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An Inverse Problem for the Magnetic Schrödinger Equation in Infinite Cylindrical Domains

by

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Abstract

We study the inverse problem of determining the magnetic field and the electric potential entering the Schrödinger equation in an infinite 3D cylindrical domain, by the Dirichlet-to-Neumann map. The cylindrical domain we consider is a closed waveguide in the sense that the cross section is a bounded domain of the plane. We prove that knowledge of the Dirichlet-to-Neumann map determines uniquely and even Hölder-stably the magnetic field and the electric potential. Moreover, if the maximal strength of both the magnetic field and the electric potential is attained in a fixed bounded subset of the domain, we extend the above results by taking only finitely extended boundary observations of the solution.

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§1. Introduction

§1.1. Statement of the problem

Let ω be a bounded and simply connected domain of $\mathbb{R}^2$ with C^2 boundary $\partial\omega$. We set $\Omega := \omega \times \mathbb{R}$ and for $T > 0$, we consider the initial boundary value problem (IBVP)

$$\begin{cases} (i\partial_t + \Delta_A + q)\, u = 0 & \text{in } Q := (0,T) \times \Omega, \\ u(0,\cdot) = 0 & \text{in } \Omega, \\ u = f & \text{on } \Sigma := (0,T) \times \Gamma, \end{cases} \tag{1.1}$$

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where Δ_A is the Laplace operator associated with the magnetic potential $A \in W^{1,\infty}(\Omega)^3$, i.e.,

$$\Delta_A := \sum_{j=1}^{3} \left(\partial_{x_j} + ia_j\right)^2 = \Delta + 2iA \cdot \nabla + i(\nabla \cdot A) - |A|^2 \tag{1.2}$$

and $q \in L^\infty(\Omega)$. We define the Dirichlet-to-Neumann (DN) map associated with (1.1) as

$$\Lambda_{A,q}(f) := (\partial_\nu + iA \cdot \nu)u, \quad f \in L^2(\Sigma), \tag{1.3}$$

where $\nu(x)$ denotes the unit outward normal vector to $\partial\Omega$ at x and u is the solution to (1.1).

In the remaining part of this text, two magnetic potentials $A_j \in W^{1,\infty}(\Omega)^3$, $j = 1, 2$ are said to be gauge equivalent if there exists $\Psi \in W^{2,\infty}(\Omega)$ obeying $\Psi|_\Gamma = 0$ such that

$$A_2 = A_1 + \nabla\Psi. \tag{1.4}$$

In this paper we examine the uniqueness and stability issues in the inverse problem of determining the electric potential q and the gauge class of A from knowledge of $\Lambda_{A,q}$.

§1.2. Physical motivations

System (1.1) describes the quantum motion of a charged particle (the various physical constants are taken equal to 1) constrained by the unbounded domain Ω, under the influence of the electric potential q and the magnetic field generated by A. Carbon nanotubes, with length-to-diameter ratio up to $10^8/1$, are commonly modeled by infinite waveguides such as Ω. In this context, the inverse problem under consideration in this paper can be rephrased as whether the strength of the electromagnetic quantum disorder (modeled by the magnetic field and the electric impurity potential q; see, e.g., [17, 30]) can be determined by boundary measurement of the wave function u.

§1.3. State of the art

Inverse coefficient problems for partial differential equations such as the Schrödinger equation are the source of challenging mathematical problems that have attracted a lot of attention over recent decades. For instance, using the Bukhgeim–Klibanov method (see [15, 36, 37]), [4] claims Lipschitz stable determination of the time-independent electric potential perturbing the dynamic (i.e., nonstationary) Schrödinger equation, from a single boundary measurement of the solution. In this case, the observation is performed on a subboundary fulfilling the geometric optics condition for the observability derived by Bardos, Lebeau and Rauch in [3]. This geometrical condition was removed by [9] for potentials that are a priori known in a neighborhood of the boundary, at the expense of weaker stability. In the same spirit, using the Bukhgeim–Klibanov method, [22] determines Lipschitz-stably the magnetic potential in the Coulomb gauge class, from a finite number of boundary measurements of the solution. Uniqueness results in inverse problems for the DN map related to the magnetic Schrödinger equation are also available in [25], but they are based on a different approach involving geometric optics (GO) solutions. The stable recovery of the magnetic field by the DN map of the dynamic magnetic Schrödinger equation is established in [10] by combining the approach used for determining the potential in hyperbolic equations (see [6, 8, 12, 29, 45, 48, 50]) with the one employed for the identification of the magnetic field in elliptic equations (see [23, 46, 51]). Notice that in the one-dimensional case, [2] proved with the boundary control method introduced in [5] that the DN map uniquely determines the time-independent electric potential of the Schrödinger equation. In [11] the time-independent electric potential is stably determined by the DN map associated with the dynamic magnetic Schrödinger equation on a Riemannian manifold. This result was recently extended by [7] to simultaneous determination of both the magnetic field and the electric potential. As for inverse coefficient problems of the Schrödinger equation with either Neumann, spectral or scattering data, we refer to [23, 24, 26, 33, 38, 39, 46, 47, 51, 53].

All the above-mentioned results are obtained in a bounded domain. Actually, there are only a small number of mathematical papers dealing with inverse coefficient problems in unbounded domains. One of them, [44], examines the problem of determining a potential appearing in the wave equation in the half-space. Assuming that the potential is known outside a fixed compact set, the author proves that it is uniquely determined by the DN map. Unique determination of compactly supported potentials appearing in the stationary Schrödinger equation in an infinite slab from partial DN measurements is established in [40]. The same problem is addressed by [38] for the stationary magnetic Schrödinger equation, and by [54] for biharmonic operators with perturbations of order zero or one. The inverse problem of determining the twisting function of an infinite twisted waveguide by the DN map is addressed in [21]. The analysis carried out in [29, 45, 48, 50] is adapted to unbounded cylindrical domains in [21] for time-independent potentials with prescribed behavior outside a compact set. In [35], electric potentials with suitable exponential decay along the infinite direction of the waveguide are stably recovered from a single boundary measurement of the solution. This is by means of a specifically designed Carleman estimate for the dynamic Schrödinger equation in infinite cylindrical domains, derived in [34]. The geometrical condition satisfied by the boundary data measurements in [35] is relaxed in [13] for potentials that are known in a neighborhood of the boundary. In [18], time-dependent potentials that are periodic in the translational direction of the waveguide are stably retrieved by the DN map of the Schrödinger equation, and periodic potentials are stably recovered in [31] from the asymptotics of the boundary spectral data of the Dirichlet Laplacian. As for the Calderón problem in a waveguide, translationally invariant unknown coefficients are uniquely determined by the DN map in [28], whereas the case of periodic coefficients is treated by [19, 20].

§1.4. Well-posedness

We start by examining the well-posedness of the IBVP (1.1) in the functional space $\mathcal{C}([0,T],H^1(\Omega))\cap\mathcal{C}^1([0,T],H^{-1}(\Omega))$. Namely, we are aiming for sufficient conditions on the coefficients A, q and the nonhomogeneous Dirichlet data f, ensuring that (1.1) admits a unique solution in the transposition sense. We say that $u\in L^\infty(0,T;H^{-1}(\Omega))$ is a solution to (1.1) in the transposition sense if the identity

$$\langle u,F\rangle_{L^\infty(0,T;H^{-1}(\Omega)),L^1(0,T;H_0^1(\Omega))}=\langle f,\partial_\nu v\rangle_{L^2(\Sigma)}$$

holds for any $F\in L^1(0,T;H_0^1(\Omega))$. Here, v denotes the unique $\mathcal{C}([0,T],H^1(\Omega))$-solution to the transposition system

$$\begin{cases}(i\partial_t v+\Delta_A+q)v=F & \text{in } Q,\\ v(T,\cdot)=0 & \text{in } \Omega,\\ v=0 & \text{on } \Sigma.\end{cases}\tag{1.5}$$

We refer to Section 2.3 for the full definition and description of transposition solutions to (1.1).

Since $\partial\Omega$ is not bounded, we introduce the following notation. First, we set

$$H^s(\partial\Omega):=H^s_{x_3}(\mathbb{R},L^2(\partial\omega))\cap L^2_{x_3}(\mathbb{R},H^s(\partial\omega)),\quad s>0,$$

where x_3 denotes the longitudinal variable of Ω. Next we put

$$H^{r,s}((0,T)\times X):=H^r(0,T;L^2(X))\cap L^2(0,T;H^s(X)),\quad r,s>0,$$

where X is either Ω or $\partial\Omega$. For the sake of shortness, we write $H^{r,s}(Q)$ (resp., $H^{r,s}(\Sigma)$) instead of $H^{r,s}((0,T)\times\Omega)$ (resp., $H^{r,s}((0,T)\times\partial\Omega)$). Finally, we define

$$H_0^{2,1}(\Sigma):=\{f\in H^{2,1}(\Sigma);\ f(0,\cdot)=\partial_t f(0,\cdot)=0\}$$

and state the existence and uniqueness result of solutions to (1.1) in the transposition sense, as follows.

Theorem 1.1. *For $M>0$, let $A\in W^{1,\infty}(\Omega,\mathbb{R})^3$ and $q\in W^{1,\infty}(\Omega,\mathbb{R})$ satisfy the condition*

$$\|A\|_{W^{1,\infty}(\Omega)^3}+\|q\|_{W^{1,\infty}(\Omega)}\leqslant M. \tag{1.6}$$

Then, for each $f\in H_0^{2,1}(\Sigma)$, the IBVP (1.1) admits a unique solution $u\in H^1(0,T;H^1(\Omega))$ in the transposition sense and the estimate

$$\|u\|_{H^1(0,T;H^1(\Omega))}\leqslant C\|f\|_{H^{2,1}(\Sigma)} \tag{1.7}$$

holds for some positive constant C depending only on T, ω and M. Moreover, the normal derivative $\partial_\nu u\in L^2(\Sigma)$ and we have

$$\|\partial_\nu u\|_{L^2(\Sigma)}\leqslant C\|f\|_{H^{2,1}(\Sigma)}. \tag{1.8}$$

It is clear from definition (1.3) and the continuity property (1.8) that the DN map $\Lambda_{A,q}$ belongs to $\mathcal{B}(H_0^{2,1}(\Sigma),L^2(\Sigma))$, the set of linear bounded operators from $H_0^{2,1}(\Sigma)$ into $L^2(\Sigma)$.

§1.5. Nonuniqueness

There is a natural obstruction to the identification of A by $\Lambda_{A,q}$, arising from the invariance of the DN map under gauge transformation. More precisely, if $\Psi\in W^{2,\infty}(\Omega)$ verifies $\Psi|_\Gamma=0$ then we have $u_{A+\nabla\Psi}=e^{-i\Psi}u_A$, where u_A (resp., $u_{A+\nabla\Psi}$) denotes the solution to (1.1) associated with the magnetic potential A (resp., $A+\nabla\Psi$), $q\in L^\infty(\Omega)$ and $f\in H_0^{2,1}(\Sigma)$. Further, as

$$(\partial_\nu+i(A+\nabla\Psi)\cdot\nu)u_{A+\nabla\Psi}=e^{-i\Psi}(\partial_\nu+iA\cdot\nu)u_A=(\partial_\nu+iA\cdot\nu)u_A\quad\text{on }\Sigma,$$

by direct calculation, we get $\Lambda_{A,q}=\Lambda_{A+\nabla\Psi,q}$, despite the fact that the two potentials A and $A+\nabla\Psi$ do not coincide in Ω (unless ψ is uniformly zero).

This shows that the best we can expect from knowledge of the DN map is to identify (A,q) modulo gauge transformation of A. When $A_{|\partial\Omega}$ is known, this may be equivalently reformulated as whether the magnetic field defined by the 2-form associated with the vector curl A,

$$\mathrm{d}A:=\frac{1}{2}\sum_{i,j=1}^{3}\left(\partial_{x_j}a_i-\partial_{x_i}a_j\right)\mathrm{d}x_j\wedge\mathrm{d}x_i,$$

and the electric potential q can be retrieved by $\Lambda_{A,q}$. This is the inverse problem that we examine in the remaining part of this article.

§1.6. Main results

We define the set of admissible magnetic potentials as

$$\mathcal{A} := \big\{ A = (a_i)_{1 \leqslant i \leqslant 3};\ a_1, a_2 \in L^\infty_{x_3}(\mathbb{R}, H_0^2(\omega)) \cap W^{2,\infty}(\Omega) \\ \text{and } a_3 \in C^3(\overline{\Omega}) \text{ fulfills (1.9)–(1.10)} \big\},$$

where

$$\sup_{x \in \Omega} \left(\sum_{\alpha \in \mathbb{N}_0^3,\ |\alpha| \leqslant 3} \langle x_3 \rangle^d |\partial_x^\alpha a_3(x)| \right) < \infty \quad \text{for some } d > 1, \tag{1.9}$$

and

$$\partial_x^\alpha a_3(x) = 0, \quad x \in \partial\Omega,\ \alpha \in \mathbb{N}_0^3 \text{ such that } |\alpha| \leqslant 2. \tag{1.10}$$

Here and henceforth, $H_0^2(\omega)$ denotes the closure of $\mathcal{C}_0^\infty(\omega)$ in the $H^2(\omega)$-topology, $\langle x_3 \rangle := (1 + x_3^2)^{1/2}$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

As will appear in Section 5 below, the technical conditions (1.9)–(1.10) are useful for reducing the analysis of the inverse problem under investigation to the particular case of unknown transverse magnetic potentials, i.e., magnetic potentials whose third component is a priori known, which is a cornerstone of the strategy used for proving the stability results of this article. This is made possible by (5.1), showing that any magnetic potential $A \in \mathcal{A}$ admits a transverse magnetic potential which is gauge equivalent to A.

The first result of this paper claims stable determination of the magnetic field $\mathrm{d}A$ and unique identification of electric potential q from knowledge of the full data, i.e., the DN map defined by (1.3) where both the Dirichlet and Neumann measurements are performed on the whole boundary Σ.

Theorem 1.2. *Fix $A_* := (a_{i,*})_{1 \leqslant i \leqslant 3} \in W^{2,\infty}(\Omega, \mathbb{R})^3$ and for $j = 1, 2$, let $q_j \in W^{1,\infty}(\Omega)$ and $A_j := (a_{i,j})_{1 \leqslant i \leqslant 3} \in A_* + \mathcal{A}$ satisfy the condition*

$$\sum_{i=1}^{2} \partial_{x_i} \left(\partial_{x_3}(a_{i,1} - a_{i,2}) - \partial_{x_i}(a_{3,1} - a_{3,2}) \right) = 0 \quad \text{in } \Omega. \tag{1.11}$$

Then $\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2}$ yields $\mathrm{d}A_1 = \mathrm{d}A_2$ and $q_1 = q_2$.

Assume moreover that the estimate

$$\sum_{j=1}^{2} \left(\|A_j\|_{W^{2,\infty}(\Omega)} + \|q_j\|_{W^{1,\infty}(\Omega)} + \|e_j\|_{W^{3,\infty}(\Omega)} \right) + \|A_*\|_{W^{2,\infty}(\Omega)} \leqslant M \tag{1.12}$$

holds for some $M>0$*, with*

$$e_j(x',x_3) := \int_{-\infty}^{x_3} (a_{3,j}(x',y_3) - a_{3,*}(x',y_3))\mathrm{d}y_3, \quad (x',x_3)\in\Omega.$$

Then, there exist two constants $\mu_0 \in (0,1)$ *and* $C>0$*, both of them depending only on* T*,* ω *and* M*, such that we have*

$$\|\mathrm{d}A_1 - \mathrm{d}A_2\|_{L^\infty_{x_3}(\mathbb{R},L^2(\omega))} \leqslant C\|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\|^{\mu_0}. \tag{1.13}$$

In (1.13) and in the remaining part of this text, $\|\cdot\|$ denotes the usual norm in $\mathcal{B}(H^{2,1}(\Sigma), L^2(\Sigma))$, the space of linear bounded operators from $H^{2,1}(\Sigma)$ into $L^2(\Sigma)$.

Notice that condition (1.11) imposes that the transverse magnetic field induced by $A_1 - A_2$ (i.e., the vector defined by the two first components of $\mathrm{d}A_1 - \mathrm{d}A_2$) has a magnetic field strength gradient. We point out that many applications, such as magnetic resonance imaging techniques, need magnetic structures having a permanent transverse magnetic field with a magnetic field strength gradient; see [43, Sect. 11.2.1].

In Theorem 1.2 we make use of the full DN map, as the magnetic field $\mathrm{d}A$ and the electric potential q are recovered by observing the solution to (1.1) on the entire lateral boundary Σ. In this case we may consider general unknown coefficients, in the sense that the behavior of A and q with respect to the infinite variable is not prescribed (we assume only that these coefficients and their derivatives are uniformly bounded in Ω). In order to achieve the same result by measuring on a bounded subset of Σ only, we need some extra information on the behavior of the unknown coefficients with respect to x_3. Namely, we impose that the strength of the magnetic field generated by $A=(a_i)_{1\leqslant i\leqslant 3}$ reaches its maximum in the bounded subset $(-r,r)\times\omega$ of Ω, for some fixed $r>0$, i.e., that

$$\|\partial_{x_i}a_j - \partial_{x_j}a_i\|_{L^\infty_{x_3}(\mathbb{R},L^2(\omega))} = \|\partial_{x_i}a_j - \partial_{x_j}a_i\|_{L^\infty_{x_3}(-r,r;L^2(\omega))}, \quad i,j=1,2,3. \tag{1.14}$$

Thus, with reference to (1.14), we set $\Gamma_r := \partial\omega\times(-r,r)$, introduce the space

$$\begin{aligned} H_0^{2,1}((0,T)\times\Gamma_r) := \big\{ & f\in H^{2,1}(\Sigma);\, f(0,\cdot) = \partial_t f(0,\cdot) = 0 \\ & \text{and supp } f\subset [0,T]\times\partial\omega\times[-r,r]\big\}, \end{aligned}$$

and define the partial DN map $\Lambda_{A,q,r}$ by

$$\Lambda_{A,q,r}(f) := (\partial_\nu + iA\cdot\nu)u_{|(0,T)\times\Gamma_r}, \quad f\in H_0^{2,1}((0,T)\times\Gamma_r),$$

where u denotes the solution to (1.1). The following result states, for each $r>0$, that the magnetic field induced by potentials belonging (up to an additive $W^{2,\infty}(\Omega,\mathbb{R})^3$-term) to

$$\mathcal{A}_r := \{A = (a_i)_{1\leqslant i\leqslant 3} \in \mathcal{A} \text{ satisfying } (1.14)\}$$

can be retrieved from knowledge of the partial DN map $\Lambda_{A,q,r'}$, provided we have $r' > r$.

Theorem 1.3. *For $j = 1,2$, let $q_j \in W^{1,\infty}(\Omega,\mathbb{R})$ and let $A_j \in W^{2,\infty}(\Omega,\mathbb{R})^3$ satisfy $A_1 - A_2 \in \mathcal{A}_r$, for some $r > 0$. Suppose that there exists $r' > r$ such that $\Lambda_{A_1,q_1,r'} = \Lambda_{A_2,q_2,r'}$. Then, we have $\mathrm{d}A_1 = \mathrm{d}A_2$. Furthermore, if*

$$\|q_1 - q_2\|_{L^\infty_{x_3}(\mathbb{R},H^{-1}(\omega))} = \|q_1 - q_2\|_{L^\infty_{x_3}(-r,r;H^{-1}(\omega))},$$

then we have in addition $q_1 = q_2$.

Assume moreover that (1.11)–(1.12) hold. Then, the estimate

$$\|\mathrm{d}A_1 - \mathrm{d}A_2\|_{L^\infty_{x_3}(\mathbb{R},L^2(\omega))^3} \leqslant C\|\Lambda_{A_1,q_1,r'} - \Lambda_{A_2,q_2,r'}\|^{\mu_1} \tag{1.15}$$

holds with two constants $C > 0$ and $\mu_1 \in (0,1)$ that depend only on T, ω, M, r and r'.

We stress that Theorem 1.3 applies not only to magnetic (resp., electric) potentials A_j (resp., q_j), $j = 1,2$, which coincide outside $\omega \times (-r,r)$, but to a fairly general class of magnetic potentials containing, e.g., $2r$-periodic potentials with respect to x_3. More generally, if $g \in W^{2,\infty}(\mathbb{R},\mathbb{R}_+)$ (resp., $g \in W^{1,\infty}(\mathbb{R},\mathbb{R}_+)$) is an even and nonincreasing function in $\mathbb{R}_+$ then it is easy to see that potentials of the form $g \times A_j$ (resp., $g \times q_j$), where A_j (resp., q_j) are suitable $2r$-periodic magnetic (resp., electric) potentials with respect to x_3, fulfill the conditions of Theorem 1.3.

Notice that the absence of stability for the electric potential q, manifested in both Theorems 1.2 and 1.3, arises from the infinite extension of the spatial domain Ω in the x_3-direction. Indeed, the usual derivation of a stability equality for q, from estimates such as (1.13) or (1.15), requires that the differential operator d be invertible in Ω. Such a property is true in bounded domains (see, e.g., [53]) but, to the best of our knowledge, it is not known whether it can be extended to unbounded waveguides. One way to overcome this technical difficulty is to impose a certain gauge condition on the magnetic potentials, by prescribing their divergence. In this case, we establish in Theorem 1.4 below, that the electric and magnetic potentials can be simultaneously and stably determined by the DN map.

