \setminus Z$, and we set $\mathcal{F} := \mathcal{O}_Y \oplus \mathcal{O}_V$ and $\mathcal{F}_0 := \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_0}$. By (1) and (2), we have isomorphisms

$$j_*(\mathcal{F}|_U) \simeq j_*\mathcal{O}_U \oplus j_*\mathcal{O}_{U\cap V} \simeq \mathcal{O}_Y \oplus \mathcal{O}_V \quad \text{and} \tag{III-5}$$

$$j_*(\mathcal{F}_0|_{U\cap Y_0}) \simeq j_*\mathcal{O}_{U\cap Y_0} \simeq \mathcal{O}_{Y_0}, \tag{III-6}$$

since $U \cap V \cap Y_0 = \emptyset$, $\operatorname{depth}_Z \mathcal{O}_Y \geq 2$, $\operatorname{depth}_{Z\cap V} \mathcal{O}_V \geq 2$ and $\operatorname{depth}_{Z\cap Y_0} \mathcal{O}_{Y_0} \geq 2$. Thus, we have

(4) $\mathcal{F}|_U = \mathcal{O}_U \oplus \mathcal{O}_{U\cap V}$ is flat over T by (3);

(5) $j_*(\mathcal{F}|_U)$ is not flat over T by (3) and (III-5);

(6) $j_*(\mathcal{F}_0|_{U\cap Y_0})$ satisfies $\mathbf{S}_3$ by (III-6);

(7) the canonical homomorphism

$$j_*(\mathcal{F}|_U) \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_0} \to j_*(\mathcal{F}_0|_{U\cap Y_0})$$

is not an isomorphism by (III-5) and (III-6).

Thus, $f\colon Y \to T$, $\mathcal{F}$, and U give a counterexample to [28, Thm. 12]: The required assumptions are satisfied by (2), (4), and (6), but the conclusion is denied by (5) and (7).

The kernel $\mathcal{J}$ of $\mathcal{O}_Y \to \mathcal{O}_V$ has also an interesting infinitesimal property. Let $A = \Bbbk[\mathtt{y}_1, \mathtt{y}_2, \mathtt{y}_3]$ be the coordinate ring of T, $\mathfrak{m} = (\mathtt{y}_1, \mathtt{y}_2, \mathtt{y}_3)$ the maximal ideal at the origin $0 \in T$, and set

$$A_n = A/\mathfrak{m}^{n+1}, \quad T_n = \operatorname{Spec} A_n, \quad Y_n = Y \times_T T_n,$$
$$V_n = V \times_T T_n, \quad \mathcal{J}_n = \mathcal{J} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_n}$$

for each $n \geq 0$ as in Notation 3.3. Then we can prove

(III-7) $\mathcal{J} \not\simeq \mathcal{O}_Y, \quad \mathcal{J}|_U \not\simeq \mathcal{O}_U, \quad \mathcal{J} \simeq \mathcal{J}_* = j_*(\mathcal{J}|_U), \quad \text{and} \quad \mathcal{J}_n|_{U \cap Y_n} \simeq \mathcal{O}_{U \cap Y_n}$

for any $n \geq 0$. In fact, the first two of (III-7) are consequences of the property that the ideal sheaf $\mathcal{J}|_U$ of $V \cap U$ is not an invertible $\mathcal{O}_U$-module, and it is derived from $\operatorname{codim}(V \cap U, U) = 4 > 1$. The third isomorphism of (III-7) follows from $\operatorname{depth}_Z \mathcal{O}_Y \geq 2$ and $\operatorname{depth}_Z \mathcal{O}_V \geq 2$ (cf. (2)), and the last one from the property that the kernel of $\mathcal{J}_n \to \mathcal{O}_{Y_n}$ is isomorphic to $\mathcal{T}or_1^{\mathcal{O}_Y}(\mathcal{O}_V, \mathcal{O}_{Y_n})$, which is supported on $V \cap Y_0 \subset Y \setminus U$.

Remark 3.13. In the situation of Notation 3.3, we consider the following assertion:

(∗) *If $\phi\colon \mathcal{F} \to j_*(\mathcal{F}|_U)$ is an isomorphism, then the canonical morphism*

$$\phi_\infty\colon \varprojlim\nolimits_n \mathcal{F}_n \to \varprojlim\nolimits_n j_*(j^*\mathcal{F}_n) \simeq j_*(\varprojlim\nolimits_n \mathcal{F}_n|_{U_n})$$

induced by $\phi_n\colon \mathcal{F}_n \to \mathcal{F}_{n*} = j_*(j^*\mathcal{F}_n)$ is also an isomorphism.

Here $\varprojlim$ stands for the projective limit in the category of $\mathcal{O}_Y$-modules. In Remark 3.8 above, we have proved (∗) for the sheaf $\mathcal{F}$ in Proposition 3.7. But (∗) is not true in general. We shall show here that the ideal sheaf $\mathcal{J}$ of V in Example 3.12 provides a counterexample of (∗). In the situation of Example 3.12, we have

$$H^0(Y, \varprojlim\nolimits_n \mathcal{J}_n) \simeq \varprojlim\nolimits_n H^0(Y_n, \mathcal{J}_n) \simeq H^0(Y, \mathcal{J}) \otimes_R \widehat{R}$$

for $R = \Bbbk[\mathtt{y}_1, \ldots, \mathtt{y}_8]$ and for the formal completion $\widehat{R}$ of R along the ideal $\mathfrak{m}R = (\mathtt{y}_1, \mathtt{y}_2, \mathtt{y}_3)R$. On the other hand, we can show that

$$\begin{aligned} H^0(Y, j_*(\varprojlim\nolimits_n \mathcal{J}_n|_{U_n})) &\simeq \varprojlim\nolimits_n H^0(U_n, \mathcal{J}_n|_{U_n}) \\ &\simeq \varprojlim\nolimits_n H^0(U_n, \mathcal{O}_{U_n}) \simeq \varprojlim\nolimits_n H^0(Y_n, \mathcal{O}_{Y_n}) \simeq \widehat{R}. \end{aligned}$$

In fact, $\mathcal{J}_n|_{U_n} \simeq \mathcal{O}_{U_n}$ for any $n \geq 0$ by (III-7), and we have $\mathcal{O}_{Y_n} \simeq j_*\mathcal{O}_{U_n}$ for any $n \geq 0$ by $\operatorname{depth}_{Z \cap Y_0} \mathcal{O}_{Y_0} \geq 2$ and by applying Lemma 3.5(2) to $\mathcal{O}_Y$. Hence, if (∗)

holds true for $\mathcal{J}$, then $H^0(Y, \mathcal{J}) \otimes_R \widehat{R} \simeq \widehat{R}$, and it implies that $\mathcal{J}_y = \mathcal{O}_{Y,y}$ for any point $y \in Y_0 = \operatorname{Spec} R/\mathfrak{m}R$, i.e., the closed subscheme V defined by $\mathcal{J}$ does not contain any point of Y_0. This is a contradiction, since $V \cap Y_0 \simeq \mathbb{A}^2$.

In the situation of $(*)$, let $\mathfrak{Y}$ be the formal scheme defined as the $\mathfrak{m}\mathcal{O}_Y$-adic completion of Y, and let $\mathfrak{U}$ be the $\mathfrak{m}\mathcal{O}_U$-adic completion of U with an open immersion $\hat{j} \colon \mathfrak{U} \hookrightarrow \mathfrak{Y}$ of formal schemes. Then, for the $\mathfrak{m}\mathcal{O}_Y$-adic completion $\mathfrak{F}$ of $\mathcal{F}$, the canonical morphism

$$\hat{\phi} \colon \mathfrak{F} \to \hat{j}_*(\mathfrak{F}|_{\mathfrak{U}})$$

is not an isomorphism in general, since the direct image of $\hat{\phi}$ by $\mathfrak{Y} \to Y$ is isomorphic to the morphism ϕ_∞ of $(*)$.

In the rest of Section 3.1, in Lemmas 3.14 and 3.15 below, we shall give sufficient conditions for $\mathcal{F}$ to admit an exact sequence of Proposition 3.7.

Lemma 3.14. *Suppose that $f \circ j \colon U \to T$ is flat and*

$$\operatorname{depth}_{Y_t \cap Z} \mathcal{O}_{Y_t} \geq 2$$

for any $t \in f(Z)$. If $\mathcal{F}$ is a reflexive $\mathcal{O}_Y$-module and if $\mathcal{F}|_U$ is locally free, then there exists an exact sequence $0 \to \mathcal{F} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{G} \to 0$ locally on Y *which satisfies conditions* (i) *and* (ii) *of Proposition* 3.7.

Proof. The morphism f is flat by Corollary 3.6. Since $\mathcal{F}$ is coherent, locally on Y, we have a finite presentation

$$\mathcal{O}_Y^{\oplus m} \xrightarrow{h} \mathcal{O}_Y^{\oplus n} \to \mathcal{F}^\vee \to 0$$

of the dual $\mathcal{O}_Y$-module $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y)$. Let $\mathcal{K}$ be the kernel of the left homomorphism h. Then $\mathcal{K}|_U$ is locally free, since so is $\mathcal{F}|_U$. We have an exact sequence

$$0 \to \mathcal{F} \simeq \mathcal{F}^{\vee\vee} \to \mathcal{O}_Y^{\oplus n} \xrightarrow{h^\vee} \mathcal{O}_Y^{\oplus m}$$

by taking the dual. Let $\mathcal{G}$ be the cokernel of $h^\vee$. Then $\mathcal{G}|_U$ is isomorphic to the locally free sheaf $\mathcal{K}^\vee|_U$. Thus, the exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_Y^{\oplus n} \xrightarrow{h^\vee} \mathcal{O}_Y^{\oplus m} \to \mathcal{G} \to 0$$

satisfies conditions (i) and (ii) of Proposition 3.7. □

Lemma 3.15. *Suppose that $f \colon Y \to T$ is a flat morphism and*

$$\operatorname{depth}_{Y_t \cap Z} \mathcal{O}_{Y_t} \geq 2 \qquad \text{(III-8)}$$

for any $t \in f(Z)$. Moreover, suppose that there is a bounded complex

$$\mathcal{E}^\bullet = [\cdots \to \mathcal{E}^i \to \mathcal{E}^{i+1} \to \cdots]$$

of locally free $\mathcal{O}_Y$-modules of finite rank satisfying the following four conditions:

(i) $\mathcal{H}^i(\mathcal{E}^\bullet)|_{Y \setminus Z} = 0$ *for any $i > 0$;*

(ii) $\mathcal{F} \simeq \mathcal{H}^0(\mathcal{E}^\bullet)$*;*

(iii) $\mathcal{H}^i(\mathcal{E}^\bullet_{(t)}) = 0$ *for any $i < 0$ and any $t \in T$, where $\mathcal{E}^\bullet_{(t)}$ stands for the complex*

$$[\cdots \to \mathcal{E}^i_{(t)} \to \mathcal{E}^{i+1}_{(t)} \to \cdots] \simeq_{\mathrm{qis}} \mathcal{E}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{O}_{Y_t};$$

(iv) *the local cohomology group $\mathbb{H}^i_y(M^\bullet)$ at the maximal ideal $\mathfrak{m}_{Y,y}$ for the complex*

$$M^\bullet = \left(\tau^{\le 1}\mathcal{E}^\bullet_{(t)}\right)_y$$

of $\mathcal{O}_{Y,y}$-modules is zero for any $i \le 1$ and any $y \in Z$, where $t = f(y)$.

Then $\mathcal{H}^i(\mathcal{E}^\bullet) = 0$ for any $i < 0$, and $\mathcal{F}$ admits an exact sequence satisfying conditions (i) and (ii) of Proposition 3.7.

Proof. For an integer k, the truncated complex $\tau^{\ge k}(\mathcal{E}^\bullet)$ is expressed as

$$[\cdots \to 0 \to \mathcal{C}^k \to \mathcal{E}^{k+1} \to \mathcal{E}^{k+2} \to \cdots],$$

where $\mathcal{C}^k$ is the cokernel of $\mathcal{E}^{k-1} \to \mathcal{E}^k$. First we shall show that $\mathcal{E}^\bullet \simeq_{\mathrm{qis}} \tau^{\ge 0}(\mathcal{E}^\bullet)$ and $\mathcal{C}^0$ is flat over T. Note that it implies that $\mathcal{H}^i(\mathcal{E}^\bullet) = 0$ for any $i < 0$. Since $\mathcal{E}^\bullet$ is bounded, we have an integer $k < 0$ such that $\mathcal{E}^\bullet \simeq_{\mathrm{qis}} \tau^{\ge k}(\mathcal{E}^\bullet)$ and $\mathcal{C}^k$ is flat over T. Then, by (iii), one has

$$\mathcal{H}^k(\mathcal{E}^\bullet_{(t)}) \simeq \mathrm{Ker}(\mathcal{C}^k_{(t)} \to \mathcal{E}^{k+1}_{(t)}) = 0$$

for any $t \in T$. Hence, $\mathcal{C}^k \to \mathcal{E}^{k+1}$ is injective and $\mathcal{C}^{k+1} \simeq \mathcal{E}^{k+1}/\mathcal{C}^k$ is flat over T by a version of local criterion of flatness (cf. Corollary A.2). Thus, $\mathcal{E}^\bullet \simeq_{\mathrm{qis}} \tau^{\ge k+1}(\mathcal{E}^\bullet)$, and we can increase k by one. Therefore, we can take $k = 0$, and consequently, $\mathcal{E}^\bullet \simeq_{\mathrm{qis}} \tau^{\ge 0}(\mathcal{E}^\bullet)$, and $\mathcal{C}^0$ is flat over T. We write $\mathcal{C} := \mathcal{C}^0$. Then

$$\mathcal{E}^\bullet_{(t)} \simeq_{\mathrm{qis}} [\cdots \to 0 \to \mathcal{C}_{(t)} \to \mathcal{E}^1_{(t)} \to \mathcal{E}^2_{(t)} \to \cdots]$$

for any $t \in T$, since $\mathcal{C}$ and $\mathcal{E}^i$ are all flat over T.

Second we shall prove that

$$\mathrm{depth}_{Y_t \cap Z}\, \mathcal{C}_{(t)} \ge 2 \tag{III-9}$$

for any $t \in f(Z)$. We define $\mathcal{K}_t$ to be the kernel of $\mathcal{E}^1_{(t)} \to \mathcal{E}^2_{(t)}$. Then

$$\mathrm{depth}_{Y_t \cap Z}\, \mathcal{E}^i_{(t)} \ge 2 \quad \text{and} \quad \mathrm{depth}_{Y_t \cap Z}\, \mathcal{K}_t \ge 2$$

for $i = 0, 1$, and for any $t \in f(Z)$, by (III-8) and by Lemma 2.17(1). In particular, for any $y \in Z \cap Y_t$, we have the vanishing

$$\mathbb{H}^i_y((\mathcal{K}_t)_y) = 0 \tag{III-10}$$

of the local cohomology group at y for any $i \leq 1$ (cf. Property 2.6). By construction, we have a quasi-isomorphism

$$\tau^{\leq 1}(\mathcal{E}^\bullet_{(t)}) \simeq_{\mathrm{qis}} [\cdots \to 0 \to \mathcal{C}_{(t)} \to \mathcal{K}_t \to 0 \to \cdots].$$

In view of the induced exact sequence

$$\cdots \to \mathbb{H}^i_y(M^\bullet) \to \mathbb{H}^i_y((\mathcal{C}_{(t)})_y) \to \mathbb{H}^i_y((\mathcal{K}_t)_y) \to \cdots$$

of local cohomology groups, we have

$$\mathbb{H}^i_y((\mathcal{C}_{(t)})_y) = 0$$

for any $i \leq 1$ by (iv) and (III-10). Thus, we have (III-9) (cf. Property 2.6).

Finally, we consider the cokernel $\mathcal{G}$ of $\mathcal{C} \to \mathcal{E}^1$. Then $\mathcal{G}|_U$ is flat over T by (i). Therefore, the exact sequence $0 \to \mathcal{F} \to \mathcal{C} \to \mathcal{E}^1 \to \mathcal{G} \to 0$ satisfies conditions (i) and (ii) of Proposition 3.7. □

§3.2. Applications of the key proposition

First we shall prove the following criterion for a sheaf to be invertible.

Theorem 3.16. *Let $f : Y \to T$ be a morphism of locally Noetherian schemes, Z a closed subset of Y, $\mathcal{F}$ a coherent $\mathcal{O}_Y$-module, and t a point of $f(Z)$. We set $U = Y \setminus Z$, and write $j : U \hookrightarrow Y$ for the open immersion. Assume that*

(i) $\operatorname{depth}_Z \mathcal{O}_Y \geq 1$,

(ii) *$\mathcal{F}|_U$ is flat over T, $\mathcal{F}|_U$ is invertible,* $\operatorname{depth}_Z \mathcal{F} \geq 2$, *and*

(iii) *the direct image sheaf*

$$\mathcal{F}_{(t)*} = j_*((\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t})|_{U \cap Y_t})$$

(*cf. Definition* 3.2) *is an invertible $\mathcal{O}_{Y_t}$-module.*

Assume furthermore that one of conditions (a) *and* (b) *below is satisfied:*

(a) $\operatorname{depth}_{Z \cap Y_t} \mathcal{O}_{Y_t} \geq 3$;

(b) *the double-dual $\mathcal{F}^{[r]}$ of $\mathcal{F}^{\otimes r}$ is invertible along Y_t for a positive integer r coprime to the characteristic of the residue field $\Bbbk(t)$.*

Then f is flat along Y_t, and $\mathcal{F}$ is invertible along Y_t.

Proof. We may replace Y with its open subset, since the assertions are local on Y. By (ii), U is flat over T. Moreover,

$$\operatorname{depth}_{Z\cap Y_t} \mathcal{O}_{Y_t} \geq 2 \tag{III-11}$$

by (iii), since the isomorphism $\mathcal{F}_{(t)*} \simeq j_*(\mathcal{F}_{(t)*}|_{U\cap Y_t})$ implies that $\operatorname{depth}(\mathcal{F}_{(t)*})_y = \operatorname{depth}\mathcal{O}_{Y_t,y} \geq 2$ for any $y \in Z \cap Y_t$. Hence, $f\colon Y \to T$ is flat along Y_t by Corollary 3.6 (cf. Property 2.6). Then $\mathcal{F}$ is a reflexive $\mathcal{O}_Y$-module by Lemma 2.21(3), since we have assumed $\operatorname{depth}_Z \mathcal{O}_Y \geq 1$ and $\operatorname{depth}_Z \mathcal{F} \geq 2$ in (i) and (ii). Therefore, by (III-11) and Lemma 3.14, we may assume that $\mathcal{F}$ admits an exact sequence of Proposition 3.7.

By Fact 2.26(2), we see that $\mathcal{F}$ is invertible along Y_t if the two conditions below are both satisfied:

(1) $\mathcal{F}$ is flat over T along Y_t;

(2) $\phi_t\colon \mathcal{F}_{(t)} \to \mathcal{F}_{(t)*}$ is an isomorphism.

Here (1) is a consequence of (2) by Proposition 3.7(3). When (a) holds, we have

$$\operatorname{depth}_{Y_t\cap Z} \mathcal{F}_{(t)*} \geq 3$$

by (iii), and hence, condition (2) is satisfied by Corollary 3.10. Thus, it remains to prove (2) assuming condition (b).

We use Notation 3.3 for t. By replacing Y with its open subset, we may assume that Y is affine, and there exist isomorphisms

$$\mathcal{O}_{Y_t} = \mathcal{O}_{Y_0} \simeq \mathcal{F}_{(t)*} = \mathcal{F}_{0*} \quad \text{and} \quad \mathcal{F}^{[r]} \simeq \mathcal{O}_Y$$

in (iii) and in (b), respectively. Note that we have

$$\operatorname{depth}_{Y_n\cap Z} \mathcal{O}_{Y_n} \geq 2$$

for any $n \geq 0$: this follows from (III-11) by Lemma 2.32(3) applied to the flat morphism $Y_n \to T_n$. As a consequence,

$$\mathrm{H}^0(Y_n, \mathcal{O}_{Y_n}) \simeq \mathrm{H}^0(U_n, \mathcal{O}_{U_n})$$

for any $n \geq 0$, and the restriction homomorphism

$$\mathrm{H}^0(U_n, \mathcal{O}_{U_n}) \to \mathrm{H}^0(U_{n-1}, \mathcal{O}_{U_{n-1}}) \tag{III-12}$$

is surjective for any $n > 0$, since we have assumed that Y is affine.

We set $\mathcal{N}_n := \mathcal{F}_n|_{U_n}$. It is enough to show that $\mathcal{N}_n \simeq \mathcal{O}_{U_n}$ for all n. In fact, if this is true, then we have an isomorphism

$$\mathcal{F}_{n*} = j_*(\mathcal{N}_n) \simeq j_*(\mathcal{O}_{U_n}) \simeq \mathcal{O}_{Y_n}$$

and, as a consequence, the restriction homomorphism $\varphi_n\colon (\mathcal{F}_{n*})_{(t)} \to \mathcal{F}_{0*}$ is an isomorphism for any $n \geq 0$. Hence, in this case $\phi_t\colon \mathcal{F}_{(t)} \to \mathcal{F}_{(t)*}$ is an isomorphism by (b′) ⇒ (b) of Proposition 3.7(3).

We shall prove $\mathcal{N}_n \simeq \mathcal{O}_{U_n}$ by induction on n. When $n = 0$, we have the isomorphism from the isomorphism $\mathcal{F}_{0*} \simeq \mathcal{O}_{Y_0}$ above. Assume that $\mathcal{N}_{n-1} \simeq \mathcal{O}_{U_{n-1}}$ for an integer $n > 0$. Let $\mathcal{J}$ be the kernel of $\mathcal{O}_{Y_n} \to \mathcal{O}_{Y_{n-1}}$. Then $\mathcal{J}^2 = 0$ as an ideal of $\mathcal{O}_{Y_n}$, and

$$\mathcal{J} \simeq \mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_{\Bbbk} \mathcal{O}_{Y_0}.$$

We have an exact sequence

$$0 \to \mathcal{J} \to \mathcal{O}^{\star}_{Y_n} \to \mathcal{O}^{\star}_{Y_{n-1}} \to 1$$

of sheaves on $|Y_n| = |Y_0|$ with respect to the Zariski topology, where * stands for the subsheaf of invertible sections of a sheaf of rings, and where a local section ζ of $\mathcal{J}$ is mapped to the invertible section $1 + \zeta$ of $\mathcal{O}_{Y_n}$. It induces a long exact sequence

$$\mathrm{H}^0(U_n, \mathcal{O}^{\star}_{U_n}) \xrightarrow{\mathrm{res}^0} \mathrm{H}^0(U_{n-1}, \mathcal{O}^{\star}_{U_{n-1}}) \to \mathrm{H}^1(U_0, \mathcal{J}) \to \mathrm{Pic}(U_n) \xrightarrow{\mathrm{res}^1} \mathrm{Pic}(U_{n-1}),$$

where res^0 and res^1 are restriction homomorphisms to U_{n-1}. Note that res^0 is surjective, since so is (III-12). Hence, the kernel of res^1 is a $\Bbbk$-vector space isomorphic to $\mathrm{H}^1(U_0, \mathcal{J})$. Now, the isomorphism class of $\mathcal{N}_n$ in $\mathrm{Pic}(U_n)$ belongs to the kernel by $\mathcal{N}_{n-1} \simeq \mathcal{O}_{U_{n-1}}$, and its multiple by r is zero by (b), where r is coprime to $\mathrm{char}(\Bbbk)$. Thus, $\mathcal{N}_n \simeq \mathcal{O}_{U_n}$, and we are done. □

We have the following by a direct application of Proposition 3.7.

Lemma 3.17. *Let $f\colon Y \to T$ be a flat morphism of locally Noetherian schemes and let Z be a closed subset of Y such that*

$$\operatorname{depth}_{Y_t \cap Z} \mathcal{O}_{Y_t} \geq 2$$

for any fiber Y_t. Let $q\colon T' \to T$ be a morphism from another locally Noetherian scheme T' such that $Y' = Y \times_T T'$ is also locally Noetherian. We write $f'\colon Y' \to T'$ and $p\colon Y' \to Y$ for the projections. Let $0 \to \mathcal{F} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{G} \to 0$ be an exact sequence of coherent $\mathcal{O}_Y$-modules such that $\mathcal{F}|_U$, $\mathcal{E}^0$, $\mathcal{E}^1$, and $\mathcal{G}|_U$ are locally free, where $U = Y \setminus Z$. Then $\mathcal{F}$ is a reflexive $\mathcal{O}_Y$-module, and

$$(p^*\mathcal{F})^{\vee\vee} \simeq \mathrm{Ker}(p^*\mathcal{E}^0 \to p^*\mathcal{E}^1) \simeq j'_*(p^*\mathcal{F}|_{U'})$$

for the open immersion $j'\colon U' = p^{-1}(U) \hookrightarrow Y'$. Moreover, $(p^*\mathcal{F})^{\vee\vee}$ satisfies relative $\mathbf{S}_2$ over T' if and only if $p^*\mathcal{G}$ is flat over T'.

Proof. The exact sequence satisfies the assumptions of Proposition 3.7 for $Y \to T$. Hence, $\mathcal{F} \simeq j_*(\mathcal{F}|_U)$, i.e., $\operatorname{depth}_Z \mathcal{F} \geq 2$, by Proposition 3.7(1). Moreover, $\mathcal{F}$ is reflexive by Lemma 2.21(3), since we have $\operatorname{depth}_Z \mathcal{O}_Y \geq 2$ by Lemma 2.32(3). Let $\mathcal{F}'$ be the kernel of $p^*\mathcal{E}^0 \to p^*\mathcal{E}^1$. Then the exact sequence

$$0 \to \mathcal{F}' \to p^*\mathcal{E}^0 \to p^*\mathcal{E}^1 \to p^*\mathcal{G} \to 0$$

on Y' satisfies the assumptions of Proposition 3.7 for $f' \colon Y' \to T'$, since

$$\operatorname{depth}_{Y'_{t'} \cap p^{-1}(Z)} \mathcal{O}_{Y'_{t'}} = \operatorname{depth}_{Y_t \cap Z} \mathcal{O}_{Y_t} \geq 2$$

for any $t' \in T'$ and $t = q(t')$, by Lemma 2.31(1). Hence, $\mathcal{F}' \simeq j'_*(\mathcal{F}'|_{U'})$ by Proposition 3.7(1). Since $\mathcal{F}'|_{U'} \simeq p^*\mathcal{F}|_{U'}$, we have $\mathcal{F}' \simeq (p^*\mathcal{F})^{\vee\vee}$ by Lemma 2.34. Furthermore, by Proposition 3.7(3), we see that $\mathcal{F}'$ satisfies relative $\mathbf{S}_2$ over T' if and only if $p^*\mathcal{G}$ is flat over T'. □

Here we introduce the following notion useful for stating results in the rest of Section 3.2.

Definition 3.18. Let $f \colon Y \to T$ be a morphism of locally Noetherian schemes and let $\mathcal{F}$ be a coherent $\mathcal{O}_Y$-module. We say that $\mathcal{F}$ is *locally free in codimension one on each fiber* of f if there is an open subset $U \subset Y$ such that $\mathcal{F}|_U$ is locally free and $\operatorname{codim}(Y_t \setminus U, Y_t) \geq 2$ for any fiber $Y_t = f^{-1}(t)$.

Proposition 3.19 (Infinitesimal and valuative criteria). *Let $f \colon Y \to T$ be a flat morphism locally of finite type between locally Noetherian schemes and let $y \in Y$ be a point such that f satisfies relative $\mathbf{S}_2$ over T at y. Let $\mathcal{F}$ be a reflexive $\mathcal{O}_Y$-module which is locally free in codimension one on each fiber of f. Then $\mathcal{F}$ satisfies relative $\mathbf{S}_2$ over T at y if one of the following two conditions* (I) *and* (II) *is satisfied, where $Y_A = Y \times_T \operatorname{Spec} A$ and $\mathcal{F}_A = p_A^*\mathcal{F}$ for the projection $p_A \colon Y_A \to Y$:*

(I) *Let $\operatorname{Spec} A \to T$ be a morphism defined by a surjective local ring homomorphism $\mathcal{O}_{T,f(y)} \to A$ to an Artinian local ring A. Then the double-dual $(\mathcal{F}_A)^{\vee\vee}$ satisfies relative $\mathbf{S}_2$ over $\operatorname{Spec} A$ at the point $y_A = p_A^{-1}(y)$.*

(II) *The local ring $\mathcal{O}_{T,f(y)}$ is reduced. Let $\operatorname{Spec} A \to T$ be a morphism defined by a local ring homomorphism $\mathcal{O}_{T,f(y)} \to A$ to a discrete valuation ring A. Then the double-dual $(\mathcal{F}_A)^{\vee\vee}$ satisfies relative $\mathbf{S}_2$ over $\operatorname{Spec} A$ at any point $z \in Y_A$ lying over $y \in Y$ and the closed point $\mathfrak{m}_A$ of $\operatorname{Spec} A$.*

Proof. We may assume that $T = \operatorname{Spec} B$ for the local ring $B = \mathcal{O}_{T,f(y)}$ and we can localize Y freely. Thus, we may assume that $Y = \operatorname{Spec} C$ for a finitely generated B-algebra C and, moreover, that $Y \to T$ is an $\mathbf{S}_2$-morphism by Fact 2.29(3). By assumption, there is a closed subset $Z \subset Y$ such that $\mathcal{F}|_{Y \setminus Z}$ is locally free and $\operatorname{codim}(Y_t \cap Z, Y_t) \geq 2$ for any $t \in T$. In particular,

$$\operatorname{depth}_{Y_t \cap Z} \mathcal{O}_{Y_t} \geq 2$$

for any $t \in T$ (cf. Lemma 2.15(2)). As in the proof of Lemma 3.14, we have an exact sequence $0 \to \mathcal{F} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{G} \to 0$ of coherent $\mathcal{O}_Y$-modules such that $\mathcal{E}^0$, $\mathcal{E}^1$ and $\mathcal{G}|_U$ are locally free. By Proposition 3.7(3), we see that $\mathcal{F}$ satisfies relative $\mathbf{S}_2$ over T at y if and only if the stalk $\mathcal{G}_y$ is flat over B. Let $\operatorname{Spec} A \to T$ be a morphism in (I) or (II). Then $Y_A = \operatorname{Spec} C \otimes_B A$ is Noetherian, since $C \otimes_B A$ is a finitely generated A-algebra. By Proposition 3.7(3) and Lemma 3.17, we see also that $(\mathcal{F}_A)^{\vee\vee}$ satisfies relative $\mathbf{S}_2$ over $\operatorname{Spec} A$ at a point z lying over y and $\mathfrak{m}_A$ if and only if the stalk $\mathcal{G}_{A,z}$ is flat over A for the pullback $\mathcal{G}_A = p_A^* \mathcal{G}$, where $\mathcal{G}_{A,z} \simeq (\mathcal{G}_y \otimes_B A)_z$. Therefore, the assertions in cases (I) and (II), respectively, follow from the local criterion of flatness (cf. Proposition A.1(iv)) for $\mathcal{G}_y$ over B and from the valuative criterion of flatness (cf. [12, IV, Thm. (11.8.1)]) for $\mathcal{G}$ over T at y. □

Definition 3.20 (Relative $\mathbf{S}_2$ refinement). Let $Y \to T$ be an $\mathbf{S}_2$-morphism of locally Noetherian schemes and let $\mathcal{F}$ be a reflexive $\mathcal{O}_Y$-module which is locally free in codimension one on each fiber. A morphism $S \to T$ from a locally Noetherian scheme S is called a *relative* $\mathbf{S}_2$ *refinement* for $\mathcal{F}$ over T if the following conditions are satisfied:

(i) $S \to T$ is a monomorphism in the category of schemes (cf. Fact 3.23);

(ii) for any morphism $T' \to T$ of locally Noetherian schemes, and for the pullback $\mathcal{F}'$ of $\mathcal{F}$ to the fiber product $Y \times_T T'$, the double dual $(\mathcal{F}')^{\vee\vee}$ satisfies relative $\mathbf{S}_2$ over T' if and only if $T' \to T$ factors through $S \to T$.

Remark. The fiber product $Y \times_T T'$ in (ii) is locally Noetherian, since it is locally of finite type over T'. Thus, we can consider the relative $\mathbf{S}_2$-condition for $(\mathcal{F}')^{\vee\vee}$ (cf. Definition 2.28). By (i) and (ii), $S \to T$ is unique up to unique isomorphism.

Remark 3.21. In the situation of Definition 3.20(ii), we write $\mathcal{F} \times_T T'$ for $\mathcal{F}'$, and we set

$$F(T'/T) = \begin{cases} \star & \text{if } (\mathcal{F} \times_T T')^{\vee\vee} \text{ satisfies relative } \mathbf{S}_2 \text{ over } T', \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\star$ denotes a one-point set. For any morphism $T'' \to T'$ from a locally Noetherian scheme T'', we can show that if $F(T'/T) = \star$, then $F(T''/T) = \star$. In fact, we have an isomorphism

$$(\mathcal{F} \times_T T'')^{\vee\vee} \simeq (\mathcal{F} \times_T T')^{\vee\vee} \times_{T'} T''$$

by Lemma 2.34, and this sheaf satisfies relative $\mathbf{S}_2$ over T'' by Lemma 2.31(3). Therefore, F is regarded as a functor $(\mathsf{LNSch}/T)^{\mathrm{op}} \to \mathbf{Set}$ for the category LNSch/T of locally Noetherian T-schemes, and the relative $\mathbf{S}_2$ refinement is a T-scheme representing F.

Remark 3.22. If $T' = \operatorname{Spec} \Bbbk$ for a field $\Bbbk$, then $F(T'/T) = \star$, since $(\mathcal{F}')^{\vee\vee}$ satisfies $\mathbf{S}_2$ by Corollary 2.22. In particular, if the relative $\mathbf{S}_2$ refinement $S \to T$ exists, then it is bijective.

Fact 3.23. Let $h\colon S \to T$ be a morphism locally of finite type between locally Noetherian schemes. Then h is a morphism *locally of finite presentation* (cf. [12, IV, §1.4]), and we have the following properties:

(1) The morphism h is a monomorphism in the category of schemes if and only if h is radicial and unramified, by [12, IV, Prop. (17.2.6)].
(2) If h is an unramified morphism, then it is étale locally a closed immersion, i.e., for any point $s \in S$, there exists an open neighborhood V of s such that the induced morphism $V \to T$ is written as the composite of a closed immersion $V \to W$ and an étale morphism $W \to T$ (cf. [12, IV, Cor. (18.4.7)], [13, I, Cor. 7.8]).

Example 3.24. For a Noetherian scheme T and a finite number of locally closed subschemes $S_1, S_2, \dots, S_k$ of T, assume that T is equal to the disjoint union $\bigsqcup_{i=1}^k S_i$ as a set; the collection $\{S_i\}$ is called a *stratification* in [39, Lect. 8]. Then immersions $S_i \subset T$ define a morphism $h\colon S \to T$ from the scheme-theoretic disjoint union $S = \bigsqcup_{i=1}^k S_i$. This h is a separated surjective monomorphism of finite type and is a *local immersion* (cf. [12, I, Déf. (4.5.1)]), i.e., a closed immersion Zariski-locally.

Example 3.25. There is a separated monomorphism $h\colon X \to Y$ of finite type of Noetherian schemes such that X is connected but h is not an immersion. An example is given as follows. For an algebraically closed field $\Bbbk$, let D be a reduced effective divisor of degree three on the projective plane $Y = \mathbb{P}^2_{\Bbbk}$ having a node P. For the blowing up $M \to Y$ at P, let $\overline{X}$ be the proper transform of D in M and let $Q \in \overline{X}$ be one of the two points lying over P. We set $X := \overline{X} \setminus \{Q\}$. Then X is connected, the induced morphism $h\colon X \to Y$ is a separated monomorphism of finite type inducing a bijection $X \to D$, and $h^{-1}(Y \setminus \{P\}) \simeq D \setminus \{P\}$. However, h is not a closed immersion, since X is not isomorphic to D. If D is irreducible, then

h is not a local immersion. On the other hand, if D is reducible, then h is a local immersion. In fact, for another node P' of D, the inverse image $h^{-1}(Y \setminus \{P'\})$ is isomorphic to a disjoint union of two locally closed subschemes of Y.

The following is analogous to the flattening stratification theorem by Mumford in [39, Lect. 8] or to the representability theorem of unramified functors by Murre [41]: A similar result is stated by Kollár in [29, Thm. 2] in the case where f is projective but is not required to satisfy the $\mathbf{S}_2$-condition.

Theorem 3.26. *Let $f\colon Y \to T$ be an $\mathbf{S}_2$-morphism of locally Noetherian schemes. Let $\mathcal{F}$ be a reflexive $\mathcal{O}_Y$-module which is locally free in codimension one on each fiber (cf. Definition 3.18). If the following condition (i) is satisfied, then there is a relative $\mathbf{S}_2$ refinement for $\mathcal{F}$ over T as a separated morphism $S \to T$ locally of finite type:*

(i) *$\mathcal{F}|_{Y\setminus\Sigma}$ satisfies relative $\mathbf{S}_2$ over T for a closed subset $\Sigma \subset Y$ such that $\Sigma \to T$ is proper.*

Furthermore, the morphism $S \to T$ is a local immersion of finite type if

(ii) *f is a projective morphism locally over T.*

Proof. For the first assertion, by Fact 3.23(1), it is enough to prove that the functor F in Remark 3.21 is representable by a separated morphism $S \to T$ locally of finite type. We may replace T freely by an open subset, since $S \to T$ is unique up to unique isomorphism and since the second assertion is also local on T. Thus, we assume that T is an affine Noetherian scheme. We set U to be an open subset of Y such that $\mathcal{F}|_U$ is locally free and $\operatorname{codim}(Y_t \setminus U, Y_t) \geq 2$ for any fiber Y_t.

We first consider case (ii). We may assume that Y is a closed subscheme of $\mathbb{P}^N \times T$ for some $N > 0$. Let $\mathcal{A}$ be the f-ample invertible $\mathcal{O}_Y$-module defined as the pullback of $\mathcal{O}(1)$ on $\mathbb{P}^N$. Then we can construct an exact sequence

$$(\mathcal{A}^{\otimes -l'})^{\oplus m'} \to (\mathcal{A}^{\otimes -l})^{\oplus m} \to \mathcal{F}^\vee \to 0$$

on Y for some positive integers m, m', l, and l', where the kernel of the left homomorphism is locally free on U, since $\mathcal{F}^\vee$ is so. Taking the dual, we have an exact sequence $0 \to \mathcal{F} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{G} \to 0$ of coherent $\mathcal{O}_Y$-modules such that $\mathcal{E}^0$, $\mathcal{E}^1$ and $\mathcal{G}|_U$ are locally free (cf. the proof of Lemma 3.14). Let $T' \to T$ be an arbitrary morphism from another locally Noetherian scheme T'. Then $F(T'/T) = \star$ if and only if $\mathcal{G} \times_T T'$ is flat over T', by Lemma 3.17. Hence, the functor F is nothing but the "universal flattening functor" $G\colon (\mathsf{Sch}/T)^{\mathrm{op}} \to \mathbf{Set}$ for $\mathcal{G}$ (cf.

Remark 3.27 below) restricted to the category LNSch/T. Here

$$G(T'/T) = \begin{cases} \star & \text{if } \mathcal{G} \times_T T' \text{ is flat over } T', \\ \emptyset & \text{otherwise,} \end{cases}$$

for any T-scheme T'. By the theorem of [39, Lect. 8], it is represented by a separated morphism $S \to T$ of finite type which is a local immersion. Thus, we have proved the assertion in case (ii).

In case (i), we can cover Σ by finitely many open affine subsets Y_λ. We may assume that $Y = \bigcup Y_\lambda$, since $\mathcal{F}|_{Y \setminus \Sigma}$ satisfies relative $\mathbf{S}_2$ over T. By Lemma 3.14, we may also assume that there exists an exact sequence

$$0 \to \mathcal{F}|_{Y_\lambda} \to \mathcal{E}^0_\lambda \to \mathcal{E}^1_\lambda \to \mathcal{G}_\lambda \to 0$$

on each Y_λ such that $\mathcal{E}^0_\lambda$ and $\mathcal{E}^1_\lambda$ are free $\mathcal{O}_{Y_\lambda}$-modules of finite rank, and that $\mathcal{G}_\lambda$ is locally free on $U_\lambda = U \cap Y_\lambda$. Let $T' \to T$ be an arbitrary morphism from a locally Noetherian scheme T'. By Lemma 3.17, we see that $F(T'/T) = \star$ if and only if $\mathcal{G}_\lambda \times_T T'$ is flat over T' for any λ. Let $G_\lambda \colon (\mathsf{Sch}/T)^{\mathrm{op}} \to \mathbf{Set}$ be the universal flattening functor for $\mathcal{G}_\lambda$, which is defined by

$$G_\lambda(T'/T) = \begin{cases} \star & \text{if } \mathcal{G}_\lambda \times_T T' \text{ is flat over } T', \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $G \colon (\mathsf{Sch}/T)^{\mathrm{op}} \to \mathbf{Set}$ be the "intersection" functor of all G_λ, i.e., $G(T'/T) = \bigcap G_\lambda(T'/T)$ for any T'/T. By the argument above, F is the restriction of G to LNSch/T. Every functor G_λ satisfies conditions (F_1)–(F_8) of [41] except (F_3), by the proof of [41, Thm. 2]. Hence, the intersection functor G satisfies the same conditions except possibly (F_3) and (F_8). By [41, Thm. 1], we are reduced to checking these two conditions for G. Since the two conditions concern only Noetherian schemes, we may take $F = G$. We write $F(T') = F(T'/T)$ for simplicity for a morphism $T' \to T$.

We shall show that F satisfies (F_3) (cf. [41, (F_3), p. 244]). Let A be a Noetherian complete local ring with maximal ideal $\mathfrak{m}_A$ and let $\operatorname{Spec} A \to T$ be a morphism. What we have to prove is the bijectivity of the canonical map

$$F(\operatorname{Spec} A) \to \varprojlim\nolimits_n F(\operatorname{Spec} A/\mathfrak{m}_A^n),$$

or equivalently that $F(\operatorname{Spec} A) = \star$ if $F(\operatorname{Spec} A/\mathfrak{m}_A^n) = \star$ for all $n > 0$. Assume the latter condition. By Corollary 3.9 applied to $Y_\lambda \times_T \operatorname{Spec} A \to \operatorname{Spec} A$ for each λ, we have an open neighborhood W_λ of the closed fiber $Y_\lambda \times_T \operatorname{Spec} A/\mathfrak{m}_A$ in $Y_\lambda \times_T \operatorname{Spec} A$ such that $(\mathcal{F} \times_T \operatorname{Spec} A)^{\vee\vee}|_{W_\lambda}$ satisfies relative $\mathbf{S}_2$ over $\operatorname{Spec} A$. On the other hand, the restriction of $(\mathcal{F} \times_T \operatorname{Spec} A)^{\vee\vee}$ to $(Y \setminus \Sigma) \times_T \operatorname{Spec} A$ also satisfies relative $\mathbf{S}_2$ over $\operatorname{Spec} A$. Then the union $\bigcup W_\lambda \cup ((Y \setminus \Sigma) \times_T \operatorname{Spec} A)$ equals $Y \times_T \operatorname{Spec} A$, since the complement of the union is proper over $\operatorname{Spec} A$ but its image does not contain the closed point $\mathfrak{m}_A$. Therefore, $F(\operatorname{Spec} A) = \star$.

Next we shall show that F satisfies (F_8) (cf. [41, (F_8), p. 246]). Let A be a Noetherian ring containing a unique minimal prime ideal $\mathfrak{p}$ and let I be a nilpotent ideal of A such that $I\mathfrak{p} = 0$. Note that $\mathfrak{p} = \sqrt{0}$. Let $\operatorname{Spec} A \to T$ be a morphism and assume that $F(\operatorname{Spec} A/I) = \star$ but $F(\operatorname{Spec} A_\mathfrak{p}/I') = \emptyset$ for any ideal I' of $A_\mathfrak{p}$ such that $I' \subsetneq I_\mathfrak{p}$. What we have to prove is the existence of an element $a \in A \setminus \mathfrak{p}$ having the following property:

($\diamond$) For any element $b \in A \setminus \mathfrak{p}$ and for any ideal J of $A_{ab} = A[(ab)^{-1}]$, if $J \subset IA_{ab}$ and if $F(\operatorname{Spec} A_{ab}/J) = \star$, then $J = IA_{ab}$.

