

A Microlocal Characterization of Lipschitz Continuity

by

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Abstract

We study continuous maps between differential manifolds from a microlocal point of view. In particular, we characterize the Lipschitz continuity of these maps in terms of the microsupport of the constant sheaf on their graph. Furthermore, we give lower and upper bounds on the microsupport of the graph of a continuous map and use these bounds to characterize strict differentiability in microlocal terms.

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§1. Introduction

Microlocal analysis is the study of phenomena occurring on differential manifolds via a study in their cotangent bundle, such as the study of the singularities of solutions of a partial differential equation on a manifold M via the study of their wavefront set in T^*M . A general setting for microlocal analysis is the microlocal theory of sheaves, developed by Kashiwara and Schapira (see [6]). In [7], Vichery used this theory to study, from a microlocal viewpoint, continuous real-valued functions on differential manifolds, and to define for these functions a good notion of subdifferential. We extend this study to continuous maps between differential manifolds. We study simultaneously the tangent aspects of the subject to emphasize the parallelism between the tangent and cotangent sides.

Specifically, let $f: M \rightarrow N$ be a continuous map between differential manifolds. We denote its graph by $\Gamma_f \subseteq M \times N$. We define its *Whitney cone* C_f as the Whitney cone of its graph and its *conormal* Λ_f as the microsupport of the

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constant sheaf on its graph, that is,

$$(1.1) \quad C_f := C(\Gamma_f, \Gamma_f) \subseteq T(M \times N)$$

$$(1.2) \quad \text{and } \Lambda_f := \mu\text{supp}(\mathbb{k}_{\Gamma_f}) \subseteq T^*(M \times N),$$

where \mathbb{k} is any nonzero commutative ring of finite global dimension (for instance \mathbb{Z} or a field). All these terms and pieces of notation are precisely defined in the article.

The Whitney cone C_f is a closed symmetric cone and the conormal Λ_f is a coisotropic closed symmetric cone. If f is C^1 , then its Whitney cone is equal to the tangent bundle of its graph and its conormal is equal to the conormal bundle of its graph, that is, $C_f = T\Gamma_f$ and $\Lambda_f = (T\Gamma_f)^\perp$.

We prove that f is Lipschitz if and only if its Whitney cone contains no nonzero “vertical” vectors, that is, $C_f \cap (0_M \times TN) \subseteq 0_{MN}$, if and only if its conormal contains no nonzero “horizontal” covectors, that is, $\Lambda_f \cap (T^*M \times 0_N^*) \subseteq 0_{MN}^*$.

To prove these results, we use the microlocal theory of sheaves of Kashiwara and Schapira, which we review in Section 2.

In Section 3, we review the main properties of microsupports associated with subsets. If A is a locally closed subset of M , we set $\mu\text{supp}(A) := \mu\text{supp}(\mathbb{k}_A)$ and denote its tangent cone by $C(A)$ and its strict tangent cone by $N(A)$. If C is a cone in TM , we denote its polar by C° . We give a direct proof of the following known bounds: if $Z \subseteq M$ is closed, then $\pi_M^{-1}(Z) \cap C(Z)^\circ \subseteq \mu\text{supp}(Z) \subseteq N(Z)^\circ$.

In Section 4, we define the Whitney cone of a continuous map and give its first properties. In particular, we characterize Lipschitz continuity and strict differentiability in terms of the Whitney cone and extend these characterizations to topological submanifolds. We also prove the following chain rule, which will be needed later. Let $f_i: M_i \rightarrow M_{i+1}$ be continuous maps between differential manifolds for $i \in \{1, 2\}$. If $(0_1 \circ C_{f_1}) \cap (C_{f_2} \circ 0_3) \subseteq 0_2$, for instance if f_1 is Lipschitz, then

$$(1.3) \quad C_{f_2 \circ f_1} \subseteq C_{f_1} \circ C_{f_2}$$

with equality if f_2 is C^1 .

In Section 5, we define the conormal of a continuous map and give its first properties. We use the convolution of kernels to extend to continuous maps the functorial properties of the microsupport for the four image operations. We also use it to prove the following chain rule. With the notation above, if $(0_1^* \circ \Lambda_{f_1}) \cap (\Lambda_{f_2} \circ 0_3^*) \subseteq 0_2^*$, for instance if f_2 is Lipschitz, then

$$(1.4) \quad \Lambda_{f_2 \circ f_1} \subseteq \Lambda_{f_1} \overset{a}{\circ} \Lambda_{f_2}$$

with equality if f_1 is a C^1 -submersion (and if f_1 and f_2 are both C^1).

In Section 6, we study the case of real-valued functions. We define directional Dini derivatives in order to describe more precisely the various cones associated with a function. We also study local extrema and more generally “extrema at first order”, which we define in that section. Namely, if $x \in M$, we set $C_x(f) := (C_f)_{(x, f(x))}$ and $\Lambda_x(f) := (\Lambda_f)_{(x, f(x))}$. Then we prove the following generalization of Fermat’s lemma to continuous functions: if $f: M \rightarrow \mathbb{R}$ is continuous and has a first-order extremum at $x \in M$, then

$$(1.5) \quad T_x M \times \{0\} \subseteq C_x(f)$$

$$(1.6) \quad \text{and } \{0\} \times \mathbb{R} \subseteq \Lambda_x(f).$$

Finally, we relate the conormal of a function to the microsupport of the constant sheaf on its epigraph, studied by Vichery in [7], and we prove that the two points of view are equivalent for Lipschitz functions.

Section 7 is the main section of the paper, where the claimed characterizations of Lipschitz continuity and strict differentiability are proved. First, we prove analogues of Rolle’s lemma and the mean value theorem for continuous maps between vector spaces. This allows us to give the following upper bound on the Whitney cone of a continuous map in terms of its conormal. To state it, we define the following analogues of the directional derivatives. If $(x, u) \in TM$ and $(x, \eta) \in M \times_N T^*N$, we set

$$(1.7) \quad C_x^u(f) := C_x(f) \cap (\mathbb{R}_{\geq 0}u \times T_{f(x)}N)$$

$$(1.8) \quad \text{and } \Lambda_x^\eta(f) := \Lambda_x(f) \cap (T_x^*M \times \mathbb{R}_{\geq 0}\eta).$$

For a subset A of a vector space, we set $\dot{A} := A \setminus \{0\}$ and $A^\top := \bigcup_{v \in \dot{A}} v^\perp$. Then,

$$(1.9) \quad C_x(f) \subseteq \bigcap_{\eta \in \dot{T}_{f(x)}^*M} \Lambda_x^\eta(f)^\top$$

with equality if $\dim N = 1$, in which case $C_f = \Lambda_f^\top$. We use this bound to prove the microlocal characterization of Lipschitz continuity. This allows us to prove the following upper bound:

$$(1.10) \quad \Lambda_x(f) \subseteq \bigcap_{u \in \dot{T}_x M} C_x^u(f)^\top$$

with equality if $\dim M = 1$, in which case $\Lambda_f = C_f^\top$. This in turn allows us to characterize strict differentiability in terms of the Whitney cone and of the conormal. We give applications of these results to the theory of causal manifolds.

In Section 8, we generalize some of these results to topological submanifolds, and we give conditions in terms of the Whitney cone and of the conormal in order that such submanifolds be locally graphs of Lipschitz or strictly differentiable maps. For instance, if M is a closed topological submanifold of a differential manifold, then

$$(1.11) \quad \mu\text{supp}(M) \subseteq C(M, M)^\top$$

with equality if $\dim M = 1$.

Some results (but not all) also hold if one replaces “Lipschitz” (resp. “strictly differentiable”, Lip , C_f , Λ_f) with “pointwise Lipschitz” (resp. “differentiable”, Lip^{pw} , $C(\Gamma_f)$, $\mathbb{C}\text{Ppg}(f)$), but we do not state them.

Erratum. As explained in Remark 7.15, Proposition 1.12 of [4] is misstated. We give in that remark the correct statement and explain why this has no consequences for the rest of [4].

§2. Background material

§2.1. Notation and conventions

Unless otherwise specified,

- the symbol \mathbb{k} denotes a nonzero commutative ring of finite global dimension (for instance \mathbb{Z} or a field);
- vector spaces and manifolds are real and finite-dimensional;
- manifolds are paracompact Hausdorff;
- manifolds, morphisms of manifolds, and submanifolds are smooth, that is, of class C^∞ , and submanifolds are embedded (hence locally closed);
- topological (sub)manifolds are called C^0 -(sub)manifolds, and C^0 -submanifolds are locally flatly embedded (that is, their inclusion is locally C^0 -isomorphic to a linear inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$), hence locally closed.

We use the terms “function” and “map” interchangeably.

Sets. Given some sets X_i , we set for short $X_{ij} := X_i \times X_j$ and similarly for X_{ijk} , and we write p_i (resp. p_{ij}) for any projection from a product of the X_j ’s (which will be clear from the context) to X_i (resp. to X_{ij}). For a product of the form $X \times Y$, we also denote the projections by $p_X: X \times Y \rightarrow X$ and $p_Y: X \times Y \rightarrow Y$, and use the same notation for a pullback $X \times_Z Y$. The diagonal map of X is denoted by $\delta_X: X \rightarrow X \times X$, and the diagonal of X by $\Delta_X := \delta_X(X)$, or simply by δ and Δ if there is no risk of confusion.

If $R_i \subseteq X_i \times X_{i+1}$ for $i \in \{1, 2\}$ are relations, we define the *composite relation*

$$(2.1) \quad R_1 \circ R_2 := p_{13}(p_{12}^{-1}(R_1) \cap p_{23}^{-1}(R_2)).$$

If a is an involution of X_2 , we set $p_{12^a} := (\text{id}_{X_1} \times a) \circ p_{12}$, and similarly for other indices, and we set

$$(2.2) \quad R_1 \overset{a}{\circ} R_2 := p_{13}(p_{12^a}^{-1}(R_1) \cap p_{23}^{-1}(R_2)) = p_{13}(p_{12}^{-1}(R_1) \cap p_{2^a 3}^{-1}(R_2)).$$

If there is a risk of confusion, we write the composition as $\overset{\circ}{\circ}$ and the twisted composition as $\overset{a}{\circ}$.

If $A_i \subseteq X_i$ for $i \in \{1, 2\}$ and $R \subseteq X_1 \times X_2$, then we define $A_1 \circ R := p_2(p_1^{-1}(A_1) \cap R)$ and $R \circ A_2 := p_1(R \cap p_2^{-1}(A_2))$, and if a is an involution of X_1 or X_2 respectively, $A_1 \overset{a}{\circ} R := p_2(p_{1^a}^{-1}(A_1) \cap R)$ and $R \overset{a}{\circ} A_2 := p_1(R \cap p_{2^a}^{-1}(A_2))$. These definitions can be considered as special cases of the previous paragraph, by identifying, for instance, A_1 with the relation $R_0 = \{(\emptyset, x) \in X_0 \times X_1 \mid x \in A_1\}$ with $X_0 = \{\emptyset\}$.

If X and Y are two sets and $f: X \rightarrow Y$ is a function, we denote by $\Gamma_f \subseteq X \times Y$ its *graph*. We will often use implicitly the isomorphism $p_X|_{\Gamma_f}: \Gamma_f \xrightarrow{\sim} X$, with inverse (id_X, f) . This is also an isomorphism of manifolds if f is a morphism of manifolds. If $f_i: X_i \rightarrow X_{i+1}$ for $i \in \{1, 2\}$, then $\Gamma_{f_{i+1} \circ f_i} = \Gamma_{f_i} \circ \Gamma_{f_{i+1}}$ (note the usual backward composition). If $f: X_1 \rightarrow X_2$ and $A_i \subseteq X_i$ for $i \in \{1, 2\}$, then $f(A_1) = A_1 \circ \Gamma_f$ and $f^{-1}(A_2) = \Gamma_f \circ A_2$. If $f: X \rightarrow \mathbb{R}$, we set $\{f \leq 0\} := \{x \in X \mid f(x) \leq 0\}$, and similarly for “ < 0 ”, etc.

Given a real-valued function $f: X \rightarrow \mathbb{R}$, we denote its *epigraph* and *hypograph* by

$$(2.3) \quad \Gamma_f^\pm := \{(x, t) \in X \times \mathbb{R} \mid \pm(t - f(x)) \geq 0\}.$$

Topological spaces. Given any subset A of a topological space, we denote by \overline{A} its closure, by $\text{Int}(A)$ its interior and by $\partial A := \overline{A} \setminus \text{Int}(A)$ its boundary.

A *topological embedding* is a continuous map that is an isomorphism onto its image. A *topological immersion* is a map that is locally a topological embedding. A continuous map is *proper* if it is universally closed (that is, all its pullbacks are closed) or equivalently if it is closed with compact fibers.

Let $R_i \subseteq X_i \times X_{i+1}$ for $i \in \{1, 2\}$ be relations on topological spaces. If p_{13} is proper on $p_{12}^{-1}(R_1) \cap p_{23}^{-1}(R_2)$, then $\overline{R_1 \circ R_2} \subseteq \overline{R_1} \circ \overline{R_2}$. In particular, if R_1 and R_2 are closed, then under the above condition, $R_1 \circ R_2$ is closed.

Given an extended real-valued function $f: X \rightarrow \overline{\mathbb{R}}$, we define the function $\liminf f: X \rightarrow \overline{\mathbb{R}}, x \mapsto \liminf_{y \rightarrow x} f(y)$. It is characterized by $\Gamma_{\liminf f}^+ = \overline{\Gamma_f^+}$. An extended real-valued function is lower semicontinuous if and only if its epigraph is closed, if and only if it is equal to its \liminf . We define similarly $\limsup f$, characterized by $\Gamma_{\limsup f}^- = \overline{\Gamma_f^-}$.

Vector spaces. Let \mathbb{V} be a vector space and let $A \subseteq \mathbb{V}$. We set $A^a := -A$ and $\dot{A} := A \setminus \{0\}$. The subset A is *conic* (or is a *cone*) if $\mathbb{R}_{>0}A = A$ and is *symmetric* if $A = A^a$. Note that a nonempty symmetric convex cone is a vector subspace.

We denote respectively the *orthogonal* and the *polar* of A by

$$(2.4) \quad A^\perp := \{\xi \in \mathbb{V}^* \mid \forall v \in A, \langle \xi, v \rangle = 0\}$$

$$(2.5) \quad \text{and } A^\circ := \{\xi \in \mathbb{V}^* \mid \forall v \in A, \langle \xi, v \rangle \geq 0\}$$

and we define

$$(2.6) \quad A^\top := \{\xi \in \mathbb{V}^* \mid \exists v \in \dot{A}, \langle \xi, v \rangle = 0\}.$$

Setting $I_{\mathbb{V}} := \{(v, \xi) \in \mathbb{V} \times \mathbb{V}^* \mid \langle \xi, v \rangle = 0\}$, one has $A^\top = \dot{A} \circ I_{\mathbb{V}}$. Note that $A^\top = \bigcup_{v \in \dot{A}} v^\perp$, so $(-)^\top$ is increasing. One has $\mathbb{V}^* \setminus A^\top = \{\xi \in \mathbb{V}^* \mid \xi^\perp \cap A \subseteq \{0\}\}$. If A is compact or is a closed cone, then A^\top is closed.

Vector bundles. Let $p: E \rightarrow B$ be a vector bundle. One denotes by $a_E: E \rightarrow E$ the *antipodal map* (that is, the fiberwise opposite) and by 0_E the *zero section* (or its image in E). If $A \subseteq E$ and $x \in B$, we set $A_x := A \cap p^{-1}(x)$. A subset A of E is defined to be a cone (resp. to be symmetric, convex, a vector subspace) if all the A_x 's are. Note however that A being closed (resp. open) implies that all the A_x 's are, but the converse is false in general. Similarly, the vector space operations, the polar, orthogonal, removal of the origin and the operation $(-)^{\top}$ are done fiberwise (but not the operations of closure, interior and boundary). The polar of an open subset of a vector bundle is a closed convex cone (see for instance [4, Lem. 1.2]).

Manifolds. Let M be a manifold. We denote by $\tau_M: TM \rightarrow M$ the tangent bundle of M and by $\pi_M: T^*M \rightarrow M$ its cotangent bundle, and simply write τ and π if there is no risk of confusion. For short, we denote the antipodal maps a_{TM} and a_{T^*M} by a_M and the zero sections 0_{TM} and 0_{T^*M} by 0_M and 0_M^* respectively. For various projections, we may write p_1 instead of, for instance, p_{TM_1} , etc.

For a submanifold N of M , we denote by $T_N M := (N \times_M TM)/TN \rightarrow N$ its *normal bundle* and by $T_N^* M \rightarrow N$ its *conormal bundle* (the subbundle of $N \times_M T^*M \rightarrow N$ orthogonal to $TN \rightarrow N$).

Let $f: M \rightarrow N$ be a morphism of manifolds. One has the following commutative diagram, with the obvious maps.

$$(2.7) \quad \begin{array}{ccccc} & & Tf & & \\ & \nearrow & \text{---} & \searrow & \\ TM & \xrightarrow{f'} & M \times_N TN & \xrightarrow{f\tau} & TN \\ & \searrow \tau_M & \downarrow p_M & & \downarrow \tau_N \\ & & M & \xrightarrow{f} & N \end{array}$$

Note that $T\Gamma_f = \Gamma_{Tf}$ under the identification $T(M \times N) \simeq TM \times TN$. We denote by

$$(2.8) \quad \Lambda_f := T_{\Gamma_f}^*(M \times N)$$

the conormal bundle of the graph of f . The fiberwise transpose of f' is denoted by f_d . The restrictions of the projections $p_{T^*M}: T^*(M \times N) \rightarrow T^*M$ and $p_{T^*N}: T^*(M \times N) \rightarrow T^*N$ to Λ_f will be denoted by p_M and p_N respectively. We write $p_M^a := a_M \circ p_M$. The map $\pi_M \times \text{id}_{T^*N}: T^*(M \times N) \simeq T^*M \times T^*N \rightarrow M \times T^*N$ induces an isomorphism $\Lambda_f \xrightarrow{\simeq} M \times_N T^*N$. We have the following commutative diagram.

$$(2.9) \quad \begin{array}{ccccc} & & \Lambda_f & & \\ & \swarrow p_M^a & \uparrow \wr & \searrow p_N & \\ T^*M & \xleftarrow{f_d} & M \times_N T^*N & \xrightarrow{f\pi} & T^*N \\ & \searrow \pi_M & \downarrow p_M & & \downarrow \pi_N \\ & & M & \xrightarrow{f} & N \end{array}$$

For $A \subseteq T^*M$ and $B \subseteq T^*N$, one has

$$(2.10) \quad f_\pi f_d^{-1}(A) = A \circlearrowleft \Lambda_f \quad \text{and} \quad f_d f_\pi^{-1}(B) = \Lambda_f \circlearrowleft B.$$

Note that f is a *submersion* if and only if

$$(2.11) \quad \Lambda_f \cap (0_M^* \times T^*N) \subseteq 0_{M \times N}^*.$$

More generally, if $B \subseteq T^*N$ is a closed cone, one says that f is *noncharacteristic for B* if

$$(2.12) \quad \Lambda_f \cap (0_M^* \times B) \subseteq 0_{M \times N}^*.$$

§2.2. Sheaves

We recall in this and the next two subsections a few basic results on sheaves, kernels and their microlocal theory, and we refer to [6] for a complete treatment.

The *support* of a presheaf is the complement of the union of the open subsets sent by this presheaf to zero; in particular, it is a closed subset. Let X be a topological space. We denote by \mathbb{k}_X the *constant sheaf on X associated with \mathbb{k}* . If $f: X \rightarrow Y$ is a continuous map, then $f^{-1}(\mathbb{k}_Y) = \mathbb{k}_X$.

In the rest of this subsection, A, A_1, A_2 (resp. B) denote locally closed subsets of the topological space X (resp. Y), and i will denote the inclusion of that subset. We define the *constant sheaf on A associated with \mathbb{k} extended by zero to X* by $\mathbb{k}_{X,A} := i_! i^{-1} \mathbb{k}_X$. We will also denote it by \mathbb{k}_A if there is no risk of confusion. It is characterized by $\mathbb{k}_{X,A}|_A = \mathbb{k}_X|_A$ and $\mathbb{k}_{X,A}|_{X \setminus A} = 0$. If $A_1, A_2 \subseteq X$, then

$$(2.13) \quad \mathbb{k}_{X,A_1} \otimes \mathbb{k}_{X,A_2} = \mathbb{k}_{X,A_1 \cap A_2}.$$

Let $D^b(\mathbb{k}_X)$ denote the *bounded derived category* of the category of sheaves of \mathbb{k} -modules on X . Its objects will still be called sheaves. For $F \in D^b(\mathbb{k}_X)$ and $A \subseteq X$, one sets

$$(2.14) \quad F_A := F \otimes^L \mathbb{k}_{X,A}$$

$$(2.15) \quad \text{and } R\Gamma_A(F) := R\mathcal{H}om(\mathbb{k}_{X,A}, F).$$

The functor $(-)_A$ is exact.

By the Grothendieck spectral sequence, the derived functors of $\Gamma(X; -) \circ (-)_A$ and of $\Gamma_A(X; -) := \Gamma(X; -) \circ \Gamma_A(-)$ are respectively

$$(2.16) \quad R\Gamma(X; (-)_A) \simeq R\Gamma(X; -) \circ (-)_A$$

$$(2.17) \quad \text{and } R\Gamma_A(X; -) \simeq R\Gamma(X; -) \circ R\Gamma_A(-).$$

If Z is closed and U is open in X and i denotes either inclusion, then, for a genuine sheaf F , one has $F_Z = i_* i^{-1} F$ and $\Gamma_U F = i_* i^{-1} F$. Therefore, $\Gamma(X; F_Z) = \Gamma(Z; F)$ and $\Gamma(X; \Gamma_U(F)) = \Gamma(U; F)$. Therefore, in the derived category,

$$(2.18) \quad R\Gamma(X; F_Z) \simeq R\Gamma(Z; F)$$

$$(2.19) \quad \text{and } R\Gamma(X; R\Gamma_U(F)) \simeq R\Gamma(U; F).$$

If $f: X \rightarrow Y$ is continuous and $G \in D^b(\mathbb{k}_Y)$ and $B \subseteq Y$, then $f^{-1}(G_B) = (f^{-1}G)_{f^{-1}(B)}$ (see [6, (2.3.19)]), and in particular,

$$(2.20) \quad f^{-1}(\mathbb{k}_{Y,B}) = \mathbb{k}_{X,f^{-1}(B)}.$$

In the rest of this subsection, assumptions are made on the topological spaces involved (hausdorffness, local compactness, finite c-soft dimension) and their morphisms (finite cohomological dimension). All of these properties are satisfied by topological manifolds and their morphisms.