1.6.1. Simultaneous stable recovery of magnetic and electric potentials. We first introduce the set of divergence-free transverse magnetic potentials,

$$\mathcal{A}_0 := \big\{A = (a_1, a_2, 0);\ a_1, a_2 \in L^\infty_{x_3}(\mathbb{R}, H^2_0(\omega)) \cap W^{2,\infty}(\Omega),\\ \partial_{x_1} a_1 + \partial_{x_2} a_2 = 0 \text{ in } \Omega\big\}$$

in such a way that we have $\nabla \cdot A = \nabla \cdot A_*$ for any $A \in A_* + \mathcal{A}_0$, where $A_* \in W^{2,\infty}(\Omega)^3$ is an arbitrary fixed magnetic potential. Since determining $A \in A_* + \mathcal{A}_0$ from knowledge of the DN map amounts to recovering the magnetic field $\mathrm{d}A$, then we have the following result.

Theorem 1.4. *Let $M > 0$ and let $A_* \in W^{2,\infty}(\Omega, \mathbb{R})^3$. For $j = 1, 2$, let $q_j \in W^{1,\infty}(\Omega, \mathbb{R})$ and let $A_j \in A_* + \mathcal{A}_0$ satisfy (1.12). Then, there exist two constants $\mu_2 \in (0, 1)$ and $C = C(T, \omega, M) > 0$ such that we have*

$$\|A_1 - A_2\|_{L^\infty_{x_3}(\mathbb{R},L^2(\omega))^3} + \|q_1 - q_2\|_{L^\infty_{x_3}(\mathbb{R},H^{-1}(\omega))} \leqslant C\|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\|^{\mu_2}. \tag{1.16}$$

Assume moreover that the two conditions

$$\|A_1 - A_2\|_{L^\infty_{x_3}(\mathbb{R},L^2(\omega))^3} = \|A_1 - A_2\|_{L^\infty_{x_3}(-r,r;L^2(\omega))^3} \tag{1.17}$$

and

$$\|q_1 - q_2\|_{L^\infty_{x_3}(\mathbb{R},H^{-1}(\omega))} = \|q_1 - q_2\|_{L^\infty_{x_3}(-r,r;H^{-1}(\omega))} \tag{1.18}$$

hold simultaneously for some $r > 0$. Then, for each $r' > r$, we have

$$\|A_1 - A_2\|_{L^\infty_{x_3}(\mathbb{R},L^2(\omega))} + \|q_1 - q_2\|_{L^\infty_{x_3}(\mathbb{R},H^{-1}(\omega))} \leqslant C\|\Lambda_{A_1,q_1,r'} - \Lambda_{A_2,q_2,r'}\|^{\mu_2}, \tag{1.19}$$

where C is a positive constant depending only on T, ω, M, r and r'.

We recall from [1, 53] that the DN map associated with the stationary magnetic Schrödinger equation $(-\Delta_A + q)u = 0$ in a bounded domain, logarithmic-stably determines the electric potential q and the magnetic field $\mathrm{d}A$. Moreover, we know from [42] that this logarithmic stability rate is the best we can expect in this context. Therefore, it might seem surprising at first that the stability estimates (1.12), (1.16) and (1.19) are of Hölder type. Actually, there is no surprise here as the upgrade of the stability rate from logarithmic to Hölder, when substituting (1.1) for the stationary Schrödinger equation, arises from the presence of the time variable in the dynamic Schrödinger equation. This fact, which was previously used by [10], provides an additional level of freedom that enables us to build GO solutions of the form (3.3), which are well suited for revealing the Hölder stable dependency manifested in (1.13), (1.16) and (1.19) of the electromagnetic coefficients of (1.1) with respect to the full DN map.

It is not known whether a partial DN map associated with Neumann data taken on $(0,T)\times\gamma\times\mathbb{R}$, where γ is an arbitrary subboundary of $\partial\omega$ with nonzero Lebesgue measure, instead of Σ, still stably determines the electromagnetic coefficients of (1.1). Nevertheless, it is reasonable to think that the best we can expect by adapting the analysis carried out in [9, 13] to the framework of (1.1) is a logarithmic stable recovery of these coefficients, provided they are a priori known in a neighborhood of the boundary $\partial\Omega$.

1.6.2. Comments. The key ingredient in the analysis of the inverse problem under examination is a suitable set of GO solutions to the magnetic Schrödinger equation appearing in (1.1). These functions are specifically designed for the waveguide geometry of Ω, in such a way that the unknown coefficients can be recovered by a separation of variables argument. More precisely, we seek GO solutions that are functions of $x=(x',x_3)\in\Omega$, but where the transverse variable $x'\in\omega$ and the translational variable $x_3\in\mathbb{R}$ are separated. This approach was previously used in [32] for determining zeroth-order unknown coefficients of the wave equation. Since we consider first-order unknown coefficients in this paper, the main issue here is to take into account both the cylindrical shape of Ω and the presence of the magnetic potential in the design of the GO solutions.

When the domain Ω is bounded, we know from [10] that the magnetic field $\mathrm{d}A$ is uniquely determined by the DN map associated with (1.1). The main achievement of the present paper is to extend the above statement to unbounded cylindrical domains. Actually, we also improve the results of [10] in two directions. First, we prove simultaneous determination of the magnetic field $\mathrm{d}A$ and the electric potential q. Second, the regularity condition imposed on admissible magnetic potentials entering the Schrödinger equation of (1.1) is weakened from $W^{3,\infty}(\Omega)$ to $W^{2,\infty}(\Omega)$.

To the best of our knowledge, this is the first mathematical paper claiming identification by boundary measurements, of non-compactly supported magnetic field and electric potential. Moreover, in contrast to the other works of the mathematical literature [13, 18, 34] dealing with the stability issue of inverse problems for the Schrödinger equation in an infinite cylindrical domain, we no longer require that the various unknown coefficients be periodic or decay exponentially fast with respect to the translational direction of the waveguide.

Finally, since conditions (1.14) and (1.17)–(1.18) are imposed in $\omega\times(-r,r)$ only and since the solution to (1.1) lives in the infinitely extended cylinder $(0,T)\times\Omega$, we point out that the results of Theorems 1.3 and 1.4 cannot be derived from similar statements derived in a bounded domain.

§1.7. Outline

The paper is organized as follows. In Section 2 we examine the forward problem associated with (1.1) by rigorously defining the transposition solutions to (1.1) and proving Theorem 1.1. In Section 3 we build the GO solutions to the Schrödinger equation appearing in (1.1), which are the key ingredient in the analysis of the inverse problem carried out in the two last sections of this paper. In Section 4 we estimate the X-ray transform of first-order partial derivatives of the transverse magnetic potential and the Fourier transform of the aligned magnetic field, in terms of the DN map. Finally, Section 5 contains the proofs of Theorems 1.2, 1.3 and 1.4.

§2. Analysis of the forward problem

In this section we study the forward problem associated with (1.1), i.e., we prove the statement of Theorem 1.1. Although this problem is very well documented when Ω is bounded (see, e.g., [10]), to the best of our knowledge, it cannot be directly derived from any published mathematical work in the framework of the unbounded waveguide Ω under consideration in this paper.

The proof of Theorem 1.1, which is presented in Section 2.4, deals with transposition solutions to (1.1) that are rigorously defined in Section 2.3. As a preliminary, we start by examining the elliptic part of the dynamic magnetic Schrödinger operator appearing in (1.1) in Section 2.1 and we establish an existence and uniqueness result for the corresponding system in Section 2.2.

§2.1. Elliptic magnetic Schrödinger operator

For $A \in W^{1,\infty}(\Omega,\mathbb{R})^3$ we set $\nabla_A := \nabla + iA$, where iA denotes the multiplier by iA, and we notice for all $u \in H^1(\Omega)$ that

$$|\nabla_A u(x)|^2 \geqslant (1-\epsilon)|\nabla u(x)| + (1-\epsilon^{-1})|Au(x)|^2, \quad \epsilon > 0, \quad x \in \Omega. \tag{2.1}$$

Next, for $q \in L^\infty(\Omega;\mathbb{R})$, we introduce the sesquilinear form

$$\mathbf{h}_{A,q}(u,v) := \int_\Omega \nabla_A u(x) \cdot \overline{\nabla_A v}(x)\mathrm{d}x - \int_\Omega q(x)u(x)\overline{v}(x)\mathrm{d}x, \\ u,v \in \mathcal{D}(\mathbf{h}_{A,q}) := H_0^1(\Omega)$$

and consider the self-adjoint operator $\mathscr{H}_{A,q}$ in $L^2(\Omega)$, generated by $\mathbf{h}_{A,q}$. In light of [35, Prop. 2.5], $\mathscr{H}_{A,q}$ acts on its domain $\mathcal{D}(\mathscr{H}_{A,q}) := H_0^1(\Omega) \cap H^2(\Omega)$ as the operator $-(\Delta_A + q)$, where $\Delta_A := \nabla_A \cdot \nabla_A$ is expressed by (1.2).

Further, for all $x \in \Omega$ fixed, taking $\epsilon = |A(x)|^2/(1+|A(x)|^2)$ in (2.1), we get $|\nabla_A u(x)|^2 \geqslant |\nabla u(x)|^2/(1+|A(x)|^2) - |u(x)|^2$, whence

$$\mathbf{h}_{A,0}(u,u) + \|u\|^2_{L^2(\Omega)} \geqslant \frac{\|\nabla u\|^2_{L^2(\Omega)^3}}{1+\|A\|^2_{L^\infty(\Omega)}}, \quad u \in H^1_0(\Omega),$$

where $\mathbf{h}_{A,0}$ stands for $\mathbf{h}_{A,q}$ in the particular case where q is uniformly zero. Thus, we deduce from the Poincaré inequality and the Lax–Milgram theorem that for any $v \in H^{-1}(\Omega)$ there exists a unique $\phi_v \in H^1_0(\Omega)$ satisfying

$$-\Delta_A \phi_v + \phi_v = v. \tag{2.2}$$

Next, for u and v in $H^{-1}(\Omega)$, we put

$$\langle u, v\rangle_{-1} := \mathrm{Re}\left(\int_\Omega \nabla_A \phi_u(x) \cdot \overline{\nabla_A \phi_v}(x)\mathrm{d}x + \int_\Omega \phi_u(x)\overline{\phi_v}(x)\mathrm{d}x \right)$$

and check that the space $H^{-1}(\Omega)$ endowed with the above scalar product is Hilbertian. Having said that, we may now prove the following technical result.

Lemma 2.1. *For each $A \in W^{1,\infty}(\Omega,\mathbb{R})^3$, the linear operator $\mathscr{B}_A := \Delta_A$ with domain $\mathcal{D}(\mathscr{B}_A) := H^1_0(\Omega)$ is self-adjoint and negative in $H^{-1}(\Omega)$.*

Proof. We proceed as in the proof of [16, Prop. 2.6.14 and Cor. 2.6.15]. Namely, we pick u and v in $\mathcal{C}^\infty_0(\Omega)$, and write

$$\langle \mathscr{B}_A u, v\rangle_{-1} = \langle w, v\rangle_{-1} + \langle u, v\rangle_{-1},$$

with $w := \mathscr{B}_A u - u$. Taking into account that $\phi_w = -u$, we obtain

$$\begin{aligned}\langle \mathscr{B}_A u, v\rangle_{-1} = -\mathrm{Re}&\left(\int_\Omega \nabla_A u(x) \cdot \overline{\nabla_A \phi_v}(x)\mathrm{d}x + \int_\Omega u(x)\overline{\phi_v}(x)\mathrm{d}x \right) \\ &+ \langle u, v\rangle_{-1}.\end{aligned} \tag{2.3}$$

Next, integrating by parts, we get

$$\begin{aligned}-\mathrm{Re}\left(\int_\Omega \nabla_A u(x) \cdot \overline{\nabla_A \phi_v}(x)\mathrm{d}x + \int_\Omega u(x)\overline{\phi_v}(x)\mathrm{d}x \right) &= -\mathrm{Re}\langle u, -\Delta_A \phi_v + \phi_v\rangle_{L^2(\Omega)} \\ &= -\mathrm{Re}\langle u, v\rangle_{L^2(\Omega)},\end{aligned}$$

so (2.3) yields

$$\langle \mathscr{B}_A u, v\rangle_{-1} = -\mathrm{Re}\langle u, v\rangle_{L^2(\Omega)} + \langle u, v\rangle_{-1}. \tag{2.4}$$

Further, since $\langle u, u\rangle_{-1} = \mathrm{Re}\langle \phi_u, (-\Delta_A + 1)\phi_u\rangle_{L^2(\Omega)} = \mathrm{Re}\langle \phi_u, u\rangle_{L^2(\Omega)}$ and $\|\phi_u\|^2_{L^2(\Omega)} \leqslant \langle u, u\rangle_{-1}$, we see that $\langle u, u\rangle_{-1} \leqslant \|u\|^2_{L^2(\Omega)}$. Therefore, we obtain

$$\langle \mathscr{B}_A u, u\rangle_{-1} = -\|u\|^2_{L^2(\Omega)} + \langle u, u\rangle_{-1} \leqslant 0 \tag{2.5}$$

by taking $v = u$ in (2.4).

By density of $\mathcal{C}_0^\infty(\Omega)$ in $H_0^1(\Omega)$, both estimates (2.4) and (2.5) remain valid for all u and v in $H_0^1(\Omega)$. As a consequence, the operator $\mathscr{B}_A$ is dissipative. Furthermore, $1 - \mathscr{B}_A$ being surjective from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$, by (2.2), we get that $\mathscr{B}_A$ is m-dissipative. Moreover, it follows readily from (2.4) that

$$\langle \mathscr{B}_A u, v\rangle_{-1} = \langle u, \mathscr{B}_A v\rangle_{-1}, \quad u, v \in H_0^1(\Omega).$$

Hence the graph of $\mathscr{B}_A$ is contained in the one of its adjoint $\mathscr{B}_A^*$ and thus $\mathscr{B}_A$ is self-adjoint by virtue of [16, Cor. 2.4.10]. □

§2.2. Existence and uniqueness result

For further use, we establish the following existence and uniqueness result for the system

$$\begin{cases} (i\partial_t + \Delta_A + q)v = F & \text{in } Q, \\ v(0, \cdot) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Sigma, \end{cases} \tag{2.6}$$

with homogeneous Dirichlet boundary condition and suitable source term F.

Lemma 2.2. *Let M, A and q be the same as in Theorem 1.1.*

(i) *Assume that $F \in L^1(0, T; H_0^1(\Omega))$. Then, system (2.6) admits a unique solution $v \in \mathcal{C}([0, T], H_0^1(\Omega))$, satisfying*

$$\|v\|_{\mathcal{C}([0,T],H^1(\Omega))} \leqslant C\|F\|_{L^1(0,T;H^1(\Omega))} \tag{2.7}$$

for some constant $C > 0$, depending only on T, ω and M.

(ii) *If $F \in W^{1,1}(0, T; L^2(\Omega))$, then (2.6) admits a unique solution*

$$v \in \mathcal{Z} := \mathcal{C}^1([0, T], L^2(\Omega)) \cap \mathcal{C}([0, T], H_0^1(\Omega) \cap H^2(\Omega))$$

and there exists $C = C(T, \omega, M) > 0$ such that we have

$$\|v\|_{\mathcal{Z}} \leqslant C\|F\|_{W^{1,1}(0,T;L^2(\Omega))}.$$

Proof. The proof boils down to the statement, borrowed from [18, Lem. 2.1], claiming for any Banach space X, any m-dissipative operator U in X with dense domain $D(U)$ and any $B \in \mathcal{C}([0, T], \mathcal{B}(D(U)))$, that for all $v_0 \in D(U)$ and all $f \in \mathcal{C}([0, T], X) \cap L^1(0, T; D(U))$ (resp., $f \in W^{1,1}(0, T; X)$) there exists a unique solution $v \in Z_0 = \mathcal{C}([0, T], D(U)) \cap C^1([0, T], X)$ to the Cauchy problem

$$\begin{cases} v'(t) = Uv(t) + B(t)v(t) + f(t), \\ v(0) = v_0, \end{cases}$$

such that

$$\|v\|_{Z_0} = \|v\|_{C^0([0,T],D(U))} + \|v\|_{C^1([0,T],X)} \leqslant C(\|v_0\|_{D(U)} + \|f\|_*).$$

Here C is some positive constant depending only on T and $\|B\|_{C([0,T],\mathcal{B}(D(U)))}$, and $\|f\|_*$ stands for the norm $\|f\|_{C([0,T],X)\cap L^1(0,T;D(U))}$ (resp., $\|f\|_{W^{1,1}(0,T;X)}$).

Notice that the operator $i\mathscr{B}_A$ is skew-adjoint as $\mathscr{B}_A$ is self-adjoint in $H^{-1}(\Omega)$. Hence, $i\mathscr{B}_A$ is m-dissipative with dense domain in $H^{-1}(\Omega)$. Further, the multiplier by iq being bounded in $\mathcal{C}([0,T],H_0^1(\Omega))$, we obtain (i) by applying the above result with $X = H^{-1}(\Omega)$, $U = i\mathscr{B}_{A,q}$, $f = iF$, $B(t) = iq$ and $v_0 = 0$.

Similarly, as $\mathscr{H}_{A,q}$ is self-adjoint in $L^2(\Omega)$, then the operator $-i\mathscr{H}_{A,q}$ is m-dissipative with dense domain in $L^2(\Omega)$ and we derive (ii) by applying [18, Lem. 2.1] with $X = L^2(\Omega)$, $U = -i\mathscr{H}_{A,q}$, $f = iF$, $B(t) = 0$ and $v_0 = 0$. $\square$

Remark 2.3. Put $w(t,x) := \overline{v(T-t,x)}$ for $(t,x) \in Q$. Since w is a solution to the system

$$\begin{cases} (i\partial_t + \Delta_A + q)w = F & \text{in } Q, \\ w(T,\cdot) = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \Sigma, \end{cases} \tag{2.8}$$

whenever v is a solution to the IBVP (2.6) where the function $(t,x) \mapsto \overline{F(T-t,x)}$ is substituted for F, we infer from Lemma 2.2 that the transposed system (2.8) admits a unique solution w in $\mathcal{C}^0([0,T],H_0^1(\Omega))$ (resp., $\mathcal{Z}$), provided F is in $L^1(0,T;H_0^1(\Omega))$ (resp., $W^{1,1}(0,T;L^2(\Omega))$).

§2.3. Transposition solutions

As preamble to the definition of transposition solutions to (1.1), we establish that the normal derivative of the $\mathcal{C}([0,T],H_0^1(\Omega))$-solution to (2.6) lies in $L^2(\Sigma)$.

Lemma 2.4. *Let M, A and q be as in Lemma 2.2. Then, the linear map $F \mapsto \partial_\nu v$, where v denotes the $\mathcal{C}([0,T],H_0^1(\Omega))$-solution to (2.6) associated with $F \in L^1(0,T;H_0^1(\Omega))$, given by Lemma 2.2, is bounded from $L^1(0,T;H_0^1(\Omega))$ into $L^2(\Sigma)$.*

Proof. Since $\|v\|_{\mathcal{C}([0,T],H^1(\Omega))} \leqslant C\|F\|_{L^1(0,T;H^1(\Omega))}$, by (2.7), we may assume without loss of generality that $A = 0$ and $q = 0$.