For each λ, we set B_λ to be an A-algebra such that $\operatorname{Spec} B_\lambda \simeq Y_\lambda \times_T \operatorname{Spec} A$ over $\operatorname{Spec} A$ and let M_λ be a finitely generated B_λ-module such that the quasi-coherent sheaf $M_\lambda^\sim$ on $\operatorname{Spec} B_\lambda$ is isomorphic to $\mathcal{G}_\lambda \times_T \operatorname{Spec} A$. Note that

$$M_\lambda \otimes_A A_\mathfrak{p}/IA_\mathfrak{p}$$

is a free $A_\mathfrak{p}/IA_\mathfrak{p}$-module, since it is flat over $A_\mathfrak{p}/IA_\mathfrak{p}$ by $G_\lambda(\operatorname{Spec} A_\mathfrak{p}/IA_\mathfrak{p}) = \star$ and since $A_\mathfrak{p}$ is an Artinian local ring. Hence,

$$(M_\lambda \otimes_A A/I) \otimes_A A_a = M_\lambda \otimes_A A_a/IA_a$$

is a free A_a/IA_a-module for an element $a \in A \setminus \mathfrak{p}$. For each λ, let $\mathcal{S}_\lambda$ be the set of ideals J of A_a such that $G_\lambda(\operatorname{Spec} A_a/J) = \star$, or equivalently, that $M_\lambda \otimes_A A_a/J$ is a flat A_a/J-module. By [12, IV, Cor. (11.4.4)], there exists a unique minimal element $I_\lambda = I_{\lambda,(a)}$ in $\mathcal{S}_\lambda$, and

(†) for any A_a-algebra A', if $M_\lambda \otimes_A A'$ is a flat A'-module, then A' is an A_a/I_λ-algebra.

Note that I_λ is nilpotent, since the nilpotent ideal IA_a belongs to $\mathcal{S}_\lambda$. We define $I_{(a)} := \sum I_{\lambda,(a)}$ as an ideal of A_a. Then it has the following property:

(‡) For any A_a-algebra A', it is an $A_a/I_{(a)}$-algebra if and only if $M_\lambda \otimes_A A'$ is flat over A' for any λ, i.e., $F(\operatorname{Spec} A') = \star$.

By the assumption of $I_\mathfrak{p}$, we have $(I_{(a)})_\mathfrak{p} = I_{(a)}A_\mathfrak{p} = IA_\mathfrak{p}$. Thus, there is an element $a' \in A \setminus \mathfrak{p}$ such that $I_{(a)}A_{aa'} = IA_{aa'}$. Here $I_{\lambda,(aa')} = I_{\lambda,(a)}A_{aa'}$ for any λ by the property (†) of I_λ. Thus, $I_{(aa')} = I_{(a)}A_{aa'} = IA_{aa'}$. Therefore, aa' satisfies condition ($\diamond$) by the property (‡). Thus, we have checked conditions (F_3) and (F_8), and the assertion in case (i) has been proved. □

Remark 3.27. The (universal) flattening functor is introduced by Murre in [41], but its origin seems to go back to Grothendieck as the subtitle says. Murre gives a criterion of the representability of the functor in [41, §3, (A)], whose prototype seems to be [12, IV, Prop. (11.4.5)]. Mumford considers the case of a projective morphism in [39, Lect. 8], and proves the representability by using Hilbert polynomials, where the representing scheme is called the "flattening stratification". He also mentioned that Grothendieck has proved a weaker result by a much deeper method. Raynaud [47, Chap. 3] and Raynaud–Gruson [48, Part 1, §4] give further criteria for the representability of the universal flattening functor by another method. One of them is used in proving the following "local version" of the existence of relative $\mathbf{S}_2$ refinement.

Theorem 3.28. *Let $f\colon Y \to T$ be an $\mathbf{S}_2$-morphism of locally Noetherian schemes and $\mathcal{F}$ a reflexive $\mathcal{O}_Y$-module which is* locally free in codimension one on each fiber (*cf. Definition* 3.18). *Assume that $T = \operatorname{Spec} R$ for a Henselian local ring R and let $o \in T$ be the closed point. Then, for any point y of the closed fiber $Y_o = f^{-1}(o)$, there is a closed subscheme $S \subset T$ having the following universal property. Let $T' = \operatorname{Spec} R' \to T = \operatorname{Spec} R$ be a morphism defined by a local ring homomorphism $R \to R'$ for a Noetherian local ring R' and let $o' \in T'$ be the closed point. Let $f'\colon Y' = Y \times_T T' \to T'$ and $p\colon Y' \to Y$ be the induced morphisms. Then, for the pullback $\mathcal{F}' := p^*\mathcal{F}$, its double-dual $(\mathcal{F}')^{\vee\vee}$ on Y' satisfies relative $\mathbf{S}_2$ over T' at any point y' of Y' with $p(y') = y$ and $f'(y') = o'$ if and only if $T' \to T$ factors through S.*

Proof. Replacing Y with an open neighborhood of y, we may assume that Y is an affine R-scheme of finite type. Then, by the same argument as in the proof of Lemma 3.14, we have an exact sequence $0 \to \mathcal{F} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{G} \to 0$ of coherent $\mathcal{O}_Y$-modules such that $\mathcal{E}^0$ and $\mathcal{E}^1$ are locally free and $\mathcal{G}|_U$ is also locally free for the maximal open subset U such that $\mathcal{F}|_U$ is locally free. By Lemma 3.17, $(\mathcal{F}')^{\vee\vee}$ satisfies relative $\mathbf{S}_2$ over T' at a point y' lying over y if and only if $(p^*\mathcal{G})_{y'}$ is flat over T'. Therefore, the universal closed subscheme $S \subset T$ exists by [48, Part 1, Thm. (4.1.2)] or [47, Chap. 3, Thm. 1] applied to $\mathcal{G}$. □

§4. Grothendieck duality

We shall explain the theory of Grothendieck duality with some base change theorems by referring to [17], [7], [35], etc. We do not prove the main part of the duality theory but show several consequences. Some of them are useful for studying $\mathbb{Q}$-Gorenstein schemes and $\mathbb{Q}$-Gorenstein morphisms in Sections 6 and 7.

Some well-known properties on the dualizing complex are mentioned in Sections 4.1 and 4.2 based on arguments in [17] and [7]. Section 4.1 explains some basic properties and results on a locally Noetherian scheme admitting a dualizing complex, mainly on the codimension function associated with the dualizing complex and on interpretation of $\mathbf{S}_k$-conditions for a coherent sheaf via the dualizing complex. In Section 4.2 we introduce the useful notion of an *ordinary dualizing complex* for locally equi-dimensional locally Noetherian schemes, and study cohomology sheaves of ordinary dualizing complexes. Section 4.3 explains the notion of twisted inverse image and the relative duality theory referring mainly to [17], [7], [35]. Our original base change result for the relative dualizing complex to the fiber is proved in Corollary 4.38. In Section 4.4 we explain the *relative dualizing sheaf* for a *Cohen–Macaulay morphism* (cf. Definition 2.30) and its base change property referring to [7], [51], etc.

§4.1. Dualizing complex

We shall begin by recalling the notion of a dualizing complex, which is introduced in [17, V].

Definition 4.1. A dualizing complex $\mathcal{R}^\bullet$ of a locally Noetherian scheme X is defined to be a complex of $\mathcal{O}_X$-modules bounded below such that

- it has coherent cohomology and has finite injective dimension, i.e., $\mathcal{R}^\bullet \in \mathbf{D}^+_{\mathrm{coh}}(X)_{\mathrm{fid}}$ in the sense of [17], and
- the natural morphism
$$\mathcal{O}_X \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}^\bullet, \mathcal{R}^\bullet)$$
is a quasi-isomorphism (cf. [17, V, Prop. 2.1]).

Remark. Every complex in $\mathbf{D}^+_{\mathrm{coh}}(X)_{\mathrm{fid}}$ is quasi-isomorphic to a bounded complex of quasi-coherent injective $\mathcal{O}_X$-modules when X is quasi-compact (cf. [17, II, Prop. 7.20]). The derived functor $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}$ of the bi-functor $\mathcal{H}om_{\mathcal{O}_X}$ is considered as a functor

$$\mathbf{D}(X)^{\mathrm{op}} \times \mathbf{D}(X) \ni (\mathcal{F}^\bullet, \mathcal{G}^\bullet) \mapsto \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \in \mathbf{D}(X)$$

(cf. [17, I, §6], [54, Thm. A(ii)]).

Example. A Noetherian local ring A is said to be *Gorenstein* if there is a finite injective resolution of A. In particular, $\mathcal{O}_X$ is a dualizing complex for $X = \operatorname{Spec} A$. Several conditions for a local ring A to be Gorenstein are known (e.g. [17, V, Thm. 9.1], [37, Thm. 18.1]): for example, A is Gorenstein if and only if A is Cohen–Macaulay and $\operatorname{Ext}^n(A/\mathfrak{m}_A, A) \simeq A/\mathfrak{m}_A$ for the maximal ideal $\mathfrak{m}_A$ and $n = \dim A$.

A locally Noetherian scheme Y is said to be *Gorenstein* if every local ring $\mathcal{O}_{Y,y}$ is Gorenstein. For a locally Noetherian scheme Y, it is Gorenstein of finite Krull dimension if and only if $\mathcal{O}_Y$ is a dualizing complex (cf. [17, II, Prop. 7.20]).

Example (Cf. [17, V, Prop. 3.4], [37, Thm. 18.6]). For an Artinian local ring A, let I be an injective hull of the residue field $A/\mathfrak{m}_A$. Then the associated quasi-coherent sheaf $I^\sim$ on $\operatorname{Spec} A$ is a dualizing complex.

Remark 4.2 ([17, V, §10]). Let X be a locally Noetherian scheme. If there is a morphism $X \to Y$ of finite type to a locally Noetherian scheme Y admitting a dualizing complex in which the dimensions of fibers are bounded, then X also admits a dualizing complex [17, VI, Cor. 3.5]. In particular, any scheme of finite type over a Noetherian Gorenstein scheme of finite Krull dimension admits a dualizing complex. When X is connected, the dualizing complex is unique up to quasi-isomorphism, shift, and up to tensor product with invertible sheaves (cf. [17, V, Thm. 3.1], [7, (3.1.30)]).

Fact. For a Noetherian ring A, the affine scheme $\operatorname{Spec} A$ admits a dualizing complex if and only if there is a surjection $B \to A$ from a Gorenstein ring B of finite Krull dimension. This is conjectured by Sharp [53, Conj. (4.4)] and has been proved by Kawasaki [25, Cor. 1.4].

We shall explain the notion of codimension function.

Definition 4.3 (Cf. [17, V, p. 283]). Let X be a scheme such that every local ring $\mathcal{O}_{X,x}$ has finite Krull dimension. A function $d\colon X \to \mathbb{Z}$ is called a *codimension function* if

$$d(x) = d(y) + \operatorname{codim}(\overline{\{x\}}, \overline{\{y\}})$$

for any points x and y such that $x \in \overline{\{y\}}$.

Remark. Let X be a scheme whose local rings $\mathcal{O}_{X,x}$ all have finite Krull dimension. If X admits a codimension function, then X is catenary (cf. Property 2.3). In fact,

$$\operatorname{codim}(\overline{\{x\}}, \overline{\{z\}}) = \operatorname{codim}(\overline{\{x\}}, \overline{\{y\}}) + \operatorname{codim}(\overline{\{y\}}, \overline{\{z\}})$$

holds for any x, y, $z \in X$ satisfying $x \in \overline{\{y\}}$ and $y \in \overline{\{z\}}$. Moreover, if the codimension function is bounded, then X has finite Krull dimension.

Lemma 4.4. *Let X be a scheme such that every local ring $\mathcal{O}_{X,x}$ has finite Krull dimension, and let $d\colon X \to \mathbb{Z}$ be a codimension function. Then*

$$d(y) - \dim \mathcal{O}_{X,y} \geq d(x) - \dim \mathcal{O}_{X,x}$$

holds for any points x, $y \in X$ *with* $x \in \overline{\{y\}}$. *Moreover, the following three conditions are equivalent to each other:*

(i) *the equality*
$$d(y) - \dim \mathcal{O}_{X,y} = d(x) - \dim \mathcal{O}_{X,x}$$
holds for any points x, $y \in X$ *with* $x \in \overline{\{y\}}$;

(ii) *the function* $X \ni x \mapsto d(x) - \dim \mathcal{O}_{X,x} \in \mathbb{Z}$ *is locally constant;*

(iii) X *is locally equi-dimensional* (*cf. Definition* 2.2(3)).

Proof. The first inequality is derived from the well-known inequality
$$\dim \mathcal{O}_{X,x} \geq \dim \mathcal{O}_{Y,y} + \operatorname{codim}(\overline{\{x\}}, \overline{\{y\}})$$
(cf. Property 2.1(1), [12, 0_{IV}, Prop. (14.2.2)]). To show the equivalence of the three conditions (i)–(iii), we may assume that X is connected. Let $\mathcal{S}$ be the set of generic points of irreducible components of X and, for a point $x \in X$, let $\mathcal{S}(x)$ be the subset consisting of $y \in \mathcal{S}$ with $x \in \overline{\{y\}}$. Note that $\mathcal{O}_{X,x}$ is equi-dimensional if and only if

$$\operatorname{codim}(\overline{\{x\}}, \overline{\{y\}}) = \operatorname{codim}(\overline{\{x\}}, X) \tag{IV-1}$$

for any $y \in \mathcal{S}(x)$. In fact, a point $y \in \mathcal{S}(x)$ corresponds to a minimal prime ideal $\mathfrak{p}$ of $\mathcal{O}_{X,x}$ via the natural morphism $\operatorname{Spec} \mathcal{O}_{X,x} \to X$, and (IV-1) is written as
$$\dim \mathcal{O}_{X,x}/\mathfrak{p} = \dim \mathcal{O}_{X,x}$$
(cf. Property 2.1(1)). The implication (ii) $\Rightarrow$ (i) is trivial, and (i) $\Rightarrow$ (iii) is shown by the equality $\dim \mathcal{O}_{X,x} = \operatorname{codim}(\overline{\{x\}}, \overline{\{y\}})$ for any $y \in \mathcal{S}(x)$, which holds by (i). It suffices to prove (iii) $\Rightarrow$ (ii). In the situation of (iii), by (IV-1), we have $d(x) - \dim \mathcal{O}_{X,x} = d(y) = d(y) - \dim \mathcal{O}_{X,y}$ for any $x \in X$ and $y \in \mathcal{S}(x)$. This implies that $x \mapsto d(x) - \dim \mathcal{O}_{X,x}$ is a constant function with value $d(y)$ on $\overline{\{y\}}$ for any $y \in \mathcal{S}$, and $d(y) = d(y')$ for any points y, $y' \in \mathcal{S}$ with $\overline{\{y\}} \cap \overline{\{y'\}} \neq \emptyset$. Consequently, $x \mapsto d(x) - \dim \mathcal{O}_{X,x}$ is constant on X, since X is connected. Thus, we are done. □

The importance of the codimension function comes from the following.

Fact 4.5. Let X be a locally Noetherian scheme with a dualizing complex $\mathcal{R}^\bullet$. Then we can define a function $d \colon X \to \mathbb{Z}$ by
$$\operatorname{Ext}^i_{\mathcal{O}_{X,x}}(\Bbbk(x), \mathcal{R}^\bullet_x) = \mathrm{H}^i(\mathbf{R}\mathrm{Hom}_{\mathcal{O}_{X,x}}(\Bbbk(x), \mathcal{R}^\bullet_x)) = \begin{cases} 0 & \text{for } i \neq d(x), \\ \Bbbk(x) & \text{for } i = d(x), \end{cases}$$

where $\Bbbk(x)$ denotes the residue field at x and $\mathcal{R}^\bullet_x$ denotes the stalk at x (cf. [17, V, Prop. 3.4]). The function d is a bounded codimension function (cf. [17, V, Cor. 7.2]), and we call d the *codimension function associated with* $\mathcal{R}^\bullet$. In particular, X is catenary and has finite Krull dimension.

The following result and Lemma 4.8 below are useful for checking $\mathbf{S}_k$-conditions for coherent sheaves.

Proposition 4.6. *Let X be a locally Noetherian scheme admitting a dualizing complex $\mathcal{R}^\bullet$ with codimension function $d\colon X \to \mathbb{Z}$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. For an integer j, we set*

$$\mathcal{G}^{(j)} := \mathcal{E}xt^j_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}^\bullet) := \mathcal{H}^j(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}^\bullet)).$$

Then $\mathcal{G}^{(j)}$ is a coherent $\mathcal{O}_X$-module and

$$\mathcal{G}^{(j)}_x \simeq \mathbb{E}\mathrm{xt}^j_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{R}^\bullet_x)$$

for the stalk $\mathcal{G}^{(j)}_x = (\mathcal{G}^{(j)})_x$ at any point $x \in X$. Moreover, the following hold for a point $x \in X$:

(1) *if $j - d(x) < -\dim \mathcal{F}_x$ or $j - d(x) > 0$, then $\mathcal{G}^{(j)}_x = 0$;*
(2) *for an integer k,* depth $\mathcal{F}_x \geq k$ *if and only if $\mathcal{G}^{(j)}_x = 0$ for any $j > d(x) - k$;*
(3) *for an integer k, $\mathcal{F}$ satisfies $\mathbf{S}_k$ at x if and only if $\mathcal{G}^{(j)}_y = 0$ for any point $y \in X$ with $x \in \overline{\{y\}}$ and for any $j > d(y) - \inf\{k, \dim \mathcal{F}_y\}$;*
(4) *$\mathcal{F}_x$ is a Cohen–Macaulay $\mathcal{O}_{X,x}$-module if and only if $\mathcal{G}^{(j)}_x = 0$ for any $j \neq d(x) - \dim \mathcal{F}_x$;*
(5) *if $x \in \operatorname{Supp} \mathcal{F}$, then $\mathcal{G}^{(i)}_x \neq 0$ for $i = d(x) - \dim \mathcal{F}_x$.*

Proof. The first assertion is derived from $\mathcal{R}^\bullet \in \mathbf{D}^+_{\mathrm{coh}}(X)_{\mathrm{fid}}$. Assertions (1) and (2) are essentially proved in [17, V]: (1) is shown in the proof of [17, V, Prop. 3.4], and (2) follows from the local duality theorem [17, V, Cor. 6.3]. Assertion (3) follows from (2) and Definition 2.9. Assertion (4) is a consequence of (1) and (2), since $\mathcal{F}_x$ is Cohen–Macaulay if and only if depth $\mathcal{F}_x = \dim \mathcal{F}_x$ unless $\mathcal{F}_x = 0$. Assertion (5) is shown as follows. For the given point $x \in \operatorname{Supp} \mathcal{F}$, we can find a point $y \in \operatorname{Supp} \mathcal{F}$ such that $\overline{\{y\}}$ is an irreducible component of $\operatorname{Supp} \mathcal{F}$ containing x and $\dim \mathcal{F}_x = \operatorname{codim}(\overline{\{x\}}, \overline{\{y\}})$. Then $d(x) - \dim \mathcal{F}_x = d(y)$. If $\mathcal{G}^{(d(y))}_x = 0$, then $\mathcal{G}^{(d(y))}_y = 0$, since $x \in \overline{\{y\}}$. But in this case $\mathcal{G}^{(j)}_y = 0$ for any $j \in \mathbb{Z}$ by (1), i.e., $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}^\bullet)_y \simeq_{\mathrm{qis}} 0$. This is a contradiction, since $\mathcal{F}_y \neq 0$ and

$$\mathcal{F} \simeq_{\mathrm{qis}} \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}^\bullet), \mathcal{R}^\bullet)$$

by [17, V, Prop. 2.1]. Therefore, $\mathcal{G}^{(i)}_x \neq 0$ for $i = d(x) - \dim \mathcal{F}_x = d(y)$. □

Corollary 4.7. *Let X be a locally Noetherian scheme admitting a dualizing complex $\mathcal{R}^\bullet$ and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

(1) *Assume that* $\operatorname{Supp}\mathcal{F}$ *is connected. Then $\mathcal{F}$ is a Cohen–Macaulay $\mathcal{O}_X$-module if and only if*

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}^\bullet) \simeq_{\mathrm{qis}} \mathcal{G}[-c]$$

for a coherent $\mathcal{O}_X$-module $\mathcal{G}$ and a constant $c \in \mathbb{Z}$. In this case, $\mathcal{G}$ is also a Cohen–Macaulay $\mathcal{O}_X$-module and $\operatorname{Supp}\mathcal{G} = \operatorname{Supp}\mathcal{F}$.

(2) *Assume that X is connected. Then X is Cohen–Macaulay if and only if $\mathcal{R}^\bullet \simeq_{\mathrm{qis}} \mathcal{L}[-c]$ for a coherent $\mathcal{O}_X$-module $\mathcal{L}$ and a constant $c \in \mathbb{Z}$. In this case, $\mathcal{L}$ is also a Cohen–Macaulay $\mathcal{O}_X$-module and* $\operatorname{Supp}\mathcal{L} = X$.

(3) *Assume that $\mathcal{F}$ is a Cohen–Macaulay $\mathcal{O}_X$-module and let S be a closed subscheme of X such that $S =$* $\operatorname{Supp}\mathcal{F}$ *as a set. Then S is locally equi-dimensional.*

Proof. It suffices to prove (1) and (3), since (2) is a special case of (1). Let $d\colon X \to \mathbb{Z}$ be the codimension function associated with $\mathcal{R}^\bullet$. First we shall prove the "if" part of (1). The quasi-isomorphism in (1) implies that $\mathcal{G}^{(j)} := \mathcal{E}xt^j_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}^\bullet) = 0$ for any $j \neq c$ and $\mathcal{G} \simeq \mathcal{G}^{(c)}$. Then $\mathcal{F}$ is Cohen–Macaulay and $c = d(x) - \dim \mathcal{F}_x$ for any $x \in X$ by (4) and (5) of Proposition 4.6. Second we shall prove the remaining part of (1) and (3). For the proof of (3), we may also assume that $\operatorname{Supp}\mathcal{F}$ is connected. Suppose that $\mathcal{F}$ is Cohen–Macaulay. Then $d(x) - \dim \mathcal{F}_x = d(y) - \dim \mathcal{F}_y$ holds for any points x, $y \in S$ with $x \in \overline{\{y\}}$ by (4) and (5) of Proposition 4.6, where we use the property that $\mathcal{G}^{(j)}_x = 0$ implies $\mathcal{G}^{(j)}_y = 0$. As a consequence, $c := d(x) - \dim \mathcal{F}_x$ is constant on $S = \operatorname{Supp}\mathcal{F}$. We have $\dim \mathcal{F}_x = \dim \mathcal{O}_{S,x}$ for any $x \in S$ by Property 2.1(1). Thus, S is locally equi-dimensional by Lemma 4.4, and this proves (3). Furthermore, $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}^\bullet) \simeq \mathcal{G}[-c]$ for the cohomology sheaf $\mathcal{G} = \mathcal{G}^{(c)}$. We have also

$$\mathcal{F} \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}[-c], \mathcal{R}^\bullet)$$

by [17, V, Prop. 2.1]. Thus, $\operatorname{Supp}\mathcal{G} = \operatorname{Supp}\mathcal{F}$, and $\mathcal{G}$ is also a Cohen–Macaulay $\mathcal{O}_X$-module by the "if" part of (1). Thus, we are done. □

Lemma 4.8. *Let X, $\mathcal{R}^\bullet$, $\mathcal{F}$, and $\mathcal{G}^{(j)}$ be as in Proposition 4.6. Then $\mathcal{G}^{(j)} = 0$ except for finitely many j. For a positive integer k, the following hold:*

(1) *$\mathcal{F}$ satisfies $\mathbf{S}_k$ at a point $x \in$* $\operatorname{Supp}\mathcal{F}$ *if and only if*

$$\operatorname{codim}_x(\operatorname{Supp}\mathcal{G}^{(i)} \cap \operatorname{Supp}\mathcal{G}^{(j)}, \operatorname{Supp}\mathcal{F}) \geq k + i - j$$

for any $i > j$;

(2) *$\mathcal{F}$ satisfies $\mathbf{S}_k$ if and only if*

$$\operatorname{codim}(\operatorname{Supp}\mathcal{G}^{(i)} \cap \operatorname{Supp}\mathcal{G}^{(j)}, \operatorname{Supp}\mathcal{F}) \geq k + i - j$$

for any $i > j$.

Proof. The first assertion follows from Proposition 4.6(1), since $d\colon X \to \mathbb{Z}$ is bounded and $\dim X < \infty$ by Fact 4.5. For integers i, j with $i > j$, we set

$$Z^{(i,j)} := \operatorname{Supp}\mathcal{G}^{(i)} \cap \operatorname{Supp}\mathcal{G}^{(j)}.$$

Note that $\operatorname{codim}(Z^{(i,j)}, \operatorname{Supp}\mathcal{F}) = +\infty$ if $Z^{(i,j)} = \emptyset$. Assertion (1) is derived from (2) applied to the coherent sheaf $(\mathcal{F}_x)^\sim$ on $\operatorname{Spec}\mathcal{O}_{X,x}$ associated with $\mathcal{F}_x$ (cf. Remark 2.10), since

$$\operatorname{codim}_x(Z^{(i,j)}, \operatorname{Supp}\mathcal{F}) = \operatorname{codim}(\operatorname{Supp}\mathcal{G}^{(i)}_x \cap \operatorname{Supp}\mathcal{G}^{(j)}_x, \operatorname{Supp}\mathcal{F}_x)$$

(cf. Property 2.1(3)). Hence, it is enough to prove (2). Assume first that $\mathcal{F}$ satisfies $\mathbf{S}_k$. For integers $i > j$ with $Z^{(i,j)} \neq \emptyset$, we can find a generic point x of $Z^{(i,j)}$ such that

$$\operatorname{codim}(Z^{(i,j)}, \operatorname{Supp}\mathcal{F}) = \operatorname{codim}(\overline{\{x\}}, \operatorname{Supp}\mathcal{F}) = \dim \mathcal{F}_x.$$

If $\dim \mathcal{F}_x \leq k$, then $i = j = d(x) - \dim \mathcal{F}_x$ by (1) and (3) of Proposition 4.6. This is a contradiction, since $i > j$. Thus, $\dim \mathcal{F}_x > k$, and

$$d(x) - \dim \mathcal{F}_x \leq j < i \leq d(x) - k$$

also by (1) and (3) of Proposition 4.6. Hence, $i - j \leq \dim \mathcal{F}_x - k$, and this is equivalent to the inequality in (2).

Conversely, assume that the inequality in (2) holds for any $i > j$. For a point $x \in \operatorname{Supp}\mathcal{F}$, we set $c(x) := d(x) - \dim \mathcal{F}_x$. By (1) and (5) of Proposition 4.6, we know that $x \in \operatorname{Supp}\mathcal{G}^{(c(x))}$ and $x \notin \operatorname{Supp}\mathcal{G}^{(i)}$ for any $i < c(x)$. If $\mathcal{G}^{(i)}_x \neq 0$ for some $i \neq c(x)$, then $i > c(x)$ and

$$\dim \mathcal{F}_x \geq \operatorname{codim}(Z^{(i,c(x))}, \operatorname{Supp}\mathcal{F}) \geq k + i - c(x) = k + i - d(x) + \dim \mathcal{F}_x.$$

Hence, $i \leq d(x) - k$ and $\dim \mathcal{F}_x > k$. Thus, $\mathcal{F}$ satisfies $\mathbf{S}_k$ by Proposition 4.6(3). Therefore, (2) has been proved, and we are done. □

Corollary 4.9. *Let X, $\mathcal{R}^\bullet$, $\mathcal{F}$, and $\mathcal{G}^{(j)}$ be as in Proposition 4.6. Let k be a positive integer.*

(1) *Assume that $\operatorname{Supp}\mathcal{F}$ is connected and locally equi-dimensional. Then there is a positive integer c such that $c = d(x) - \dim \mathcal{F}_x$ for any $x \in X$. For the integer c, one has $\operatorname{Supp}\mathcal{G}^{(c)} = \operatorname{Supp}\mathcal{F}$. Moreover, $\mathcal{F}$ satisfies $\mathbf{S}_k$ if and only if*

$$\operatorname{codim}(\operatorname{Supp}\mathcal{G}^{(j)}, \operatorname{Supp}\mathcal{F}) \geq k + j - c$$

for any $j > c$.

(2) *Assume that* $\operatorname{Supp}\mathcal{F}$ *is connected, equi-dimensional and* equi-codimensional (*cf.* [12, 0_{IV}, Déf. (14.2.1)]). *Furthermore, assume that* $\operatorname{Supp}\mathcal{F}$ *is Noetherian. Let* c *be the integer in* (1). *Then* $\mathcal{F}$ *satisfies* $\mathbf{S}_k$ *if and only if*

$$\dim \operatorname{Supp} \mathcal{G}^{(j)} \le \dim \operatorname{Supp} \mathcal{F} + c - j - k$$

for any $j > c$.

(3) *Assume that* $\mathcal{F}_x \neq 0$ *and* $\mathcal{F}_x$ *is equi-dimensional* (*cf. Definition* 2.2). *Then* $\mathcal{F}$ *satisfies* $\mathbf{S}_k$ *at* x *if and only if* $\dim \mathcal{G}^{(j)}_x \le d(x) - j - k$ *for any* $j \neq d(x) - \dim \mathcal{F}_x$.

Proof. (1): For a closed subscheme S with $S = \operatorname{Supp}\mathcal{F}$, we have the integer c such that $c = d(x) - \dim \mathcal{O}_{S,x} = d(x) - \dim \mathcal{F}_x$ for any $x \in \operatorname{Supp}\mathcal{F}$ by Lemma 4.4. Then $\operatorname{Supp}\mathcal{G}^{(c)} = \operatorname{Supp}\mathcal{F}$ by Proposition 4.6(5). Assume that $\mathcal{F}$ satisfies $\mathbf{S}_k$. Then

$$\operatorname{codim}(\operatorname{Supp}\mathcal{G}^{(j)}, \operatorname{Supp}\mathcal{F}) = \operatorname{codim}(\operatorname{Supp}\mathcal{G}^{(j)} \cap \operatorname{Supp}\mathcal{G}^{(c)}, \operatorname{Supp}\mathcal{F}) \ge k + j - c$$

for any $j > c$ by Lemma 4.8. Conversely, assume that the inequality in (1) holds for any $j > c$. If $\mathcal{G}^{(j)}_x \neq 0$ for some $j > c$, then

$$\dim \mathcal{F}_x \ge \operatorname{codim}(\operatorname{Supp}\mathcal{G}^{(j)}, \operatorname{Supp}\mathcal{F}) \ge k + j - c = k + j - d(x) + \dim \mathcal{F}_x$$

as in the proof of Lemma 4.8(2). Hence, $\mathcal{G}^{(j)}_x \neq 0$ implies that $\dim \mathcal{F}_x > k$ and $j \le d(x) - k$. This means that $\mathcal{F}$ satisfies $\mathbf{S}_k$ by Proposition 4.6(3).

(2): The closed subset $S = \operatorname{Supp}\mathcal{F}$ is a bi-equi-dimensional Kolmogorov Noetherian space in the sense of EGA (cf. [12, 0_{IV}, Prop. (14.3.3)]), since it is catenary and has finite Krull dimension (cf. Fact 4.5). Then

$$x \mapsto \dim \mathcal{O}_{S,x} = \operatorname{codim}(\overline{\{x\}}, S)$$

is a codimension function on S by [12, 0_{IV}, (14.3.3.2)], and it implies that S is locally equi-dimensional by Lemma 4.4. Moreover, $\dim Z + \operatorname{codim}(Z, S) = \dim S$ for any closed subset $Z \subset S$ by [12, 0_{IV}, Cor. (14.3.5)]. Thus, (2) follows from (1).

(3): The closed subset $\operatorname{Supp}\mathcal{F}_x$ of $\operatorname{Spec}\mathcal{O}_{X,x}$ is equi-codimensional, since x is the unique closed point of it. Hence, $\operatorname{Supp}\mathcal{F}_x$ is also a connected bi-equi-dimensional Kolmogorov Noetherian space and it is locally equi-dimensional by the same reason as above. Thus, we can apply (1) to the coherent sheaf $\mathcal{F}^{\sim}_x$ on $\operatorname{Spec}\mathcal{O}_{X,x}$ associated with $\mathcal{F}_x$. Hence, $\mathcal{F}$ satisfies $\mathbf{S}_k$ at x (cf. Remark 2.10) if and only if

$$\operatorname{codim}(\operatorname{Supp}\mathcal{G}^{(j)}_x, \operatorname{Supp}\mathcal{F}_x) \ge k + j - c(x)$$

for any $j > c(x)$, where $c(x) := d(x) - \dim \mathcal{F}_x$. Here the left-hand side equals $\dim \mathcal{F}_x - \dim \mathcal{G}^{(j)}_x$ by [12, 0_{IV}, Cor. (14.3.5)]. Therefore, the $\mathbf{S}_k$-condition at x is equivalent to

$$\dim \mathcal{G}_x^{(j)} \leq c(x) - k - j + \dim \mathcal{F}_x = d(x) - k - j$$

for any $j > c(x) = d(x) - \dim \mathcal{F}_x$. Thus, we have (3) by Proposition 4.6(1), and we are done. □

Definition 4.10 ($\mathrm{Gor}(X)$). The *Gorenstein locus* $\mathrm{Gor}(X)$ of a locally Noetherian scheme X is defined to be the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is Gorenstein.

Note that X is Gorenstein if and only if $X = \mathrm{Gor}(X)$. The following is a generalization of [12, IV, Prop. (6.11.2)(ii)] (cf. [52, Prop. (3.2)] for $\mathrm{Gor}(X)$).

Proposition 4.11. *Let X be a locally Noetherian scheme admitting a dualizing complex locally on X and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\mathbf{S}_k(\mathcal{F})$ for all $k \geq 1$ and $\mathrm{CM}(\mathcal{F})$ are open subsets of X. In particular, $\mathrm{CM}(X)$ is open. Moreover, $\mathrm{Gor}(X)$ is also open.*

Proof. Localizing X, we may assume that X is an affine Noetherian scheme with a dualizing complex $\mathcal{R}^\bullet$. The openness of $\mathrm{Gor}(X)$ follows from that of $\mathrm{CM}(X)$. In fact, if X is Cohen–Macaulay, then we may assume that $\mathcal{R}^\bullet \simeq_{\mathrm{qis}} \mathcal{L}$ for a coherent $\mathcal{O}_X$-module $\mathcal{L}$ by Corollary 4.7(2), and $\mathrm{Gor}(X)$ is the maximal open subset on which $\mathcal{L}$ is invertible. The openness of $\mathrm{CM}(\mathcal{F})$ is derived from Corollary 4.7(1). This follows also from the openness of $\mathbf{S}_k(\mathcal{F})$ for all $k \geq 1$. In fact, $\mathrm{CM}(\mathcal{F}) = \mathbf{S}_k(\mathcal{F})$ for $k \gg 0$, since $\dim \mathcal{F} \leq \dim X < \infty$ (cf. Fact 4.5 and Remark 2.12). The openness of $\mathbf{S}_k(\mathcal{F})$ is derived from Lemma 4.8(1), since $x \mapsto \mathrm{codim}_x(Z, \mathrm{Supp}\,\mathcal{F})$ is lower semi-continuous for any closed subset $Z \subset \mathrm{Supp}\,\mathcal{F}$ (cf. Property 2.1(3)). □

Remark. In the situation of Proposition 4.11, all $\mathbf{S}_k(\mathcal{F})$ are open if and only if the map

$$\mathrm{Supp}\,\mathcal{F} \ni x \mapsto \mathrm{codepth}\,\mathcal{F}_x := \dim \mathcal{F}_x - \mathrm{depth}\,\mathcal{F}_x \in \mathbb{Z}_{\geq 0}$$

is upper semi-continuous (cf. [12, IV, Rem. (6.11.4)]).

The following analogy of Fact 2.26(6) for $\mathcal{G} = \mathcal{O}_Y$ is known.

Fact 4.12 (Cf. [37, Thm. 23.4], [17, V, Prop. 9.6]). Let $Y \to T$ be a flat morphism of locally Noetherian schemes. Then Y is Gorenstein if and only if T and every fiber are Gorenstein.

§4.2. Ordinary dualizing complex

We introduce the notion of an ordinary dualizing complex $\mathcal{R}^\bullet$ and that of a dualizing sheaf as the cohomology sheaf $\mathcal{H}^0(\mathcal{R}^\bullet)$ for locally Noetherian schemes which are *locally equi-dimensional* (cf. Definition 2.2(3)), especially for locally Noetherian schemes satisfying $\mathbf{S}_2$. In many articles, the dualizing sheaf is usually defined for a Cohen–Macaulay scheme, and it coincides with the dualizing sheaf in our sense (cf. Remark 4.15 below).

Definition 4.13. Let X be a locally Noetherian scheme.

(1) A dualizing complex $\mathcal{R}^\bullet$ of X is said to be *ordinary* if the codimension function d associated with $\mathcal{R}^\bullet$ satisfies $d(x) = \dim \mathcal{O}_{X,x}$ for any $x \in X$.
(2) A coherent sheaf $\mathcal{L}$ on X is called a *dualizing sheaf* of X if $\mathcal{L} \simeq \mathcal{H}^0(\mathcal{R}^\bullet)$ for an ordinary dualizing complex $\mathcal{R}^\bullet$ of X.

As a corollary to Lemma 4.4 above, we have the following.

Lemma 4.14. *Let X be a locally Noetherian scheme admitting a dualizing complex. Then X admits an ordinary dualizing complex if and only if X is locally equi-dimensional. In particular, X admits an ordinary dualizing complex if X satisfies $\mathbf{S}_2$.*

Proof. We may assume that X is connected. Let $\mathcal{R}^\bullet$ be a dualizing complex of X with codimension function $d\colon X \to \mathbb{Z}$. If it is ordinary, then X is locally equi-dimensional by Lemma 4.4. Conversely, if X is locally equi-dimensional, then $d(x) - \dim \mathcal{O}_{X,x}$ is a constant c by Lemma 4.4, and hence, the shift $\mathcal{R}^\bullet[c]$ is an ordinary dualizing complex. The last assertion follows from Facts 4.5 and 2.23(1). □

Remark. For a locally Noetherian scheme, the ordinary dualizing complex is unique up to quasi-isomorphism and tensor product with an invertible sheaf (cf. Remark 4.2). Similarly, the dualizing sheaf is unique up to isomorphism and tensor product with an invertible sheaf.

Remark 4.15. Let X be a locally Noetherian Cohen–Macaulay scheme admitting a dualizing complex. Then X has an ordinary dualizing complex $\mathcal{R}^\bullet$ which is quasi-isomorphic to the dualizing sheaf $\mathcal{L} = \mathcal{H}^0(\mathcal{R}^\bullet)$. Here $\mathcal{L}$ is also a Cohen–Macaulay $\mathcal{O}_X$-module. These are derived from Proposition 4.6(4) and Corollary 4.7(2). In many articles, $\mathcal{L}$ is called a “dualizing sheaf” for a locally Noetherian Cohen–Macaulay scheme.

Lemma 4.16. *Let X be a locally Noetherian scheme admitting an ordinary dualizing complex $\mathcal{R}^\bullet$. Let $Z^{(i)}$ be the support of the cohomology sheaf $\mathcal{H}^i(\mathcal{R}^\bullet)$ for any $i \in \mathbb{Z}$. Then $Z^{(i)} = \emptyset$ for any $i < 0$, $Z^{(0)} = X$, and the following hold for any $x \in X$:*

(1) *$x \notin Z^{(i)}$ for any $i > \dim \mathcal{O}_{X,x}$;*

(2) $\operatorname{depth} \mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x} - \sup\{j \mid x \in Z^{(j)}\}$;

(3) *for an integer* $k \geq 1$, X *satisfies* $\mathbf{S}_k$ *at* x *if and only if*

$$\operatorname{codim}_x(Z^{(j)}, X) \geq k + j$$

for any $j > 0$*; this is also equivalent to*

$$\dim_x Z^{(j)} \leq \dim \mathcal{O}_{X,x} - k - j$$

for any $j > 0$*;*

(4) $\mathcal{O}_{X,x}$ *is Cohen–Macaulay if and only if* $x \notin Z^{(j)}$ *for any* $j > 0$.

Proof. Now, $d(x) = \dim \mathcal{O}_{X,x}$ for the codimension function $d\colon X \to \mathbb{Z}$ associated with $\mathcal{R}^\bullet$, and X is locally equi-dimensional by Lemma 4.4. Thus, applying Proposition 4.6 to $\mathcal{F} = \mathcal{O}_X$, we have the assertions except (3). The remaining assertion (3) is obtained by Corollary 4.9(3) (cf. Property 2.1). □

Remark. Let X be a connected locally Noetherian scheme with a dualizing complex $\mathcal{R}^\bullet$ such that $\mathcal{H}^i(\mathcal{R}^\bullet) = 0$ for any $i < 0$ and $\mathcal{H}^0(\mathcal{R}^\bullet) \neq 0$. The sheaf $\mathcal{H}^0(\mathcal{R}^\bullet)$ is called the "canonical module" in many articles. But as in Example 4.17 below, the support of the sheaf $\mathcal{H}^0(\mathcal{R}^\bullet)$ is not always X. This is one of the reasons why we do not consider $\mathcal{H}^0(\mathcal{R}^\bullet)$ as the dualizing sheaf for arbitrary locally Noetherian schemes.

Example 4.17. Let P be a polynomial ring $\Bbbk[\mathtt{x}, \mathtt{y}, \mathtt{z}]$ of three variables over a field $\Bbbk$. For the ideals $I = (\mathtt{x}, \mathtt{y})$ and $J = (\mathtt{z})$ of P, we set $A := P/IJ$ and $R^\bullet := \mathbf{R}\mathrm{Hom}_P(A, P[1])$. Then we have a Noetherian affine scheme $X = \operatorname{Spec} A$ and a dualizing complex $\mathcal{R}^\bullet$ on X associated with $R^\bullet$ (cf. Example 4.23 below). The X is a union of a plane $\operatorname{Spec} P/J$ and a line $\operatorname{Spec} P/I$ in the three-dimensional affine space $\operatorname{Spec} P \simeq \mathbb{A}^3_{\Bbbk}$, where the plane and the line intersect at the origin O corresponding to the maximal ideal $(\mathtt{x}, \mathtt{y}, \mathtt{z})$. Note that the local ring $\mathcal{O}_{X,O}$ is not equi-dimensional. We can calculate the cohomology modules of $R^\bullet$ as

$$\mathrm{H}^i(R^\bullet) \simeq \operatorname{Ext}^{i+1}_P(A, P) \simeq \begin{cases} 0 & \text{for any } i < 0 \text{ and } i > 1, \\ P/J & \text{for } i = 0, \\ P/I & \text{for } i = 1, \end{cases}$$

by the free resolution

$$0 \to P \xrightarrow{g} P^{\oplus 2} \xrightarrow{f} P \to A \to 0,$$

where f and g are defined by

$$f(a, b) = \mathtt{xz}a + \mathtt{yz}b \quad \text{and} \quad g(c) = (\mathtt{y}c, -\mathtt{x}c)$$

for any a, b, and $c \in P$. Consequently, $\operatorname{Supp} \mathcal{H}^0(\mathcal{R}^\bullet) = \operatorname{Spec} R/J$ is a proper subset of X.

Lemma 4.18. *Let X be a locally Noetherian scheme admitting an ordinary dualizing complex $\mathcal{R}^\bullet$. We set*

$$\mathcal{G}^{(j)}_{\le b} := \boldsymbol{\mathcal{E}}xt^j_{\mathcal{O}_X}(\tau^{\le b}(\mathcal{R}^\bullet), \mathcal{R}^\bullet) \quad \text{and} \quad \mathcal{G}^{(j)}_{\ge b} := \boldsymbol{\mathcal{E}}xt^j_{\mathcal{O}_X}(\tau^{\ge b}(\mathcal{R}^\bullet), \mathcal{R}^\bullet)$$

for integers $b \ge 0$ and j, where $\tau^{\le b}$ and $\tau^{\ge b}$ stand for the truncations of a complex (cf. Notation and conventions, (1)). Then the following hold:

(1) *One has $\mathcal{G}^{(0)}_{\ge 0} \simeq \mathcal{O}_X$ and $\mathcal{G}^{(i)}_{\ge 0} = 0$ for any $i \ne 0$.*

(2) *There exist an exact sequence*

$$0 \to \mathcal{G}^{(-1)}_{\le b} \to \mathcal{G}^{(0)}_{\ge b+1} \to \mathcal{O}_X \to \mathcal{G}^{(0)}_{\le b} \to \mathcal{G}^{(1)}_{\ge b+1} \to 0$$

and an isomorphism

$$\mathcal{G}^{(j)}_{\le b} \simeq \mathcal{G}^{(j+1)}_{\ge b+1}$$

for any $j \ne \{0, -1\}$. Moreover, $\mathcal{G}^{(j)}_{\le 0} = 0$ for any $j < 0$.

(3) *For any integers $b \ge 0$ and j, one has*

- $\operatorname{codim}(\operatorname{Supp} \mathcal{G}^{(j)}_{\ge b}, X) \ge j + b$ *for any $j \in \mathbb{Z}$,*
- $\operatorname{codim}(\operatorname{Supp} \mathcal{G}^{(j)}_{\le b}, X) \ge j + b + 2$ *for any $j \ne 0$, and*
- $\operatorname{codim}(\operatorname{Supp} \mathcal{G}^{(0)}_{\le b}, X) = 0$.