We recall the following fundamental result without proof.

Proposition 2.1 (Proper base change [6, Prop. 2.6.7, 3.1.9]). *Let $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ be continuous maps between Hausdorff locally compact spaces. Denote their pullback as*

$$(2.21) \quad \begin{array}{ccc} T & \xrightarrow{q} & Y \\ p \downarrow & \lrcorner & \downarrow g \\ X & \xrightarrow{f} & Z \end{array} .$$

Then one has a natural isomorphism of functors

$$(2.22) \quad Rq_! \circ p^{-1} \cong g^{-1} \circ Rf_! .$$

If $g_!$ has finite cohomological dimension, then so does $p_!$, and one has a natural isomorphism of functors

$$(2.23) \quad Rq_* \circ p^! \cong g^! \circ Rf_* .$$

If $f: X \rightarrow Y$ is a continuous map of finite cohomological dimension between topological spaces, then the *relative dualizing complex* (see [6, Def. 3.1.16(i)]) of f is $\omega_{X/Y} := f^! \mathbb{k}_Y$. There is a natural transformation $f^{-1}(-) \otimes^L \omega_{Y/X} \Rightarrow f^!(-)$. It is an isomorphism if f is a topological submersion between Hausdorff locally compact spaces (see [6, Prop. 3.3.2(ii)]) and under microlocal conditions that we give in Propositions 2.8(2b) and 5.8(2). We write $\omega_X := \omega_{X/\{*\}}$. If X is a C^0 -manifold, then ω_X is isomorphic to the orientation sheaf shifted by the dimension of X .

Now, assume that X is Hausdorff locally compact and has finite c-soft dimension. The *duality functors* on X (see [6, Def. 3.1.16(ii)]) are defined by

$$(2.24) \quad D_X := R\mathcal{H}om(-, \omega_X)$$

$$(2.25) \quad \text{and } D'_X := R\mathcal{H}om(-, \mathbb{k}_X) .$$

Recall that if $F \in D^b(\mathbb{k}_X)$ is cohomologically constructible (see [6, Def. 3.4.1]), then so are $D_X(F)$ and $D'_X(F)$, and $F \simeq D_X(D_X(F)) \simeq D'_X(D'_X(F))$ (this is [6, Prop. 3.4.3]), that is, cohomologically constructible sheaves are reflexive.

§2.3. Kernels

Let X_i be Hausdorff locally compact spaces for $i \in \{1, 2, 3\}$. A *kernel* from X_1 to X_2 is an object of $D^b(\mathbb{k}_{X_{12}})$. We consider the bifunctor of *convolution of kernels* (see [6, Prop. 3.6.4]) defined on objects by

$$(2.26) \quad \begin{aligned} \circ: D^b(\mathbb{k}_{M_{12}}) \times D^b(\mathbb{k}_{M_{23}}) &\longrightarrow D^b(\mathbb{k}_{M_{13}}), \\ (K_1, K_2) &\longmapsto K_1 \circ K_2 := R p_{13!} (p_{12}^{-1}(K_1) \otimes^L p_{23}^{-1}(K_2)) \end{aligned}$$

and similarly on morphisms. If there is a risk of confusion, we write this convolution as \circ_2 . Recall that with our notation, p_i , for instance, can stand for p_{M_i} or p_{TM_i} or $p_{T^*M_i}$, and the signification is clear from the context, for instance in the above formula.

The convolution of kernels is associative. We spell out the special cases when either X_1 or X_3 is a point. Adapting the notation, for $K \in D^b(\mathbb{k}_{M \times N})$ and $F \in D^b(\mathbb{k}_M)$ and $G \in D^b(\mathbb{k}_N)$, one has

$$(2.27) \quad F \circ K \simeq R p_{N!} (p_M^{-1} F \otimes^L K) \quad \text{and} \quad K \circ G \simeq R p_{M!} (K \otimes^L p_N^{-1} G).$$

The following two standard results show that the convolution of kernels is a generalization of the composition of functions.

If $f: X \rightarrow Y$ is a function between topological spaces, we set

$$(2.28) \quad K_f := \mathbb{k}_{X \times Y, \Gamma_f}$$

for the constant sheaf on the graph of f associated with \mathbb{k} extended by zero to $X \times Y$.

Proposition 2.2. *Let $f: X \rightarrow Y$ be a continuous map between Hausdorff locally compact spaces.*

- (1) *If $F \in D^b(\mathbb{k}_X)$, then $F \circ K_f \simeq R f_! F$.*
- (2) *If $G \in D^b(\mathbb{k}_Y)$, then $K_f \circ G \simeq f^{-1} G$.*

Proof. This follows easily from the proper base change theorem applied to the pullback of f and id_Y . \square

Proposition 2.3. *If $f_i: X_i \rightarrow X_{i+1}$ for $i \in \{1, 2\}$ are continuous maps between Hausdorff locally compact spaces, then*

$$(2.29) \quad K_{f_2 \circ f_1} = K_{f_1} \circ K_{f_2}.$$

Proof. This follows easily from Equations (2.20) and (2.13) and the fact that p_{13} induces an isomorphism $p_{12}^{-1}(\Gamma_{f_1}) \cap p_{23}^{-1}(\Gamma_{f_2}) = \Gamma_{f_1} \times_{M_2} \Gamma_{f_2} \xrightarrow[p_{13}]{\simeq} \Gamma_{f_1} \circ \Gamma_{f_2} = \Gamma_{f_2 \circ f_1}$. \square

§2.4. Microsupport of sheaves

Let M be a manifold and let $F \in D^b(\mathbb{k}_M)$. We define the *propagation set* of F by

$$(2.30) \quad \text{Ppg}(F) := \left\{ (x, \xi) \in T^*M \mid \text{for all } \phi \in C^\infty(U), \text{ where } U \text{ is an open neighborhood of } x, \text{ with } \phi(x) = 0 \text{ and } d\phi(x) = \xi, \text{ one has } (\text{R}\Gamma_{\{\phi \geq 0\}}(F))_x = 0 \right\}.$$

We set $\text{Ppg}_x(F) := \text{Ppg}(F)_x$. The *microsupport* of F (see [6, Def. 5.1.2(i)]) is defined by

$$(2.31) \quad \mu\text{supp}(F) := \overline{T^*M \setminus \text{Ppg}(F)}.$$

Therefore, the microsupport of a sheaf is the closure of the set of codirections of nonpropagation. It is a coisotropic closed conic subset of T^*M such that $\pi_M(\mu\text{supp}(F)) = \text{supp}(F)$ and $\mu\text{supp}(F[i]) = \mu\text{supp}(F)$ for $i \in \mathbb{Z}$, and satisfies the following triangular inequality: if $F_1 \rightarrow F_2 \rightarrow F_3 \xrightarrow{+1}$ is a distinguished triangle in $D^b(\mathbb{k}_M)$ and $i, j, k \in \{1, 2, 3\}$ with $j \neq k$, then $\mu\text{supp}(F_i) \subseteq \mu\text{supp}(F_j) \cup \mu\text{supp}(F_k)$ (see [6, Prop. 5.1.3 and Thm. 6.5.4]). We set $\mu\text{supp}_x(F) := \mu\text{supp}(F)_x$.

The following lemma gives a useful criterion for belonging to the propagation set of a sheaf.

Lemma 2.4. *Let M be a manifold, let $F \in D^b(\mathbb{k}_M)$ and let $x \in M$. Then $\xi \in \text{Ppg}_x(F)$ if and only if for all $\phi \in C^\infty(U)$, with U an open neighborhood of x , such that $\phi(x) = 0$ and $d\phi(x) = \xi$, the morphism*

$$(2.32) \quad F_x \longrightarrow (\text{R}\Gamma_{\{\phi < 0\}}(F))_x$$

induced by the inclusions $(i_V : V \cap \{\phi < 0\} \hookrightarrow V)_{V \ni x}$ is an isomorphism.

Proof. The result follows by applying the stalk functor to the distinguished triangle $\text{R}\Gamma_{\{\phi \geq 0\}}(F) \rightarrow \text{R}\Gamma_U(F) \rightarrow \text{R}\Gamma_{\{\phi < 0\}}(F) \xrightarrow{+1}$. \square

The following lemma will simplify some proofs below. The definition of strict differentiability is recalled in Appendix A.

Lemma 2.5. *Let M be a manifold and let $F \in D^b(\mathbb{k}_M)$. If $(x, \xi) \notin \mu\text{supp}(F)$, then for all functions $\phi: U \rightarrow \mathbb{R}$, with U an open neighborhood of x , which are strictly differentiable at x with $d\phi(x) = \xi$, one has $(\text{R}\Gamma_{\{\phi \geq \phi(x)\}}(F))_x = 0$.*

Proof. In the proof of [6, Prop. 5.1.1], the part (2) \Rightarrow (1)₁ proves the lemma without any change. Indeed, with the notation there, if $\phi: U \rightarrow \mathbb{R}$ is strictly differentiable at x with $d\phi(x) \in \text{Int}(\gamma^{\circ a})$, then $\{\phi < 0\}$ coincides with a γ -open set in a neighborhood of x by Lemma A.3. \square

Remark 2.6. An analogous statement assuming only differentiability of ϕ is false. Indeed, let $Z = \{(x, y) \in \mathbb{R}^2 \mid x + y = 0 \text{ or } \exists n \in \mathbb{N}_{>0} x + y = 1/n\}$. Then $\mu\text{supp}(\mathbb{k}_Z) = Z \times \mathbb{R}(1, 1)$. Let $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto 2x^2 \sin(\pi/x) + y$. Then $\phi(0, 0) = 0$ and $d\phi(0, 0) = (0, 1) \notin \mu\text{supp}_{(0,0)}(\mathbb{k}_Z)$. For $n \in \mathbb{N}_{>0}$, set $U_n =]-1/n, 1/n[^2$. Then the inclusions $i_n: U_n \cap Z \cap \{\phi < 0\} \hookrightarrow U_n \cap Z$ do not induce an isomorphism in cohomology of the inductive limit. Indeed, setting $Z_m := \{x + y = 1/m\}$, the subsets $Z_m \cap U_n$ are connected, but for any $n, m \in \mathbb{N}$ with $2m > n$, the subset $Z_{2m} \cap U_n \cap \{\phi < 0\}$ is not.

Proposition 2.7 ([6, Exe. V.13]). *Let M be a manifold and let $F \in \text{D}^b(\mathbb{k}_M)$ be cohomologically constructible. Then,*

$$(2.33) \quad \mu\text{supp}(\text{D}'_M(F)) = \mu\text{supp}(F)^a.$$

Proof. Since constructible sheaves are reflexive, it suffices to prove $\mu\text{supp}(\text{D}'_M(F)) \subseteq \mu\text{supp}(F)^a$. This is a special case of [6, Prop. 5.4.2] where one factor is reduced to a point. \square

We recall the following functoriality properties of the microsupport. A morphism is said to be noncharacteristic for a sheaf if it is noncharacteristic for its microsupport.

Proposition 2.8 ([6, Prop. 5.4.4–5, 5.4.13–14]). *Let M be a manifold.*

- (1) *Let $F_1, F_2 \in \text{D}^b(\mathbb{k}_M)$.*
 - (a) *Assume that $\mu\text{supp}(F_1) \cap \mu\text{supp}(F_2)^a \subseteq 0_M^*$. Then $\mu\text{supp}(F_1 \otimes^L F_2) \subseteq \mu\text{supp}(F_1) + \mu\text{supp}(F_2)$.*
 - (b) *Assume that $\mu\text{supp}(F_1) \cap \mu\text{supp}(F_2) \subseteq 0_M^*$. Then $\mu\text{supp}(\mathbb{R}\mathcal{H}om(F_1, F_2)) \subseteq \mu\text{supp}(F_1)^a + \mu\text{supp}(F_2)$.*
- (2) *Let $f: M \rightarrow N$ be a morphism of manifolds.*
 - (a) *Let $F \in \text{D}^b(\mathbb{k}_M)$ and assume that f is proper on $\text{supp}(F)$. Then $\mu\text{supp}(\mathbb{R}f_*F) \subseteq \mu\text{supp}(F) \overset{a}{\circ} \Lambda_f$ with equality if f is a closed embedding.*
 - (b) *Let $G \in \text{D}^b(\mathbb{k}_N)$ and assume that f is noncharacteristic for G . Then the morphism $f^{-1}G \otimes^L \omega_{M/N} \rightarrow f^!G$ is an isomorphism and $\mu\text{supp}(f^{-1}G) \subseteq \Lambda_f \overset{a}{\circ} \mu\text{supp}(G)$ with equality if f is a submersion.*

Remark 2.9. The proof of the inclusion in item (2a) given in [6, Prop. 5.4.4] actually proves that if $f: M \rightarrow N$ is a morphism of manifolds and $F \in D^b(\mathbb{k}_M)$, and if f is proper on $\text{supp}(F)$, then $TN \setminus \text{Ppg}(Rf_*F) \subseteq (TM \setminus \text{Ppg}(F)) \overset{a}{\circ} \Lambda_f$ with equality if f is an isomorphism.

Remark 2.10. By Lemma 2.5, the inclusion of item (2a) and the result of the previous remark still hold at a point if the map f is required only to be strictly differentiable at that point.

Finally, we give a standard upper bound on the microsupport of the convolute of two kernels.

Proposition 2.11 ([2, §Kernels]). *Let M_i be manifolds for $i \in \{1, 2, 3\}$. Let $K_1 \in D^b(\mathbb{k}_{M_{12}})$ and $K_2 \in D^b(\mathbb{k}_{M_{23}})$. Assume that*

- (1) p_{13} is proper on $p_{12}^{-1}(\text{supp}(K_1)) \cap p_{23}^{-1}(\text{supp}(K_2))$;
- (2) $(\mu\text{supp}(K_1)^a \times 0_3^*) \cap (0_1^* \times \mu\text{supp}(K_2)) \subseteq 0_{123}^*$.

Then,

$$(2.34) \quad \mu\text{supp}(K_1 \circ K_2) \subseteq \mu\text{supp}(K_1) \overset{a}{\circ} \mu\text{supp}(K_2).$$

Proof. The two assumptions of the proposition allow us to apply items (1a), (2a) and (2b) of Proposition 2.8 to conclude. □

Remark 2.12. Note that the two assumptions of Proposition 2.11 are equivalent to the following condition: p_{13} is proper on $p_{12}^{-1}(\mu\text{supp}(K_1)) \cap p_{23}^{-1}(\mu\text{supp}(K_2))$.

§3. Microsupports associated with subsets

In this section, we study the microsupports of (constant sheaves on) subsets. We first give a criterion for belonging to such microsupports and give results on the microsupports of closed submanifolds and closed C^0 -submanifolds. We then offer direct proofs of lower and upper bounds on the microsupport of a closed set.

§3.1. General properties

Let M be a manifold. If $A \subseteq M$ is locally closed, we set for short $\text{Ppg}(A) := \text{Ppg}(\mathbb{k}_{M,A})$ and

$$(3.1) \quad \mu\text{supp}(A) := \mu\text{supp}(\mathbb{k}_{M,A}),$$

and also $\text{Ppg}_x(A) := \text{Ppg}(A)_x$ and $\mu\text{supp}_x(A) := \mu\text{supp}(A)_x$ for $x \in M$. Note that

$$(3.2) \quad \pi_M(\mu\text{supp}(A)) = \overline{A}.$$

The following lemma gives a useful criterion for belonging to the propagation set of a sheaf associated with a subset.

Lemma 3.1. *Let M be a manifold, let $Z \subseteq M$ be a closed subset and let $x \in M$. Then $\xi \in \text{Ppg}_x(Z)$ if and only if for all $\phi \in C^\infty(U)$, with U an open neighborhood of x , such that $\phi(x) = 0$ and $d\phi(x) = \xi$, the morphism*

$$(3.3) \quad (\mathbb{k}_Z)_x \longrightarrow (\text{R}\Gamma_{Z \cap \{\phi < 0\}} \mathbb{k}_Z)_x$$

induced by the inclusions $(i_V : Z \cap \{\phi < 0\} \cap V \hookrightarrow Z \cap V)_{V \ni x}$ is an isomorphism.

Proof. As for Lemma 2.4, the result follows by applying the stalk functor to the distinguished triangle $\text{R}\Gamma_{Z \cap \{\phi \geq 0\}}(\mathbb{k}_Z) \rightarrow \text{R}\Gamma_{Z \cap U}(\mathbb{k}_Z) \rightarrow \text{R}\Gamma_{Z \cap \{\phi < 0\}}(\mathbb{k}_Z) \xrightarrow{+1} \cdot$. □

As for inverse images, Proposition 2.8(2b) and Equation (2.20) show that if $f : M \rightarrow N$ is a morphism of manifolds and $B \subseteq N$ is locally closed, and if f is noncharacteristic for $\mu\text{supp}(B)$, then

$$(3.4) \quad \mu\text{supp}(f^{-1}(B)) \subseteq \Lambda_f \overset{a}{\circ} \mu\text{supp}(B)$$

with equality if f is a submersion. If f is an isomorphism and $A \subseteq M$ is locally closed, then Remark 2.9 gives

$$(3.5) \quad \text{Ppg}(f(A)) = \text{Ppg}(A) \overset{a}{\circ} \Lambda_f.$$

In particular, if $\Phi : \mathbb{V} \rightarrow \mathbb{V}'$ is a linear isomorphism, $A \subseteq \mathbb{V}$ is locally closed and $x \in \mathbb{V}$, then

$$(3.6) \quad \Phi^\top(\text{Ppg}_{\Phi(x)}(\Phi(A))) = \text{Ppg}_x(A).$$

We will also need the following result.

Proposition 3.2 ([6, Prop. 5.3.2]). *Let N be a closed submanifold of a manifold M . Then*

$$(3.7) \quad \mu\text{supp}(N) = T_N^*M.$$

For closed C^0 -submanifolds, we have the following result.

Proposition 3.3. *Let N be a closed C^0 -submanifold of a manifold M . Then \mathbb{k}_N is a cohomologically constructible sheaf, self-dual up to a locally constant sheaf of rank 1, and its microsupport is symmetric, that is,*

$$(3.8) \quad \mu\text{supp}(N) = \mu\text{supp}(N)^\alpha.$$

Proof. The cohomological constructibility and self-duality results hold if N is a vector subspace of a vector space M (in that case, $D_M(\mathbb{k}_{M,N}) = R\mathcal{H}om(\mathbb{k}_{M,N}, \omega_M) = \omega_N$ by [6, Exa. 3.4.5(i)]), and they are of a local and topological nature. Therefore, they hold for closed C^0 -submanifolds, and we can apply Proposition 2.7. \square

§3.2. Bounds on the microsupports associated with closed subsets

We have the following bounds on the microsupport associated with a closed subset. For the tangent and strict tangent cones appearing in this proposition, we refer to Appendix B.

Proposition 3.4. *Let M be a manifold. If $Z \subseteq M$ is closed, then*

$$(3.9) \quad \overline{\pi_M^{-1}(Z) \cap C(Z)^\circ} \subseteq \mu\text{supp}(Z) \subseteq N(Z)^\circ.$$

The upper bound is [6, Prop. 5.3.8]. The lower bound was proved in [5, Prop. 3.1], where it was shown that $\overline{\pi_M^{-1}(Z) \cap C(Z)^\circ}$ is equal to the 0-truncated microsupport of \mathbb{k}_Z (see the definition there). We give direct proofs of both bounds.

The lower bound is not widely applicable, since $C_x(Z)^\circ$ is nonzero only if Z is “at first order” contained in a half-space of T_xM . On the other hand, the upper bound is trivial if Z is a C^0 -submanifold. We will give another upper bound in that case (Proposition 8.4).

For the proof, we will need the following two lemmas.

Lemma 3.5. *Let M be a manifold, $A \subseteq M$ and $x \in M$. Let U be an open neighborhood of x and $\phi \in C^\infty(U)$ with $\phi(x) = 0$ and $d\phi(x) \notin N_x(A)^\circ$. Then there exist an open neighborhood $V \subseteq U$ of x and $\alpha > 0$ and for all $\beta \in]-\alpha, \alpha[$, a strong deformation retraction $h_\beta: V \times [0, 1] \rightarrow V$ of V onto $V \cap \{\phi \leq \beta\}$ such that $h_\beta((V \cap A) \times [0, 1]) \subseteq V \cap A$.*

Proof. Let $u \in N_x(A)$ be such that $\langle d\phi(x), u \rangle < 0$. We can fix a chart at x with domain $V \subseteq U$ such that in that chart, $x = 0 \in \mathbb{R}^m$ and $\phi = d\phi(0)$ (since $d\phi(0) \neq 0$). Since $N(A)$ is open, we can suppose, reducing V if necessary, that for all $y \in V$, one has $u \in N_y(A)$. We can also suppose that the intersection of V with any line parallel to u is connected, for instance by assuming that V is of the form $]-\alpha, \alpha[\times V'$ and $u \in \mathbb{R} \times \{0\}$.

If $\beta \in]-\alpha, \alpha[$ and $y \in V$ and $t \in [0, 1]$, we set $h_\beta(y, t) := y - t \frac{(\phi(y) - \beta)^+}{\phi(u)} u$ where $x^+ := \max(x, 0)$. This defines $h_\beta: V \times [0, 1] \rightarrow V$. If $\phi(y) \leq \beta$ and $t \in [0, 1]$, then $h_\beta(y, t) = y$. If $y \in V$, then $h_\beta(y, 0) = y$ and $\phi(h_\beta(y, 1)) = \phi(y) - (\phi(y) - \beta)^+ = \min(\phi(y), \beta) \leq \beta$. Finally, if $y \in V \cap A$ and $t \in [0, 1]$, then the assumptions $-t \frac{(\phi(y) - \beta)^+}{\phi(u)} \geq 0$ and $u \in N_{h_\beta(y,s)}(A)$ for any $s \in [0, t]$ imply $h_\beta(y, t) \in A$. \square

Lemma 3.6. *Let A be a subset of a manifold M . Then,*

$$(3.10) \quad C(A)^\circ = \{(x, \xi) \in \bar{A} \times_M T^*M \mid \text{there exist an open neighborhood } U \text{ of } x \text{ and } \phi \in C^1(U) \text{ such that } \phi(x) = 0 \text{ and } d\phi(x) = \xi \text{ and } \phi(A \cap U) \subseteq \mathbb{R}_{\geq 0}\}$$

and this set is equal to the set defined similarly with the function ϕ required only to be continuous on U and differentiable at x .