Assume that $F \in W^{1,1}(0,T;L^2(\Omega))$ in such a way that $v \in \mathcal{Z}$, by virtue of Lemma 2.2. Let $N_1 \in \mathcal{C}^2(\overline{\omega})^2$ satisfy $N_1 = \nu_1$ on $\partial\omega$, where ν_1 denotes the unit outward normal vector to $\partial\omega$. Put $N(x',x_3) := (N_1(x'),0)$ for all $x' \in \omega$ and $x_3 \in \mathbb{R}$, so that $N \in \mathcal{C}^2(\overline{\Omega})^3 \cap W^{2,\infty}(\Omega)^3$ verifies $N = \nu$ on $\partial\Omega$. Then, we have

$$\langle i\partial_t v + \Delta v, N \cdot \nabla v\rangle_{L^2(Q)} = \langle F, N \cdot \nabla v\rangle_{L^2(Q)}. \tag{2.9}$$

By integrating by parts with respect to t, we get

$$\begin{aligned}\langle \partial_t v, N\cdot\nabla v\rangle_{L^2(Q)} &= \langle v(T,\cdot), N\cdot\nabla v(T,\cdot)\rangle_{L^2(\Omega)} - \langle v, N\cdot\nabla\partial_t v\rangle_{L^2(Q)}\\ &= \langle v(T,\cdot), N\cdot\nabla v(T,\cdot)\rangle_{L^2(\Omega)} + \langle N\cdot\nabla v, \partial_t v\rangle_{L^2(Q)} - I,\end{aligned} \tag{2.10}$$

where $I := \int_Q N\cdot\nabla(v\overline{\partial_t v})\mathrm{d}x\,\mathrm{d}t$. Taking into account that $N\cdot\nabla = N_1\cdot\nabla_{x'}$, where $\nabla_{x'}$ denotes the gradient operator with respect to $x'\in\omega$, we have $I = \int_Q N_1\cdot\nabla_{x'}(v\overline{\partial_t v})\mathrm{d}x\,\mathrm{d}t$ and hence

$$\begin{aligned} I &= \int_Q \nabla_{x'}\cdot(v(t,x)\overline{\partial_t v}(t,x)N_1(x'))\mathrm{d}x'\,\mathrm{d}x_3\,\mathrm{d}t - \langle(\nabla\cdot N)v, \overline{v}\partial_t v\rangle_{L^2(Q)}\\ &= \int_\Sigma v(t,x)\overline{\partial_t v}(t,x)N_1(x')\cdot\nu_1(x')\mathrm{d}x'\,\mathrm{d}x_3\,\mathrm{d}t - \langle(\nabla\cdot N)v, \partial_t v\rangle_{L^2(Q)}\\ &= -\langle(\nabla\cdot N)v, \partial_t v\rangle_{L^2(Q)},\end{aligned} \tag{2.11}$$

by Green's formula, since $v_{|\Sigma}=0$. Putting (2.10)–(2.11) together, we obtain

$$\begin{aligned} 2\mathrm{Re}\langle i\partial_t v, N\cdot\nabla v\rangle_{L^2(Q)} &= i\langle v(T,\cdot), N\cdot\nabla v(T,\cdot)\rangle_{L^2(\Omega)} - \langle(\nabla\cdot N)v, i\partial_t v\rangle_{L^2(Q)}\\ &= i\langle v(T,\cdot), N\cdot\nabla v(T,\cdot)\rangle_{L^2(\Omega)} + \langle(\nabla\cdot N)v, \Delta v\rangle_{L^2(Q)}\\ &\quad - \langle(\nabla\cdot N)v, F\rangle_{L^2(Q)}.\end{aligned} \tag{2.12}$$

Applying Green's formula with respect to $x'\in\omega$ and integrating by parts with respect to $x_3\in\mathbb{R}$, we find

$$\langle(\nabla\cdot N)v, \Delta v\rangle_{L^2(Q)} = -\langle(\nabla\cdot N)\nabla v, \nabla v\rangle_{L^2(Q)^3} - \langle v\nabla(\nabla\cdot N), \nabla v\rangle_{L^2(Q)^3},$$

so (2.12) entails

$$\begin{aligned} 2\mathrm{Re}\langle i\partial_t v, N\cdot\nabla v\rangle_{L^2(Q)} &= i\langle v(T,\cdot), N\cdot\nabla v(T,\cdot)\rangle_{L^2(\Omega)} - \langle(\nabla\cdot N)\nabla v, \nabla v\rangle_{L^2(Q)^3}\\ &\quad - \langle v\nabla(\nabla\cdot N), \nabla v\rangle_{L^2(Q)^3} - \langle(\nabla\cdot N)v, F\rangle_{L^2(Q)}.\end{aligned}$$

This and (2.7) yield

$$\begin{aligned}\left|\mathrm{Re}\langle i\partial_t v, N\cdot\nabla v\rangle_{L^2(Q)}\right| &\leqslant C\|v\|_{\mathcal{C}([0,T],H^1(\Omega))}\left(\|v\|_{\mathcal{C}([0,T],H^1(\Omega))} + \|F\|_{L^1(0,T;H^1(\Omega))}\right)\\ &\leqslant C\|F\|^2_{L^1(0,T;H^1(\Omega))}.\end{aligned}$$

From this and (2.9), it then follows that

$$\left|\mathrm{Re}\langle \Delta v, N\cdot\nabla v\rangle_{L^2(Q)}\right| \leqslant C\|F\|^2_{L^1(0,T;H^1(\Omega))}. \tag{2.13}$$

On the other hand, upon applying Green's formula with respect to $x'\in\omega$ and integrating by parts with respect to $x_3\in\mathbb{R}$, we get

$$\begin{aligned}\langle \Delta v, N\cdot\nabla v\rangle_{L^2(Q)} &= -\langle\nabla v, \nabla(N\cdot\nabla v)\rangle_{L^2(Q)^3} + \langle\nabla v\cdot\nu, N\cdot\nabla v\rangle_{L^2(\Sigma)}\\ &= -\langle\nabla v, \nabla(N\cdot\nabla v)\rangle_{L^2(Q)^3} + \|\partial_\nu v\|^2_{L^2(\Sigma)}.\end{aligned} \tag{2.14}$$

Moreover, since $\mathrm{Re}\big(\nabla v \cdot \nabla(N \cdot \overline{\nabla v})\big) = \mathrm{Re}\big((H\nabla v) \cdot \overline{\nabla v}\big) + \frac{1}{2} N \cdot \nabla |\nabla v|^2$ with $H := (\partial_{x_i} N_j)_{1 \leqslant i,j \leqslant 3}$ and $N := (N_j)_{1 \leqslant j \leqslant 3}$, we infer from (2.14) that

$$\begin{aligned}\mathrm{Re}\langle \Delta v, N \cdot \nabla v\rangle_{L^2(Q)} = \|\partial_\nu v\|^2_{L^2(\Sigma)} - \mathrm{Re}\langle H\nabla v, \nabla v\rangle_{L^2(Q)^3} \\ - \frac{1}{2}\int_Q N \cdot \nabla |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.15}$$

Further, by applying Green's formula with respect to $x' \in \omega$ once more, we find for a.e. $(t, x_3) \in (0,T) \times \mathbb{R}$, that

$$\begin{aligned}\int_\omega N(x', x_3) \cdot \nabla |\nabla v(t,x',x_3)|^2 \,\mathrm{d}x' &= \int_\omega N_1(x') \cdot \nabla_{x'} |\nabla v(t,x',x_3)|^2 \,\mathrm{d}x' \\ &= \|\nabla v(t,\cdot,x_3)\|^2_{L^2(\partial\omega)^3} \\ &\quad - \langle (\nabla \cdot N)\nabla v(t,\cdot,x_3), \nabla v(t,\cdot,x_3)\rangle_{L^2(\omega)^3}.\end{aligned} \tag{2.16}$$

Bearing in mind that $v_{|\Sigma} = 0$, we have $|\nabla v|^2 = |\partial_\nu v|^2$ on Σ, so we deduce from (2.16) that

$$\int_Q N \cdot \nabla |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}t = \|\partial_\nu v\|^2_{L^2(\Sigma)} - \langle (\nabla \cdot N)\nabla v, \nabla v\rangle_{L^2(Q)^3}.$$

We infer from this and (2.15) that

$$\begin{aligned}\|\partial_\nu v\|^2_{L^2(\Sigma)} = 2\mathrm{Re}\langle \Delta v, N \cdot \nabla v\rangle_{L^2(Q)} + 2\mathrm{Re}\langle H\nabla v, \nabla v\rangle_{L^2(Q)^3} \\ - \langle (\nabla \cdot N)\nabla v, \nabla v\rangle_{L^2(Q)^3}\end{aligned}$$

and hence

$$\|\partial_\nu v\|_{L^2(\Sigma))} \leqslant C\left(\|F\|_{L^1(0,T;H^1(\Omega))} + \|v\|_{\mathcal{C}([0,T],H^1(\Omega))}\right) \leqslant C\|F\|_{L^1(0,T;H^1(\Omega))},$$

according to (2.7) and (2.13). By density of $W^{1,1}(0,T;H_0^1(\Omega))$ in $L^1(0,T;H_0^1(\Omega))$ we extend the above estimate to every $F \in L^1(0,T;H_0^1(\Omega))$, proving the desired result. □

Armed with Lemma 2.4, we now introduce the transposition solution to (1.1). For $F \in L^1(0,T;H_0^1(\Omega))$, we denote by $v \in \mathcal{C}^0([0,T],H_0^1(\Omega))$ the solution to (2.8) given by Remark 2.3. Since $(t,x) \mapsto \overline{v(T-t,x)}$ is a solution to (2.6) associated with the source term $(t,x) \mapsto \overline{F(T-t,x)}$, we infer from Lemma 2.4 that the mapping $F \mapsto \partial_\nu v$ is bounded from $L^1(0,T;H_0^1(\Omega))$ into $L^2(\Sigma)$. Therefore, for each $f \in L^2(\Sigma)$, the mapping

$$\ell_f : F \mapsto \langle f, \partial_\nu v\rangle_{L^2(\Sigma)}$$

is an antilinear form on $L^1(0,T;H_0^1(\Omega))$. Thus, there exists a unique $u \in L^\infty(0,T;H^{-1}(\Omega))$ such that we have

$$\langle u, F\rangle_{L^\infty(0,T;H^{-1}(\Omega)),L^1(0,T;H_0^1(\Omega))} = \ell_f(F), \quad F \in L^1(0,T;H_0^1(\Omega)), \tag{2.17}$$

according to Riesz's representation theorem. The function u, characterized by (2.17), is named the solution in the transposition sense to (1.1).

§2.4. Proof of Theorem 1.1

Let $w \in L^\infty(0,T;H^{-1}(\Omega))$ be the solution in the transposition sense to the system

$$\begin{cases} (i\partial_t + \Delta_A + q)\, w = 0 & \text{in } Q, \\ w(0,\cdot) = 0 & \text{in } \Omega, \\ w = \partial_t^2 f & \text{on } \Sigma. \end{cases}$$

For any $t \in (0,T)$ we put $v(t,\cdot) := \int_0^t w(s,\cdot)\mathrm{d}s$ in such a way that v is the solution in the transposition sense to the system

$$\begin{cases} (i\partial_t + \Delta_A + q)\, v = 0 & \text{in } Q, \\ v(0,\cdot) = 0 & \text{in } \Omega, \\ v = \partial_t f & \text{on } \Sigma. \end{cases} \tag{2.18}$$

We have $v = \partial_t f \in H^{1,1/2}(\Sigma)$ by [41, Sect. 4, Prop. 2.3]. Next, since $H^{1,1/2}(\Sigma) \subset L^2(0,T;H^{1/2}(\partial\Omega))$ from the very definition of $H^{1,1/2}(\Sigma)$, and $-\Delta_A v = iw + qv$ in Q, from the first line of (2.18), then $v \in L^2(0,T;H^1(\Omega)) \cap W^{1,\infty}(0,T;H^{-1}(\Omega))$. Moreover, we have the estimate

$$\begin{aligned} \|v\|_{L^2(0,T;H^1(\Omega))} \leqslant C\Big(&\|w\|_{L^2(0,T;H^{-1}(\Omega))} + \|qv\|_{L^2(0,T;H^{-1}(\Omega))} \\ &+ \|\partial_t f\|_{L^2(0,T;H^{1/2}(\partial\Omega))}\Big), \end{aligned} \tag{2.19}$$

where the constant $C > 0$ depends only on T, ω and M.

On the other hand, from the very definition of the transposition solution w, we obtain

$$\begin{aligned} \|w\|_{L^2(0,T;H^{-1}(\Omega))} &\leqslant T^{1/2}\|w\|_{L^\infty(0,T;H^{-1}(\Omega))} \\ &\leqslant C\|\partial_t^2 f\|_{L^2(\Sigma)} \leqslant C\|f\|_{H^{2,1}(\Sigma)}, \end{aligned} \tag{2.20}$$

with the aid of Lemma 2.4. As a consequence we have

$$\|qv\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant \|q\|_{W^{1,\infty}(\Omega)} T\|w\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant C\|f\|_{H^{2,1}(\Sigma)}. \tag{2.21}$$

Putting (2.19)–(2.21) together, we find

$$\|v\|_{L^2(0,T;H^1(\Omega))} \leqslant C\|f\|_{H^{2,1}(\Sigma)}, \tag{2.22}$$

for some constant $C = C(T, \omega, M) > 0$.

Finally, as $u(t) = \int_0^t v(s)ds$ is a solution to (1.1) in the transposition sense, we have

$$\|u\|_{H^1(0,T;H^1(\Omega))} \leqslant (1+T)^{1/2}\|v\|_{L^2(0,T;H^1(\Omega))}$$

and (1.7) follows from this and (2.22).

We turn now to proving (1.8). To do that, we pick $f \in \mathcal{C}^\infty([0,T] \times \partial\Omega) \cap H_0^{2,1}(\Sigma)$ and proceed as in the derivation of Lemma 2.4. We get

$$\|\partial_\nu u\|_{L^2(\Sigma)} \leqslant C\left(\|u\|_{H^1(0,T;H^1(\Omega))} + \|f\|_{H^{2,1}(\Sigma)}\right),$$

for some constant $C = C(T, \omega, M) > 0$, so we deduce from (1.7) that

$$\|\partial_\nu u\|_{L^2(\Sigma)} \leqslant C\|f\|_{H^{2,1}(\Sigma)}.$$

The desired result follows from this by invoking the density of $\mathcal{C}^\infty([0,T] \times \partial\Omega) \cap H_0^{2,1}(\Sigma)$ in $H_0^{2,1}(\Sigma)$.

§3. GO solutions

In this section we build GO solutions to the magnetic Schrödinger equation in Ω. These functions are essential tools in the proofs of Theorems 1.2, 1.3 and 1.4. As in [33], we take advantage of the translational invariance of Ω with respect to the longitudinal direction x_3, in order to adapt the method suggested by Bellassoued and Choulli in [10] for building GO solutions to the magnetic Schrödinger equation in a bounded domain, to the framework of the unbounded waveguide Ω. Moreover, as we aim to reduce the regularity assumption imposed on the magnetic potential by the GO solutions construction method, we follow the strategy developed in [23, 38, 39, 46] for magnetic Laplace operators, and rather build GO solutions to the Schrödinger equation associated with a suitable smooth approximation of the magnetic potential.

Throughout the entire section, we consider two magnetic potentials

$$A_j = (A_j^\sharp, a_{j,3}) \in W^{2,\infty}(\Omega, \mathbb{R})^2 \times W^{2,\infty}(\Omega, \mathbb{R}), \quad j = 1, 2$$

and two electric potentials $q_j \in W^{1,\infty}(\Omega, \mathbb{R})$, obeying the conditions

$$\|A_j\|_{W^{2,\infty}(\Omega)^3} + \|q_j\|_{W^{1,\infty}(\Omega)} \leqslant M, \quad j = 1, 2 \tag{3.1}$$

and

$$\partial_x^\alpha A_1 = \partial_x^\alpha A_2 \quad \text{on } \partial\Omega, \text{ for all } \alpha \in \mathbb{N}_0^3 \text{ such that } |\alpha| \leqslant 1. \tag{3.2}$$

For $\sigma > 0$, we denote by $A_{j,\sigma}^\sharp$ a suitable $\mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R})^2 \cap W^{\infty,\infty}(\mathbb{R}^3,\mathbb{R})^2$-approximation of $A_j^\sharp$ that we shall make precise in Lemma 3.3 below. We seek solutions $u_{j,\sigma}$ to the magnetic Schrödinger equation of (1.1) where (A_j, q_j) is substituted for (A, q), of the form

$$u_{j,\sigma}(t,x',x_3) := \Phi_j(2\sigma t, x) b_{j,\sigma}(2\sigma t, x) e^{i\sigma(x'\cdot\theta - \sigma t)} + \psi_{j,\sigma}(t,x), \tag{3.3}$$
$$t \in \mathbb{R},\ x = (x', x_3) \in \omega \times \mathbb{R}.$$

Here, $\theta \in \mathbb{S}^1 := \{y \in \mathbb{R}^2 :\ |y| = 1\}$ is fixed,

$$b_{j,\sigma}(t,x) := \exp\left(-i \int_0^t \theta \cdot A_{j,\sigma}^\sharp(x' - s\theta, x_3)\mathrm{d}s\right), \tag{3.4}$$
$$t \in \mathbb{R},\ x = (x', x_3) \in \omega \times \mathbb{R},$$

Φ_j is a solution to the transport equation

$$(\partial_t + \theta \cdot \nabla_{x'})\, \Phi_j = 0 \quad \text{in } \mathbb{R} \times \Omega \tag{3.5}$$

and we impose that the remainder term $\psi_{j,\sigma} \in L^2(Q)$ scales at best like $\sigma^{-1/2}$ when σ is large, i.e.,

$$\lim_{\sigma \to +\infty} \sigma^{1/2} \|\psi_{j,\sigma}\|_{L^2(Q)} = 0. \tag{3.6}$$

As it will appear in the coming subsection, such a construction requires that $A_{j,\sigma}^\sharp$ be sufficiently close to $A_j^\sharp$.

§3.1. Magnetic potential mollification

We aim to define a suitable smooth approximation

$$A_{j,\sigma}^\sharp \in \mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R})^2 \cap W^{\infty,\infty}(\mathbb{R}^3,\mathbb{R})^2, \quad j = 1,2$$

of $A_j^\sharp = (a_{1,j}, a_{2,j})$. This preliminarily requires that $A_j^\sharp$ be appropriately extended to a larger domain than Ω.

Lemma 3.1. *Let $A_j^\sharp$, for $j = 1,2$ be in $W^{2,\infty}(\Omega,\mathbb{R})^2$ and fulfill (3.2). Let $\tilde\omega$ be a smooth open bounded subset of $\mathbb{R}^2$ containing $\overline{\omega}$. Then, there exist two potentials $\tilde A_1^\sharp$ and $\tilde A_2^\sharp$ in $W^{2,\infty}(\mathbb{R}^3,\mathbb{R})^2$, both of them supported in $\tilde\Omega := \tilde\omega \times \mathbb{R}$, such that we have*

$$\tilde A_j^\sharp = A_j^\sharp \quad \text{in } \Omega,\ \text{for } j = 1,2 \qquad \text{and} \qquad \tilde A_1^\sharp = \tilde A_2^\sharp \quad \text{in } \tilde\Omega \setminus \Omega. \tag{3.7}$$

Moreover, the two estimates

$$\|\tilde{A}_j^\sharp\|_{W^{2,\infty}(\mathbb{R}^3)^2} \leqslant C \max\left(\|A_1^\sharp\|_{W^{2,\infty}(\Omega)^2}, \|A_2^\sharp\|_{W^{2,\infty}(\Omega)^2}\right),\ j=1,2 \tag{3.8}$$

hold for some constant $C>0$, depending only on ω and $\tilde{\omega}$.

Proof. By [49, Sect. 3, Thm. 5] and [35, Lem. 2.7], there exists $\tilde{A}_1^\sharp \in W^{2,\infty}(\mathbb{R}^3,\mathbb{R})^2$ such that $\tilde{A}_1^\sharp = A_1^\sharp$ in Ω and (3.8) holds true for $j=1$. Then, upon possibly substituting $\chi\tilde{A}_1^\sharp$ for $\tilde{A}_1^\sharp$, where $\chi \in C^\infty(\mathbb{R}^3,\mathbb{R})$ is supported in $\tilde{\Omega}$ and verifies $\chi(x)=1$ for all $x\in\Omega$, we may assume that $\tilde{A}_1^\sharp$ is supported in $\tilde{\Omega}$ as well.

Next, putting

$$\tilde{A}_2^\sharp(x) := \begin{cases} A_2^\sharp(x) & \text{if } x\in\Omega, \\ \tilde{A}_1^\sharp(x) & \text{if } x\in\mathbb{R}^3\setminus\Omega, \end{cases} \tag{3.9}$$

it is clear from (3.2) that $\tilde{A}_2^\sharp \in W^{2,\infty}(\mathbb{R}^3,\mathbb{R})^2$ and that it satisfies (3.8) with $j=2$. $\square$

Having seen this, we define for each $\sigma>0$ the smooth approximation $a_\sigma \in \mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R}) \cap W^{\infty,\infty}(\mathbb{R}^3,\mathbb{R})$ of a function $\tilde{a}\in W^{2,\infty}(\mathbb{R}^3,\mathbb{R})$, supported in $\tilde{\Omega}$, by

$$a_\sigma(x) := \int_{\mathbb{R}^3} \chi_\sigma(x-y)\left(\tilde{a}(y) + (x-y)\cdot\nabla\tilde{a}(y)\right)\mathrm{d}y, \quad x\in\mathbb{R}^3. \tag{3.10}$$

Here we have set $\chi_\sigma(x) := \sigma\chi(\sigma^{1/3}x)$ for all $x\in\mathbb{R}^3$, where $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^3,\mathbb{R}_+)$ is such that

$$\operatorname{supp}\chi \subset \{x\in\mathbb{R}^3;\ |x|\leqslant 1\} \quad \text{and} \quad \int_{\mathbb{R}^3}\chi(x)\mathrm{d}x = 1.$$

As can be seen from the following result, the function a_σ gets closer to $\tilde{a}$ as the parameter σ becomes large.