(4) *If X satisfies $\mathbf{S}_k$ for some $k \ge 1$, then*

- $\mathcal{G}^{(j)}_{\ge b} = 0$ *for any $b > 0$ and $j < k$, and*
- $\mathcal{G}^{(i)}_{\le b} = 0$ *for any $0 < i < k - 1$.*

Proof. We have a quasi-isomorphism $\mathcal{R}^\bullet \simeq_{\mathrm{qis}} \tau^{\ge 0}(\mathcal{R}^\bullet)$ by Lemma 4.16. Hence, the first assertion (1) interprets the quasi-isomorphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}^\bullet, \mathcal{R}^\bullet) \simeq_{\mathrm{qis}} \mathcal{O}_X.$$

The exact sequence and the isomorphism in the second assertion (2) are derived from the canonical distinguished triangle

$$\cdots \to \tau^{\le b}(\mathcal{R}^\bullet) \to \mathcal{R}^\bullet \to \tau^{\ge b+1}(\mathcal{R}^\bullet) \to \tau^{\le b}(\mathcal{R}^\bullet)[1] \to \cdots .$$

The last vanishing in (2) is expressed as $\boldsymbol{\mathcal{E}}xt^j_{\mathcal{O}_X}(\mathcal{L}, \mathcal{R}^\bullet) = 0$ for any $j < 0$, where $\mathcal{L} := \mathcal{H}^0(\mathcal{R}^\bullet)$, and this is a consequence of Proposition 4.6(1) applied to $\mathcal{F} = \mathcal{L}$

with the property $X = \operatorname{Supp} \mathcal{L}$ shown in Lemma 4.16. For the remaining assertions (3) and (4), it is enough to consider only the sheaves $\mathcal{G}^{(j)}_{\geq b}$. In fact, by (2), we have an injection $\mathcal{G}^{(i)}_{\leq b} \to \mathcal{G}^{(i+1)}_{\geq b+1}$ for any $i \neq 0$, and an exact sequence $\mathcal{G}^{(0)}_{\geq b+1} \to \mathcal{O}_X \to \mathcal{G}^{(0)}_{\leq b}$, where $\operatorname{codim}(\operatorname{Supp} \mathcal{G}^{(0)}_{\geq b+1}, X) > 0$ by the assertion for $\mathcal{G}^{(0)}_{\geq b+1}$. Hence,

$$\operatorname{codim}(\operatorname{Supp} \mathcal{G}^{(i)}_{\leq b}, X) \geq \operatorname{codim}(\operatorname{Supp} \mathcal{G}^{(i+1)}_{\geq b+1}, X)$$

for any $i \neq 0$ and $\operatorname{codim}(\operatorname{Supp} \mathcal{G}^{(0)}_{\leq b}, X) = 0$. In order to prove (3) and (4) for $\mathcal{G}^{(j)}_{\geq b}$, let us consider the spectral sequence

$$\text{(IV-2)} \qquad \mathcal{E}^{p,q}_2 = \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{H}^{-q}(\tau^{\geq b}(\mathcal{R}^\bullet)), \mathcal{R}^\bullet) \Rightarrow \mathcal{E}^{p+q} = \mathcal{G}^{(p+q)}_{\geq b}$$

of $\mathcal{O}_X$-modules (cf. Remark 4.19 below). Assume that $(\mathcal{E}^{p,q}_2)_x \neq 0$ for a point $x \in X$. Then $-q \geq b$, and

$$\text{(IV-3)} \quad \dim \mathcal{O}_{X,x} \geq p \geq \dim \mathcal{O}_{X,x} - \dim \mathcal{H}^{-q}(\mathcal{R}^\bullet)_x = \operatorname{codim}_x(\operatorname{Supp} \mathcal{H}^{-q}(\mathcal{R}^\bullet), X)$$

by Proposition 4.6(1), since $d(x) = \dim \mathcal{O}_{X,x}$ for the codimension function d of $\mathcal{R}^\bullet$. In particular, $p + q \leq \dim \mathcal{O}_{X,x} - b$. Therefore, if $j + b > \dim \mathcal{O}_{X,x}$, then $x \notin \operatorname{Supp} \mathcal{G}^{(j)}_{\geq b}$, since $(\mathcal{E}^{p,q}_2)_x = 0$ for any integers p, q with $p + q = j$. Thus, we have (3). Assume that X satisfies $\mathbf{S}_k$. If $(\mathcal{E}^{p,q}_2)_x \neq 0$ and $q < 0$, then $p + q \geq k$ by (IV-3), since

$$\operatorname{codim}_x(\operatorname{Supp} \mathcal{H}^{-q}(\mathcal{R}^\bullet), X) \geq k - q$$

for any $q < 0$ by Lemma 4.16(3). Hence, $\mathcal{G}^{(j)}_{\geq b} = 0$ for any $b > 0$ and $j < k$, since $\mathcal{E}^{p,q}_2 = 0$ for any integers p, q with $p + q = j$. This proves (4), and we are done. □

Remark 4.19. The spectral sequence (IV-2) is obtained by the same method as follows. Let A be a commutative ring and let $M^\bullet$ and $N^\bullet$ be complexes of A-modules such that $N^\bullet$ is bounded below. We shall construct a spectral sequence

$$E^{p,q}_2 = \mathbb{E}\mathrm{xt}^p_A(\mathrm{H}^{-q}(M^\bullet), N^\bullet) \Rightarrow E^{p+q} = \mathbb{E}\mathrm{xt}^{p+q}_A(M^\bullet, N^\bullet),$$

where $\mathbb{E}\mathrm{xt}^p_A$ denotes the p-th hyper-ext group. Since there is a quasi-isomorphism from $N^\bullet$ into a complex of injective A-modules bounded below, we may assume that $N^\bullet$ itself is a complex of injective A-modules bounded below. We consider the hom complex $K^\bullet = \mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$ (cf. [17, I, Thm. 6.4], [10, Exem. 1.1.10(ii)], [35, (1.5.3)]): this is the total complex of a double complex $K^{\bullet,\bullet}$ such that $K^{p,q} = \mathrm{Hom}_A(M^{-q}, N^p)$ for p, $q \in \mathbb{Z}$ and that the differentials $d^{p,q}_{\mathrm{I}} \colon K^{p,q} \to K^{p+1,q}$ and $d^{p,q}_{\mathrm{II}} \colon K^{p,q} \to K^{p,q+1}$ are induced by $d^p_N \colon N^p \to N^{p+1}$ and $(-1)^q d^{-q-1}_M \colon M^{-q-1} \to M^{-q}$, respectively. Then $\mathbb{E}\mathrm{xt}^k_A(M^\bullet, N^\bullet) \simeq \mathrm{H}^k(K^\bullet)$ for any k. We have

$$\mathrm{H}^q_{\mathrm{II}}(K^{p,\bullet}) \simeq \mathrm{Hom}_A(\mathrm{H}^{-q}(M^\bullet), N^p)$$

for any p and q, since N^p is assumed to be injective. Thus, we have the spectral sequence above as the well-known spectral sequence $\mathrm{H}^p_{\mathrm{I}}\,\mathrm{H}^q_{\mathrm{II}}(K^{\bullet,\bullet}) \Rightarrow \mathrm{H}^{p+q}(K^\bullet)$ associated with the double complex $K^{\bullet,\bullet}$.

Corollary 4.20. *Let X be a locally Noetherian scheme admitting an ordinary dualizing complex $\mathcal{R}^\bullet$. For a point $x \in X$ and for an integer $b \geq 0$, the vanishing*

$$\mathbb{H}^i_x(\tau^{\leq b}(\mathcal{R}^\bullet)_x) = 0$$

holds for any $i < b+2$ except $i = \dim \mathcal{O}_{X,x}$, where $\mathbb{H}^i_x(M^\bullet)$ stands for the local cohomology group at the maximal ideal $\mathfrak{m}_x$ for a complex $M^\bullet$ of $\mathcal{O}_{X,x}$-modules bounded below.

Proof. By the local duality theorem [17, V, Thm. 6.2], we have

$$\mathbb{H}^i_x(\tau^{\leq b}(\mathcal{R}^\bullet)_x) \simeq \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathbb{E}\mathrm{xt}^{-i}_{\mathcal{O}_{X,x}}(\tau^{\leq b}(\mathcal{R}^\bullet)_x, \mathcal{R}^\bullet_x[d(x)]), I_x)$$

for the injective $\mathcal{O}_{X,x}$-module $I_x = \mathbb{H}^{d(x)}_x(\mathcal{R}^\bullet_x)$, where $d(x) = \dim \mathcal{O}_{X,x}$. In particular,

$$\mathbb{H}^i_x(\tau^{\leq b}(\mathcal{R}^\bullet)_x) \neq 0 \quad \text{if and only if} \quad x \in \mathrm{Supp}\, \mathcal{G}^{(d(x)-i)}_{\leq b}.$$

If $d(x) - i \neq 0$, then the non-vanishing above implies that

$$d(x) = \dim \mathcal{O}_{X,x} \geq \mathrm{codim}(\mathrm{Supp}\, \mathcal{G}^{(d(x)-i)}_{\leq b}, X) \geq d(x) - i + b + 2$$

by Lemma 4.18(3). Thus, we have the vanishing for $i < b+2$ except $i = d(x)$. $\square$

Proposition 4.21. *Let X be a locally Noetherian scheme admitting an ordinary dualizing complex $\mathcal{R}^\bullet$. Then the dualizing sheaf $\mathcal{L} = \mathcal{H}^0(\mathcal{R}^\bullet)$ satisfies $\mathbf{S}_2$ and* $\mathrm{Supp}\,\mathcal{L} = X$. *If X satisfies $\mathbf{S}_2$, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \simeq \mathcal{O}_X$. If X satisfies $\mathbf{S}_3$, then $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) = 0$.*

Proof. We have $\mathrm{Supp}\,\mathcal{L} = X$ by Lemma 4.16. Hence,

$$\dim \mathcal{L}_x = \mathrm{codim}(\overline{\{x\}}, \mathrm{Supp}\,\mathcal{L}) = \dim \mathcal{O}_{X,x}$$

for any $x \in X$. Applying Corollary 4.9(1) to $\mathcal{L}$, where $c = d(x) - \dim \mathcal{L}_x = 0$, we see that $\mathcal{L}$ satisfies $\mathbf{S}_2$ by Lemma 4.18(3), since $\mathrm{codim}(\mathrm{Supp}\, \mathcal{G}^{(i)}_{\leq 0}, X) \geq i+2$ for any $i > 0$, where $\mathcal{G}^{(i)}_{\leq 0} = \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{L}, \mathcal{R}^\bullet)$. For the remaining assertions, we assume that X satisfies $\mathbf{S}_2$ or $\mathbf{S}_3$. Note that we have an isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \simeq \boldsymbol{\mathcal{E}xt}^0_{\mathcal{O}_X}(\mathcal{L}, \mathcal{R}^\bullet) = \mathcal{G}^{(0)}_{\leq 0}$$

and an injection

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \to \boldsymbol{\mathcal{E}xt}^1_{\mathcal{O}_X}(\mathcal{L}, \mathcal{R}^\bullet) = \mathcal{G}^{(1)}_{\leq 0}$$

by $\mathcal{L} \simeq_{\text{qis}} \tau^{\leq 0}(\mathcal{R}^\bullet)$. If X satisfies $\mathbf{S}_2$, then $\mathcal{G}^{(0)}_{\geq 1} = \mathcal{G}^{(1)}_{\geq 1} = 0$ by Lemma 4.18(4), and hence, $\mathcal{O}_X \simeq \mathcal{G}^{(0)}_{\leq 0}$ by Lemma 4.18(2); thus, $\mathcal{O}_X \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})$. If X satisfies $\mathbf{S}_3$, then $\mathcal{G}^{(1)}_{\leq 0} = 0$ by Lemma 4.18(4), and consequently, $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) = 0$. □

Remark. The $\mathbf{S}_2$-condition for $\mathcal{L}$ is proved in [55, (46), Lem. 23.3] by another method.

Remark ($\mathbf{S}_2$-ification). For a locally Noetherian scheme X admitting an ordinary dualizing complex $\mathcal{R}^\bullet$ and for the dualizing sheaf $\mathcal{L} = \mathcal{H}^0(\mathcal{R}^\bullet)$, we consider the coherent $\mathcal{O}_X$-module $\mathcal{A} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})$. Then we can show that

- $\mathcal{A}$ has the structure of an $\mathcal{O}_X$-algebra,
- $\mathcal{O}_X \to \mathcal{A}$ is an isomorphism on the $\mathbf{S}_2$-locus $\mathbf{S}_2(X)$ (cf. Definition 2.13), and
- $\mathcal{A}$ satisfies $\mathbf{S}_2$.

Therefore, the finite morphism $\mathcal{S}pec_X \mathcal{A} \to X$ is regarded as the so-called "$\mathbf{S}_2$-ification" of X (cf. [12, IV, (5.10.11), Prop. (5.11.1)], [4, Prop. 2], [5, Thm. 3.2], [22, Prop. 2.7]). The three properties above are shown as follows: We know that $\mathcal{L}$ satisfies $\mathbf{S}_2$, $U := \mathbf{S}_2(X)$ is an open subset by Proposition 4.11 and that $\mathcal{O}_X \to \mathcal{A}$ is an isomorphism on U by Proposition 4.21. In particular, $\mathcal{A} \simeq j_*(\mathcal{A}|_U)$ for the open immersion $j\colon U \hookrightarrow X$, since it is expressed as

$$\mathcal{A} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \to j_*(\mathcal{A}|_U) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, j_*(\mathcal{L}|_U)).$$

Thus, $\mathcal{A}$ satisfies $\mathbf{S}_2$ by Corollary 2.16, and consequently, $\mathcal{A} \simeq j_*\mathcal{O}_U$ has an $\mathcal{O}_X$-algebra structure.

Corollary 4.22. *Let X be a locally Noetherian scheme admitting a dualizing complex $\mathcal{R}^\bullet$, and set $\mathcal{L} := \mathcal{H}^0(\mathcal{R}^\bullet)$. Let $X^\circ \subset X$ be an open subset such that*

$$\operatorname{codim}(X \setminus X^\circ, X) \geq 1 \quad \textit{and} \quad \mathcal{R}^\bullet|_{X^\circ} \simeq_{\text{qis}} \mathcal{L}|_{X^\circ}.$$

Then $\mathcal{R}^\bullet$ is ordinary and $\mathcal{L}$ satisfies $\mathbf{S}_2$. In particular, if $\operatorname{codim}(X \setminus X^\circ, X) \geq 2$, *then $\mathcal{L} \simeq j_*(\mathcal{L}|_{X^\circ})$ for the open immersion $j\colon X^\circ \hookrightarrow X$.*

Proof. It is enough to prove that $\mathcal{R}^\bullet$ is ordinary. In fact, if so, then the dualizing sheaf $\mathcal{L}$ satisfies $\mathbf{S}_2$ by Proposition 4.21, and we have the isomorphism $\mathcal{L} \simeq j_*(\mathcal{L}|_{X^\circ})$ by Corollary 2.16 when $\operatorname{codim}(X \setminus X^\circ, X) \geq 2$. Let $d\colon X \to \mathbb{Z}$ be the codimension function associated with $\mathcal{R}^\bullet$. Then $d(x) = \dim \mathcal{O}_{X,x}$ for any $x \in X^\circ$ by Proposition 4.6(5) applied to $\mathcal{F} = \mathcal{O}_{X^\circ}$. For a point $x \in X \setminus X^\circ$, we have a generic point y of X such that $x \in \overline{\{y\}}$ and $\operatorname{codim}(\overline{\{x\}}, \overline{\{y\}}) = \dim \mathcal{O}_{X,x}$. Then $d(y) = 0$, since $y \in X^\circ$, and we have

$$d(x) = d(y) + \operatorname{codim}(\overline{\{x\}}, \overline{\{y\}}) = \dim \mathcal{O}_{X,x}.$$

Thus, $\mathcal{R}^\bullet$ is ordinary. □

§4.3. Twisted inverse image

We shall explain the twisted inverse image functor, the relative duality theorem, and some base change theorems referring to [17], [7], [35]. Let $f\colon Y \to T$ be a morphism of locally Noetherian schemes which is locally of finite type. In the theory of Grothendieck duality, the "twisted inverse image functor" $f^!$ plays an essential role, which is unfortunately defined only when some suitable conditions are satisfied (cf. [17, III, Thm. 8.7; VII, Cor. 3.4], [9, no. 4] [44], [7], [35]). However, $f^!\mathcal{O}_T$ has a unique meaning at least locally on Y, where $f^!\mathcal{O}_T$ is expressed as a complex of $\mathcal{O}_Y$-modules with coherent cohomology which vanish in sufficiently negative degree, i.e., $f^!\mathcal{O}_T \in \mathbf{D}^+_{\mathrm{coh}}(Y)$. We write $\omega^\bullet_{Y/T} := f^!\mathcal{O}_T$ whenever $f^!\mathcal{O}_T$ is defined, and call it the *relative dualizing complex for* Y/T (or, with respect to f). When $T = \operatorname{Spec} A$, we write $\omega^\bullet_{Y/A}$ for $\omega^\bullet_{Y/\operatorname{Spec} A}$.

Example 4.23. For a scheme S, an S-morphism $f\colon Y \to T$ of locally Noetherian schemes over S is called an S-*embeddable morphism* if $f = p \circ i$ for a finite morphism $i\colon Y \to P \times_S T$ and the second projection $p\colon P \times_S T \to T$ for a locally Noetherian S-scheme P such that $P \to S$ is a smooth separated morphism of pure relative dimension (cf. [7, (2.8.1)], [17, III, p. 189]). When $S = T$, an S-embeddable morphism is called simply an *embeddable morphism*. There is a theory of $f^!\colon \mathbf{D}^+_{\mathrm{qcoh}}(T) \to \mathbf{D}^+_{\mathrm{qcoh}}(Y)$ (resp. $f^!\colon \mathbf{D}^+_{\mathrm{coh}}(T) \to \mathbf{D}^+_{\mathrm{coh}}(Y)$) for the S-embeddable morphisms $f\colon Y \to T$ of locally Noetherian S-schemes as in [17, III, Thm. 8.7] (cf. [7, Thm. 2.8.1]). For a complex $\mathcal{G}^\bullet \in \mathbf{D}^+_{\mathrm{qcoh}}(T)$, if f is separated and smooth of pure relative dimension d (cf. Definition 2.36), then

$$f^!(\mathcal{G}^\bullet) = \Omega^d_{Y/T}[d] \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathbf{L}f^*(\mathcal{G}^\bullet),$$

and if f is a finite morphism, then $f^!(\mathcal{G}^\bullet)$ is defined by

$$\mathbf{R}f_*(f^!(\mathcal{G}^\bullet)) = \mathbf{R}\mathcal{H}om_{\mathcal{O}_T}(f_*\mathcal{O}_Y, \mathcal{G}^\bullet).$$

In both cases, $f^!(\mathcal{G}^\bullet) \in \mathbf{D}^+_{\mathrm{coh}}(Y)$ if $\mathcal{G}^\bullet \in \mathbf{D}^+_{\mathrm{coh}}(T)$. If $f = g \circ h$ for two S-embeddable morphisms $h\colon Y \to Z$ and $g\colon Z \to T$, then $f^! \simeq h^! \circ g^!$ as functors $\mathbf{D}^+_{\mathrm{qcoh}}(T) \to \mathbf{D}^+_{\mathrm{qcoh}}(Y)$ (resp. $\mathbf{D}^+_{\mathrm{coh}}(T) \to \mathbf{D}^+_{\mathrm{coh}}(Y)$).

Example 4.24. Let $f\colon Y \to T$ be a morphism of finite type between Noetherian schemes. Then the dimensions of fibers are bounded. Assume that T admits a dualizing complex $\mathcal{R}^\bullet_T$. In this situation, we have the twisted inverse image functor $f^!\colon \mathbf{D}^+_{\mathrm{coh}}(T) \to \mathbf{D}^+_{\mathrm{coh}}(Y)$ as follows (cf. [17, VI], [7, §3]). For the dualizing complex $\mathcal{R}^\bullet_T$ of T, we have the corresponding *residual complex* $E(\mathcal{R}^\bullet_T)$ on T (cf. [17, VI, Prop. 1.1], [7, Lem. 3.2.1]) and the "twisted inverse image" $f^\triangle(E(\mathcal{R}^\bullet_T))$ on Y as a residual complex on Y (cf. [17, VI, Thm. 3.1, Cor. 3.5], [7, §3.2]), which corresponds to a dualizing complex

$$\mathcal{R}^{\bullet}_{Y} := f^{!}(\mathcal{R}^{\bullet}_{T}) := Q(f^{\triangle}(E(\mathcal{R}^{\bullet}_{T})))$$

of Y (cf. [17, VI, Prop. 1.1, Remarks in p. 306], [7, §3.3]). Then one can define $f^{!}\colon \mathbf{D}^{+}_{\mathrm{coh}}(T) \to \mathbf{D}^{+}_{\mathrm{coh}}(Y)$ by

$$f^{!}(\mathcal{G}^{\bullet}) = \mathfrak{D}_{Y}(\mathbf{L}f^{*}(\mathfrak{D}_{T}(\mathcal{G}^{\bullet}))),$$

where $\mathfrak{D}_Y$ and $\mathfrak{D}_T$ are the dualizing functors defined by

$$\mathfrak{D}_{Y}(\mathcal{F}^{\bullet}) := \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}^{\bullet}, \mathcal{R}^{\bullet}_{Y}) \quad \text{and} \quad \mathfrak{D}_{T}(\mathcal{G}^{\bullet}) := \mathbf{R}\mathcal{H}om_{\mathcal{O}_T}(\mathcal{G}^{\bullet}, \mathcal{R}^{\bullet}_{T}).$$

The definition of $f^{!}$ does not depend on the choice of $\mathcal{R}^{\bullet}_{T}$ (cf. [7, §3.3]), and $f^{!}$ satisfies expected compatible properties in [17, VII, Cor. 3.4] (cf. [7, Thm. 3.3.1]). Moreover, when f is an embeddable morphism, this $f^{!}$ is isomorphic to the functor $f^{!}$ defined in Example 4.23 (cf. [17, VI, Thm. 3.1; VII, Cor. 3.4], [7, §3.3]).

The following is shown in [17, V, Cor. 8.4; VI, Prop. 3.4] but with an error concerning $\pm$ (cf. [7, (3.1.25), (3.2.4)]).

Lemma 4.25. *Let $f\colon Y \to T$ be a morphism of finite type between Noetherian schemes such that T admits a dualizing complex $\mathcal{R}^{\bullet}_{T}$. Let $\mathcal{R}^{\bullet}_{Y}$ be the induced dualizing complex $f^{!}(\mathcal{R}^{\bullet}_{T})$ of Y. Let $d_T\colon T \to \mathbb{Z}$ and $d_Y\colon Y \to \mathbb{Z}$ be the codimension functions associated with $\mathcal{R}^{\bullet}_{T}$ and $\mathcal{R}^{\bullet}_{Y}$, respectively. Then*

$$d_Y(y) = d_T(t) - \operatorname{tr.deg} \Bbbk(y)/\Bbbk(t)$$

for any $y \in Y$ with $t = f(y)$, where $\Bbbk(t)$ and $\Bbbk(y)$ denote the residue fields of $\mathcal{O}_{T,t}$ and $\mathcal{O}_{Y,y}$, respectively.

Proof. Since the assertion is local on Y, we may assume that $Y \to T$ is an embeddable morphism. Hence, it is enough to prove assuming that f is a finite morphism or a smooth and separated morphism. Assume first that f is finite. Then

$$\mathbf{R}f_{*}\,\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{R}^{\bullet}_{Y}) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_T}(f_{*}\mathcal{F}, \mathcal{R}^{\bullet}_{T})$$

for any coherent $\mathcal{O}_Y$-module $\mathcal{F}$ by [17, III, Thm. 6.7] (cf. Theorem 4.30 below). Applying this to $\mathcal{F} = \mathcal{O}_Z$ for the closed subscheme $Z = \overline{\{y\}}$ with reduced structure, and localizing Y, we have

$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_{Y,y}}(\Bbbk(y), (\mathcal{R}^{\bullet}_{Y})_{y}) \simeq_{\mathrm{qis}} \mathbf{R}\mathrm{Hom}_{\mathcal{O}_{T,t}}(\Bbbk(y), (\mathcal{R}^{\bullet}_{T})_{t}).$$

Since $\Bbbk(y)$ is a finite-dimensional $\Bbbk(t)$-vector space, we have $\operatorname{tr.deg} \Bbbk(y)/\Bbbk(t) = 0$ and $d_Y(y) = d_T(t)$. Thus, we are done in the case where f is finite. Assume next that f is smooth and separated. We may assume furthermore that f has pure relative dimension d by localizing Y. Then

$$\mathcal{R}_Y^\bullet \simeq_{\mathrm{qis}} \Omega^d_{Y/T}[d] \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathbf{L}f^*(\mathcal{R}_T^\bullet)$$

as in Example 4.23, and it implies that

$$\mathcal{H}^i(\mathcal{R}_Y^\bullet)_y \simeq \mathcal{H}^{i+d}(\mathcal{R}_T^\bullet)_t \otimes_{\mathcal{O}_{T,t}} \mathcal{O}_{Y,y},$$

since f is flat. Here $\mathcal{H}^i(\mathcal{R}_Y^\bullet)_y \neq 0$ if and only if $\mathcal{H}^{i+d}(\mathcal{R}_T^\bullet)_t \neq 0$, since f is faithfully flat. We know that

$$d_T(t) - \dim \mathcal{O}_{T,t} = \inf\{i \mid \mathcal{H}^i(\mathcal{R}_T^\bullet)_t \neq 0\}$$

by (1) and (5) of Proposition 4.6. A similar formula holds also for (Y, y) and $\mathcal{R}_Y^\bullet$. Thus,

$$d_Y(y) - \dim \mathcal{O}_{Y,y} = d_T(t) - \dim \mathcal{O}_{T,t} - d.$$

Since f is flat, we have

$$\dim \mathcal{O}_{Y,y} = \dim \mathcal{O}_{T,t} + \dim \mathcal{O}_{Y_t,y}$$

for the fiber $Y_t = f^{-1}(t)$ by (II-1). Furthermore, we have

$$d = \dim_y Y_t = \dim \mathcal{O}_{Y_t,y} + \operatorname{tr.deg} \Bbbk(y)/\Bbbk(t),$$

since Y_t is algebraic over $\Bbbk(t)$ (cf. [12, IV, Cor. (5.2.3)]). Therefore,

$$d_Y(y) = d_T(t) - d + \dim \mathcal{O}_{Y,y} - \dim \mathcal{O}_{T,t} = d_T(t) - \operatorname{tr.deg} \Bbbk(y)/\Bbbk(t).$$

□

Definition 4.26 (Canonical dualizing complex). Let X be an algebraic scheme over a field $\Bbbk$, i.e., a $\Bbbk$-scheme of finite type. We define the *canonical dualizing complex* $\omega^\bullet_{X/\Bbbk}$ of X to be the twisted inverse image $f^!(\Bbbk)$ for the structure morphism $f\colon X \to \operatorname{Spec} \Bbbk$.

The dualizing complex $\omega^\bullet_{X/\Bbbk}$ has the following property related to Serre's conditions $\mathbf{S}_k$.

Lemma 4.27. *Let X be an n-dimensional algebraic scheme over a field $\Bbbk$. For an integer i, let Z_i be the support of $\mathcal{H}^{-i}(\omega^\bullet_{X/\Bbbk})$. Then $Z_i = \emptyset$ for any $i > n$, and Z_n is the union of irreducible components of X of dimension n. If X is equi-dimensional, then $\omega^\bullet_{X/\Bbbk}[-n]$ is an ordinary dualizing complex, and the following holds for integers $k \geq 1$: X satisfies $\mathbf{S}_k$ if and only if* $\dim Z_i \leq i - k$ *for any $i \neq n$.*

Proof. By Lemma 4.25, $d(x) = -\operatorname{tr.deg} \Bbbk(x)/\Bbbk$ for the codimension function $d\colon X \to \mathbb{Z}$ associated with the dualizing complex $\omega^{\bullet}_{X/\Bbbk}$ (cf. Example 4.24). Moreover,

$$n \geq \dim_x X = \dim \mathcal{O}_{X,x} + \operatorname{tr.deg} \Bbbk(x)/\Bbbk \tag{IV-4}$$

by [12, IV, Cor. (5.2.3)]. Thus, $d(x) - \dim \mathcal{O}_{X,x} = -\dim_x X \geq -n$, and $\mathcal{H}^{-i}(\omega^{\bullet}_{X/\Bbbk}) = 0$ for any $i > n$ by Proposition 4.6(1) applied to the case where $(\mathcal{R}^{\bullet}, \mathcal{F}) = (\omega^{\bullet}_{X/\Bbbk}, \mathcal{O}_X)$. Thus, $Z_i = \emptyset$ for any $i > n$. If $\dim_x X = n$, then $\mathcal{H}^{-n}(\omega^{\bullet}_{X/\Bbbk})_x \neq 0$ by Proposition 4.6(5). If $\dim_x X < n$, then $\mathcal{H}^{-n}(\omega^{\bullet}_{X/\Bbbk})_x = 0$ by Proposition 4.6(1). Therefore, Z_n is just the union of irreducible components of X of dimension n.

Assume that X is equi-dimensional, i.e., $\dim_x X = n$ for any $x \in X$. Then $\omega^{\bullet}_{X/\Bbbk}[-n]$ is an ordinary dualizing complex, since the associated codimension function is $x \mapsto d(x) + n = \dim \mathcal{O}_{X,x}$. Moreover, X is equi-codimensional, since $n = \dim_z X = \dim \mathcal{O}_{X,z}$ for any closed point z of X by (IV-4). Thus, the assertion on $\mathbf{S}_k$ is a consequence of Corollary 4.9(2), since $d(x) - \dim \mathcal{O}_{X,x} = -n$ for any $x \in X$. □

Definition 4.28 (Canonical sheaf). Let X be an algebraic scheme over a field $\Bbbk$. Assume that X is locally equi-dimensional. This is satisfied for example when X satisfies $\mathbf{S}_2$ (cf. Fact 2.23(1)). Then we define the canonical sheaf $\omega_{X/\Bbbk}$ by

$$\omega_{X/\Bbbk}|_{X_\alpha} := \mathcal{H}^{-\dim X_\alpha}(\omega^{\bullet}_{X/\Bbbk})|_{X_\alpha}$$

for any connected component X_α of X.

Remark. The canonical sheaf $\omega_{X/\Bbbk}$ is a dualizing sheaf of X in the sense of Definition 4.13. In fact,

$$\omega^{\bullet}_{X_\alpha/\Bbbk}[-\dim X_\alpha] = \omega^{\bullet}_{X/\Bbbk}[-\dim X_\alpha]|_{X_\alpha}$$

is an ordinary dualizing complex of the connected component X_α by Lemma 4.27. In particular, if X is connected and Cohen–Macaulay, then $\omega^{\bullet}_{X/\Bbbk} \simeq_{\mathrm{qis}} \omega_{X/\Bbbk}[\dim X]$.

By Corollary 4.22, we have the following.

Corollary 4.29. *For an algebraic scheme X over $\Bbbk$, if it is locally equi-dimensional, then $\omega_{X/\Bbbk}$ satisfies $\mathbf{S}_2$.*

For a proper morphism $f\colon Y \to T$ of Noetherian schemes, we have the following general result on the twisted inverse image functor $f^!$, which is derived from [35, Thm. 4.1.1].

Theorem 4.30 (Grothendieck duality for a proper morphism). *Let $f\colon Y \to T$ be a proper morphism of Noetherian schemes. Then there is a triangulated functor*

$f^!\colon \mathbf{D}_{\text{qcoh}}(T) \to \mathbf{D}_{\text{qcoh}}(Y)$ which induces $\mathbf{D}^+_{\text{coh}}(T) \to \mathbf{D}^+_{\text{coh}}(Y)$ and which is right adjoint to the derived functor $\mathbf{R}f_*\colon \mathbf{D}_{\text{qcoh}}(Y) \to \mathbf{D}_{\text{qcoh}}(T)$ in the sense that there is a functorial isomorphism

$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_T}(\mathbf{R}f_*(\mathcal{F}^\bullet), \mathcal{G}^\bullet) \simeq_{\text{qis}} \mathbf{R}\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}^\bullet, f^!(\mathcal{G}^\bullet))$$

for $\mathcal{F}^\bullet \in \mathbf{D}_{\text{qcoh}}(Y)$ and $\mathcal{G}^\bullet \in \mathbf{D}_{\text{qcoh}}(T)$.

Remark. In [35, Thm. 4.1.1], the existence of a similar right adjoint $f^\times$ is proved for a quasi-compact and quasi-separated morphism $f\colon Y \to T$ of quasi-compact and quasi-separated schemes Y and T. When f is proper, it is written as $f^!$ (cf. the paragraph just before [35, Cor. 4.2.2]). By [54, Thm. A], the total derived functor $\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}$ of $\mathrm{Hom}_{\mathcal{O}_X}$ exists for any scheme X as a bi-functor $\mathbf{D}(X)^{\text{op}} \times \mathbf{D}(X) \to \mathbf{D}(\mathbb{Z})$, and there exists also the total right derived functor $\mathbf{R}f_*\colon \mathbf{D}(Y) \to \mathbf{D}(T)$ of the direct image functor f_*. When $f\colon Y \to T$ is a proper morphism of Noetherian schemes, we have

- $\mathbf{R}f_*(\mathbf{D}_{\text{qcoh}}(Y)) \subset \mathbf{D}_{\text{qcoh}}(T)$ by [35, Prop. 3.9.1],
- $\mathbf{R}f_*(\mathbf{D}^+_{\text{coh}}(Y)) \subset \mathbf{D}^+_{\text{coh}}(T)$ by [17, II, Prop. 2.2], and
- $\mathbf{R}f_*(\mathbf{D}^-_{\text{coh}}(Y)) \subset \mathbf{D}^-_{\text{coh}}(T)$ by the explanation just before [35, Cor. 4.2.2].

The functor $f^!$ is bounded below (cf. [35, Def. 11.1.1]). Thus, $f^!(\mathbf{D}^+_{\text{qcoh}}(T)) \subset \mathbf{D}^+_{\text{qcoh}}(Y)$. The inclusion $f^!(\mathbf{D}^+_{\text{coh}}(T)) \subset \mathbf{D}^+_{\text{coh}}(Y)$ is proved by first reducing to the case where T is the spectrum of a Noetherian local ring by the base change isomorphism (cf. [35, Cor. 4.4.3]), and second applying [57, Lem. 1].

Remark. When T admits a dualizing complex (or a residual complex), Theorem 4.30 for $\mathcal{G}^\bullet \in \mathbf{D}^+_{\text{coh}}(T)$ is a consequence of [17, VII, Cor. 3.4]. In [9, Thm. 2], Deligne proved Theorem 4.30 for $\mathcal{F}^\bullet \in \mathbf{D}^b_{\text{coh}}(Y)$ without assuming the existence of a dualizing complex of T. These results are summarized by Verdier as [57, Thm. 1], which is almost the same as Theorem 4.30 in the case where T has finite Krull dimension. Neeman [44] gives a new idea toward the proof of Theorem 4.30 by using Brown representability. He generalizes Theorem 4.30 to the case where Y and T are only quasi-compact and separated schemes but $\mathbf{D}_{\text{qcoh}}(T)$ and $\mathbf{D}_{\text{qcoh}}(Y)$ are replaced with $\mathbf{D}(\mathsf{QCoh}(\mathcal{O}_T))$ and $\mathbf{D}(\mathsf{QCoh}(\mathcal{O}_Y))$, respectively (cf. [44, Exam. 4.2]). Neeman's idea is used in Lipman's article [35], which contains generalizations of Theorem 4.30 to non-proper and non-Noetherian cases.

The sheafified form of the duality theorem is as follows (cf. [35, Thm. 4.2]).

Corollary 4.31. *In the situation of Theorem 4.30, there exists a canonical quasi-isomorphism*

$$\mathbf{R}f_* \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet) \simeq_{\mathrm{qis}} \mathbf{R}\mathcal{H}om_{\mathcal{O}_T}(\mathbf{R}f_*\mathcal{F}^\bullet, \mathcal{G}^\bullet)$$

for any $\mathcal{F}^\bullet \in \mathbf{D}_{\mathrm{qcoh}}(Y)$ *and* $\mathcal{G}^\bullet \in \mathbf{D}_{\mathrm{qcoh}}(T)$.

As a special case of Theorem 4.30, we have the following, which is called the Serre duality theorem for coherent sheaves.

Corollary 4.32. *Let X be a projective scheme over a field $\Bbbk$. Then there is a canonical quasi-isomorphism*

$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}^\bullet, \omega^\bullet_{X/\Bbbk}) \simeq_{\mathrm{qis}} \mathbf{R}\mathrm{Hom}_{\Bbbk}(\mathbf{R}\Gamma(X, \mathcal{F}^\bullet), \Bbbk)$$

for any $\mathcal{F}^\bullet \in \mathbf{D}^+_{\mathrm{coh}}(X)$*. In particular,*

$$\mathbb{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{F}^\bullet, \omega^\bullet_{X/\Bbbk}) \simeq \mathrm{Hom}_{\Bbbk}(\mathbb{H}^i(X, \mathcal{F}^\bullet), \Bbbk)$$

for any i, where $\mathbb{E}\mathrm{xt}^i$ and $\mathbb{H}^i$ stand for the i-th hyper-Ext group and i-th hyper cohomology group, respectively.

Example 4.33. Let $f\colon Y \to T$ be a separated morphism of finite type between Noetherian schemes. By the Nagata compactification theorem (cf. [42], [43], [36], [8], [11]), f is expressed as the composite $\pi \circ j$ of an open immersion $j\colon Y \hookrightarrow Z$ and a proper morphism $\pi\colon Z \to T$. Using the functor $\pi^!\colon \mathbf{D}^+_{\mathrm{qcoh}}(T) \to \mathbf{D}^+_{\mathrm{qcoh}}(Z)$ in Theorem 4.30, we define the twisted inverse image functor $f^!\colon \mathbf{D}^+_{\mathrm{qcoh}}(T) \to \mathbf{D}^+_{\mathrm{qcoh}}(Y)$ as $\mathbf{L}j^* \circ \pi^!$. This is well defined up to functorial isomorphism, i.e., it is independent of the choice of factorization $f = \pi \circ j$, by [9, Thm. 2], [57, Cor. 1]. Deligne [9] defines a functor $\mathbf{R}f_!\colon \operatorname{pro}\mathbf{D}^b_{\mathrm{coh}}(Y) \to \operatorname{pro}\mathbf{D}^b_{\mathrm{coh}}(T)$ and shows in [9, Thm. 2] that $f^!$ above is a right adjoint of $\mathbf{R}f_!$.

Fact 4.34. The twisted inverse image functors in Example 4.33 have the following properties. Let $f\colon Y \to T$ be a separated morphism of finite type between Noetherian schemes.

(1) *Let $h\colon X \to Y$ be a separated morphism of finite type from another Noetherian scheme X. Then there is a functorial isomorphism $(f \circ h)^! \simeq h^! \circ f^!$.*
(2) *If $f\colon Y \to T$ is a smooth morphism of pure relative dimension d, then $f^!(\mathcal{G}^\bullet) \simeq_{\mathrm{qis}} \Omega^d_{Y/T}[d] \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathbf{L}f^*(\mathcal{G}^\bullet)$.*
(3) *If T admits a dualizing complex, then $f^!$ is functorially isomorphic to the twisted inverse image functor $\mathfrak{D}_Y \circ \mathbf{L}f^* \circ \mathfrak{D}_T$ in Example 4.24.*

(4) *For a flat morphism $g\colon T' \to T$ from a Noetherian scheme T', let Y' be the fiber product $Y \times_T T'$ and let $f'\colon Y' \to T'$ and $g'\colon Y' \to Y$ be the induced morphisms. Then $g'^* \circ f^! \simeq f'^! \circ g^*$.*

Property (1) is derived from the isomorphism $\mathbf{R}(f \circ h)_! \simeq \mathbf{R}f_! \circ \mathbf{R}h_!$ shown in [9, no. 3]. This is also proved in [35, Thm. 4.8.1]. Properties (2) and (3) are proved in [57, Thm. 3, Cor. 3] and [35, (4.9.4.2), Prop. 4.10.1]. In order to prove property (4), we may assume that f is proper, and in this case, this is shown in [35, Cor. 4.4.3] (cf. [57, Thm. 2]). As a refinement of property (1), $f \mapsto f^!$ can be regarded as a pseudo-functor, and Lipman proves in [35, Thm. 4.8.1] the uniqueness of $f \mapsto f^!$ under three conditions corresponding to

- $f^!$ is right adjoint to $\mathbf{R}f_*$ when f is proper (Theorem 4.30);
- property (2) for étale f;
- property (4) for proper f and étale g.

Fact 4.35. The following are also known for a flat separated morphism $f\colon Y \to T$ of finite type between Noetherian schemes:

(1) The twisted inverse image $f^!\mathcal{O}_T$ is an f*-perfect* complex in $\mathbf{D}_{\mathrm{coh}}(Y)$ (cf. [23, III, Prop. 4.9], [35, Thm. 4.9.4]). For the definition of "f-perfect", see [23, III, Déf. 4.1] (cf. Remark 4.36 below). Note that a coherent $\mathcal{O}_Y$-module flat over T is f-perfect.

(2) For an f-perfect complex $\mathcal{E}^\bullet$,

$$\mathfrak{D}_{Y/T}(\mathcal{E}^\bullet) := \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{E}^\bullet, f^!\mathcal{O}_T)$$

is also f-perfect and the canonical morphism

$$\mathcal{E}^\bullet \to \mathfrak{D}_{Y/T}(\mathfrak{D}_{Y/T}(\mathcal{E}^\bullet))$$

is a quasi-isomorphism (cf. [23, III, Cor. 4.9.2]). In particular,

$$\mathcal{O}_Y \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(f^!\mathcal{O}_T, f^!\mathcal{O}_T) \tag{IV-5}$$

is a quasi-isomorphism (cf. [35, p. 234]).

(3) There is a quasi-isomorphism

$$f^!(\mathcal{F}^\bullet) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathbf{L}f^*(\mathcal{G}^\bullet) \simeq_{\mathrm{qis}} f^!(\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_T} \mathcal{G}^\bullet)$$

for any $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathbf{D}^+_{\mathrm{qcoh}}(T)$ with $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_T} \mathcal{G}^\bullet \in \mathbf{D}^+_{\mathrm{qcoh}}(T)$ (cf. [35, Thm. 4.9.4]). In particular,

$$f^!\mathcal{O}_T \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathbf{L}f^*(\mathcal{G}^\bullet) \simeq_{\mathrm{qis}} f^!(\mathcal{G}^\bullet) \tag{IV-6}$$

for any $\mathcal{G}^\bullet \in \mathbf{D}^+_{\mathrm{qcoh}}(T)$. Similar results are proved in [17, V, Cor. 8.6], [57, Cor. 2], and [44, Thm. 5.4].

Remark 4.36 (Cf. [23, III, Prop. 4.4]). Let $f\colon Y \to T$ be a morphism of finite type between Noetherian schemes and let $\mathcal{F}^\bullet$ be an object of $\mathbf{D}_{\mathrm{qcoh}}(Y)$. Assume that f is the composite $g \circ i$ of a closed immersion $i\colon Y \to P$ and a smooth separated morphism $g\colon P \to T$. Then $\mathcal{F}^\bullet$ is f-perfect if and only if $\mathbf{R}i_*(\mathcal{F}^\bullet)$ is perfect (cf. [23, I, Déf. 4.7]), i.e., locally on P, it is quasi-isomorphic to a bounded complex of free $\mathcal{O}_P$-modules.