Proof. (i) Let $(x, \xi) \in \bar{A} \times_M T^*M$. Let U be an open neighborhood of x and $\phi \in C^0(U)$ be such that $\phi(x) = 0$ and $d\phi(x) = \xi$ and $\phi(A \cap U) \subseteq \mathbb{R}_{\geq 0}$. Let $u \in C_x(A)$. There are sequences $(x_n) \in A^\mathbb{N}$ and $(c_n) \in (\mathbb{R}_{>0})^\mathbb{N}$ such that $x_n \xrightarrow{n} x$ and $c_n(x_n - x) \xrightarrow{n} u$. One has $0 \leq c_n(\phi(x_n) - \phi(x)) = \langle d\phi(x), c_n(x_n - x) \rangle + o(c_n(x_n - x))$. Therefore, $\langle \xi, u \rangle \geq 0$. Therefore, $(x, \xi) \in C(A)^\circ$.

(ii) Conversely, let $(x, \xi) \in C(A)^\circ$. Fix a chart centered at x , with image in \mathbb{R}^m . If $m = 0$, the result is trivial, so we suppose $m > 0$. Denote by $\|-\|$ the Euclidean norm on \mathbb{R}^m . If $\xi = 0$, then set $\phi := \|-\|^2: \mathbb{R}^m \rightarrow \mathbb{R}$. Now, suppose $\xi \neq 0$. We can suppose that $\xi = (-1, 0, \dots, 0)$.

Let $n \in \mathbb{N}_{>0}$. We will prove that there exists $\alpha_n > 0$ such that, setting $E_n := \{(u_1, u') \in \mathbb{R} \times \mathbb{R}^{m-1} \mid \frac{1}{n}\|u'\| \leq u_1 \leq \alpha_n\} \setminus \{0\}$, one has $E_n \cap A = \emptyset$. Indeed, suppose that there is a sequence of points $(x_m) = ((x_{m,1}, x'_m)) \in A^\mathbb{N}$ such that $x_{m,1} \geq \frac{1}{n}\|x'_m\| > 0$ for all $m \in \mathbb{N}$ and $x_{m,1} \xrightarrow{m} 0$. Then $x_m \xrightarrow{m} 0$, so up to extracting a subsequence, we can suppose that $\frac{x_m}{\|x_m\|} \xrightarrow{m} u \in \dot{C}_0(A)$. Since $\xi \in C_x(A)^\circ$, one has $\langle \xi, u \rangle \geq 0$. But, writing $u = (u_1, u')$, one has $\langle \xi, u \rangle = -u_1 \leq -\frac{1}{n}\|u'\| \leq 0$, and actually, $\langle \xi, u \rangle < 0$, since otherwise, one would have $u_1 = u' = 0$, so $u = 0$. This is a contradiction. This proves the existence of the desired $\alpha_n > 0$. We can suppose that for $n \in \mathbb{N}_{>0}$, one has $(n+1)\alpha_{n+2} < n\alpha_{n+1}$, and in particular the sequence (α_n) decreases to 0.

We can assume that the broken line connecting the points $A_n = (n\alpha_{n+1}, \alpha_{n+1}) \in \mathbb{R}^2$ defines a convex function $f: [0, \alpha_2[\rightarrow \mathbb{R}$. Indeed, each line $(A_n A_{n+1})$ crosses the x -axis at some $\lambda_n > 0$, and one can ensure recursively that $(n+2)\alpha_{n+2} \leq \lambda_n$. It is elementary to construct a function $\psi \in C^1(]-\alpha_2, \alpha_2[)$ with $\psi(t) \geq f(|t|)$ for $t \in]-\alpha_2, \alpha_2[$ and $\psi(0) = \psi'(0) = 0$. For instance, construct smooth functions with graphs in the triangles formed by A_{n+1} and the midpoints of $[A_n A_{n+1}]$ and $[A_{n+1} A_{n+2}]$, which connect in a C^1 fashion. Finally, setting $\phi: B(0, \alpha_2) \rightarrow \mathbb{R}$, $(x_1, x') \mapsto \psi(\|x'\|) - x_1$, one has $\phi(0) = 0$ and $d\phi(0) = \xi$ and $\phi(A \cap B(0, \alpha_2)) \subseteq \mathbb{R}_{\geq 0}$. □

Remark 3.7. In view of Lemma 2.5, we will need in the applications of the lemma only that the function ϕ constructed in the proof is strictly differentiable at x , so we could simply have defined $\psi(t) := f(|t|)$.

Remark 3.8. In the right-hand side in the lemma, we cannot require that $\phi \in C^2(U)$. Consider for example the graph Γ_f of the function $f: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto |x|^{3/2}$. Then $(0, 1) \in C_{(0,0)}(\Gamma_f)^\circ$, but the Taylor–Lagrange formula shows that for any neighborhood $U \subseteq \mathbb{R}$ of 0, there cannot be a function $\phi \in C^2(U)$ with $\phi(0) = 0$ and $d\phi(0) = 0$ and $f|_U \leq \phi$. As a consequence, we see that in the definition of the propagation set of a sheaf, the differentiability class of the test functions matters, as opposed to the situation for the microsupport. Namely, in this example, $(0, 1) \in \text{Ppg}_{(0,0)}^2(\Gamma_f) \setminus \text{Ppg}_{(0,0)}^1(\Gamma_f)$, where the superscript denotes the differentiability class of the test functions.

Proof of Proposition 3.4. (i) Upper bound. Since $N(Z)$ is open, its polar cone is closed, so it is enough to prove that $T^*M \setminus \text{Ppg}(Z) \subseteq N(Z)^\circ$. If $x \notin Z$, then $\mu\text{supp}_x(Z) = \emptyset$, so the inclusion is true. Let $x \in Z$ and $\xi \in T_x^*M \setminus N_x(Z)^\circ$. Let $\phi \in C^\infty(U)$ with U an open neighborhood of x be such that $\phi(x) = 0$ and $d\phi(x) = \xi$. We will prove that the natural morphism $(\mathbb{k}_Z)_x \rightarrow (\text{R}\Gamma_{Z \cap \{\phi < 0\}} \mathbb{k}_Z)_x$ is an isomorphism. Then, by Lemma 3.1, we will have $\xi \in \text{Ppg}_x(Z)$.

By Lemma 3.5, there exist an open neighborhood $V \subseteq U$ of x and $\alpha > 0$ and for all $\beta \in (-\alpha, 0)$, a deformation retraction $h_\beta: V \times [0, 1] \rightarrow V$ of V onto $V \cap \{\phi \leq \beta\}$ such that $h_\beta((V \cap Z) \times [0, 1]) \subseteq V \cap Z$.

This proves that the inclusions $i_{\beta,V}: Z \cap V \cap \{\phi \leq \beta\} \hookrightarrow Z \cap V$ induce isomorphisms $i_{\beta,V}^\sharp: \text{R}\Gamma(Z \cap V; \mathbb{k}_M) \xrightarrow{\sim} \text{R}\Gamma(Z \cap V \cap \{\phi \leq \beta\}; \mathbb{k}_M)$. One has $\text{R}\Gamma(Z \cap V \cap \{\phi < 0\}; \mathbb{k}_M) = \lim_{\beta \rightarrow 0} \text{R}\Gamma(Z \cap V \cap \{\phi \leq \beta\}; \mathbb{k}_M)$, and if $\beta < \beta'$, then $i_{\beta',V} \circ i_{\beta,V} = i_{\beta,V}$. Therefore, the $i_{\beta,V}$'s induce an isomorphism which is $i_V^\sharp: \text{R}\Gamma(Z \cap V; \mathbb{k}_M) \xrightarrow{\sim} \text{R}\Gamma(Z \cap V \cap \{\phi < 0\}; \mathbb{k}_M)$.

Since in Lemma 3.5, the homotopies h_β can be restricted to arbitrarily small neighborhoods V of x , this proves that the natural morphism $(\mathbb{k}_Z)_x \rightarrow (\text{R}\Gamma_{Z \cap \{\phi < 0\}} \mathbb{k}_Z)_x$ is an isomorphism.

(ii) Lower bound. Let $x \in Z$ and $\xi \in C_x(Z)^\circ$. Then, by Lemma 3.6, there exist an open neighborhood U of x and a function $\phi \in C^1(U)$ such that $\phi(x) = 0$ and $d\phi(x) = \xi$ and $\phi(Z \cap U) \subseteq \mathbb{R}_{\geq 0}$. Therefore, $Z \cap U \cap \{\phi < 0\} = \emptyset$, so $(\text{R}\Gamma_{Z \cap \{\phi < 0\}} \mathbb{k}_Z)_x = 0 \not\cong (\mathbb{k}_Z)_x = \mathbb{k}$, so by Lemma 3.1, $\xi \notin \text{Ppg}_x(Z)$. \square

Remark 3.9. The proof actually shows that $\pi_M^{-1}(Z) \cap C(Z)^\circ \subseteq T^*M \setminus \text{Ppg}^1(Z)$.

Remark 3.10. Taking polars of the inclusions (3.9), we see that the fiberwise closure of $N(Z)$ is contained in $\mu\text{supp}(Z)^\circ$, which is contained in $C(Z)$, but the inclusion $\overline{N(Z)} \subseteq \mu\text{supp}(Z)^\circ$ need not hold: if $x \in \partial(\text{Int}(Z))$, then $\overline{N(Z)}_x = T_x M$ but $\mu\text{supp}_x(Z) \not\supseteq \{0\}$.

§4. Whitney cones of maps

In this section, we study the Whitney cones of continuous maps. The results are elementary and their proofs do not require any sheaf theory. After the definitions and general properties, we give characterizations of Lipschitz continuity and strict differentiability in terms of the Whitney cone and extend these characterizations to C^0 -submanifolds. Then we prove a chain rule involving Whitney cones. Finally, we introduce directional Dini derivatives and relate them to the Whitney cone, which will be used to prove an upper bound on the conormal of a continuous map in Section 7.

§4.1. Definitions and first properties

The definitions of the strict tangent cone $N(A)$, the tangent cone $C(A)$, and the Whitney cone $C(A, B)$ of subsets A, B of a manifold, and their main properties, are recalled in Appendix B. If $f: M \rightarrow N$ is a continuous map between manifolds, we define its *Whitney cone* as the Whitney cone of its graph, that is,

$$(4.1) \quad C_f := C(\Gamma_f, \Gamma_f).$$

This is a closed symmetric cone in $T(M \times N)$.

Furthermore, if $x \in M$, we set

$$(4.2) \quad C_x(f) := C_{(x, f(x))}(\Gamma_f, \Gamma_f).$$

This is a closed cone in $T_{(x, f(x))}(M \times N)$. We also set $C_x(\Gamma_f) := C_{(x, f(x))}(\Gamma_f)$ and similarly for $N_x(\Gamma_f^+)$.

Furthermore, if $u \in T_x M$, we define the following analogue of the directional derivatives:

$$(4.3) \quad C_x^u(f) := C_x(f) \cap (\mathbb{R}_{\geq 0}u \times T_{f(x)}N).$$

§4.2. Characterizations of Lipschitz continuity and strict differentiability

We begin with two straightforward lemmas whose proofs are left to the reader. For notions related to Lipschitz continuity and strict differentiability, we refer to Appendix A. In particular, the notation $\text{Lip}_x(f)$ is defined in Definition A.1(3).

Lemma 4.1. *Let $f: \mathbb{V} \rightarrow \mathbb{W}$ be a map between normed vector spaces. Let $x \in \mathbb{V}$. Then,*

$$(4.4) \quad \text{Lip}_x(f) = \min \left\{ C \in \overline{\mathbb{R}} \mid C_x(f) \subseteq \{(v, w) \in \mathbb{V} \times \mathbb{W} \mid \|w\| \leq C\|v\|\} \right\}.$$

Lemma 4.2. *If $f: M \rightarrow N$ is a continuous map between manifolds that is point-wise Lipschitz at $x \in M$, then $p_M(C_x(f)) = T_x M$.*

The following proposition provides geometric characterizations of Lipschitz continuity and strict differentiability.

Proposition 4.3. *Let $f: M \rightarrow N$ be a continuous map between manifolds and let $x \in M$.*

- (1) *The map f is Lipschitz on a neighborhood of x if and only if $C_x^0(f) = \{0\}$.*
- (2) *The map f is strictly differentiable at x if and only if $C_x^0(f) = \{0\}$ and $C_x(f)$ is contained in a $(\dim_x M)$ -dimensional vector subspace, and in that case, $C_x(f) = \Gamma_{T_x f}$.*

Proof. (1) Fixing charts at x and $f(x)$, one has $C_x^0(f) = \{0\}$ if and only if there exists $C \in \mathbb{R}$ such that $C_x(f) \subseteq \{(u, v) \in \mathbb{R}^{m+n} \mid \|v\| \leq C\|u\|\}$. Therefore, the result follows from Lemma 4.1.

(2) Necessity is obvious. We prove sufficiency. We deduce from $C_x^0(f) = \{0\}$ and (1) that f is Lipschitz at x . It follows from Lemma 4.2 that $p_M(C_x(f)) = T_x M$. Therefore, $C_x(f)$ is a $(\dim_x M)$ -dimensional vector space and is the graph of the linear map $L := p_N \circ (p_M|_{C_x(f)})^{-1}: T_x M \rightarrow T_{f(x)} N$. From Lemma 4.1 applied to $f - L$, we see that f is strictly differentiable at x with $T_x f = L$. □

These characterizations extend to C^0 -submanifolds as follows.

Proposition 4.4. *Let M be a C^0 -submanifold of a manifold P and let $x \in M$.*

- (1) *Let F be a $(\text{codim}_x M)$ -dimensional vector subspace of $T_x P$. The C^0 -submanifold M is locally at x the graph of a Lipschitz map with codomain tangent to F if and only if $C_x(M, M) \cap F = \{0\}$.*
- (2) *The C^0 -submanifold M is locally at x the graph of a map f strictly differentiable at x if and only if $C_x(M, M)$ is contained in a $(\dim_x M)$ -dimensional vector subspace of $T_x P$, and in that case, $C_x(M, M) = \Gamma_{T_x f}$.*

The condition that M be locally at x a graph means that there exist an open neighborhood U of x and a chart $\phi = (\phi_M, \phi_N): U \xrightarrow{\simeq} U_M \times U_N \subseteq \mathbb{R}^m \times \mathbb{R}^n$ and a map $f: U_M \rightarrow U_N$ such that $\phi(M \cap U) = \Gamma_f$. The condition “ f strictly differentiable at x ” then means “ f strictly differentiable at $\phi_M(x)$ ”. One identifies $C_x(M, M)$ with its image in the chart $T_x \phi(C_x(M, M)) = C_{\phi(x)}(\phi(M \cap U), \phi(M \cap U))$. That the “codomain of f is tangent to F ” means that $T_x \phi(F) = \{0\} \times \mathbb{R}^n$.

Proof. In both cases, necessity is straightforward. As for sufficiency, fix a chart ϕ of P at x . We set $m := \dim_x M$ and $n := \text{codim}_x M$. For the first equivalence of the

proposition, F is given, and for the second equivalence, we let F be a complement of a $(\dim_x M)$ -dimensional vector subspace of $T_x P$ in which $C_x(M, M)$ is included. We can suppose that the chart ϕ is of the form $\phi = (\phi_M, \phi_N): U \xrightarrow{\sim} U_M \times U_N \subseteq \mathbb{R}^m \times \mathbb{R}^n$ with $T_x \phi(F) = \{0\} \times \mathbb{R}^n$, hence $T_x \phi(C_x(M, M)) \cap (\{0\} \times \mathbb{R}^n) = \{0\}$. Being a C^0 -submanifold, M is locally at x the image of a continuous injection $\tilde{f}: \mathbb{R}^m \rightarrow \mathbb{R}^{m+n}$ with $\tilde{f}(0) = x$ and $\tilde{f}(\mathbb{R}^m) = M \cap U'$ for $U' \subseteq P$ an open neighborhood of x . We can suppose that $U' = U$.

The relation $T_x \phi(C_x(M, M)) \cap (\{0\} \times \mathbb{R}^n) = \{0\}$ implies that $\phi_M|_{M \cap U}: M \cap U \rightarrow \mathbb{R}^m$ is injective in a neighborhood of x which we can suppose to be $\tilde{f}(\mathbb{R}^m) = M \cap U$. Therefore, $\phi_M \circ \tilde{f}: \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a continuous injection, so by the theorem of invariance of domain, it is a topological isomorphism onto its image, say V , which is open in \mathbb{R}^m .

Then define $f := (\phi_N \circ \tilde{f}) \circ (\phi_M \circ \tilde{f})^{-1} = \phi_N \circ (\phi_M|_{M \cap U})^{-1}: V \rightarrow \mathbb{R}^n$. In particular, $\phi_N = f \circ \phi_M$ on $\tilde{f}(\mathbb{R}^m) = M \cap U$. Therefore, $\phi(M \cap U) = \Gamma_f$. Therefore, $C_x(M, M) = C_{\phi_M(x)}(f)$. By Proposition 4.3, the hypothesis of item (1) (resp. item (2)) on $C_x(M, M)$ implies that f is Lipschitz (resp. strictly differentiable at 0 with derivative 0). □

Remark 4.5. The analogous statements with the tangent cone $C_x(M)$ and pointwise Lipschitz continuity (resp. differentiability) are false, as $M = \Gamma_f$ where $f = \sqrt{|\cdot|}: \mathbb{R} \rightarrow \mathbb{R}$ shows: one has $C_0(\Gamma_f) = \{0\} \times \mathbb{R}_{\geq 0}$, which intersects $\mathbb{R} \times \{0\}$ trivially. Another example is any wild enough curve contained in $\{(x, y) \in \mathbb{R}^2 \mid |y| \leq x^2\}$. Of course, one still has one implication.

Note that we did not assume that the embedding \tilde{f} in the proof was locally flat. The local flatness of M at x is a consequence of the hypothesis on $C_x(M, M)$.

If $\text{codim}_x M = 1$, then the condition of item (1) is equivalent to $C_x(M, M) \neq T_x P$.

Corollary 4.6. *Let $f: M \rightarrow N$ be a continuous map between manifolds and let $x \in M$. If $C_x(f)$ is contained in a $(\dim_x M)$ -dimensional vector subspace, then it is equal to it.*

In particular, if $\dim_x M > 0$, then $C_x(f)$ cannot be contained in a $(\dim_x M - 1)$ -dimensional vector subspace. Contrast this to the case of $C_x(\Gamma_f)$: if $f = \sqrt{\|\cdot\|}: \mathbb{R}^m \rightarrow \mathbb{R}$, then $C_0(\Gamma_f) = \{0\} \times \mathbb{R}_{\geq 0}$ is contained in a 1-dimensional vector subspace.

We end this subsection with a result proving the lower hemicontinuity of $C_x(f)$ considered as a multivalued function from $T_x M$ to $T_{f(x)} N$.

Proposition 4.7. *Let $f: M \rightarrow N$ be a Lipschitz map between manifolds and let $x \in M$. For any $u \in T_x M$ and any open set $V \subseteq T_{f(x)} N$ such that $C_x(f) \cap (\{u\} \times$*

$V) \neq \emptyset$, there exists an open neighborhood $U \subseteq T_x M$ of u such that for all $u' \in U$, one has $C_x(f) \cap (\{u'\} \times V) \neq \emptyset$.

Proof. Let $(u, v) \in C_x(f)$ and let $(u_n) \in (T_x M)^\mathbb{N}$ be a sequence converging to u . We are going to construct a sequence $(v_n) \in (T_{f(x)} N)^\mathbb{N}$ converging to v such that $(u_n, v_n) \in C_x(f)$, which proves the proposition. Fix charts at x and $f(x)$ in which f is C -Lipschitz. There exist sequences $x_n, y_n \xrightarrow{n} x$ and $c_n > 0$ such that $c_n(y_n - x_n, f(y_n) - f(x_n)) \xrightarrow{n} (u, v)$. Set $y_{m,n} := y_n + (u_m - u)/c_n$. One has $c_n(y_{m,n} - x_n) \xrightarrow{n} u_m$ for all $m \in \mathbb{N}$.

One has $\|c_n(f(y_{m,n}) - f(x_n)) - c_n(f(y_n) - f(x_n))\| = \|c_n(f(y_{m,n}) - f(y_n))\| \leq C\|u_m - u\|$. Therefore, $c_n(f(y_{m,n}) - f(x_n)) \xrightarrow{m} c_n(f(y_n) - f(x_n))$ uniformly in n . By Lipschitz continuity of f , for any $m \in \mathbb{N}$, the sequence $(c_n(f(y_{m,n}) - f(x_n)))_n$ has a converging subsequence, indexed by ρ , and we set $v_m := \lim_n c_{\rho(n)}(f(y_{m,\rho(n)}) - f(x_{\rho(n)}))$. Since the above convergence is uniform, one has $v_m \xrightarrow{m} v$. For any $m \in \mathbb{N}$, one has $c_{\rho(n)}(y_{m,\rho(n)} - x_{\rho(n)}, f(y_{m,\rho(n)}) - f(x_{\rho(n)})) \xrightarrow{n} (u_m, v_m)$, so $(u_m, v_m) \in C_x(f)$. \square

Remark 4.8. The Lipschitz continuity of f is needed, as the function $\mathbb{R}^2 \rightarrow \mathbb{R}$, $x \mapsto (x_1)^{1/3}$ shows. One also needs to fix the point $x \in M$. In other words, C_f is not lower hemicontinuous, as the function $|\cdot|: \mathbb{R} \rightarrow \mathbb{R}$ shows.

§4.3. Chain rule for Whitney cones

We begin with a “tangent analogue” of Proposition 2.11.