Lemma 3.2. *Let $\tilde{a}\in W^{2,\infty}(\mathbb{R}^3,\mathbb{R})$ be supported in $\tilde{\Omega}$ and satisfy $\|\tilde{a}\|_{W^{2,\infty}(\mathbb{R}^3)} \leqslant M$, for some $M>0$. Then, there exists a constant $C>0$, depending only on ω, $\tilde{\omega}$ and M, such that for all $\sigma>0$ we have*

$$\|a_\sigma - \tilde{a}\|_{W^{k,\infty}(\mathbb{R}^3)} \leqslant C\sigma^{(k-2)/3}, \quad k=0,1, \tag{3.11}$$

where $W^{0,\infty}(\Omega)$ stands for $L^\infty(\Omega)$, and

$$\|a_\sigma\|_{W^{k,\infty}(\mathbb{R}^3)} \leqslant C\sigma^{(k-2)/3}, \quad k\geqslant 2. \tag{3.12}$$

Proof. We establish only (3.11), the estimate (3.12) being obtained in a similar fashion. For $x\in\mathbb{R}^3$ fixed, we make the change of variable $\eta=\sigma^{1/3}(x-y)$ in (3.10) and get

$$a_\sigma(x) = \int_{\mathbb{R}^3} \chi(\eta)\tilde{a}(x-\sigma^{-1/3}\eta)\mathrm{d}\eta \\ + \sigma^{-1/3}\int_{\mathbb{R}^3} \chi(\eta)\left(\eta\cdot\nabla\tilde{a}(x-\sigma^{-1/3}\eta)\right)\mathrm{d}\eta. \tag{3.13}$$

On the other hand, we have

$$\int_{\mathbb{R}^3} \chi(\eta)\tilde{a}(x-\sigma^{-1/3}\eta)\mathrm{d}\eta - \tilde{a}(x) = \int_{\mathbb{R}^3} \chi(\eta)\left(\tilde{a}(x-\sigma^{-1/3}\eta)-\tilde{a}(x)\right)\mathrm{d}\eta \\ = -\sigma^{-1/3}\int_{\mathbb{R}^3} \chi(\eta)\left(\int_0^1 \eta\cdot\nabla\tilde{a}(x-s\sigma^{-1/3}\eta)\mathrm{d}s\right)\mathrm{d}\eta,$$

so we infer from (3.13) that

$$a_\sigma(x)-\tilde{a}(x) = \sigma^{-1/3}\int_{\mathbb{R}^3} \chi(\eta)\left(\int_0^1 \eta\cdot\left(\nabla\tilde{a}(x-\sigma^{-1/3}\eta)-\nabla\tilde{a}(x-s\sigma^{-1/3}\eta)\right)\mathrm{d}s\right)\mathrm{d}\eta. \tag{3.14}$$

By the Sobolev embedding theorem (see, e.g., [27, Thm. 1.4.4.1] and [52, Lem. 3.13] we know that $\tilde{a}\in\mathcal{C}^{1,1}(\mathbb{R}^3)$ satisfies the estimate $\|\tilde{a}\|_{\mathcal{C}^{1,1}(\mathbb{R}^3)} \leqslant C\|\tilde{a}\|_{W^{2,\infty}(\mathbb{R}^3)}$ for some constant $C>0$ that is independent of $\tilde{a}$. Thus, (3.14) yields

$$|a_\sigma(x)-\tilde{a}(x)| \leqslant C\|\tilde{a}\|_{W^{2,\infty}(\mathbb{R}^3)}\left(\int_{\mathbb{R}^3}\chi(\eta)|\eta|^2\mathrm{d}\eta\right)\sigma^{-2/3}$$

and (3.11) with $k=0$ follows readily from this and the estimate $\|\tilde{a}\|_{W^{2,\infty}(\mathbb{R}^3)} \leqslant M$. Further, upon differentiating (3.14) with respect to x_i, for $i=1,2,3$, and upper bounding the integrand function $(\eta,s)\mapsto \nabla\partial_i\tilde{a}(x-\sigma^{-1/3}\eta)-\nabla\partial_i\tilde{a}(x-s\sigma^{-1/3}\eta)$ by $2\|\tilde{a}\|_{W^{2,\infty}(\mathbb{R}^3)}$, uniformly over $\mathbb{R}^3\times(0,1)$, we obtain (3.11) for $k=1$. □

We notice, for further use, from (3.10) and the expression of χ_σ, that

$$a_\sigma(x) = \int_{\mathbb{R}^3}\left(\chi_\sigma(x-y)-\nabla\cdot((x-y)\chi_\sigma(x-y))\right)\tilde{a}(y)\mathrm{d}y \\ = \int_{\mathbb{R}^3}\left(4\sigma\chi(\sigma^{1/3}(x-y))+\sigma^{4/3}(x-y)\cdot\nabla\chi(\sigma^{1/3}(x-y))\right)\tilde{a}(y)\mathrm{d}y, \quad x\in\mathbb{R}^3.$$

Making the change of variable $z=\sigma^{1/3}(x-y)$ in the above integral, we find

$$a_\sigma(x) = \int_{\mathbb{R}^3}\left(4\chi(z)+z\cdot\nabla\chi(z)\right)\tilde{a}(\sigma^{-1/3}z-x)\mathrm{d}z, \quad x\in\mathbb{R}^3.$$

Since χ is compactly supported in $\mathbb{R}^3$, this entails that

$$\|a_\sigma\|_{L^\infty(\mathbb{R}^3)} \leqslant C\|\tilde{a}\|_{L^\infty(\mathbb{R}^3)}, \quad \sigma>0, \tag{3.15}$$

where the constant $C > 0$ depends only on χ.

Let $\tilde{A}_j^\sharp = (\tilde{a}_{1,j}, \tilde{a}_{2,j})$, $j = 1, 2$ be given by Lemma 3.1. With reference to (3.10), we define the smooth magnetic potentials

$$A_{j,\sigma}^\sharp = (a_{1,j,\sigma}, a_{2,j,\sigma}) \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})^2 \cap W^{\infty,\infty}(\mathbb{R}^3, \mathbb{R})^2$$

by setting

$$a_{i,j,\sigma}(x) := \int_{\mathbb{R}^3} \chi_\sigma(x-y)\,(\tilde{a}_{i,j}(y) + (x-y)\cdot\nabla\tilde{a}_{i,j}(y))\,\mathrm{d}y, \quad x \in \mathbb{R}^3,\ i,j = 1,2. \tag{3.16}$$

Thus, applying Lemma 3.2 with $\tilde{a} = \tilde{a}_{i,j}$ for $i, j = 1, 2$, we obtain the following result.

Lemma 3.3. *For $j = 1, 2$, let $A_j^\sharp$ be the same as in Lemma 3.1 and fulfill (3.1). Then, there exists a constant $C > 0$, depending only on ω and M, such that for all $\sigma > 0$ we have*

$$\begin{aligned}\|A_{j,\sigma}^\sharp - A_j^\sharp\|_{W^{k,\infty}(\Omega)^2} &\leqslant \|A_{j,\sigma}^\sharp - \tilde{A}_j^\sharp\|_{W^{k,\infty}(\mathbb{R}^3)^2} \\ &\leqslant C\sigma^{(k-2)/3}, \quad j = 1,2,\ k = 0,1,\end{aligned} \tag{3.17}$$

where $\tilde{A}_j^\sharp$ is given by Lemma 3.1, and

$$\|A_{j,\sigma}^\sharp\|_{W^{k,\infty}(\mathbb{R}^3)^2} \leqslant C\sigma^{(k-2)/3}\|\tilde{A}_j^\sharp\|_{W^{2,\infty}(\mathbb{R}^3)} \leqslant C\sigma^{(k-2)/3}, \quad k \geqslant 2. \tag{3.18}$$

For further use, we notice from (3.4) and from (3.18) with $k = 2$, that the estimate

$$\|b_{j,\sigma}\|_{W^{2,\infty}(\mathbb{R}\times\Omega)} + \|\partial_t b_{j,\sigma}\|_{W^{2,\infty}(\mathbb{R}\times\Omega)} \leqslant C, \quad j = 1,2 \tag{3.19}$$

holds uniformly in $\sigma > 0$, for some constant $C > 0$ that is independent of σ. Moreover, it can be checked from (3.4) through direct calculation, that

$$\begin{aligned}\theta\cdot\nabla_{x'} b_{j,\sigma}(t,x) &= -i\left(\sum_{m=1}^2 \theta_m \int_0^t \sum_{k=1}^2 \theta_k \partial_{x_k} a_{j,m,\sigma}(x' - s\theta, x_3)\mathrm{d}s\right) b_j(t,x) \\ &= i\left(\sum_{k=1}^2 \theta_k \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} a_{j,k,\sigma}(x' - s\theta, x_3)\mathrm{d}s\right) b_j(t,x) \\ &= i\left(\theta\cdot A_{j,\sigma}^\sharp(x' - t\theta, x_3) - \theta\cdot A_{j,\sigma}^\sharp(x', x_3)\right) b_j(t,x), \quad (t,x) \in Q.\end{aligned}$$

Therefore, $b_{j,\sigma}$ is a solution to the transport equation

$$(\partial_t + \theta\cdot\nabla_{x'} + i\theta\cdot A_{j,\sigma}^\sharp) b_{j,\sigma} = 0 \quad \text{in } Q,\ \sigma \in \mathbb{R}_+^*,\ j = 1,2. \tag{3.20}$$

We turn now to building suitable GO solutions to the magnetic Schrödinger equation of (1.1).

§3.2. Building GO solutions to magnetic Schrödinger equations

For $j = 1, 2$, we seek GO solutions to the magnetic Schrödinger equation of (1.1) where (A, q) is replaced by (A_j, q_j). We assume that (A_j, q_j) fulfills the conditions (3.3)–(3.6), where the function $A^\sharp_{j,\sigma}$, appearing in (3.4), is the smooth magnetic potential described by Lemma 3.3. This requires that the functions Φ_j in (3.3) be preliminarily defined. To do that, we set $B(0, r) := \{x' \in \mathbb{R}^2;\ |x'| < r\}$ for all $r > 0$ and take $R > 1$ so large that $\overline{\tilde{\omega}} \subset B(0, R - 1)$, where $\tilde{\omega}$ is the same as in Lemma 3.1. Next we pick $\phi_j \in \mathcal{C}^\infty_0(\mathbb{R}^3)$ such that

$$\operatorname{supp} \phi_j(\cdot, x_3) \subset \mathcal{D}_R := B(0, R + 1)\backslash\overline{B(0, R)}, \quad x_3 \in \mathbb{R}, \tag{3.21}$$

and put

$$\Phi_j(t, x) := \phi_j(x' - t\theta, x_3), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3. \tag{3.22}$$

It is clear from (3.21) and the embedding $\omega \subset B(0, R - 1)$ that

$$\operatorname{supp} \phi_j(\cdot, x_3) \cap \omega = \emptyset, \quad x_3 \in \mathbb{R}, \tag{3.23}$$

and from (3.22) that Φ is a solution to the transport equation (3.5).

In the sequel, we choose $\sigma > \sigma_* := (R + 1)/T$ in such a way that

$$\begin{aligned} \operatorname{supp} \Phi_j(\pm 2\sigma t, \cdot, x_3) \cap \omega &= \operatorname{supp} \phi_j(\cdot \mp 2\sigma t\theta, x_3) \cap \omega \\ &= \emptyset, \quad (t, x_3) \in [T, +\infty) \times \mathbb{R}. \end{aligned} \tag{3.24}$$

Notice that upon possibly enlarging R, we may assume that $\sigma_* \geqslant 1$, which will always be the case in the remaining part of this text.

Next we introduce

$$\mathcal{H}^k_\theta := \big\{\phi \in H^k(\mathbb{R}^3);\ \theta \cdot \nabla_{x'} \cdot \phi \in H^k(\mathbb{R}^3) \text{ and } \operatorname{supp} \phi(\cdot, x_3) \subset \mathcal{D}_R \text{ for a.e. } x_3 \in \mathbb{R}\big\},$$

a subspace of $H^k(\mathbb{R}^3)$ for $k \in \mathbb{N}_0$, endowed with the norm

$$N_{k,\theta}(\phi) := \|\phi\|_{H^k(\mathbb{R}^3)} + \|\theta \cdot \nabla_{x'}\phi\|_{H^k(\mathbb{R}^3)}, \quad \phi \in \mathcal{H}^2_\theta. \tag{3.25}$$

For notational simplicity, we put

$$\mathcal{N}_{\theta,\sigma}(\phi) := N_{2,\theta}(\phi) + \sigma^{1/3} N_{0,\theta}(\phi). \tag{3.26}$$

The coming statement claims existence of GO solutions $u_{j,\sigma}$, given by (3.3), where the $L^2(0, T; H^k(\Omega))$-norm of the correction term $\psi_{j,\sigma}$ is bounded by $\mathcal{N}_{\theta,\sigma}(\phi_j)/\sigma^{1-k}$ for $k = 0, 1$.

Proposition 3.4. *Let $M>0$ and let $A_j \in W^{2,\infty}(\Omega,\mathbb{R}^3)$ and $q_j \in W^{1,\infty}(\Omega,\mathbb{R})$, $j=1,2$ fulfill (3.1)–(3.2). Then, for all $\sigma>\sigma_*$, there exists $u_{j,\sigma} \in \mathcal{C}^1([0,T], L^2(\Omega)) \cap \mathcal{C}([0,T],H^2(\Omega))$ obeying (3.3)–(3.6), where Φ_j is defined by (3.21)–(3.22), such that we have*

$$\left(i\partial_t + \Delta_{A_j} + q_j\right) u_{j,\sigma} = 0 \quad \text{in } Q$$

and the correction term satisfies $\psi_{j,\sigma}=0$ on Σ, for $j=1,2$, and $\psi_{1,\sigma}(T,\cdot) = \psi_{2,\sigma}(0,\cdot)=0$ in Ω.

Moreover, the estimate

$$\sigma\|\psi_{j,\sigma}\|_{L^2(Q)} + \|\nabla\psi_{j,\sigma}\|_{L^2(Q)^3} \leqslant C\mathcal{N}_{\theta,\sigma}(\phi_j), \quad j=1,2 \tag{3.27}$$

holds for some constant $C>0$ depending only on T, ω and M, where the function $\phi_j \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ fulfills (3.21).

Proof. We prove the result for $j=2$, the case $j=1$ being obtained in the same way.

In light of (3.3)–(3.5) and the identity $(i\partial_t + \Delta_{A_2} + q_2)u_{2,\sigma}=0$ imposed on $u_{2,\sigma}$ in Q, we seek a solution $\psi_{2,\sigma}$ to the IBVP

$$\begin{cases} \left(i\partial_t + \Delta_{A_2} + q_2\right)\psi_{2,\sigma} = g_\sigma & \text{in } Q, \\ \psi_2(0,\cdot) = 0 & \text{in } \Omega, \\ \psi_2 = 0 & \text{on } \Sigma, \end{cases} \tag{3.28}$$

where

$$g_\sigma := -\left(i\partial_t + \Delta_{A_2} + q_2\right)\left(w_\sigma \varphi_\sigma\right),$$

with

$$w_\sigma(t,x') := e^{i\sigma(x'\cdot\theta - \sigma t)} \quad \text{and} \quad \varphi_\sigma(t,x) := \vartheta_\sigma(2\sigma t, x), \quad \text{where } \vartheta_\sigma := \Phi_2 b_{2,\sigma}. \tag{3.29}$$

Next, taking into account that $(i\partial_t + \Delta_{A_2} + q_2)w_\sigma = (i\nabla\cdot A_2 - |A_2|^2 - 2\sigma\theta\cdot A_2^\sharp + q_2)w_\sigma$ and recalling from (3.5) and (3.20) that $i(\partial_t + 2\sigma\theta\cdot\nabla_{x'})\varphi_\sigma = 2\sigma\theta\cdot A_{2,\sigma}^\sharp \varphi_\sigma$, we get by straightforward computations that

$$g_\sigma(t,x) = -w_\sigma(t,x)\sum_{m=0,1} g_{m,\sigma}(2\sigma t, x), \tag{3.30}$$

$$\text{with } g_{0,\sigma} := (\Delta_{A_2} + q_2)\vartheta_\sigma,\ g_{1,\sigma} := 2\sigma\theta\cdot(A_{2,\sigma}^\sharp - A_2^\sharp)\vartheta_\sigma.$$

As $g_\sigma \in W^{1,1}(0,T;L^2(\Omega))$, by (3.21)–(3.22), we know from Lemma 2.2 that (3.28) admits a unique solution $\psi_{2,\sigma} \in \mathcal{C}^1([0,T],L^2(\Omega)) \cap \mathcal{C}([0,T],H_0^1(\Omega)\cap H^2(\Omega))$. Moreover, since

$$\psi_{2,\sigma}(t,x) = -i\int_0^t e^{-i(t-s)\mathscr{H}_{A_2,q_2}} g_\sigma(s,x)\mathrm{d}s, \quad (t,x)\in Q,$$

where $\mathscr{H}_{A_2,q_2}$ is the self-adjoint operator acting in $L^2(\Omega)$ which is defined in Section 2.1, we have

$$\|\psi_{2,\sigma}(t,\cdot)\|_{L^2(\Omega)} \leqslant \int_0^t \|e^{-i(t-s)\mathscr{H}_{A_2,q_2}} g_\sigma(s,\cdot)\|_{L^2(\Omega)} \mathrm{d}s \leqslant \|g_\sigma\|_{L^1(0,T;L^2(\Omega))},$$

uniformly in $t \in (0,T)$. This entails $\|\psi_{2,\sigma}\|_{L^2(Q)} \leqslant T^{1/2}\|g_\sigma\|_{L^1(0,T;L^2(\Omega))}$ so (3.30) yields

$$\begin{aligned} \|\psi_{2,\sigma}\|_{L^2(Q)} &\leqslant T^{1/2} \sum_{m=0,1} \int_0^T \|g_{m,\sigma}(2\sigma t,\cdot)\|_{L^2(\Omega)} \mathrm{d}t \\ &\leqslant \sigma^{-1} T^{1/2} \sum_{m=0,1} \|g_{m,\sigma}\|_{L^1(\mathbb{R},L^2(\Omega))}. \end{aligned} \tag{3.31}$$

We are left with the task of bounding each term $\|g_{m,\sigma}\|_{L^1(\mathbb{R},L^2(\Omega))}$, for $m = 0,1$, separately. We start with $m = 0$ and obtain

$$\begin{aligned} \|g_{0,\sigma}\|_{L^1(\mathbb{R},L^2(\Omega))} &= \int_{\mathbb{R}} \|(\Delta_{A_2} + q_2)(\Phi_2 b_{2,\sigma})(s,\cdot)\|_{L^2(\Omega)} \mathrm{d}s \\ &\leqslant C\|b_{2,\sigma}\|_{W^{2,\infty}(\mathbb{R}\times\Omega)} \|\phi_2\|_{H^2(\mathbb{R}^3)} \leqslant C\|\phi_2\|_{H^2(\mathbb{R}^3)} \end{aligned} \tag{3.32}$$

by combining estimate (3.19) with definitions (3.21)–(3.22) and (3.30). Next, applying (3.17) with $k = 0$, we get

$$\|g_{1,\sigma}\|_{L^1(\mathbb{R},L^2(\Omega))} \leqslant C\sigma \|A_{2,\sigma}^\sharp - A_2^\sharp\|_{L^\infty(\Omega)} \|\phi_2\|_{L^2(\mathbb{R}^3)} \leqslant C\sigma^{1/3}\|\phi_2\|_{L^2(\mathbb{R}^3)}. \tag{3.33}$$

Putting this together with (3.31)–(3.32), we find with the aid of (3.25) that

$$\begin{aligned} \sigma\|\psi_{2,\sigma}\|_{L^2(Q)} &\leqslant C\left(\|\phi_2\|_{H^2(\mathbb{R}^3)} + \sigma^{1/3}\|\phi_2\|_{L^2(\mathbb{R}^3)}\right) \\ &\leqslant C\left(N_{2,\theta}(\phi_2) + \sigma^{1/3} N_{0,\theta}(\phi_2)\right). \end{aligned} \tag{3.34}$$

It remains to bound $\|\nabla\psi_{2,\sigma}\|_{L^1(\mathbb{R},L^2(\Omega))}$ from above. To do that, we apply [11, Lem. 3.2], which is permitted since $g_\sigma(0,\cdot) = 0$, with $\varepsilon = \sigma^{-1}$. In light of (3.29)–(3.30), we get

$$\begin{aligned} \|\nabla\psi_{2,\sigma}(t,\cdot)\|_{L^2(\Omega)} &\leqslant C\left(\sigma\|g_\sigma\|_{L^1(0,T;L^2(\Omega))} + \sigma^{-1}\|\partial_t g_\sigma\|_{L^1(0,T;L^2(\Omega))}\right) \\ &\leqslant C\sum_{m=0,1}\left(\sigma\int_0^T \|g_{m,\sigma}(2\sigma t,\cdot)\|_{L^2(\Omega)}\mathrm{d}t \right. \\ &\qquad \left. + \int_0^T \|\partial_t g_{m,\sigma}(2\sigma t,\cdot)\|_{L^2(\Omega)}\mathrm{d}t\right) \\ &\leqslant C\sum_{m=0,1}\left(\|g_{m,\sigma}\|_{L^1(\mathbb{R},L^2(\Omega))} + \|\partial_t g_{m,\sigma}\|_{L^1(\mathbb{R},L^2(\Omega))}\right), \quad t\in(0,T). \end{aligned} \tag{3.35}$$

Further, as we have

$$\partial_t g_{0,\sigma}(t,x) = -(\Delta_{A_2}+q_2)\theta\cdot\left(\nabla_{x'}\phi_2 + iA^\sharp_{2,\sigma}\phi_2\right)(x'-t\theta,x_3)b_{2,\sigma}(t,x),$$
$$(t,x)\in\mathbb{R}\times\Omega,$$

we obtain

$$\|\partial_t g_{0,\sigma}\|_{L^1(\mathbb{R},L^2(\Omega))} \leqslant CN_{2,\theta}(\phi_2) \tag{3.36}$$

from (3.4), (3.18) with $k=2$, (3.19) with $j=2$, (3.21)–(3.22) and (3.29)–(3.30). Similarly, as

$$\partial_t g_{1,\sigma}(t,x) = -2\sigma\theta\cdot(A^\sharp_2 - A^\sharp_{2,\sigma})(x)\theta\cdot\left(\nabla_{x'}\phi_2 + iA^\sharp_{2,\sigma}\phi_2\right)(x'-t\theta,x_3)b_{2,\sigma}(t,x),$$
$$(t,x)\in\mathbb{R}\times\Omega,$$

we find

$$\|\partial_t g_{1,\sigma}\|_{L^1(\mathbb{R},L^2(\Omega))} \leqslant C\sigma\|A^\sharp_2 - A^\sharp_{2,\sigma}\|_{L^\infty(\Omega)}N_{0,\theta}(\phi_2) \leqslant C\sigma^{1/3}N_{0,\theta}(\phi_2) \tag{3.37}$$

by virtue of (3.17) with $j=2$ and $k=0$. Thus, we infer from (3.32)–(3.33) and (3.35)–(3.37) that

$$\|\nabla\psi_{2,\sigma}(t,\cdot)\|_{L^2(\Omega)^3} \leqslant C\left(N_{2,\theta}(\phi_2) + \sigma^{1/3}N_{0,\theta}(\phi_2)\right),\quad t\in(0,T),\ \sigma>\sigma_*.$$

This and (3.34) yield (3.27) with $j=2$, upon recalling definition (3.26). □

Let us now prove that we may substitute $\sigma^{-1/6}u_{j,\sigma}$ for $\psi_{j,\sigma}$ in estimate (3.27).

Corollary 3.5. *For $j=1,2$, let q_j, A_j, ϕ_j and $u_{j,\sigma}$ be the same as in Proposition 3.4. Then, there exists a constant $C>0$, depending only on T, ω and M, such that the estimate*

$$\sigma\|u_{j,\sigma}\|_{L^2(Q)} + \|\nabla u_{j,\sigma}\|_{L^2(Q)^3} \leqslant C\sigma^{1/6}\mathcal{N}_{\theta,\sigma}(\phi_j),\quad j=1,2 \tag{3.38}$$

holds for all $\sigma>\sigma_*$.