Lemma 4.37. *Let $f\colon Y \to T$ be a flat separated morphism of finite type between Noetherian schemes in which T admits a dualizing complex. Let $g\colon T' \to T$ be a finite morphism from another Noetherian scheme T'. For the fiber product $Y' = Y \times_T T'$, let $f\colon Y' \to T'$ and $g'\colon Y' \to Y$ be the projections. Thus, we have a Cartesian diagram:*

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{g} & T. \end{array}$$

In this situation, there is a natural quasi-isomorphism

$$\mathbf{L}g'^*(f^!\mathcal{O}_T) \simeq_{\mathrm{qis}} f'^!\mathcal{O}_{T'}.$$

Proof. Let $\mathfrak{D}_T$, $\mathfrak{D}_Y$, $\mathfrak{D}_{T'}$, and $\mathfrak{D}_{Y'}$, respectively, be the dualizing functors on T, Y, T', and Y' defined by a dualizing complex on T and their transforms by $f^!$, $g^!$, and $(f \circ g')^! \simeq (g \circ f')^!$ (cf. Fact 4.34(1)) as in Example 4.24. For any $\mathcal{G}^\bullet \in \mathbf{D}^+_{\mathrm{coh}}(T')$, we have

$$\begin{aligned} f^!(\mathbf{R}g_*(\mathcal{G}^\bullet)) &\simeq_{\mathrm{qis}} \mathfrak{D}_Y \circ \mathbf{L}f^* \circ \mathfrak{D}_T(\mathbf{R}g_*(\mathcal{G}^\bullet)) \simeq_{\mathrm{qis}} \mathfrak{D}_Y \circ \mathbf{L}f^* \circ \mathbf{R}g_*(\mathfrak{D}_{T'}(\mathcal{G}^\bullet)) \\ &\simeq_{\mathrm{qis}} \mathfrak{D}_Y \circ \mathbf{R}g'_* \circ \mathbf{L}f'^*(\mathfrak{D}_{T'}(\mathcal{G}^\bullet)) \simeq_{\mathrm{qis}} \mathbf{R}g'_* \circ \mathfrak{D}_{Y'}(\mathbf{L}f'^*(\mathfrak{D}_{T'}(\mathcal{G}^\bullet))) \\ &\simeq_{\mathrm{qis}} \mathbf{R}g'_*(f'^!(\mathcal{G}^\bullet)), \end{aligned}$$

where we use the flat base change isomorphism $\mathbf{L}f^* \circ \mathbf{R}g_* \simeq_{\mathrm{qis}} \mathbf{R}g'_* \circ \mathbf{L}f'^*$ (cf. Proposition A.10), and the duality isomorphisms $\mathfrak{D}_T \circ \mathbf{R}g_* \simeq_{\mathrm{qis}} \mathbf{R}g_* \circ \mathfrak{D}_{T'}$ and $\mathfrak{D}_Y \circ \mathbf{R}g'_* \simeq_{\mathrm{qis}} \mathbf{R}g'_* \circ \mathfrak{D}_{Y'}$ for the finite morphisms g and g' (cf. Corollary 4.31). On the other hand, we have

$$f^!(\mathbf{R}g_*(\mathcal{G}^\bullet)) \simeq_{\mathrm{qis}} f^!\mathcal{O}_T \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathbf{L}f^*(\mathbf{R}g_*\mathcal{G}^\bullet) \simeq_{\mathrm{qis}} f^!\mathcal{O}_T \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathbf{R}g'_*(\mathbf{L}f'^*(\mathcal{G}^\bullet))$$

by the quasi-isomorphism (IV-6) in Fact 4.35 and by the flat base change isomorphism. Substituting $\mathcal{G}^\bullet = \mathcal{O}_{T'}$, we have a quasi-isomorphism

$$f^!\mathcal{O}_T \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathbf{R}g'_*\mathcal{O}_{Y'} \simeq_{\mathrm{qis}} \mathbf{R}g'_*(f'^!\mathcal{O}_{T'}).$$

It is associated with a morphism $\mathbf{L}g'^*(f^!\mathcal{O}_T) \to f'^!\mathcal{O}_{T'}$ in $\mathbf{D}^+_{\mathrm{coh}}(Y')$ which induces a quasi-isomorphism by taking $\mathbf{R}g'_*$. Hence, $\mathbf{L}g'^*(f^!\mathcal{O}_T) \simeq_{\mathrm{qis}} f'^!\mathcal{O}_{T'}$. □

Corollary 4.38 (Cf. [45, Prop. 3.3(1)]). *Let $f\colon Y \to T$ be a flat separated morphism of finite type between Noetherian schemes. For a point $t \in T$, let $\phi_t\colon \operatorname{Spec}\Bbbk(t) \to T$ be the canonical morphism for the residue field $\Bbbk(t)$, and let $\psi_t\colon Y_t = f^{-1}(t) \to Y$ be the base change of ϕ_t by $f\colon Y \to T$. Then the canonical dualizing complex $\omega^\bullet_{Y_t/\Bbbk(t)}$ defined in Definition 4.26 is quasi-isomorphic to $\mathbf{L}\psi_t^*(f^!\mathcal{O}_T)$.*

Proof. Let $\operatorname{Spec}\mathcal{O}_{T,t} \to T$ be the canonical morphism from the spectrum of the local ring $\mathcal{O}_{T,t}$. Considering the completion $\widehat{\mathcal{O}}_{T,t}$ of $\mathcal{O}_{T,t}$ and the surjection $\widehat{\mathcal{O}}_{T,t} \to \Bbbk(t)$ to the residue field, we have a flat morphism

$$\tau\colon T^\flat := \operatorname{Spec}\widehat{\mathcal{O}}_{T,t} \to \operatorname{Spec}\mathcal{O}_{T,t} \to T$$

and a closed immersion $\iota\colon \operatorname{Spec}\Bbbk(t) \hookrightarrow T^\flat$. Let $Y^\flat$ be the fiber product $Y \times_T T^\flat$ and let $f^\flat\colon Y^\flat \to T^\flat$ and $\tau'\colon Y^\flat \to Y$ be projections, which make a Cartesian diagram:

$$\begin{array}{ccc} Y^\flat & \xrightarrow{\tau'} & Y \\ {\scriptstyle f^\flat}\downarrow & & \downarrow{\scriptstyle f} \\ T^\flat & \xrightarrow{\tau} & T. \end{array}$$

By Fact 4.34(4), we have a quasi-isomorphism

$$\mathbf{L}\tau'^*(f^!\mathcal{O}_T) \simeq_{\mathrm{qis}} f^{\flat!}\mathcal{O}_{T^\flat}.$$

Hence, we may assume from the beginning that $T = T^\flat$. Then ϕ_t is the closed immersion ι. Now, T admits a dualizing complex, since we have a surjection to $\widehat{\mathcal{O}}_{T,t}$ from a complete regular local ring by Cohen's structure theorem. Thus, we are done by Lemma 4.37. □

§4.4. Cohen–Macaulay morphisms and Gorenstein morphisms

The notions of a Cohen–Macaulay morphism and a Gorenstein morphism are introduced in [12, IV, Déf. (6.8.1)] and [17, V, Ex. 9.7]. By [7, Sect. 3.5] or [51, Thm. 2.2.3], one can define the relative dualizing sheaf for a Cohen–Macaulay morphism (cf. Definition 4.43 below), and prove a base change property (cf. Theorem 4.46 below). We shall explain these facts.

We defined the notion of a Cohen–Macaulay morphism in Definition 2.30. The notion of a Gorenstein morphism is defined as follows.

Definition 4.39 (Gor(Y/T)). Let Y and T be locally Noetherian schemes and $f\colon Y \to T$ a flat morphism locally of finite type. We define

$$\mathrm{Gor}(Y/T) := \bigcup\nolimits_{t \in T} \mathrm{Gor}(Y_t),$$

and call it the *relative Gorenstein locus for* f. The flat morphism f is called a *Gorenstein morphism* if $\mathrm{Gor}(Y/T) = Y$.

Remark. The Gorenstein locus $\mathrm{Gor}(Y/T)$ is open. In fact, this is characterized as the maximal open subset of the relative Cohen–Macaulay locus $Y^\flat = \mathrm{CM}(Y/T)$ on which the relative dualizing sheaf $\omega_{Y^\flat/T}$ is invertible (cf. Lemma 4.40 below), where $Y^\flat$ is open by Fact 2.29(1).

The following characterizations of a Cohen–Macaulay morphism and a Gorenstein morphism are known.

Lemma 4.40 ([17, V, Exer. 9.7], [7, Thm. 3.5.1]). *Let $f\colon Y \to T$ be a flat morphism locally of finite type between locally Noetherian schemes. Then f is Cohen–Macaulay if and only if, locally on Y, the twisted inverse image $f^!\mathcal{O}_T$ is quasi-isomorphic to an f-flat coherent $\mathcal{O}_Y$-module $\omega_{Y/T}$ up to shift. Here f is Gorenstein if and only if $\omega_{Y/T}$ is invertible.*

Proof. We may assume that f is a separated morphism of finite type between affine Noetherian schemes by localizing Y and T. Assume first that $f^!\mathcal{O}_T \simeq_{\mathrm{qis}} \omega_{Y/T}[d]$ for a coherent $\mathcal{O}_Y$-module $\omega_{Y/T}$ flat over T and for an integer d. For an arbitrary fiber Y_t, the dualizing complex $\omega^\bullet_{Y_t/\Bbbk(t)}$ is quasi-isomorphic to $\omega_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t}[d]$ by Corollary 4.38. Hence, Y_t is Cohen–Macaulay by Corollary 4.7(2) or Lemma 4.16(4).

Conversely, assume that every fiber Y_t is Cohen–Macaulay. Then we may assume that f has pure relative dimension d by Lemma 2.38. We shall show that

$$f^!\mathcal{O}_T \simeq_{\mathrm{qis}} \omega_{Y/T}[d]$$

for the cohomology sheaf $\omega_{Y/T} := \mathcal{H}^{-d}(f^!\mathcal{O}_T)$ and that $\omega_{Y/T}$ is flat over T. For a point $t \in T$ and the inclusion morphism $\psi_t\colon Y_t \to Y$, we have a quasi-isomorphism

$$\mathbf{L}\psi_t^*(f^!\mathcal{O}_T) \simeq_{\mathrm{qis}} \omega_{Y_t/\Bbbk(t)}[d] \tag{IV-7}$$

for the canonical sheaf $\omega_{Y_t/\Bbbk(t)}$ by Corollary 4.38. Now, $f^!\mathcal{O}_T$ belongs to $\mathbf{D}^-_{\mathrm{coh}}(\mathcal{O}_Y)$. In fact, $f^!\mathcal{O}_T$ is f-perfect by Fact 4.35(1). For the stalk $(f^!\mathcal{O}_T)_y$ at a point $y \in Y_t$, we have

$$(f^!\mathcal{O}_T)_y[-d] \otimes^{\mathbf{L}}_{\mathcal{O}_{T,t}} \Bbbk(t) \simeq_{\mathrm{qis}} (\omega_{Y_t/\Bbbk(t)})_y$$

by (IV-7). By applying Lemma 4.41 below to $(f^!\mathcal{O}_T)_y[-d]$ and $\mathcal{O}_{T,t} \to \mathcal{O}_{Y,y}$, we see that $\mathcal{H}^i(f^!\mathcal{O}_T)_y = 0$ for any $i \neq -d$ and $\mathcal{H}^{-d}(f^!\mathcal{O}_T)_y$ is a flat $\mathcal{O}_{T,t}$-module with an isomorphism

$$\mathcal{H}^{-d}(f^!\mathcal{O}_T)_y \otimes_{\mathcal{O}_{T,t}} \Bbbk(t) \simeq (\omega_{Y_t/\Bbbk(t)})_y.$$

Since these hold for an arbitrary point $y \in Y$, there is a quasi-isomorphism $f^!\mathcal{O}_T \simeq_{\mathrm{qis}} \omega_{Y/T}[d]$ and $\omega_{Y/T}$ is flat over T. Therefore, we have proved the first assertion on a characterization of a Cohen–Macaulay morphism. For the second assertion, we assume that f is a Cohen–Macaulay morphism. Then $\omega_{Y/T}$ is flat over T. Thus, $\omega_{Y/T}$ is invertible along a fiber Y_t if and only if $\omega_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t}$ is invertible (cf. Fact 2.26(2)). By the isomorphism $\omega_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq \omega_{Y_t/\Bbbk(t)}$, we see that Y_t is Gorenstein if and only if $\omega_{Y/T}$ is invertible along Y_t. Thus, the second assertion follows, and we are done. □

The following is used in the proof of Lemma 4.40 above.

Lemma 4.41. *Let A be a Noetherian local ring with residue field $\Bbbk$ and let $A \to B$ be a local ring homomorphism to another Noetherian local ring B. Let $L^\bullet$ be a complex of B-modules such that $\mathrm{H}^l(L^\bullet) = 0$ for $l \gg 0$ and $\mathrm{H}^i(L^\bullet)$ is a finitely generated B-module for any $i \in \mathbb{Z}$, i.e., $L^\bullet \in \mathbf{D}^-_{\mathrm{coh}}(B)$. Assume that*

$$\mathrm{H}^i(L^\bullet \otimes^{\mathbf{L}}_A \Bbbk) = 0$$

for any $i > 0$. Then $\mathrm{H}^i(L^\bullet) = 0$ for any $i > 0$, and there exist an isomorphism

$$\mathrm{H}^0(L^\bullet) \otimes_A \Bbbk \simeq \mathrm{H}^0(L^\bullet \otimes^{\mathbf{L}}_A \Bbbk)$$

and an exact sequence

$$\mathrm{Tor}_2^A(\mathrm{H}^0(L^\bullet), \Bbbk) \to \mathrm{H}^{-1}(L^\bullet) \otimes_A \Bbbk \xrightarrow{h} \mathrm{H}^{-1}(L^\bullet \otimes^{\mathbf{L}}_A \Bbbk) \to \mathrm{Tor}_1^A(\mathrm{H}^0(L^\bullet), \Bbbk) \to 0.$$

Consequently, the following hold:

(1) *$\mathrm{H}^0(L^\bullet)$ is flat over A if and only if the homomorphism h above is surjective;*
(2) *If $\mathrm{H}^i(L^\bullet \otimes^{\mathbf{L}}_A \Bbbk) = 0$ for any $i \neq 0$, then $L^\bullet$ is quasi-isomorphic to a flat A-module.*

Proof. There is a standard spectral sequence

$$E_2^{p,q} = \mathrm{Tor}^A_{-p}(\mathrm{H}^q(L^\bullet), \Bbbk) \Rightarrow E^{p+q} = \mathrm{H}^{p+q}(L^\bullet \otimes^{\mathbf{L}}_A \Bbbk)$$

(cf. [12, III, (6.3.2.2)]), where $E_2^{p,q} = 0$ for any $p > 0$. Let a be an integer such that $\mathrm{H}^l(L^\bullet) = 0$ for any $l > a$. Then $E_2^{p,q} = 0$ for any $q > a$, and we have $E^a \simeq E_2^{0,a}$

and an exact sequence

$$E_2^{-2,a} \to E_2^{0,a-1} \to E^{a-1} \to E_2^{-1,a} \to 0.$$

Hence, if $a > 0$, then $\mathrm{H}^a(L^\bullet) = 0$ by $E_2^{0,a} = 0$, and we may decrease a by 1. Thus, we can choose $a = 0$, and we have the required isomorphism and exact sequence. Assertion (1) is derived from the local criterion of flatness (cf. Proposition A.1), since $\mathrm{H}^0(L^\bullet)$ is flat over A if and only if $\mathrm{Tor}_1^A(\mathrm{H}^0(L^\bullet), \Bbbk) = 0$. Assertion (2) follows from (1) and $\tau^{\le -1}(L^\bullet) \simeq_{\mathrm{qis}} 0$, the latter of which is obtained by applying the result above to the complex $\tau^{\le -1}(L^\bullet)$ instead of $L^\bullet$. □

Fact 4.42. Let $f\colon Y \to T$ be a Cohen–Macaulay morphism having pure relative dimension d (cf. Definition 2.36). In [7, Sect. 3.5], Conrad defines a sheaf ω_f, called the *dualizing sheaf* for f, on Y such that

$$\omega_f|_U \simeq \mathcal{H}^{-d}((f|_U)^!\mathcal{O}_T)$$

for any open subset $U \subset Y$ such that the restriction $f|_U\colon U \to T$ factors as a closed immersion $U \hookrightarrow P$ followed by a smooth separated morphism $P \to T$ with pure relative dimension. Here the sheaf ω_f is obtained by gluing the sheaves $\mathcal{H}^{-d}((f|_U)^!\mathcal{O}_T)$ along natural isomorphisms, where the compatibility of gluing is checked by explicit calculation of Ext groups. In [51, Thms. 2.3.3, 2.3.5], Sastry defines the same sheaf ω_f by another method: this is obtained by gluing $\mathcal{H}^{-d}((f|_V)^!\mathcal{O}_T)$ for open subsets $V \subset Y$ such that $f|_V$ factors as an open immersion $V \hookrightarrow \overline{V}$ followed by a d-proper morphism $\overline{V} \to T$ in the sense of [51, Def. 2.2.1].

Definition 4.43 (Relative dualizing sheaf). Let $f\colon Y \to T$ be a Cohen–Macaulay morphism. For any connected component Y_α of Y, it is shown in Lemma 2.38 that the restriction morphism $f_\alpha = f|_{Y_\alpha}\colon Y_\alpha \to T$ has pure relative dimension. Thus, one can consider the dualizing sheaf ω_{f_α} in Fact 4.42 for f_α. We define the *relative dualizing sheaf* $\omega_{Y/T}$ of Y over T by

$$\omega_{Y/T}|_{Y_\alpha} = \omega_{f_\alpha}$$

for any connected component Y_α. The $\omega_{Y/T}$ is also called the *relative dualizing sheaf* for f or the *relative canonical sheaf* of Y over T (cf. Definition 5.3 below). We sometimes write ω_f for $\omega_{Y/T}$.

By Corollary 4.7(2) and Lemma 4.40, we have the following.

Corollary 4.44. *For a Cohen–Macaulay morphism $f\colon Y \to T$, the relative dualizing sheaf $\omega_{Y/T}$ is relatively Cohen–Macaulay over T (cf. Definition 2.28) and* $\operatorname{Supp}\omega_{Y/T} = Y$.

By Lemma 2.33(5), we also have the following.

Corollary 4.45. *For a Cohen–Macaulay morphism* $f\colon Y \to T$*, let* Y° *be an open subset of* Y *such that* $\operatorname{codim}(Y_t \setminus Y^{\circ}, Y_t) \geq 2$ *for any fiber* $Y_t = f^{-1}(t)$*. Then* $\omega_{Y/T} \simeq j_*(\omega_{Y^{\circ}/T})$ *for the open immersion* $j\colon Y^{\circ} \hookrightarrow Y$*.*

The following base change property is known for the relative dualizing sheaves (cf. [7, Thm. 3.6.1], [26, Prop. (9)], [51, Thm. 2.3.5]).

Theorem 4.46. *Let* $f\colon Y \to T$ *be a Cohen–Macaulay morphism. For an arbitrary morphism* $T' \to T$ *from a locally Noetherian scheme* T'*, let* Y' *be the fiber product* $Y \times_T T'$ *and let* $p\colon Y' \to Y$ *be the projection. Then* $p^*(\omega_{Y/T}) \simeq \omega_{Y'/T'}$*.*

Remark. Conrad [7, Thm. 3.6.1] and Sastry [51, Thm. 2.3.5] prove Theorem 4.46 assuming that f has pure relative dimension, but it is enough for the proof, since the restriction of f to any connected component of Y has pure relative dimension (cf. Lemma 2.38). When f is proper, Theorem 4.46 is shown by Kleiman [26, Prop. (9)(iii)], whose proof uses another version of the twisted inverse image functor $f^!$. The proof of [7, Thm. 3.6.1] is based on arguments in [17, V], while the proof of [51, Thm. 2.3.5] is based on arguments in [9], [57], [26], and [35].

§5. Relative canonical sheaves

As a generalization of the relative dualizing sheaf for a Cohen–Macaulay morphism, we introduce the notion of *relative canonical sheaf* for an arbitrary $\mathbf{S}_2$*-morphism* (cf. Definition 2.30). We give some base change properties of the relative canonical sheaf and its “multiple”. These are used for studying ℚ-Gorenstein morphisms in Section 7. In Section 5.1 we shall study the relative canonical sheaf and the conditions for it to satisfy relative $\mathbf{S}_2$. Section 5.2 is devoted to proving Theorem 5.10 which shows that a certain sheaf related to the relative canonical sheaf is invertible. This provides sufficient conditions for the base change homomorphism of the relative canonical sheaf to the fiber to be an isomorphism.

§5.1. Relative canonical sheaf for an $\mathbf{S}_2$-morphism

First of all, we shall give a partial generalization of the notion of canonical sheaf in Definition 4.28 as follows.

Definition 5.1. Let X be a $\Bbbk$-scheme locally of finite type for a field $\Bbbk$. Assume that

- X is locally equi-dimensional and
- $\operatorname{codim}(X \setminus X^{\flat}, X) \geq 2$ for the Cohen–Macaulay locus $X^{\flat} = \operatorname{CM}(X)$.

Note that this assumption is verified when X satisfies $\mathbf{S}_2$. For the relative dualizing sheaf $\omega_{X^\flat/\Bbbk}$ over $\operatorname{Spec}\Bbbk$ in Definition 4.43 and for the open immersion $j^\flat\colon X^\flat \hookrightarrow X$, we set

$$\omega_{X/\Bbbk} := j^\flat_*(\omega_{X^\flat/\Bbbk})$$

and call it the *canonical sheaf* of X.

Remark. By Corollaries 4.22 and 4.29, we have the following properties in the situation of Definition 5.1:

(1) Let U be an arbitrary open subset of X which is of finite type over $\operatorname{Spec}\Bbbk$. Then $\omega_{X/\Bbbk}|_U$ is isomorphic to the canonical sheaf $\omega_{U/\Bbbk}$ defined in Definition 4.28. Thus, the use of the same symbol $\omega_{X/\Bbbk}$ for the canonical sheaf causes no confusion.
(2) The canonical sheaf $\omega_{X/\Bbbk}$ is coherent and satisfies $\mathbf{S}_2$.

Lemma 5.2. *Let X be a scheme locally of finite type over a field $\Bbbk$. Assume that X is Gorenstein in codimension one and satisfies $\mathbf{S}_2$. Then $\omega_{X/\Bbbk}$ is reflexive, and every reflexive $\mathcal{O}_X$-module satisfies $\mathbf{S}_2$. In particular, the double-dual $\omega^{[m]}_{X/\Bbbk}$ of $\omega^{\otimes m}_{X/\Bbbk}$ satisfies $\mathbf{S}_2$ for any $m \in \mathbb{Z}$.*

Proof. Let Z be the complement of the Gorenstein locus of X (cf. Definition 4.10). Then $\operatorname{codim}(Z, X) \geq 2$ and $\omega_{X/\Bbbk}|_{X\setminus Z}$ is invertible. Hence, $\omega_{X/\Bbbk}$ is reflexive by Corollary 2.22, since $\omega_{X/\Bbbk}$ satisfies $\mathbf{S}_2$ and $\operatorname{Supp}\omega_{X/\Bbbk} = X$. Every reflexive $\mathcal{O}_X$-module satisfies $\mathbf{S}_2$ by Lemma 2.21(2). □

Remark. Lemma 5.2 is derived also from [21, Prop. 3.5, Cor. 3.7].

The definition of the canonical sheaf above is partially extended to the relative situation as follows.

Definition 5.3 (Relative canonical sheaf). Let $f\colon Y \to T$ be an $\mathbf{S}_2$-morphism of locally Noetherian schemes. Let $j\colon Y^\flat \hookrightarrow Y$ be the open immersion from the relative Cohen–Macaulay locus $Y^\flat = \operatorname{CM}(Y/T)$. Note that $\operatorname{codim}(Y_t \setminus Y^\flat, Y_t) \geq 3$ for any fiber $Y_t = f^{-1}(t)$, since Y_t satisfies $\mathbf{S}_2$. In this situation, we define

$$\omega_{Y/T} := j_*(\omega_{Y^\flat/T})$$

for the relative dualizing sheaf $\omega_{Y^\flat/T}$ for $f|_{Y^\flat}$ in the sense of Definition 4.43. We also call $\omega_{Y/T}$ the *relative canonical sheaf* of Y over T.

Lemma 5.4. *Let $f\colon Y \to T$ be an $\mathbf{S}_2$-morphism of locally Noetherian schemes and let*

$$\begin{array}{ccc} Y' & \xrightarrow{p} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow \\ T' & \longrightarrow & T \end{array}$$

be a Cartesian diagram such that T' is a locally Noetherian scheme flat over T. Then $\omega_{Y'/T'} \simeq p^*(\omega_{Y/T})$.

Proof. Let $Y^\flat$ (resp. $Y'^\flat$) be the relative Cohen–Macaulay locus for f (resp. f') and let $j\colon Y^\flat \hookrightarrow Y$ (resp. $j'\colon Y'^\flat \hookrightarrow Y'$) be the open immersion. Then $Y'^\flat = p^{-1}(Y^\flat)$ by Lemma 2.31(3) for $\mathcal{F} = \mathcal{O}_Y$, and j' is induced from j. Let $p^\flat\colon Y'^\flat \to Y^\flat$ be the restriction of p. Then $\omega_{Y'^\flat/T'} \simeq p^{\flat *}(\omega_{Y^\flat/T})$ by Theorem 4.46. Thus, we have

$$\omega_{Y'/T'} \simeq j'_*(p^{\flat *}(\omega_{Y^\flat/T})) \simeq p^*(j_*(\omega_{Y^\flat/T})) \simeq p^*\omega_{Y/T}$$

by the flat base change isomorphism (cf. Lemma A.9) for the Cartesian diagram composed of p, $p^\flat$, j, and j'. □

Proposition 5.5. *Let $f\colon Y \to T$ be an $\mathbf{S}_2$-morphism of locally Noetherian schemes. Then the relative canonical sheaf $\omega_{Y/T}$ defined in Definition 5.3 is coherent, and moreover, if f is a separated morphism of pure relative dimension d, then*

$$\mathcal{H}^i(f^!\mathcal{O}_T) \simeq \begin{cases} 0 & \text{if } i < -d, \\ \omega_{Y/T} & \text{if } i = -d, \end{cases}$$

for the twisted inverse image $f^!\mathcal{O}_T$. Let Y° be an open subset of $\mathrm{CM}(Y/T)$ such that $\mathrm{codim}(Y_t \setminus Y^\circ, Y_t) \geq 2$ for any fiber $Y_t = f^{-1}(t)$. For a point $t \in T$, let

$$\phi_t\colon \omega_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \to \omega_{Y_t/\Bbbk(t)} = j_{t*}(\omega_{Y^\circ \cap Y_t/\Bbbk(t)})$$

be the homomorphism induced from the base change isomorphism

$$\omega_{Y^\circ/T} \otimes_{\mathcal{O}_{Y^\circ}} \mathcal{O}_{Y^\circ \cap Y_t} \simeq \omega_{Y^\circ \cap Y_t/\Bbbk(t)} \tag{V-1}$$

(cf. Theorem 4.46), where $j_t\colon Y^\circ \cap Y_t \hookrightarrow Y_t$ denotes the open immersion. Then, for any point $y \in Y$, the following three conditions are equivalent to each other:

(a) *$\phi_{f(y)}$ is surjective at y;*
(b) *$\phi_{f(y)}$ is an isomorphism at y;*
(c) *there is an open neighborhood U of y in Y such that $\omega_{Y/T}|_U$ satisfies* relative $\mathbf{S}_2$ over T (*cf. Definition* 2.28).

Proof. The coherence of $\omega_{Y/T}$ and conditions (a)–(c) are local on Y. Hence, we may assume that f is a separated morphism of pure relative dimension d by Lemma 2.38(1). Then we have the twisted inverse image $f^!\mathcal{O}_T$ with a quasi-isomorphism

$$(f^!\mathcal{O}_T)|_{Y^\flat} \simeq_{\mathrm{qis}} \omega_{Y^\flat/T}[d]$$

for $Y^\flat = \mathrm{CM}(Y/T)$ by Lemma 4.40, and we have a canonical homomorphism

$$\phi\colon \mathcal{H}^{-d}(f^!\mathcal{O}_T) \to j^\flat_*(\omega_{Y^\flat/T}) = \omega_{Y/T}$$

for the open immersion $j^\flat\colon Y^\flat \hookrightarrow Y$. In order to prove that ϕ is an isomorphism, since it is a local condition, we may replace Y with an open subset freely. Thus, we may assume that

- f is the composite $p \circ \iota$ of a closed immersion $\iota\colon Y \hookrightarrow P$ and a smooth affine morphism $p\colon P \to T$.

By Fact 4.35(1) and Remark 4.36, we know that $\mathbf{R}\iota_*(f^!\mathcal{O}_T)$ is perfect. Hence, by localizing Y, we may assume that

- $\mathbf{R}\iota_*(f^!\mathcal{O}_T)$ is quasi-isomorphic to a bounded complex $\mathcal{E}^\bullet = [\cdots \to \mathcal{E}^i \to \mathcal{E}^{i+1} \to \cdots]$ of free $\mathcal{O}_P$-modules of finite rank.

Then we have an isomorphism $\mathcal{H}^i(\mathcal{E}^\bullet) \simeq \iota_*\mathcal{H}^i(f^!\mathcal{O}_T)$ for any $i \in \mathbb{Z}$. Moreover, there exist quasi-isomorphisms

$$\begin{aligned}\mathcal{E}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_P} \mathcal{O}_{P_t} &\simeq_{\mathrm{qis}} \mathbf{R}\iota_*(f^!\mathcal{O}_T \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathbf{L}\iota^*\mathcal{O}_{P_t}) \simeq_{\mathrm{qis}} \mathbf{R}\iota_*(f^!\mathcal{O}_T \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathbf{L}f^*\Bbbk(t)) \\ &\simeq_{\mathrm{qis}} \mathbf{R}\iota_*(f^!\mathcal{O}_T \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{O}_{Y_t}) \simeq_{\mathrm{qis}} \mathbf{R}\iota_{t*}(\omega^\bullet_{Y_t/\Bbbk(t)})\end{aligned}$$

for any $t \in T$ and for the induced closed immersion $\iota_t\colon Y_t \hookrightarrow P_t = p^{-1}(t)$. In fact, the first quasi-isomorphism is known as the projection formula (cf. [17, II, Prop. 5.6]), the quasi-isomorphisms

$$\mathcal{O}_{P_t} \simeq_{\mathrm{qis}} \mathbf{L}p^*\Bbbk(t) \quad \text{and} \quad \mathbf{L}f^*\Bbbk(t) \simeq_{\mathrm{qis}} \mathcal{O}_{Y_t}$$

are derived from the flatness of p and f, and the quasi-isomorphism

$$f^!\mathcal{O}_T \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq_{\mathrm{qis}} \omega^\bullet_{Y_t/\Bbbk(t)}$$

is obtained by Corollary 4.38. We shall show that the three data

$$\mathcal{E}^\bullet[-d], \quad Z := \iota(Y \setminus Y^\circ), \quad \mathcal{F} := \mathcal{H}^0(\mathcal{E}^\bullet[-d]) \simeq \iota_*\mathcal{H}^{-d}(f^!\mathcal{O}_T)$$

satisfy the conditions of Lemma 3.15 for the morphism $P \to T$. The required inequality (III-8) of Lemma 3.15 is derived from

$$\operatorname{depth}_{P_t \cap Z} \mathcal{O}_{P_t} = \operatorname{codim}(P_t \cap Z, P_t) = \operatorname{codim}(Y_t \cap Z, P_t) \geq \operatorname{codim}(Y_t \cap Z, Y_t) \geq 2$$

(cf. Lemma 2.14). Condition (i) of Lemma 3.15 is derived from (cf. Lemma 4.40)

$$\mathcal{H}^i(\mathcal{E}^\bullet)|_{P\setminus Z} \simeq \iota_*(\mathcal{H}^i(f^!\mathcal{O}_T))|_{P\setminus Z} \simeq \begin{cases} 0 & \text{if } i \neq -d, \\ \iota_*\omega_{Y^\circ/T} & \text{if } i = -d, \end{cases}$$

and the next condition (ii) has no meaning now. Condition (iii) follows from

$$\mathcal{H}^i(\mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_P} \mathcal{O}_{P_t}) \simeq \iota_{t*}(\mathcal{H}^i(\omega^\bullet_{Y_t/\Bbbk(t)})) = 0$$

for any $i < -d$ (cf. Lemma 4.27). The last condition (iv) of Lemma 3.15 is a consequence of Corollary 4.20 applied to the ordinary dualizing complex $\omega^\bullet_{Y_t/\Bbbk(t)}[-d]$ (cf. Lemma 4.27) and to $b = 1$, since

- the complex $M^\bullet$ in Lemma 3.15(iv) is quasi-isomorphic to the stalk of

$$\tau^{\leq 1}(\mathbf{R}\iota_*\omega^\bullet_{Y_t/\Bbbk(t)}[-d]) \simeq_{\mathrm{qis}} \mathbf{R}\iota_*(\tau^{\leq 1}(\omega^\bullet_{Y_t/\Bbbk(t)}[-d])) \quad \text{and}$$

- $\dim \mathcal{O}_{P_t,z} \geq \operatorname{codim}(Z \cap Y_t, Y_t) \geq 2$ for any $z \in Z$ with $t = f(z)$.

Therefore, all the conditions of Lemma 3.15 are satisfied, and consequently,

$$\mathcal{H}^i(\mathcal{E}^\bullet) \simeq \iota_*\mathcal{H}^i(f^!\mathcal{O}_T) = 0$$

for any $i < -d$, and we can apply Proposition 3.7 to $\mathcal{F}$ via Lemma 3.15. Then $\mathcal{F} \simeq j_*(\mathcal{F}|_{P\setminus Z})$ for the open immersion $j\colon P\setminus Z \hookrightarrow P$ by Proposition 3.7(1), and it implies that the morphism ϕ above is an isomorphism. Moreover, the three conditions (a)–(c) are equivalent to each other by Proposition 3.7(3) and Corollary 3.9. Thus, we are done. □

Proposition 5.6. *Let $f\colon Y \to T$ be an $\mathbf{S}_2$-morphism of locally Noetherian schemes and let $j\colon Y^\circ \hookrightarrow Y$ be the open immersion from an open subset Y° of the relative Gorenstein locus* $\operatorname{Gor}(Y/T)$ *for f. Assume that*

- $\operatorname{codim}(Y_t \setminus Y^\circ, Y_t) \geq 2$ *for any fiber $Y_t = f^{-1}(t)$.*

For an integer m and for the relative canonical sheaf $\omega_{Y/T}$, let $\omega^{[m]}_{Y/T}$ denote the double-dual of $\omega^{\otimes m}_{Y/T}$. Then

$$\omega^{[m]}_{Y/T} \simeq j_*(\omega^{\otimes m}_{Y^\circ/T})$$

for any m. In particular, $\omega_{Y/T}$ is reflexive. For an integer m and a point $t \in T$, let

$$\phi^{[m]}_t\colon \omega^{[m]}_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \to \omega^{[m]}_{Y_t/\Bbbk(t)} = j_{t*}(\omega^{\otimes m}_{Y^\circ \cap Y_t/\Bbbk(t)})$$

be the homomorphism induced from the base change isomorphism (V-1), where $j_t\colon Y^\circ \cap Y_t \hookrightarrow Y_t$ denotes the open immersion. Then, for any integer m and any point $y \in Y$, the following three conditions are equivalent to each other:

(a) $\phi^{[m]}_{f(y)}$ *is surjective at* y*;*

(b) $\phi^{[m]}_{f(y)}$ *is an isomorphism at* y*;*

(c) *there is an open neighborhood* V *of* y *in* Y *such that* $\omega^{[m]}_{Y/T}|_V$ *satisfies relative* $\mathbf{S}_2$ *over* T*.*

Proof. We apply some results in Section 3.1 to the reflexive sheaf $\mathcal{F} = \omega^{[m]}_{Y/T}$ and the closed subset $Z := Y \setminus Y^\circ$. By assumption, $\mathcal{F}|_{Y\setminus Z}$ is invertible and $\operatorname{depth}_{Y_t\cap Z}\mathcal{O}_{Y_t} \geq 2$ (cf. Lemma 2.14(ii)). Thus, we can apply Lemma 3.14, and consequently, we can assume that $\mathcal{F}$ has an exact sequence of Proposition 3.7, by replacing Y with its open subset. Then $\omega^{[m]}_{Y/T} \simeq j_*(\omega^{\otimes m}_{Y^\circ/T})$ by Proposition 3.7(1). In the case $m = 1$, we have $\omega^{[1]}_{Y/T} \simeq \omega_{Y/T} \simeq j_*(\omega_{Y^\circ/T})$ by Corollary 4.45 and Definition 5.3, and as a consequence, $\omega_{Y/T}$ is reflexive. The equivalence of the three conditions (a)–(c) is derived from Proposition 3.7(3) and Corollary 3.9. □

Corollary 5.7. *Let us consider a Cartesian diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{p} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{q} & T \end{array}$$

of locally Noetherian schemes in which f *is a flat morphism locally of finite type. Then* $p^{-1}\operatorname{CM}(Y/T) = \operatorname{CM}(Y'/T')$ *and* $p^{-1}\operatorname{Gor}(Y/T) = \operatorname{Gor}(Y'/T')$*. Assume in addition that* f *is an* $\mathbf{S}_2$*-morphism.*

(1) *If* $\omega_{Y/T}$ *satisfies relative* $\mathbf{S}_2$ *over* T*, then* $p^*\omega_{Y/T} \simeq \omega_{Y'/T'}$*.*

(2) *If every fiber* $Y_t = f^{-1}(t)$ *is Gorenstein in codimension one, then for any* $m \in \mathbb{Z}$*, there is a canonical isomorphism*

$$(p^*\omega^{[m]}_{Y/T})^{\vee\vee} \simeq \omega^{[m]}_{Y'/T'}.$$

Here, if $\omega^{[m]}_{Y/T}$ *satisfies relative* $\mathbf{S}_2$ *over* T*, then* $p^*\omega^{[m]}_{Y/T} \simeq \omega^{[m]}_{Y'/T'}$*.*

Proof. The equality for CM is derived from Lemma 2.31(3) for $\mathcal{F} = \mathcal{O}_Y$. If f is a Cohen–Macaulay morphism, then $p^*\omega_{Y/T} \simeq \omega_{Y'/T'}$ by Theorem 4.46. This implies the equality for Gor by the remark of Definition 4.39. Assume that f is an $\mathbf{S}_2$-morphism. Then f' is so by Lemma 2.31(5). For open subsets $Y^\flat := \operatorname{CM}(Y/T)$ and $Y'^\flat := p^{-1}(Y^\flat)$, we have

$$\operatorname{codim}(Y'_{t'} \setminus Y'^\flat, Y'_{t'}) = \operatorname{codim}(Y_t \setminus Y^\flat, Y_t) \geq 3$$

for any $t' \in T'$ and $t = q(t)$ by Lemma 2.31(1) and by the $\mathbf{S}_2$-condition of Y_t. If $\omega_{Y/T}$ satisfies relative $\mathbf{S}_2$ over T, then the canonical base change isomorphism

$$p^*\omega_{Y^\flat/T} \simeq \omega_{Y'^\flat/T'} \tag{V-2}$$

in Theorem 4.46 induces an isomorphism

$$p^*\omega_{Y/T} \simeq j'_*(p^*\omega_{Y/T}|_{Y'^\flat}) \simeq j'_*\omega_{Y'^\flat/T'} = \omega_{Y'/T'}$$

for the open immersion $j' \colon Y'^\flat \hookrightarrow Y'$, by Lemma 2.32(2) applied to $(\mathcal{F}, Z) = (p^*\omega_{Y/T}, Y' \setminus Y'^\flat)$. This proves (1). In the situation of (2), $\operatorname{codim}(Y_t \setminus Y^\circ, Y_t) \geq 2$ for any $t \in T$, where $Y^\circ = \operatorname{Gor}(Y/T)$. In particular,

$$\operatorname{depth}_{Y_t \setminus Y^\circ} \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \geq 2$$

for any coherent $\mathcal{O}_Y$-module $\mathcal{F}$ satisfying relative $\mathbf{S}_2$ over T, by Lemma 2.15(2). Thus, (2) is a consequence of Lemma 2.34 via the isomorphism (V-2). □

Proposition 5.8. *Let $f \colon Y \to T$ be an $\mathbf{S}_2$-morphism of locally Noetherian schemes. Then*

$$\mathcal{H}om_{\mathcal{O}_Y}(\omega_{Y/T}, \omega_{Y/T}) \simeq \mathcal{O}_Y$$

for the relative canonical sheaf $\omega_{Y/T}$ in the sense of Definition 5.3. If every fiber satisfies $\mathbf{S}_3$, then

$$\mathcal{E}xt^1_{\mathcal{O}_Y}(\omega_{Y/T}, \omega_{Y/T}) = 0.$$

Proof. Let $j \colon Y^\flat \hookrightarrow Y$ be the open immersion from the relative Cohen–Macaulay locus $Y^\flat = \operatorname{CM}(Y/T)$. Now, we have a quasi-isomorphism

$$\mathcal{O}_{Y^\flat} \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_{Y^\flat}}(\omega_{Y^\flat/T}, \omega_{Y^\flat/T})$$

by (IV-5) in Fact 4.35(2). This induces another quasi-isomorphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\omega_{Y/T}, \mathbf{R}j_*(\omega_{Y^\flat/T})) \simeq_{\mathrm{qis}} \mathbf{R}j_*\, \mathbf{R}\mathcal{H}om_{\mathcal{O}_{Y^\flat}}(\omega_{Y^\flat/T}, \omega_{Y^\flat/T}) \simeq \mathbf{R}j_*\mathcal{O}_{Y^\flat}$$

and the spectral sequence

$$\mathcal{E}_2^{p,q} = \mathcal{E}xt^p_{\mathcal{O}_Y}(\omega_{Y/T}, R^q j_*(\omega_{Y^\flat/T})) \Rightarrow \mathcal{E}^{p+q} = R^{p+q}j_*\mathcal{O}_{Y^\flat}.$$

Since $\omega_{Y/T} = j_*(\omega_{Y^\flat/T})$, the isomorphism $\mathcal{E}_2^{0,0} \simeq \mathcal{E}^0$ and the injection $\mathcal{E}_2^{1,0} \hookrightarrow \mathcal{E}^1$, respectively, correspond to an isomorphism $\mathcal{H}om_{\mathcal{O}_Y}(\omega_{Y/T}, \omega_{Y/T}) \simeq j_*\mathcal{O}_{Y^\flat}$ and an injection $\mathcal{E}xt^1_{\mathcal{O}_Y}(\omega_{Y/T}, \omega_{Y/T}) \hookrightarrow R^1j_*\mathcal{O}_{Y^\flat}$. Therefore, it suffices to prove that

(1) $\mathcal{O}_Y \simeq j_*\mathcal{O}_{Y^\flat}$ and
(2) if every fiber satisfies $\mathbf{S}_3$, then $R^1j_*\mathcal{O}_{Y^\flat} = 0$.

Here (1) (resp. (1) with the conclusion of (2)) is equivalent to $\operatorname{depth}_Z \mathcal{O}_Y \geq 2$ (resp. ≥ 3) for $Z := Y \setminus Y^\flat$ (cf. Property 2.6). If a fiber Y_t satisfies $\mathbf{S}_k$, then $\operatorname{codim}(Z \cap Y_t, Y_t) > k$, and $\operatorname{depth}_{Z\cap Y_t} \mathcal{O}_{Y_t} \geq k$ by Lemma 2.15(2). Hence, we have $\operatorname{depth}_Z \mathcal{O}_Y \geq 2$ (resp. ≥ 3) by Lemma 2.32(3) when every fiber Y_t satisfies $\mathbf{S}_2$ (resp. $\mathbf{S}_3$). Thus, we are done. □

§5.2. Some base change theorems for the relative canonical sheaf

For an $\mathbf{S}_2$-morphism $f\colon Y \to T$ of locally Noetherian schemes and for a fiber $Y_t = f^{-1}(t)$, let

$$\phi_t(\omega_{Y/T})\colon \omega_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \to \omega_{Y_t/\Bbbk(t)} = j^\flat_*(\omega_{Y^\flat_t/\Bbbk(t)})$$

be the canonical homomorphism induced from the base change isomorphism

$$\omega_{Y^\flat/T} \otimes_{\mathcal{O}_{Y^\flat}} \mathcal{O}_{Y^\flat_t} \simeq \omega_{Y^\flat_t/\Bbbk(t)}$$

(cf. Theorem 4.46), where $Y^\flat = \operatorname{CM}(Y/T)$, $Y^\flat_t = Y^\flat \cap Y_t$ and $j^\flat$ is the open immersion $Y^\flat \hookrightarrow Y$. The homomorphism $\phi_t(\omega_{Y/T})$ is not necessarily an isomorphism (e.g., Fact 7.7 below). We shall give a sufficient condition for $\phi_t(\omega_{Y/T})$ to be an isomorphism in Theorem 5.10 below.

Lemma 5.9. *Let $f\colon Y \to T$ be a Cohen–Macaulay morphism of locally Noetherian schemes. Let $\mathcal{L}$ be a coherent $\mathcal{O}_Y$-module flat over T with an isomorphism*

$$\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq \omega_{Y_t/\Bbbk(t)} \tag{V-3}$$

for the fiber $Y_t = f^{-1}(t)$ over a given point $t \in T$. Then, for the sheaf $\mathcal{M} := \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}, \omega_{Y/T})$, the canonical homomorphism $\mathcal{L} \otimes \mathcal{M} \to \omega_{Y/T}$ is an isomorphism along Y_t, and $\mathcal{M}$ is an invertible sheaf along Y_t with an isomorphism $\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq \mathcal{O}_{Y_t}$.

