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Pro-p Grothendieck Conjecture for Hyperbolic Polycurves

by

Koichiro SAWADA

Abstract

In the present paper, we study the geometrically pro-p fundamental groups of hyperbolic polycurves, i.e., successive extensions of families of hyperbolic curves. Among other results, we show that the isomorphism class of a hyperbolic polycurve of dimension ≤ 4 over a sub-p-adic field satisfying a certain group-theoretic condition is completely determined by the geometrically pro-p fundamental group equipped with surjection onto the absolute Galois group of the base field.

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§1. Introduction

Let k be a field of characteristic zero, $\overline{k}$ an algebraic closure of k and X a variety over k. (In this paper, a variety over k is defined to be a scheme that is of finite type, separated and geometrically connected over k (cf. Definition 2.4).) Write $G_k := \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group of k and Π_X for the étale fundamental group of X. Then the structure morphism $X \to \mathrm{Spec}\, k$ induces a natural surjection $\Pi_X \twoheadrightarrow G_k$. Write $\Delta_{X/k}$ for the kernel of this surjection $\Pi_X \twoheadrightarrow G_k$. Grothendieck proposed the following philosophy (cf. [7], [8]):

> For certain types of k, if X is an "anabelian variety" over k, then the isomorphism class of X is completely determined by the fundamental group Π_X as a profinite group equipped with the surjection $\Pi_X \twoheadrightarrow G_k$.

We often call this philosophy the "Grothendieck conjecture". Although we do not have any general definition of the notion of an "anabelian variety", successive

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K. Sawada: Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan;
e-mail: sawada@kurims.kyoto-u.ac.jp

extensions of families of hyperbolic curves (hereinafter called "hyperbolic polycurves"; cf. Definition 3.1(ii)) have been regarded as typical examples of anabelian varieties. The Grothendieck conjecture for hyperbolic polycurves of dimension ≤ 2 was proved in [11] (cf. [11, Thms. 16.5, a2.4]), and thereafter, in [10], it was extended to the case of hyperbolic polycurves of dimension ≤ 4 (cf. [10, Cor. 3.18]).

On the other hand, we may consider a pro-p version of the Grothendieck conjecture. Let p be a prime number and $X \to Y$ a morphism between connected noetherian schemes. Write $\Delta_{X/Y}$ for the kernel of the (outer) homomorphism $\Pi_X \to \Pi_Y$ induced by the morphism $X \to Y$, $\Delta^p_{X/Y}$ for the maximal pro-p quotient of $\Delta_{X/Y}$ and $\Pi^p_{X/Y} := \Pi_X/\ker(\Delta_{X/Y} \twoheadrightarrow \Delta^p_{X/Y})$. Then let us consider the following:

> For certain types of k, if X is an "anabelian variety" over k, then is the isomorphism class of X completely determined by the geometrically pro-p fundamental group $\Pi^p_{X/k}$ as a profinite group equipped with the surjection $\Pi^p_{X/k} \twoheadrightarrow G_k$?

In [11], a very strong form of the pro-p Grothendieck conjecture for hyperbolic curves was proved (cf. [11, Thm. 16.5]). In the present paper, we consider the pro-p Grothendieck conjecture for hyperbolic polycurves. Let

$$X = X_n \to X_{n-1} \to \cdots \to X_2 \to X_1 \to \operatorname{Spec} k = X_0$$

be a sequence of parametrizing morphisms of a hyperbolic polycurve X over k (cf. Definition 3.1(ii)). Then for any triplet of integers (i, j, l) such that $0 \leq i < j < l \leq n$, we have an exact sequence of profinite groups

$$1 \to \Delta_{X_l/X_j} \to \Delta_{X_l/X_i} \to \Delta_{X_j/X_i} \to 1$$

(cf. Remark 3.8), which plays an important role in the study of the Grothendieck conjecture for hyperbolic polycurves of [10]. However, since the operation of taking the maximal pro-p quotient of a profinite group is not exact, the sequence

$$1 \to \Delta^p_{X_l/X_j} \to \Delta^p_{X_l/X_i} \to \Delta^p_{X_j/X_i} \to 1$$

is not exact in general. For this reason, let us introduce a condition that the above sequence is exact, which we call $(*)_p$ (cf. Definition 3.10), and consider the pro-p Grothendieck conjecture for hyperbolic polycurves satisfying condition $(*)_p$. The following is one of the main results of the present paper.