Proof. Notice from (3.22) and (3.24) that

$$\int_0^T \|\Phi_j(2\sigma t,\cdot)\|^2_{H^k(\Omega)}\mathrm{d}t = \int_0^{+\infty}\|\Phi_j(2\sigma t,\cdot)\|^2_{H^k(\Omega)}\mathrm{d}t$$
$$= (2\sigma)^{-1}\int_0^{2R}\|\Phi_j(s,\cdot)\|^2_{H^k(\Omega)}\mathrm{d}s,$$

so we have

$$\|\Phi_j(2\sigma\cdot,\cdot)\|_{L^2(0,T;H^k(\Omega))} \leqslant R^{1/2}\sigma^{-1/2}\|\phi_j\|_{H^k(\mathbb{R}^3)},\quad j=1,2,\ k\in\mathbb{N}_0. \tag{3.39}$$

From this, (3.3), (3.19) and (3.25)–(3.27), it follows for each $j = 1, 2$ that

$$\begin{aligned}\|u_{j,\sigma}\|_{L^2(Q)} &\leqslant \|b_{j,\sigma}\|_{L^\infty(\mathbb{R}\times\Omega)}\|\Phi_j(2\sigma\cdot,\cdot)\|_{L^2(Q)} + \|\psi_{j,\sigma}\|_{L^2(Q)}\\ &\leqslant C\left(\sigma^{-1/2}\|\phi_j\|_{L^2(\mathbb{R}^3)} + \sigma^{-1}\mathcal{N}_{\theta,\sigma}(\phi_j)\right) \leqslant C\sigma^{-5/6}\mathcal{N}_{\theta,\sigma}(\phi_j)\end{aligned}$$

and

$$\begin{aligned}\|\nabla u_{j,\sigma}\|_{L^2(Q)^3} &\leqslant \|b_{j,\sigma}\|_{W^{1,\infty}(\mathbb{R}\times\Omega)}\left(\sigma\|\Phi_j(2\sigma\cdot,\cdot)\|_{L^2(Q)} + \|\Phi_j(2\sigma\cdot,\cdot)\|_{L^2(0,T;H^1(\Omega))}\right)\\ &\quad + \|\nabla\psi_{j,\sigma}\|_{L^2(Q)^3}\\ &\leqslant C\left(\sigma^{1/2}\|\phi_j\|_{L^2(\mathbb{R}^3)} + \sigma^{-1/2}\|\phi_j\|_{H^1(\mathbb{R}^3)} + \mathcal{N}_{\theta,\sigma}(\phi_j)\right)\\ &\leqslant C\sigma^{1/6}\mathcal{N}_{\theta,\sigma}(\phi_j),\end{aligned}$$

which yields (3.38). □

In the coming subsection we probe the medium with the GO solutions described in Proposition 3.4 in order to upper bound the transverse magnetic potential in terms of a suitable norm of the DN map.

§3.3. Probing the medium with GO solutions

Let us introduce

$$\tilde{A}^\sharp := \tilde{A}_2^\sharp - \tilde{A}_1^\sharp \quad \text{and} \quad A_\sigma^\sharp := A_{2,\sigma}^\sharp - A_{1,\sigma}^\sharp \quad \text{for } \sigma > 0, \tag{3.40}$$

where the functions $\tilde{A}_j^\sharp$ and $A_{j,\sigma}^\sharp$, $j = 1, 2$ are defined in Lemmas 3.1 and 3.3, respectively. Evidently, $\tilde{A}^\sharp$ is the function $A_2^\sharp - A_1^\sharp$ extended by zero outside Ω, and we have

$$\|A_\sigma^\sharp - \tilde{A}^\sharp\|_{W^{1,\infty}(\mathbb{R}^3)^2} \leqslant \sum_{j=1,2}\|A_{j,\sigma}^\sharp - \tilde{A}_j^\sharp\|_{W^{1,\infty}(\mathbb{R}^3)^2} \leqslant 2C\sigma^{-1/3}, \quad \sigma > 0, \tag{3.41}$$

from (3.17) with $k = 1$. Thus, writing $A_\sigma^\sharp = (a_{1,\sigma}, a_{2,\sigma})$ and $\tilde{A}^\sharp = (\tilde{a}_1, \tilde{a}_2)$, it follows readily from (3.16) that

$$a_{i,\sigma}(x) = \int_{\mathbb{R}^3}\chi_\sigma(x-y)\left(\tilde{a}_i(y) + (x-y)\cdot\nabla\tilde{a}_i(y)\right)\mathrm{d}y, \quad x \in \mathbb{R}^3,\ i = 1, 2. \tag{3.42}$$

The main purpose of this subsection is the following technical result.

Lemma 3.6. *Let $M > 0$ and $\theta \in \mathbb{S}^1$ be fixed. For $j = 1, 2$, let $A_j \in W^{2,\infty}(\Omega, \mathbb{R})^3$, let $q_j \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ fulfill (3.1)–(3.2) and let ϕ_j be defined by (3.21). Then, for every $\sigma > \sigma_*$, there exists a constant $C > 0$ depending only on T, ω and M, such* *that we have*

$$\begin{aligned}&\sigma\left|\int_{(0,T)\times\mathbb{R}^3}\theta\cdot\tilde{A}^\sharp(x)(\overline{\phi}_1\phi_2)(x'-2\sigma t\theta,x_3)(\overline{b_{1,\sigma}}b_{2,\sigma})(2\sigma t,x)\mathrm{d}x'\,\mathrm{d}x_3\,\mathrm{d}t\right|\\ &\qquad\leqslant C\left(\sigma^5\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|+\sigma^{-5/6}\right)\mathcal{N}_{\theta,\sigma}(\phi_1)\mathcal{N}_{\theta,\sigma}(\phi_2),\end{aligned}\tag{3.43}$$

where $\|\cdot\|$ *stands for the usual norm in* $\mathcal{B}(H^{2,1}(\Sigma),L^2(\Sigma))$ *and* $\tilde{A}^\sharp$ *is given by* (3.40).

Proof. We proceed in two steps. The first step is to establish a suitable orthogonality identity for $A:=A_2-A_1$ and $V:=i\nabla\cdot A-(|A_2|^2-|A_1|^2)+q_2-q_1$, which is the key ingredient in the derivation of estimate (3.43), presented in the second step.

Step 1: Orthogonality identity. We probe the system with the GO functions $u_{j,\sigma}$, $j=1,2$, given by Proposition 3.4. We recall that $u_{j,\sigma}\in\mathcal{C}^1([0,T],L^2(\Omega))\cap\mathcal{C}([0,T],H^2(\Omega))$ is expressed by (3.3) and satisfies the equation

$$\left(i\partial_t+\Delta_{A_j}+q_j\right)u_{j,\sigma}=0\quad\text{in } Q.\tag{3.44}$$

Since $A^\sharp_{2,\sigma}\in W^{2,\infty}(\Omega)^2$ and $\phi_2\in C_0^\infty(\mathbb{R}^3)$, it follows readily from (3.3)–(3.4) and (3.22) that $u_{2,\sigma}-\psi_{2,\sigma}\in C^\infty([0,T],W^{2,\infty}(\Omega))$. Thus, we have $F:=-\left(i\partial_t+\Delta_{A_1}+q_1\right)(u_{2,\sigma}-\psi_{2,\sigma})\in W^{1,1}(0,T;L^2(\Omega))$ and there is consequently a unique solution $z\in\mathcal{C}^1([0,T],L^2(\Omega))\cap\mathcal{C}([0,T],H_0^1(\Omega)\cap H^2(\Omega))$ to the IBVP

$$\begin{cases}\left(i\partial_t+\Delta_{A_1}+q_1\right)z=F & \text{in } Q,\\ z(0,\cdot)=0 & \text{in } \Omega,\\ z=0 & \text{on } \Sigma,\end{cases}\tag{3.45}$$

according to Lemma 2.2. Further, as $(u_{2,\sigma}-\psi_{2,\sigma})(0,\cdot)=0$ in Ω, by (3.22)–(3.23), we infer from (3.45) that $v:=z+u_{2,\sigma}-\psi_{2,\sigma}\in\mathcal{C}^1([0,T],L^2(\Omega))\cap\mathcal{C}([0,T],H^2(\Omega))$ verifies

$$\begin{cases}\left(i\partial_t+\Delta_{A_1}+q_1\right)v=0 & \text{in } Q,\\ v(0,\cdot)=0 & \text{in } \Omega,\\ v=f_\sigma & \text{on } \Sigma,\end{cases}\tag{3.46}$$

where we have set

$$\begin{aligned}f_\sigma(t,x):=u_{2,\sigma}(t,x)&=u_{2,\sigma}(t,x)-\psi_{2,\sigma}(t,x)\\ &=(\Phi_2b_{2,\sigma})(2\sigma t,x)e^{i\sigma(x'\cdot\theta-\sigma t)},\quad(t,x)\in\Sigma.\end{aligned}\tag{3.47}$$

From this and Proposition 3.4, it then follows that $w := v - u_{2,\sigma}$ is the $\mathcal{C}^1([0,T], L^2(\Omega)) \cap \mathcal{C}([0,T], H_0^1(\Omega) \cap H^2(\Omega))$-solution to the IBVP

$$\begin{cases} (i\partial_t + \Delta_{A_1} + q_1)\, w = 2iA \cdot \nabla u_{2,\sigma} + V u_{2,\sigma} & \text{in } Q, \\ w(0,\cdot) = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \Sigma. \end{cases} \tag{3.48}$$

In light of (3.48), we deduce from (3.44) with $j = 1$, upon applying Green's formula, that

$$\begin{aligned} \langle 2iA \cdot \nabla u_{2,\sigma} + V u_{2,\sigma}, u_{1,\sigma} \rangle_{L^2(Q)} &= \langle (i\partial_t + \Delta_{A_1} + q_1)\, w, u_{1,\sigma} \rangle_{L^2(Q)} \\ &= \langle (\partial_\nu + iA_1 \cdot \nu)\, w, u_{1,\sigma} \rangle_{L^2(\Sigma)}. \end{aligned} \tag{3.49}$$

Next, taking into account that $A_1 = A_2$ on $\partial\Omega$, by (3.2), we see that

$$\begin{aligned} (\partial_\nu + iA_1 \cdot \nu)\, w &= (\partial_\nu + iA_1 \cdot \nu)\, v - (\partial_\nu + iA_1 \cdot \nu)\, u_{2,\sigma} \\ &= (\partial_\nu + iA_1 \cdot \nu)\, v - (\partial_\nu + iA_2 \cdot \nu)\, u_{2,\sigma} \\ &= (\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}) f_\sigma \end{aligned}$$

from (3.47) and the last line of (3.46). This and (3.49) yield the orthogonality identity

$$\begin{aligned} &2i\langle A \cdot \nabla u_{2,\sigma}, u_{1,\sigma} \rangle_{L^2(Q)} + \langle V u_{2,\sigma}, u_{1,\sigma} \rangle_{L^2(Q)} \\ &\qquad = \langle (\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}) f_\sigma, g_\sigma \rangle_{L^2(\Sigma)}, \end{aligned} \tag{3.50}$$

where

$$\begin{aligned} g_\sigma(t,x) := u_{1,\sigma}(t,x) &= u_{1,\sigma}(t,x) - \psi_{1,\sigma}(t,x) \\ &= (\Phi_1 b_{1,\sigma})(2\sigma t, x) e^{i\sigma(x' \cdot \theta - 2\sigma t)}, \quad (t,x) \in \Sigma. \end{aligned} \tag{3.51}$$

Having established (3.50), we turn now to proving estimate (3.43).

Step 2: Derivation of (3.43). In light of (3.3), we have

$$\begin{aligned} &\langle A \cdot \nabla u_{2,\sigma}, u_{1,\sigma} \rangle_{L^2(Q)} \\ &\qquad = I_\sigma + i\sigma \int_Q \theta \cdot A^\sharp(x) (\overline{\Phi}_1 \Phi_2)(2\sigma t, x) (\overline{b_{1,\sigma}} b_{2,\sigma})(2\sigma t, x) \mathrm{d}x\, \mathrm{d}t, \end{aligned} \tag{3.52}$$

with

$$\begin{aligned}
I_\sigma := &\int_Q A\cdot\nabla(\Phi_2 b_{2,\sigma})(2\sigma t,x)\left(\overline{(\Phi_1 b_{1,\sigma})}(2\sigma t,x)+e^{i\sigma(x'\cdot\theta-\sigma t)}\overline{\psi_{1,\sigma}}(t,x)\right)\mathrm{d}x\,\mathrm{d}t\\
&+\int_Q A\cdot\nabla\psi_{2,\sigma}(t,x)\left(e^{i\sigma(x'\cdot\theta-\sigma t)}\overline{(\Phi_1 b_{1,\sigma})}(2\sigma t,x)+\overline{\psi_{1,\sigma}}(t,x)\right)\mathrm{d}x\,\mathrm{d}t\\
&+i\sigma\int_Q \theta\cdot A^\sharp(x)(\Phi_2 b_{2,\sigma})(2\sigma t,x)\overline{\psi_{1,\sigma}}(t,x)e^{i\sigma(x'\cdot\theta-\sigma t)}\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We infer from (3.19), (3.27) and (3.39) that

$$|I_\sigma|\leqslant C\sigma^{-5/6}\mathcal{N}_{\theta,\sigma}(\phi_1)\mathcal{N}_{\theta,\sigma}(\phi_2),\quad \sigma>\sigma_*.$$

Putting this together with (3.50) and (3.52), we find

$$\begin{aligned}
\sigma&\left|\int_Q\theta\cdot A^\sharp(x)(\phi_2\overline{\phi}_1)(x'-2\sigma t\theta,x_3)(b_{2,\sigma}\overline{b_{1,\sigma}})(2\sigma t,x)\mathrm{d}x'\,\mathrm{d}x_3\,\mathrm{d}t\right|\\
&\leqslant C\Big(\left|\langle Vu_{2,\sigma},u_{1,\sigma}\rangle_{L^2(Q)}\right|+\left|\langle(\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2})f_\sigma,g_\sigma\rangle_{L^2(\Sigma)}\right|\\
&\qquad+\sigma^{-5/6}\mathcal{N}_{\theta,\sigma}(\phi_1)\mathcal{N}_{\theta,\sigma}(\phi_2)\Big).
\end{aligned}\tag{3.53}$$

Next we notice from (3.38) that

$$\left|\langle Vu_{2,\sigma},u_{1,\sigma}\rangle_{L^2(Q)}\right|\leqslant C\sigma^{-5/3}\mathcal{N}_{\theta,\sigma}(\phi_1)\mathcal{N}_{\theta,\sigma}(\phi_2).\tag{3.54}$$

Moreover, in view of (3.47) and (3.51), we have

$$\begin{aligned}
&\left|\langle(\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2})f_\sigma,g_\sigma\rangle_{L^2(\Sigma)}\right|\\
&\qquad\leqslant\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\|f_\sigma\|_{H^{2,1}(\Sigma)}\|g_\sigma\|_{L^2(\Sigma)}\\
&\qquad\leqslant\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\|u_{2,\sigma}-\psi_{2,\sigma}\|_{H^{2,1}(\Sigma)}\|u_{1,\sigma}-\psi_{1,\sigma}\|_{L^2(\Sigma)},
\end{aligned}$$

with

$$\begin{aligned}
\|u_{1,\sigma}-\psi_{1,\sigma}\|_{L^2(\Sigma)}&\leqslant\|u_{1,\sigma}-\psi_{1,\sigma}\|_{L^2(0,T;H^1(\Omega))}\\
&\leqslant C\sigma\|\Phi_1(2\sigma\cdot,\cdot)\|_{L^2(0,T;H^1(\Omega))}\|b_{1,\sigma}\|_{W^{1,\infty}(\mathbb{R}\times\Omega)}\\
&\leqslant C\sigma^{1/2}\mathcal{N}_{\theta,\sigma}(\phi_1)
\end{aligned}$$

and

$$\begin{aligned}
&\|u_{2,\sigma}-\psi_{2,\sigma}\|_{H^{2,1}(\Sigma)}\\
&\quad\leqslant C\left(\|u_{2,\sigma}-\psi_{2,\sigma}\|_{H^2(0,T;H^1(\Omega))}+\|u_{2,\sigma}-\psi_{2,\sigma}\|_{L^2(0,T;H^2(\Omega))}\right)\\
&\quad\leqslant C\sigma^5\|\Phi_2(2\sigma\cdot,\cdot)\|_{L^2(0,T;H^2(\Omega))}\left(\|b_{2,\sigma}\|_{W^{2,\infty}(\mathbb{R}\times\Omega)}+\|\partial_t b_{2,\sigma}\|_{W^{2,\infty}(\mathbb{R}\times\Omega)}\right)\\
&\quad\leqslant C\sigma^{9/2}\mathcal{N}_{\theta,\sigma}(\phi_2),
\end{aligned}$$

according to (3.3), (3.19), (3.25) and (3.39). As a consequence, we have

$$\left|\langle(\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2})f_\sigma, g_\sigma\rangle_{L^2(\Sigma)}\right| \leqslant C\sigma^5\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|\mathcal{N}_{\theta,\sigma}(\phi_1)\mathcal{N}_{\theta,\sigma}(\phi_2). \tag{3.55}$$

This and (3.53)–(3.54) yield (3.43). □

§4. Preliminary estimates

§4.1. X-ray transform

In this subsection we estimate the partial X-ray transform of the functions

$$\tilde{\rho}_j(x', x_3) := \theta \cdot \frac{\partial \tilde{A}^\sharp}{\partial x_j}(x) = \sum_{i=1,2} \theta_i \frac{\partial \tilde{a}_i}{\partial x_j}(x), \quad x \in \mathbb{R}^3,\ j = 1, 2, 3 \tag{4.1}$$

in terms of the DN map. We recall that the partial X-ray transform of a function

$$f \in \mathscr{X} := \left\{\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^3);\ x' \mapsto \varphi(x', x_3) \in L^1(\mathbb{R}^2) \text{ for a.e. } x_3 \in \mathbb{R}\right\} \tag{4.2}$$

in the direction $\theta \in \mathbb{S}^1$ is defined by

$$\mathcal{P}(f)(\theta, x', x_3) := \int_{\mathbb{R}} f(x' + s\theta, x_3)\mathrm{d}s, \quad x' \in \mathbb{R}^2,\ x_3 \in \mathbb{R}. \tag{4.3}$$

The X-ray transform stability estimate is as follows.

Lemma 4.1. *Let $M > 0$ and let A_j and q_j, for $j = 1, 2$ be as in Proposition 3.4. Then, there exists a constant $C > 0$, depending only on T, ω and M, such that for all $\theta \in \mathbb{S}^1$, all $\xi' \in \mathbb{R}^2$ and all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ fulfilling* $\operatorname{supp}\,\phi(\cdot, x_3) \subset \mathcal{D}_R^-(\theta) := \{x' \in \mathcal{D}_R,\ x' \cdot \theta \leqslant 0\}$ *for every $x_3 \in \mathbb{R}$, the estimate*

$$\begin{aligned}&\left|\int_{\mathbb{R}^3} \phi^2(x)\mathcal{P}(\tilde{\rho}_j)(\theta, x', x_3)\exp\left(-i\int_{\mathbb{R}} \theta \cdot A_\sigma^\sharp(x' + s\theta, x_3)\mathrm{d}s\right)\mathrm{d}x\right| \\ &\qquad \leqslant C\left(\sigma^5\|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\| + \sigma^{-5/6}\right)\mathcal{N}_{\theta,\sigma}(\phi)\mathcal{N}_{\theta,\sigma}(\partial_{x_j}\phi)\end{aligned} \tag{4.4}$$

holds uniformly in $\sigma > \sigma_*$ and $j = 1, 2, 3$.