Proof. Since the assertions are local on Y_t, we may assume that

(1) f has pure relative dimension d (cf. Lemma 2.38) and
(2) f is the composite $p \circ \iota$ of a closed immersion $\iota\colon Y \hookrightarrow P$ and a smooth affine morphism $p\colon P \to T$ of pure relative dimension e.

Then $f^!\mathcal{O}_T \simeq \omega_{Y/T}[d]$ and $\omega_{Y/T}$ is flat over T by Lemma 4.40. The complex $\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}, f^!\mathcal{O}_T)$ is f-perfect by Fact 4.35(2), and there is a quasi-isomorphism

$$\mathbf{R}\iota_*\, \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}, f^!\mathcal{O}_T) \simeq_{\mathrm{qis}} \mathbf{R}\mathcal{H}om_{\mathcal{O}_P}(\iota_*\mathcal{L}, \omega_{P/T}[e])$$

by Corollary 4.31, where $p^!\mathcal{O}_T = \omega_{P/T}[e]$ by (2) above. Localizing Y, by Remark 4.36, we may assume furthermore that

(3) $\mathbf{R}\iota_* \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}, f^!\mathcal{O}_T)$ is quasi-isomorphic to a bounded complex $\mathcal{E}^\bullet = [\cdots \to \mathcal{E}^i \to \mathcal{E}^{i+1} \to \cdots]$ of free $\mathcal{O}_P$-modules of finite rank.

Note that we have an isomorphism

$$\mathcal{H}^{-d}(\mathcal{E}^\bullet) \simeq \iota_* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}, \omega_{Y/T}) \simeq \iota_*\mathcal{M}.$$

For the closed immersion $\iota\colon Y \hookrightarrow P$ and the induced closed immersion $\iota_t\colon Y_t \hookrightarrow P_t = p^{-1}(t)$, we have quasi-isomorphisms

$$\begin{aligned}\mathcal{E}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_P} \mathcal{O}_{P_t} &\simeq_{\text{qis}} \mathbf{R}\mathcal{H}om_{\mathcal{O}_{P_t}}((\iota_*\mathcal{L}) \otimes^{\mathbf{L}}_{\mathcal{O}_P} \mathcal{O}_{P_t}, \omega_{P/T}[e] \otimes^{\mathbf{L}}_{\mathcal{O}_P} \mathcal{O}_{P_t}) \\ &\simeq_{\text{qis}} \mathbf{R}\mathcal{H}om_{\mathcal{O}_{P_t}}(\iota_{t*}(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t}), \omega_{P_t/\Bbbk(t)}[e])\end{aligned}$$

by [23, I, Prop. 7.1.2], since $\mathcal{L}$ is flat over T, $\iota_*\mathcal{L}$ is perfect (cf. Fact 4.35(1) and Remark 4.36), and since $P \to T$ is smooth. From the isomorphism (V-3) and the base change isomorphism

$$\phi_t(\omega_{Y/T})\colon \omega_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq \omega_{Y_t/\Bbbk(t)}$$

(cf. Theorem 4.46), by duality for ι_t (cf. Corollary 4.31) we have quasi-isomorphisms

$$\begin{aligned}\mathcal{E}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_T} \Bbbk(t) \simeq_{\text{qis}} \mathcal{E}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_P} \mathcal{O}_{P_t} &\simeq_{\text{qis}} \mathbf{R}\iota_{t*} \mathbf{R}\mathcal{H}om_{\mathcal{O}_{Y_t}}(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t}, \omega_{Y_t/\Bbbk(t)}[d]) \\ &\simeq_{\text{qis}} \mathbf{R}\iota_{t*} \mathbf{R}\mathcal{H}om_{\mathcal{O}_{Y_t}}(\omega_{Y_t/\Bbbk(t)}, \omega_{Y_t/\Bbbk(t)}[d]) \simeq_{\text{qis}} \iota_{t*}\mathcal{O}_{Y_t}[d],\end{aligned}$$

where the last quasi-isomorphism follows from the fact that $\omega_{Y_t/\Bbbk(t)}[d]$ is a dualizing complex of Y_t. Then, by Lemma 4.41, we see that

(4) $\mathcal{E}^\bullet[-d]$ is quasi-isomorphic to $\mathcal{H}^{-d}(\mathcal{E}^\bullet) \simeq \iota_*\mathcal{M}$ along Y_t,

(5) $\iota_*\mathcal{M}$ is flat over T along Y_t, and

(6) there is an isomorphism

$$\iota_*\mathcal{M} \otimes_{\mathcal{O}_P} \mathcal{O}_{P_t} \simeq \mathcal{H}^{-d}(\mathcal{E}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_T} \Bbbk(t)) \simeq \iota_{t*}\mathcal{O}_{Y_t}.$$

Hence, $\mathcal{M}$ is flat over T along Y_t with an isomorphism $\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq \mathcal{O}_{Y_t}$ by (5) and (6). As a consequence, $\mathcal{M}$ is an invertible $\mathcal{O}_Y$-module along Y_t by Fact 2.26(2). Now, we have a quasi-isomorphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}, \omega_{Y/T}) \simeq_{\text{qis}} \mathcal{M}$$

along Y_t by (3) and (4). By the duality quasi-isomorphism

$$\mathcal{L} \simeq_{\text{qis}} \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}, \omega_{Y/T}), \omega_{Y/T})$$

(cf. Fact 4.35(2)), we have an isomorphism

$$\mathcal{L} \simeq \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, \omega_{Y/T}) \simeq \omega_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{M}^{-1}$$

along Y_t, since $\mathcal{M}$ is invertible along Y_t. Thus, we are done. □

Theorem 5.10. *For an* $\mathbf{S}_2$*-morphism* $f\colon Y \to T$ *of locally Noetherian schemes, let* $\mathcal{L}$ *be a coherent* $\mathcal{O}_Y$*-module and set* $\mathcal{M} := \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}, \omega_{Y/T})$*. For an open subset* U *of* Y *and for the fiber* $Y_t = f^{-1}(t)$ *over a given point* $t \in T$*, assume that*

(i) $\operatorname{codim}(Y_t \setminus U, Y_t) \geq 2$,

(ii) $\mathcal{L}$ *is flat over* T *with an isomorphism* $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq \omega_{Y_t/\Bbbk(t)}$*, and*

(iii) *one of the following two conditions is satisfied:*

  (a) Y_t *satisfies* $\mathbf{S}_3$ *and* $\operatorname{codim}(Y_t \setminus U, Y_t) \geq 3$*;*

  (b) *there is a positive integer* r *coprime to the characteristic of* $\Bbbk(t)$ *such that* $\mathcal{L}^{[r]} = (\mathcal{L}^{\otimes r})^{\vee\vee}$ *and* $\omega^{[r]}_{Y/T} = (\omega^{\otimes r}_{Y/T})^{\vee\vee}$ *are invertible* $\mathcal{O}_Y$*-module along* Y_t*.*

Then $\mathcal{M}$ *is an invertible* $\mathcal{O}_Y$*-module along* Y_t *with an isomorphism* $\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq \mathcal{O}_{Y_t}$*, and the canonical homomorphism* $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{M} \to \omega_{Y/T}$ *is an isomorphism along* Y_t*. Moreover, the "base change homomorphism"*

$$\phi_t(\omega_{Y/T})\colon \omega_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \to \omega_{Y_t/\Bbbk(t)}$$

is an isomorphism.

Proof. Since the assertions are local on Y_t, we may replace Y with an open subset freely. Let $Y^\flat$ be the relative Cohen–Macaulay locus $\operatorname{CM}(Y/T)$, which is an open subset by Fact 2.29(1). Then $\operatorname{codim}(Y_t \setminus Y^\flat, Y_t) \geq 3$ (resp. ≥ 4 in case (a)), since Y_t satisfies $\mathbf{S}_2$ (resp. $\mathbf{S}_3$). We set $U^\flat := U \cap Y^\flat$. Then

(V-4) $$\operatorname{codim}(Y_t \setminus U^\flat, Y_t) = \operatorname{codim}((Y_t \setminus U) \cup (Y_t \setminus Y^\flat), Y_t) \geq 2 \quad (\text{resp. } \geq 3).$$

By Lemma 5.9 applied to the Cohen–Macaulay morphism $U^\flat \to T$, there is an isomorphism

(1) $\mathcal{M}|_{U^\flat} \otimes_{\mathcal{O}_{U^\flat}} \mathcal{O}_{U^\flat \cap Y_t} \simeq \mathcal{O}_{U^\flat \cap Y_t}$,

and there is an open neighborhood U' of $U^\flat \cap Y_t$ in $U^\flat$ such that

(2) $\mathcal{M}|_{U'}$ is an invertible sheaf, and

(3) the canonical homomorphism $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{M} \to \omega_{Y/T}$ is an isomorphism on U'.

We set $Z = Y \setminus U'$. Then $\operatorname{codim}(Y_t \cap Z, Y_t) = \operatorname{codim}(Y_t \setminus U^\flat, Y_t) \geq 2$ by (V-4). Since f is an $\mathbf{S}_2$-morphism, by Lemma 2.38, we may assume that $\operatorname{codim}(Y_{t'} \cap Z, Y_{t'}) \geq 2$ for any $t' \in T$ by replacing Y with an open subset. Then $\operatorname{depth}_Z \mathcal{O}_Y \geq 2$ by Lemma 2.32(3), and

$$\omega_{Y/T} \simeq j_*(\omega_{U/T}) \simeq j'_*(\omega_{U'/T})$$

for the open immersion $j' \colon U' \hookrightarrow Y$ by Corollary 4.45. In particular, $\operatorname{depth}_Z \mathcal{M} \geq 2$, i.e., $\mathcal{M} \simeq j'_*(\mathcal{M}|_{U'})$, by the isomorphism

$$\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}, \omega_{Y/T}) \simeq \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}, j'_*(\omega_{U'/T})) \simeq j'_* \mathcal{H}om_{\mathcal{O}_{U'}}(\mathcal{L}|_{U'}, \omega_{U'/T}).$$

By (ii), $\mathcal{L}$ satisfies relative $\mathbf{S}_2$ over T along Y_t, since $\omega_{Y_t/\Bbbk(t)}$ satisfies $\mathbf{S}_2$ by Corollary 4.29. Hence, we have also an isomorphism $\mathcal{L} \simeq j'_*(\mathcal{L}|_{U'})$ by Lemma 2.33(5).

We shall show that $\mathcal{M}$ is invertible along Y_t by applying Theorem 3.16 to $Y \to T$, the closed subset $Z = Y \setminus U'$, and to the sheaf $\mathcal{M}$ as $\mathcal{F}$. By the previous argument, we have checked conditions (i) and (ii) of Theorem 3.16. Condition (iii) is derived from (1): in fact, we have

$$\mathcal{M}_{(t)*} = j'_*((\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t})|_{U' \cap Y_t}) \simeq j'_*(\mathcal{O}_{U' \cap Y_t}) \simeq \mathcal{O}_{Y_t}, \tag{V-5}$$

since we have $\operatorname{depth}_{Y_t \cap Z} \mathcal{O}_{Y_t} \geq 2$ by the $\mathbf{S}_2$-condition on Y_t and by $\operatorname{codim}(Y_t \cap Z, Y_t) \geq 2$ (cf. Lemma 2.14). Similarly, in the situation of (a), we can verify condition (a) of Theorem 3.16 by the $\mathbf{S}_3$-condition on Y_t and by $\operatorname{codim}(Y_t \cap Z, Y_t) = \operatorname{codim}(Y_t \setminus U^\flat, Y_t) \geq 3$ (cf. (V-4)). In the situation of (b), $\mathcal{M}^{[r]}$ is an invertible $\mathcal{O}_Y$-module along Y_t. In fact, the restriction homomorphisms

$$\mathcal{M}^{[r]} \to j'_*(\mathcal{M}^{[r]}|_{U'}) \quad \text{and} \quad \omega^{[r]}_{Y/T} \to j'_*(\omega^{[r]}_{U'/T})$$

are isomorphisms by Lemma 2.33(4), since $\mathcal{M}^{[r]}$ and $\omega^{[r]}_{Y/T}$ are reflexive, and the isomorphism

$$\mathcal{L}^{[r]}|_{U'} \otimes_{\mathcal{O}_{U'}} \mathcal{M}^{[r]}|_{U'} \simeq \omega^{[r]}_{U'/T}$$

obtained by (2) and (3) induces an isomorphism

$$\begin{aligned} \mathcal{M}^{[r]} \simeq j'_*(\mathcal{M}^{[r]}|_{U'}) &\simeq j'_* \mathcal{H}om_{\mathcal{O}_{U'}}(\mathcal{L}^{[r]}|_{U'}, \omega^{[r]}_{U'/T}) \\ &\simeq \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}^{[r]}, j'_*(\omega^{[r]}_{U'/T})) \simeq \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}^{[r]}, \omega^{[r]}_{Y/T}). \end{aligned}$$

Thus, condition (b) of Theorem 3.16 is also satisfied in the situation of (b). Hence, we can apply Theorem 3.16, and as a result, we see that $\mathcal{M}$ is an invertible sheaf along Y_t.

Then we have an isomorphism $\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq \mathcal{O}_{Y_t}$ by (V-5), and the canonical homomorphism $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{M} \to \omega_{Y/T}$ is an isomorphism along Y_t by (3). In fact, it is expressed as the composite

$$\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{M} \simeq j'_*(\mathcal{L}|_{U'}) \otimes_{\mathcal{O}_Y} \mathcal{M} \to j'_*(\mathcal{L}|_{U'} \otimes \mathcal{M}|_{U'}) \simeq j'_*(\omega_{U'/T}) \simeq \omega_{Y/T},$$

where the middle arrow is an isomorphism along Y_t by the projection formula, since $\mathcal{M}$ is invertible along Y_t. In particular, $\omega_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t}$ satisfies $\mathbf{S}_2$, and as a consequence, $\phi_t(\omega_{Y/T})$ is an isomorphism by (V-4). Thus, we are done. □

§6. $\mathbb{Q}$-Gorenstein schemes

A normal algebraic variety defined over a field is said to be $\mathbb{Q}$-Gorenstein if some positive multiple of the canonical divisor is Cartier. We shall generalize the notion of $\mathbb{Q}$-Gorenstein to locally Noetherian schemes. In Section 6.1 a $\mathbb{Q}$-Gorenstein scheme is defined and its basic properties are given. In Section 6.2 we consider the case of affine cones over polarized projective schemes over a field, and determine when it is a $\mathbb{Q}$-Gorenstein scheme.

§6.1. Basic properties of $\mathbb{Q}$-Gorenstein schemes

Definition 6.1 ($\mathbb{Q}$-Gorenstein scheme). Let X be a locally Noetherian scheme admitting a dualizing complex locally on X and assume that X is *Gorenstein in codimension one*, i.e., $\mathrm{codim}(X \setminus X^\circ) \geq 2$ for the Gorenstein locus $X^\circ = \mathrm{Gor}(X)$ (cf. Definition 4.10).

(1) The scheme X is said to be *quasi-Gorenstein* (or 1-*Gorenstein*) *at* a point P if there exist an open neighborhood U of P and a dualizing complex $\mathcal{R}^\bullet$ of U such that $\mathcal{H}^0(\mathcal{R}^\bullet)$ is invertible at P. If X is quasi-Gorenstein at every point, then X is said to be quasi-Gorenstein (or 1-Gorenstein).

(2) The scheme X is said to be $\mathbb{Q}$-*Gorenstein at* P if there exist an open neighborhood U of P, a dualizing complex $\mathcal{R}^\bullet$ of U, and an integer $r > 0$ such that $\mathcal{L} = \mathcal{H}^0(\mathcal{R}^\bullet)$ is invertible on the Gorenstein locus $U^\circ = U \cap X^\circ$ and

$$j_*\left(\mathcal{L}^{\otimes r}|_{U^\circ}\right)$$

is invertible at P, where $j\colon U^\circ \hookrightarrow U$ denotes the open immersion. If X is $\mathbb{Q}$-Gorenstein at every point, then X is said to be $\mathbb{Q}$-Gorenstein.

Definition 6.2 (Gorenstein index). For a $\mathbb{Q}$-Gorenstein scheme X, the *Gorenstein index* of X at $P \in X$ is defined to be the smallest positive integer r satisfying condition (2) of Definition 6.1 for an open neighborhood of P. The least common multiple of Gorenstein indices of X at all the points is called the *Gorenstein index* of X, which might be $+\infty$.

Remark. Conditions (1) and (2) of Definition 6.1 do not depend on the choice of $\mathcal{R}^\bullet$ by the essential uniqueness of the dualizing complex (cf. Remark 4.2).

Lemma 6.3. (1) *A quasi-Gorenstein* (1-*Gorenstein*) *scheme is nothing but a* $\mathbb{Q}$-*Gorenstein scheme of Gorenstein index one.*

(2) *Every* $\mathbb{Q}$-*Gorenstein scheme satisfies* $\mathbf{S}_2$.

Proof. (1): Let X be a locally Noetherian scheme admitting a dualizing complex $\mathcal{R}^\bullet$ such that it is Gorenstein in codimension one and that $\mathcal{L} := \mathcal{H}^0(\mathcal{R}^\bullet)$ is invertible on the Gorenstein locus X°. Then $\mathcal{L}$ satisfies $\mathbf{S}_2$ and $\mathcal{L} \to j_*(\mathcal{L}|_{X^\circ})$ is an isomorphism by Corollary 4.22. Hence, X is $\mathbb{Q}$-Gorenstein with Gorenstein index one if and only if $\mathcal{L}$ is invertible, equivalently, X is quasi-Gorenstein.

(2): We may assume that X admits a dualizing complex $\mathcal{R}^\bullet$ such that $\mathcal{L} = \mathcal{H}^0(\mathcal{R}^\bullet)$ is invertible on the Gorenstein locus X°, since the $\mathbf{S}_2$-condition is local. Then $\mathcal{M}_r := j_*(\mathcal{L}^{\otimes r}|_{X^\circ})$ is invertible for some r by Definition 6.1(2). Hence, $\mathcal{M}_r$ satisfies $\mathbf{S}_2$ by Corollary 2.16. Therefore, X satisfies $\mathbf{S}_2$. □

Lemma 6.4. *Let X be a locally Noetherian scheme admitting a dualizing complex $\mathcal{R}^\bullet$. For the cohomology sheaf $\mathcal{L} := \mathcal{H}^0(\mathcal{R}^\bullet)$ and for an open subset U with* $\operatorname{codim}(X \setminus U, X) \geq 2$, *assume that $\mathcal{L}|_U$ is invertible and $\mathcal{R}^\bullet|_U \simeq_{\mathrm{qis}} \mathcal{L}|_U$. Then the following hold:*

(1) *If X satisfies $\mathbf{S}_1$, then $\mathcal{R}^\bullet$ is an ordinary dualizing complex of X and the dualizing sheaf $\mathcal{L}$ is a reflexive $\mathcal{O}_X$-module satisfying $\mathbf{S}_2$.*

(2) *If X satisfies $\mathbf{S}_2$, then the double-dual $\mathcal{L}^{[m]}$ of $\mathcal{L}^{\otimes m}$ satisfies $\mathbf{S}_2$ for any integer m, and in particular,*
$$\mathcal{L}^{[m]} \simeq j_*(\mathcal{L}^{\otimes m}|_U)$$
for the open immersion $j\colon U \hookrightarrow X$.

(3) *The scheme X is $\mathbb{Q}$-Gorenstein if and only if X satisfies $\mathbf{S}_2$ and, locally on X, there is a positive integer r such that $\mathcal{L}^{[r]}$ is invertible.*

Proof. (1): This follows from Corollary 4.22 with Lemmas 2.14 and 2.21(3).

(2): Since $\operatorname{depth}_{X \setminus U} \mathcal{O}_X \geq 2$ by the $\mathbf{S}_2$-condition, we have the isomorphism $\mathcal{L}^{[m]} \simeq j_*(\mathcal{L}^{\otimes m}|_U)$ by Lemma 2.21(1). Hence, $\mathcal{L}^{[m]}$ satisfies $\mathbf{S}_2$ by Corollary 2.16, since $\mathcal{L}|_U$ is invertible.

(3): This is a consequence of (2) above and Lemma 6.3(2) by the uniqueness of the dualizing complex, explained in Remark 4.2. □

Example 6.5. Let X be a $\Bbbk$-scheme locally of finite type for a field $\Bbbk$. Assume that X satisfies $\mathbf{S}_2$ and $\operatorname{codim}(X \setminus X^\circ, X) \geq 2$ for the Gorenstein locus $X^\circ = \operatorname{Gor}(X)$. Let $\omega_{X/\Bbbk}$ be the canonical sheaf defined in Definition 5.1 and let $\omega_{X/\Bbbk}^{[m]}$ denote the double-dual of $\omega_{X/\Bbbk}^{\otimes m}$ for any $m \in \mathbb{Z}$ (cf. Proposition 5.6). Then X is $\mathbb{Q}$-Gorenstein at a point x if and only if $\omega_{X/\Bbbk}^{[r]}$ is invertible at x for some $r > 0$.

Example 6.6. Let X be a normal algebraic $\Bbbk$-variety for a field $\Bbbk$, i.e., a normal integral separated scheme of finite type over $\Bbbk$. Then X is $\mathbb{Q}$-Gorenstein if and only if the multiple rK_X of the canonical divisor K_X is Cartier for some $r > 0$. In fact, X satisfies $\mathbf{S}_2$, $\omega_{X^\circ/\Bbbk} \simeq \mathcal{O}_{X^\circ}(K_X)$ for the Gorenstein locus $X^\circ = \operatorname{Gor}(X)$, where $\operatorname{codim}(X \setminus X^\circ, X) \geq 2$, and hence $\omega_{X/\Bbbk}^{[m]} \simeq \mathcal{O}_X(mK_X)$ for any $m \in \mathbb{Z}$.

Lemma 6.7. *Let X be a locally Noetherian scheme and let $\pi\colon Y \to X$ be a smooth surjective morphism. Then, for any integer $k \geq 1$, Y satisfies $\mathbf{S}_k$ if and only if X satisfies $\mathbf{S}_k$. In particular, Y is Cohen–Macaulay if and only if X is so. Moreover, Y is Gorenstein if and only if X is so. Assume that X admits a dualizing complex locally on X. Then Y is quasi-Gorenstein (resp. $\mathbb{Q}$-Gorenstein of index r) if and only if X is so.*

Proof. The first assertion follows from Fact 2.26(6). In particular, we have the equivalence for the Cohen–Macaulay property (cf. Remark 2.12). The Gorenstein case follows from Fact 4.12. It remains to prove the case of the $\mathbb{Q}$-Gorenstein property, since "quasi-Gorenstein" is nothing but "$\mathbb{Q}$-Gorenstein of index one" (cf. Lemma 6.3(1)). Since the $\mathbb{Q}$-Gorenstein property is local and it implies $\mathbf{S}_2$, we may assume that

- X has a dualizing complex $\mathcal{R}^{\bullet}_X$,
- X and Y are affine schemes satisfying $\mathbf{S}_2$, and
- $\pi = p \circ \lambda$ for an étale morphism $\lambda\colon Y \to X \times \mathbb{A}^d$ and the first projection $p\colon X \times \mathbb{A}^d \to X$ for the "d-dimensional affine space" $\mathbb{A}^d = \operatorname{Spec} \mathbb{Z}[\mathtt{x}_1, \ldots, \mathtt{x}_d]$ for some integer $d \geq 0$ (cf. [12, IV, Cor. (17.11.4)]).

In particular, π has pure relative dimension d. We may assume also that $\mathcal{R}^{\bullet}_X$ is an ordinary dualizing complex by Lemma 4.14. We set $\mathcal{L}_X$ to be the dualizing sheaf $\mathcal{H}^0(\mathcal{R}^{\bullet}_X)$.

By Examples 4.23 and 4.24, we see that $\mathcal{R}^{\bullet}_Y := \pi^!(\mathcal{R}^{\bullet}_X)$ is a dualizing complex of Y, and we have an isomorphism

$$\omega_{Y/X} \simeq \Omega^d_{Y/X} \simeq \lambda^*(\omega_{X \times \mathbb{A}^d/X}) \simeq \mathcal{O}_Y$$

for the relative dualizing sheaf $\omega_{Y/X}$. Thus, $\pi^!(\mathcal{O}_X) \simeq_{\mathrm{qis}} \mathcal{O}_Y[d]$, and

$$\mathcal{R}^{\bullet}_Y \simeq_{\mathrm{qis}} \pi^!(\mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathbf{L}\pi^*(\mathcal{R}^{\bullet}_X) \simeq_{\mathrm{qis}} \mathbf{L}\pi^*(\mathcal{R}^{\bullet}_X)[d]$$

(cf. Example 4.23, Fact 4.34(2)). Since Y satisfies $\mathbf{S}_2$, the shift $\mathcal{R}^{\bullet}_Y[-d]$ is an ordinary dualizing complex on Y by the proof of Lemma 4.14. Here the associated dualizing sheaf $\mathcal{L}_Y := \mathcal{H}^0(\mathcal{R}^{\bullet}_Y[-d])$ is isomorphic to $\pi^*(\mathcal{L}_X)$. Since π is faithfully flat, we see that $\mathcal{L}_Y$ is invertible if and only if $\mathcal{L}_X$ is so (cf. Lemma A.7). For an integer m, let $\mathcal{L}^{[m]}_X$ (resp. $\mathcal{L}^{[m]}_Y$) be the double-dual of $\mathcal{L}^{\otimes m}_X$ (resp. $\mathcal{L}^{\otimes m}_Y$). Then $\mathcal{L}^{[m]}_Y \simeq \pi^*(\mathcal{L}^{[m]}_X)$ for any $m \in \mathbb{Z}$ by Remark 2.20. Hence, for a given integer r, $\mathcal{L}^{[r]}_Y$ is invertible if and only if $\mathcal{L}^{[r]}_X$ is invertible by the same argument as above. Therefore, by Lemma 6.4(3), Y is $\mathbb{Q}$-Gorenstein of index r if and only if X is so. Thus, we are done. □

Remark 6.8. By Lemma 6.7, we see that the $\mathbb{Q}$-Gorenstein property is local even in the étale topology. More precisely, for an étale morphism $X' \to X$, for a point $P \in X$, and for a point $P' \in X'$ lying over P, X is $\mathbb{Q}$-Gorenstein of index r at P if and only if X' is so at P'.

§6.2. Affine cones of polarized projective schemes over a field

For an affine cone over a projective scheme over a field $\Bbbk$, we shall determine when it is Cohen–Macaulay, Gorenstein, $\mathbb{Q}$-Gorenstein, etc., under suitable conditions. We fix a field $\Bbbk$ which is not necessarily algebraically closed.

Definition 6.9 (Affine cone). A *polarized projective scheme* over $\Bbbk$ is a pair $(S, \mathcal{A})$ consisting of a projective scheme S over $\Bbbk$ and an ample invertible sheaf $\mathcal{A}$ on S. The polarized projective scheme $(S, \mathcal{A})$ is said to be *connected* if S is connected. For a connected polarized projective scheme $(S, \mathcal{A})$, the *affine cone* of $(S, \mathcal{A})$ is defined to be $\operatorname{Spec} R$ for the graded $\Bbbk$-algebra

$$R = R(S, \mathcal{A}) := \bigoplus\nolimits_{m \geq 0} \mathrm{H}^0(S, \mathcal{A}^{\otimes m}).$$

We denote the affine cone by $\operatorname{Cone}(S, \mathcal{A})$. Note that the closed subscheme of $\operatorname{Cone}(S, \mathcal{A}) = \operatorname{Spec} R$ defined by the ideal

$$R_+ = \bigoplus\nolimits_{m > 0} \mathrm{H}^0(S, \mathcal{A}^{\otimes m})$$

of R is isomorphic to $\operatorname{Spec} \mathrm{H}^0(S, \mathcal{O}_S)$, and the support of the closed subscheme is a point, since the finite-dimensional $\Bbbk$-algebra $\mathrm{H}^0(S, \mathcal{O}_S)$ is an Artinian local ring by the connectedness of S. The point is called the *vertex* of $\operatorname{Cone}(S, \mathcal{A})$.

Remark. The $\Bbbk$-algebra $R(S, \mathcal{A})$ above is finitely generated, since S is projective and $\mathcal{A}$ is ample. Moreover, $S \simeq \operatorname{Proj} R(S, \mathcal{A})$. In some articles, the affine cone of $(S, \mathcal{A})$ is defined to be $\operatorname{Spec} R'$ for the graded subring R' of R such that $R'_n = R_n$ for any $n > 0$ and $R'_0 = \Bbbk$.

Similar results to the following are well known on the structure of affine cones (cf. [12, II, Props. (8.6.2), (8.8.2)]).

Lemma 6.10. *For a connected polarized projective scheme $(S, \mathcal{A})$ over $\Bbbk$, let X be the affine cone* $\operatorname{Cone}(S, \mathcal{A})$. *Let $\pi\colon Y \to S$ be the geometric line bundle associated with $\mathcal{A}$, i.e., $Y = \mathbb{V}(\mathcal{A}) = \mathcal{S}pec_S \mathcal{R}$, where $\mathcal{R} = \bigoplus_{m \geq 0} \mathcal{A}^{\otimes m}$. Let E be the zero-section of π corresponding to the projection $\mathcal{R} \to \mathcal{O}_S$ to the component of degree zero. Then E is a* relative Cartier divisor over S (*cf.* [12, IV, Déf. (21.15.2)]) *with an isomorphism $\mathcal{O}_Y(-E) \simeq \pi^*\mathcal{A}$. Moreover, there exists a projective $\Bbbk$-morphism $\mu\colon Y \to X$ such that*

(1) *$\mathcal{O}_X \to \mu_*\mathcal{O}_Y$ is an isomorphism,*
(2) *$\pi^*\mathcal{A}$ is μ-ample,*
(3) *$\mu^{-1}(P) = E$ as a closed subset of Y for the vertex P of X, and*
(4) *μ induces an isomorphism $Y \setminus E \simeq X \setminus P$.*

Proof. For an open subset $U = \operatorname{Spec} B$ of S with an isomorphism $\varepsilon \colon \mathcal{A}|_U \simeq \mathcal{O}_U$, we have an isomorphism $\varphi \colon \pi^{-1}(U) \simeq \operatorname{Spec} B[\mathtt{t}]$ for the polynomial B-algebra $B[\mathtt{t}]$ of one variable such that φ induces an isomorphism

$$\mathrm{H}^0(\pi^{-1}(U), \mathcal{O}_Y) = \bigoplus\nolimits_{m\geq 0} \mathrm{H}^0(U, \mathcal{A}^{\otimes m}) \simeq B[\mathtt{t}] = \bigoplus\nolimits_{m\geq 0} B\mathtt{t}^m$$

of graded B-algebras. Then $E|_{\pi^{-1}(U)}$ is a Cartier divisor corresponding to $\operatorname{div}(\mathtt{t})$ on $\operatorname{Spec} B[\mathtt{t}]$, which is relatively Cartier over $\operatorname{Spec} B$ (cf. [12, IV, (21.15.3.3)]). Thus, E is a relative Cartier divisor over S, since such open subsets U cover S. The exact sequence $0 \to \mathcal{O}_Y(-E) \to \mathcal{O}_Y \to \mathcal{O}_E \to 0$ induces an isomorphism

$$\pi_*\mathcal{O}_Y(-E) \simeq \bigoplus\nolimits_{m\geq 1} \mathcal{A}^{\otimes m} \simeq \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{R}(-1)$$

of graded $\mathcal{R}$-modules, where $\mathcal{R}(-1)$ denotes the twisted graded module. In particular, $\mathcal{O}_Y(-E) \simeq \pi^*\mathcal{A}$.

The canonical homomorphisms $\mathrm{H}^0(S, \mathcal{A}^{\otimes m}) \otimes_{\Bbbk} \mathcal{O}_S \to \mathcal{A}^{\otimes m}$ induce a graded homomorphism $\Phi \colon R \otimes_{\Bbbk} \mathcal{O}_S \to \mathcal{R}$ of graded $\mathcal{O}_S$-algebras, where $R := R(S, \mathcal{A})$. The cokernel of Φ is a finitely generated $\mathcal{O}_S$-module, since $\mathcal{A}^{\otimes m}$ is generated by global sections for $m \gg 0$. Hence, $\mathcal{R}$ is a finitely generated $R \otimes_{\Bbbk} \mathcal{O}_S$-module. Therefore, Φ defines a finite morphism

$$\nu \colon Y = \mathcal{S}pec_S\, \mathcal{R} \to \mathcal{S}pec_S(R \otimes_{\Bbbk} \mathcal{O}_S) \simeq X \times_{\operatorname{Spec}\Bbbk} S$$

over S. Let $p_1 \colon X \times_{\operatorname{Spec}\Bbbk} S \to X$ and $p_2 \colon X \times_{\operatorname{Spec}\Bbbk} S \to S$ be the first and second projections. Then $\mu := p_1 \circ \nu \colon Y \to X$ is a projective morphism, since S is projective over $\Bbbk$. Here, $\mathcal{O}_X \simeq \mu_*\mathcal{O}_Y$, since $\mathrm{H}^0(Y, \mathcal{O}_Y) \simeq \mathrm{H}^0(S, \mathcal{R}) \simeq R$. Moreover, $\pi^*\mathcal{A}$ is μ-ample, since $p_2^*\mathcal{A}$ is relatively ample over X and $\pi^*\mathcal{A}$ is the pullback by the finite morphism ν. Thus, μ satisfies conditions (1) and (2). Since the projection $\mathcal{R} \to \mathcal{O}_S$ defining E induces the projection $R = \mathrm{H}^0(S, \mathcal{R}) \to \mathrm{H}^0(S, \mathcal{O}_S)$ to the component of degree zero, the scheme-theoretic image $\mu(E)$ is the zero-dimensional closed subscheme $\operatorname{Spec} \mathrm{H}^0(S, \mathcal{O}_S)$ of X defined by the ideal $R_+ = \bigoplus_{m>0} \mathrm{H}^0(S, \mathcal{A}^{\otimes m})$ of R. Hence, the image $\mu(E)$ is set-theoretically the vertex P. We shall show that the morphism

$$\mu' \colon Y' := Y \setminus \mu^{-1}(P) \to X' := X \setminus P$$

induced by μ is an isomorphism. Since μ is proper, so is μ'. Moreover, the structure sheaf $\mathcal{O}_{Y'}$ is μ'-ample, since $\pi^*\mathcal{A} \simeq \mathcal{O}_Y(-E)$ is μ-ample by (2). Hence, μ' is a finite morphism. Thus, μ' is an isomorphism by (1), since $\mathcal{O}_{X'} \simeq \mu'_*\mathcal{O}_{Y'}$. As a consequence, (4) is derived from (3), and it remains to prove (3) for μ and P.

For a global section f of $\mathcal{A}^{\otimes m}$ for some $m > 0$, we set $V(f)$ to be the closed subscheme $\operatorname{Spec}(R/fR)$ of $X = \operatorname{Spec} R$ by regarding f as a homogeneous element of R of degree m. We also set a closed subscheme $W(f)$ of S to be the "zero-subscheme" of f, i.e., it is defined by the exact sequence

$$\mathcal{A}^{\otimes -m} \xrightarrow{\otimes f} \mathcal{O}_S \to \mathcal{O}_{W(f)} \to 0.$$

Condition (3) is derived from the following (∗) for any f and for any affine open subsets $U = \operatorname{Spec} B$ with an isomorphism $\varepsilon\colon \mathcal{A}|_U \simeq \mathcal{O}_U$:

(∗) $\mu^{-1}V(f) \cap \pi^{-1}(U) = (\pi^{-1}W(f) \cup E) \cap \pi^{-1}(U)$ as a subset of $\pi^{-1}(U)$.

In fact, if (∗) holds for all U and f, then $\mu^{-1}V(f) = \pi^{-1}W(f) \cup E$ for any f, and we have $\mu^{-1}(P) = E$ by $\bigcap_f V(f) = P$ and $\bigcap_f W(f) = \emptyset$. Here $\bigcap_f V(f) = P$ and $\bigcap_f W(f) = \emptyset$ hold, since all of such $f \in R$ generate the ideal R_+ and since $\mathcal{A}$ is ample. We shall prove (∗) as follows. Let $\varphi\colon \pi^{-1}(U) \simeq \operatorname{Spec} B[\mathtt{t}]$ be the isomorphism above defined by ε. We set

$$b = \varepsilon^{\otimes m}(f|_U) \in \mathrm{H}^0(U, \mathcal{O}_U) = B$$

for the induced isomorphism $\varepsilon^{\otimes m}\colon \mathcal{A}^{\otimes m}|_U \simeq \mathcal{O}_U$. Then $W(f) \cap U = \operatorname{Spec} B/bB$, and φ induces isomorphisms $\mu^{-1}V(f) \cap \pi^{-1}(U) \simeq \operatorname{Spec} B[\mathtt{t}]/(b\mathtt{t}^m)$ and $E \cap \pi^{-1}(U) \simeq \operatorname{Spec} B[\mathtt{t}]/(\mathtt{t})$. This implies (∗), and we are done. □

Corollary 6.11. *In the situation of Lemma 6.10, for an integer $k \geq 1$, S satisfies $\mathbf{S}_k$ if and only if $X \setminus P$ satisfies $\mathbf{S}_k$. Moreover, S is Cohen–Macaulay (resp. Gorenstein, resp. quasi-Gorenstein, resp. $\mathbb{Q}$-Gorenstein of Gorenstein index r) if and only if $X \setminus P$ is so.*

Proof. This is a consequence of Lemmas 6.7 and 6.10, since $X \setminus P \simeq Y \setminus E$ is smooth and surjective over S. □

The following result is essentially well known (cf. [38, Prop. 1.7], [45, Lem. 4.3]).

Proposition 6.12. *Let X be the affine cone of a connected polarized projective scheme $(S, \mathcal{A})$ over $\Bbbk$ and let P be the vertex of X. For a coherent $\mathcal{O}_S$-module $\mathcal{G}$, we set $\mathcal{F} = \mu_*(\pi^*\mathcal{G})$ for the morphisms $\mu\colon Y \to X$ and $\pi\colon Y \to S$ in Lemma 6.10 for the geometric line bundle $Y = \mathbb{V}_S(\mathcal{A})$ over S. We define also $\widetilde{\mathcal{F}} := j_*(\mathcal{F}|_{X\setminus P})$ for the open immersion $j\colon X \setminus P \hookrightarrow X$, and for simplicity we define*

$$\mathrm{H}^i(\mathcal{G}(m)) := \mathrm{H}^i(S, \mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{A}^{\otimes m})$$

for $m \in \mathbb{Z}$ and $i \geq 0$. Then the following hold:

(0) *If* $\mathcal{G} = \mathcal{O}_S$, *then* $\mathcal{F} \simeq \mathcal{O}_X$.

(1) *The inequality* $\operatorname{depth} \mathcal{F}_P \geq 1$ *holds; equivalently,* $\mathcal{F} \hookrightarrow \widetilde{\mathcal{F}}$ *is injective.*

(2) *The inequality* $\operatorname{depth} \mathcal{F}_P \geq 2$ *holds if and only if* $\mathrm{H}^0(\mathcal{G}(m)) = 0$ *for any* $m < 0$. *This condition is also equivalent to* $\mathcal{F} \simeq \widetilde{\mathcal{F}}$.

(3) *The quasi-coherent* $\mathcal{O}_X$*-module* $\widetilde{\mathcal{F}}$ *is coherent if and only if* $\mathrm{H}^0(\mathcal{G}(m)) = 0$ *for* $m \ll 0$. *In particular,* $\widetilde{\mathcal{F}}$ *is coherent if* $\mathcal{G}$ *satisfies* $\mathbf{S}_1$ *and every irreducible component of* $\operatorname{Supp} \mathcal{G}$ *has positive dimension.*

(4) *Assume that* $\widetilde{\mathcal{F}}$ *is coherent. Then, for an integer* $k \geq 3$, $\operatorname{depth} \widetilde{\mathcal{F}}_P \geq k$ *holds if and only if* $\mathrm{H}^i(\mathcal{G}(m)) = 0$ *for any* $m \in \mathbb{Z}$ *and* $0 < i < k - 1$.

(5) *The* $\mathcal{F}$ *satisfies* $\mathbf{S}_1$ *if and only if* $\mathcal{G}$ *satisfies* $\mathbf{S}_1$.

(6) *The* $\mathcal{F}$ *satisfies* $\mathbf{S}_2$ *if and only if* $\mathcal{G}$ *satisfies* $\mathbf{S}_2$ *and* $\mathrm{H}^0(\mathcal{G}(m)) = 0$ *for any* $m < 0$.

(7) *Assume that* $\widetilde{\mathcal{F}}$ *is coherent. Then, for an integer* $k \geq 3$, $\widetilde{\mathcal{F}}$ *satisfies* $\mathbf{S}_k$ *if and only if* $\mathcal{G}$ *satisfies* $\mathbf{S}_k$ *and* $\mathrm{H}^i(\mathcal{G}(m)) = 0$ *for any* $m \in \mathbb{Z}$ *and* $0 < i < k - 1$.

(8) *Assume that* $\widetilde{\mathcal{F}}$ *is coherent. Then* $\widetilde{\mathcal{F}}$ *is a Cohen–Macaulay* $\mathcal{O}_X$*-module if and only* $\mathcal{G}$ *is a Cohen–Macaulay* $\mathcal{O}_S$*-module and* $\mathrm{H}^i(\mathcal{G}(m)) = 0$ *for any* $m \in \mathbb{Z}$ *and* $0 < i < \dim \operatorname{Supp} \mathcal{G}$.

Proof. Assertion (0) is a consequence of Lemma 6.10(1). We consider the local cohomology sheaves $\mathcal{H}^i_P(\mathcal{F}')$ with support in P for $\mathcal{F}' = \mathcal{F}$ or $\mathcal{F}' = \widetilde{\mathcal{F}}$. These are quasi-coherent sheaves on X supported on P (cf. [18, Prop. 2.1]). Thus,

$$\mathrm{H}^i_P(X, \mathcal{F}') \simeq \mathrm{H}^0(X, \mathcal{H}^i_P(\mathcal{F}'))$$

and it is also isomorphic to the stalk $(\mathcal{H}^i_P(\mathcal{F}'))_P$ at P. Note that, for a positive integer k, when $\mathcal{F}'$ is coherent, $\operatorname{depth} \mathcal{F}'_P \geq k$ if and only if $(\mathcal{H}^i_P(\mathcal{F}'))_P = 0$ for any $i < k$ (cf. Property 2.6). There exist an exact sequence

$$0 \to \mathrm{H}^0_P(X, \mathcal{F}') \to \mathrm{H}^0(X, \mathcal{F}') \to \mathrm{H}^0(X \setminus P, \mathcal{F}') \to \mathrm{H}^1_P(X, \mathcal{F}') \to 0$$

and isomorphisms $\mathrm{H}^i(X \setminus P, \mathcal{F}') \simeq \mathrm{H}^{i+1}_P(X, \mathcal{F}')$ for all $i \geq 1$ (cf. [18, Prop. 2.2]). Hence, if $\mathcal{F}'$ is a coherent $\mathcal{O}_X$-module, then $\operatorname{depth} \mathcal{F}'_P \geq k$ if and only if

(i) $\mathrm{H}^0(X, \mathcal{F}') \to \mathrm{H}^0(X \setminus P, \mathcal{F}')$ is injective when $k = 1$,

(ii) $\mathrm{H}^0(X, \mathcal{F}') \to \mathrm{H}^0(X \setminus P, \mathcal{F}')$ is an isomorphism when $k = 2$, and

(iii) $\mathrm{H}^0(X, \mathcal{F}') \to \mathrm{H}^0(X \setminus P, \mathcal{F}')$ is an isomorphism, and $\mathrm{H}^i(X \setminus P, \mathcal{F}') = 0$ for any $0 < i < k - 1$ when $k \geq 3$.

By construction and by Lemma 6.10(4), we have isomorphisms

$$\mathrm{H}^0(X, \mathcal{F}) \simeq \mathrm{H}^0(Y, \pi^* \mathcal{G}) \simeq \bigoplus\nolimits_{m \geq 0} \mathrm{H}^0(\mathcal{G}(m)), \quad \text{and}$$

$$\mathrm{H}^i(X \setminus P, \mathcal{F}) \simeq \mathrm{H}^i(Y \setminus E, \pi^*\mathcal{G}) \simeq \bigoplus\nolimits_{m\in\mathbb{Z}} \mathrm{H}^i(\mathcal{G}(m))$$

for any $i \geq 0$, where the homomorphism $\mathrm{H}^0(Y, \pi^*\mathcal{G}) \to \mathrm{H}^0(Y \setminus E, \pi^*\mathcal{G})$ is an injection and is the identity on each component $\mathrm{H}^0(\mathcal{G}(m))$ of degree $m \geq 0$. We have (1), (2), and (4) by considering conditions (i)–(iii) above. Moreover, (3) holds, since $\widetilde{\mathcal{F}}$ is coherent if and only if $\widetilde{\mathcal{F}}_P/\mathcal{F}_P$ is a finite-dimensional $\Bbbk$-vector space, and since we have an isomorphism

$$\widetilde{\mathcal{F}}_P/\mathcal{F}_P \simeq \bigoplus\nolimits_{m<0} \mathrm{H}^0(\mathcal{G}(m))$$

by the argument above. This implies the first half of (3), and the second half follows from Lemma 2.18.