Theorem 1.1 (Cf. Theorems 4.4, 4.17, Corollaries 4.19, 4.21). *Let p be a prime number, n a positive integer, k a sub-p-adic field (cf. Definition 4.1), X a hyperbolic polycurve of dimension n over k satisfying condition $(*)_p$, Y a normal variety over*

k and $\phi : \Pi^p_{Y/k} \to \Pi^p_{X/k}$ an open homomorphism. Suppose that one of the following conditions (1), (2), (3), (4) *is satisfied:*

(1) $n = 1$.

(2) *The following conditions are satisfied:*

- (2-i) $n = 2$.
- (2-ii) *The kernel of ϕ is topologically finitely generated.*

(3) *The following conditions are satisfied:*

- (3-i) $n = 3$.
- (3-ii) *The kernel of ϕ is finite.*
- (3-iii) *Y is of p-LFG-type (cf. Definition 3.25).*
- (3-iv) $3 \leq \dim(Y)$.

(4) *The following conditions are satisfied:*

- (4-i) $n = 4$.
- (4-ii) *ϕ is injective.*
- (4-iii) *Y is a hyperbolic polycurve over k satisfying condition $(*)_p$.*
- (4-iv) $4 \leq \dim(Y)$.

Then ϕ arises from a uniquely determined dominant morphism $Y \to X$ over k.

The next result follows from Theorem 1.1.

Theorem 1.2 (Cf. Corollary 4.22). *Let p be a prime number, k a sub-p-adic field and X, Y hyperbolic polycurves over k satisfying condition $(*)_p$. Suppose that either X or Y is of dimension ≤ 4. Then the natural map*

$$\mathrm{Isom}_k(Y, X) \to \mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k})/\mathrm{Inn}(\Delta^p_{X/k})$$

is bijective.

This implies that the isomorphism class of a hyperbolic polycurve of dimension ≤ 4 over a sub-p-adic field satisfying condition $(*)_p$ is completely determined by the geometrically pro-p fundamental group equipped with surjection onto the absolute Galois group of the sub-p-adic field. Condition $(*)_p$ is (at least, in order to perform the proofs in the present paper) essential. The majority of the proof of Theorem 1.1 is analogous to the proof of the Grothendieck conjecture for hyperbolic polycurves in [10], together with Theorem 1.1 in the case where condition (1) is satisfied, which was essentially proved in [11] (cf. [11, Thm. 16.5]). However, the difference between the pro-p version and the original (profinite) version is the necessity of considering base schemes. In fact, $\Pi^p_{X/Y}$ depends on the base scheme Y, although Π_X does not depend on Y. It seems (to the author) that choosing a suitable base scheme to complete the proof is difficult. In the present paper, to avoid this problem, first we assume a certain condition stronger than $(*)_p$ and use the maximal pro-p quotient Π^p_X of Π_X (cf. Theorem 4.16, Corollaries 4.18, 4.20), which is independent of the base scheme, instead of "$\Pi^p_{X/Y}$", which essentially depends on the base scheme "Y". Then, by replacing the base field k by a suitable Galois extension and then descending, we complete the proof of Theorem 1.1.

Next, recall that, if X and Y are hyperbolic polycurves over a field k, it follows that $\mathrm{Isom}_k(Y, X)$ is finite (cf. Proposition 5.5). Thus, if the natural map discussed in Theorem 1.2 is bijective without the assumption that "either X or Y is of dimension ≤ 4" holds, then $\mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k})/\mathrm{Inn}(\Delta^p_{X/k})$ is finite. In general, it is not known that the map discussed in Theorem 1.2 is bijective. However, we can prove the finiteness of $\mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k})/\mathrm{Inn}(\Delta^p_{X/k})$.

Theorem 1.3 (Cf. Theorem 5.6). *Let p be a prime number, k a sub-p-adic field and X, Y hyperbolic polycurves over k. Suppose that at least one of X/k, Y/k satisfies condition $(*)_p$. Then the set*

$$\mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k})/\mathrm{Inn}(\Delta^p_{X/k})$$

is finite.