Proposition 4.9. *Let M_i be manifolds for $i \in \{1, 2, 3\}$. Let $A_1, B_1 \subseteq M_{12}$ and $A_2, B_2 \subseteq M_{23}$. Assume that*

- (1) p_{13} is injective and proper on $\overline{p_{12}^{-1}(A_1) \cap p_{23}^{-1}(A_2)} \cup \overline{p_{12}^{-1}(B_1) \cap p_{23}^{-1}(B_2)}$;
- (2) $(C(A_1, B_1) \times 0_3) \cap (0_1 \times C(A_2, B_2)) \subseteq 0_{123}$.

Then,

$$(4.5) \quad C(A_1 \circ A_2, B_1 \circ B_2) \subseteq C(A_1, B_1) \circ C(A_2, B_2).$$

The reverse inclusion holds if $A_2 = B_2$ is the graph of a C^1 -map f (without the above two assumptions); furthermore, if $M_1 = \{*\}$, it reads $Tf(C(A_1, B_1)) \subseteq C(f(A_1), f(B_1))$.

Proof. (1) Let $(u^1, u^3) \in C_{(x^1, x^3)}(A_1 \circ A_2, B_1 \circ B_2)$. There exist sequences $(x_n^1, x_n^3) \in A_1 \circ A_2$ and $(y_n^1, y_n^3) \in B_1 \circ B_2$ both converging to (x^1, x^3) , and a sequence $c_n > 0$ such that $c_n(y_n^1 - x_n^1, y_n^3 - x_n^3) \xrightarrow{n} (u^1, u^3)$. There exists a sequence $(x_n^2) \in M_2^\mathbb{N}$ such that $(x_n^1, x_n^2, x_n^3) \in p_{12}^{-1}(A_1) \cap p_{23}^{-1}(A_2)$, and similarly a sequence $(y_n^2) \in M_2^\mathbb{N}$

such that $(y_n^1, y_n^2, y_n^3) \in p_{12}^{-1}(B_1) \cap p_{23}^{-1}(B_2)$. Since p_{13} is injective and proper on $p_{12}^{-1}(A_1) \cap p_{23}^{-1}(A_2) \cup p_{12}^{-1}(B_1) \cap p_{23}^{-1}(B_2)$, one can extract subsequences $(x_{\rho(n)}^2)$ and $(y_{\rho(n)}^2)$ both converging to some $x^2 \in M_2$.

If the sequence $(c_n(y_n^2 - x_n^2))$ has a converging subsequence, let $u^2 \in T_{x^2}M_2$ be a limit of a converging subsequence. Then $(u^1, u^2) \in C_{(x^1, x^2)}(A_1, B_1)$ and $(u^2, u^3) \in C_{(x^2, x^3)}(A_2, B_2)$. If not, then $c_n \|y_n^2 - x_n^2\| \xrightarrow{n} +\infty$ (in some chart of M_2 at x^2). Let $d_n := \|y_n^2 - x_n^2\|^{-1}$. We extract a converging subsequence of $(d_n(y_n^2 - x_n^2))$ and call $u^2 \in T_{x^2}M_2$ its limit. One has $\frac{d_n}{c_n} \xrightarrow{n} 0$, so $d_n(y_n^i - x_n^i) = \frac{d_n}{c_n} c_n(y_n^i - x_n^i) \xrightarrow{n} 0$ for $i \in \{1, 3\}$. Therefore, the sequences (x_n^1, x_n^2) , (y_n^1, y_n^2) and d_n show that $(0, u^2) \in C_{(x^1, x^2)}(A_1, B_1)$ and the sequences (x_n^2, x_n^3) , (y_n^2, y_n^3) and d_n show that $(u^2, 0) \in C_{(x^2, x^3)}(A_2, B_2)$ with $u^2 \neq 0$. This contradicts the second assumption.

(2) Suppose that $A_2 = B_2 = \Gamma_f$ with f a C^1 -map, so that $C(A_2, B_2) = \Gamma_{Tf}$. Let $(u^1, u^3) \in C_{(x^1, x^2)}(A_1, B_1) \circ C_{(x^2, x^3)}(A_2, B_2)$. Therefore, there exists $u^2 \in T_{x^2}M_2$ such that $(u^1, u^2) \in C_{(x^1, x^2)}(A_1, B_1)$ and $T_{x^2}f(u^2) = u^3$. Therefore, there exist sequences $((x_n^1, x_n^2)) \in A_1^{\mathbb{N}}$ and $((y_n^1, y_n^2)) \in B_1^{\mathbb{N}}$ both converging to (x^1, x^2) , and $(c_n) \in (\mathbb{R}_{>0})^{\mathbb{N}}$ such that $c_n(y_n^1 - x_n^1, y_n^2 - x_n^2) \xrightarrow{n} (u^1, u^2)$. Set $x_n^3 := f(x_n^2)$ and $y_n^3 := f(y_n^2)$. The sequences $((x_n^1, x_n^3))$ and $((y_n^1, y_n^3))$ and (c_n) show that $(u^1, u^3) \in C_{(x^1, x^3)}(A_1 \circ \Gamma_f, B_1 \circ \Gamma_f)$. \square

Note that the first assumption of the proposition is satisfied if $A_1 = B_1$ is the graph of a continuous map f . Furthermore, if $M_3 = \{*\}$, then the second assumption means that f is Lipschitz for $C(A_2, B_2)$ (see the definition below) and the conclusion reads $C(f^{-1}(A_2), f^{-1}(B_2)) \subseteq C_f \circ C(A_2, B_2)$. The consideration of the second assumption motivates the following definition.

Definition 4.10. Let $f_i: M_i \rightarrow M_{i+1}$ for $i \in \{1, 2\}$ be continuous maps between manifolds. The pair (f_1, f_2) is *Whitney-regular* if

$$(4.6) \quad (C_{f_1} \times 0_3) \cap (0_1 \times C_{f_2}) \subseteq 0_{123}.$$

It will also be convenient to use the following definitions.

Definition 4.11. Let $f: M \rightarrow N$ be a continuous map between manifolds and $A \subseteq TM$ and $B \subseteq TN$ be closed cones. The map f is

- (1) *Whitney-immersive* if $C_f \cap (TM \times 0_N) \subseteq 0_{M \times N}$;
- (2) *Whitney-immersive for A* if $C_f \cap (A \times 0_N) \subseteq 0_{M \times N}$;
- (3) *Lipschitz for B* if $C_f \cap (0_M \times B) \subseteq 0_{M \times N}$.

By Proposition 4.3(1), a continuous map $f: M \rightarrow N$ is Lipschitz if and only if it is Lipschitz for TN . One can characterize Whitney immersions via a reversed Lipschitz inequality, using arguments similar to the proof of Proposition 4.3(1). A Whitney immersion is a topological immersion, and a C^1 -map is a Whitney immersion if and only if it is a C^1 -immersion. However, a topological immersion which is smooth need not be a Whitney immersion, as the function $\mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto x^3$ shows.

In view of Proposition 4.3(1), the next proposition is obvious.

Proposition 4.12. *Let $f_i: M_i \rightarrow M_{i+1}$ for $i \in \{1, 2\}$ be continuous maps between manifolds. If f_1 is Lipschitz or f_2 is a Whitney immersion, then the pair (f_1, f_2) is Whitney-regular.*

We can now prove the chain rule for Whitney cones.

Proposition 4.13 (Chain rule). *Let $f_i: M_i \rightarrow M_{i+1}$ for $i \in \{1, 2\}$ be continuous maps between manifolds. If the pair (f_1, f_2) is Whitney-regular, then*

$$(4.7) \quad C_{f_2 \circ f_1} \subseteq C_{f_1} \circ C_{f_2}.$$

The reverse inclusion holds if f_2 is C^1 (without assuming the pair Whitney-regular).

Proof. We apply Proposition 4.9 with $A_i = B_i = \Gamma_{f_i}$ for $i \in \{1, 2\}$. □

Example 4.14. Let $f_2: \mathbb{R} \rightarrow \mathbb{R}$, $t \mapsto t^3$ and $f_1 = f_2^{-1}$. Then f_2 is C^1 and $C_0(f_2 \circ f_1) \supsetneq C_0(f_1) \circ C_0(f_2)$. This shows that the hypothesis that (f_1, f_2) is Whitney-regular is needed. Similarly, (f_2, f_1) is Whitney-regular and $C_0(f_1 \circ f_2) \subsetneq C_0(f_2) \circ C_0(f_1)$. This shows that for the reverse inclusion, the hypothesis that f_2 is C^1 is needed.

Corollary 4.15. *The set of Whitney immersions is closed under composition.*

Proof. If $f_i: M_i \rightarrow M_{i+1}$ for $i \in \{1, 2\}$ are Whitney immersions, then $C_{f_2 \circ f_1} \circ 0_3 \subseteq C_{f_1} \circ C_{f_2} \circ 0_3 \subseteq C_{f_1} \circ 0_2 \subseteq 0_1$, so $f_2 \circ f_1$ is a Whitney immersion. □

We end this section with a characterization of Whitney immersions. By invariance of domain, a Whitney immersion between manifolds of the same dimension is a homeomorphism with Lipschitz inverse. If the dimensions of the domain and codomain differ, we proceed as follows. We say that a continuous map $f: M \rightarrow N$ between manifolds has Lipschitz local retractions if for any $x \in M$, there are open neighborhoods U of x and V of $f(x)$ such that $f(U) \subseteq V$ and a Lipschitz map $r: V \rightarrow U$ such that $r \circ f|_U = \text{id}_U$.

Proposition 4.16. *A continuous map between manifolds is a Whitney immersion if and only if it has Lipschitz local retractions.*

Proof. Let $f: M \rightarrow N$ be a Whitney immersion and let $x \in M$. A Whitney immersion is locally injective, so there is an open neighborhood U of x such that f is injective on U . One can shrink U so that $f(U)$ be included in a chart of N with domain W . The map $f|_U^{-1}$ is defined on $f(U) \subseteq W$, which need not be open, and is not known yet to be continuous. However, the definition of the Whitney cone and Proposition 4.3(1) still hold in this case. The cone $C_{f|_U^{-1}}$ is the image of $C_{f|_U}$ by the “flip” $T(U \times W) \rightarrow T(W \times U)$. Therefore, $C^0(f|_U^{-1}) \subseteq 0_{W \times U}$, so $f|_U^{-1}: f(U) \rightarrow U$ is Lipschitz. Therefore, there exists an open neighborhood V_1 of $f(x)$ such that $f|_U^{-1}$ is globally Lipschitz on $f(U) \cap V_1$, and we can assume $f(U) \subseteq V_1$. Therefore, we can extend $f|_U^{-1}$ to a Lipschitz map $r_1: V_1 \rightarrow M$ (see for instance [3, §2]). The set $V := r_1^{-1}(U)$ is open and contains $f(U)$. We set $r := r_1|_V$. One has $r \circ f|_U = \text{id}_U$. The converse is clear, since the relation $r \circ f|_U = \text{id}_U$ implies $C_{f|_U} \subseteq \text{flip}(C_r)$ (as seen by applying the definition of the Whitney cone). \square

§5. Conormals of maps

In this section, we define the conormal of a continuous map and give its first properties. We use the convolution of kernels to extend to continuous maps the functorial properties of the microsupport for the four image operations, and we prove a chain rule involving conormals.

§5.1. Definition and first properties

If $f: M \rightarrow N$ is a continuous map between manifolds, we set $\text{Ppg}(f) := \text{Ppg}(\Gamma_f)$ and

$$(5.1) \quad \Lambda_f := \mu\text{supp}(\Gamma_f)$$

and also $\text{Ppg}_x(f) := \text{Ppg}_{(x, f(x))}(\Gamma_f)$ and $\Lambda_x(f) := \mu\text{supp}_{(x, f(x))}(\Gamma_f)$ for $x \in M$. We call Λ_f the *conormal of f* . This definition is consistent with Equation (2.8), as Proposition 3.2 shows.

Proposition 5.1. *The conormal of $f: M \rightarrow N$ is a coisotropic closed symmetric cone in $T^*(M \times N)$ satisfying $\pi_{M \times N}(\Lambda_f) = \Gamma_f$.*

Proof. A microsupport is always a coisotropic closed cone. Since Γ_f is closed, one has $\pi_{M \times N}(\Lambda_f) = \Gamma_f$ by Equation (3.2). The graph of a continuous map is a closed C^0 -submanifold, so its conormal is symmetric by Proposition 3.3. \square

Remark 5.2. The conormal of a map need not be a C^0 -submanifold of half dimension (that is, a C^0 -Lagrangian), as the following two examples show. There is a 1-Lipschitz function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $\Lambda_f = \Gamma_f \times \{(\xi, \eta) \in \mathbb{R}^2 \mid |\eta| \leq |\xi|\}$. In particular, $\dot{\Lambda}_f$ is a Lipschitz submanifold with boundary of dimension 3. There is a continuous function $g: \mathbb{R} \rightarrow \mathbb{R}$ such that $\Lambda_g = \Gamma_g \times \mathbb{R}^2$. In particular, Λ_g is a C^0 -submanifold of dimension 3. The idea of the following constructions is taken from a talk by David Preiss. Let $(U_i)_{i \in \mathbb{N}}$ be a decreasing sequence of open subsets of \mathbb{R} with $\bigcap_i U_i = \mathbb{Q}$ (or any dense subset of \mathbb{R} of measure 0) such that each U_{n+1} has at most half measure in each connected component of U_n . For $x \in \mathbb{R}$, let $\psi(x)$ be the largest index n such that $x \in U_n$ and 0 if $x \notin \bigcup_i U_i$ or $x \in \mathbb{Q}$. Let $f(x) := \int_0^x (-1)^{\psi(x)} dx$ and $g(x) := \int_0^x (-3/2)^{\psi(x)} dx$. Then f is 1-Lipschitz, g is continuous and their conormals are as claimed by the case of equality of the upper bound on the conormal (Theorem 7.9).

The following lemma gives a useful criterion for belonging to the propagation set of a sheaf associated with a continuous map.

Lemma 5.3. *Let $f: M \rightarrow N$ be a continuous map between manifolds and let $x \in M$. Then $\nu \in \text{Ppg}_x(f)$ if and only if for all $\phi \in C^\infty(W)$, with W an open neighborhood of $(x, f(x))$, such that $\phi(x, f(x)) = 0$ and $d\phi(x, f(x)) = \nu$, the morphism*

$$(5.2) \quad \mathbb{k} \longrightarrow (\text{R}\Gamma_{\Gamma_f \cap \{\phi < 0\}} \mathbb{k}_{\Gamma_f})_{(x, f(x))}$$

induced by the topological embeddings $(p_M \circ i_{W'}: \Gamma_f \cap \{\phi < 0\} \cap W' \hookrightarrow p_M(W'))_{W' \ni (x, f(x))}$ is an isomorphism. This implies in particular that the germ of $\Gamma_f \cap \{\phi < 0\}$ at $(x, f(x))$ has the cohomology of a point.

Proof. We apply Lemma 3.1 with $Z = \Gamma_f$. The result follows since the isomorphism $p_M|_{\Gamma_f}: \Gamma_f \rightarrow M$ induces an isomorphism $\mathbb{k} = (\mathbb{k}_M)_x \xrightarrow{\sim} (\mathbb{k}_{\Gamma_f})_{(x, f(x))}$. \square

Remark 5.4. In most of the proofs below, we use only the fact that having the cohomology of a point implies being nonempty. So in a sense, we use only crude estimates, which shows the efficiency of sheaf theoretical methods.

We will need the following result in later sections. It shows that adding a strictly differentiable map to a continuous map shears its tangent cone and its conormal.

Lemma 5.5 (Shearing lemma). *Let $f, g: M \rightarrow \mathbb{R}^n$ be continuous maps on a manifold with g strictly differentiable at $x \in M$. Then,*

$$(5.3) \quad C_x(f+g) = (p_M, p_N + g'(x) \circ p_M)(C_x(f))$$

$$(5.4) \quad \text{and } \Lambda_x(f+g) = (p_M - g'(x)^\top \circ p_N, p_N)(\Lambda_x(f)).$$

Proof. (i) The case of the Whitney cone is straightforward (directly or using Proposition B.1(8)).

(ii) Fix a chart of M at x . We introduce the topological automorphism $\Phi: \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n}$, $(x, y) \mapsto \Phi(x, y) = (x, y - g(x))$. Then $\Phi(\Gamma_f) = \Gamma_{f+g}$. Since Φ is strictly differentiable on $\{x\} \times \mathbb{R}^n$ as well as its inverse, it follows from Remark 2.10 that $\Lambda_x(f) = \Lambda_{(x, f(x))}(\Phi) \circ \Lambda_x(f+g)$ and a simple computation completes the proof. \square

§5.2. Chain rule for conormals

In this subsection, we will apply Proposition 2.11 to kernels associated with maps, in order to derive upper bounds on microsupports of direct and inverse images of sheaves, and then on the conormal of a composite map.

By analogy with the smooth case (cf. Equations (2.11) and (2.12)), we make the following definitions (compare with Definition 4.11).

Definition 5.6. Let $f: M \rightarrow N$ be a continuous map between manifolds, and $A \subseteq T^*M$ and $B \subseteq T^*N$ be closed cones. The map f is

- (1) a *microlocal submersion* if $\Lambda_f \cap (0_M^* \times T^*N) \subseteq 0_{M \times N}^*$;
- (2) *noncharacteristic for B* if $\Lambda_f \cap (0_M^* \times B) \subseteq 0_{M \times N}^*$;
- (3) *Lipschitz for A* if $\Lambda_f \cap (A \times 0_N^*) \subseteq 0_{M \times N}^*$.

A C^1 -map is obviously Lipschitz for T^*M . We will see in Theorem 7.8 that “Lipschitz for T^*M ” is equivalent to “Lipschitz”.

Example 5.7. For $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, $x \mapsto (x_1)^{1/3}$, one has $\Lambda_x(f) = \mathbb{R}(-1, 0, 3|x_1|^{2/3})$, so f is Lipschitz for the constant cone $\mathbb{R}^2 \times \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid |\xi_1| \leq C|\xi_2|\}$ for any $C \in \mathbb{R}$.

We first prove the following generalization of Proposition 2.8(2) to continuous maps. A continuous map is said to be Lipschitz (resp. noncharacteristic) for a sheaf if it is Lipschitz (resp. noncharacteristic) for its microsupport.

Proposition 5.8. *Let $f: M \rightarrow N$ be a continuous map between manifolds.*

- (1) Let $F \in D^b(\mathbb{k}_M)$ and assume that f is Lipschitz for F and proper on $\text{supp}(F)$. Then $\mu\text{supp}(Rf_*F) \subseteq \mu\text{supp}(F) \overset{a}{\circ} \Lambda_f$.
- (2) Let $G \in D^b(\mathbb{k}_N)$ and assume that f is noncharacteristic for G . Then the morphism $f^{-1}G \otimes^L \omega_{M/N} \rightarrow f^!G$ is an isomorphism and $\mu\text{supp}(f^{-1}G) \subseteq \Lambda_f \overset{a}{\circ} \mu\text{supp}(G)$.

*Proof.*¹ The bounds on the microsupport are straightforward applications of Proposition 2.11 (using Proposition 2.2). As for the morphism $f^{-1}G \otimes^L \omega_{M/N} \rightarrow f^!G$, we first note that it is functorial in f . Therefore, the lemma following this proof shows that it is enough to prove that it is an isomorphism when f is a closed embedding and when f is a submersion. Indeed, we use the decomposition $f = p_N \circ (\text{id}_M, f)$, that is, the inclusion in the graph of f (which is a closed embedding) followed by the projection on the second factor (which is a smooth submersion). The submersion case is treated in [6, Prop. 3.3.2].

Suppose that f is a closed embedding of M into N . For a conic sheaf $F \in D^b(\mathbb{k}_{T^*N})$, one has the Sato distinguished triangle $R\pi_!F \rightarrow R\pi_*F \rightarrow R\hat{\pi}_*F \xrightarrow{+1}$. When $F = \mu\text{hom}(\mathbb{k}_M, G)$ (with $\mathbb{k}_M = \mathbb{k}_{N,M} \in D^b(\mathbb{k}_N)$), one has $R\pi_! \mu\text{hom}(\mathbb{k}_M, G) = R\mathcal{H}om(\mathbb{k}_M, \mathbb{k}_N) \otimes^L G$ and $R\pi_* \mu\text{hom}(\mathbb{k}_M, G) = R\mathcal{H}om(\mathbb{k}_M, G) = R\Gamma_M G$ (by [6, Prop. 4.4.2(i)]). Noncharacteristicness says that $\mu\text{supp}(G) \cap \mu\text{supp}(\mathbb{k}_M) \subseteq 0_N^*$. By [6, Cor. 5.4.10(ii)], one has $\text{supp}(\mu\text{hom}(\mathbb{k}_M, G)) \subseteq \mu\text{supp}(G) \cap \mu\text{supp}(\mathbb{k}_M) \subseteq 0_N^*$, so $R\hat{\pi}_* \mu\text{hom}(\mathbb{k}_M, G) = 0$. Therefore, $R\Gamma_M G \simeq R\mathcal{H}om(\mathbb{k}_M, \mathbb{k}_N) \otimes^L G$.

Since f is a closed embedding, one has $f^!G \simeq f^{-1}(R\Gamma_M G)$ by [6, Prop. 3.1.12]. Applying this formula to \mathbb{k}_N , we also have $\omega_{M/N} = f^! \mathbb{k}_N \simeq f^{-1}(R\mathcal{H}om(\mathbb{k}_M, \mathbb{k}_N))$. Therefore, $f^!G \simeq f^{-1}(R\Gamma_M G) \simeq f^{-1}(R\mathcal{H}om(\mathbb{k}_M, \mathbb{k}_N) \otimes^L G) \simeq \omega_{M/N} \otimes^L f^{-1}G$. □

Lemma 5.9. *Let $f: M \rightarrow N$ be a continuous map between manifolds. Let $s: T^*M \times_M T^*M \rightarrow T^*M$ be the fiberwise subtraction. Then $\Lambda_{(\text{id}_M, f)} \subseteq (s \times \text{id}_{T^*N})^{-1}(\Lambda_f)$. In particular, if f is noncharacteristic for $G \in D^b(\mathbb{k}_N)$, then the closed embedding (id_M, f) is noncharacteristic for $p_N^{-1}(G)$.*

Proof. We set $\hat{f} := (\text{id}_M, f)$ and $P := \Delta_M \times N$. The latter is a closed submanifold of $M \times M \times N$. One has $\Gamma_{\hat{f}} = P \cap p_{MN}^{-1}(\Gamma_f)$, so $K_{\hat{f}} = (p_{MN}^{-1}K_f)_P$. One has $\mu\text{supp}(p_{MN}^{-1}K_f) = \Lambda_{p_{MN}} \overset{a}{\circ} \Lambda_f = 0_M^* \times \Lambda_f$ and $T_P^*(M \times M \times N) = \{(\xi, -\xi, 0) \mid \xi \in T^*M\}$. Therefore, $\mu\text{supp}(p_{MN}^{-1}K_f) \cap T_P^*(M \times M \times N) \subseteq 0_{MMN}^*$, so by [6, Cor. 5.4.11(i)], one has $\Lambda_{\hat{f}} \subseteq \mu\text{supp}(p_{MN}^{-1}K_f) + T_P^*(M \times M \times N) = (s \times \text{id}_{T^*N})^{-1}(\Lambda_f)$.