For an integer $k > 0$, $\mathcal{F}|_{X\setminus P}$ satisfies $\mathbf{S}_k$ if and only if $\mathcal{G}$ satisfies $\mathbf{S}_k$ by [12, IV, Cor. (6.4.2)], since $Y \setminus E \simeq X \setminus P$. Thus, assertion (5) (resp. (6), resp. (7)) follows from (1) (resp. (2), resp. (4)) by the equivalence (i) ⇔ (iv) in Lemma 2.14 applied to $Z = P$. The last assertion (8) is a consequence of (7), since $\dim \widetilde{\mathcal{F}}_P = \dim \operatorname{Supp} \mathcal{G} + 1$. □

Proposition 6.13. *Let $(S, \mathcal{A})$ be a connected polarized projective scheme over $\Bbbk$ and let X be the affine cone $\operatorname{Cone}(S, \mathcal{A})$. Let $\pi\colon Y \to S$ and $\mu\colon Y \to X$ be the morphisms in Lemma 6.10. Assume that X satisfies $\mathbf{S}_2$ and $n := \dim S > 0$. Then*

(0) *S and Y also satisfy $\mathbf{S}_2$, and the schemes S, Y, and X are all equi-dimensional.*

Let $\omega_{X/\Bbbk}$ (resp. $\omega_{Y/\Bbbk}$, resp. $\omega_{S/\Bbbk}$) be the canonical sheaf of X (resp. Y, resp. S) in the sense of Definition 4.28, and let $\omega^{[r]}_{X/\Bbbk}$ (resp. $\omega^{[r]}_{S/\Bbbk}$) denote the double-dual of $\omega^{\otimes r}_{X/\Bbbk}$ (resp. $\omega^{\otimes r}_{S/\Bbbk}$) for an integer r.

(1) *There exist isomorphisms*

(VI-1) $$\omega_{Y/\Bbbk} \simeq \pi^*(\omega_{S/\Bbbk} \otimes_{\mathcal{O}_S} \mathcal{A}) \quad \textit{and}$$

(VI-2) $$\omega^{[r]}_{Y/\Bbbk} \simeq \pi^*(\omega^{[r]}_{S/\Bbbk} \otimes_{\mathcal{O}_S} \mathcal{A}^{\otimes r})$$

for any integer r. Moreover, $\omega^{[r]}_{X/\Bbbk}$ is isomorphic to the double-dual of $\mu_*(\omega^{[r]}_{Y/\Bbbk})$ for any integer r.

(2) *For any integer r and for any integer $k \geq 3$,*

$$\operatorname{depth}(\omega^{[r]}_{X/\Bbbk})_P \geq k$$

holds for the vertex P of X if and only if

$$\mathrm{H}^i(S, \omega^{[r]}_{S/\Bbbk} \otimes_{\mathcal{O}_S} \mathcal{A}^{\otimes m}) = 0$$

for any $m \in \mathbb{Z}$ *and any* $0 < i < k-1$. *Moreover,* $\omega^{[r]}_{X/\Bbbk}$ *satisfies* $\mathbf{S}_k$ *for the same* r *and* k *if and only if* $\omega^{[r]}_{S/\Bbbk}$ *satisfies* $\mathbf{S}_k$ *and*

$$\mathrm{H}^i(S, \omega^{[r]}_{S/\Bbbk} \otimes_{\mathcal{O}_S} \mathcal{A}^{\otimes m}) = 0$$

for any $m \in \mathbb{Z}$ *and any* $0 < i < k-1$.

(3) *For any positive integer* r*, the following three conditions are equivalent to each other:*

(i) $\omega^{[r]}_{X/\Bbbk} \simeq \mathcal{O}_X$*;*

(ii) $\omega^{[r]}_{X/\Bbbk}$ *is invertible;*

(iii) $\omega^{[r]}_{S/\Bbbk} \simeq \mathcal{A}^{\otimes l}$ *for an integer* l*.*

Proof. Assertion (0) is a consequence of Proposition 6.12(6) for $\mathcal{G} = \mathcal{O}_S$, Lemma 6.7, and Fact 2.23(1). Let $\omega^{\bullet}_{S/\Bbbk}$ (resp. $\omega^{\bullet}_{Y/\Bbbk}$, resp. $\omega^{\bullet}_{X/\Bbbk}$) be the canonical dualizing complex of S (resp. Y, resp. X) in the sense of Definition 4.26. Note that $\omega^{\bullet}_{S/\Bbbk}[-n]$ (resp. $\omega^{\bullet}_{Y/\Bbbk}[-n-1]$, resp. $\omega^{\bullet}_{X/\Bbbk}[-n-1]$) is an ordinary dualizing complex by Lemma 4.27 for $n = \dim S$. Then

$$\omega^{\bullet}_{Y/\Bbbk} \simeq_{\mathrm{qis}} \Omega^1_{Y/S}[1] \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathbf{L}\pi^*(\omega^{\bullet}_{S/\Bbbk}) \simeq_{\mathrm{qis}} \mathbf{L}\pi^*(\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{O}_S} \omega^{\bullet}_{S/\Bbbk})[1],$$

since π is separated and smooth (cf. Example 4.23) and since there is an isomorphism $\Omega^1_{Y/S} \simeq \pi^*\mathcal{A}$ (cf. [12, IV, Cor. 16.4.9]). Thus, we have the isomorphism (VI-1). By taking the double-dual of tensor powers of both sides of (VI-1), we have the isomorphism (VI-2) for any integer r by Remark 2.20. Since X satisfies $\mathbf{S}_2$, any reflexive $\mathcal{O}_X$-module $\mathcal{F}$ satisfies $\mathbf{S}_2$ by Corollary 2.22, and moreover, $\operatorname{depth}_P \mathcal{F} \geq 2$, since $\operatorname{codim}(P, X) = \dim X = n + 1 \geq 2$. Thus, we have isomorphisms

$$\omega^{[r]}_{X/\Bbbk} \simeq j_*(\omega^{[r]}_{X\setminus P/\Bbbk}) \simeq j_*(\mu_*(\omega^{[r]}_{Y/\Bbbk})|_{X\setminus P}) \simeq (\mu_*(\omega^{[r]}_{Y/\Bbbk}))^{\vee\vee}$$

for any integer r and for the open immersion $j\colon X \setminus P \hookrightarrow X$. This proves (1).

By (1), we see that (2) is a consequence of (4) and (7) of Proposition 6.12 applied to the case $\mathcal{G} = \omega^{[r]}_{S/\Bbbk} \otimes \mathcal{A}^{\otimes r}$, where $\widetilde{\mathcal{F}} \simeq \omega^{[r]}_{X/\Bbbk}$. It remains to prove the equivalence of conditions (i)–(iii) of (3). Since (i) $\Rightarrow$ (ii) is trivial, it is enough to prove (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i).

Proof of (iii) $\Rightarrow$ (i): Assume that $\omega^{[r]}_{S/\Bbbk} \simeq \mathcal{A}^{\otimes l}$ for some $r > 0$ and $l \in \mathbb{Z}$. Since $\mathcal{O}_Y(-E) \simeq \pi^*\mathcal{A}$ for the zero-section E of Lemma 6.10, we have

$$\omega^{[r]}_{Y/\Bbbk} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y((r+l)E) \simeq \pi^*(\omega^{[r]}_{S/\Bbbk} \otimes \mathcal{A}^{\otimes r} \otimes_{\mathcal{O}_S} \mathcal{A}^{\otimes -(r+l)}) \simeq \mathcal{O}_Y$$

from the isomorphism in (1). By taking μ_*, we have $\omega^{[r]}_{X/\Bbbk} \simeq \pi_*\mathcal{O}_Y \simeq \mathcal{O}_X$.

Proof of (ii) $\Rightarrow$ (iii): Assume that $\omega^{[r]}_{X/\Bbbk}$ is invertible. Then $\omega^{[r]}_{Y/\Bbbk}$ is invertible on $Y \setminus E$, since $Y \setminus E \simeq X \setminus P$. Moreover, $\omega^{[r]}_{S/\Bbbk}$ is invertible by (VI-2), since $Y \setminus E \to S$ is faithfully flat (cf. Lemma A.7). Thus, $\omega^{[r]}_{Y/\Bbbk}$ is also invertible again by (VI-2). There is an injection

$$\phi\colon \omega^{[r]}_{Y/\Bbbk} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-bE) \hookrightarrow \mu^*(\omega^{[r]}_{X/\Bbbk})$$

for some integer b such that the cokernel of ϕ is supported on E. In fact, for any integer b, we have a canonical homomorphism

$$\begin{aligned}\mu_*(\omega^{[r]}_{Y/\Bbbk} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-bE)) &\hookrightarrow j_*(\mu_*(\omega^{[r]}_{Y/\Bbbk} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-bE))|_{X\setminus P}) \\ &\simeq j_*(\mu_*(\omega^{[r]}_{Y/\Bbbk})|_{X\setminus P}) \simeq \omega^{[r]}_{X/\Bbbk},\end{aligned}$$

whose cokernel is supported on P, and if b is sufficiently large, then

$$\mu^*\mu_*(\omega^{[r]}_{Y/\Bbbk} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-bE)) \to \omega^{[r]}_{Y/\Bbbk} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-bE)$$

is surjective, since $\mathcal{O}_Y(-E) \simeq \pi^*\mathcal{A}$ is relatively ample over X. Thus,

$$\mu^*\mu_*(\omega^{[r]}_{Y/\Bbbk} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-bE)) \to \mu^*(\omega^{[r]}_{X/\Bbbk})$$

induces the injection ϕ, since the invertible sheaf $\mu^*(\omega^{[r]}_{X/\Bbbk})$ does not contain a non-zero coherent $\mathcal{O}_Y$-submodule whose support is contained in E, by the $\mathbf{S}_1$-condition on Y. Let b be a minimal integer with an injection ϕ above. Then ϕ is an isomorphism. This is shown as follows. The homomorphism

$$\phi|_E\colon (\omega^{[r]}_{Y/\Bbbk} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-bE)) \otimes_{\mathcal{O}_Y} \mathcal{O}_E \to \mu^*(\omega^{[r]}_{X/\Bbbk}) \otimes_{\mathcal{O}_Y} \mathcal{O}_E$$

is not zero by the minimality of b. Here $\phi|_E$ corresponds to a non-zero homomorphism

$$\omega^{[r]}_{S/\Bbbk} \otimes_{\mathcal{O}_S} \mathcal{A}^{\otimes(r+b)} \to \mathcal{O}_S$$

by the isomorphism $\pi|_E\colon E \simeq S$ and by (VI-2). In particular, there is a non-empty open subset $U \subset S$ such that ϕ is an isomorphism on $\pi^{-1}(U)$. On the other hand, since ϕ is an injection between invertible sheaves, there is an effective Cartier divisor D on Y such that the cokernel of ϕ is isomorphic to $\mathcal{O}_D \otimes_{\mathcal{O}_Y} \pi^*(\omega^{[r]}_{X/\Bbbk})$ and that $\operatorname{Supp} D \subset E$. Then D is a relative Cartier divisor over S, since every fiber of π is $\mathbb{A}^1$ (cf. [12, IV, (21.15.3.3)]). Thus, $\pi|_D\colon D \to S$ is a flat and finite morphism. If $D \neq 0$, then $\pi(D) = S$ by the connectedness of S, and it contradicts $\operatorname{Supp} D \cap \pi^{-1}(U) = \emptyset$. Thus, $D = 0$, and consequently, ϕ is an isomorphism.

Therefore, we have an isomorphism

$$\omega^{[r]}_{S/\Bbbk} \otimes_{\mathcal{O}_S} \mathcal{A}^{\otimes(r+b)} \simeq \mathcal{O}_S$$

corresponding to the isomorphism $\phi|_E$, and condition (iii) is satisfied for $l = -(r + b)$. Thus, we have proved the equivalence of (i)–(iii), and we are done. □

Corollary 6.14. *Let X be the affine cone of a connected polarized scheme $(S, \mathcal{A})$ over $\Bbbk$. Assume that $n = \dim S > 0$ and $\mathrm{H}^0(S, \mathcal{A}^{\otimes m}) = 0$ for any $m < 0$. Then the following hold:*

(1) *The scheme X is Gorenstein if and only if*
- *S is Gorenstein,*
- *$\mathrm{H}^i(S, \mathcal{A}^{\otimes m}) = 0$ for any $0 < i < n$ and any $m \in \mathbb{Z}$ and*
- *$\omega_{S/\Bbbk} \simeq \mathcal{A}^{\otimes l}$ for some integer l.*

(2) *The scheme X is quasi-Gorenstein if and only if S is quasi-Gorenstein and $\omega_{S/\Bbbk} \simeq \mathcal{A}^{\otimes l}$ for some integer l.*

(3) *The scheme X is $\mathbb{Q}$-Gorenstein if and only if S is $\mathbb{Q}$-Gorenstein and $\omega^{[r]}_{S/\Bbbk} \simeq \mathcal{A}^{\otimes l}$ for some integers $r > 0$ and l.*

Proof. Assertion (1) follows from (2) and Proposition 6.12(8). The "only if" parts of (2) and (3) are shown as follows. Assume that X is $\mathbb{Q}$-Gorenstein of Gorenstein index r. Note that X is quasi-Gorenstein if and only if $r = 1$ by Lemma 6.3(1). Then S is $\mathbb{Q}$-Gorenstein by Corollary 6.11. Moreover, $\omega^{[r]}_{S/\Bbbk} \simeq \mathcal{A}^{\otimes l}$ for some $l \in \mathbb{Z}$ by the implication (ii) $\Rightarrow$ (iii) of Proposition 6.13(3). Thus, the "only if" parts are proved. The "if" parts of (2) and (3) are shown as follows. Assume that S is $\mathbb{Q}$-Gorenstein. Then $X \setminus P$ is $\mathbb{Q}$-Gorenstein by Corollary 6.11. In particular, $\mathrm{codim}(X \setminus X^\circ, X) \geq 2$ for the Gorenstein locus $X^\circ = \mathrm{Gor}(X)$. Moreover, X satisfies $\mathbf{S}_2$ by Proposition 6.12(6), since S satisfies $\mathbf{S}_2$ and $\mathrm{H}^0(S, \mathcal{A}^{\otimes m}) = 0$ for any $m < 0$ by assumption. If $\omega^{[r]}_{S/\Bbbk} \simeq \mathcal{A}^{\otimes l}$ for integers $r > 0$ and l, then $\omega^{[r]}_{X/\Bbbk}$ is invertible by the implication (iii) $\Rightarrow$ (ii) of Proposition 6.13(3). Thus, X is $\mathbb{Q}$-Gorenstein. This proves the "if" part of (3). The "if" part of (2) follows also from the argument above by setting $r = 1$. Thus, we are done. □

Corollary 6.15. *Let X be the affine cone of a connected polarized scheme $(S, \mathcal{A})$ over $\Bbbk$. Assume that S is Cohen–Macaulay, $n := \dim S > 0$, and*

$$\mathrm{H}^i(S, \mathcal{A}^{\otimes m}) = \mathrm{H}^i(S, \omega_{S/\Bbbk} \otimes \mathcal{A}^{\otimes m}) = 0$$

for any $i > 0$ and $m > 0$. Then the following hold:

(1) *The affine cone X satisfies $\mathbf{S}_2$. In particular, S is reduced (resp. normal) if and only if X is so.*

(2) *The following conditions are equivalent to each other for an integer $k \geq 3$:*

(a) $\operatorname{depth} \mathcal{O}_{X,P} \geq k$;
(b) *X satisfies* $\mathbf{S}_k$;
(c) $\mathrm{H}^i(S, \mathcal{O}_S) = 0$ *for any* $0 < i < k-1$.

(3) *The affine cone X is Cohen–Macaulay if and only if* $\mathrm{H}^i(S, \mathcal{O}_S) = 0$ *for any* $0 < i < n$.

(4) *The following conditions are equivalent to each other for an integer* $k \geq 3$:

(a) $\operatorname{depth}(\omega_{X/\Bbbk})_P \geq k$;
(b) $\omega_{X/\Bbbk}$ *satisfies* $\mathbf{S}_k$;
(c) $\mathrm{H}^i(S, \mathcal{O}_S) = 0$ *for any* $n-k+1 < i < n$.

(5) *When S is Gorenstein, X is $\mathbb{Q}$-Gorenstein if and only if* $\omega_{S/\Bbbk}^{\otimes r} \simeq \mathcal{A}^{\otimes l}$ *for some integers* $r > 0$ *and* l.

(6) *When S is Gorenstein, X is Gorenstein if and only if* $\omega_{S/\Bbbk} \simeq \mathcal{A}^{\otimes l}$ *for some* $l \in \mathbb{Z}$ *and if* $\mathrm{H}^i(S, \mathcal{O}_S) = 0$ *for any* $0 < i < n$.

Proof. By duality (cf. Corollary 4.32), we have

$$\mathrm{H}^i(S, \mathcal{A}^{\otimes m}) \simeq \mathrm{H}^{n-i}(S, \omega_{S/\Bbbk} \otimes_{\mathcal{O}_S} \mathcal{A}^{\otimes -m})^\vee$$

for any integers m and i, and by assumption, this is zero either if $m > 0$ and $i > 0$ or if $m < 0$ and $i < n$. Thus, X satisfies $\mathbf{S}_2$ by considering the case $m < 0$ and $i = 0$ and by Proposition 6.12(6) applied to $\mathcal{G} = \mathcal{O}_S$. This proves (1). Assertion (2) (resp. (4)) is a consequence of (4) and (7) of Proposition 6.12 applied to $\mathcal{G} = \mathcal{O}_S$ (resp. $\mathcal{G} = \omega_{S/\Bbbk} \otimes \mathcal{A}$). Similarly, assertion (3) is a consequence of Proposition 6.12(8) applied to $\mathcal{G} = \mathcal{O}_S$. Moreover, assertion (5) (resp. (6)) is derived from (3) (resp. (1)) of Corollary 6.14. Thus, we are done. □

§7. $\mathbb{Q}$-Gorenstein morphisms

Section 7 introduces the notion of "$\mathbb{Q}$-Gorenstein morphisms" and their weak forms: "naively $\mathbb{Q}$-Gorenstein morphisms" and "virtually $\mathbb{Q}$-Gorenstein morphisms". We inspect relations between these three notions, and prove basic properties and several theorems on $\mathbb{Q}$-Gorenstein morphisms.

In Sections 7.1 and 7.2 we define the notions of a $\mathbb{Q}$-Gorenstein morphism, a naively $\mathbb{Q}$-Gorenstein morphism, and a virtually $\mathbb{Q}$-Gorenstein morphism, and we discuss their properties giving some criteria for a morphism to be $\mathbb{Q}$-Gorenstein. A $\mathbb{Q}$-Gorenstein morphism is always naively and virtually $\mathbb{Q}$-Gorenstein. In Section 7.1 we provide a new example of naively $\mathbb{Q}$-Gorenstein morphisms which are not $\mathbb{Q}$-Gorenstein, by Lemma 7.8 and Example 7.9, and we discuss the relative

Gorenstein index for a naively $\mathbb{Q}$-Gorenstein morphism in Proposition 7.11. In Section 7.2 we shall prove Theorem 7.18, which is one of our main results and which shows that a virtually $\mathbb{Q}$-Gorenstein morphism is a $\mathbb{Q}$-Gorenstein morphism under some mild conditions. In Section 7.3 several basic properties including base change of $\mathbb{Q}$-Gorenstein morphisms and of their variants are discussed.

Finally, in Section 7.4, we shall prove notable theorems. We prove three criteria for a morphism to be $\mathbb{Q}$-Gorenstein: an infinitesimal criterion (Theorem 7.25), a valuative criterion (Theorem 7.29), and a criterion by $\mathbf{S}_3$-conditions on fibers (Theorem 7.30). Moreover, we prove the existence theorem of $\mathbb{Q}$-Gorenstein refinement (Theorem 7.32) and its variants (Theorems 7.34 and 7.35).

§7.1. $\mathbb{Q}$-Gorenstein morphisms and naively $\mathbb{Q}$-Gorenstein morphisms

Definition 7.1. Let $f\colon Y \to T$ be an $\mathbf{S}_2$-morphism of locally Noetherian schemes such that every fiber is $\mathbb{Q}$-Gorenstein. Let $\omega_{Y/T}$ denote the relative canonical sheaf in the sense of Definition 5.3 and let $\omega_{Y/T}^{[m]}$ denote the double-dual of $\omega_{Y/T}^{\otimes m}$ for $m \in \mathbb{Z}$.

(1) The morphism f is said to be *naively $\mathbb{Q}$-Gorenstein* at a point $y \in Y$ if $\omega_{Y/T}^{[r]}$ is invertible at y for some integer $r > 0$. If f is naively $\mathbb{Q}$-Gorenstein at every point of Y, then it is called a *naively $\mathbb{Q}$-Gorenstein morphism*.

(2) If $\omega_{Y/T}^{[m]}$ satisfies relative $\mathbf{S}_2$ over T (cf. Definition 2.28) for any $m \in \mathbb{Z}$, then f is called a *$\mathbb{Q}$-Gorenstein morphism*. If $f|_U\colon U \to T$ is a $\mathbb{Q}$-Gorenstein morphism for an open neighborhood U of a point $y \in Y$, then f is said to be $\mathbb{Q}$-Gorenstein at y.

Remark. We explain origins of (naively) $\mathbb{Q}$-Gorenstein morphisms (cf. Section 1). The origin of naively $\mathbb{Q}$-Gorenstein morphisms goes back to the $\mathbb{Q}$-Gorenstein one-parameter deformations studied in [31, §3]. It is generalized in the situation of a moduli problem of polarized $\mathbb{Q}$-Gorenstein schemes in [58, Def. 1.23], which is referred to as Viehweg's functor in [21, §2]. The $\mathbb{Q}$-Gorenstein deformation satisfying the Kollár condition is the source of our definition of a $\mathbb{Q}$-Gorenstein morphism. The origin of the Kollár condition seems to be in [27, 2.1.2]. In [21, §2], Kollár's functor is defined as a moduli functor of polarized $\mathbb{Q}$-Gorenstein schemes satisfying the Kollár condition. In these references (and most references for moduli of $\mathbb{Q}$-Gorenstein schemes in the 2000s), $\mathbb{Q}$-Gorenstein schemes are assumed to be Cohen–Macaulay.

Remark 7.2. For an $\mathbf{S}_2$-morphism $f\colon Y \to T$ of locally Noetherian schemes, if every fiber is Gorenstein in codimension one and if $\omega_{Y/T}$ is an invertible $\mathcal{O}_Y$-module, then f is a $\mathbb{Q}$-Gorenstein morphism. In fact, $\omega_{Y/T}^{[m]} \simeq \omega_{Y/T}^{\otimes m}$ satisfies relative

$\mathbf{S}_2$ over T for any $m \in \mathbb{Z}$ and every fiber $Y_t = f^{-1}(t)$ is $\mathbb{Q}$-Gorenstein of Gorenstein index one, since $\omega_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq \omega_{Y_t/\Bbbk(t)}$ (cf. Proposition 5.5).

The $\mathbb{Q}$-Gorenstein morphism and the naively $\mathbb{Q}$-Gorenstein morphism are characterized as follows.

Lemma 7.3. *Let Y and T be locally Noetherian schemes and $f \colon Y \to T$ a flat morphism locally of finite type. Let $j \colon Y^\circ \hookrightarrow Y$ be the open immersion from an open subset Y° of the relative Gorenstein locus $\mathrm{Gor}(Y/T)$. For a point $y \in Y$, the fibers $Y_t = f^{-1}(t)$ and $Y_t^\circ = Y^\circ \cap Y_t$ over $t = f(y)$, and for a positive integer r, let us consider the following conditions:*

(i) *The fiber Y_t satisfies $\mathbf{S}_2$ at y and $\mathrm{codim}_y(Y_t \setminus Y^\circ, Y_t) \geq 2$.*

(ii) *The direct image sheaf $j_*(\omega_{Y^\circ/T}^{\otimes r})$ is invertible at y.*

(iii) *The fiber Y_t is $\mathbb{Q}$-Gorenstein at y, and r is divisible by the Gorenstein index of Y_t at y.*

(iv) *For any $0 < k \leq r$, the base change homomorphism*

$$\phi_t^{[k]} \colon j_*(\omega_{Y^\circ/T}^{\otimes k}) \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \to \omega_{Y_t/\Bbbk(t)}^{[k]} = j_*(\omega_{Y_t^\circ/\Bbbk(t)}^{\otimes k})$$

induced from the base change isomorphism $\omega_{Y^\circ/T} \otimes \mathcal{O}_{Y_t} \simeq \omega_{Y_t^\circ/\Bbbk(t)}$ (cf. Proposition 5.6) is surjective at y.

(v) *There is an open neighborhood U of y such that $f|_U \colon U \to T$ is a naively $\mathbb{Q}$-Gorenstein morphism and $\omega_{U/T}^{[r]}$ is invertible.*

(vi) *There is an open neighborhood U of y such that $f|_U \colon U \to T$ is a $\mathbb{Q}$-Gorenstein morphism and $\omega_{U/T}^{[r]}$ is invertible.*

Then one has the following equivalences and implication on these conditions:

- (i) + (ii) $\Leftrightarrow$ (v)*;*
- (i) + (ii) $\Rightarrow$ (iii)*;*
- (iii) + (iv) $\Leftrightarrow$ (vi).

Proof. First, we shall prove (i) + (ii) $\Rightarrow$ (iii). We set $\mathcal{M}_r := j_*(\omega_{Y^\circ/T}^{\otimes r})$. Then $\mathcal{M}_r \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t}$ is invertible at y by (ii), and

$$\mathcal{M}_r \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \to j_* \left((\mathcal{M}_r \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t})|_{Y^\circ} \right) \simeq j_*(\omega_{Y_t^\circ/\Bbbk(t)}^{\otimes r}) \simeq \omega_{Y_t/\Bbbk(t)}^{[r]}$$

is an isomorphism at y by (i). In particular, $\omega_{Y_t/\Bbbk(t)}^{[r]}$ is invertible at y. Thus, (iii) holds (cf. Definitions 6.1(2) and 6.2).

Second, we shall prove (v) $\Rightarrow$ (i) + (ii) and (vi) $\Rightarrow$ (iii) + (iv). We may assume that f is naively $\mathbb{Q}$-Gorenstein. Since every fiber Y_t is a $\mathbb{Q}$-Gorenstein scheme, we have (i) (cf. Definition 6.1). Moreover,

$$\omega^{[k]}_{Y/T} \simeq j_*(\omega^{\otimes k}_{Y^\circ/T})$$

for any $k \in \mathbb{Z}$ by Proposition 5.6. Hence, (ii) is also satisfied, since $\omega^{[r]}_{Y/T}$ is invertible for an integer $r > 0$. If f is $\mathbb{Q}$-Gorenstein, then $\omega^{[k]}_{Y/T}$ is flat over T and $\omega^{[k]}_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t}$ satisfies the $\mathbf{S}_2$-condition for any $t \in T$ (cf. Definition 7.1(2)); thus, $\phi^{[k]}_t$ is an isomorphism for any $t \in T$ and $k \in \mathbb{Z}$, and in particular, (iii) and (iv) are satisfied.

Finally, we shall prove (i) + (ii) $\Rightarrow$ (v) and (iii) + (iv) $\Rightarrow$ (vi). Assume that (i) holds. By Lemma 2.38, there is an open neighborhood U of y such that $f|_U : U \to T$ is an $\mathbf{S}_2$-morphism having pure relative dimension and $\operatorname{codim}(U_{t'} \setminus Y^\circ, U_{t'}) \geq 2$ for any $t' \in f(U)$, where $U_{t'} = U \cap Y_{t'}$; thus,

$$\omega^{[k]}_{U/T} \simeq j_*(\omega^{\otimes k}_{U \cap Y^\circ/T})$$

for any $k \in \mathbb{Z}$ by Lemma 2.33(4). Therefore, if (ii) also holds, then $\omega^{[r]}_{U'/T}$ is invertible for an open neighborhood U' of y in U, and $f|_{U'} \colon U' \to T$ is a naively $\mathbb{Q}$-Gorenstein morphism. This proves (i) + (ii) $\Rightarrow$ (v). Next assume that (iii) and (iv) hold. Note that (iii) implies (i). Thus, we have the same open neighborhood U of y as above. For any integer $0 < k \leq r$, there is an open neighborhood U'_k of y in U' above such that $\omega^{[k]}_{U'_k/T}$ satisfies relative $\mathbf{S}_2$ over T, by (iv) and by Proposition 5.6. In particular, $\omega^{[r]}_{Y/T}$ is invertible at y by Fact 2.26(2). In fact, it is flat over T at y and its restriction to the fiber Y_t is invertible at y. Then $\omega^{[r]}_{Y/T}$ is invertible on an open neighborhood U''_r of y in U'_r. We set U'' to be the intersection of U'_k for all $0 < k < r$ and U''_r. Then $\omega^{[l]}_{U''/T}$ satisfies relative $\mathbf{S}_2$ over T for any $l \in \mathbb{Z}$, since

$$\omega^{[l]}_{U''/T} \simeq (\omega^{[r]}_{U''/T})^{\otimes m} \otimes \omega^{[k]}_{U''/T}$$

for integers m and k such that $l = mr + k$ and $0 \leq k < r$. This means that $f|_{U''} \colon U'' \to T$ is a $\mathbb{Q}$-Gorenstein morphism, and it proves (iii) + (iv) $\Rightarrow$ (vi). Thus, we are done. □

Remark. For $f \colon Y \to T$ and $j \colon Y^\circ \hookrightarrow Y$ in Lemma 7.3, we have the following properties:

(1) The set of points $y \in Y$ satisfying condition (i) of Lemma 7.3 is open.

(2) If every fiber of f satisfies $\mathbf{S}_2$ and is Gorenstein in codimension one, then $\mathcal{O}_Y \simeq j_*\mathcal{O}_{Y^\circ}$ and $\operatorname{codim}(Y \setminus Y^\circ, Y) \geq 2$. Here, if Y is connected in addition, then f has pure relative dimension.

(3) The set of points $y \in Y$ at which f is naively $\mathbb{Q}$-Gorenstein is open.

(4) The set of points $y \in Y$ at which f is $\mathbb{Q}$-Gorenstein is open.

In fact, property (1) is mentioned in the proof of Lemma 7.3, and property (2) is derived from Lemmas 2.33(4), 2.35, and 2.38. Properties (3) and (4) are deduced from Definition 7.1.

An $\mathbf{S}_2$-morphism of locally Noetherian schemes is not necessarily naively $\mathbb{Q}$-Gorenstein even if every fiber is $\mathbb{Q}$-Gorenstein. The following example is well known.

Example 7.4. Let $S = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(4))$ be the Hirzebruch surface of degree four over an algebraically closed field $\Bbbk$. By contracting of the unique (-4)-curve Γ, we have a birational morphism $S \to X$ to the weighted projective plane $X = \mathbb{P}(1, 1, 4)$. Note that X is $\mathbb{Q}$-Gorenstein and its Gorenstein index is two. Let η be the extension class in $\mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}(4), \mathcal{O}_{\mathbb{P}^1})$ of an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}(2) \oplus \mathcal{O}(2) \to \mathcal{O}(4) \to 0 \tag{VII-1}$$

on $\mathbb{P}^1$. We set $T = \mathbb{A}^1 = \operatorname{Spec} \Bbbk[\mathtt{t}]$ and $P = \mathbb{P}^1 \times_{\Bbbk} T$, and let $p\colon P \to \mathbb{P}^1$ and $q\colon P \to T$ be projections. Let us consider the element η_P of $\mathrm{Ext}^1_P(p^*\mathcal{O}(4), \mathcal{O}_P)$ corresponding to $\eta \otimes \mathtt{t}$ by the isomorphism

$$\mathrm{Ext}^1_P(p^*\mathcal{O}(4), \mathcal{O}_P) \simeq \mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}(4), \mathcal{O}_{\mathbb{P}^1}) \otimes_{\Bbbk} \mathrm{H}^0(T, \mathcal{O}_T),$$

and let

$$0 \to \mathcal{O}_P \to \mathcal{E} \to p^*\mathcal{O}(4) \to 0 \tag{VII-2}$$

be an exact sequence on P whose extension class is η_P. Let $\pi\colon V \to P$ be the $\mathbb{P}^1$-bundle associated with $\mathcal{E}$, and let $f\colon Y \to T$ be the T-scheme defined as $\mathit{Proj}_T\, q_*(\mathit{Sym}(\mathcal{E}))$ for the symmetric $\mathcal{O}_P$-algebra $\mathit{Sym}(\mathcal{E})$. Then Y is a normal projective variety and there is a birational morphism $\mu\colon V \to Y$ over T such that the induced morphism $\mu_t\colon V_t \to Y_t$ of fibers over $t \in T$ is described as follows:

- μ_0 is isomorphic to the contraction morphism $S \to X$ of Γ, and
- μ_t is isomorphic to the identity morphism of $\mathbb{P}^1_{\Bbbk(t)} \times_{\Bbbk(t)} \mathbb{P}^1_{\Bbbk(t)}$ for any $t \neq 0$.

In fact, $\mathcal{L} \simeq \mu^*\mathcal{H}$ for the tautological invertible sheaf $\mathcal{L}$ on V associated with $\mathcal{E}$ and for the f-ample tautological invertible sheaf $\mathcal{H}$ on Y associated with the graded algebra $q_*(\mathit{Sym}\,\mathcal{E})$. In particular, f is a flat projective morphism whose fibers are all $\mathbb{Q}$-Gorenstein. However, f is not naively $\mathbb{Q}$-Gorenstein. For, if the canonical divisor K_Y is $\mathbb{Q}$-Cartier, then $K_{Y_t}^2$ is constant for $t \in T$, but we have $K_{Y_0}^2 = 9$ and $K_{Y_t}^2 = 8$ for $t \neq 0$.

Lemma 7.5. *In the situation of Example 7.4, let* $\operatorname{Spec} A \subset T = \mathbb{A}^1$ *be the closed immersion defined by* $\Bbbk[\mathtt{t}] \to A = \Bbbk[\mathtt{t}]/(\mathtt{t}^2)$*. Then, for the base change* $f_A \colon Y_A \to \operatorname{Spec} A$ *of* $f \colon Y \to T$ *by the closed immersion, the reflexive sheaf* $\omega_{Y_A/A}^{[2]}$ *on* Y_A *is not invertible. Moreover,* $\omega_{Y_A/A}^{[2]}$ *does not satisfy relative* $\mathbf{S}_2$ *over* $\operatorname{Spec} A$*, and* $Y_A \to \operatorname{Spec} A$ *is not a* $\mathbb{Q}$*-Gorenstein morphism.*

Proof. The last assertion follows from the previous one by Fact 2.26(2) and Proposition 5.6, since $\omega_{X/\Bbbk}^{[2]}$ is invertible. Let $g_A \colon V_A \to \operatorname{Spec} A$ be the base change of $g := q \circ \pi \colon V \to T$, and let $\mu_A \colon V_A \to Y_A$ and $\pi_A \colon V_A \to P_A := P \times_{\operatorname{Spec} T} \operatorname{Spec} A \simeq \mathbb{P}^1_A$ be the induced morphisms from the morphisms μ and π over T, respectively. Note that the further base change by $\operatorname{Spec} \Bbbk \to \operatorname{Spec} A$ produces the contraction morphism $S \to X$ and the ruling $S \to \mathbb{P}^1$ from μ_A and π_A, respectively. Assume that $\omega_{Y_A/A}^{[2]}$ is invertible. Then

$$\mathcal{M} := \omega_{V_A/A}^{\otimes 2} \otimes \mu_A^*(\omega_{Y_A/A}^{[2]})^{-1}$$

is invertible on V_A. We have canonical homomorphisms

$$(\mu_A)_*\omega_{V_A/A}^{\otimes 2} \to \omega_{Y_A/A}^{[2]} \quad \text{and} \quad \mu_A^*\left((\mu_A)_*\omega_{V_A/A}^{\otimes 2}\right) \to \omega_{V_A/A}^{\otimes 2}.$$

The first one is obtained by taking double-dual. The second one is surjective, since

$$\omega_{V_A/A} \simeq \omega_{V/T} \otimes_{\mathcal{O}_V} \mathcal{O}_{V_A} \simeq \left(\pi^* p^* \mathcal{O}_{\mathbb{P}^1}(2) \otimes_{\mathcal{O}_V} \mathcal{L}^{\otimes -2}\right) \otimes_{\mathcal{O}_V} \mathcal{O}_{V_A}$$

and since $\mathcal{L} \simeq \mu^*\mathcal{H}$. Therefore, there is a homomorphism $\mathcal{M} \to \mathcal{O}_{V_A}$ which is an isomorphism outside Γ. Since $V_A \times_{\operatorname{Spec} A} \operatorname{Spec} \Bbbk \simeq S$, we have $\operatorname{depth}_\Gamma \mathcal{M} \geq 1$ by Lemma 2.32(3), and it implies that $\mathcal{M} \to \mathcal{O}_{V_A}$ is injective and $\mathcal{M} \otimes_{\mathcal{O}_{V_A}} \mathcal{O}_S \to \mathcal{O}_S$ is also injective. Therefore, the closed subscheme D of V_A defined by the ideal sheaf $\mathcal{M}$ is an effective Cartier divisor, and it is flat over $\operatorname{Spec} A$ by the local criterion of flatness (cf. Proposition A.1(ii)). Moreover, $D \times_{\operatorname{Spec} A} \operatorname{Spec} \Bbbk \simeq \Gamma$ by the isomorphism

$$\omega_{S/\Bbbk}^{\otimes 2} \otimes_{\mathcal{O}_S} \mathcal{O}_S(\Gamma) \simeq \mu_0^*(\omega_{X/\Bbbk}^{[2]}).$$

Hence, the composite $D \subset V_A \to P_A$ is a finite morphism, and the corresponding ring homomorphism $\mathcal{O}_{P_A} \to \pi_{A*}\mathcal{O}_D$ is an isomorphism, since its base change by $A \to \Bbbk$ is isomorphic to the isomorphism $\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_\Gamma$ and since $\pi_{A*}\mathcal{O}_D$ is flat over A. Therefore, D is a section of $\pi_A \colon V_A \to P_A$. Then the pullback of the exact sequence (VII-2) to P_A is split by the surjection

$$\mathcal{E} \otimes_{\mathcal{O}_P} \mathcal{O}_{P_A} \simeq \pi_{A*}(\mathcal{L} \otimes_{\mathcal{O}_V} \mathcal{O}_{V_A}) \to \pi_{A*}(\mathcal{L} \otimes_{\mathcal{O}_V} \mathcal{O}_D) \simeq \pi_{A*}\mathcal{O}_D \simeq \mathcal{O}_{P_A}.$$

This means that the morphism $\operatorname{Spec} A \to T$ factors through $\operatorname{Spec} \Bbbk \subset T$; this is a contradiction. Therefore, $\omega_{Y_A/A}^{[2]}$ is not invertible. ∎

Remark 7.6. Let $f\colon Y \to T$ be an $\mathbf{S}_2$-morphism of locally Noetherian schemes whose fibers are all Gorenstein in codimension one. The *Kollár condition for f along a fiber $Y_t = f^{-1}(t)$* is a condition that the base change homomorphism

$$\phi_t^{[m]}\colon \omega_{Y/T}^{[m]} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \to \omega_{Y_t/\Bbbk(t)}^{[m]}$$

is an isomorphism for any $m \in \mathbb{Z}$. By Proposition 5.6 the Kollár condition is equivalent to the condition that $\omega_{Y/T}^{[m]}$ satisfies relative $\mathbf{S}_2$ over T along Y_t for any $m \in \mathbb{Z}$. Therefore, when Y_t is $\mathbb{Q}$-Gorenstein, the Kollár condition for f is satisfied along Y_t if and only if f is $\mathbb{Q}$-Gorenstein along Y_t. The Kollár condition has been considered for deformations of $\mathbb{Q}$-Gorenstein algebraic varieties of characteristic zero in [27, 2.1.2], [21, §2, Property **K**], etc.

Fact 7.7. Some naively $\mathbb{Q}$-Gorenstein morphisms are not $\mathbb{Q}$-Gorenstein. Kollár gives an example of a naively $\mathbb{Q}$-Gorenstein morphism which is not $\mathbb{Q}$-Gorenstein in the positive characteristic case (cf. [16, 14.7], [32, Exam. 7.6]). See Remark 7.28 below for detail. Patakfalvi has constructed an example of characteristic zero in [45, Thm. 1.2] using some example of projective cones (cf. [45, Prop. 5.4]): This is a projective flat morphism $\mathcal{H} \to B$ of normal algebraic varieties over a field $\Bbbk$ of characteristic zero such that

- B is an open subset of $\mathbb{P}^1_{\Bbbk}$,
- a closed fiber $\mathcal{H}_0$ has a unique singular point, but other fibers are all smooth of dimension ≥ 3,
- $\omega_{\mathcal{H}/B}^{[r]}$ is invertible for some $r > 0$, but

$$\omega_{\mathcal{H}/B} \otimes_{\mathcal{O}_{\mathcal{H}}} \mathcal{O}_{\mathcal{H}_0} \not\simeq \omega_{\mathcal{H}_0/\Bbbk}.$$

Recently, Altmann and Kollár [3] constructed several examples of natively $\mathbb{Q}$-Gorenstein morphisms which are not $\mathbb{Q}$-Gorenstein as infinitesimal deformations of two-dimensional cyclic quotient singularities.

We can construct another example by the following lemma, which is inspired by Patakfalvi's work [45].

Lemma 7.8. *Let S be a non-singular projective variety of dimension ≥ 2 over an algebraically closed field $\Bbbk$ of characteristic zero, and let $\mathcal{L}$ be an invertible $\mathcal{O}_S$-module of order $l > 1$, i.e., l is the smallest positive integer such that $\mathcal{L}^{\otimes l} \simeq \mathcal{O}_S$. Assume that $\mathrm{H}^1(S, \mathcal{O}_S) = 0$, $\mathrm{H}^1(S, \mathcal{L}) \neq 0$ and that K_S is ample. For an integer $r \geq 2$, we set*

$$\mathcal{A} := \mathcal{O}_S(rK_S) \otimes \mathcal{L}^{-1} = \omega_{S/\Bbbk}^{\otimes r} \otimes \mathcal{L}^{-1},$$

and let X be the affine cone $\mathrm{Cone}(S, \mathcal{A})$ with a vertex P. Then

(1) *X is a normal $\mathbb{Q}$-Gorenstein variety with one isolated singularity P of Gorenstein index lr.*

Moreover, for any non-constant function $f\colon X \to T := \mathbb{A}^1_{\Bbbk}$, the following hold:

(2) *f is a naively $\mathbb{Q}$-Gorenstein morphism along the fiber $F = f^{-1}(f(P))$;*

(3) *$\omega^{[r]}_{X/T} \simeq \omega^{[r]}_{X/\Bbbk}$ does not satisfy relative $\mathbf{S}_2$ over T at P, in particular, f is not $\mathbb{Q}$-Gorenstein at P.*

Proof. (1): The affine cone X is $\mathbb{Q}$-Gorenstein by Corollary 6.14(3). Here $X \setminus P$ is a non-singular variety by Lemma 6.10(4). Therefore, X is a normal variety. We have $\omega^{[lr]}_{X/\Bbbk} \simeq \mathcal{O}_X$ by Proposition 6.13(3). If $\omega^{[m]}_{X/\Bbbk}$ is invertible for some $m > 0$, then $\omega^{\otimes m}_{S/\Bbbk} \simeq \mathcal{A}^{\otimes l'}$ for some integer l' by Proposition 6.13(3), but it implies that $m = l'r$, and $\mathcal{L}^{\otimes l'} \simeq \mathcal{O}_S$. Hence, the Gorenstein index of X is lr.

(2): For any $i > 0$ and $m > 0$, we have

$$\mathrm{H}^i(S, \mathcal{A}^{\otimes m}) = \mathrm{H}^i(S, \omega_{S/\Bbbk} \otimes_{\mathcal{O}_S} \mathcal{A}^{\otimes m}) = 0$$

by the Kodaira vanishing theorem, since

$$\mathcal{A}^{\otimes m} \otimes_{\mathcal{O}_S} \omega^{-1}_{S/\Bbbk} \simeq \mathcal{A}^{\otimes m-1} \otimes_{\mathcal{O}_S} \omega^{\otimes r-1}_{S/\Bbbk} \otimes \mathcal{L}^{-1}$$

is ample. Then we can apply Corollary 6.15(2). As a consequence, X satisfies $\mathbf{S}_3$, since $\mathrm{H}^1(S, \mathcal{O}_S) = 0$. Now, f is a flat morphism, since X is irreducible and dominates T. Hence, F satisfies $\mathbf{S}_2$ by the equality

$$\operatorname{depth} \mathcal{O}_{F,x} = \operatorname{depth} \mathcal{O}_{X,x} - \operatorname{depth} \mathcal{O}_{T,f(x)} = \operatorname{depth} \mathcal{O}_{X,x} - 1$$

for any closed point $x \in F$ (cf. (II-2) in Fact 2.26). Thus, f is an $\mathbf{S}_2$-morphism along F, and f is a naively $\mathbb{Q}$-Gorenstein morphism along F (cf. Definition 7.1(1)), since $\omega^{[lr]}_{X/T} \simeq \omega^{[lr]}_{X/\Bbbk}$ is invertible by (1).