Remark. A morphism (resp. k-morphism) $Y \to X$ between connected noetherian schemes (resp. k-schemes) induces an outer homomorphism $\Pi_Y \to \Pi_X$ (resp. outer homomorphism $\Pi_Y \to \Pi_X$ over G_k), i.e., a Π_X-conjugacy class of homomorphisms $\Pi_Y \to \Pi_X$ (resp. $\Delta_{X/k}$-conjugacy class of homomorphisms $\Pi_Y \to \Pi_X$ over G_k). However, we sometimes choose one homomorphism belonging to the Π_X- (resp. $\Delta_{X/k}$-) conjugacy class of homomorphisms $\Pi_Y \to \Pi_X$ induced by $Y \to X$, and we call it the homomorphism induced by $Y \to X$.

§2. Étale fundamental groups of varieties

In the present Section 2, we study étale fundamental groups of algebraic varieties. Let k be a field of characteristic zero, $\overline{k}$ an algebraic closure of k, $G_k := \mathrm{Gal}(\overline{k}/k)$ and $\mathfrak{Primes}$ the set of all prime numbers.

Definition 2.1. Let X be a connected noetherian scheme.

(i) We shall write

$$\Pi_X$$

for the étale fundamental group of X (for some choice of basepoint).

(ii) Let Y be a connected noetherian scheme and $f : X \to Y$ a morphism. Then we shall write

$$\Delta_f = \Delta_{X/Y} \subset \Pi_X$$

for the kernel of the outer homomorphism $\Pi_X \to \Pi_Y$ induced by f. If $Y = \operatorname{Spec} A$, then by abuse of notation we often write

$$\Delta_{X/A}$$

instead of $\Delta_{X/Y}$. (Similar notation will be used for $\Pi^p_{X/S}$, $\Delta^p_{f/S} = \Delta^p_{X\to Y/S}$, $\Delta^{(p)}_{X/Y}$, which are defined below.)

Lemma 2.2 ([10, Lem. 1.2]). *Let X be a connected noetherian normal scheme. Write $\eta \to X$ for the generic point of X. Then the outer homomorphism $\Pi_\eta \to \Pi_X$ induced by the morphism $\eta \to X$ is surjective.*

Lemma 2.3 ([10, Lem. 1.3]). *Let X, Y be connected noetherian schemes and $f : X \to Y$ a morphism. Suppose that Y is normal and that f is dominant and of finite type. Then the outer homomorphism $\Pi_X \to \Pi_Y$ induced by f is open.*

Definition 2.4. Let X be a scheme over k. Then we shall say that X is a *variety* over k if X is of finite type, separated and geometrically connected over k.

Lemma 2.5 ([10, Lem. 1.5]). *Let X be a variety over k. Then the sequence of schemes $X \times_k \overline{k} \overset{\mathrm{pr}_1}{\to} X \to \operatorname{Spec} k$ determines an exact sequence of profinite groups*

$$1 \to \Pi_{X\times_k \overline{k}} \to \Pi_X \to G_k \to 1.$$

In particular, we obtain an isomorphism $\Pi_{X\times_k\overline{k}} \overset{\sim}{\to} \Delta_{X/k}$ (which is well defined up to Π_X-conjugation).

Lemma 2.6 ([10, Lem. 1.6]). *Let X, Y be connected noetherian schemes and $f : X \to Y$ a morphism. Suppose that f is of finite type, separated, dominant and generically geometrically connected. Suppose, moreover, that Y is normal. Then the outer homomorphism $\Pi_X \to \Pi_Y$ induced by f is surjective.*

Lemma 2.7 ([10, Lem. 1.7]). *Let X be a variety over k. Suppose that G_k is topologically finitely generated (e.g., the case where $k = \overline{k}$). Then the profinite group Π_X is topologically finitely generated.*

Definition 2.8. Let X, Y be integral noetherian schemes and $f : X \to Y$ a dominant morphism of finite type. Then we shall write

$$\operatorname{Nor}(f) = \operatorname{Nor}(X/Y) \to Y$$

for the normalization of Y in the finite extension of the function field of Y obtained by forming the algebraic closure of the function field of Y in the function field of X. Note that $\mathrm{Nor}(f) = \mathrm{Nor}(X/Y)$ is integral and normal and the morphism $\mathrm{Nor}(f) = \mathrm{Nor}(X/Y) \to Y$ is dominant and affine.