¹This proof was obtained jointly with Pierre Schapira.

As for the second claim, let $G \in D^b(\mathbb{k}_N)$. Then $\mu\text{supp}(p_N^{-1}G) = \Lambda_{p_N} \circ \mu\text{supp}(G) = 0_M^* \times \mu\text{supp}(G)$. Therefore, $\Lambda_f \cap (0_M^* \times \mu\text{supp}(p_N^{-1}G)) \subseteq (s \times \text{id}_{T^*N})^{-1}(\Lambda_f) \cap (0_{MM}^* \times \mu\text{supp}(G)) \subseteq 0_M^* \times (\Lambda_f \cap (0_M^* \times \mu\text{supp}(G)))$. \square

As an application, let $f: M \rightarrow N$ be a closed topological embedding between manifolds. If f is a closed C^1 -embedding, we know that $T_{f(M)}^*N = 0_M^* \circ \Lambda_f$. In the continuous case, we have $Rf_*\mathbb{k}_M \simeq \mathbb{k}_{f(M)}$, so the proposition tells us that $\mu\text{supp}(f(M)) \subseteq 0_M^* \circ \Lambda_f$.

Now, we will apply Proposition 2.11 when both kernels are associated with maps, in order to derive an upper bound on the conormal of a composite map. This constitutes an analogue for continuous maps of the usual chain rule for differentiable maps. The hypotheses of Proposition 2.11 applied to $K_i := K_{f_i}$ (notation introduced in Equation (2.28)) with $f_i: M_i \rightarrow M_{i+1}$ continuous maps between manifolds for $i \in \{1, 2\}$, read

- (1) p_{13} is proper on $p_{12}^{-1}(\Gamma_{f_1}) \cap p_{23}^{-1}(\Gamma_{f_2})$;
- (2) $(\Lambda_{f_1} \times 0_3^*) \cap (0_1^* \times \Lambda_{f_2}) \subseteq 0_{123}^*$

(one can remove the antipodal map from the second condition since conormals of maps are symmetric). Since f_1 is continuous, the first hypothesis is satisfied. This motivates the following definition.

Definition 5.10. Let $f_i: M_i \rightarrow M_{i+1}$ for $i \in \{1, 2\}$ be continuous maps between manifolds. The pair (f_1, f_2) is *regular* if

$$(5.5) \quad (\Lambda_{f_1} \times 0_3^*) \cap (0_1^* \times \Lambda_{f_2}) \subseteq 0_{123}^*.$$

It is convenient to define

$$(5.6) \quad \Lambda_x^\eta(f) := \Lambda_x(f) \cap (T_x^*M \times \mathbb{R}_{\geq 0}\eta)$$

for $(x, \eta) \in M \times_N T^*N$ and

$$(5.7) \quad \Lambda^0(f) := \Lambda_f \cap (T^*M \times 0_N^*).$$

Therefore, a map is Lipschitz for T^*M if and only if

$$(5.8) \quad \Lambda^0(f) \subseteq 0_{M \times N}^*.$$

Example 5.11. The function $f: \mathbb{R} \rightarrow \mathbb{R}, t \mapsto t^3$ is a topological automorphism which is smooth but is not microlocally submersive and its inverse is not Lipschitz for $T^*\mathbb{R}$, and the pair (f, f^{-1}) is not regular. In particular, a topological submersion need not be a microlocal submersion, even if it is smooth.

The next proposition is obvious.

Proposition 5.12. *Let $f_i: M_i \rightarrow M_{i+1}$ for $i \in \{1, 2\}$ be continuous maps between manifolds. If f_1 is microlocally submersive or if f_2 is Lipschitz for T^*M_3 , then the pair (f_1, f_2) is regular.*

We can now prove the chain rule for conormals.

Proposition 5.13 (Chain rule). *Let $f_i: M_i \rightarrow M_{i+1}$ for $i \in \{1, 2\}$ be continuous maps between manifolds. If the pair (f_1, f_2) is regular, then*

$$(5.9) \quad \Lambda_{f_2 \circ f_1} \subseteq \Lambda_{f_1} \overset{a}{\circ} \Lambda_{f_2}$$

with equality if f_1 is a C^1 -submersion (and if f_1 and f_2 are both C^1).

Proof. As we have seen above, both hypotheses of Proposition 2.11 are satisfied when the pair (f_1, f_2) is regular, so the inclusion follows from Proposition 2.3.

If f_1 is a C^1 -submersion, then $\tilde{f}_1 := f_1 \times \text{id}_3: M_{13} \rightarrow M_{23}$ is a C^1 -submersion and $\Gamma_{f_2 \circ f_1} = \tilde{f}_1^{-1}(\Gamma_{f_2})$. By Equation (3.4) (case of equality), one has $\Lambda_{f_2 \circ f_1} = \mu\text{supp}(\Gamma_{f_2 \circ f_1}) = \mu\text{supp}(\tilde{f}_1^{-1}(\Gamma_{f_2})) = \Lambda_{\tilde{f}_1} \overset{a}{\circ} \mu\text{supp}(\Gamma_{f_2}) = \Lambda_{\tilde{f}_1} \overset{a}{\circ} \Lambda_{f_2}$. The result follows since for any $B \subseteq T^*M_{23}$, one has $\Lambda_{\tilde{f}_1} \overset{a}{\circ} B = \Lambda_{f_1} \overset{a}{\circ} B$ (since f_1 is C^1). \square

Example 5.14. The inclusion in the chain rule may be strict: take $f_2: \mathbb{R} \rightarrow \mathbb{R}$, $t \mapsto t^3$ and $f_1 = f_2^{-1}$.

If the pair (f_1, f_2) is not regular, then the chain rule need not hold. For example, take $f_1: \mathbb{R} \rightarrow \mathbb{R}$, $t \mapsto t^3$ and $f_2 = f_1^{-1}$.

Corollary 5.15. *The set of microlocal submersions is closed under composition.*

Proof. If $f_i: M_i \rightarrow M_{i+1}$ for $i \in \{1, 2\}$ are microlocal submersions, then $0_1^* \circ \Lambda_{f_2 \circ f_1} \subseteq 0_1^* \circ \Lambda_{f_1} \overset{a}{\circ} \Lambda_{f_2} \subseteq 0_2^* \circ \Lambda_{f_2} \subseteq 0_3^*$, so $f_2 \circ f_1$ is a microlocal submersion. \square

We end this section with a result giving sufficient conditions for the direct and inverse images of closed cones in a (co)tangent bundle to be closed.

Proposition 5.16. *Let $f: M \rightarrow N$ be a continuous map between manifolds.*

- (1) *If f is Lipschitz for the closed cone $B \subseteq TN$, then $C_f \circ B$ is closed in TM .*
- (2) *If f is noncharacteristic for the closed cone $B \subseteq T^*N$, then $\Lambda_f \circ B$ is closed in T^*M .*
- (3) *If f is Whitney-immersive for the closed cone $A \subseteq TM$ and proper on $\tau_M(A)$, then $A \circ C_f$ is closed in TN .*

- (4) If f is Lipschitz for the closed cone $A \subseteq T^*M$ and proper on $\pi_M(A)$, then $A \circ \Lambda_f$ is closed in T^*N .

In the smooth case, the proof of item (2) is sketched in the paragraph following the definition of noncharacteristic morphisms in [6, Def. 5.4.12].

Proof. The four claims have similar proofs, so we give a proof of the second one only. Let $K \subseteq M \times N$ be compact and contained in a chart. Since $\Lambda_f \cap (0_M^* \times B) \subseteq 0_{M \times N}^*$ and Λ_f is closed, there exists $\alpha_K > 0$ such that $\pi_{M \times N}^{-1}(K) \cap \Lambda_f \cap (T^*M \times B) \subseteq \{(\xi, \eta) \in T^*(M \times N) \mid \|\xi\| \geq \alpha_K \|\eta\|\}$. Therefore, p_M is proper on $\pi_{M \times N}^{-1}(K) \cap \Lambda_f \cap (T^*M \times B)$. This implies that $p_M(\pi_{M \times N}^{-1}(K)) \cap (\Lambda_f \circ B)$ is closed. The requirement that K be contained in a chart can be dropped. Note that $p_M \circ \pi_{M \times N}^{-1} = \pi_M^{-1} \circ p_M$.

Let $x \in M$. Let K_N be a compact neighborhood of $f(x)$. There exists a compact neighborhood K_M of x included in $f^{-1}(K_N)$. Set $U := p_M(\pi_{M \times N}^{-1}(K_M \times K_N)) = \pi_M^{-1}(K_M)$. Then $U \cap (\Lambda_f \circ B)$ is closed. Therefore, any point $(x, \xi) \in T^*M$ has a neighborhood U such that $U \cap (\Lambda_f \circ B)$ is closed. This implies that $\Lambda_f \circ B$ is closed. \square

Remark 5.17. With the notation of Definition 5.10, if $x_1 \in M_1$, one says that the pair (f_1, f_2) is regular at x_1 if $(\Lambda_{x_1}(f_1) \times \{0\}) \cap (\{0\} \times \Lambda_{f(x_1)}(f_2)) = \{0\}$. If (f_1, f_2) is regular at x_1 , then it is so in a neighborhood of x_1 , since conormals are closed cones. The same remark applies for the notions of a Whitney-regular pair and of microlocally submersive, noncharacteristic, Whitney immersive and Lipschitz maps.

As for a characterization of microlocal submersions, in view of Proposition 4.16 a reasonable conjecture is that the microlocal submersions are the continuous maps which have Lipschitz local sections. Here is a result in that direction.²

Proposition 5.18. *A microlocal submersion with 1-dimensional codomain is an open map.*

Proof. Let $f: M \rightarrow \mathbb{R}$ be a microlocal submersion and let $x_0 \in M$. We can suppose that M is open in \mathbb{R}^m . Define $\phi: M \times \mathbb{R} \rightarrow \mathbb{R}$, $(x, y) \mapsto y - f(x_0)$. Then $\phi(x_0, f(x_0)) = 0$ and $d\phi(x_0, f(x_0)) = (0, 1) \in \text{Ppg}_{x_0}(f)$. Therefore, by Lemma 5.3, for any neighborhood U of $(x_0, f(x_0))$, one has $U \cap \Gamma_f \cap \{\phi < 0\} \neq \emptyset$. This means that arbitrarily close to x_0 , there are points x with $f(x) < f(x_0)$, and similarly,

²Pierre Schapira showed me how an adaptation of the proof of the microlocal Bertini–Sard theorem [6, Prop. 8.3.12] shows that subanalytic microlocal submersions are open maps.

points y with $f(y) > f(x_0)$. Since M is locally connected, this implies that f is open. \square

§6. Real-valued functions

In this section, we study more precisely the case of real-valued functions. We introduce directional Dini derivatives, which permit us to give precise descriptions of the Whitney cones related to the graph and epigraph of a function. Then we study extrema of real-valued functions and prove an analogue of Fermat’s lemma for continuous functions. In the third subsection, we relate the conormal of a continuous function to the microsupport of the constant sheaf on its epigraph.

§6.1. Directional Dini derivatives

If \mathbb{V} is a vector space, $f: \mathbb{V} \rightarrow \mathbb{R}$ and $(x, u) \in T\mathbb{V}$, we define the *supremal derivative* and *supremal quotient* of f at (x, u) respectively by

$$(6.1) \quad \overline{D}f(x, u) := \limsup_{\substack{t \rightarrow 0^+ \\ v \rightarrow u}} \frac{f(x + tv) - f(x)}{t}$$

$$(6.2) \quad \text{and } \overline{Q}f(x, u) := \limsup_{\substack{t \rightarrow 0^+ \\ (y,v) \rightarrow (x,u)}} \frac{f(y + tv) - f(y)}{t}.$$

One has $\overline{Q}f = \limsup \overline{D}f: T\mathbb{V} \rightarrow \overline{\mathbb{R}}$. The functions $\overline{D}f(x, -)$ and $\overline{Q}f(x, -)$ are $\mathbb{R}_{>0}$ -homogeneous on $T_x\mathbb{V}$ for any $x \in \mathbb{V}$. The functions $\underline{D}f$ and $\underline{Q}f$ are defined similarly with \liminf . One has $\overline{Q}f(x, -u) = -\underline{Q}f(x, u)$ for any $(x, u) \in T\mathbb{V}$. If $\mathbb{V} = \mathbb{R}$, we recover the usual Dini derivatives: for instance, $\overline{D}f(x, 1) = D^+f(x)$ and $\overline{D}f(x, -1) = -D_-f(x)$.

Once a norm is fixed in \mathbb{V} , one has (with the notation introduced in Definition A.1; the maxima are in $\overline{\mathbb{R}}$)

$$(6.3) \quad \text{Lip}^{\text{pw}} f(x) = \max_{\substack{u \in \mathbb{V} \\ \|u\|=1}} (|\overline{D}f(x, u)|, |\underline{D}f(x, u)|)$$

$$(6.4) \quad \text{and } \text{Lip } f(x) = \max_{\substack{u \in \mathbb{V} \\ \|u\|=1}} |\overline{Q}f(x, u)|.$$

The following proposition, which relates the Whitney cone to the directional Dini derivatives, will be needed in the proof of the upper bound on the conormal of a map.

Proposition 6.1. *Let \mathbb{V} be a vector space and $f: \mathbb{V} \rightarrow \mathbb{R}$. For any $x \in \mathbb{V}$, one has*

$$(6.5) \quad C_x(\Gamma_f) = \left\{ (u, t) \in T_x\mathbb{V} \times \mathbb{R} \mid \underline{D}f(x, u) \leq t \leq \overline{D}f(x, u) \right\},$$

$$(6.6) \quad C_x(f) = \left\{ (u, t) \in T_x\mathbb{V} \times \mathbb{R} \mid \underline{Q}f(x, u) \leq t \leq \overline{Q}f(x, u) \right\}$$

$$(6.7) \quad \text{and } N_x(\Gamma_f^+) = \left\{ (u, t) \in T_x\mathbb{V} \times \mathbb{R} \mid t > \overline{Q}f(x, u) \right\}.$$

Proof. We treat the case of extremal quotients, the case of extremal derivatives being similar.

(i) Inclusion “ \subseteq ”. Let $(u, t) \in C_x(f)$. There are sequences $x_n, y_n \xrightarrow{n} x$ and $c_n > 0$ such that $c_n(y_n - x_n, f(y_n) - f(x_n)) \xrightarrow{n} (u, t)$. Since $y_n = x_n + \frac{1}{c_n}(c_n(y_n - x_n))$, one has $\overline{Q}f(x, u) \geq \liminf_n c_n(f(y_n) - f(x_n)) = t$. One proves similarly that $\underline{Q}f(x, u) \leq t$.

(ii) Inclusion “ \supseteq ”. If $u \in T_x\mathbb{V}$ and $t \in [\underline{Q}f(x, u), \overline{Q}f(x, u)] \cap \mathbb{R}$, then by definition of the lim inf and lim sup, for any $\epsilon > 0$, there exist sequences $y_n, z_n \xrightarrow{n} x$ and $v_n, w_n \xrightarrow{n} u$ and $a_n, b_n \xrightarrow{n} 0^+$ such that

$$(6.8) \quad \frac{f(y_n + a_n v_n) - f(y_n)}{a_n} - \epsilon \leq t \leq \frac{f(z_n + b_n w_n) - f(z_n)}{b_n} + \epsilon$$

for all $n \in \mathbb{N}$. By the intermediate value theorem, there exists $t_n \in [0, 1]$ such that for $(x_n, d_n, u_n) := (1 - t_n)(y_n, a_n, v_n) + t_n(z_n, b_n, w_n)$, one has

$$(6.9) \quad \left| \frac{f(x_n + d_n u_n) - f(x_n)}{d_n} - t \right| \leq \epsilon$$

for all $n \in \mathbb{N}$. Then, with $c_n := d_n^{-1}$ and $s_n := x_n + d_n u_n$, one has $x_n, s_n \xrightarrow{n} x$ and $c_n > 0$ and $c_n(s_n - x_n) = u_n \xrightarrow{n} u$ and $\limsup_n |c_n(f(s_n) - f(x_n)) - t| \leq \epsilon$. In particular, there exists $a \in [-\epsilon, \epsilon]$ such that $(u, t + a) \in C_x(f)$. Since this is true for any $\epsilon > 0$, the result follows from the closedness of $C_x(f)$.

(iii) The proof for $N_x(\Gamma_f^+)$ is similar. □

Remark 6.2. In particular, if f is Lipschitz at x , then $\overline{Q}f(x, 0) = \underline{Q}f(x, 0) = 0$, else $\overline{Q}f(x, 0) = -\underline{Q}f(x, 0) = +\infty$.

Corollary 6.3. *Let $f: M \rightarrow \mathbb{R}$ be a function on a manifold. For any $x \in M$, one has a partition*

$$(6.10) \quad C_x(f) \sqcup N_x(\Gamma_f^+) \sqcup N_x(\Gamma_f^-) = T_x M \times \mathbb{R}.$$

Proof. It follows easily from the proposition and Proposition B.3. □

§6.2. First-order extrema

Since Whitney cones and microsupports depend only on the C^1 -structure of a manifold, it is natural to introduce the notion of a first-order extremum. If \mathbb{V} is a normed vector space, $f: \mathbb{V} \rightarrow \mathbb{R}$ and $x \in \mathbb{V}$, we define

$$(6.11) \quad \underline{d}f(x) := \liminf_{v \rightarrow 0} \frac{f(x+v) - f(x)}{\|v\|},$$

$$(6.12) \quad \bar{d}f(x) := \limsup_{v \rightarrow 0} \frac{f(x+v) - f(x)}{\|v\|}.$$

Let M be a manifold. If $f: M \rightarrow \mathbb{R}$ and $x \in M$ and $\|-\|$ is a norm on $T_x M$, then $\liminf_{v \rightarrow 0} \frac{1}{\|v\|} (f(\phi^{-1}(\phi(x) + \phi'(x)v)) - f(x))$ does not depend on the chart ϕ at x . Therefore, when no norm is specified, the extended reals $\underline{d}f(x)$ and $\bar{d}f(x)$ are well defined up to multiplication by a strictly positive real number. Therefore, properties like “ $\underline{d}f(x) > 0$ ” still make sense.

Using implicitly the canonical identifications $T(M \times \mathbb{R}) \simeq TM \times T\mathbb{R}$ and $T\mathbb{R} \simeq \mathbb{R} \times \mathbb{R}$, we define

$$(6.13) \quad T_{\geq 0}(M \times \mathbb{R}) := TM \times (\mathbb{R} \times \mathbb{R}_{\geq 0})$$

and similarly for the cotangent bundle, for $T_{=0}(M \times \mathbb{R})$, etc.

Definition 6.4 (First-order extremum). Let $f: M \rightarrow \mathbb{R}$ be a function on a manifold. A point $x \in M$ is a *first-order minimum* (or f-o minimum) of f if $\underline{d}f(x) \geq 0$. It is an f-o maximum if $\bar{d}f(x) \leq 0$, and an f-o extremum if it is either an f-o minimum or an f-o maximum.

A local minimum (resp. maximum, extremum) is obviously an f-o minimum (resp. maximum, extremum). A point which is both an f-o minimum and an f-o maximum is a *stationary point*: the function is differentiable at that point with derivative zero. We have the following characterization of f-o extrema.

Proposition 6.5. *Let M be a manifold, $x \in M$ and $f: M \rightarrow \mathbb{R}$ be a function. Then the following are equivalent:*

- (1) x is an f-o minimum of f ;
- (2) $C_x(\Gamma_f) \subseteq T_{\geq 0}(M \times \mathbb{R})$;
- (3) $(0, 1) \in C_x(\Gamma_f)^\circ$;
- (4) *there exist an open neighborhood U of x and a function $\psi \in C^1(U)$ such that $\psi(x) = f(x)$ and $d\psi(x) = 0$ and $\psi \leq f|_U$.*

Proof. (4) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) are obvious.

(3) \Rightarrow (4). Since $(0, 1) \in C_x(\Gamma_f)^\circ$, Lemma 3.6 gives an open neighborhood W of $(x, f(x))$ and $\phi \in C^1(W)$ with $\phi(x, f(x)) = 0$ and $d\phi(x, f(x)) = (0, 1)$ and $\phi(\Gamma_f \cap W) \subseteq \mathbb{R}_{\geq 0}$. The implicit function theorem gives an open neighborhood $U \times V \subseteq W$ of $(x, f(x))$ and $\psi \in C^1(U)$ with $\psi(x) = 0$ and $d\psi(x) = 0$ such that for $(x, y) \in U \times V$, $\phi(x, y) \geq 0$ is equivalent to $y \geq \psi(x)$. Therefore, $\psi \leq f|_U$. \square

We obtain the following corollary of independent interest.

Corollary 6.6. *Let M be a manifold and \mathbb{W} be a normed vector space. Let $x \in M$ and $f: M \rightarrow \mathbb{W}$ be a function differentiable at x with $df(x) = 0$. Then there exist an open neighborhood U of x and a function $\psi \in C^1(U)$ with $\psi(x) = 0$ and $d\psi(x) = 0$ such that $\|f|_U - f(x)\|_{\mathbb{W}} \leq \psi$ (that is, for any $y \in U$, one has $\|f(y) - f(x)\|_{\mathbb{W}} \leq \psi(y)$).*

Proof. The hypothesis $df(x) = 0$ implies that x is an f-o maximum of $\|f - f(x)\|_{\mathbb{W}}$ and the result follows from Proposition 6.5. \square

Remark 6.7. One cannot strengthen the conclusion of this corollary nor of Proposition 6.5(4) to $\psi \in C^2(U)$, as the example of Remark 3.8 shows.