(3): By assumption, we have

$$\mathrm{H}^1(S, \omega^{[r]}_{S/\Bbbk} \otimes \mathcal{A}^{-1}) \simeq \mathrm{H}^1(S, \mathcal{L}) \neq 0.$$

Then $\operatorname{depth}(\omega^{[r]}_{X/\Bbbk})_P = 2$ by Proposition 6.13(2). Since $\omega^{[r]}_{X/\Bbbk}$ is flat over T, we have

$$\operatorname{depth}(\omega^{[r]}_{X/\Bbbk} \otimes_{\mathcal{O}_X} \mathcal{O}_F)_P = \operatorname{depth}(\omega^{[r]}_{X/\Bbbk})_P - \operatorname{depth} \mathcal{O}_{T,f(P)} = 1$$

by (II-2) in Fact 2.26. This implies that $\omega^{[r]}_{X/\Bbbk} \simeq \omega^{[r]}_{X/T}$ does not satisfy relative $\mathbf{S}_2$ over T at P. Therefore, f is not $\mathbb{Q}$-Gorenstein at P (cf. Definition 7.1(2)). $\square$

We have the following example of non-singular projective varieties S with invertible $\mathcal{O}_S$-module $\mathcal{L}$ of order $l = 2$ in Lemma 7.8.

Example 7.9. Let V be an abelian variety of dimension $d \geq 3$ and let $\iota\colon V \to V$ be the involution defined by $\iota(v) = -v$ with respect to the group structure on V. Let W be the quotient variety $V/\langle\iota\rangle$. Then W is a normal projective variety with only isolated singular points, and

$$\text{(VII-3)} \qquad \mathrm{H}^1(W, \mathcal{O}_W) = 0,$$

since it is isomorphic to the invariant part of $\mathrm{H}^1(V, \mathcal{O}_V)$ by the induced action of ι, which is just the multiplication map by -1. The quotient morphism $\pi\colon V \to W$ is a double-cover étale outside the singular locus of W, and we have isomorphisms $\pi_*\mathcal{O}_V \simeq \mathcal{O}_W \oplus \omega_{W/\Bbbk}$ and $\omega^{[2]}_{W/\Bbbk} \simeq \mathcal{O}_W$. In particular,

$$\text{(VII-4)} \qquad \mathrm{H}^1(W, \omega_{W/\Bbbk}) \simeq \mathrm{H}^1(V, \mathcal{O}_V) \simeq \Bbbk^{\oplus d}$$

by (VII-3). We can take a smooth ample divisor S on W away from the singular locus of W. Then $\dim S = d - 1 \geq 2$. By the Kodaira vanishing theorem applied to the ample divisor π^*S on V, we have $\mathrm{H}^i(V, \pi^*\mathcal{O}_W(-S)) = 0$ for any $0 < i < d = \dim W$. Hence,

$$\text{(VII-5)} \qquad \mathrm{H}^i(W, \mathcal{O}_W(-S)) = \mathrm{H}^i(W, \omega_{W/\Bbbk} \otimes_{\mathcal{O}_W} \mathcal{O}_W(-S)) = 0$$

for $i = 1$ and 2. The canonical divisor K_S is ample by

$$\omega^{\otimes 2}_{S/\Bbbk} \simeq (\omega^{[2]}_{W/\Bbbk} \otimes_{\mathcal{O}_W} \mathcal{O}_W(2S)) \otimes_{\mathcal{O}_W} \mathcal{O}_S \simeq \mathcal{O}_S(2S).$$

We define $\mathcal{L} := \omega_{W/\Bbbk} \otimes_{\mathcal{O}_W} \mathcal{O}_S$. This is invertible and $\mathcal{L}^{\otimes 2} \simeq \mathcal{O}_S$. We have

$$\mathrm{H}^1(S, \mathcal{O}_S) = 0 \quad \text{and} \quad \mathrm{H}^1(S, \mathcal{L}) \simeq \Bbbk^{\oplus d}$$

by applying (VII-3), (VII-4), and (VII-5) to the cohomology long exact sequences derived from the exact sequences

$$0 \to \mathcal{O}_W(-S) \to \mathcal{O}_W \to \mathcal{O}_S \to 0,$$
$$0 \to \omega_{W/\Bbbk} \otimes_{\mathcal{O}_W} \mathcal{O}_W(-S) \to \omega_{W/\Bbbk} \to \mathcal{L} \to 0.$$

The order of $\mathcal{L}$ is two by $\mathrm{H}^1(S, \mathcal{L}) \not\simeq \mathrm{H}^1(S, \mathcal{O}_S)$. Therefore, S and $\mathcal{L}$ satisfy the conditions of Lemma 7.8.

Definition 7.10 (Relative Gorenstein index). For a naively $\mathbb{Q}$-Gorenstein morphism $f\colon Y \to T$ and for a point $y \in Y$, the *relative Gorenstein index* of f *at* y is the smallest positive integer r such that $\omega^{[r]}_{Y/T}$ is invertible at y. The least common multiple of relative Gorenstein indices at all the points is called the *relative Gorenstein index* of f, which might be $+\infty$.

Proposition 7.11. *Let $f\colon Y \to T$ be a naively $\mathbb{Q}$-Gorenstein morphism. For a point $y \in Y$, let m be the relative Gorenstein index of f at y and let r be the Gorenstein index of $Y_t = f^{-1}(t)$ at y, where $t = f(y)$. Then $m = r$ in the following three cases:*

(i) *f is $\mathbb{Q}$-Gorenstein at y;*
(ii) *Y_t is Gorenstein in codimension two and satisfies $\mathbf{S}_3$ at y;*
(iii) *m is coprime to the characteristic of $\Bbbk(t)$.*

Proof. Note that m is divisible by r. In fact, the base change homomorphism

$$\omega_{Y/T}^{[m]} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \to \omega_{Y_t/\Bbbk(t)}^{[m]}$$

is an isomorphism at y, since the source is invertible at y and since Y_t satisfies $\mathbf{S}_2$. We set $\mathcal{M} := \omega_{Y/T}^{[r]}$. It is enough to prove that $\mathcal{M}$ is invertible at y. Let Z be the complement of the relative Gorenstein locus $\mathrm{Gor}(Y/T)$ and let $j\colon Y \setminus Z \hookrightarrow Y$ be the open immersion. Note that $\mathrm{codim}(Z \cap Y_t, Y_t) \geq 2$ (≥ 3 in case (ii)) and $\mathrm{codim}(Z, Y) \geq 2$. If f is $\mathbb{Q}$-Gorenstein, then $\mathcal{M}$ satisfies relative $\mathbf{S}_2$ over T; in particular,

$$\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq j_*\left((\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t})|_{Y_t \setminus Z}\right) \simeq \omega_{Y_t/\Bbbk(t)}^{[r]}$$

and hence, $\mathcal{M}$ is invertible at y by Fact 2.26(2). Thus, it is enough to consider cases (ii) and (iii). By replacing Y with an open neighborhood of y, we may assume the following:

(1) $\mathrm{depth}_Z \mathcal{O}_Y \geq 2$ (cf. Lemma 2.32(3));
(2) $\mathcal{M}|_{Y \setminus Z}$ is invertible and $\mathrm{depth}_Z \mathcal{M} \geq 2$ (cf. Proposition 5.6);
(3) $j_*(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t}|_{Y_t \setminus Z}) \simeq \omega_{Y_t/\Bbbk(t)}^{[r]}$ is invertible;
(4) one of the following holds:
   (a) $\mathrm{depth}_{Z \cap Y_t} \mathcal{O}_{Y_t} \geq 3$;
   (b) $\mathcal{M}^{[m/r]} \simeq \omega_{Y/T}^{[m]}$ is invertible, where m/r is coprime to the characteristic of $\Bbbk(t)$.

Then $\mathcal{M}$ is invertible by Theorem 3.16, and we are done. □

Remark 7.12. A special case of Proposition 7.11 for naively $\mathbb{Q}$-Gorenstein morphisms is stated in [31, Lem. 3.16], where T is the spectrum of a complete Noetherian local $\mathbb{C}$-algebra and the closed fiber Y_t is a normal complex algebraic surface. However, the proof of [31, Lem. 3.16] has two problems. We explain them using the notation there, where $(X \to S, 0 \in S)$ corresponds to $(Y \to T, t \in T)$ in our situation, and 0 is the closed point of S. The central fiber X_0 is only a germ of a complex algebraic surface in [31, §3], but here, for simplicity, we consider X_0 as a usual algebraic surface and hence consider $X \to S$ as a morphism of finite type. The authors of [31] write X^0 for $\mathrm{Gor}(X/S)$ and write $Y^0 \to X^0$ for the cyclic étale cover associated with an isomorphism $\omega_{X/S}^{[m]} \simeq \mathcal{O}_X$. They want to prove that m is equal to the Gorenstein index r of the fiber X_0 of $X \to S$ over 0.

The first problem lies in the case where $S = \operatorname{Spec} A$ is Artinian. This is minor and is caused by omitting an explanation of the isomorphism $\omega_{X/S}^{[m]} \simeq \mathcal{O}_X$. In this situation, they assert that it is enough to prove the fiber Y_0^0 of $Y^0 \to S$ over 0 to be connected. However, Y_0^0 is connected even if $r \neq m$. In fact, for isomorphisms $u\colon \omega_{X_0/\mathbb{C}}^{[r]} \simeq \mathcal{O}_{X_0}$ and $v\colon \omega_{X/S}^{[m]} \simeq \mathcal{O}_X$, we have an invertible element θ of $\mathcal{O}_{X_0}$ such that

$$v|_{X_0} = \theta u^{\otimes m/r}$$

as an isomorphism $\omega_{X/S}^{[m]} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0} \simeq \mathcal{O}_{X_0}$. Here we can take v so that θ has no k-th roots in $\mathcal{O}_{X_0}$ for any integer k dividing r. Then Y_0^0 is connected for such a v. Of course, this problem is resolved by replacing the isomorphism v with $v(\tilde{\theta})^{-1}$ for a function $\tilde{\theta} \in \mathcal{O}_X$ which is a lift of $\theta \in \mathcal{O}_{X_0}$.

The second problem lies in the reduction to the Artinian case. They set $A_n = A/\mathfrak{m}^n$, $S_n = \operatorname{Spec} A_n$, and $X_n^0 = X^0 \times_S S_n$, for $n \geq 1$ and for the maximal ideal $\mathfrak{m}$ of A, and they obtain an isomorphism

$$\Phi_n \colon \omega_{X_n^0/S_n}^{\otimes r} \simeq \mathcal{O}_{X_n^0}$$

for any n by applying the assertion $m = r$, to the Artinian case. However, just after the isomorphism Φ_n, they deduce an isomorphism $\omega_{X^0/S}^{\otimes r} \simeq \mathcal{O}_{X^0}$ without mentioning any reason. This is thought of as a lack of a proof. In fact, the ideal sheaf $\mathcal{J}$ in Example 3.12 is a counterexample of a similar deduction, where we have isomorphisms $\mathcal{J}|_{U_n} \simeq \mathcal{O}_{U_n}$ for all $n \geq 0$, but $\mathcal{J}|_U \not\simeq \mathcal{O}_U$ (cf. (III-7)). In the hidden argument, the authors of [31] might apply assertion (∗) of Remark 3.13 without any doubt, but (∗) does not hold true in general.

§7.2. Virtually $\mathbb{Q}$-Gorenstein morphisms

Definition 7.13. Let $f\colon Y \to T$ be a morphism locally of finite type between locally Noetherian schemes. For a given point $y \in Y$ and the image $o = f(y)$, the morphism f is said to be *virtually* $\mathbb{Q}$-*Gorenstein at* y if

- f is flat at y,
- the fiber $Y_o = f^{-1}(o)$ is $\mathbb{Q}$-Gorenstein at y,

and if there exist an open neighborhood U of y in Y and a reflexive $\mathcal{O}_U$-module $\mathcal{L}$ satisfying the following conditions:

(i) $\mathcal{L} \otimes_{\mathcal{O}_U} \mathcal{O}_{U_o} \simeq \omega_{U_o/\Bbbk(o)}$, where $U_o = U \cap Y_o$;

(ii) for any integer m, the double-dual $\mathcal{L}^{[m]}$ of $\mathcal{L}^{\otimes m}$ satisfies relative $\mathbf{S}_2$ over T at y.

If f is virtually $\mathbb{Q}$-Gorenstein at every point of Y, then it is called a *virtually $\mathbb{Q}$-Gorenstein morphism.*

Remark 7.14. If the morphism f above is virtually $\mathbb{Q}$-Gorenstein at y, then there exist an open neighborhood U of y in Y and a reflexive $\mathcal{O}_U$-module $\mathcal{L}$ such that

(1) $f|_U : U \to T$ is an $\mathbf{S}_2$-morphism of pure relative dimension,

(2) every non-empty fiber $U_t = U \cap Y_t$ of $f|_U$ is Gorenstein in codimension one, i.e., $\operatorname{codim}(U_t \setminus Y^\circ, U_t) \geq 2$ for any $t \in f(U)$, where $Y^\circ = \operatorname{Gor}(Y/T)$,

(3) $\mathcal{L} \otimes_{\mathcal{O}_U} \mathcal{O}_{U_o} \simeq \omega_{U_o/\Bbbk(o)}$,

(4) $\mathcal{L}|_{U \cap Y^\circ}$ is invertible,

(5) $\mathcal{L}^{[r]}$ is invertible for some integer $r > 0$, and

(6) $\mathcal{L}^{[m]}$ satisfies relative $\mathbf{S}_2$ over T for any integer m.

In fact, we have an open neighborhood U satisfying (1) and (2) by Lemma 2.38. By shrinking U and by Fact 2.26(2), we may assume the existence of $\mathcal{L}$ satisfying (3), (4), and (5), where r is a multiple of the Gorenstein index of Y_o at y. Then, for any point $t \in f(U)$, the coherent sheaf $\mathcal{L}^{[m]}_{(t)} = \mathcal{L}^{[m]} \otimes \mathcal{O}_{U_t}$ is locally equi-dimensional by Fact 2.23(1), since $\operatorname{Supp} \mathcal{L}^{[m]} = U$, $\operatorname{Supp} \mathcal{L}^{[m]}_{(t)} = U_t$, and since U_t is catenary satisfying $\mathbf{S}_2$. Hence, the relative $\mathbf{S}_2$-locus $\mathbf{S}_2(\mathcal{L}^{[m]}/T)$ is an open subset of U by Fact 2.29(2), and now, $y \in \mathbf{S}_2(\mathcal{L}^{[m]}/T)$ for any $m \in \mathbb{Z}$. We have $\mathbf{S}_2(\mathcal{L}^{[m+r]}/T) = \mathbf{S}_2(\mathcal{L}^{[m]}/T)$ for any m by $\mathcal{L}^{[m+r]} \simeq \mathcal{L}^{[r]} \otimes \mathcal{L}^{[m]}$, and hence the intersection of $\mathbf{S}_2(\mathcal{L}^{[m]}/T)$ for all m is still an open neighborhood of y. Thus, we can also assume (6). As a consequence of (1)–(6), we see that

(7) $U_o = U \cap Y_o$ is $\mathbb{Q}$-Gorenstein and

(8) $\mathcal{L}^{[m]} \otimes_{\mathcal{O}_U} \mathcal{O}_{U_o} \simeq \omega^{[m]}_{U_o/\Bbbk(o)}$ for any $m \in \mathbb{Z}$.

In fact, $\mathcal{L}^{[m]} \otimes_{\mathcal{O}_U} \mathcal{O}_{U_o}$ satisfies $\mathbf{S}_2$ by (6) and its depth along $U_o \setminus Y^\circ$ is ≥ 2 by (1) and (2) (cf. Lemma 2.15(2)); this implies (8). Condition (7) follows from (5) and (8).

Remark. The set of points $y \in T$ at which f is virtually $\mathbb{Q}$-Gorenstein is not open in general. Even if a morphism $f : Y \to T$ is virtually $\mathbb{Q}$-Gorenstein at any point of a fiber Y_o, the other fibers Y_t are not necessarily $\mathbb{Q}$-Gorenstein even if $t \in T$ is sufficiently close to the point o. The following gives such an example.

Example 7.15. Let X be a non-singular projective variety over an algebraically closed field $\Bbbk$ of characteristic zero such that the dualizing sheaf $\omega_{X/\Bbbk}$ is ample, $\mathrm{H}^1(X, \mathcal{O}_X) \neq 0$, and $\mathrm{H}^1(X, \omega_{X/\Bbbk}) = 0$. Then $n := \dim X \geq 3$. As an example of X, we can take the product $C \times S$ of a non-singular projective curve C of genus ≥ 2 and a non-singular projective surface S such that $\omega_{S/\Bbbk}$ is ample and $\mathrm{H}^1(S, \mathcal{O}_S) = \mathrm{H}^2(S, \mathcal{O}_S) = 0$. Let us take a positive-dimensional non-singular affine subvariety $T = \operatorname{Spec} A$ of the Picard scheme $\operatorname{Pic}^0(X)$ which contains the origin 0 of $\operatorname{Pic}^0(X)$. Then there is an invertible sheaf $\mathcal{N}$ on $X_A := X \times_{\operatorname{Spec}\Bbbk} T$ such that

- $\mathcal{N}_{(t)}$ is algebraically equivalent to zero for any $t \in T$ and
- $\mathcal{N}_{(t)} \simeq \mathcal{O}_{X_t}$ if and only if $t = 0$,

where $X_t = X \times_{\operatorname{Spec}\Bbbk} \operatorname{Spec}\Bbbk(t)$ and $\mathcal{N}_{(t)} = \mathcal{N} \otimes_{\mathcal{O}_{X_A}} \mathcal{O}_{X_t}$ (cf. Notation 2.24). We define a $\mathbb{Z}_{\geq 0}$-graded A-algebra $R = \bigoplus_{m \geq 0} R_m$ by

$$R_m := \mathrm{H}^0(X_A, (p^*(\omega_{X/\Bbbk}) \otimes_{\mathcal{O}_{X_A}} \mathcal{N})^{\otimes m})$$

for the projection $p\colon X_A \to X$, and let $f\colon Y := \operatorname{Spec} R \to T = \operatorname{Spec} A$ be the induced affine morphism. We shall prove the following by replacing T with a suitable open neighborhood of 0:

(1) *f is a flat morphism*;
(2) *for any $t \in T$, the fiber $Y_t = f^{-1}(t)$ is isomorphic to the affine cone of the polarized scheme $(X_t, \omega_{X_t/\Bbbk(t)} \otimes \mathcal{N}_{(t)})$*;
(3) *the set of points $t \in T$ such that Y_t is $\mathbb{Q}$-Gorenstein, is a countable set*;
(4) *f is virtually $\mathbb{Q}$-Gorenstein at any point of the fiber Y_0.*

For the proof, we consider a graded $\Bbbk(t)$-algebra $R^t = \bigoplus_{m \geq 0} R^t_m$ defined by

$$R^t_m = \mathrm{H}^0(X_t, (\omega_{X_t/\Bbbk(t)} \otimes_{\mathcal{O}_{X_t}} \mathcal{N}_{(t)})^{\otimes m}).$$

Then $\operatorname{Spec} R^t$ is the affine cone associated with $(X_t, \omega_{X_t} \otimes \mathcal{N}_{(t)})$. On the other hand, $Y_t = \operatorname{Spec}(R \otimes_A \Bbbk(t))$, and we have a natural homomorphism

$$\varphi^t\colon R \otimes_A \Bbbk(t) \to R^t$$

of graded $\Bbbk(t)$-algebras, since $(p^*\omega_{X/\Bbbk}) \otimes_{\mathcal{O}_{X_A}} \mathcal{O}_{X_t} \simeq \omega_{X_t/\Bbbk(t)}$. Let φ^t_m be the homomorphism $R_m \otimes_A \Bbbk(t) \to R^t_m$ of the m-th graded piece of φ^t. Note that

$$\mathrm{H}^1(X_t, (\omega_{X_t/\Bbbk(t)} \otimes_{\mathcal{O}_{X_t}} \mathcal{N}_{(t)})^{\otimes m}) = 0$$

for any $m \geq 2$ by the Kodaira vanishing theorem, since $\omega_{X/\Bbbk}$ is ample and $\mathcal{N}_{(t)}$ is algebraically equivalent to zero. Moreover, there is an open neighborhood U of 0 in T such that

$$\mathrm{H}^1(X_t, \omega_{X_t/\Bbbk(t)} \otimes_{\mathcal{O}_{X_t}} \mathcal{N}_{(t)}) = 0$$

for any $t \in U$ by the upper semi-continuity theorem (cf. [12, III, Thm. (7.7.5) I], [40, §5, Cor., p. 50]), since we have assumed that $\mathrm{H}^1(X, \omega_{X/\Bbbk}) = 0$. We may replace T with U. Then φ^t_m is an isomorphism for any $m \geq 1$ and for any $t \in T$ by [12, III, Thm. (7.7.5) II] (cf. [40, §5, Cor. 3, p. 53]). Since φ^t_0 is obviously an isomorphism, φ^t is an isomorphism and $Y_t \simeq \operatorname{Spec} R^t$ for any $t \in T$. Moreover, R_m is a flat A-module for any $m \geq 0$ by [12, III, Cor. (7.5.5)] (cf. [19, III, Thm. 12.11]), and it implies that $Y = \operatorname{Spec} R$ is flat over T. This proves (1) and (2).

By Corollary 6.15(5), Y_t is $\mathbb{Q}$-Gorenstein if and only if $\mathcal{N}_{(t)}^{\otimes r} \simeq \mathcal{O}_{X_t}$ for some $r > 0$. For an integer $r > 0$, let F_r be the kernel of the r-th power map $\operatorname{Pic}^0(X) \to \operatorname{Pic}^0(X)$ which sends an invertible sheaf $\mathcal{L}$ to $\mathcal{L}^{\otimes r}$. Then F_r is a finite set, and $F_r \cap T$ is just the set of points $t \in T$ such that $\mathcal{N}_{(t)}^{\otimes r} \simeq \mathcal{O}_{X_t}$. Thus, Y_t is $\mathbb{Q}$-Gorenstein if and only if t is contained in the countable set $\bigcup_{r>0} F_r \cap T$. This proves (3).

Note that $\omega_{Y_0/\Bbbk} \simeq \mathcal{O}_{Y_0}$ by Proposition 6.13(3). Hence, $f\colon Y \to T$ is virtually $\mathbb{Q}$-Gorenstein at any point of Y_0, since $\mathcal{O}_Y$ plays the role of $\mathcal{L}$ in Definition 7.13. This proves (4).

Lemma 7.16. *Let $f\colon Y \to T$ be a flat morphism locally of finite type between locally Noetherian schemes and let $o \in T$ be a point such that $Y_o = f^{-1}(o)$ is $\mathbb{Q}$-Gorenstein. For a given isomorphism $u\colon \omega^{[r]}_{Y_o/\Bbbk(o)} \to \mathcal{O}_{Y_o}$ for a positive integer r, we set*

$$\mathcal{R} = \bigoplus\nolimits_{i=0}^{r-1} \omega^{[i]}_{Y_o/\Bbbk(o)}$$

to be the $\mathbb{Z}/r\mathbb{Z}$-graded $\mathcal{O}_{Y_o}$-algebra defined by the isomorphism u. Then the following two conditions are equivalent to each other:

(1) *Locally on Y, there exists a $\mathbb{Z}/r\mathbb{Z}$-graded coherent $\mathcal{O}_Y$-algebra $\mathcal{R}^\sim$ flat over T with an isomorphism*
$$\mathcal{R}^\sim \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_o} \simeq \mathcal{R}$$
as a $\mathbb{Z}/r\mathbb{Z}$-graded $\mathcal{O}_{Y_o}$-algebra.

(2) *The morphism f is virtually $\mathbb{Q}$-Gorenstein along Y_o.*

Proof. We write $X = Y_o$ and $\Bbbk = \Bbbk(o)$ for short. First, we shall show (1) $\Rightarrow$ (2). We may assume that $\mathcal{R}^\sim$ is defined on Y. Thus, there exist coherent $\mathcal{O}_Y$-modules $\mathcal{L}_i$ for $0 \leq i \leq r-1$ such that

$$\mathcal{R}^\sim = \bigoplus\nolimits_{i=0}^{r-1} \mathcal{L}_i$$

as a $\mathbb{Z}/r\mathbb{Z}$-graded $\mathcal{O}_Y$-algebra. Hence, $\mathcal{L}_i$ are all flat over T, and moreover,

- $\mathcal{L}_i \otimes_{\mathcal{O}_Y} \mathcal{O}_X \simeq \omega^{[i]}_{X/\Bbbk}$ for any $0 \leq i \leq r-1$,
- the multiplication map $\mathcal{L}_1^{\otimes i} \to \mathcal{L}_i$ restricts to the canonical homomorphism $\omega^{\otimes i}_{X/\Bbbk} \to \omega^{[i]}_{X/\Bbbk}$ for any $1 \leq i \leq r-1$, and

- the multiplication map $\mathcal{L}_1^{\otimes r} \to \mathcal{O}_Y$ induces the isomorphism $u\colon \omega_{X/\Bbbk}^{[r]} \to \mathcal{O}_X$.

We shall show that $\mathcal{L}_1^{[r]} \simeq \mathcal{O}_Y$ and $\mathcal{L}_i \simeq \mathcal{L}_1^{[i]}$ for any $1 \leq i \leq r-1$ along $X = Y_o$. Now, $\mathcal{L}_i$ satisfies relative $\mathbf{S}_2$ over T along X for any $0 \leq i \leq r-1$, since $\omega_{X/\Bbbk}^{[i]}$ satisfies $\mathbf{S}_2$ (cf. Lemma 5.2). Thus, there is a closed subset Z of Y such that

- $\mathrm{Gor}(X) \subset X \setminus Z$,
- $\mathcal{L}_i|_{Y \setminus Z}$ is invertible for any $0 \leq i \leq r-1$ (cf. Fact 2.26(2)),
- the multiplication maps $\mathcal{L}_1^{\otimes i} \to \mathcal{L}_i$ and $\mathcal{L}_1^{\otimes r} \to \mathcal{O}_Y$ are isomorphisms on $Y \setminus Z$.

By replacing Y with its open subset, we may assume that $\mathrm{codim}(Y_t \cap Z, Y_t) \geq 2$ for any $t \in T$ by Lemma 2.38, since $\mathrm{codim}(Y_o \cap Z, Y_o) \geq \mathrm{codim}(X \setminus \mathrm{Gor}(X), X) \geq 2$ and we may assume that $\mathcal{L}_i$ satisfies relative $\mathbf{S}_2$ over T for all i (cf. Fact 2.29(2)). Then, for any $m \geq 1$ and any $1 \leq i \leq r-1$, we have

$$\mathcal{L}_1^{[m]} \simeq j_*(\mathcal{L}_1^{\otimes m}|_{Y \setminus Z}) \quad \text{and} \quad \mathcal{L}_i \simeq j_*(\mathcal{L}_i|_{Y \setminus Z})$$

for the open immersion $j\colon Y \setminus Z \hookrightarrow Y$ by (4) and (5) of Lemma 2.33, respectively. This argument shows that $\mathcal{L}_i \simeq \mathcal{L}_1^{[i]}$ and $\mathcal{O}_Y \simeq \mathcal{L}_1^{[r]}$ along $X = Y_o$.

As a consequence, $\mathcal{L}_1$ satisfies the conditions in Definition 7.13 for any point of Y_o, and we have proved (1) $\Rightarrow$ (2).

Next, we shall show (2) $\Rightarrow$ (1). We may assume the existence of a reflexive $\mathcal{O}_Y$-module $\mathcal{L}$ which satisfies the conditions of Remark 7.14 for $U = Y$ and for the fiber $Y_o = X$. By replacing Y with an open neighborhood of an arbitrary point of Y_o, we may assume that there is an isomorphism $u^{\sim}\colon \mathcal{L}^{[r]} \to \mathcal{O}_Y$ which restricts to the composite of the isomorphism $\mathcal{L}^{[r]} \otimes_{\mathcal{O}_Y} \mathcal{O}_X \simeq \omega_{X/\Bbbk}^{[r]}$ and the isomorphism $u\colon \omega_{X/\Bbbk}^{[r]} \to \mathcal{O}_X$. Then $u^{\sim}$ defines a $\mathbb{Z}/r\mathbb{Z}$-graded $\mathcal{O}_Y$-algebra

$$\mathcal{R}^{\sim} = \bigoplus\nolimits_{i=0}^{r-1} \mathcal{L}^{[i]},$$

which satisfies condition (1). Thus, we are done. □

Remark 7.17. The $\mathbb{Q}$-Gorenstein deformation in the sense of Hacking [15, Def. 3.1] is considered as a virtually $\mathbb{Q}$-Gorenstein deformation by Lemma 7.16. Hacking's notion is generalized to the notion of a *Kollár family of* $\mathbb{Q}$-*line bundles* by Abramovich–Hassett (cf. [1, Def. 5.2.1]). This is related to the notion of a virtually $\mathbb{Q}$-Gorenstein morphism as follows. Let $f\colon Y \to T$ be an $\mathbf{S}_2$-morphism between Noetherian schemes such that every fiber is connected, reduced and $\mathbb{Q}$-Gorenstein. Let $\mathcal{L}$ be a reflexive $\mathcal{O}_Y$-module. Then $\mathcal{L}$ satisfies conditions (i) and (ii) of Definition 7.13 for $U = Y$ and for any $y \in Y$, if and only if $(Y \to T, \mathcal{L})$ is a Kollár family of $\mathbb{Q}$-line bundles with $\mathcal{L} \otimes \mathcal{O}_{Y_t} \simeq \omega_{Y_t/\Bbbk(t)}$ for all $t \in T$. However, in their study of Kollár families $(Y \to T, \mathcal{L})$ for $\mathcal{L} = \omega_{Y/T}$, every fiber and every $\omega^{[m]}_{Y/T}$ are assumed to be Cohen–Macaulay (cf. [1, Rems. 5.3.9, 5.3.10]).

A $\mathbb{Q}$-Gorenstein morphism is always virtually $\mathbb{Q}$-Gorenstein. The following theorem shows, conversely, that a virtually $\mathbb{Q}$-Gorenstein morphism is a $\mathbb{Q}$-Gorenstein morphism under some mild conditions. In particular, we see that a virtually $\mathbb{Q}$-Gorenstein morphism is $\mathbb{Q}$-Gorenstein if it is a Cohen–Macaulay morphism.

Theorem 7.18. *Let Y and T be locally Noetherian schemes and $f\colon Y \to T$ a flat morphism locally of finite type. For a point $t \in T$, assume that f is virtually $\mathbb{Q}$-Gorenstein at any point of the fiber $Y_t = f^{-1}(t)$ and that one of the following two conditions is satisfied:*

(a) *Y_t satisfies $\mathbf{S}_3$;*

(b) *there is a positive integer r coprime to the characteristic of $\Bbbk(t)$ such that $\omega^{[r]}_{Y/T}$ is invertible along Y_t.*

Then f is $\mathbb{Q}$-Gorenstein along Y_t.

Proof. Since the assertion is local, by Remark 7.14, we may assume that f is an $\mathbf{S}_2$-morphism and there is a reflexive $\mathcal{O}_Y$-module $\mathcal{L}$ satisfying the following two conditions:

(1) $\mathcal{L}^{[m]} = (\mathcal{L}^{\otimes m})^{\vee\vee}$ satisfies relative $\mathbf{S}_2$ over T for any integer m;

(2) there is an isomorphism $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq \omega_{Y_t/\Bbbk(t)}$.

We can prove the following for $\mathcal{M} := \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}, \omega_{Y/T})$ applying Theorem 5.10:

(3) *$\mathcal{M}$ is an invertible $\mathcal{O}_Y$-module along Y_t*;

(4) *$\mathcal{L} \simeq \omega_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{M}^{-1}$ along Y_t.*

In fact, condition (ii) of Theorem 5.10 holds by (1) and (2) above, and condition (i) of Theorem 5.10 holds for $U = \mathrm{CM}(Y/T)$ (resp. $U = \mathrm{Gor}(Y/T)$) in case (a) (resp. (b)). The remaining condition (iii) of Theorem 5.10 is checked as follows. In case (a), condition (iii)(a) of Theorem 5.10 is satisfied for U above. In case (b), $\mathcal{L}^{[r]}$ is invertible along Y_t by (1) and (2), since

$$\mathcal{L}^{[r]} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq \omega^{[r]}_{Y_t/\Bbbk(t)} \simeq \omega^{[r]}_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t}$$

is invertible (cf. Fact 2.26(2)); thus, condition (iii)(b) of Theorem 5.10 is satisfied in this case. Therefore, we can apply Theorem 5.10 and obtain (3) and (4).

As a consequence, we have an isomorphism

$$\omega^{[m]}_{Y/T} \simeq \mathcal{L}^{[m]} \otimes_{\mathcal{O}_Y} \mathcal{M}^{\otimes m}$$

for any $m \in \mathbb{Z}$ along Y_t. Therefore, $\omega_{Y/T}^{[m]}$ satisfies relative $\mathbf{S}_2$ over T along Y_t by (1), and hence $f\colon Y \to T$ is $\mathbb{Q}$-Gorenstein along Y_t. □

Corollary 7.19. *Let Y and T be locally Noetherian schemes and $f\colon Y \to T$ a flat morphism locally of finite type. For a point $t \in T$, assume that the fiber $Y_t = f^{-1}(t)$ is quasi-Gorenstein. If $\omega_{Y/T}^{[r]}$ is invertible for a positive integer r coprime to the characteristic of $\Bbbk(t)$, then f is $\mathbb{Q}$-Gorenstein along Y_t.*

Proof. The morphism f is virtually $\mathbb{Q}$-Gorenstein at any point of Y_t, since $\mathcal{O}_Y$ plays the role of $\mathcal{L}$ in Definition 7.13. Thus, we are done by Theorem 7.18 in case (b). □

§7.3. Basic properties of $\mathbb{Q}$-Gorenstein morphisms

We shall prove some basic properties of $\mathbb{Q}$-Gorenstein morphisms and their variants. The following is a criterion for a morphism to be naively $\mathbb{Q}$-Gorenstein.

Lemma 7.20. *Let $f\colon Y \to T$ be an $\mathbf{S}_2$-morphism of locally Noetherian schemes. Assume that T is $\mathbb{Q}$-Gorenstein and that every fiber of f is Gorenstein in codimension one. Then f is a naively $\mathbb{Q}$-Gorenstein morphism if and only if Y is $\mathbb{Q}$-Gorenstein.*

Proof. Since the $\mathbb{Q}$-Gorenstein properties are local, we may assume that T and Y are affine and that f is of finite type with pure relative dimension (cf. Lemma 2.38). Since the $\mathbb{Q}$-Gorenstein scheme T satisfies $\mathbf{S}_2$ (cf. Lemma 6.3(2)), we may assume the following (cf. Lemma 6.4):

- T admits an ordinary dualizing complex $\mathcal{R}^\bullet$ (cf. Lemma 4.14) with the dualizing sheaf $\omega_T := \mathcal{H}^0(\mathcal{R}^\bullet)$;
- the double-dual $\omega_T^{[m]}$ of $\omega_T^{\otimes m}$ satisfies $\mathbf{S}_2$ for any integer m;
- $\omega_T^{[r]}$ is invertible for a positive integer r.

For the Gorenstein locus $T^\circ := \mathrm{Gor}(T)$ and the relative Gorenstein locus $Y^\circ := \mathrm{Gor}(Y/T)$, we set $U := f^{-1}(T^\circ)$ and $U^\circ := U \cap Y^\circ$. Then $\mathrm{codim}(Y \setminus U, Y) \geq 2$ by (II-1) in Fact 2.26 and Property 2.1(3), since f is flat and $\mathrm{codim}(T \setminus T^\circ, T) \geq 2$. Hence, $\mathrm{codim}(Y \setminus U^\circ, Y) \geq 2$ by $\mathrm{codim}(Y \setminus Y^\circ, Y) \geq 2$, since f is an $\mathbf{S}_2$-morphism (cf. Lemma 2.35). The twisted inverse image $\mathcal{R}_Y^\bullet := f^!(\mathcal{R}^\bullet)$ is a dualizing complex of Y (cf. Example 4.24) with a quasi-isomorphism

$$\mathcal{R}_Y^\bullet \simeq_{\mathrm{qis}} f^!\mathcal{O}_T \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathbf{L}f^*(\mathcal{R}^\bullet)$$

by (IV-6) in Fact 4.35, where

$$\omega_{Y^\circ/T}[d] \simeq_{\mathrm{qis}} f^!\mathcal{O}_T|_{Y^\circ}$$

for the relative dimension d of f. Note that Y satisfies $\mathbf{S}_2$ by Fact 2.26(6). Thus, $\mathcal{R}^{\bullet}_Y[-d]$ is an ordinary dualizing complex of Y, and $\omega_Y := \mathcal{H}^{-d}(\mathcal{R}^{\bullet}_Y)$ is a dualizing sheaf of Y. In particular, U° is a Gorenstein scheme with the dualizing sheaf

$$\omega_Y|_{U^{\circ}} = \mathcal{H}^{-d}(\mathcal{R}^{\bullet}_Y)|_{U^{\circ}} \simeq \omega_{Y^{\circ}/T}|_{U^{\circ}} \otimes_{\mathcal{O}_{U^{\circ}}} (f|_{U^{\circ}})^*(\omega_{T^{\circ}}).$$

By Lemma 6.4, we have an isomorphism

$$\omega_Y^{[m]} \simeq \omega_{Y/T}^{[m]} \otimes_{\mathcal{O}_Y} f^*(\omega_T^{[m]}) \tag{VII-6}$$

for any integer m. For a point $y \in Y$, Y is $\mathbb{Q}$-Gorenstein at y if and only if $\omega_Y^{[m]}$ is invertible at y for some $m > 0$. On the other hand, f is naively $\mathbb{Q}$-Gorenstein at y if and only if $\omega_{Y/T}^{[m]}$ is invertible at y for some $m > 0$. Since $\omega_T^{[r]}$ is invertible, the isomorphism (VII-6) implies that Y is $\mathbb{Q}$-Gorenstein if and only if f is naively $\mathbb{Q}$-Gorenstein. □

The following is a criterion for a morphism to be $\mathbb{Q}$-Gorenstein.

Proposition 7.21. *Let $f\colon Y \to T$ be a flat morphism locally of finite type between locally Noetherian schemes. For a point $t \in T$, assume that the fiber $Y_t = f^{-1}(t)$ is a $\mathbb{Q}$-Gorenstein scheme. If there exist coherent $\mathcal{O}_Y$-modules $\mathcal{M}_m$ for $m \geq 1$ such that*

$$\mathcal{M}_m \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \simeq \omega_{Y_t/\Bbbk(t)}^{[m]} \quad \text{and} \quad \mathcal{M}_m|_{Y^{\circ}} \simeq \omega_{Y^{\circ}/T}^{\otimes m},$$

where Y° is the relative Gorenstein locus $\operatorname{Gor}(Y/T)$, then f is a $\mathbb{Q}$-Gorenstein morphism along Y_t.

Proof. We set $\mathcal{M}_0 = \mathcal{O}_Y$. Then $\mathcal{M}_{m,(t)} = \mathcal{M}_m \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t}$ satisfies $\mathbf{S}_2$ along Y_t for any $m \geq 0$. For the complement $Z = Y \setminus Y^{\circ}$, we have $\operatorname{codim}(Z \cap Y_t, Y_t) \geq 2$, since Y_t is $\mathbb{Q}$-Gorenstein. Hence, $\mathcal{M}_m$ is flat over T along Y_t by Lemma 3.5(1), since $\mathcal{M}_m|_{Y^{\circ}} \simeq \omega_{Y^{\circ}/T}^{\otimes m}$ is flat over T and

$$\operatorname{depth}_{Z \cap Y_t} \mathcal{M}_{m,(t)} \geq 2$$

(cf. Lemma 2.15(2)). As a consequence, $\mathcal{M}_m$ satisfies relative $\mathbf{S}_2$ over T along Y_t for any $m \geq 0$. In particular, f is an $\mathbf{S}_2$-morphism along Y_t by considering the case $m = 0$. By replacing Y with an open neighborhood of Y_t, we may assume that f is an $\mathbf{S}_2$-morphism and that $\operatorname{codim}(Z \cap Y_{t'}, Y_{t'}) \geq 2$ for any $t' \in f(Y)$, by Lemma 2.38.

Now, $\operatorname{Supp} \mathcal{M}_m = Y$, since it contains the dense open subset Y°. Hence, $\operatorname{Supp} \mathcal{M}_{m,(t')} = Y_{t'}$ for any $t' \in T$, and it is locally equi-dimensional by Fact 2.23(1). Thus, $U_m := \mathbf{S}_2(\mathcal{M}_m)$ is open by Fact 2.29(2), and

$$\operatorname{depth}_{Z \cap U_m} \mathcal{M}_m|_{U_m} \geq 2$$

by Lemma 2.32(1). It implies that, for the open immersion $j\colon Y^\circ \hookrightarrow Y$,

$$\mathcal{M}_m \to j_*(\mathcal{M}_m|_{Y^\circ}) \simeq j_*(\omega_{Y^\circ/T}^{\otimes m}) = \omega_{Y/T}^{[m]}$$

is an isomorphism along Y_t. As a consequence, $\omega_{Y/T}^{[m]}$ satisfies relative $\mathbf{S}_2$ over T along Y_t for any $m \geq 0$. Therefore, f is a $\mathbb{Q}$-Gorenstein morphism along Y_t. □

We have the following base change properties for $\mathbb{Q}$-Gorenstein morphisms and for their variants.

Proposition 7.22. *Let $f\colon Y \to T$ be a flat morphism locally of finite type between locally Noetherian schemes and let*

$$\begin{array}{ccc} Y' & \xrightarrow{p} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{q} & T \end{array}$$

be a Cartesian diagram of schemes such that T' is also locally Noetherian.

(1) *If f is a naively $\mathbb{Q}$-Gorenstein morphism, then so is f'. Here, if $\omega_{Y/T}^{[r]}$ is invertible, then $\omega_{Y'/T'}^{[r]} \simeq p^*(\omega_{Y/T}^{[r]})$.*
(2) *In the case $q\colon T' \to T$ is a flat and surjective morphism, if f' is naively $\mathbb{Q}$-Gorenstein, then so is f.*
(3) *If every fiber of f is $\mathbb{Q}$-Gorenstein, then every fiber of f' is so. The converse holds if q is surjective.*
(4) *If f is virtually $\mathbb{Q}$-Gorenstein at a point $y \in Y$, then f' is so at any point of $p^{-1}(y)$.*
(5) *If f is $\mathbb{Q}$-Gorenstein, then f' is so and $\omega_{Y'/T'}^{[m]} \simeq p^*(\omega_{Y/T}^{[m]})$ for any $m \in \mathbb{Z}$.*
(6) *In the case $q\colon T' \to T$ is a flat and surjective morphism, if f' is $\mathbb{Q}$-Gorenstein, then so is f.*

Proof. Note that $Y'^\circ = p^{-1}(Y^\circ)$ for $Y'^\circ := \mathrm{Gor}(Y'/T')$ (cf. Corollary 5.7) and that

$$\text{(VII-7)} \qquad \operatorname{codim}(Y_t \setminus Y^\circ, Y_t) = \operatorname{codim}(Y'_{t'} \setminus Y'^\circ, Y')$$

for any $t' \in T'$ and $t = q(t')$ (cf. Lemma 2.31(1)).

(1): The base change f' is an $\mathbf{S}_2$-morphism by Lemma 2.31(5), and we have an isomorphism $\omega_{Y'/T'}^{[r]} \simeq p^*(\omega_{Y/T}^{[r]})$ by Corollary 5.7(2). In particular, f' is a naively $\mathbb{Q}$-Gorenstein morphism.

(2): The morphism f is an $\mathbf{S}_2$-morphism by Lemma 2.31(3) applied to $\mathcal{F} = \mathcal{O}_Y$, since $p\colon Y' \to Y$ is surjective. Moreover, every fiber of f is Gorenstein in codimension one by (VII-7). Now, $p^*(\omega_{Y/T}^{[m]})$ is reflexive for any m by Remark 2.20, since p is flat. Hence, $p^*(\omega_{Y/T}^{[m]}) \simeq \omega_{Y'/T'}^{[m]}$ for any m by Corollary 5.7(2). If $p^*(\omega_{Y/T}^{[r]})$ is invertible, then so is $\omega_{Y/T}^{[r]}$, since p is fully faithful (cf. Lemma A.7). Therefore, f is naively $\mathbb{Q}$-Gorenstein.