Lemma 2.9 ([10, Lem. 1.9]). *Let X, Y be integral noetherian schemes and $f : X \to Y$ a dominant morphism of finite type. Suppose that X is normal. Then f factors through the natural morphism $\mathrm{Nor}(f) \to Y$ and the resulting morphism $X \to \mathrm{Nor}(f)$ is dominant and generically geometrically irreducible. If, moreover, X and Y are varieties over k and f is a morphism over k, then the natural morphism $\mathrm{Nor}(f) \to Y$ is finite and surjective and $\mathrm{Nor}(f)$ is a normal variety over k.*

Lemma 2.10 ([10, Prop. 1.10(i)]). *Let S, X and Y be connected noetherian normal schemes, $Y \to X \to S$ morphisms of schemes and $\overline{s} \to S$ a geometric point of S. Suppose that the following conditions are satisfied:*

(1) *$Y \to X$ is dominant and induces an outer surjection $\Pi_Y \twoheadrightarrow \Pi_X$.*
(2) *$X \to S$ is surjective, of finite type, separated and generically geometrically integral.*
(3) *$Y \to S$ is of finite type, separated, faithfully flat, geometrically normal and generically geometrically connected.*
(4) *For any connected finite étale covering $X' \to X$ and any geometric point $\overline{s}' \to \mathrm{Nor}(X'/S)$ of $\mathrm{Nor}(X'/S)$ that lifts the geometric point $\overline{s}$ of S, the geometric fiber $X' \times_{\mathrm{Nor}(X'/S)} \overline{s}'$ of $X' \to \mathrm{Nor}(X'/S)$ at $\overline{s}' \to \mathrm{Nor}(X'/S)$ is connected. (Note that it follows from Lemma 2.9 that condition (4) is satisfied if the image of the geometric point $\overline{s} \to S$ is the generic point of S).*

Then the sequence of connected schemes $X \times_S \overline{s} \overset{\mathrm{pr}_1}{\to} X \to S$ determines an exact sequence of profinite groups

$$\Pi_{X \times_S \overline{s}} \to \Pi_X \to \Pi_S \to 1.$$

Lemma 2.11 ([10, Cor. 1.11]). *Let S, X be connected noetherian normal schemes and $X \to S$ a morphism of schemes that is surjective, of finite type, separated and generically geometrically irreducible. Suppose that the function field of S is of characteristic zero. Suppose, moreover, that one of the following conditions is satisfied:*

(1) *There exists an open subscheme $U \subset X$ of X such that the composite $U \hookrightarrow X \to S$ is surjective and smooth.*

(2) *There exist a connected normal scheme* Y *and a morphism* $Y \to X$ *that is proper, surjective and that induces an isomorphism between the respective function fields, such that the composite* $Y \to X \to S$ *is smooth.*

Then $\Delta_{X/S}$ *is topologically finitely generated.*

Definition 2.12. Let G be a profinite group and Σ a subset of $\mathfrak{Primes}$. Then we shall write

$$G^{\Sigma}$$

for the maximal pro-Σ quotient of G. Let p be a prime number. Then we shall write simply

$$G^{p}$$

for the pro-p group $G^{\{p\}}$.

Remark 2.13. The right exactness of $G \mapsto G^{\Sigma}$ is well known. Moreover, one verifies easily that if $U \subset G^{\Sigma}$ is an open subgroup of G^{Σ} and V is the inverse image of $U \subset G^{\Sigma}$ by the natural surjection $G \twoheadrightarrow G^{\Sigma}$, then the natural surjection $G \twoheadrightarrow G^{\Sigma}$ induces an isomorphism $V^{\Sigma} \xrightarrow{\sim} U$.

Definition 2.14. Let p be a prime number, S, X connected noetherian schemes and $X \to S$ a morphism of schemes. Then we shall write

$$\Pi^{p}_{X/S}$$

for the quotient of Π_X by the kernel of the natural surjection $\Delta_{X/S} \twoheadrightarrow \Delta^{p}_{X/S}$ (which is a characteristic subgroup of $\Delta_{X/S}$).

Remark 2.15. We shall use not only $\Pi^{p}_{X/S}$ but also the maximal pro-p quotient of Π_X, which we shall write Π^{p}_{X} (as Definition 2.12 above).