As in standard calculus, the following Fermat lemma will be used to prove Rolle's lemma and the mean value theorem for continuous functions.

Proposition 6.8 (Fermat lemma). *Let M be a manifold and $f: M \rightarrow \mathbb{R}$ be a continuous function. If f has an f-o extremum at $x \in M$, then*

$$(6.14) \quad T_x M \times \{0\} \subseteq C_x(f)$$

$$(6.15) \quad \text{and } \{0\} \times \mathbb{R} \subseteq \Lambda_x(f).$$

Proof. (i) Whitney cone. We can suppose that x is an f-o minimum. First, suppose that $M = \mathbb{R}$ and $x = 0$. Let $\epsilon > 0$ and set $f_\epsilon(t) := f(t) + \epsilon|t|$. Then 0 is a local strict minimum of f_ϵ . We can suppose that it is a global strict minimum.

If $f_\epsilon(-1) \leq f_\epsilon(1)$, then set $x_0 := -1$ and $y_0 := \min\{z \in]0, 1] \mid f_\epsilon(z) = f_\epsilon(x_0)\}$ (which exists by the intermediate value theorem and is a minimum by continuity of f_ϵ). If $f_\epsilon(-1) > f_\epsilon(1)$, then set $y_0 := 1$ and $x_0 := \max\{z \in [-1, 0[\mid f_\epsilon(z) = f_\epsilon(y_0)\}$. Use the same procedure with the points ± 1 replaced with the points $\pm \min(-x_0, y_0)/2$, to construct (x_1, y_1) . This way, one constructs sequences $x_n, y_n \xrightarrow{n} 0$ that show that $(1, 0) \in C_0(f_\epsilon)$, so there exists $\alpha_\epsilon \in [-\epsilon, \epsilon]$ such that $(1, \alpha_\epsilon) \in C_0(f)$. Since this is true for any $\epsilon > 0$, the closedness of $C_0(f)$ implies that $\mathbb{R} \times \{0\} \subseteq C_0(f)$.

In the general case, suppose that M is open in \mathbb{R}^m . Let $u \in \dot{T}_x M$ and set $\gamma:]-\alpha, \alpha[\rightarrow M$, $t \mapsto x + tu$. By the previous paragraph, $\mathbb{R} \times \{0\} \subseteq C_0(f \circ \gamma)$,

and by the chain rule for Whitney cones (γ being Lipschitz), one has $C_0(f \circ \gamma) \subseteq C_0(\gamma) \circ C_x(f)$. But $C_0(\gamma) = \mathbb{R}(1, u)$, so $(u, 0) \in C_x(f)$.

(ii) Conormal. If x is an f-o minimum of f , then $(0, 1) \in C_x(\Gamma_f)^\circ$ by Proposition 6.5, so the result follows from Proposition 3.4. If x is an f-o maximum of f , then it is an f-o minimum of $-f$, so by the above, one has $\{0\} \times \mathbb{R} \subseteq \Lambda_x(-f)$. Then the result follows from the fact that Λ_f and Λ_{-f} are mapped onto each other by the involutive automorphism of $T^*(M \times \mathbb{R})$ given by $(x, t; u, \tau) \mapsto (x, -t; u, -\tau)$. \square

§6.3. Microsupports of epigraphs

In [7], Vichery studied the microsupport of the constant sheaf on the epigraph of a real-valued function, rather than on its graph. In this subsection, we show that the two points of view are equivalent for Lipschitz functions.

Let M be a manifold and let $f: M \rightarrow \mathbb{R}$ be a continuous function. We introduce the microsupports

$$(6.16) \quad \Lambda_f^\pm := \mu\text{supp}(\Gamma_f^\pm).$$

One has the exact sequences (by [6, Prop. 2.3.6(v-vi)])

$$(6.17) \quad 0 \rightarrow \mathbb{k}_{M \times \mathbb{R}} \rightarrow \mathbb{k}_{\Gamma_f^+} \oplus \mathbb{k}_{\Gamma_f^-} \rightarrow \mathbb{k}_{\Gamma_f} \rightarrow 0$$

$$(6.18) \quad \text{and } 0 \rightarrow \mathbb{k}_{\text{Int}(\Gamma_f^\pm)} \rightarrow \mathbb{k}_{M \times \mathbb{R}} \rightarrow \mathbb{k}_{\Gamma_f^\mp} \rightarrow 0.$$

It follows from [6, Exe. III.4] that if U is a convex open subset of a vector space, then \mathbb{k}_U is cohomologically constructible and

$$(6.19) \quad D'_M(\mathbb{k}_U) \simeq \mathbb{k}_{\bar{U}},$$

so by Proposition 2.7 one has

$$(6.20) \quad \mu\text{supp}(\bar{U}) = \mu\text{supp}(U)^a.$$

Note that these properties are of a topological and local nature.

Proposition 6.9. *Let M be a manifold and let $f: M \rightarrow \mathbb{R}$ be a continuous function. Then,*

- (1) $\dot{\Lambda}_f = \dot{\Lambda}_f^+ \cup \dot{\Lambda}_f^-;$
- (2) $\dot{\Lambda}_f^- = (\dot{\Lambda}_f^+)^a;$
- (3) $\Lambda_f^+ \subseteq T_{\geq 0}^*(M \times \mathbb{R}).$

This proposition implies that the knowledge of Λ_f is equivalent to that of Λ_f^+ outside $T_{=0}^*(M \times \mathbb{R})$, and in particular it is equivalent for Lipschitz functions. Note that item (2) was proved in essence by Vichery (see [7, Lem. 4.12]).

Proof. (1) This follows from the triangular inequality applied to the exact sequence (6.17).

(2) Consider the topological automorphism of $M \times \mathbb{R}$ given by $\Phi(x, t) := (x, t - f(x))$. Then $\Phi(\Gamma_f^\pm) = M \times (\pm\mathbb{R}_{\geq 0})$. Therefore, since $\overline{\text{Int}(\Gamma_f^+)} = \Gamma_f^+$, we have $\mu\text{supp}(\Gamma_f^+) = \mu\text{supp}(\text{Int}(\Gamma_f^+))^\alpha$ by Equation (6.20) and the fact that it is a topological and local property. On the other hand, the triangular inequality applied to the exact sequence (6.18) implies that $\dot{\Lambda}_f^- = \mu\text{supp}(\text{Int}(\Gamma_f^+))$.

(3) The result is clear if f is C^1 . Now, let f be continuous. Since the result to prove is local, we can assume that M is compact. We define an increasing sequence $(f_n)_{n \in \mathbb{N}}$ of smooth real-valued functions on M converging pointwise (indeed, uniformly) to f as follows. Let f_0 be the constant function equal everywhere to $(\min_M f) - 1$, and given $f_n < f$, let $\epsilon_n := \min_M(f - f_n) > 0$ and by density of smooth functions in the space of continuous functions with the compact-open topology, let f_{n+1} be a smooth function such that $f - \epsilon_n/2 < f_{n+1} < f$.

Then $\Gamma_f^+ \subseteq \Gamma_{f_{n+1}}^+ \subseteq \Gamma_{f_n}^+$ and $\bigcap_{n \in \mathbb{N}} \Gamma_{f_n}^+ = \Gamma_f^+$. It follows that the inclusions induce an isomorphism $\varinjlim_n \mathbb{k}_{\Gamma_{f_n}^+} \xrightarrow{\sim} \mathbb{k}_{\Gamma_f^+}$. Applying [6, Exe. V.7], one gets $\Lambda_f^+ \subseteq \overline{\bigcup_n \Lambda_{f_n}^+} \subseteq T_{\geq 0}^*(M \times \mathbb{R})$. \square

Example 6.10. It can happen that $(\Lambda_f^+)_x = T_x^*M \times \mathbb{R}_{\geq 0}$. An example is given by $f: \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto x \sin(1/x)$, for which $(\Lambda_f^+)_0 = \mathbb{R} \times \mathbb{R}_{\geq 0}$. In particular, the union in item (1) need not be disjoint (although it is disjoint outside $T_{=0}^*(M \times \mathbb{R})$ in view of items (2) and (3), and in particular for Lipschitz functions).

§7. Main results

This is the main section of the paper, where we prove the characterizations of Lipschitz continuity and strict differentiability in terms of the conormal. We also prove upper bounds on the Whitney cone and on the conormal of a continuous map.

§7.1. Mean value theorem

In the case of a continuous map between vector spaces, we can give a lower bound on the conormal in the form of a mean value theorem. As in the classical treatment, we prove it first for maps of a real variable. Recall that the conormal Λ_f of a continuous map f was defined in Equation (5.1).

Lemma 7.1 (Rolle's lemma). *Let $a, b \in \mathbb{R}$ with $a < b$. If $f: [a, b] \rightarrow \mathbb{R}$ is a continuous function with $f(a) = f(b)$, then there exists $c \in]a, b[$ such that $\{0\} \times \mathbb{R} \subseteq \Lambda_c(f)$.*

Proof. Apply Fermat’s lemma (Proposition 6.8) at a local extremum of f in $]a, b[$. □

Lemma 7.2 (Mean value theorem for real-valued functions of a real variable). *Let $a, b \in \mathbb{R}$ with $a < b$. If $f: [a, b] \rightarrow \mathbb{R}$ is a continuous function, then there exists $c \in]a, b[$ such that $(f(b) - f(a), a - b) \in \Lambda_c(f)$.*

Proof. As in the classical case, we apply Rolle’s lemma to the function $x \mapsto f(x) - \frac{f(b)-f(a)}{b-a}x$ and we use the shearing lemma (Lemma 5.5). □

The previous Rolle lemma is not true for real-valued functions of several variables. For instance, consider $p_2: \mathbb{R}^2 \rightarrow \mathbb{R}$. Then $p_2(0, 0) = p_2(1, 0) = 0$, but for all $t \in [0, 1]$, one has $\Lambda_{(t,0)}(p_2) = \mathbb{R}(-1, 1)$, which intersects $\{0\} \times \mathbb{R}$ only at $\{(0, 0)\}$. However, if we slightly relax the conclusion, there is a mean value theorem for continuous maps, which we prove first in the case of real-valued functions. Recall that we defined $\dot{A} := A \setminus \{0\}$.

Lemma 7.3 (Mean value theorem for real-valued functions). *Let U be an open subset of a vector space. Let $f: U \rightarrow \mathbb{R}$ be a continuous function. Let $a, b \in U$ be such that $a \neq b$ and $[a, b] \subseteq U$. Then there exists $c \in]a, b[$ such that $\dot{\Lambda}_c(f) \cap (b - a, f(b) - f(a))^\perp \neq \emptyset$.*

Proof. Set $u := b - a$ and $v := f(b) - f(a)$. Define $\gamma:]0, 1[\rightarrow U, t \mapsto (1 - t)a + tb$. One has $\Lambda_t(\gamma) = \Gamma_\gamma^\perp = (1, u)^\perp$ for any $t \in]0, 1[$.

If the pair (γ, f) is not regular, then there exists a nonzero vector $\xi \in (0_{\mathbb{R}}^* \circ \Lambda_\gamma) \cap (\Lambda_f \circ 0_{\mathbb{R}}^*)$ with $\xi \in T_c^*U$ for some $c = \gamma(s) \in]a, b[$. Therefore, $(0, \xi) \in \Lambda_s(\gamma) = (1, u)^\perp$, so $\langle \xi, u \rangle = 0$, so $(\xi, 0) \in \dot{\Lambda}_c(f) \cap (u, v)^\perp$, which completes the proof.

If the pair (γ, f) is regular, then the chain rule for conormals (Proposition 5.13) applies, and one has $\Lambda_{f \circ \gamma} \subseteq \Lambda_\gamma \overset{\circ}{\circ} \Lambda_f$. We apply the mean value theorem for real-valued functions of a real variable (Lemma 7.2) to $f \circ \gamma$ extended by continuity to $[0, 1]$. It gives an $s \in]0, 1[$ such that $(v, -1) \in \Lambda_s(f \circ \gamma)$. Let $c := \gamma(s)$. By the chain rule, there exists $\xi \in T_c^*U$ such that $(v, \xi) \in \Lambda_s(\gamma) = (1, u)^\perp$ and $(\xi, 1) \in \Lambda_c(f)$. One has $\langle (\xi, 1), (u, v) \rangle = \langle \xi, u \rangle + v = 0$. □

We can now prove a mean value theorem for continuous maps between vector spaces. Recall that the notation $\dot{\Lambda}_c^{\eta_0}(f)$ was defined in Equation (5.6).

Theorem 7.4 (Mean value theorem). *Let \mathbb{V} and \mathbb{W} be vector spaces and $U \subseteq \mathbb{V}$ be open. Let $f: U \rightarrow \mathbb{W}$ be a continuous map. Let $a, b \in U$ be such that $a \neq b$ and $[a, b] \subseteq U$. Let $\eta_0 \in \dot{\mathbb{W}}^*$. Then there exists $c \in]a, b[$ such that*

$$(7.1) \quad \dot{\Lambda}_c^{\eta_0}(f) \cap (b - a, f(b) - f(a))^\perp \neq \emptyset.$$

In particular, if f is Lipschitz at c for $(b - a)^\perp$, then there exists $\xi_0 \in \mathbb{V}^*$ such that $(\xi_0, \eta_0) \in \Lambda_c(f) \cap (b - a, f(b) - f(a))^\perp$.

Proof. Set $u := b - a$ and $v := f(b) - f(a)$ and $v_0 := \langle \eta_0, v \rangle$. Since $\eta_0: \mathbb{W} \rightarrow \mathbb{R}$ is Lipschitz for $T^*\mathbb{R}$, the chain rule gives $\Lambda_{\eta_0 \circ f} \subseteq \Lambda_f \circ \Lambda_{\eta_0}$. One has $\Lambda_x(\eta_0) = \Gamma_{\eta_0}^\perp = \mathbb{R}(-\eta_0, 1)$ for all $x \in]a, b[$. Applying Lemma 7.3 to this real-valued function, we obtain $c \in]a, b[$ and a nonzero vector $(\xi, \tau) \in \dot{\Lambda}_c(\eta_0 \circ f) \cap (u, v_0)^\perp$.

By the chain rule, there exists $\eta \in \mathbb{W}^*$ such that $(\xi, \eta) \in \Lambda_c(f)$ and $(-\eta, \tau) \in \Lambda_{f(c)}(\eta_0) = \mathbb{R}(-\eta_0, 1)$. This implies $\eta = \tau\eta_0$. Therefore, $(\xi, \tau\eta_0) \in \Lambda_c^{\eta_0}(f)$, and it is nonzero since $(\xi, \tau) \neq 0$. Moreover, $\langle (\xi, \tau\eta_0), (u, v) \rangle = \langle (\xi, \tau), (u, v_0) \rangle = 0$.

Finally, if f is Lipschitz at c for $(b - a)^\perp$, then $\tau \neq 0$, and since the conormal of f is symmetric, we can suppose by \mathbb{R} -homogeneity that $\eta = \eta_0$. □

§7.2. Lower bound on the conormal

We immediately obtain from the mean value theorem an upper bound on the Whitney cone of a continuous map in terms of its conormal. We will also consider it as a sort of lower bound on its conormal in terms of its Whitney cone. Recall that we defined $A^\top := \bigcup_{v \in \dot{A}} v^\perp$.

Theorem 7.5 (Lower bound on the conormal). *Let $f: M \rightarrow N$ be a continuous map between manifolds and let $x \in M$. Then,*

$$(7.2) \quad C_x(f) \subseteq \bigcap_{\eta \in \dot{T}_{f(x)}^* N} \Lambda_x^\eta(f)^\top$$

with equality if $\dim_{f(x)} N = 1$, in which case it reads

$$(7.3) \quad C_x(f) = \Lambda_x(f)^\top.$$

In particular, if $w \in C_x(f)$ and $\eta \in T_{f(x)}^* N$ and f is Lipschitz at x for $p_M(w)^\perp$, then there exists $\xi \in T_x^* M$ such that $(\xi, \eta) \in \Lambda_x(f) \cap w^\perp$. In particular, if f is Lipschitz at x for $T_x^* M$, then $p_N(\Lambda_x(f)) = T_{f(x)}^* N$.

Proof. (i) Let $w \in \dot{C}_x(f)$ and $\eta \in \dot{T}_{f(x)}^* N$ and fix charts at x and $f(x)$. There exist sequences $y_n, z_n \xrightarrow{n} x$ and $c_n > 0$ such that $c_n(z_n - y_n, f(z_n) - f(y_n)) \xrightarrow{n} w$. Since $w \neq 0$, we can suppose that $y_n \neq z_n$ for all $n \in \mathbb{N}$. By the mean value theorem (Theorem 7.4), there exist sequences $x_n \in]y_n, z_n[$ and $\nu_n \in \dot{\Lambda}_{x_n}^\eta(f) \cap (z_n - y_n, f(z_n) - f(y_n))^\perp$. Since $x_n \xrightarrow{n} x$, up to extracting a subsequence and normalizing ν_n , we can suppose that (ν_n) converges. Its limit, say ν , is in $\dot{\Lambda}_x^\eta(f) \cap w^\perp$.

(ii) For the case of equality, suppose that $\dim_{f(x)} N = 1$, and let $w = (u, v) \in \Lambda_x(f)^\top \setminus \{0\}$.

Suppose first that f is Lipschitz. There exists $(\xi, \eta) \in \dot{\Lambda}_x(f)$ such that $\langle \xi, u \rangle + \eta v = 0$. Since f is Lipschitz, $u \neq 0$, so by the upper bound on the conormal (Theorem 7.9³), there exists $v' \in \mathbb{R}$ such that $(u, v') \in \dot{C}_x(f)$ and $\langle \xi, u \rangle + \eta v' = 0$. So $\eta(v - v') = 0$. Since f is Lipschitz, $\eta \neq 0$, so $v = v'$, so $w = (u, v') \in C_x(f)$.

In the general case, suppose that $w \notin C_x(f)$. By Proposition 4.4(1), this implies that Γ_f is the graph (in other coordinates) of a Lipschitz map. The result then follows from the previous paragraph, since $C_x(f)$ and $\Lambda_x(f)$ depend on f only via its graph.

(iii) For the last claim, let $w \in C_x(f)$ and $\eta \in T_{f(x)}^*N$. The case $\eta = 0$ is trivial since $0 \in \Lambda_x(f) \cap w^\perp$, so we suppose $\eta \neq 0$. By (i), there exist $\xi \in T_x^*M$ and $t \in \mathbb{R}_{\geq 0}$ such that $(\xi, t\eta) \in \dot{\Lambda}_x(f) \cap w^\perp$. Since f is Lipschitz at x for $p_M(w)^\perp$, we have $t \neq 0$, so we can suppose $t = 1$, so $(\xi, \eta) \in \Lambda_x(f) \cap w^\perp$. \square

Remark 7.6. The condition in the theorem that f be Lipschitz at x for $p_M(w)^\perp$ is necessary, as the function $\sqrt[3]{-}: \mathbb{R} \rightarrow \mathbb{R}$ shows.

§7.3. Characterization of Lipschitz continuity

In this subsection, we prove that a continuous map between manifolds $f: M \rightarrow N$ is Lipschitz if and only if it is “Lipschitz for T^*M ” (Definition 5.6(3) or Equation (5.8)). The definitions and properties we use related to Lipschitz continuity are recalled in Appendix A. We first need a technical lemma.

Lemma 7.7. *Let A be a topological space and $a_0 \in A$. Let $n \in \mathbb{N}$ and $C, D \in \mathbb{R}_{>0}$ with $CD < 1$. Let $f: \mathbb{R} \times A \rightarrow \mathbb{R}^n$ be a continuous map with $f(0, a_0) = 0$ which is C -Lipschitz in its first variable in a neighborhood of $(0, a_0)$. Let $\psi: A \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous map with $\psi(a_0, 0) = 0$ which is D -Lipschitz in its last n variables in a neighborhood of $(a_0, 0)$. Then there exist $\eta > 0$, an open neighborhood U of a_0 , and a continuous map $\Psi: U \rightarrow \mathbb{R}$ such that for all $(t, a) \in]-\eta, \eta[\times U$, one has $t = \psi(a, f(t, a)) \Leftrightarrow t = \Psi(a)$.*

Proof. It suffices to apply the Banach fixed-point theorem with continuous parameter to the map $g: \mathbb{R} \times A \rightarrow \mathbb{R}$, $(t, a) \mapsto \psi(a, f(t, a))$ in a suitable neighborhood of $(0, a_0)$ of the form $]-\eta, \eta[\times U$ with $\eta > 0$ and U an open neighborhood of a_0 . \square

Recall that we defined $C_x^0(f) := C_x(f) \cap (\{0\} \times T_{f(x)}N)$ and $\Lambda_x^0(f) := \Lambda_x(f) \cap (T_x^*M \times \{0\})$.

Theorem 7.8. *Let $f: M \rightarrow N$ be a continuous map between manifolds and let $x \in M$. The following are equivalent:*

³This does not create any circular argument.

- (1) f is Lipschitz on a neighborhood of x ;
- (2) $C_x^0(f) = \{0\}$;
- (3) $\Lambda_x^0(f) = \{0\}$.

The implication (1) \Rightarrow (3) for real-valued maps was proved in essence in [7, Thm. 3.9(6)]. We recall the proof here: if $f: M \rightarrow \mathbb{R}$ is, say, C -Lipschitz at $x_0 \in M$ in a given chart, then, setting $\gamma := \{(x, t) \in \mathbb{R}^{m+1} \mid t \geq C\|x\|\}$, one has $\Gamma_f^+ + \gamma \subseteq \Gamma_f^+$, so $N_{x_0}(\Gamma_f^+) \supseteq \text{Int}(\gamma)$, so by Proposition 3.4, $\Lambda_{x_0}^+(f) \subseteq N_{x_0}(\Gamma_f^+)^\circ \subseteq \gamma^\circ$. Therefore, $\Lambda_{x_0}^+(f) \circ \{0\} \subseteq \gamma^\circ \circ \{0\} = \{0\}$, and we conclude by Proposition 6.9(1).

Proof of Theorem 7.8.

(2) \Rightarrow (1) is (one implication of) Proposition 4.3(1).