(3): This is obtained by applying (1) and (2) to the case where $T = \operatorname{Spec} \Bbbk(t)$ and $T' = \operatorname{Spec} \Bbbk(t')$ and by Lemma 7.20.

(4): We may assume that the conditions of Remark 7.14 are satisfied for $U = Y$, a certain reflexive $\mathcal{O}_Y$-module $\mathcal{L}$, and for $o = f(y)$. Then conditions (1) and (2) of Remark 7.14 imply

$$\operatorname{depth}_{Y_t \setminus Y^\circ} \mathcal{O}_{Y_t} \geq 2$$

for any $t \in f(Y)$, by Lemma 2.15(2). Hence, $p^*(\mathcal{L}^{[m]})$ is a reflexive $\mathcal{O}_{Y'}$-module and $(p^*\mathcal{L})^{[m]} \simeq p^*(\mathcal{L}^{[m]})$ for any m, by Lemma 2.34 applied to $Z = Y \setminus Y^\circ$ and to $\mathcal{L}^{[m]}$. Here $(p^*\mathcal{L})^{[m]}$ satisfies relative $\mathbf{S}_2$ over T' by Remark 7.14(6) and Lemma 2.31(4). Furthermore, for any point $t' \in T'$ and $t = q(t')$, we have isomorphisms

$$p^*\mathcal{L} \otimes_{\mathcal{O}_{Y'}} \mathcal{O}_{Y'_{t'}} \simeq (\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t}) \otimes_{\Bbbk(t)} \Bbbk(t') \simeq \omega_{Y_t/\Bbbk(t)} \otimes_{\Bbbk(t)} \Bbbk(t') \simeq \omega_{Y'_{t'}/\Bbbk(t')},$$

by applying Lemma 5.4 to $\operatorname{Spec} \Bbbk(t') \to \operatorname{Spec} \Bbbk(t)$. Therefore, f' is virtually $\mathbb{Q}$-Gorenstein at any point of $p^{-1}(y)$, since $p^*\mathcal{L}$ plays the role of $\mathcal{L}$ in Definition 7.13.

(5): By (1), f' is an $\mathbf{S}_2$-morphism whose fibers are all $\mathbb{Q}$-Gorenstein. If $\omega_{Y/T}^{[m]}$ satisfies relative $\mathbf{S}_2$ over T, then $p^*\omega_{Y/T}^{[m]}$ does so over T' by Lemma 2.31(4), and $p^*\omega_{Y/T}^{[m]} \simeq \omega_{Y'/T'}^{[m]}$ by Corollary 5.7(2). Therefore, f' is $\mathbb{Q}$-Gorenstein (cf. Definition 7.1(2)).

(6): By (2) above, f is naively $\mathbb{Q}$-Gorenstein. By Lemma 7.3, it is enough to prove that the base change homomorphism

$$\phi_t^{[m]} \colon \omega_{Y/T}^{[m]} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \to \omega_{Y_t/\Bbbk(t)}^{[m]}$$

is an isomorphism for any $m \in \mathbb{Z}$ and any point $t \in T$. For any point $t' \in q^{-1}(t)$, the base change morphism

$$\phi_{t'}^{[m]} \colon \omega_{Y'/T'}^{[m]} \otimes_{\mathcal{O}_{Y'}} \mathcal{O}_{Y'_{t'}} \to \omega_{Y'_{t'}/\Bbbk(t')}^{[m]}$$

is an isomorphism, since f' is $\mathbb{Q}$-Gorenstein. Now, $\phi_{t'}^{[m]}$ is isomorphic to the homomorphism $p_{t'}^*(\phi_t^{[m]})$ for the morphism $p_{t'} \colon Y'_{t'} \to Y_t$ induced from p, since we have an isomorphism $\omega_{Y'/T'}^{[m]} \simeq p^*\omega_{Y/T}^{[m]}$ as in the proof of (2). Since $p_{t'}$ is faithfully flat, $\phi_t^{[m]}$ is an isomorphism for any $m \in \mathbb{Z}$ and $t \in T$. Therefore, f is $\mathbb{Q}$-Gorenstein. □

We have the following properties for compositions of $\mathbb{Q}$-Gorenstein morphisms and of their variants.

Proposition 7.23. *Let $f\colon Y \to T$ and $g\colon X \to Y$ be flat morphisms of locally Noetherian schemes.*

(1) *If f and g are naively $\mathbb{Q}$-Gorenstein, then $f \circ g$ is so, and*
$$\omega^{[r]}_{X/T} \simeq \omega^{[r]}_{X/Y} \otimes_{\mathcal{O}_X} g^*(\omega^{[r]}_{Y/T})$$
for an integer $r > 0$ such that $\omega^{[r]}_{X/Y}$ and $\omega^{[r]}_{Y/T}$ are invertible.

(2) *Assume that g is a $\mathbb{Q}$-Gorenstein morphism. If f is virtually $\mathbb{Q}$-Gorenstein at a point y, then $f \circ g$ is virtually $\mathbb{Q}$-Gorenstein at any point of $g^{-1}(y)$.*

(3) *If f and g are $\mathbb{Q}$-Gorenstein morphisms, then $f \circ g$ is so, and*
$$\omega^{[m]}_{X/T} \simeq \omega^{[m]}_{X/Y} \otimes_{\mathcal{O}_X} g^*(\omega^{[m]}_{Y/T})$$
for any integer m.

Proof. (1): Every fiber of the composite $f \circ g$ is $\mathbb{Q}$-Gorenstein by Lemma 7.20 and by Proposition 7.22(1). In particular, $f \circ g$ is an $\mathbf{S}_2$-morphism. For the relative Gorenstein loci $Y^\circ := \mathrm{Gor}(Y/T)$ and $X^\circ := \mathrm{Gor}(X/Y)$, let V be the intersection $X^\circ \cap g^{-1}(Y^\circ)$. Then $V \subset \mathrm{Gor}(X/T)$ and $\mathrm{codim}(X_t \setminus V, X_t) \geq 2$ for any fiber $X_t = (f \circ g)^{-1}(t)$ of $f \circ g$. We set
$$\mathcal{M}_r := \omega^{[r]}_{X/Y} \otimes g^*(\omega^{[r]}_{Y/T})$$
for an integer $r > 0$ such that $\omega^{[r]}_{X/Y}$ and $\omega^{[r]}_{Y/T}$ are invertible. Then $\mathcal{M}_r|_V \simeq \omega^{\otimes r}_{V/T}$ and
$$\mathcal{M}_r \simeq j_*(\omega^{\otimes r}_{V/T}) = \omega^{[r]}_{Y/T}$$
for the open immersion $j\colon V \hookrightarrow X$, since $f \circ g$ is an $\mathbf{S}_2$-morphism. Thus, $f \circ g$ is naively $\mathbb{Q}$-Gorenstein.

(2): We may assume that the conditions of Remark 7.14 are satisfied for $U = Y$, a certain reflexive $\mathcal{O}_Y$-module $\mathcal{L}$, and for $o = f(y)$. We set
$$\mathcal{N}_m := \omega^{[m]}_{X/Y} \otimes_{\mathcal{O}_X} g^*(\mathcal{L}^{[m]})$$
for an integer m. This is flat over T, since $\mathcal{L}^{[m]}$ is so over T and $\omega^{[m]}_{X/Y}$ is so over Y. Let $g_o = g|_{X_o}\colon X_o \to Y_o$ be the induced $\mathbb{Q}$-Gorenstein morphism (cf. Proposition 7.22(5)). Then $X_o = g^{-1}(Y_o)$ is $\mathbb{Q}$-Gorenstein by Remark 7.14(7) and Lemma 7.20, and we have isomorphisms
$$\begin{aligned}\mathcal{N}_m \otimes_{\mathcal{O}_X} \mathcal{O}_{X_o} &\simeq (\omega^{[m]}_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_o}) \otimes_{\mathcal{O}_{X_o}} g_o^*(\omega^{[m]}_{Y_o/\Bbbk(o)}) \\ &\simeq \omega^{[m]}_{X_o/Y_o} \otimes_{\mathcal{O}_{X_o}} g_o^*(\omega^{[m]}_{Y_o/\Bbbk(o)}) \simeq \omega^{[m]}_{X_o/\Bbbk(o)},\end{aligned}$$

where the first isomorphism is derived from Remark 7.14(8) and the last one from (VII-6) in the proof of Lemma 7.20. In particular, $\mathcal{N}_m$ satisfies relative $\mathbf{S}_2$ over T along X_o. Then, for $\mathcal{N} := \mathcal{N}_1$, we have isomorphisms

$$\mathcal{N}_m \simeq j_*(\mathcal{N}_m|_V) \simeq j_*\left(\omega_{V/Y}^{\otimes m} \otimes_{\mathcal{O}_V} (g^*\mathcal{L})^{\otimes m}|_V\right) \simeq j_*(\mathcal{N}^{\otimes m}|_V) = \mathcal{N}^{[m]}$$

along X_o by Lemma 2.33(5), where $j\colon V \hookrightarrow X$ is the open immersion in the proof of (1). Hence, $\mathcal{N}^{[m]}$ satisfies relative $\mathbf{S}_2$ over T along X_o for any m, and $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_o} \simeq \omega_{X_o/\Bbbk(o)}$. Therefore, $f \circ g$ is virtually $\mathbb{Q}$-Gorenstein at any point of $g^{-1}(y)$, since $\mathcal{N}$ plays the role of $\mathcal{L}$ in Definition 7.13.

(3): We can apply the argument in the proof of (2) by setting $\mathcal{L} = \omega_{Y/T}$. Then

$$\mathcal{N}^{[m]} \simeq j_*(\mathcal{N}_m|_V) \simeq j_*\left(\omega_{V/Y}^{\otimes m} \otimes_{\mathcal{O}_V} (g^*\omega_{Y/T}^{\otimes m})|_V\right) \simeq j_*(\omega_{V/T}^{\otimes m}) = \omega_{X/T}^{[m]}$$

along X_o. Hence, $\omega_{X/T}^{[m]}$ satisfies relative $\mathbf{S}_2$ over T for any m. Consequently, $f \circ g$ is $\mathbb{Q}$-Gorenstein with an isomorphism $\omega_{X/T}^{[m]} \simeq \omega_{X/Y}^{[m]} \otimes_{\mathcal{O}_X} g^*(\omega_{Y/T}^{[m]})$ for any $m \in \mathbb{Z}$. Thus, we are done. □

Corollary 7.24. *Let Y and T be locally Noetherian schemes and $f\colon Y \to T$ a flat morphism locally of finite type. Let $g\colon X \to Y$ be a smooth separated surjective morphism from a locally Noetherian scheme X. Then f is $\mathbb{Q}$-Gorenstein if and only if $f \circ g\colon X \to Y \to T$ is so.*

Proof. For the relative Gorenstein loci $Y^\circ := \mathrm{Gor}(Y/T)$ and $X^\circ := \mathrm{Gor}(X/T)$, we have $X^\circ = g^{-1}(Y^\circ)$ by Lemma 6.7. Let $g^\circ\colon X^\circ \to Y^\circ$ be the induced smooth morphism. Then

$$\text{(VII-8)} \qquad \omega_{X^\circ/T} \simeq \omega_{X^\circ/Y^\circ} \otimes_{\mathcal{O}_{X^\circ}} g^{\circ *}(\omega_{Y^\circ/T})$$

for the relative canonical sheaves $\omega_{Y^\circ/T}$, $\omega_{X^\circ/T}$, and ω_{X°/Y° (cf. (1) and (2) of Fact 4.34). For a point $t \in T$, let $g_t\colon X_t \to Y_t$ be the smooth morphism induced on the fibers $Y_t = f^{-1}(t)$ and $X_t = (f \circ g)^{-1}(t)$.

By Proposition 7.23(2), it is enough to prove the "if" part. Assume that $f \circ g$ is $\mathbb{Q}$-Gorenstein. Then every fiber Y_t is $\mathbb{Q}$-Gorenstein by Lemma 6.7. In particular, Y_t satisfies $\mathbf{S}_2$ and $\mathrm{codim}(Y_t \setminus Y^\circ, Y_t) \geq 2$. Hence, by Lemma 2.33(4),

$$\omega_{Y/T}^{[m]} \simeq j_*(\omega_{Y^\circ/T}^{\otimes m})$$

for any $m \in \mathbb{Z}$, where $j\colon Y^\circ \hookrightarrow Y$ is the open immersion. For the open immersion $j_X\colon X^\circ \hookrightarrow X$, we have an isomorphism

$$g^*(\omega_{Y/T}^{[m]}) \simeq g^*(j_*(\omega_{Y^\circ/T}^{\otimes m})) \simeq j_{X*}(g^{\circ *}(\omega_{Y^\circ/T}^{\otimes m}))$$

by the flat base change isomorphism (cf. Lemma A.9). Thus,

$$\omega^{[m]}_{X/T} \simeq j_{X*}(\omega^{\otimes m}_{X^\circ/T}) \simeq j_{X*}(j^*_X(\omega^{\otimes m}_{X/Y}) \otimes_{\mathcal{O}_{X^\circ}} g^{\circ *}(\omega^{\otimes m}_{Y^\circ/T^\circ})) \\ \simeq \omega^{\otimes m}_{X/Y} \otimes_{\mathcal{O}_Y} g^*(\omega^{[m]}_{Y/T})$$

for any $m \in \mathbb{Z}$ by (VII-8). In particular, $\omega^{[m]}_{Y/T}$ is flat over T, since g is faithfully flat (cf. Lemma A.6). Moreover,

$$g_t^*(\omega^{[m]}_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t}) \simeq \omega^{\otimes -m}_{X_t/Y_t} \otimes_{\mathcal{O}_{X_t}} (\omega^{[m]}_{X/T} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_t}) \simeq \omega^{\otimes -m}_{X_t/Y_t} \otimes_{\mathcal{O}_{X_t}} \omega^{[m]}_{X_t/\Bbbk(t)}$$

satisfies $\mathbf{S}_2$ for any $t \in T$ (cf. Lemma 6.4(2)). As a consequence, $\omega^{[m]}_{Y/T} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t}$ satisfies $\mathbf{S}_2$ by Fact 2.26(6). Therefore, $\omega^{[m]}_{Y/T}$ satisfies relative $\mathbf{S}_2$ over T for any m, and $Y \to T$ is a $\mathbb{Q}$-Gorenstein morphism. □

Remark. Considering an étale morphism g in Corollary 7.24, we see that, for a given flat morphism $f\colon Y \to T$ locally of finite type between locally Noetherian schemes, the $\mathbb{Q}$-Gorenstein condition at a point of Y is not only Zariski local but also étale local (cf. Remark 6.8).

§7.4. Theorems on ℚ-Gorenstein morphisms

First of all, we shall prove some infinitesimal criteria for a morphism to be $\mathbb{Q}$-Gorenstein or naively $\mathbb{Q}$-Gorenstein.

Theorem 7.25 (Infinitesimal criterion). *Let $f\colon Y \to T$ be a flat morphism locally of finite type between locally Noetherian schemes. For a point $y \in Y$ and its image $t = f(y)$, assume that the fiber $Y_t = f^{-1}(t)$ is $\mathbb{Q}$-Gorenstein at y. Then, for a positive integer m, the following two conditions (i) and (ii) are equivalent to each other:*

- (i) *The sheaf $\omega^{[m]}_{Y/T}$ satisfies relative $\mathbf{S}_2$ over T at y.*
- (ii) *Let $\mathcal{O}_{T,t} \to A$ be a surjective local ring homomorphism to an Artinian local ring A and let $Y_A = Y \times_T \operatorname{Spec} A \to \operatorname{Spec} A$ be the base change of f by the associated morphism $\operatorname{Spec} A \to T$. Then $\omega^{[m]}_{Y_A/A}$ satisfies relative $\mathbf{S}_2$ over A at the point $y_A \in Y_A$ lying over y.*

Moreover, the following two conditions (iii) and (iv) are also equivalent to each other:

- (iii) *The sheaf $\omega^{[m]}_{Y/T}$ is invertible at y.*
- (iv) *For the same morphism $Y_A \to \operatorname{Spec} A$ in (ii), $\omega^{[m]}_{Y_A/A}$ is invertible at y_A.*

Proof. Since the fiber Y_t satisfies $\mathbf{S}_2$ at y, by localizing Y, we may assume that f is an $\mathbf{S}_2$-morphism (cf. Fact 2.29(3)). Moreover, we may assume that every fiber of f is Gorenstein in codimension one by Lemma 2.38(2), since

$$\operatorname{codim}_y(Y_t \setminus U, Y_t) \geq 2$$

for the relative Gorenstein locus $U = \operatorname{Gor}(Y/T)$. Let $p_A : Y_A \to Y$ be the projection for a morphism $\operatorname{Spec} A \to T$ in (ii). Then, by Corollary 5.7(2), we have an isomorphism

$$(p_A^* \omega_{Y/T}^{[m]})^{\vee\vee} \simeq \omega_{Y_A/A}^{[m]},$$

and, moreover, the base change homomorphism

$$p_A^* \omega_{Y/T}^{[m]} \to \omega_{Y_A/A}^{[m]}$$

is an isomorphism at y_A when (i) holds. In particular, we have (i) $\Rightarrow$ (ii), and the converse (ii) $\Rightarrow$ (i) is a consequence of Proposition 3.19 in case (I) applied to $\mathcal{F} = \omega_{Y/T}^{[m]}$. If (i) or (ii) holds, then the base change homomorphisms

$$\omega_{Y/T}^{[m]} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_t} \to \omega_{Y_t/\Bbbk(t)}^{[m]} \quad \text{and} \quad \omega_{Y_A/A}^{[m]} \otimes_{\mathcal{O}_{Y_A}} \mathcal{O}_{Y_t} \to \omega_{Y_t/\Bbbk(t)}^{[m]}$$

are isomorphisms at y or y_A, again by Corollary 5.7(2). Hence, by Fact 2.26(2), we have equivalences (iii) $\Leftrightarrow$ (i) + (v) and (iv) $\Leftrightarrow$ (ii) + (v) for the following condition:

(v) The sheaf $\omega_{Y_t/\Bbbk(t)}^{[m]}$ is invertible at y.

Thus, we have (iii) $\Leftrightarrow$ (iv). □

By the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 7.25 for all $m \in \mathbb{Z}$ and for any $y \in Y_t$, we have the following infinitesimal criterion for a morphism to be $\mathbb{Q}$-Gorenstein.

Corollary 7.26 (Infinitesimal criterion). *Let $f : Y \to T$ be a flat morphism locally of finite type between locally Noetherian schemes. Then, for a given point $t \in T$, the morphism f is $\mathbb{Q}$-Gorenstein along the fiber $Y_t = f^{-1}(t)$ if and only if the base change $f_A : Y_A = Y \times_T \operatorname{Spec} A \to \operatorname{Spec} A$ is $\mathbb{Q}$-Gorenstein for any morphism $\operatorname{Spec} A \to T$ defined by a surjective local ring homomorphism $\mathcal{O}_{T,t} \to A$ to any Artinian local ring A.*

Remark. For an Artinian local ring A, a flat morphism $Y_A \to \operatorname{Spec} A$ of finite type is not necessarily a $\mathbb{Q}$-Gorenstein morphism even if Y_A is $\mathbb{Q}$-Gorenstein and A is Gorenstein. For example, let us consider a naively $\mathbb{Q}$-Gorenstein morphism $f : Y \to T = \operatorname{Spec} R$ for a discrete valuation ring R such that f is not $\mathbb{Q}$-Gorenstein along the closed fiber $Y_o = f^{-1}(o)$, where o is the closed point of T corresponding to the maximal ideal $\mathfrak{m}_R$. See Fact 7.7 or Lemma 7.8 and Example 7.9 for such an example of f. Let $\operatorname{Spec} A \to T$ be a closed immersion for a local Artinian ring A. Then A is Gorenstein, and the base change $f_A \colon Y_A = Y \times_T \operatorname{Spec} A \to \operatorname{Spec} A$ of f is a naively $\mathbb{Q}$-Gorenstein morphism by Proposition 7.22(1). Hence, Y_A is $\mathbb{Q}$-Gorenstein by Lemma 7.20. However, f_A is not a $\mathbb{Q}$-Gorenstein morphism for some A by Corollary 7.26.

By the equivalence (iii) $\Leftrightarrow$ (iv) in Theorem 7.25 for any $y \in Y_t$, we have the following version of an infinitesimal criterion for naively $\mathbb{Q}$-Gorenstein morphisms with bounded relative Gorenstein index.

Corollary 7.27 (Infinitesimal criterion). *Let $f \colon Y \to T$ be a flat morphism locally of finite type between locally Noetherian schemes. For a point $t \in T$ and a positive integer m, the following two conditions are equivalent to each other:*

(i) *The morphism f is naively $\mathbb{Q}$-Gorenstein along Y_t and the relative Gorenstein index of f along Y_t is a divisor of m.*

(ii) *The sheaf $\omega^{[m]}_{Y_A/A}$ is invertible for the base change $f_A \colon Y_A = Y \times_T \operatorname{Spec} A \to \operatorname{Spec} A$ by any morphism $\operatorname{Spec} A \to T$ defined by a surjective local ring homomorphism $\mathcal{O}_{T,t} \to A$ to any Artinian local ring A.*

Remark 7.28. The infinitesimal criterion does not hold for naively $\mathbb{Q}$-Gorenstein morphisms without boundedness conditions for the relative Gorenstein index. Let $f \colon Y \to T$ be a flat morphism of finite type of Noetherian schemes such that $T = \operatorname{Spec} R$ for a discrete valuation ring R. For an integer n, we set $R_n := R/\mathfrak{m}_R^{n+1}$, $T_n = \operatorname{Spec} R_n$ and let $Y_n = Y \times_T T_n \to T_n$ be the base change of f by the closed immersion $T_n \subset T$. Assume that the special fiber Y_0 is a $\mathbb{Q}$-Gorenstein scheme and the residue field $\Bbbk = R/\mathfrak{m}_R$ has characteristic $p > 0$. Then $Y_n \to T_n$ is naively $\mathbb{Q}$-Gorenstein for any $n \geq 0$ by an argument of Kollár in [16, 14.7] or [32, Exam. 7.6]. But, there is an example of $f \colon Y \to T$ above such that f is not naively $\mathbb{Q}$-Gorenstein (cf. Example 7.4). Let r_n be the relative Gorenstein index of $Y_n \to T_n$; this equals the Gorenstein index of Y_n. Then $\{r_n\}_{n \geq 0}$ is not bounded by Corollary 7.27 if f is not naively $\mathbb{Q}$-Gorenstein.

By a similar argument in the proof of Theorem 7.25 and by Proposition 3.19 in the case (II) instead of (I), we have the following valuative criterion for a morphism to be $\mathbb{Q}$-Gorenstein.

Theorem 7.29 (Valuative criterion). *Let $f \colon Y \to T$ be a flat morphism locally of finite type between locally Noetherian schemes. Assume that T is reduced. Then f is a $\mathbb{Q}$-Gorenstein morphism if and only if the base change $f_R \colon Y_R = Y \times_T \operatorname{Spec} R \to \operatorname{Spec} R$ is a $\mathbb{Q}$-Gorenstein morphism for any discrete valuation ring R and for any morphism $\operatorname{Spec} R \to T$.*

Proof. It is enough to check the "if" part by Proposition 7.22(5). Then every fiber Y_t is $\mathbb{Q}$-Gorenstein, since we can consider R as the localization at the prime ideal $(\mathtt{x})$ of the polynomial ring $\Bbbk(t)[\mathtt{x}]$ for the residue field $\Bbbk(t)$ and consider the morphism $\operatorname{Spec} R \to T$ defined by the composite $\mathcal{O}_{T,t} \to \Bbbk(t) \subset R$. Therefore, it is enough to prove that $\omega^{[m]}_{Y/T}$ satisfies relative $\mathbf{S}_2$ over T for any $m \in \mathbb{Z}$. For the base change morphism $f_R \colon Y_R \to \operatorname{Spec} R$ and the projection $p \colon Y_R \to Y$, we have an isomorphism

$$(p^*\omega^{[m]}_{Y/T})^{\vee\vee} \simeq \omega^{[m]}_{Y_R/R}$$

for any m by Corollary 5.7(2). Therefore, the assertion is a consequence of Proposition 3.19 in case (II). □

The following theorem gives a criterion for a morphism to be $\mathbb{Q}$-Gorenstein only by conditions on fibers.

Theorem 7.30. *Let Y and T be locally Noetherian schemes and $f \colon Y \to T$ be a flat morphism locally of finite type. For a point $t \in T$, if the following three conditions are all satisfied, then f is $\mathbb{Q}$-Gorenstein along the fiber $Y_t = f^{-1}(t)$:*

(i) *Y_t is $\mathbb{Q}$-Gorenstein;*

(ii) *Y_t is Gorenstein in codimension two;*

(iii) *$\omega^{[m]}_{Y_t/\Bbbk(t)}$ satisfies $\mathbf{S}_3$ for any $m \in \mathbb{Z}$.*

Proof. As in the first part of the proof of Theorem 7.25, by (i) and (ii), we may assume that $Y \to T$ is an $\mathbf{S}_2$-morphism and its fibers are all Gorenstein in codimension one. Then it is enough to prove that $\mathcal{F} = \omega^{[m]}_{Y/T}$ satisfies relative $\mathbf{S}_2$ over T along Y_t for any m. Since $\mathcal{F}$ is reflexive, we can apply Proposition 3.7 and its corollaries to the morphism $Y \to T$ and the closed subset $Z = Y \setminus \operatorname{Gor}(Y/T)$. Then (ii) and (iii) imply inequality (III-4) of Corollary 3.10. Thus, $\mathcal{F}$ satisfies relative $\mathbf{S}_2$ over T along Y_t by Corollaries 3.9 and 3.10. □

Definition 7.31 ($\mathbb{Q}$-Gorenstein refinement). Let $f \colon Y \to T$ be an $\mathbf{S}_2$-morphism of locally Noetherian schemes such that every fiber is $\mathbb{Q}$-Gorenstein. A morphism $S \to T$ from a locally Noetherian scheme S is called a $\mathbb{Q}$-*Gorenstein refinement* of f if the following two conditions are satisfied:

(i) $S \to T$ is a monomorphism in the category of schemes;

(ii) for any morphism $T' \to T$ from a locally Noetherian scheme T', the base change $Y' \times_T T' \to T'$ is a $\mathbb{Q}$-Gorenstein morphism if and only if $T' \to T$ factors through $S \to T$.

Remark. If the $\mathbb{Q}$-Gorenstein refinement $S \to T$ exists, then it is bijective, since every fiber is $\mathbb{Q}$-Gorenstein.

Theorem 7.32. *Let $f\colon Y \to T$ be an $\mathbf{S}_2$-morphism of locally Noetherian schemes whose fibers are all $\mathbb{Q}$-Gorenstein. Assume that*

(i) *$Y \setminus \Sigma \to T$ is a $\mathbb{Q}$-Gorenstein morphism for a closed subset Σ proper over T and*

(ii) *the Gorenstein indices of all the fibers $Y_t = f^{-1}(t)$ are bounded above.*

Then f admits a $\mathbb{Q}$-Gorenstein refinement as a separated morphism $S \to T$ locally of finite type. Furthermore, $S \to T$ is a local immersion of finite type if assumption (i) is replaced with

(iii) *f is a projective morphism locally on T.*

Proof. By (ii), we have a positive integer m such that $\omega^{[m]}_{Y_t/\Bbbk(t)}$ is invertible for any $t \in T$. Let $S \to T$ be the relative $\mathbf{S}_2$ refinement for the reflexive $\mathcal{O}_Y$-module

$$\mathcal{F} = \bigoplus\nolimits_{i=1}^{m} \omega^{[i]}_{Y/T}.$$

It exists as a separated morphism $S \to T$ locally of finite type by Theorem 3.26, since $\mathcal{F}|_U$ is locally free for the relative Gorenstein locus $U = \mathrm{Gor}(Y/T)$ in which $\mathrm{codim}(Y_t \setminus U, Y_t) \geq 2$ for any $t \in T$, and since $\mathcal{F}|_{Y\setminus\Sigma}$ satisfies relative $\mathbf{S}_2$ over T by (i). Moreover, $S \to T$ is a local immersion of finite type in case (iii), also by Theorem 3.26. It is enough to show that $S \to T$ is the $\mathbb{Q}$-Gorenstein refinement. Let $T' \to T$ be a morphism from a locally Noetherian scheme T'. Then $T' \to T$ factors through $S \to T$ if and only if $\omega^{[i]}_{Y'/T'}$ satisfies relative $\mathbf{S}_2$ over T' for the base change $Y' = Y \times_T T' \to T'$ for any $0 \leq i \leq m$, since we have an isomorphism

$$\omega^{[i]}_{Y'/T'} \simeq (p^*\omega^{[i]}_{Y/T})^{\vee\vee}$$

by Corollary 5.7(2) for the projection $p\colon Y' \to Y$. This is also equivalent to the condition that $Y' \to T'$ is $\mathbb{Q}$-Gorenstein. Therefore, $S \to T$ is the relative $\mathbb{Q}$-Gorenstein refinement. □

Example 7.33. Let $f\colon Y \to T$ be the morphism in Example 7.4. Then the $\mathbb{Q}$-Gorenstein refinement $S \to T$ of f is just the disjoint union of $T \setminus \{0\}$ and the closed point $\{0\}$. This is shown as follows. Since f is smooth over $T \setminus \{0\}$ and since $K^2_{Y_0} = 9$ and $K^2_{Y_t} = 8$ for $t \neq 0$, we see that $S \simeq (T \setminus \{0\}) \sqcup \mathrm{Spec}\, A$ over T for a closed immersion $\mathrm{Spec}\, A \to T$ defined by a surjection $\Bbbk[\mathtt{t}] \to \Bbbk[\mathtt{t}]/(\mathtt{t}^n) = A$. On the other hand, if $n \geq 2$, the base change $f_A\colon Y_A \to \mathrm{Spec}\, A$ is not a $\mathbb{Q}$-Gorenstein morphism by Lemma 7.5. Therefore, $A = \Bbbk$.

The following theorem is a local version of Theorem 7.32.

Theorem 7.34. *Let $f\colon Y \to T$ be a flat morphism locally of finite type from a locally Noetherian scheme Y such that $T = \operatorname{Spec} R$ for a Noetherian Henselian local ring R. For the closed point $o \in T$ and for a point y of the closed fiber $Y_o = f^{-1}(o)$, assume that Y_o is $\mathbb{Q}$-Gorenstein at y. Then there is a closed subscheme $S \subset T$ having the following universal property: Let $T' = \operatorname{Spec} R' \to T$ be a morphism defined by a local ring homomorphism $R \to R'$ to a Noetherian local ring R'. Then $T' \to T$ factors through S if and only if the base change $Y' = Y \times_T T' \to T'$ is a $\mathbb{Q}$-Gorenstein morphism at any point $y' \in Y'$ lying over y and the closed point of T'.*

Proof. As in the first part of the proof of Theorem 7.25, we may assume that f is an $\mathbf{S}_2$-morphism and every fiber is Gorenstein in codimension one. For the Gorenstein index m of Y_o at y, we consider the reflexive sheaf

$$\mathcal{F} = \bigoplus\nolimits_{i=1}^{m} \omega_{Y/T}^{[i]}$$

on Y. This is locally free in codimension one on each fiber. Let $S \subset T$ be the universal subscheme in Theorem 3.28 for $\mathcal{F}$. We shall show that S satisfies the required condition. Let $T' = \operatorname{Spec} R \to T$ be the morphism above. Then we have an isomorphism

$$(p^*\omega_{Y/T}^{[i]})^{\vee\vee} \simeq \omega_{Y'/T'}^{[i]}$$

for any $i \in \mathbb{Z}$ by Corollary 5.7(2). Thus, $T' \to T$ factors through S if and only if $\omega_{Y'/T'}^{[i]}$ satisfies relative $\mathbf{S}_2$ over T' at any point y' lying over y and the closed point o' of T', for any $1 \le i \le m$. In this case, $\omega_{Y'/T'}^{[m]}$ is invertible at y' by Fact 2.26(2), since $\omega_{Y_o/\Bbbk(o)}^{[m]}$ is so at y and since the canonical morphism

$$\omega_{Y'/T'}^{[m]} \otimes_{\mathcal{O}_{Y'}} \mathcal{O}_{Y'_{o'}} \to \omega_{Y'_{o'}/\Bbbk(o')}^{[m]} \simeq \omega_{Y_o/\Bbbk(o)}^{[m]} \otimes_{\mathcal{O}_{Y_o}} \mathcal{O}_{Y'_{o'}}$$

is an isomorphism at y' (cf. Corollary 5.7(2)). Therefore, the latter condition at y' is equivalent to the condition that $\omega_{Y'/T'}^{[i]}$ satisfies relative $\mathbf{S}_2$ over T at y' for any $i \in \mathbb{Z}$; this means that $Y' \to T'$ is a $\mathbb{Q}$-Gorenstein morphism at y'. Thus, S satisfies the required condition. □

The following theorem is similar to Theorem 7.32, and it links a projective $\mathbf{S}_2$-morphism Gorenstein in codimension one in each fiber, to a naively $\mathbb{Q}$-Gorenstein morphism by a specific base change.

Theorem 7.35. *Let $f\colon Y \to T$ be a projective $\mathbf{S}_2$-morphism of locally Noetherian schemes such that every fiber is Gorenstein in codimension one. Then, for any positive integer $r > 0$, there exists a separated monomorphism $S_r \to T$ from a locally Noetherian scheme S_r satisfying the following conditions:*

(i) *The morphism $S_r \to T$ is a local immersion of finite type.*

(ii) *Let $T' \to T$ be a morphism from a locally Noetherian scheme T'. Then it factors through $S_r \to T$ if and only if $Y \times_T T' \to T'$ is a naively $\mathbb{Q}$-Gorenstein morphism whose relative Gorenstein index is a divisor of r.*

Proof. By Theorem 3.26, there is a relative $\mathbf{S}_2$ refinement $S \to T$ for the reflexive $\mathcal{O}_Y$-module $\mathcal{F} = \omega^{[r]}_{Y/T}$. In fact, $\mathcal{F}|_U$ is locally free for the relative Gorenstein locus $U = \mathrm{Gor}(Y/T)$ and $\mathrm{codim}(Y_t \setminus U, Y_t) \geq 2$ for any $t \in T$ by assumption. Here $S \to T$ is a separated monomorphism and a local immersion of finite type, since f is a projective morphism. By the universal property of relative $\mathbf{S}_2$ refinement and by Corollary 5.7(2), we see that, for a morphism $T' \to T$ from a locally Noetherian scheme T', it factors through $S \to T$ if and only if $\omega^{[r]}_{Y'/T'}$ satisfies relative $\mathbf{S}_2$ over T' for the base change morphism $Y' = Y \times_T T' \to T'$. Note that $\omega^{[r]}_{Y'/T'}$ is invertible if and only if $Y' \to T'$ is a naively $\mathbb{Q}$-Gorenstein morphism whose relative Gorenstein index is a divisor of r. Let B_r be the set of points $P \in Y_S := Y \times_T S$ such that $\omega^{[r]}_{Y_S/S}$ is not invertible at P. Then B_r is a closed subset of Y_S. Let $S_r \subset S$ be the complement of the image of B_r in S. Then S_r is an open subset. For the morphism $T' \to T$ above, if $\omega^{[r]}_{Y'/T'}$ is invertible, then $T' \to T$ factors through $S \to T$, and for the induced morphism $h\colon Y' \to Y_S$ lying over $T' \to S$, we have an isomorphism

$$\omega^{[r]}_{Y'/T'} \simeq h^*(\omega^{[r]}_{Y_S/S})$$

by Corollary 5.7(2). This implies that $h(Y') \cap B_r = \emptyset$ and that the image of $T' \to S$ is contained in the open subset S_r. Therefore, the composite $S_r \subset S \to T$ is the required morphism. □

Remark. When $f\colon Y \to T$ is a projective morphism, similar results to Theorems 7.32 and 7.35 are found in [29, Cors. 24, 25].

Appendix A. Some basic properties in scheme theory

For readers' convenience, we collect here some famous results on the local criterion of flatness and the base change isomorphisms.

Appendix A.1. Local criterion of flatness

Here we summarize results related to the "local criterion of flatness". It is usually considered as Proposition A.1 below. But the subsequent Corollaries A.2, A.3, A.4 are also useful in the scheme theory. For detail, the reader is referred to [13, IV, §5], [6, III, §5], [12, 0_{III}, §10.2], [2, V, §3], [37, §22], etc. We also mention a "local criterion of freeness” as Lemma A.5, and explain two more results on flatness and local freeness for sheaves on schemes.

Proposition A.1 (Local criterion of flatness). *For a ring A, an ideal I of A and for an A-module M, assume that*

(1) *I is nilpotent or*
(2) *A is Noetherian and M is I-adically ideally separated, i.e., $\mathfrak{a} \otimes_A M$ is separated for the I-adic topology for all ideals $\mathfrak{a}$ of A.*

Then the following four conditions are equivalent to each other:

(i) *M is flat over A;*
(ii) *M/IM is flat over A/I and $\operatorname{Tor}_1^A(M, A/I) = 0$;*
(iii) *M/IM is flat over A/I and the canonical homomorphism*

$$M/IM \otimes_{A/I} I^k/I^{k+1} \to I^k M/I^{k+1} M$$

is an isomorphism for any $k \geq 0$;
(iv) *$M/I^k M$ is flat over A/I^k for any $k \geq 1$.*

Remark. The proof is found in [13, IV, Cor. 5.5, Thm. 5.6], [6, III, §5.2, Thm. 1], [12, 0_{III}, (10.2.1)], [2, V, Thm. (3.2)], [37, Thm. 22.3]. The second assumption, (2), is satisfied, for example, when there is a ring homomorphism $A \to B$ of Noetherian rings such that M is originally a finitely generated B-module and that IB is contained in the *Jacobson radical* $\operatorname{rad}(B)$ of B (cf. [6, III, §5.4, Prop. 2], [12, 0_{III}, (10.2.2)], [37, p. 174]).

Corollary A.2. *Let $A \to B$ be a local ring homomorphism of Noetherian local rings and let $u \colon M \to N$ be a homomorphism of B-modules such that M and N are finitely generated B-modules and that N is flat over A. Then the following two conditions are equivalent to each other:*

(i) *u is injective and the cokernel of u is flat over A;*
(ii) *$u \otimes_A \Bbbk \colon M \otimes_A \Bbbk \to N \otimes_A \Bbbk$ is injective for the residue field $\Bbbk$ of A.*

The proof is given in [13, IV, Cor. 5.7], [12, 0_{III}, (10.2.4)], [2, VII, Lem. (4.1)], [37, Thm. 22.5].

Corollary A.3 (Cf. [12, 0_{IV}, Prop. (15.1.16)], [37, Cor. to Thm. 22.5]). *Let $A \to B$ be a local ring homomorphism of Noetherian local rings and let M be a finitely generated B-module. Let $\Bbbk$ be the residue field of A and let $\bar{x}$ denote the image of $x \in B$ in $B \otimes_A \Bbbk$. For elements $x_1, \ldots, x_n$ in the maximal ideal $\mathfrak{m}_B$, the following two conditions are equivalent to each other:*

(i) $(x_1, \ldots, x_n)$ *is an* M*-regular sequence and* $M/\sum_{i=1}^n x_i M$ *is flat over* A*;*

(ii) $(\bar{x}_1, \ldots, \bar{x}_n)$ *is an* $M \otimes_A \Bbbk$*-regular sequence and* M *is flat over* A*.*

Corollary A.4. *Let* $A \to B$ *and* $B \to C$ *be local ring homomorphisms of Noetherian local rings and let* $\Bbbk$ *be the residue field of* A*. Assume that* B *is flat over* A*. Then, for a finitely generated* C*-module* M*, the following conditions are equivalent to each other:*

(i) M *is flat over* B*;*

(ii) M *is flat over* A *and* $M \otimes_A \Bbbk$ *is flat over* $B \otimes_A \Bbbk$*.*

The proof is given in [13, IV, Cor. 5.9], [6, III, §5.4, Prop. 3], [12, 0_{III}, (10.2.5)], [2, V, Prop. (3.4)].

Next, we shall give the "local criterion of freeness" as Lemma A.5 below, which is similar to Proposition A.1. This result is well known (cf. [13, IV, Prop. 4.1], [6, II, §3.2, Prop. 5], [12, 0_{III}, (10.1.2)]), but is not usually called the "local criterion of freeness" in articles.

Lemma A.5 (Local criterion of freeness). *Let* A *be a ring,* I *an ideal of* A *and* M *an* A*-module such that*

- I *is nilpotent or*
- A *is Noetherian,* $I \subset \operatorname{rad}(A)$*, and* M *is a finitely generated* A*-module.*

Then the following conditions are equivalent to each other:

(i) M *is a free* A*-module;*

(ii) M/IM *is a free* A/I*-module and* $\operatorname{Tor}_1^A(M, A/I) = 0$*;*

(iii) M/IM *is a free* A/I*-module and the canonical homomorphism*

$$M/IM \otimes_{A/I} I^k/I^{k+1} \to I^k M/I^{k+1} M$$

is an isomorphism for any $k \geq 0$*.*

Remark. Applying Lemma A.5 to the case where A is a Noetherian local ring and I is the maximal ideal, we have the equivalence of flatness and freeness for finitely generated A-modules (cf. [13, IV, Cor. 4.3], [12, 0_{III}, (10.1.3)]). On the other hand, the equivalence of flatness and freeness can be proved by other methods (cf. [37, Thm. 7.10], [2, Lem. 5.8]), and using the equivalence, we obtain Lemma A.5 for the same local ring (A, I) and for a finitely generated A-module M, as a corollary of Proposition A.1.

Remark. The equivalence explained above implies the following well-known fact: *for a locally Noetherian scheme X, a coherent flat $\mathcal{O}_X$-module is nothing but a locally free $\mathcal{O}_X$-module of finite rank.*

The following is proved immediately from the definitions of flatness and faithful flatness (cf. [6, I, §3, no. 2, Prop. 4]).

Lemma A.6. *Let $f: X \to Y$ and $g: Y \to Z$ be morphisms of schemes such that f is faithfully flat, i.e., flat and surjective. Then, for an $\mathcal{O}_Y$-module $\mathcal{G}$, it is flat over Z if and only if $f^*\mathcal{G}$ is flat over Z.*

As a corollary in the case where $Y = Z$, we have the following descent property of locally freeness by the relation with flat coherent sheaves.

Lemma A.7. *Let $f: X \to Y$ be a flat surjective morphism of locally Noetherian schemes. For a coherent $\mathcal{O}_Y$-module $\mathcal{G}$, it is locally free if and only if $f^*\mathcal{G}$ is so.*

The authors could not find a good reference for Lemma A.7. For example, we have a weaker result as a part of [13, VIII, Prop. 1.10], where f is assumed additionally to be quasi-compact; however, the quasi-compactness is related to the other part.

Appendix A.2. Base change isomorphisms

Let us consider a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

of schemes, i.e., $X' \simeq X \times_S S'$. Then, for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, one has a functorial canonical homomorphism

$$\theta(\mathcal{F}): g^*(f_*\mathcal{F}) \to f'_*(g'^*\mathcal{F})$$

of $\mathcal{O}_{S'}$-modules, and more generally, a functorial canonical homomorphism

$$\theta^i(\mathcal{F}): g^*(R^i f_*\mathcal{F}) \to R^i f'_*(g'^*\mathcal{F})$$

for each $i \geq 0$. We have the following assertions on $\theta(\mathcal{F})$ and $\theta^i(\mathcal{F})$.

Lemma A.8 (Affine base change). *If f is an affine morphism, then $\theta(\mathcal{F})$ is an isomorphism.*

Lemma A.9 (Flat base change). *Assume that g is flat and that f is quasi-compact and quasi-separated. Then $\theta^i(\mathcal{F})$ is an isomorphism for any i.*

A proof of Lemma A.8 is given [12, II, Cor. (1.5.2)], and a proof of Lemma A.9 is given in [12, III, Prop. (1.4.15)] (cf. [12, IV, (1.7.21)]). Here the morphism $f\colon X \to S$ is said to be "quasi-separated" if the diagonal morphism $X \to X \times_S X$ is quasi-compact (cf. [12, IV, Déf. (1.2.1)]).

We have also the following generalization of Lemma A.9 to the case of complexes by [17, II, Prop. 5.12], [23, IV, Prop. 3.1.0], and [35, Prop. 3.9.5].

Proposition A.10. *In the situation of Lemma A.9, let $\mathcal{F}^{\bullet}$ be a complex of $\mathcal{O}_X$-modules in $\mathbf{D}^{+}_{\mathrm{qcoh}}(X)$. Then there is a functorial quasi-isomorphism*

$$\mathbf{L}g^{*}(\mathbf{R}f_{*}(\mathcal{F}^{\bullet})) \to \mathbf{R}f'_{*}(\mathbf{L}g'^{*}(\mathcal{F}^{\bullet})).$$

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