Remark 2.16. In the notation of Definition 2.14, let U be an open subgroup of $\Pi^{p}_{X/S}$ (resp. Π^{p}_{X}). In the present paper, we shall refer to the connected finite étale covering of X corresponding to the inverse image of U by the natural surjection $\Pi_X \twoheadrightarrow \Pi^{p}_{X/S}$ (resp. $\Pi_X \twoheadrightarrow \Pi^{p}_{X}$) as the covering corresponding to U.

Definition 2.17. Let p be a prime number, S a connected noetherian scheme, X, Y connected noetherian schemes over S and $f : X \to Y$ a morphism over S. Then we shall write

$$\Delta^{p}_{f/S} = \Delta^{p}_{X\to Y/S} := \ker(\Pi^{p}_{X/S} \to \Pi^{p}_{Y/S}), \quad \Delta^{(p)}_{f} = \Delta^{(p)}_{X/Y} := \ker(\Pi^{p}_{X} \to \Pi^{p}_{Y}).$$

Note that $\Delta^{p}_{X\to Y/S} = \ker(\Delta^{p}_{X/S} \to \Delta^{p}_{Y/S})$, and that $\ker(\Delta_{X/S} \to \Delta_{Y/S} \twoheadrightarrow \Delta^{p}_{Y/S}) \subset \Delta_{X/S}$ is the inverse image of $\Delta^{p}_{X\to Y/S} \subset \Delta^{p}_{X/S}$ by the natural surjection $\Delta_{X/S} \twoheadrightarrow \Delta^{p}_{X/S}$.

Lemma 2.18. *Let p be a prime number, S, X, Y connected noetherian schemes and $X \to Y \to S$ morphisms of schemes. Suppose that the outer homomorphism $\Pi_X \to \Pi_Y$ induced by $X \to Y$ is surjective. Then $\Delta^p_{X\to Y/S}$ is the image of $\Delta_{X/Y} \subset \Pi_X$ by the natural surjection $\Pi_X \twoheadrightarrow \Pi^p_{X/S}$.*

Proof. Since $\Pi_X \twoheadrightarrow \Pi_Y$ is surjective, the sequence of profinite groups

$$1 \to \Delta_{X/Y} \to \Delta_{X/S} \to \Delta_{Y/S} \to 1$$

is exact. Thus, the sequence of pro-p groups

$$\Delta^p_{X/Y} \to \Delta^p_{X/S} \to \Delta^p_{Y/S} \to 1$$

is exact. This induces a surjection $\Delta^p_{X/Y} \twoheadrightarrow \ker(\Delta^p_{X/S} \to \Delta^p_{Y/S}) = \Delta^p_{X\to Y/S}$, hence $\Delta_{X/Y} \to \Delta^p_{X\to Y/S}$ is surjective. This completes the proof of Lemma 2.18. □

Definition 2.19. Let G be a profinite group. Then we shall say that G is *slim* if every open subgroup of G is center-free.

Lemma 2.20. *Let G be a profinite group and Π_1, Π_2 profinite groups over G. For $i = 1, 2$, write $\Delta_i = \ker(\Pi_i \to G)$. Suppose that Δ_2 is slim. Write $\mathrm{Hom}^{\mathrm{open}}_G(\Pi_1, \Pi_2)$ for the set of open homomorphisms from Π_1 to Π_2 over G. Then the natural map*

$$\mathrm{Hom}^{\mathrm{open}}_G(\Pi_1, \Pi_2) \to \mathrm{Hom}(\Delta_1, \Delta_2)$$

is injective.

Proof. Let $\varphi, \psi \in \mathrm{Hom}^{\mathrm{open}}_G(\Pi_1, \Pi_2)$ be elements of $\mathrm{Hom}^{\mathrm{open}}_G(\Pi_1, \Pi_2)$ that map to the same element $\theta \in \mathrm{Hom}(\Delta_1, \Delta_2)$ by the above map. Note that $\theta : \Delta_1 \to \Delta_2$ is an open homomorphism. Let $a \in \Pi_1$ and $b \in \Delta_1$. Then we have $\varphi(aba^{-1}) = \theta(aba^{-1}) = \psi(aba^{-1})$ and $\varphi(b) = \theta(b) = \psi(b)$, hence $\psi(a)^{-1}\varphi(a)\theta(b) = \theta(b)\psi(a)^{-1}\varphi(a)$. On the other hand, $\psi(a)^{-1}\varphi(a) \in \ker(\Pi_2 \twoheadrightarrow G) = \Delta_2$. Thus, since $b \in \Delta_1$ is arbitrary, $\psi(a)^{-1}\varphi(a) \in Z_{\Delta_2}(\mathrm{Im}\,\theta)$. Now since Δ_2 is slim and $\mathrm{Im}\,\theta \subset \Delta_2$ is an open subgroup of Δ_2, one verifies easily that $Z_{\Delta_2}(\mathrm{Im}\,\theta) = \{1\}$, which implies that $\varphi = \psi$. This completes the proof of Lemma 2.20. □