Proof. Let $\phi_j \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, $j = 1, 2$ be supported in $\mathcal{D}_R \times \mathbb{R}$. Then, bearing in mind that $\overline{\tilde{\omega}} \subset B(0, R-1)$, we infer from (3.42) that $\tilde{A}^\sharp$ and $A_\sigma^\sharp$ are both supported in $B(0, R) \times \mathbb{R}$. Further, as $|x' - 2\sigma t\theta| > 2\sigma_* T - R > R + 1$ for all $x' \in B(0, R)$ and $t > T$, we see that

$$\begin{aligned}\tilde{A}^\sharp(x)(\overline{\phi}_1\phi_2)(x' - 2\sigma t\theta, x_3) = A_\sigma^\sharp(x)(\overline{\phi}_1\phi_2)(x' - 2\sigma t\theta, x_3) = 0,& \\ x = (x', x_3) \in \mathbb{R}^3,\ t > T.&\end{aligned}$$

As a consequence we have

$$\int_0^T \int_{\mathbb{R}^3} \theta \cdot \left(\tilde{A}^\sharp(x) - A_\sigma^\sharp(x)\right) (\overline{\phi_1}\phi_2)(x' - 2\sigma t\theta, x_3)(\overline{b_{1,\sigma}} b_{2,\sigma})(2\sigma t, x) \mathrm{d}x \, \mathrm{d}t$$
$$= \int_0^{+\infty} \int_{\mathbb{R}^3} \theta \cdot \left(\tilde{A}^\sharp(x) - A_\sigma^\sharp(x)\right) (\overline{\phi_1}\phi_2)(x' - 2\sigma t\theta, x_3)(\overline{b_{1,\sigma}} b_{2,\sigma})(2\sigma t, x) \mathrm{d}x \, \mathrm{d}t.$$

Next, making the substitution $s = \sigma t$ in the above integral, we get

$$\begin{aligned}
&\sigma \left| \int_0^T \int_{\mathbb{R}^3} \theta \cdot \left(\tilde{A}^\sharp(x) - A_\sigma^\sharp(x)\right) (\overline{\phi_1}\phi_2)(x' - 2\sigma t\theta, x_3)(\overline{b_{1,\sigma}} b_{2,\sigma})(2\sigma t, x) \mathrm{d}x \, \mathrm{d}t \right| \\
&\quad = \left| \int_0^{+\infty} \int_{B(0,R)\times\mathbb{R}} \theta \cdot \left(\tilde{A}^\sharp(x) - A_\sigma^\sharp(x)\right) (\overline{\phi_1}\phi_2)(x' - 2s\theta, x_3)(\overline{b_{1,\sigma}} b_{2,\sigma})(2s, x) \mathrm{d}x \, \mathrm{d}s \right| \\
&\quad \leqslant \|\tilde{A}^\sharp - A_\sigma^\sharp\|_{L^\infty(\mathbb{R}^3)^2} \int_0^{R+1} \int_{B(0,R)\times\mathbb{R}} \left|(\overline{\phi}_1\phi_2)(x' - 2s\theta, x_3)\right| \mathrm{d}x \, \mathrm{d}s \\
&\quad \leqslant \|\tilde{A}^\sharp - A_\sigma^\sharp\|_{L^\infty(\mathbb{R}^3)^2} \int_0^{R+1} \int_{\mathbb{R}^3} \left|(\overline{\phi_1}\phi_2)(x' - 2s\theta, x_3)\right| \mathrm{d}x \, \mathrm{d}s \\
&\quad \leqslant (R+1) \|\tilde{A}^\sharp - A_\sigma^\sharp\|_{L^\infty(\mathbb{R}^3)^2} \|\phi_1\|_{L^2(\mathbb{R}^3)} \|\phi_2\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

From this, (3.17) with $k = 0$ and (3.25), it follows that

$$\begin{aligned}
&\sigma \left| \int_0^T \int_{\mathbb{R}^3} \theta \cdot \left(\tilde{A}^\sharp(x) - A_\sigma^\sharp(x)\right) (\overline{\phi_1}\phi_2)(x' - 2\sigma t\theta, x_3)(\overline{b_{1,\sigma}} b_{2,\sigma})(2\sigma t, x) \mathrm{d}x \, \mathrm{d}t \right| \\
&\quad \leqslant C\sigma^{-2/3} \|\phi_1\|_{L^2(\mathbb{R}^3)} \|\phi_2\|_{L^2(\mathbb{R}^3)} \leqslant C\sigma^{-4/3} \mathcal{N}_{\theta,\sigma}(\phi_1) \mathcal{N}_{\theta,\sigma}(\phi_2).
\end{aligned} \tag{4.5}$$

On the other hand, since

$$\begin{aligned}
(\overline{b_{1,\sigma}} b_{2,\sigma})(2\sigma t, x' + 2\sigma t\theta, x_3) &= \exp\left(-i \int_0^{2\sigma t} \theta \cdot A_\sigma^\sharp(x' + (2\sigma t - s)\theta, x_3) \mathrm{d}s \right) \\
&= \exp\left(-i \int_0^{2\sigma t} \theta \cdot A_\sigma^\sharp(x' + s\theta, x_3) \mathrm{d}s \right)
\end{aligned}$$

for a.e. $(t, x) \in (0, T) \times \mathbb{R}^3$, we have

$$\sigma \int_0^T \int_{\mathbb{R}^3} \theta \cdot A_\sigma^\sharp(x) (\overline{\phi_1}\phi_2)(x' - 2\sigma t\theta, x_3)(\overline{b_{1,\sigma}} b_{2,\sigma})(2\sigma t, x) \mathrm{d}x' \, \mathrm{d}x_3 \, \mathrm{d}t$$
$$= \sigma \int_0^T \int_{\mathbb{R}^3} \theta \cdot A_\sigma^\sharp(x' + 2\sigma t\theta, x_3) (\overline{\phi_1}\phi_2)(x) (\overline{b_{1,\sigma}} b_{2,\sigma})(2\sigma t, x' + 2\sigma t\theta, x_3) \mathrm{d}x' \, \mathrm{d}x_3 \, \mathrm{d}t$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} (\overline{\phi}_1\phi_2)(x) \Bigg(\int_0^T \sigma\theta \cdot A_\sigma^\sharp(x' + 2\sigma t\theta, x_3) \\
&\qquad \times \exp\left(-i \int_0^{2\sigma t} \theta \cdot A_\sigma^\sharp(x' + s\theta, x_3)\mathrm{d}s\right) \mathrm{d}t \Bigg) \mathrm{d}x'\, \mathrm{d}x_3 \\
&= \frac{i}{2} \int_{\mathbb{R}^3} (\overline{\phi}_1\phi_2)(x) \left(\int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \exp\left(-i \int_0^{2\sigma t} \theta \cdot A_\sigma^\sharp(x' + s\theta, x_3)\mathrm{d}s\right) \mathrm{d}t \right) \mathrm{d}x'\, \mathrm{d}x_3 \\
&= \frac{i}{2} \int_{\mathbb{R}^3} (\overline{\phi}_1\phi_2)(x) \left(\exp\left(-i \int_0^{2\sigma T} \theta \cdot A_\sigma^\sharp(x' + s\theta, x_3)\mathrm{d}s\right) - 1 \right) \mathrm{d}x'\, \mathrm{d}x_3.
\end{aligned}
\tag{4.6}
$$

As $A_\sigma^\sharp$ is supported in $B(0,R) \times \mathbb{R}$ and $|x' + s\theta| > 2\sigma_* T - (R+1) > R$ for all $x' \in \mathcal{D}_R$ and all $s > 2\sigma T$, then we have

$$
\int_0^{2\sigma T} \theta \cdot A_\sigma^\sharp(x' + s\theta, x_3)\mathrm{d}s = \int_0^{+\infty} \theta \cdot A_\sigma^\sharp(x' + s\theta, x_3)\mathrm{d}s, \quad x' \in \mathcal{D}_R,\ x_3 \in \mathbb{R}. \tag{4.7}
$$

Similarly, as $|x' + s\theta|^2 = |x'|^2 + s^2 + 2sx' \cdot \theta > R^2$ for every $x' \in \mathcal{D}_R^-(\theta)$ and $s < 0$, it holds true that

$$
\int_{-\infty}^0 \theta \cdot A_\sigma^\sharp(x' + s\theta, x_3)\mathrm{d}s = 0, \quad x' \in \mathcal{D}_R^-(\theta),\ x_3 \in \mathbb{R}.
$$

This and (4.7) entail

$$
\int_0^{2\sigma T} \theta \cdot A_\sigma^\sharp(x' + s\theta, x_3)\mathrm{d}s = \int_{\mathbb{R}} \theta \cdot A_\sigma^\sharp(x' + s\theta, x_3)\mathrm{d}s, \quad x' \in \mathcal{D}_R^-(\theta),\ x_3 \in \mathbb{R}. \tag{4.8}
$$

Having seen this, we take $\phi_1 := \partial_{x_j}\overline{\phi}$ for $j = 1,2,3$ and $\phi_2 := \phi$ in (4.6). We find

$$
\begin{aligned}
&\sigma \int_0^T \int_{\mathbb{R}^3} \theta \cdot A_\sigma^\sharp(x) (\overline{\phi_1}\phi_2)(x' - 2\sigma t\theta, x_3)(\overline{b_{1,\sigma}} b_{2,\sigma})(2\sigma t, x)\mathrm{d}x'\, \mathrm{d}x_3\, \mathrm{d}t \\
&\qquad = \frac{i}{4} \int_{\mathbb{R}^3} \partial_{x_j}\phi^2(x) \left(\exp\left(-i \int_0^{2\sigma T} \theta \cdot A_\sigma^\sharp(x' + s\theta, x_3)\mathrm{d}s\right) - 1 \right) \mathrm{d}x'\, \mathrm{d}x_3 \\
&\qquad = -\frac{1}{4} \int_{\mathbb{R}^3} \phi^2(x) \left(\int_0^{2\sigma T} \theta \cdot \partial_{x_j} A_\sigma^\sharp(x' + s\theta, x_3)\mathrm{d}s \right) \\
&\qquad\qquad \times \exp\left(-i \int_0^{2\sigma T} \theta \cdot A_\sigma^\sharp(x' + s\theta, x_3)\mathrm{d}s\right) \mathrm{d}x,
\end{aligned}
\tag{4.9}
$$

upon integrating by parts. Taking into account that ϕ is supported in $\mathcal{D}_R^-(\theta)\times\mathbb{R}$, we deduce from (4.8)–(4.9) that

$$\begin{aligned}
\sigma\int_0^T\int_{\mathbb{R}^3}&\theta\cdot A_\sigma^\sharp(x)(\overline{\phi_1}\phi_2)(x'-2\sigma t\theta,x_3)(\overline{b_{1,\sigma}}b_{2,\sigma})(2\sigma t,x)\mathrm{d}x'\,\mathrm{d}x_3\,\mathrm{d}t\\
&=-\frac{1}{4}\int_{\mathbb{R}^3}\phi^2(x)\left(\int_{\mathbb{R}}\theta\cdot\partial_{x_j}A_\sigma^\sharp(x'+s\theta,x_3)\mathrm{d}s\right)\\
&\qquad\times\exp\left(-i\int_{\mathbb{R}}\theta\cdot A_\sigma^\sharp(x'+s\theta,x_3)\mathrm{d}s\right)\mathrm{d}x'\,\mathrm{d}x_3\\
&=-\frac{1}{4}\int_{\mathbb{R}^3}\phi^2(x)\mathcal{P}(\rho_{j,\sigma})(\theta,x',x_3)\exp\left(-i\int_{\mathbb{R}}\theta\cdot A_\sigma^\sharp(x'+s\theta,x_3)\mathrm{d}s\right)\mathrm{d}x'\,\mathrm{d}x_3. \qquad (4.10)
\end{aligned}$$

Here we used (4.3) and the notation

$$\rho_{j,\sigma}(x):=\theta\cdot\partial_{x_j}A_\sigma^\sharp(x)=\sum_{i=1,2}\theta_i\partial_{x_j}a_{i,\sigma}(x),\quad x\in\mathbb{R}^3,\ j=1,2,3.$$

Finally, using once more that the functions $A_\sigma^\sharp$ and $\tilde{A}^\sharp$ are supported in $B(0,R)$, we infer from (3.41) and (4.1)–(4.3) that

$$\left|\left(\mathcal{P}(\rho_{j,\sigma})-\mathcal{P}(\tilde{\rho}_j)\right)(\theta,x',x_3)\right|\leqslant C\sigma^{-1/3},\quad (x',x_3)\in B(0,R)\times\mathbb{R},$$

for some positive constant C depending only on ω and M. This entails

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\phi^2(x)\left(\mathcal{P}(\rho_{j,\sigma})-\mathcal{P}(\tilde{\rho}_j)\right)(\theta,x',x_3)\exp\left(-i\int_{\mathbb{R}}\theta\cdot A_\sigma^\sharp(x'+s\theta,x_3)\mathrm{d}s\right)\mathrm{d}x'\,\mathrm{d}x_3\right|\\
&\qquad\leqslant C\sigma^{-1/3}\|\phi\|_{L^2(\mathbb{R}^3)}^2,
\end{aligned}$$

which, together with (3.43), (4.5) and (4.10), yields (4.4). □

As will be seen in the coming section, the result of Lemma 4.1 is a key ingredient in the estimation of the partial Fourier transform of the aligned magnetic field, in terms of the DN map. For this purpose, we recall for all $f\in\mathscr{X}$, where $\mathscr{X}$ is defined in (4.2), that the partial Fourier transform of f with respect to $x'\in\mathbb{R}^2$, is expressed as

$$\widehat{f}(\xi',x_3):=(2\pi)^{-1}\int_{\mathbb{R}^2}f(x',x_3)e^{-ix'\cdot\xi'}\mathrm{d}x',\quad \xi'\in\mathbb{R}^2,\ x_3\in\mathbb{R}. \qquad (4.11)$$

Further, setting $\theta^\perp:=\{x'\in\mathbb{R}^2;\ x'\cdot\theta=0\}$, we recall for further use from [10, Lem. 6.1] that $x'\mapsto\mathcal{P}(f)(\theta,x',x_3)\in L^1(\theta^\perp)$ for a.e. $x_3\in\mathbb{R}$, and that

$$\begin{aligned}
\widehat{\mathcal{P}(f)}(\theta,\xi',x_3)&:=(2\pi)^{-1/2}\int_{\theta^\perp}\mathcal{P}(f)(\theta,x',x_3)e^{-ix'\cdot\xi'}\mathrm{d}x'\\
&=(2\pi)^{1/2}\widehat{f}(\xi',x_3),\quad \xi'\in\theta^\perp,\ x_3\in\mathbb{R}. \qquad (4.12)
\end{aligned}$$

§4.2. Aligned magnetic field estimation

Let us now estimate the Fourier transform of the aligned magnetic field

$$\tilde{\beta}(x) := (\partial_{x_1}\tilde{a}_2 - \partial_{x_2}\tilde{a}_1)(x), \quad x \in \mathbb{R}^3, \tag{4.13}$$

with the aid of Lemma 4.1. More precisely, we aim to establish the following result.

Lemma 4.2. *Let $M > 0$ and let A_j and q_j, for $j = 1, 2$, be as in Proposition 3.4. Then, there exist two constants $\epsilon \in (0, 1)$ and $C > 0$, both of them depending only on T, ω and M, such that the estimates*

$$\|\widehat{\beta}(\xi', \cdot)\|_{L^\infty(\mathbb{R})} \leqslant C\langle\xi'\rangle^7 \left(\sigma^6\|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\| + \sigma^{-\epsilon}\right) \tag{4.14}$$

and

$$\|\partial_{x_3}\widehat{\beta}(\xi', \cdot)\|_{L^\infty(\mathbb{R})} \leqslant C\langle\xi'\rangle^8 \left(\sigma^6\|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\| + \sigma^{-\epsilon}\right) \tag{4.15}$$

hold for all $\sigma > \sigma_*$ and all $\xi' \in \mathbb{R}^2$, with $\langle\xi'\rangle := \left(1 + |\xi'|^2\right)^{1/2}$.

Proof. We shall prove (4.14) only, the derivation of (4.15) being obtained in a similar way.

We fix $\theta \in \mathbb{S}^1 \cap \xi'^{\perp}$ and we introduce the following partition of $B(0, R) \cap \theta^{\perp}$. For $N \in \mathbb{N} := \{1, 2, \ldots\}$ fixed, we pick $x'_1, \ldots, x'_N$ in $B(0, R+1/2) \cap \theta^{\perp}$ and choose $\varphi_1, \ldots, \varphi_N$ in $\mathcal{C}_0^\infty(\mathbb{R}^2, [0, 1])$ such that

$$\begin{aligned} &\operatorname{supp} \varphi_k \subset B(x'_k, 1/8) \cap \theta^{\perp} \quad \text{for } k = 1, \ldots, N \\ &\text{and} \quad \sum_{k=1}^{N} \varphi_k(x') = 1 \quad \text{for } x' \in B(0, R) \cap \theta^{\perp}. \end{aligned} \tag{4.16}$$

Next we set $r_{x'_k} := \left((R + 3/4)^2 - |x'_k|^2\right)^{1/2}$ in such a way that

$$B(x'_k - r_{x'_k}\theta, 1/4) \subset \mathcal{D}_R^-(\theta), \quad k = 1, \ldots, N. \tag{4.17}$$

In order to define a suitable set of test functions $\phi_{*,k}$, $k = 1, \ldots, N$, we fix $x_3 \in \mathbb{R}$, pick a function $\alpha \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}_+)$ which is supported in $(-1, 1)$ and normalized in $L^2(\mathbb{R})$, and put

$$\alpha_\sigma(s) := \sigma^\mu \alpha\left(\sigma^{2\mu}(x_3 - s)\right), \quad s \in \mathbb{R}, \tag{4.18}$$

where μ is a positive real parameter that we shall make precise below. Then, the test function $\phi_{*,k}$ is defined for all $y = (y', y_3) \in \mathbb{R}^3$ by

$$\begin{aligned} \phi_{*,k}(y) :=\ & h\left(y' \cdot \theta + r_{x'_k}\right) e^{-\frac{i}{2}y'\cdot\xi'} \varphi_k^{1/2}(y' - (y' \cdot \theta)\theta) \\ & \times \exp\left(\frac{i}{2}\int_{\mathbb{R}} \theta \cdot A_\sigma^\sharp(y' + s\theta, y_3)\mathrm{d}s\right) \alpha_\sigma(y_3), \end{aligned} \tag{4.19}$$

where $h \in \mathcal{C}_0^\infty(\mathbb{R})$ is supported in $(0, 1/8)$ and normalized in $L^2(\mathbb{R})$.

Further, for every $y' \in \mathbb{R}^2 \setminus B(x'_k - r_{x'_k}\theta, 1/4)$, it is easily seen from the basic inequality

$$|y' - (x'_k - r_{x'_k}\theta)| \leqslant |y' - (y' \cdot \theta)\theta - x'_k| + |y' \cdot \theta + r_{x'_k}|$$

that either of the two real numbers $|y' - (y' \cdot \theta)\theta - x'_k|$ or $|y' \cdot \theta + r_{x'_k}|$ is greater than $1/8$ and hence that $h(y' \cdot \theta + r_{x'_k})\varphi_k^{1/2}(y' - (y' \cdot \theta)\theta) = 0$. As a consequence, we have

$$\text{supp}\ \phi_{*,k}(\cdot, y_3) \subset B\left(x'_k - r_{x'_k}\theta, 1/4\right) \subset \mathcal{D}_R^-(\theta), \quad y_3 \in \mathbb{R},\ k = 1, \ldots, N, \tag{4.20}$$

directly from (4.17) and (4.19). Moreover, since

$$\theta \cdot \nabla_{y'}\left(\int_{\mathbb{R}} \theta \cdot A_\sigma^\sharp(y' + s\theta, y_3)\mathrm{d}s\right) = \theta \cdot \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}s} A_\sigma^\sharp(y' + s\theta, y_3)\mathrm{d}s = 0, \quad (y', y_3) \in \mathbb{R}^3,$$

we derive from Lemma 3.3 that for all $m \in \mathbb{N}_0$,

$$\langle \xi' \rangle \|\phi_{*,k}\|_{H^m(\mathbb{R}^3)} + \|\theta \cdot \nabla_{x'}\phi_{*,k}\|_{H^m(\mathbb{R}^3)} \leqslant C \langle \xi' \rangle^{m+1} \sigma^{2\mu m + \max(0,(m-2)/3)},$$

where C is a positive constant that is independent of σ. Therefore, we have $N_{0,\theta}(\phi_{*,k}) \leqslant C \langle \xi' \rangle$ and $N_{2,\theta}(\phi_{*,k}) \leqslant C \langle \xi' \rangle^3 \sigma^{4\mu}$, whence

$$\mathcal{N}_{\theta,\sigma}(\phi_{*,k}) \leqslant C \langle \xi' \rangle^3 \sigma^{4\mu + 1/3}. \tag{4.21}$$

Similarly, we find

$$\langle \xi' \rangle \|\partial_{x_j}\phi_{*,k}\|_{H^m(\mathbb{R}^3)} + \|\theta \cdot \nabla_{x'}\partial_{x_j}\phi_{*,k}\|_{H^m(\mathbb{R}^3)} \leqslant C \langle \xi' \rangle^{m+2} \sigma^{2\mu m + \max(0,(m-1)/3)}, \quad j = 1, 2$$

and

$$\langle \xi' \rangle \|\partial_{x_3}\phi_{*,k}\|_{H^m(\mathbb{R}^3)} + \|\theta \cdot \nabla_{x'}\partial_{x_3}\phi_{*,k}\|_{H^m(\mathbb{R}^3)} \leqslant C \langle \xi' \rangle^{m+1} \sigma^{2\mu(m+1) + \max(0,(m-1)/3)}.$$

Thus, we have $N_{0,\theta}(\partial_{x_j}\phi_{*,k}) \leqslant C \langle \xi' \rangle^2 \sigma^{2\mu}$ and $N_{2,\theta}(\partial_{x_j}\phi_{*,k}) \leqslant C \langle \xi' \rangle^4 \sigma^{6\mu + 1/3}$ for $j = 1, 2, 3$ and consequently

$$\mathcal{N}_{\theta,\sigma}(\partial_{x_j}\phi_{*,k}) \leqslant C \langle \xi' \rangle^4 \sigma^{6\mu + 1/3}, \quad j = 1, 2, 3,$$

according to (3.25). From this and (4.21) it then follows that

$$\mathcal{N}_{\theta,\sigma}(\phi_{*,k})\mathcal{N}_{\theta,\sigma}(\partial_{x_j}\phi_{*,k}) \leqslant C \langle \xi' \rangle^7 \sigma^{10\mu + 2/3}, \quad j = 1, 2, 3. \tag{4.22}$$

Having seen this, we turn now to estimating $\widehat{\tilde{\rho}_j}$, where $\tilde{\rho}_j$ is defined by (4.1). As $A_\sigma^\sharp \in W^{\infty,\infty}(\mathbb{R}^3, \mathbb{R})^2$, we infer from (4.19) that $\phi_{*,k} \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, and from

(4.20) that supp $\phi_{*,k} \subset \mathcal{D}_R \times \mathbb{R}$. Thus, by performing the change of variable $y' = x' + t\theta \in \theta^{\perp} \oplus \mathbb{R}\theta$ in the following integral, we deduce from (4.18)–(4.19) that