(3) \Rightarrow (2). Let $(0, v) \in C_x^0(f)$. Let $\eta \in \dot{T}_{f(x)}^*N$. By the lower bound on the conormal (Theorem 7.5), there exist $\xi \in T_x^*M$ and $t \in \mathbb{R}_{\geq 0}$ such that $(\xi, t\eta) \in \Lambda_x(f)$ and $\langle (\xi, t\eta), (0, v) \rangle = 0$. Condition (3) implies that $t \neq 0$. Therefore, for all $\eta \in T_{f(x)}^*N$, one has $\langle \eta, v \rangle = 0$. Therefore, $v = 0$.

(1) \Rightarrow (3). Let $(x_0, \xi_0) \in \dot{T}^*M$. We may suppose that M is open in \mathbb{R}^m , that $N = \mathbb{R}^n$ and that f is C -Lipschitz for some $C \in \mathbb{R}_{>0}$.

(1) We first prove that $(\xi_0, 0) \in \text{Ppg}_{x_0}(f)$. Let $\phi: M \times N \rightarrow \mathbb{R}$ be a function of class C^1 with $\phi(x_0, f(x_0)) = 0$ and $d\phi(x_0, f(x_0)) = (\xi_0, 0)$. We will construct a basis of open neighborhoods W of $(x_0, f(x_0))$ such that each topological embedding $p_M \circ i_W: \Gamma_f \cap \{\phi < 0\} \cap W \hookrightarrow p_M(W)$ induces an isomorphism in cohomology. Then Lemma 5.3 will imply that $(\xi_0, 0) \in \text{Ppg}_{x_0}(f)$.

Since $p_M(d\phi(x_0, f(x_0))) = \xi_0 \neq 0$, we may suppose that $\phi(x, y) = x_1 - \psi(x', y)$ in some neighborhood W_0 of $(x_0, f(x_0))$, where $x = (x_1, x') \in \mathbb{R}^m$. Therefore, for any $W \subseteq W_0$, one has

$$\Gamma_f \cap \{\phi < 0\} \cap W = \{(x, f(x)) \in W \mid x_1 < \psi(x', f(x))\}.$$

Since $\frac{\partial \psi}{\partial y}(x'_0, f(x_0)) = -p_N(d\phi(x_0, f(x_0))) = 0$, there exists an open neighborhood of $(x'_0, f(x_0))$ where ψ is $1/(C + 1)$ -Lipschitz in its last n variables. Therefore, by Lemma 7.7, there exist $\eta > 0$, an open neighborhood U' of x'_0 and a continuous map $\Psi: U' \rightarrow \mathbb{R}$ such that, setting $U_0 :=](x_0)_1 - \eta, (x_0)_1 + \eta[\times U'$, we have for $x \in U_0$,

$$x_1 < \psi(x', f(x)) \iff x_1 < \Psi(x').$$

Then, for any open neighborhood $W \subseteq W_0 \cap (U_0 \times N)$ of $(x_0, f(x_0))$, one has $\Gamma_f \cap \{\phi < 0\} \cap W = \{(x, f(x)) \in W \mid x_1 < \Psi(x')\}$. Set $U := p_M(W \cap \Gamma_f)$. The projection $p_M: \{(x, f(x)) \in W \mid x_1 < \Psi(x')\} \rightarrow \{x \in U \mid x_1 < \Psi(x')\}$ is an isomorphism. If W is convex, then so is U , and the inclusion $i_U: \{x \in U \mid$

$x_1 < \Psi(x')\} \hookrightarrow U$ induces an isomorphism in cohomology. Therefore, so does $p_M \circ i_W = i_U \circ p_M: \Gamma_f \cap \{\phi < 0\} \cap W \hookrightarrow U$.

Finally, we can find a basis of open convex neighborhoods W of $(x_0, f(x_0))$ such that $p_M(W \cap \Gamma_f) = p_M(W)$. For instance, we may set $U_n := B(x_0, 1/n)$ and $W_n := U_n \times B(f(U_n), 1/n)$ for $n \in \mathbb{N}_{>0}$, where $B(f(U_n), 1/n)$ denotes the $1/n$ -neighborhood of $f(U_n)$.

(2) We will prove that $B_\infty((\xi_0, 0), \|\xi_0\|/(C+2)) \subseteq \text{Ppg}_{x_0}(f)$, where the left-hand side denotes the open ball in \mathbb{R}^{m+n} centered at $(\xi_0, 0)$ with radius $\|\xi_0\|/(C+2)$ for the sup norm.

Let $(\xi, \eta) \in T_{(x_0, f(x_0))}^*(M \times N)$ with $\|\xi\|, \|\eta\| < \|\xi_0\|/(C+2)$. Define the linear automorphism $\Phi := \text{id} + (\xi, \eta) \otimes (e, 0)$ of $T_{(x_0, f(x_0))}(M \times N)$, where $e \in T_{x_0}M$ is such that $\|e\| = \|\xi_0\|^{-1}$ and $\langle \xi_0, e \rangle = 1$.

One has $\Phi^{-1} = \text{id} - \frac{1}{1+\langle \xi, e \rangle}(\xi, \eta) \otimes (e, 0)$. Therefore, if $v \in T_{f(x_0)}N$, then $\Phi^{-1}(0, v) = (-\frac{\langle \eta, v \rangle}{1+\langle \xi, e \rangle}e, v)$. One has $\|\frac{\langle \eta, v \rangle}{1+\langle \xi, e \rangle}e\| \leq \frac{\|\eta\|\|v\|}{1-\|\xi\|\|e\|}\|e\| \leq \frac{1/(C+2)}{1-(1/(C+2))}\|v\| = \|v\|/(C+1)$. Since f is C -Lipschitz, this implies $\Phi(C_{x_0}(f)) \cap (\{0\} \times T_{f(x_0)}N) = \{0\}$. By Proposition B.1(8), one has $C_{(x_0, f(x_0))}(\Phi(\Gamma_f), \Phi(\Gamma_f)) = \Phi(C_{x_0}(f))$. Therefore, by Proposition 4.4(1), $\Phi(\Gamma_f)$ is locally the graph of a Lipschitz map, say g . The relation $\Phi(\Gamma_f) = \Gamma_g$ and Equation (3.6) imply $\Phi^\top(\text{Ppg}_{x_0}(g)) = \text{Ppg}_{x_0}(f)$. From (1), one has $(\xi_0, 0) \in \text{Ppg}_{x_0}(g)$. Therefore, $(\xi_0 + \xi, \eta) = \Phi^\top(\xi_0, 0) \in \text{Ppg}_{x_0}(f)$.

(3) Finally, there is a neighborhood U of x_0 (contained in the fixed chart) such that f is C -Lipschitz on U and (1) and (2) apply with x_0 replaced by any $x \in U$. On the other hand, if $(x, y) \notin \Gamma_f$, then $\text{Ppg}_{(x, y)}(\Gamma_f) = T_{(x, y)}^*(M \times N)$. Therefore, there is a neighborhood W of $(x_0, f(x_0))$ such that $W \times B_\infty((\xi_0, 0), \|\xi_0\|/(C+2)) \subseteq \text{Ppg}(\Gamma_f)$, which proves that $(\xi_0, 0) \notin \Lambda_{x_0}(f)$. \square

§7.4. Upper bound on the conormal

In this subsection, we give an upper bound on the conormal of a map in terms of its Whitney cone. Recall that the notation $C_x^u(f)$ was defined in Equation (4.3).

Theorem 7.9 (Upper bound on the conormal). *Let $f: M \rightarrow N$ be a continuous map between manifolds and let $x \in M$. Then*

$$(7.4) \quad \Lambda_x(f) \subseteq \bigcap_{u \in \dot{T}_x M} C_x^u(f)^\top$$

with equality if $\dim_x M = 1$, in which case it reads

$$(7.5) \quad \Lambda_x(f) = C_x(f)^\top.$$

In particular, if $u \in T_x M$ and $v \in \Lambda_x(f)$ and f is Lipschitz at x for $p_N(v)^\perp$, then there exists $v \in T_{f(x)}N$ such that $(u, v) \in C_x(f) \cap v^\perp$.

Proof. (i) Let $\xi \in T_{(x,f(x))}^*(M \times N)$ and $u \in \dot{T}_x M$ be such that $\xi^\perp \cap C_x^u(f) = \{0\}$. We have to prove that $\xi \notin \Lambda_x(f)$. Since $u \neq 0$, there is a linear automorphism Φ of $T_{(x,f(x))}(M \times N)$ such that $\Phi^{-1}(\{0\} \times T_{f(x)}N) \subseteq \xi^\perp \cap (\mathbb{R}u \times T_{f(x)}N)$.

Let $(0, v) \in C_{(x,f(x))}(\Phi(\Gamma_f), \Phi(\Gamma_f)) = \Phi(C_x(f))$. Then $\Phi^{-1}(0, v) \in C_x(f) \cap (\mathbb{R}u \times T_{f(x)}N) \cap \xi^\perp \subseteq C_x^{\pm u}(f) \cap \xi^\perp = \{0\}$, so $v = 0$. Therefore, $\Phi(\Gamma_f)$ is locally the graph of a Lipschitz map, say g . By Equation (3.6), one has $\Phi^\top(\Lambda_x(g)) = \Lambda_x(f)$.

Let $v \in T_{f(x)}N$. One has $\langle \Phi^{\top^{-1}}(\xi), (0, v) \rangle = \langle \xi, \Phi^{-1}(0, v) \rangle = 0$, so $p_N(\Phi^{\top^{-1}}(\xi)) = 0$, so by Theorem 7.8((1) \Rightarrow (3)), one has $\Phi^{\top^{-1}}(\xi) \notin \Lambda_x(g)$. Therefore, $\xi \notin \Phi^\top(\Lambda_x(g)) = \Lambda_x(f)$.

(ii) The case of equality is a special case of Proposition 8.4.⁴

(iii) For the last claim, let $u \in T_x M$ and $\nu \in \Lambda_x(f)$. The case $u = 0$ is trivial since $0 \in C_x(f) \cap \nu^\perp$, so we suppose $u \neq 0$. By (i) and (ii), there exist $v \in T_{f(x)}M$ and $t \in \mathbb{R}_{\geq 0}$ such that $(tu, v) \in \dot{C}_x(f) \cap \nu^\perp$. Since f is Lipschitz at x for $p_N(\nu)^\perp$, we have $t \neq 0$, so we can suppose $t = 1$, so $(u, v) \in C_x(f) \cap \nu^\perp$. \square

Remark 7.10. In view of Proposition 4.3(1), the implication (1) \Rightarrow (3) of Theorem 7.8 is a special case of this upper bound on the conormal.

Example 7.11. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}, (x_1, x_2) \mapsto x_1^2 \sin(1/x_1)$. One has $C_0(f) = \{u \in \mathbb{R}^3 \mid |u_3| \leq |u_1|\}$, so $\bigcap_{v \in \dot{T}_0 M} C_0^v(f)^\top = \{(\xi_1, 0, \xi_3) \in \mathbb{R}^3 \mid |\xi_1| \leq |\xi_3|\}$. In this case, the upper bound is easily seen to be an equality.

§7.5. Characterization of strict differentiability

The lower and upper bounds on the conormal allow us to derive the following characterization of strict differentiability.

Proposition 7.12. *Let $f: M \rightarrow N$ be a continuous map between manifolds and let $x \in M$. The following are equivalent:*

- (1) f is strictly differentiable at x ;
- (2) $C_x^0(f) = \{0\}$ and $C_x(f)$ is contained in a $(\dim_x M)$ -dimensional vector subspace;
- (3) $\Lambda_x^0(f) = \{0\}$ and $\Lambda_x(f)$ is contained in a $(\dim_{f(x)} N)$ -dimensional vector subspace;

and in that case, $C_x(f) = \Gamma_{T_x f}$ and $\Lambda_x(f) = (\Gamma_{T_x f})^\perp$.

Proof. (2) \Rightarrow (1). This is (one implication of) Proposition 4.3(2), which also proves $C_x(f) = \Gamma_{T_x f}$.

⁴This does not create any circular argument.

(3) \Rightarrow (2). Condition (3) implies that f is Lipschitz, so the lower bound theorem reads as follows: for any $(u, v) \in C_x(f)$ and any $\eta \in \dot{T}_{f(x)}^*N$, there exists $\xi \in T_x^*M$ such that $(\xi, \eta) \in \Lambda_x(f)$ and $\langle \xi, u \rangle + \langle \eta, v \rangle = 0$. Therefore, $\Lambda_x(f)$ is an n -dimensional vector subspace and is the graph of a linear map $L: T_{f(x)}^*N \rightarrow T_x^*M$ such that $\langle L(\eta), u \rangle + \langle \eta, v \rangle = 0$ for all $(u, v) \in C_x(f)$ and $\eta \in T_{f(x)}^*N$. Therefore, $\langle \eta, L^\top(u) + v \rangle = 0$, so $v = -L^\top(u)$, so $C_x(f)$ is the graph of $-L^\top$, which is an m -dimensional vector subspace.

(1) \Rightarrow (3). This is a consequence of the upper bound on the conormal, since when f is strictly differentiable at x , one has $\bigcap_{u \in \dot{T}_x M} C_x^u(f)^\top = \bigcap_{u \in \dot{T}_x M} \{(u, f'(x)u)\}^\perp = (\sum_{u \in \dot{T}_x M} \mathbb{R}(u, f'(x)u))^\perp = (\Gamma_{T_x f})^\perp$. \square

§7.6. Applications to real-valued functions of a real variable

In this subsection, we present easy applications of the bounds on the conormal to the case of real-valued functions of a real variable. First, note that $(-)^\top$ is an involution on nonzero pointed symmetric cones in a two-dimensional space. Let I be an open interval of \mathbb{R} and $f: I \rightarrow \mathbb{R}$ be a continuous function. By the case of equality in the upper bound theorem, one has $\Lambda_f = C_f^\top$, hence also $C_f = \Lambda_f^\top$.

If f is Lipschitz and $x \in I$, then $\Lambda_x^+(f) = N_x(\Gamma_f^+)^\circ$, as is easily deduced from Propositions 6.1 and 6.9(3), that is, the general upper bound on the microsupport of a closed subset (Proposition 3.4) is an equality for the epigraph of a Lipschitz function of a real variable.

We set $P := \{(x, y, u, v) \in T(I \times \mathbb{R}) \mid uv \geq 0\}$ and $N := \{(x, y, \xi, \eta) \in T^*(I \times \mathbb{R}) \mid \xi\eta \leq 0\}$. Note that $N = P^\top$ and $\text{Int}(N) = \text{Int}(P)^\top \setminus 0_{I \times \mathbb{R}}^*$.

Proposition 7.13. *Let I be an open interval of \mathbb{R} and $f: I \rightarrow \mathbb{R}$ be a continuous function.*

- (1) *The following conditions are equivalent:*
 - (a) *f is nondecreasing;*
 - (b) $C_f \subseteq P$;
 - (c) $\Lambda_f \subseteq N$.
- (2) *The following conditions are equivalent:*
 - (a) *f is injective with Lipschitz inverse;*
 - (b) *f is a Whitney immersion;*
 - (c) *f is a microlocal submersion.*
- (3) *The following conditions are equivalent:*
 - (a) *f is a strictly increasing Lipschitz-embedding;*

- (b) $\dot{C}_f \subseteq \text{Int}(P)$;
 (c) $\dot{\Lambda}_f \subseteq \text{Int}(N)$.

Proof. (1) The equivalence (1b) \Leftrightarrow (1c) follows from the discussion preceding the proposition. The implication (1a) \Rightarrow (1b) is straightforward. As for the implication (1c) \Rightarrow (1a), let $a < b \in I$. By Lemma 7.2, there exists $c \in]a, b[$ such that $(f(a) - f(b), b - a) \in \Lambda_c(f)$. Therefore, the hypothesis $\dot{\Lambda}_f \subseteq \text{Int}(N)$ implies $f(a) < f(b)$. (One could prove (1b) \Rightarrow (1a) in a similar way, via an analogue of Lemma 7.2 for Whitney cones.)

(2) The equivalence (2b) \Leftrightarrow (2c) follows from the discussion preceding the proposition. Namely, one has $C_f \cap (TI \times 0_{\mathbb{R}}) \subseteq 0_{I \times \mathbb{R}}$ if and only if $\Lambda_f \cap (0_I^* \times T\mathbb{R}) \subseteq 0_{I \times \mathbb{R}}^*$, and in that case, f is injective and an open map by the Fermat lemma. If f is injective and open, then $C_{f^{-1}}$ is the image of C_f by the “flip” $T(I \times \mathbb{R}) \rightarrow T(\mathbb{R} \times I)$, so $C^0(f^{-1}) \subseteq 0_{\mathbb{R} \times I}$ is equivalent to $C_f \cap (TI \times 0_{\mathbb{R}}) \subseteq 0_{I \times \mathbb{R}}$, that is, f^{-1} is Lipschitz if and only if f is Whitney immersive.

(3) follows from (1) and (2) and Proposition 4.3(1). \square

Remark 7.14. The implication (2b) \Rightarrow (2a) follows directly from invariance of domain, but we gave the preceding proof for its elementary nature.

§7.7. Application to causal manifolds

In [4], the authors introduced the category of causal manifolds, in which the category of spacetimes (time-oriented connected Lorentzian manifolds) up to conformal isomorphisms embeds. In [4, Def. 1.7], a *causal manifold* (M, γ_M) was defined to be a connected manifold M equipped with an open convex cone $\gamma_M \subseteq TM$ which is nowhere empty ($\gamma_x \neq \emptyset$ for all $x \in M$), and a causal morphism $f: (M, \gamma_M) \rightarrow (N, \gamma_N)$ was defined to be a morphism of manifolds such that $Tf(\overline{\gamma_M}) \subseteq \overline{\gamma_N}$.

In a vector bundle, we denote by $\text{cl}_{\text{fw}}(-)$ the fiberwise closure. Note that for a nowhere empty convex cone γ in a vector bundle, one has $\gamma^{\circ\circ} = \text{cl}_{\text{fw}}(\gamma)$.

Remark 7.15. The proof of [4, Prop. 1.12] actually proves that if (M, γ_M) and (N, γ_N) are causal manifolds and $f: M \rightarrow N$ is a morphism of manifolds, then $\Lambda_f \overset{\circ}{\subseteq} \gamma_N^{\circ} \subseteq \gamma_M^{\circ}$ if and only if $Tf(\text{cl}_{\text{fw}}(\gamma_M)) \subseteq \text{cl}_{\text{fw}}(\gamma_N)$ (and not, as stated there, $Tf(\overline{\gamma_M}) \subseteq \overline{\gamma_N}$). These conditions imply that f is causal and are satisfied when f is strictly causal or when f is causal and $\text{cl}_{\text{fw}}(\gamma_N) = \overline{\gamma_N}$. Since time functions are \mathbb{R} -valued, this misstatement has no consequences for the rest of the paper, with the exception of [4, Cor. 2.10], in which the morphism f should be assumed strictly causal.

Here, we make the additional assumptions that the cone γ_M of a causal manifold (M, γ_M) is *proper*, in the sense that $(\overline{\gamma_M})_x$ does not contain any line for any $x \in M$, and is *continuous*, in the sense that $\overline{\gamma_M} = \text{cl}_{\text{fw}}(\gamma_M)$. One can check that continuity as defined here is equivalent to the continuity of the map $x \mapsto (\gamma_M)_x$ for any reasonable topology on the space of cones (as for instance defined in [1] using the Hausdorff distance). We set $\gamma_{\mathbb{R}} := \mathbb{R} \times \mathbb{R}_{>0}$, so that $(\mathbb{R}, \gamma_{\mathbb{R}})$ is a causal manifold.

We will extend some of the results of [4] from smooth maps to continuous maps.

Definition 7.16. A causal morphism $f: (M, \gamma_M) \rightarrow (N, \gamma_N)$ is a continuous map such that $\overline{\gamma_M} \circ C_f \subseteq \overline{\gamma_N}$.

Proposition 7.17. A causal morphism is Lipschitz.

Proof. Let $f: (M, \gamma_M) \rightarrow (N, \gamma_N)$ be a causal morphism and let $x \in M$. One has $0 \in (\overline{\gamma_M})_x$. If $(0, v) \in C_x(f)$, then $\mathbb{R}v \subseteq (\overline{\gamma_N})_x$, but that cone is proper, so $v = 0$. □

We have the following extension of [4, Prop. 1.12].

Proposition 7.18. Let (M, γ_M) and (N, γ_N) be causal manifolds and let $f: M \rightarrow N$ be a continuous map. Then f is a causal morphism if and only if $\Lambda_f \overset{\circ}{\circ} \gamma_N^\circ \subseteq \gamma_M^\circ$. If $(M, \gamma_M) = (\mathbb{R}, \gamma_{\mathbb{R}})$, then this is equivalent to $C_f \subseteq \mathbb{R}(\{1\} \times \overline{\gamma_N})$. If $(N, \gamma_N) = (\mathbb{R}, \gamma_{\mathbb{R}})$, then this is equivalent to $\Lambda_f \subseteq \mathbb{R}(\gamma_M^\circ \times \{-1\})$.

Proof. (i) The condition is necessary. Let $x \in M$ and $\xi \in \Lambda_x(f) \overset{\circ}{\circ} (\gamma_N^\circ)_{f(x)}$. Let $u \in (\gamma_M)_x$. There exists $\eta \in (\gamma_N^\circ)_{f(x)}$ such that $(\xi, -\eta) \in \Lambda_x(f)$. By the upper bound on the conormal, one has $(\xi, -\eta) \in C_x^u(f)^\top$. Therefore, since f is Lipschitz, there exists $v \in T_{f(x)}N$ such that $(u, v) \in \dot{C}_x(f)$ and $\langle (\xi, -\eta), (u, v) \rangle = 0$. Since f is causal and $u \in \gamma_M$, one has $v \in \overline{\gamma_N} = \gamma_N^\circ$. Therefore, since $\eta \in \gamma_N^\circ$, one has $\langle \xi, u \rangle = \langle \eta, v \rangle \geq 0$. This proves that $\xi \in \gamma_M^\circ$.

(ii) The condition is sufficient. First, note that the inclusion $\Lambda_f \overset{\circ}{\circ} \gamma_N^\circ \subseteq \gamma_M^\circ$ implies that f is Lipschitz. Indeed, for $x \in M$, one has $0 \in (\gamma_N^\circ)_{f(x)}$. If $(\xi, 0) \in \Lambda_x(f)$, then $\mathbb{R}\xi \subseteq (\gamma_M^\circ)_x$, but that cone is proper, so $\xi = 0$.