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}^2} \phi_{*,k}^2(y', y_3) \mathcal{P}(\tilde{\rho}_j)(\theta, y', y_3) \exp\left(-i \int_{\mathbb{R}} \theta \cdot A_\sigma^\sharp(y' + s\theta, y_3) \mathrm{d}s\right) \mathrm{d}y' \, \mathrm{d}y_3 \\
&\qquad = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\theta^\perp} \phi_{*,k}^2(x' + t\theta, y_3) \mathcal{P}(\tilde{\rho}_j)(\theta, x' + t\theta, y_3) \\
&\qquad\quad \times \exp\left(-i \int_{\mathbb{R}} \theta \cdot A_\sigma^\sharp(x' + s\theta, y_3) \mathrm{d}s\right) \mathrm{d}x' \, \mathrm{d}t \, \mathrm{d}y_3 \\
&\qquad = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\theta^\perp} h^2(t + r_{x'_k}) e^{-ix'\cdot\xi'} \varphi_k(x') \alpha_\sigma^2(y_3) \mathcal{P}(\tilde{\rho}_j)(\theta, x', y_3) \mathrm{d}x' \, \mathrm{d}t \, \mathrm{d}y_3 \\
&\qquad = \int_{\mathbb{R}} \int_{\theta^\perp} e^{-iy'\cdot\xi'} \varphi_k(y') \alpha_\sigma^2(y_3) \mathcal{P}(\tilde{\rho}_j)(\theta, y', y_3) \mathrm{d}y' \, \mathrm{d}y_3.
\end{aligned} \tag{4.23}$$

Thus, taking $\mu > 0$ so small that $\kappa := 1/6 - 10\mu > 0$, we deduce from this, (4.4) and (4.22) that

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \int_{\theta^\perp} e^{-iy'\cdot\xi'} \varphi_k(y') \alpha_\sigma^2(y_3) \mathcal{P}(\tilde{\rho}_j)(\theta, y', y_3) \mathrm{d}y' \, \mathrm{d}y_3 \right| \\
&\qquad \leqslant C \langle \xi' \rangle^7 \left(\sigma^6 \| \Lambda_{A_1,q_1} - \Lambda_{A_2,q_2} \| + \sigma^{-\kappa} \right), \quad x_3 \in \mathbb{R}.
\end{aligned} \tag{4.24}$$

Moreover, we see from (4.1) that $\tilde{\rho}_j \in \mathcal{C}^{0,1}(\mathbb{R}^3)$. Since supp $\tilde{\rho}_j \subset B(0,R) \times \mathbb{R}$, by Lemma 3.1, then $x \mapsto \mathcal{P}(\tilde{\rho}_j)(\theta, x) \in \mathcal{C}^{0,1}(\mathbb{R}^3)$ and we deduce from (3.18) upon making the substitution $s = \sigma^{2\mu}(x_3 - y_3)$ in the following integral, that

$$\begin{aligned}
&\Bigg| \int_{\mathbb{R}} \int_{\theta^\perp} e^{-iy'\cdot\xi'} \varphi_k(y') \mathcal{P}(\tilde{\rho}_j)(\theta, y', y_3) \alpha_\sigma^2(y_3) \mathrm{d}y' \, \mathrm{d}y_3 \\
&\qquad - \int_{\theta^\perp} e^{-iy'\cdot\xi'} \varphi_k(y') \mathcal{P}(\tilde{\rho}_j)(\theta, y', x_3) \mathrm{d}y' \Bigg| \\
&\quad = \Bigg| \int_{\mathbb{R}} \int_{\theta^\perp} e^{-iy'\cdot\xi'} \varphi_k(y') \alpha^2(s) \\
&\qquad \times \left(\mathcal{P}(\tilde{\rho}_j)(\theta, y', x_3 - \sigma^{-2\mu} s) - \mathcal{P}(\tilde{\rho}_j)(\theta, y', x_3) \right) \mathrm{d}y' \, \mathrm{d}s \Bigg| \\
&\quad \leqslant \int_{-1}^{1} \int_{\theta^\perp \cap B(0,R+1)} \alpha^2(s) \left| \mathcal{P}(\tilde{\rho}_j)(\theta, y', x_3 - \sigma^{-2\mu} s) - \mathcal{P}(\tilde{\rho}_j)(\theta, y', x_3) \right| \mathrm{d}y' \, \mathrm{d}s \\
&\quad \leqslant C \sigma^{-2\mu}
\end{aligned}$$

for some constant $C > 0$ depending only on ω and M. Here, we used the fact that $\phi_{*,k}$ and α are supported in $B(0, R+1)$ and $(-1, 1)$, respectively. This and (4.24) yield

$$
\begin{aligned}
&\left|\int_{\theta^\perp} e^{-iy'\cdot\xi'}\varphi_k(y')\mathcal{P}(\tilde{\rho}_j)(\theta,y',x_3)\mathrm{d}y'\right| \\
&\qquad \leqslant C\langle\xi'\rangle^7\left(\sigma^6\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|+\sigma^{-2\mu}+\sigma^{-\kappa}\right),
\end{aligned}
\tag{4.25}
$$

for all $x_3\in\mathbb{R}$ and $k=1,\dots,N$. Further, as $A^\sharp$ is supported in $B(0,R)\times\mathbb{R}$ by assumption, it holds true that $\partial_{x_j}A^\sharp(y'+s\theta,x_3)=0$ for all $s\in\mathbb{R}$, all $x_3\in\mathbb{R}$ and all $y'\in\theta^\perp$ such that $|y'|\geqslant R$. Therefore, we have

$$
\mathcal{P}(\tilde{\rho}_j)(\theta,y',x_3)=0,\quad y'\in\theta^\perp\cap(\mathbb{R}^2\setminus B(0,R)),\ x_3\in\mathbb{R},
$$

by virtue of (4.1), and hence

$$
\int_{\theta^\perp} e^{-iy'\cdot\xi'}\mathcal{P}(\tilde{\rho}_j)(\theta,y',x_3)\mathrm{d}y'=\int_{\theta^\perp\cap B(0,R)} e^{-iy'\cdot\xi'}\mathcal{P}(\tilde{\rho}_j)(\theta,y',x_3)\mathrm{d}y',\quad x_3\in\mathbb{R}.
$$

In light of (4.12) and (4.16), this entails that

$$
\widehat{\tilde{\rho}}_j(\xi',x_3)=\frac{1}{2\pi}\sum_{k=1}^N\int_{\theta^\perp\cap B(0,R)} e^{-iy'\cdot\xi'}\varphi_k(y')\mathcal{P}(\tilde{\rho}_j)(\theta,y',x_3)\mathrm{d}y',\quad x_3\in\mathbb{R}.
\tag{4.26}
$$

Taking $\mu\in(0,1/72]$ in such a way that we have $\kappa\geqslant 2\mu$, we infer from (4.25)–(4.26) that

$$
\begin{aligned}
\|\widehat{\tilde{\rho}}_j(\xi',\cdot)\|_{L^\infty(\mathbb{R})} &\leqslant \sum_{k=1}^N\left(\sup_{x_3\in\mathbb{R}}\left|\int_{\theta^\perp} e^{-iy'\cdot\xi'}\varphi_k(y')\mathcal{P}(\tilde{\rho}_j)(\theta,y',x_3)dy'\right|\right) \\
&\leqslant C\langle\xi'\rangle^7\left(\sigma^6\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|+\sigma^{-2\mu}\right),\quad x_3\in\mathbb{R}.
\end{aligned}
\tag{4.27}
$$

The last step of the proof is to notice from (4.1), (4.11) and the identity $\sum_{m=1,2}\theta_m\xi_m=\theta\cdot\xi'=0$ that

$$
\begin{aligned}
\widehat{\tilde{\rho}}_j(\xi',x_3)&=i\sum_{m=1,2}\theta_m\xi_j\widehat{\tilde{a}}_m(\xi',x_3)\\
&=i\sum_{m=1,2}\theta_m\left(\xi_j\widehat{\tilde{a}}_m-\xi_m\widehat{\tilde{a}}_j\right)(\xi',x_3),\quad x_3\in\mathbb{R},\ j=1,2.
\end{aligned}
$$

Thus, assuming that $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$, we get from (4.13) upon choosing $\theta=(\xi_2/|\xi'|,-\xi_1/|\xi'|)$ that

$$
\widehat{\tilde{\rho}}_j(\xi',x_3)=-\frac{\xi_j}{|\xi'|}\widehat{\tilde{\beta}}(\xi',x_3),\quad x_3\in\mathbb{R}.
$$

From this and (4.27), it then follows that

$$\begin{aligned}\|\widehat{\widehat{\beta}}(\xi',\cdot)\|_{L^\infty(\mathbb{R})} &\leqslant \frac{|\xi_1|+|\xi_2|}{|\xi'|}\|\widehat{\widehat{\beta}}(\xi',\cdot)\|_{L^\infty(\mathbb{R})}\\ &\leqslant C\langle\xi'\rangle^7\left(\sigma^6\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|+\sigma^{-2\mu}\right),\end{aligned}$$

which yields (4.14) for $\xi'\neq 0$. Since $\widehat{\widehat{\beta}}(0,x_3)=0$ for every $x_3\in\mathbb{R}$, by (4.13), then (4.14) holds for $\xi'=0$ as well and the proof is complete. □

Armed with Lemma 4.2, we turn now to proving the three main results of this paper.

§5. Proofs of Theorems 1.2, 1.3 and 1.4

Let us start by reducing the analysis of the inverse problem under investigation to the case of transverse magnetic potentials. To do that, we consider $A'=(a_i')_{1\leqslant i\leqslant 3}\in\mathcal{A}$ and put $A:=(a_1,a_2,0)$, where

$$a_i(x',x_3):=a_i'(x',x_3)-\int_{-\infty}^{x_3}\partial_{x_i}a_3'(x',s)\mathrm{d}s,\quad x=(x',x_3)\in\omega\times\mathbb{R},\ i=1,2. \tag{5.1}$$

Since $a_3'\in C^3(\overline{\Omega})$ fulfills (1.9)–(1.10), from the very definition of $\mathcal{A}$, then we have $a_3'\in L^1_{x_3}(\mathbb{R},H_0^3(\omega))$, where $H_0^3(\omega)$ denotes the closure of $\mathcal{C}_0^\infty(\omega)$ in $H^3(\omega)$. Thus $e(x):=\int_{-\infty}^{x_3}a_3'(x',s)\mathrm{d}s$ lies in $W^{3,\infty}(\Omega)\cap L^\infty_{x_3}(\mathbb{R},H_0^3(\omega))$ and we deduce from the identity $A=A'-\nabla e$ arising from (5.1) that

$$\mathrm{d}A'=\mathrm{d}A\quad\text{and}\quad\Lambda_{A_*+A',q}=\Lambda_{A_*+A,q},\quad A_*\in W^{2,\infty}(\Omega)^3,\ q\in W^{1,\infty}(\Omega).$$

Moreover, it is easy to see that A obeys (1.9) in the sense that we have

$$\partial_x^\alpha A(x)=0,\quad x\in\partial\Omega,\ \alpha\in\mathbb{N}_0^3,\ |\alpha|\leqslant 1. \tag{5.2}$$

Therefore, for each $A_*\in W^{2,\infty}(\Omega,\mathbb{R})^3$ and any $A_j\in A_*+\mathcal{A}$, for $j=1,2$, we may assume without loss of generality that the difference A_2-A_1 reads

$$A=(a_1,a_2,0) \tag{5.3}$$

and fulfills (5.2). We shall systematically assume that A verifies (5.2)–(5.3) in the sequel. For further reference, we put $A^\sharp:=(a_1,a_2)$, where a_j, $j=1,2$, are extended by zero outside Ω.

§5.1. Proof of Theorem 1.2

We establish the uniqueness result $(\mathrm{d}A_1,q_1)=(\mathrm{d}A_2,q_2)$ in Section 5.1.1, whereas the proof of the stability estimate (1.13) can be found in Section 5.1.2.

5.1.1. Uniqueness result. For $\xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$, we set $\xi'_\perp := (-|\xi'|^{-1}\xi_2, |\xi'|^{-1}\xi_1)$ and we decompose $A^\sharp$ into the sum $(A^\sharp \cdot \xi')|\xi'|^{-2}\xi' + (A^\sharp \cdot \xi'_\perp)\xi'_\perp$ in such a way that the partial Fourier transform of $\partial_{x_3} A^\sharp$ reads

$$\partial_{x_3}\widehat{A^\sharp}(\xi', x_3) = \left(\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix'\cdot\xi'}\partial_{x_3}A^\sharp(x', x_3)\cdot\xi' \mathrm{d}x'\right)\frac{\xi'}{|\xi'|^2} + i\frac{\partial_{x_3}\widehat{\beta}(\xi', x_3)}{|\xi'|}\xi'_\perp, \quad x_3 \in \mathbb{R}. \tag{5.4}$$

Next, by recalling the hypothesis $\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2}$, we get

$$\beta = \partial_{x_1}a_2 - \partial_{x_2}a_1 = 0 \quad \text{in } \Omega \tag{5.5}$$

upon sending σ to infinity in (4.14). Moreover, we have $\nabla_{x'} \cdot \partial_{x_3}A^\sharp = \nabla \cdot \partial_{x_3}A = 0$ by virtue of (1.11), whence

$$\int_{\mathbb{R}^2} e^{-ix'\cdot\xi'}\partial_{x_3}A^\sharp(x', x_3)\cdot\xi' \mathrm{d}x' = i\int_{\mathbb{R}^2}\nabla_{x'}e^{-ix'\cdot\xi'}\cdot\partial_{x_3}A^\sharp(x', x_3) = -i\int_{\mathbb{R}^2} e^{-ix'\cdot\xi'}\nabla_{x'}\cdot\partial_{x_3}A^\sharp(x', x_3)\mathrm{d}x' = 0. \tag{5.6}$$

Putting this together with (5.4)–(5.5), we find $|\xi'|\partial_{x_3}\widehat{A^\sharp}(\xi', x_3) = 0$ for a.e. $x_3 \in \mathbb{R}$. Since ξ' is arbitrary in $\mathbb{R}^2 \setminus \{0\}$, this entails that $\partial_{x_3}A^\sharp = 0$ and hence that $\partial_{x_3}a_1 = \partial_{x_3}a_2 = 0$ in $\mathbb{R}^2$. From this, (5.5) and the fact that a_3 is uniformly zero, it follows that $\mathrm{d}A_1 = \mathrm{d}A_2$.

Further, taking into account that $\partial_x^\alpha A_1 = \partial_x^\alpha A_2 = \partial_x^\alpha A_*$ on $\partial\Omega$ for every $\alpha \in \mathbb{N}_0^3$ such that $|\alpha| \leqslant 1$, we infer that $A \in W^{2,\infty}(\mathbb{R}^3, \mathbb{R})^3$. This and the identity $\mathrm{d}A = 0$ yield $A = \nabla\Psi$, where the function $\Psi(x) := \int_0^1 x \cdot A(tx)\mathrm{d}t$ lies in $W^{3,\infty}(\mathbb{R}^3, \mathbb{R})$. Moreover, since A vanishes in $\mathbb{R}^3 \setminus \Omega$ we may assume, upon possibly adding a suitable constant, that the same is true for Ψ. Therefore, $\Psi_{|\partial\Omega} = 0$ and we find $\Lambda_{A_2,q_2} = \Lambda_{A_2+\nabla\Psi,q_2} = \Lambda_{A_1,q_2}$ by combining the identity $A_1 = A_2 + \nabla\Psi$ with the gauge invariance property of the DN map. From this and the assumption $\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2}$, it then follows that

$$\Lambda_{A_1,q_2} = \Lambda_{A_1,q_1}. \tag{5.7}$$

It remains to show that the function $q = q_2 - q_1$, duly extended by zero outside Ω, is uniformly zero in $\mathbb{R}^3$. This can be done upon applying the orthogonality identity (3.50) with $A_1 = A_2$, i.e., with $A = 0$ and $V = q$. In light of (5.7), we obtain

$$\langle q u_{2,\sigma}, u_{1,\sigma}\rangle_{L^2(Q)} = 0, \quad \sigma > \sigma_*. \tag{5.8}$$

Here $u_{j,\sigma}$, for $j = 1, 2$, is given by (3.3) and since $A_1 = A_2$, we have $(\overline{b_{1,\sigma}}b_{2,\sigma})(t, x) = 1$ for all $(t, x) \in (0, T) \times \mathbb{R}^3$, by (3.4).

Next we pick $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ with support in $\{x \in \mathbb{R}^3; |x| < 1\}$ and such that $\|\phi\|_{L^2(\mathbb{R}^3)}^2 = 1$, we fix $y \in \mathcal{D}_R(\theta) \times \mathbb{R}$ and we choose $\delta > 0$ so small that $\phi_1(x) = \phi_2(x) := \delta^{-3/2}\phi(\delta^{-1}(x-y))$ is supported in $\mathcal{D}_R \times \mathbb{R}$. Thus, upon multiplying (5.8) by σ and sending σ to infinity, we find with the aid of (3.19) and (3.27) that

$$\int_0^{+\infty} \left(\int_{\mathbb{R}^3} q(\delta x' + y' + s\theta, \delta x_3 + y_3) |\phi(x', x_3)|^2 \mathrm{d}x' \, \mathrm{d}x_3 \right) \mathrm{d}s = 0, \quad \delta > 0. \tag{5.9}$$

Actually, if $y' \in \mathcal{D}_R^-(\theta)$ then we have $|y' + s\theta| > R$ for any $s \leqslant 0$ and hence $q(\delta x' + y' + s\theta, \delta x_3 + y_3) = 0$ uniformly in $|x| < 1$, provided $\delta \in (0,1)$. This and (5.9) yield

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} q(\delta x' + y' + s\theta, \delta x_3 + y_3) |\phi(x', x_3)|^2 \mathrm{d}x' \, \mathrm{d}x_3 \right) \mathrm{d}s = 0, \tag{5.10}$$
$$\delta \in (0,1),\ (y', y_3) \in \mathcal{D}_R^-(\theta) \times \mathbb{R}.$$

By performing the change of variable $t = -s$ in the above integral and then substituting $(-\theta)$ for θ in the resulting identity, we get

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} q(\delta x' + y' + s\theta, \delta x_3 + y_3) |\phi(x', x_3)|^2 \mathrm{d}x' \, \mathrm{d}x_3 \right) \mathrm{d}s = 0,$$
$$\delta \in (0,1),\ (y', y_3) \in \mathcal{D}_R^-(-\theta) \times \mathbb{R}.$$

This and (5.10) yield

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} q(\delta x' + y' + s\theta, \delta x_3 + y_3) |\phi(x', x_3)|^2 \mathrm{d}x' \, \mathrm{d}x_3 \right) \mathrm{d}s = 0,$$
$$\delta \in (0,1),\ (y', y_3) \in \mathcal{D}_R \times \mathbb{R}.$$

Next, sending δ to zero in the above identity and taking into account that ϕ is normalized in $L^2(\mathbb{R}^3)$, we obtain for each $\theta \in \mathbb{S}^1$ that

$$\mathcal{P}(q)(\theta, y', y_3) = \int_{\mathbb{R}} q(y' + s\theta, y_3) \mathrm{d}s = 0, \quad (y', y_3) \in \mathcal{D}_R \times \mathbb{R}.$$

This entails $q = 0$ since the partial X-ray transform is injective.

5.1.2. Proof of the stability estimate (1.13). We have

$$|\xi'| \left| \partial_{x_3} \widehat{A^\sharp}(\xi', x_3) \right| = \left| \partial_{x_3} \widehat{\beta}(\xi', x_3) \right|, \quad \xi' \in \mathbb{R}^2,\ x_3 \in \mathbb{R},$$

by (5.4) and (5.6), so we infer from (4.14)–(4.15) for all $\sigma > \sigma_*$ that

$$|\widehat{\beta}(\xi', x_3)| + |\xi'| |\partial_{x_3} \widehat{A^\sharp}(\xi', x_3)| \leqslant C \langle \xi' \rangle^8 \left(\sigma^6 \|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\| + \sigma^{-\epsilon} \right), \tag{5.11}$$
$$\xi' \in \mathbb{R}^2, \quad x_3 \in \mathbb{R},$$

the constants C and ϵ being the same as in Lemma 4.2.