Let $x \in M$ and $v \in (\overline{\gamma_M})_x \circ C_x(f)$. There exists $u \in (\overline{\gamma_M})_x$ such that $(u, v) \in C_x(f)$. Let $\eta \in (\gamma_N^\circ)_{f(x)} \setminus \{0\}$. By the lower bound on the conormal, one has $(u, v) \in \Lambda_x^{-\eta}(f)^\top$. Therefore, since f is Lipschitz, there exists $\xi \in T_x^*M$ such that $(\xi, -\eta) \in \Lambda_x(f)$ and $\langle (\xi, -\eta), (u, v) \rangle = 0$. By the hypothesis, one has $\xi \in (\gamma_M^\circ)_x$. Therefore, since $u \in (\overline{\gamma_M})_x = (\gamma_M)_x$, one has $\langle \eta, v \rangle = \langle \xi, u \rangle \geq 0$. This proves that $v \in \gamma_N^\circ = \overline{\gamma_N}$.

(iii) The cases where the domain or the codomain is $(\mathbb{R}, \gamma_{\mathbb{R}})$ are obvious. \square

Lemma 7.19. *Let $f: (M, \gamma_M) \rightarrow (N, \gamma_N)$ be a causal morphism and $A \subseteq T^*M$ be a closed cone. If $A \cap \gamma_M^\circ \subseteq 0_M^*$ and f is noncharacteristic for γ_N° , then $(A \overset{\alpha}{\circ} \Lambda_f) \cap \gamma_N^\circ \subseteq 0_N^*$.*

Proof. One has

$$\begin{aligned} \Lambda_f \cap (A \times \gamma_N^{\circ\alpha}) &\subseteq \Lambda_f \cap ((A \cap \gamma_M^\circ) \times \gamma_N^{\circ\alpha}) \\ &\subseteq \Lambda_f \cap (0_M^* \times \gamma_N^{\circ\alpha}) \\ &\subseteq 0_{MN}^* \end{aligned}$$

respectively because f is a causal morphism (and by Proposition 7.18), by hypothesis and because f is noncharacteristic for γ_N° . This implies $(A \overset{\alpha}{\circ} \Lambda_f) \cap \gamma_N^\circ \subseteq 0_N^*$. \square

This lemma shows that the proof of [4, Thm. 2.9] extends from the case of the causal morphism f being C^1 to f being merely continuous. Namely, we have the following theorem.

Theorem 7.20 (Extending [4, Thm. 2.9]). *Let $f: (M, \gamma_M) \rightarrow (N, \gamma_N)$ be a morphism of causal manifolds, let \preceq be a closed causal preorder on M and let $F \in D^b(\mathbb{k}_M)$. Assume that*

- (1) f is noncharacteristic for γ_N° ;
- (2) for any $x \in M$, the map f is proper on $J^-(x)$;
- (3) $\mu\text{supp}(F) \cap \gamma_M^\circ \subseteq 0_M^*$.

Then,

$$(7.6) \quad \mu\text{supp}(\mathbf{R}f_*F) \cap \text{Int}(\gamma_N^\circ) = \emptyset.$$

In [4], a time function on (M, γ_M) was defined to be a smooth submersive causal morphism $\tau: (M, \gamma_M) \rightarrow (\mathbb{R}, \gamma_{\mathbb{R}})$. By the extension of [4, Thm. 2.9] that we just proved, we see that all subsequent results of [4], in particular Theorem 2.13 and Corollary 3.8, continue to hold if the definition of a time function is weakened as follows.

Definition 7.21. A *time function* on a causal manifold (M, γ_M) is a microlocally submersive causal morphism $\tau: (M, \gamma_M) \rightarrow (\mathbb{R}, \gamma_{\mathbb{R}})$. A *Cauchy time function* is a time function which is proper on the cc-future⁵ and cc-past of any point.

⁵The cc-future (closed causal future) is the future for the cc-preorder, which is the finest closed preorder containing the causal preorder (see [4, Def. 1.44]).

Proposition 7.22. *Let $\tau: (M, \gamma_M) \rightarrow (\mathbb{R}, \gamma_{\mathbb{R}})$ be a time function. Then $\gamma_M \circ C_\tau \subseteq \gamma_{\mathbb{R}}$ and τ is an open map.*

Proof. Let $(u, v) \in C_x(f)$ with $u \in \gamma_M$. Let $\eta \in \gamma_N^\circ$. By the lower bound on the conormal, and since τ is causal hence Lipschitz, there exists $\xi \in T_x^*M$ such that $(\xi, -\eta) \in \Lambda_x(f)$ and $\langle (\xi, -\eta), (u, v) \rangle = 0$. Since f is causal, one has $\xi \in \gamma_M^\circ$ by Proposition 7.18, and since τ is microlocally submersive, one has $\xi \neq 0$. Therefore, $\langle \eta, v \rangle = \langle \xi, u \rangle > 0$. Therefore, $v \in \gamma_N$.

A time function is an open map by Proposition 5.18. □

§8. Topological submanifolds

In this section, we extend some of the previous bounds and characterizations of Lipschitz continuity and strict differentiability to topological submanifolds.

Proposition 8.1. *Let M be a C^0 -submanifold of a manifold P and let $x \in M$. The following are equivalent:*

- (1) M is locally at x the graph of a Lipschitz map;
- (2) $C_x(M, M)$ intersects trivially a $(\text{codim}_x M)$ -dimensional vector space $F \subseteq T_x P$;

and in that case,

$$(8.1) \quad \mu\text{supp}_x(M) \subseteq \bigcap_{w \in T_x P \setminus F} (C_x(M, M) \cap (w + F))^\top$$

$$(8.2) \quad \text{and } C_x(M, M) \subseteq \bigcap_{\nu \in T_x^* P \setminus F^\perp} (\mu\text{supp}_x(M) \cap (\nu + F^\perp))^\top$$

and in particular, $\mu\text{supp}_x(M) \cap F^\perp = \{0\}$.

If $\text{codim}_x M = 1$ and $C_x(M, M) \neq T_x P$, then both conditions are satisfied and

$$(8.3) \quad C_x(M, M) = \mu\text{supp}_x(M)^\top.$$

Proof. The equivalence of the two conditions is given by Proposition 4.4(1). The other claims are restatements of the upper and lower bounds on the conormal. □

We have the following strengthening of Proposition 3.2. A C^0 -submanifold is said to be *strictly differentiable* at a point if it is locally at that point the graph of a map which is strictly differentiable at that point.

Proposition 8.2. *Let M be a C^0 -submanifold of a manifold P and let $x \in M$. The following are equivalent:*

- (1) M is strictly differentiable at x ;
- (2) $C_x(M, M)$ is included in a $(\dim_x M)$ -dimensional vector subspace of $T_x P$;

and in that case, the inclusion in item (2) is an equality and $\mu\text{supp}_x(M) = C_x(M, M)^\perp$.

Proof. The equivalence of the two conditions is Proposition 4.4(2). If these conditions are fulfilled, then there exist an open neighborhood U of x and a chart $\phi = (\phi_M, \phi_N): U \xrightarrow{\sim} U_M \times U_N \subseteq \mathbb{R}^m \times \mathbb{R}^n$ and a map $f: U_M \rightarrow U_N$ strictly differentiable at $\phi_M(x)$ such that $\phi(M \cap U) = \Gamma_f$. We write $T_x f$ for $T_{\phi_M(x)} f$ and similarly for the Whitney cone and the conormal of f . The equality $\mu\text{supp}_x(M) = C_x(M, M)^\perp$ follows from $\Lambda_x(f) = C_x(f)^\perp$ (by Proposition 7.12), $T_x \phi(C_x(M, M)) = \Gamma_{T_x f} = C_x(f)$ (Propositions 4.4(2) and 7.12) and $\mu\text{supp}_x(M) = \phi'(x)^\top(\Lambda_x(f))$ (consequence of $\phi(M \cap U) = \Gamma_f$ and Equation (3.4)). \square

The following lemma is an analog ue of Rolle’s lemma (or the mean value theorem) for one-dimensional C^0 -submanifolds.

Lemma 8.3. *Let \mathbb{V} be a vector space and $f: [0, 1] \rightarrow \mathbb{V}$ be a continuous injection. Set $M := f(]0, 1[)$. If $\eta \in (f(1) - f(0))^\perp$, then there exists $t \in]0, 1[$ such that $\eta \in \mu\text{supp}_{f(t)}(M)$.*

Proof. Since $\eta \in (f(1) - f(0))^\perp$, the continuous function $\eta \circ f$ has an extremum at some $t \in]0, 1[$, say a maximum. We set $x := f(t)$. One has $M \subseteq \{v \in \mathbb{V} \mid \langle \eta, v - x \rangle \leq 0\}$, so $-\eta \in C_x(M)^\circ$. By Proposition 3.4, this implies $-\eta \in \mu\text{supp}_x(M)$, which is enough by Proposition 3.3 (since f is injective and $[0, 1]$ is compact, M is an embedded submanifold, and it is locally flat since it is 1-dimensional). \square

Proposition 8.4. *Let M be a closed C^0 -submanifold of a manifold P . Then,*

$$(8.4) \quad \mu\text{supp}(M) \subseteq C(M, M)^\top$$

with equality if $\dim M = 1$.

Recall that on the other hand, one has $\pi_P^{-1}(M) \cap \overline{C(M)^\circ} \subseteq \mu\text{supp}(M)$.

Proof. Let $(x, \eta) \in \mu\text{supp}(M)$. Then $\eta \notin C_x(M, M)^\top$ is equivalent to $C_x(M, M) \cap \eta^\perp = \{0\}$. If this is the case, then by Proposition 8.1, the submanifold M is locally at x the graph of a map f . Then, by the upper bound on the conormal, we obtain $\mu\text{supp}_x(M) = \Lambda_x(f) \subseteq C_x(f)^\top = C_x(M, M)^\top$.

In the 1-dimensional case, equality follows from Lemma 8.3. Indeed, let $u \in \dot{C}_x(M, M)$ and $\xi \in u^\perp$. Let $f:]-1, 1[\rightarrow M$ be a parametrization of M in a neighborhood of x . Then there exist sequences of reals $x_n, y_n \in]-1, 1[$ and $c_n > 0$

with $x_n, y_n \xrightarrow{n} 0$ and $c_n(f(y_n) - f(x_n)) \xrightarrow{n} u$. Since $u^\perp = -u^\perp$, we can suppose that $x_n < y_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let ξ_n be the vector of $(f(y_n) - f(x_n))^\perp$ closest to ξ . Then $\xi_n \xrightarrow{n} \xi$. Applying Lemma 8.3 in each interval $[x_n, y_n]$ gives $z_n \in]x_n, y_n[$ such that $\xi_n \in \mu\text{supp}_{f(z_n)}(M)$. One has $z_n \xrightarrow{n} 0$, so $f(z_n) \xrightarrow{n} x$, so $\xi \in \mu\text{supp}_x(M)$. \square

Appendix A. Lipschitz continuity and strict differentiability

In this appendix, we recall standard definitions related to Lipschitz continuity and differentiability, mainly to set the notation.

Definition A.1 ((Pointwise) C -Lipschitz function). Let $f: X \rightarrow Y$ be a function between metric spaces and let $C \in \mathbb{R}$.

- (1) The function f is C -Lipschitz if

$$(A.1) \quad d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)$$

for any $x_1, x_2 \in X$.

- (2) Let $x_0 \in X$. The function f is C -Lipschitz (resp. pointwise C -Lipschitz) at x_0 if for all $\epsilon > 0$ there exists a neighborhood U of x_0 such that f is $(C + \epsilon)$ -Lipschitz on U (resp. such that $d_Y(f(x_0), f(x)) \leq (C + \epsilon) d_X(x_0, x)$ for any $x \in U$).
- (3) The infimum of the numbers C such that f is C -Lipschitz (resp. pointwise C -Lipschitz) at x_0 is called the Lipschitz constant (resp. pointwise Lipschitz constant) of f at x_0 and is denoted by $\text{Lip}_{x_0}(f)$ (resp. $\text{Lip}_{x_0}^{\text{pw}}(f)$).

It is easy to prove that

$$(A.2) \quad \text{Lip}(f) = \limsup(\text{Lip}^{\text{pw}}(f)): X \rightarrow \overline{\mathbb{R}}.$$

The extended reals $\text{Lip}_{x_0}(f)$ and $\text{Lip}_{x_0}^{\text{pw}}(f)$ in Definition A.1(3) are actually minima when finite.

For maps between manifolds, one can read Lipschitz properties in charts, and because of rescaling, the notion of C -Lipschitz continuity makes no sense anymore. One can only require a map to be C -Lipschitz for some $C > 0$, or for all $C > 0$, in the following sense.

Definition A.2 ((Pointwise) Lipschitz and (strictly) differentiable maps). Let $f: M \rightarrow N$ be a continuous map between manifolds and let $x_0 \in M$.

- (1) The map f is Lipschitz (resp. pointwise Lipschitz) at x_0 if there exist charts U at x_0 and V at $f(x_0)$ and a constant $C \in \mathbb{R}$ such that in these charts f is

C -Lipschitz (resp. C -pointwise Lipschitz) at x_0 . It is *Lipschitz* if it is Lipschitz at x for all $x \in M$.

- (2) The map f is *strictly differentiable* (resp. *differentiable*) at x_0 if there exist a linear map $L: T_{x_0}M \rightarrow T_{f(x_0)}N$ and charts U at x_0 and V at $f(x_0)$ such that in these charts, for all $\epsilon > 0$, the map $f - L$ is ϵ -Lipschitz (resp. pointwise ϵ -Lipschitz) at x_0 , that is, there exists a neighborhood $U_\epsilon \subseteq U$ of x such that $f - L$ is ϵ -Lipschitz on U_ϵ .

The second item of this definition is of course a rewording of the usual definitions (if L exists, it is unique and equal to $T_{x_0}f$). It emphasizes the naturality of the notion of strict differentiability. Strict differentiability is the good notion of “ C^1 at a point”: if a map is differentiable on a neighborhood of a point, then it is strictly differentiable at that point if and only if its derivative is continuous at that point. It is also the natural hypothesis for the inverse function theorem. Strict differentiability at a point implies Lipschitz continuity in a neighborhood of that point.

The following lemma is used to prove Lemma 2.5.

Lemma A.3. *Let U be an open subset of \mathbb{R}^n and $x \in U$. Let $\gamma \subseteq \mathbb{R}^n$ be a closed cone. Let $\phi: U \rightarrow \mathbb{R}$ be a function which is strictly differentiable at x with $\phi(x) = 0$ and $d\phi(x) \in \text{Int}(\gamma^\circ)$. Then there exists an open neighborhood $V \subseteq U$ of x such that $V \cap ((V \cap \{\phi < 0\}) + \gamma) \subseteq V \cap \{\phi < 0\}$, that is, $V \cap \{\phi < 0\}$ is γ -open in V in the sense of [6, Def. 3.2.1].*

Proof. Set $L := d\phi(x)$ and $\epsilon := \min\{|\langle L, u \rangle| \mid u \in \gamma \text{ and } \|u\| = 1\}$. Let $V \subseteq U$ be an open neighborhood of x where the function $\phi - L$ is ϵ -Lipschitz. Let $y \in V \cap \{\phi < 0\}$ and $u \in \gamma$ be such that $y + u \in V$. Then $\phi(y + u) \leq \phi(y) + \langle L, u \rangle + \epsilon\|u\| < 0$. \square

Appendix B. Tangent cones

In this appendix, we recall the definitions of the tangent cone $C(A)$, the strict tangent cone $N(A)$ and the Whitney cone $C(A, B)$ of subsets A, B of a manifold, and their main properties. The (elementary) proofs can be found in [4, App. A].

Let M be a manifold. Let $x \in M$, let $x_n, y_n \xrightarrow{n} x$ and $(c_n) \in \mathbb{R}^{\mathbb{N}}$ be three sequences, and let $u \in T_x M$. We write “ $c_n(y_n - x_n) \xrightarrow{n} u$ ” to mean that in some chart ϕ at x , one has $c_n(\phi(y_n) - \phi(x_n)) \xrightarrow{n} T_x \phi(u)$. This then holds for any chart at x . If $A, B \subseteq M$, their *Whitney cone* (see [6, Def. 4.1.1 and Prop. 4.1.2]) is defined as

$$(B.1) \quad C_x(A, B) := \{u \in T_x M \mid \exists (x_n) \in A^{\mathbb{N}}, \exists (y_n) \in B^{\mathbb{N}}, \exists (c_n) \in (\mathbb{R}_{>0})^{\mathbb{N}} \text{ such that } x_n \xrightarrow{n} x \text{ and } y_n \xrightarrow{n} x \text{ and } c_n(y_n - x_n) \xrightarrow{n} u\}.$$

We write $C(A, B) := \bigcup_{x \in M} C_x(A, B)$ (and similarly for the cones defined below). We define the *tangent cone* of A as

$$(B.2) \quad C_x(A) := C_x(\{x\}, A).$$

We define the *strict tangent cone* (or strict normal cone; see [6, Def. 5.3.6]) of A as

$$(B.3) \quad N(A) := TM \setminus C(A, M \setminus A).$$

One has

$$(B.4) \quad \tau_M^{-1}(\overline{A}) \cap N(A) \subseteq C(A) \subseteq C(A, A).$$

Proposition B.1 (Elementary properties of the Whitney cone). *Let M be a manifold and $A, B, A_1, A_2 \subseteq M$.*

- (1) *The Whitney cone $C(A, B) \subseteq TM$ is a closed cone.*
- (2) *Antisymmetry: $C(A, B) = -C(B, A)$.*
- (3) *Monotony: if $A_1 \subseteq A_2$, then $C(A_1, B) \subseteq C(A_2, B)$.*
- (4) *Additivity: $C(A_1 \cup A_2, B) = C(A_1, B) \cup C(A_2, B)$.*
- (5) *Stability under closure: $C(\overline{A}, B) = C(A, B)$.*
- (6) *Projection on the manifold: $\tau_M(C(A, B)) = \overline{A} \cap \overline{B}$.*
- (7) *If $x \in \text{Int}(A) \cap \overline{B}$, then $C_x(A, B) = T_x M$.*
- (8) *If $f: M \rightarrow N$ is strictly differentiable at $x \in M$, then $T_x f(C_x(A, B)) \subseteq C_{f(x)}(f(A), f(B))$.*

These properties imply corresponding properties for the tangent cone $C(A)$ (and in that case, strict differentiability in item (8) can be replaced by differentiability). The tangent cone $C(A)$ is pointwise closed but need not be closed, and one can have $C(A) \subsetneq \overline{C(A)} \subsetneq C(A, A)$. Note also that $C(A, A)$ is a closed symmetric cone but need not be convex.

Example B.2. Let A be the graph of the function $|\cdot|: \mathbb{R} \rightarrow \mathbb{R}$. Then $C_0(A) = \overline{C(A)}_0 = A \subsetneq C_0(A, A) = \{(u, v) \in \mathbb{R}^2 \mid |v| \leq |u|\}$.

Let A be the graph of the function $\mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2 \sin(1/x)$. Then $C_0(A) = \mathbb{R} \times \{0\} \subsetneq \overline{C(A)}_0 = C_0(A, A) = \{(u, v) \in \mathbb{R}^2 \mid |v| \leq |u|\}$.

The strict tangent cone $N(A)$ of a subset A of a manifold M is an open convex cone. Let V be a chart at $x \in M$. One has $u \in N_x(A)$ if and only if there exist an open neighborhood $U \subseteq V$ of x and an open conic neighborhood $\gamma \subseteq T_x U$ of u such that in a chart, $U \cap ((U \cap A) + \gamma) \subseteq A$.

Proposition B.3. *Let A be a subset of a manifold M . One has $C(\partial A, \partial A) \cap N(A) = \emptyset$ and $N_x(A) \cap N_x(M \setminus A) = \emptyset$ for any $x \in \partial A$.*

Proof. (i) $C_x(\partial A, \partial A) \cap N_x(A) = \emptyset$ for $x \in M$. This is trivial if $x \notin \partial A$, so let $x \in \partial A$. Let $u \in C_x(\partial A, \partial A)$. There exist sequences $(x_n), (x'_n) \in (\partial A)^\mathbb{N}$ both converging to x and a sequence $(c_n) \in (\mathbb{R}_{>0})^\mathbb{N}$ such that $c_n(x'_n - x_n) \xrightarrow{n} u$. If $u = 0$, the result is trivial, so we can suppose that $x_n \neq x'_n$ for any $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exist sequences $(x_{n,m})_m \in A^\mathbb{N}$ and $(x'_{n,m})_m \in (M \setminus A)^\mathbb{N}$ converging respectively to x_n and x'_n . Let $\rho: \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function such that $c_n(x_{n,\rho(n)} - x_n - x'_{n,\rho(n)} + x'_n) \xrightarrow{n} 0$. Then the sequences $(x_{n,\rho(n)})$ and $(x'_{n,\rho(n)})$ and (c_n) show that $u \in C_x(A, M \setminus A) = T_x M \setminus N_x(A)$.

(ii) $N_x(A) \cap N_x(M \setminus A) = \emptyset$ for $x \in \partial A$. Fix a chart at x and suppose that there exists $u \in N_x(A) \cap N_x(M \setminus A)$. There exist an open neighborhood U of x and an open conic neighborhood $\gamma \subseteq T_x M$ of u such that $U \cap ((U \cap A) + \gamma) \subseteq A$ and $U \cap ((U \setminus A) + \gamma) \subseteq U \setminus A$. Therefore, $U \cap ((U \cap A) + \gamma) \cap ((U \setminus A) + \gamma) = \emptyset$. In particular, since the only conic neighborhood of 0 is $T_x M$, one has $u \neq 0$. Let $(x_n) \in (U \cap A)^\mathbb{N}$ and $(y_n) \in (U \setminus A)^\mathbb{N}$ be sequences converging to x . Rescaling u , we can suppose that $x + u \in U$. The set $(x + u) - \gamma$ is a neighborhood of x , so for n large enough, one has $x + u \in U \cap (x_n + \gamma) \cap (y_n + \gamma) \neq \emptyset$, a contradiction. \square

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