Fix $\rho \in (1, +\infty)$ and put $\mathcal{C}_\rho := \{\xi' \in \mathbb{R}^2;\ \rho^{-1} \leqslant |\xi'| \leqslant \rho\}$. Then, applying the Plancherel theorem, we obtain

$$\begin{aligned}\|\partial_{x_3} a_j(\cdot, x_3)\|^2_{L^2(\omega)} \leqslant \|\partial_{x_3}\widehat{a_j}(\xi', x_3)\|^2_{L^2(B(0,\rho^{-1}))} \\ + \rho^{-2}\int_{\mathbb{R}^2\setminus B(0,\rho)} \langle\xi'\rangle^2 |\partial_{x_3}\widehat{a_j}(\xi', x_3)|^2 \mathrm{d}\xi' \\ + \|\partial_{x_3}\widehat{a_j}(\xi', x_3)\|^2_{L^2(\mathcal{C}_\rho)}, \quad x_3 \in \mathbb{R},\ j = 1, 2.\end{aligned} \tag{5.12}$$

Further, since we have $\|\partial_{x_3}\widehat{a_j}(\xi', x_3)\|^2_{L^2(B(0,\rho^{-1}))} \leqslant |\omega|\rho^{-2}\|a_j\|^2_{W^{1,\infty}(\Omega)}$ and $\int_{\mathbb{R}^2\setminus B(0,\rho)} \langle\xi'\rangle^2 |\partial_{x_3}\widehat{a_j}(\xi', x_3)|^2 \mathrm{d}\xi' \leqslant \|a_j\|^2_{W^{1,\infty}(\Omega)}$, there exists a constant $C > 0$, depending only on M and ω, such that we have

$$\|\partial_{x_3}\widehat{a_j}(\xi', x_3)\|^2_{L^2(B(0,\rho^{-1}))} + \rho^{-2}\int_{\mathbb{R}^2\setminus B(0,\rho)} \langle\xi'\rangle^2 |\partial_{x_3}\widehat{a_j}(\xi', x_3)|^2 \mathrm{d}\xi' \leqslant \frac{M}{\rho^2}, \tag{5.13}$$

according to (1.12). On the other hand, we derive from (5.11) that

$$\begin{aligned}\|\partial_{x_3}\widehat{a_j}(\xi', x_3)|^2 \leqslant C\rho^{14}(\sigma^{12}\delta^2 + \sigma^{-2\epsilon}), \\ \xi' \in \mathcal{C}_\rho \cap B(0,\rho),\quad x_3 \in \mathbb{R},\ \sigma > \sigma_*,\ j = 1, 2,\end{aligned}$$

where $\delta := \|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\|$. Putting this and (5.12)–(5.13) together, we get for every $\sigma > \sigma_*$ that

$$\|\partial_{x_3} a_j\|^2_{L^\infty_{x_3}(\mathbb{R}, L^2(\omega))} \leqslant C\left(\rho^{16}\sigma^{12}\delta^2 + \rho^{16}\sigma^{-2\epsilon} + \rho^{-2}\right), \quad x_3 \in \mathbb{R},\ j = 1, 2. \tag{5.14}$$

Now, choosing ρ so large that $\rho > \sigma_*^{\epsilon/8}$, we get upon taking $\sigma = \rho^{8/\epsilon} > \sigma_*$ in (5.14) that

$$\|\partial_{x_3} a_j(., x_3)\|^2_{L^2(\omega)} \leqslant C\left(\rho^{M_\epsilon}\delta^2 + \rho^{-2}\right), \quad x_3 \in \mathbb{R},\ j = 1, 2, \tag{5.15}$$

with $M_\epsilon := 16 + 96/\epsilon$. Thus, if $\delta < \delta_0 := \sigma_*^{-\epsilon(M_\epsilon+2)/16}$ we have $\delta^{-2/(M_\epsilon+2)} > \sigma_*^{\epsilon/8}$ and we may apply (5.15) with $\rho = \delta^{-2/(M_\epsilon+2)}$. This leads to

$$\|\partial_{x_3} a_j\|^2_{L^\infty_{x_3}(\mathbb{R}, L^2(\omega))} \leqslant 2C\delta^{2\mu_0}, \quad \text{with } \mu_0 := \frac{2}{M_\epsilon + 2} \in (0, 1),\ j = 1, 2. \tag{5.16}$$

Then, from this and the fact, arising from (1.12), that $\|\partial_{x_3} a_j\|_{L^\infty_{x_3}(\mathbb{R}, L^2(\omega))} \leqslant (2M\delta_0^{-2\mu_0})\delta^{2\mu 0*}$ for all $\delta \geqslant \delta_0$, it follows that (5.16) remains valid for every $\delta > 0$.

Finally, arguing as before with β instead of $\partial_{x_3} A^\sharp$, we obtain in a similar way to (5.11) that $\|\beta\|_{L^\infty_{x_3}(\mathbb{R}, L^2(\omega))}$ is upper bounded, up to some multiplicative constant depending only on M and ω, by δ^{μ_0}. Finally (1.13) follows from this and (5.16).

§5.2. Proof of Theorem 1.3

The proof is an adaptation of the one of Theorem 1.2, where the adaptation is to take into account the extra information given by (1.14). Actually, since $A_j = (a_{1,j}, a_{2,j}, a_{3,*})$ and $A = (A^\sharp, 0)$ with $A^\sharp = (a_1, a_2)$, by (5.3), then (1.14) yields

$$\|\partial_{x_1} a_2 - \partial_{x_2} a_1\|_{L^\infty_{x_3}(\mathbb{R}, L^2(\omega))} = \|\partial_{x_1} a_2 - \partial_{x_2} a_1\|_{L^\infty_{x_3}(-r,r;L^2(\omega))} \tag{5.17}$$

and

$$\|\partial_{x_3} a_j\|_{L^\infty_{x_3}(\mathbb{R}, L^2(\omega))} = \|\partial_{x_3} a_j\|_{L^\infty_{x_3}(-r,r;L^2(\omega))}, \quad j = 1, 2. \tag{5.18}$$

More precisely, we still consider GO solutions $u_{1,\sigma}$ and $u_{2,\sigma}$, defined by (3.3)–(3.4) and (3.22), with $\phi_1 = \partial_{x_j}\phi$, for $j = 1, 2, 3$, and $\phi_2 = \phi$, where ϕ is given by (4.18)–(4.19). The parameter x_3 appearing in (4.18) is taken in $(-r, r)$ and we impose $\sigma > (r' - r)^{-24}$ in such a way that $\phi \in \mathcal{C}_0^\infty(\mathcal{D}_R^-(\theta) \times (-r', r'))$. Moreover, the functions

$$\begin{aligned} f_\sigma(t, x) &= \Phi_2(2\sigma t, x) b_2(2\sigma t, x) e^{i\sigma(x.\theta - \sigma t)} \\ \text{and} \qquad g_\sigma &= \Phi_1(2\sigma t, x) b_1(2\sigma t, x) e^{i\sigma(x.\theta - \sigma t)}, \quad (t, x) \in \Sigma \end{aligned}$$

lie in $H_0^{2,1}((0, T) \times \Gamma_{r'})$ and we infer from (3.50) upon arguing as for the derivation of Lemma 4.2 that

$$\|\widehat{\beta}(\xi', \cdot)\|_{L^\infty(-r',r')} \leqslant C\langle\xi'\rangle^7 \left(\sigma^6 \|\Lambda_{A_1,q_1,r'} - \Lambda_{A_2,q_2,r'}\| + \sigma^{-\epsilon}\right)$$

and that

$$\|\partial_{x_3}\widehat{\beta}(\xi', \cdot)\|_{L^\infty(r',r')} \leqslant C\langle\xi'\rangle^8 \left(\sigma^6 \|\Lambda_{A_1,q_1,r'} - \Lambda_{A_2,q_2,r'}\| + \sigma^{-\epsilon}\right)$$

for all $\xi' \in \mathbb{R}^2$ and some $\epsilon > 0$. Here, the constant C depends only on ω, T, M, r, r' and ϵ. The desired result follows from this and (5.17)–(5.18) by arguing in the same way as in the proof of Theorem 1.2.

§5.3. Proof of Theorem 1.4

We prove only (1.16), the derivation of (1.19) being similar to that of (1.15). To this end, we fix $\xi' \in \mathbb{R}^2$, recall that $A = (A^\sharp, 0) \in \mathcal{A}_0$, where $A^\sharp = (a_1, a_2)$ satisfies $\partial_{x_1} a_1 + \partial_{x_2} a_2 = 0$ in $\mathbb{R}^2$, and get

$$\begin{aligned} \widehat{A^\sharp}(\xi', x_3) \cdot \xi' &= i(2\pi)^{-1} \int_{\mathbb{R}^2} A^\sharp(x', x_3) \cdot \nabla_{x'} e^{-ix'\cdot\xi'} \mathrm{d}x' \\ &= -i(2\pi)^{-1} \int_{\mathbb{R}^2} e^{-ix'\cdot\xi'} \left(\partial_{x_1} a_1 + \partial_{x_2} a_2\right)(x', x_3) \mathrm{d}x' = 0, \quad x_3 \in \mathbb{R}, \end{aligned}$$

upon integrating by parts. Hence, remembering that $\xi'_\perp = (-|\xi'|^{-1}\xi_2, |\xi'|^{-1}\xi_1)$ whenever $\xi' \neq 0$, we obtain

$$\widehat{A^\sharp}(\xi', x_3) = (\widehat{A^\sharp}(\xi', x_3) \cdot \xi'_\perp)\xi'_\perp = (-\xi_2\widehat{a_1} + \xi_1\widehat{a_2})(\xi', x_3)\frac{\xi'_\perp}{|\xi'|}, \quad x_3 \in \mathbb{R},$$

and consequently

$$|\xi'|\widehat{A^\sharp}(\xi', x_3) = -i\widehat{\beta}(\xi', x_3), \quad x_3 \in \mathbb{R}, \tag{5.19}$$

by (4.13), the above identity being still valid for $\xi' = 0$. Therefore, arguing as in the derivation of (1.13) from (4.14), we infer from (4.15) and (5.19) that

$$\|A\|^3_{L^\infty_{x_3}(\mathbb{R}, L^2(\omega))} \leqslant C\|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\|^{\mu_1}, \tag{5.20}$$

where $C > 0$ and $\mu_1 \in (0, 1)$ are two constants depending only on T, ω and M.

We turn now to estimating $\|q\|_{L^\infty_{x_3}(\mathbb{R}, H^{-1}(\omega))^3}$, where $q = q_1 - q_2$. With reference to (4.16)–(4.17) we fix $x_3 \in \mathbb{R}$ and $k \in \{1, \dots, N\}$, pick a function $\phi_{*,k}$ expressed by (4.18)–(4.19) in the particular case where $A^\sharp_\sigma$ is uniformly zero, i.e.,

$$\begin{aligned}\phi_{*,k}(y) := h\left(y' \cdot \theta + r_{x'_k}\right) e^{-\frac{i}{2}y'\cdot\xi'} \varphi_k^{1/2}(y' - (y' \cdot \theta)\theta)\alpha_\sigma(y_3),& \\ y' \in \mathbb{R}^2,\ y_3 \in \mathbb{R},&\end{aligned} \tag{5.21}$$

and, in view of Proposition 3.4, we consider a GO solution $u_{j,\sigma}$, $j = 1, 2$ to the magnetic Schrödinger equation $(i\partial_t + \Delta_{A_j} + q_j)u_{j,\sigma} = 0$ in Q, given by (3.3) with

$$\begin{aligned}\Phi_1 = \overline{\Phi_{*,k}}, \quad \Phi_2 = \Phi_{*,k} \quad \text{and} \quad \Phi_{*,k}(t, x) := \phi_{*,k}(x' - t\theta, x_3),& \\ t \in \mathbb{R},\ x' \in \mathbb{R}^2,\ x_3 \in \mathbb{R}.&\end{aligned} \tag{5.22}$$

Bearing in mind that $\nabla \cdot A = 0$, we apply (3.50) with $V = q - A \cdot (A_1 + A_2)$. We obtain

$$\begin{aligned}\langle qu_{2,\sigma}, u_{1,\sigma}\rangle_{L^2(Q)} &= \langle A \cdot ((A_1 + A_2)u_{2,\sigma} - 2i\nabla u_{2,\sigma}), u_{1,\sigma}\rangle_{L^2(Q)} \\ &\quad + \langle(\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2})f_\sigma, g_\sigma\rangle_{L^2(\Sigma)},\end{aligned}$$

where f_σ and g_σ are given by (3.47) and (3.51), respectively. Thus, we have

$$\begin{aligned}\left|\langle qu_{2,\sigma}, u_{1,\sigma}\rangle_{L^2(Q)}\right| &\leqslant C\|A\|_{L^\infty(\Omega)^3}\|u_{1,\sigma}\|_{L^2(Q)}\|u_{2,\sigma}\|_{L^2(0,T;H^1(\Omega))} \\ &\quad + \left|\langle(\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2})f_\sigma, g_\sigma\rangle_{L^2(\Sigma)}\right|\end{aligned}$$

and hence

$$\left|\langle qu_{2,\sigma}, u_{1,\sigma}\rangle_{L^2(Q)}\right| \leqslant C\sigma^{8\mu}\left(\|A\|_{L^\infty(\Omega)^3} + \sigma^{17/3}\|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\|\right)\langle\xi'\rangle^6, \tag{5.23}$$

from (3.38), (3.55) and (4.21).

On the other hand, it follows readily from (3.3) and (5.22) that

$$\langle qu_{2,\sigma}, u_{1,\sigma}\rangle_{L^2(Q)} = \int_Q q(y)\Phi_{*,k}^2(2\sigma t, y)(\overline{b_{1,\sigma}}b_{2,\sigma})(2\sigma t, y)\mathrm{d}y\,\mathrm{d}t + R_{k,\sigma}, \tag{5.24}$$

where $b_{j,\sigma}$, $j = 1, 2$, is given by (3.4) and

$$\begin{aligned} R_{k,\sigma} := &\int_Q q(y)\Phi_{*,k}(2\sigma t, y)\Big(b_{2,\sigma}(2\sigma t, y)e^{i\sigma(y'\cdot\theta-\sigma t)}\overline{\psi_{1,\sigma}}(t, y) \\ &+ \psi_{2,\sigma}(t, y)\overline{b_{1,\sigma}}(2\sigma t, y)e^{-i\sigma(y'\cdot\theta-\sigma t)}\Big)\mathrm{d}y\,\mathrm{d}t \\ &+ \int_Q q(y)(\psi_{2,\sigma}\overline{\psi_{1,\sigma}})(t, y)\mathrm{d}y\,\mathrm{d}t. \end{aligned}$$

Therefore, we have

$$\begin{aligned} |R_{k,\sigma}| &\leqslant \|q\|_{L^\infty(\Omega)}\Big(\|\Phi_{*,k}(2\sigma\cdot,\cdot)\|_{L^2(Q)}\left(\|\psi_{1,\sigma}\|_{L^2(Q)} + \|\psi_{2,\sigma}\|_{L^2(Q)}\right) \\ &\quad + \|\psi_{1,\sigma}\|_{L^2(Q)}\|\psi_{2,\sigma}\|_{L^2(Q)}\Big) \\ &\leqslant C\sigma^{-11/6}\mathcal{N}_{\theta,\sigma}(\phi_{*,k})^2, \end{aligned}$$

according to (3.25)–(3.27) and (3.39), and consequently

$$|R_{k,\sigma}| \leqslant C\sigma^{8\mu-7/6}\left\langle \xi'\right\rangle^6, \tag{5.25}$$

by (4.21). We turn now to examining the first term in the right-hand side of (5.24). In light of (3.4), we have

$$\int_Q q(y)\Phi_{*,k}^2(2\sigma t, y)(\overline{b_{1,\sigma}}b_{2,\sigma})(2\sigma t, y)\mathrm{d}y\,\mathrm{d}t = \int_Q q(y)\Phi_{*,k}^2(2\sigma t, y)\mathrm{d}y\,\mathrm{d}t + r_{k,\sigma}, \tag{5.26}$$

with

$$r_{k,\sigma} := \int_Q q(y)\Phi_{*,k}^2(2\sigma t, y)\left(e^{-i\int_0^{2\sigma t}\theta\cdot A_\sigma^\sharp(y'-s\theta,y_3)\mathrm{d}s} - 1\right)\mathrm{d}y\,\mathrm{d}t. \tag{5.27}$$

Now, $e^{-i\int_0^{2\sigma t}\theta\cdot A_\sigma^\sharp(y'-s\theta,y_3)\mathrm{d}s} - 1 = -i\int_0^{2\sigma t}\theta\cdot A_\sigma^\sharp(y'-\tau\theta, y_3)e^{-i\int_0^\tau\theta\cdot A_\sigma^\sharp(y'-s\theta,y_3)\mathrm{d}s}\mathrm{d}\tau$, and so we have

$$\left|e^{-i\int_0^{2\sigma t}\theta\cdot A_\sigma^\sharp(y'-s\theta,y_3)\mathrm{d}s} - 1\right| \leqslant 2\sigma T\|A_\sigma^\sharp\|_{L^\infty(\mathbb{R}^3)^2} \leqslant C\sigma\|A\|_{L^\infty(\Omega)^3}, \quad (t, y) \in Q.$$

Here we used the fact, arising from (3.9) and (3.15)–(3.16), that for any $\sigma > 0$, $\|A_\sigma^\sharp\|_{L^\infty(\mathbb{R}^3)^2}$ is majorized, up to some multiplicative constant that is independent of σ, by $\|A^\sharp\|_{L^\infty(\Omega)^2}$. Therefore, we infer from (1.12), (3.39) and (4.21) that

$$\begin{aligned} |r_{k,\sigma}| &\leqslant C\sigma\|A\|_{L^\infty(\mathbb{R}^3)^3}\|\Phi_{*,k}(2\sigma\cdot,\cdot)\|_{L^2(Q)}^2 \\ &\leqslant C\|A\|_{L^\infty(\mathbb{R}^3)^3}\|\phi_{*,k}\|_{L^2(\mathbb{R}^3)}^2 \leqslant C\|A\|_{L^\infty(\mathbb{R}^3)^3}\langle\xi'\rangle^6\sigma^{8\mu}. \end{aligned} \tag{5.28}$$

We are left with the task of examining the integral

$$\begin{aligned}\int_Q q(y)\Phi_{*,k}^2(2\sigma t,y)\mathrm{d}y\,\mathrm{d}t &= \int_0^T \int_{\mathbb{R}^3} q(y)\phi_{*,k}^2(y'-2\sigma t\theta,y_3)\mathrm{d}y'\,\mathrm{d}y_3\,\mathrm{d}t\\ &= \frac{1}{2\sigma}\int_0^{2\sigma T}\int_{\mathbb{R}^3} q(y'+s\theta,y_3)\phi_{*,k}^2(y)\mathrm{d}y'\,\mathrm{d}y_3\,\mathrm{d}s,\end{aligned} \tag{5.29}$$

appearing in the right-hand side of (5.26). To do that, we notice for all $\sigma > \sigma_*$ that

$$q(y'+s\theta,y_3)\phi_{*,k}^2(y)=0,\quad s\in(-\infty,0)\cup(2\sigma T,+\infty),\ y'\in\mathbb{R}^2,\ y_3\in\mathbb{R},$$

since q and $\phi_{*,k}$ are supported in $B(0,R)\times\mathbb{R}$ and $\mathcal{D}_R^-(\theta)\times\mathbb{R}$, respectively, and that $|y'+s\theta|>R$ whenever $y'\in\mathcal{D}_R^-(\theta)$ and $s\in(-\infty,0)\cup(2\sigma T,+\infty)$. In view of (4.3) and (5.29), this entails that

$$\begin{aligned}\int_Q q(y)\Phi_{*,k}^2(2\sigma t,y)\mathrm{d}y\,\mathrm{d}t &= \frac{1}{2\sigma}\int_{\mathbb{R}}\int_{\mathbb{R}^3} q(y'+s\theta,y_3)\phi_{*,k}^2(y)\mathrm{d}y'\,\mathrm{d}y_3\,\mathrm{d}s\\ &= \frac{1}{2\sigma}\int_{\mathbb{R}^3}\mathcal{P}(q)(\theta,y',y_3)\phi_{*,k}^2(y)\mathrm{d}y'\,\mathrm{d}y_3.\end{aligned}$$

Thus, arguing in the same way as in the derivation of (4.23), we infer from (5.21) that

$$|\widehat{q}(\xi',x_3)|\leqslant C\langle\xi'\rangle^6\Big(\sigma^{20/3}\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|+\sigma^{8\mu+1}\|A\|_{L^\infty(\Omega)^3}+\sigma^{8\mu-1/6}\Big),\quad \sigma>\sigma_*. \tag{5.30}$$

The next step of the proof is to upper bound $\|A\|_{L^\infty(\Omega)^3}$ in terms of $\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|$. To do that, we pick $p>2$ and apply Sobolev's embedding theorem (see, e.g., [14, Cor. IX.14]), getting $\|A(\cdot,x_3)\|_{L^\infty(\omega)^3}\leqslant C\|A(\cdot,x_3)\|_{W^{1,p}(\omega)^3}$ for a.e. $x_3\in\mathbb{R}$, the constant $C>0$ depending only on ω. By interpolating, we obtain

$$\|A(\cdot,x_3)\|_{L^\infty(\omega)^3}\leqslant C\|A(\cdot,x_3)\|_{W^{2,p}(\omega)^3}^{1/2}\|A(\cdot,x_3)\|_{L^p(\omega)^3}^{1/2},\quad x_3\in\mathbb{R}.$$

This and (5.20) yield

$$\|A\|_{L^\infty(\Omega)^3}\leqslant C\|A\|_{L^\infty_{x_3}(\mathbb{R},L^p(\omega)^3)}^{1/2}\leqslant C\|A\|_{L^\infty_{x_3}(\mathbb{R},L^2(\omega)^3)}^{1/p}\leqslant C\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|^{\mu_1/p},$$

for some constant $C>0$ that depends only on ω, M and T. Next, by substituting the right-hand side of the above estimate for $\|A\|_{L^\infty(\Omega)^3}$ in (5.30), we get

$$\begin{aligned}|\widehat{q}(\xi',x_3)|\leqslant C\,\langle\xi'\rangle^6\Big(&\sigma^{20/3}\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|+\sigma^{8\mu+1}\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|^{\mu_1/p}\\ &+\sigma^{8\mu-1/6}\Big),\quad \sigma>\sigma_*.\end{aligned} \tag{5.31}$$

With the notation of Section 5.1.2, we infer from (5.31) and the estimate

$$\int_{\mathbb{R}^2\setminus B(0,\rho)} \langle\xi'\rangle^{-2}|\widehat{q}(\xi',x_3)|^2\mathrm{d}\xi' \leqslant \frac{M}{\rho^2}, \quad x_3\in\mathbb{R},\ \rho\in(1,+\infty)$$

that

$$\|q\|_{L^\infty_{x_3}(\mathbb{R},H^{-1}(\omega))} \leqslant C\left(\rho^6\sigma^{20/3}\delta^{\mu_1/p}+\sigma^{8\mu-1/6}+\rho^{-1}\right), \quad \sigma>\sigma_*, \tag{5.32}$$

where $\delta = \|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\| \in (0,1)$. Thus, for $\mu\in(0,1/48)$ and $\delta\in\left(0,\sigma_*^{-(41-48\mu)p/(6\mu_1)}\right)$, we obtain (1.16) with $\mu_2 := (1-48\mu)\mu_1/(7p(41-48\mu))$ by taking $\rho=\delta^{-\mu_2}$ and $\sigma=\delta^{-42\mu_2/(1-48\mu)}$ in (5.32).

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