§3. Pro-p fundamental groups of hyperbolic polycurves

In the present Section 3, we study pro-p étale fundamental groups of hyperbolic polycurves. Let k be a field of characteristic zero, $\overline{k}$ an algebraic closure of k, $G_k := \mathrm{Gal}(\overline{k}/k)$ and $\mathfrak{Primes}$ the set of all prime numbers.

Definition 3.1 (Cf. [10, Def. 2.1]). Let S be a scheme and X a scheme over S.

(i) We shall say that X is a *hyperbolic curve* (*of type*(g, r)) over S if there exist

- a pair of nonnegative integers (g, r);
- a scheme X^{cpt} which is smooth, proper, geometrically connected and of relative dimension 1 over S;
- a (possibly empty) closed subscheme $D \subset X^{\mathrm{cpt}}$ of X^{cpt} which is finite and étale over S

such that

- $2g - 2 + r > 0$;
- any geometric fiber of $X^{\mathrm{cpt}} \to S$ is (a necessarily smooth proper curve) of genus g;
- the finite étale covering $D \hookrightarrow X^{\mathrm{cpt}} \to S$ is of degree r;
- X is isomorphic to $X^{\mathrm{cpt}} \setminus D$ over S.

We shall refer to the above integer g as the genus of X over S.

(ii) We shall say that X is a *hyperbolic polycurve* (*of relative dimension* n) over S if there exist a positive integer n and a (not necessarily unique) factorization of the structure morphism $X \to S$,

$$X = X_n \to X_{n-1} \to \cdots \to X_2 \to X_1 \to S = X_0$$

such that, for each $i = 1, \ldots, n$, $X_i \to X_{i-1}$ is a hyperbolic curve. We shall refer to the above morphism $X \to X_{n-1}$ as a *parametrizing morphism* for X and refer to the above factorization of $X \to S$ as a *sequence of parametrizing morphisms* (cf. Remark 3.3).

Remark 3.2. In the notation of Definition 3.1(ii), suppose that S is a normal (resp. smooth) variety of dimension m over k. Then any hyperbolic polycurve of relative dimension n over S is a normal (resp. smooth) variety of dimension $n+m$ over k.

Remark 3.3. A sequence of parametrizing morphisms of $X \to S$,

$$X = X_n \to X_{n-1} \to \cdots \to X_2 \to X_1 \to S = X_0,$$

is not necessarily unique. In the present paper, a hyperbolic polycurve is always assumed to be equipped with a fixed sequence of parametrizing morphisms of the hyperbolic polycurve unless otherwise specified.

Definition 3.4 (Cf. [10, Def. 2.2]). In the notation of Definition 3.1(i), suppose that S is normal. Then the pair "(X^{cpt}, D)" is uniquely determined up to canonical isomorphism over S (cf. [12, §0]). We shall refer to X^{cpt} as the *smooth compactification* of X over S and refer to D as the *divisor of cusps* of X over S.

Proposition 3.5 ([10, Prop. 2.3]). *Let n be a positive integer, S a connected noetherian separated normal scheme over k, X a hyperbolic polycurve of relative dimension n over S and $Y \to X$ a connected finite étale covering of X. For each $i = 0, \ldots, n$, write $Y_i := \mathrm{Nor}(Y/X_i)$. Then the following hold:*

- (i) *For each integer i such that $1 \leq i \leq n$, Y_i is a hyperbolic curve over Y_{i-1}. Moreover, if we write Y_i^{cpt} for the smooth compactification of the hyperbolic curve Y_i over Y_{i-1}, then the composite $Y_i^{\mathrm{cpt}} \to Y_{i-1} \to X_{i-1}$ is proper and smooth. Furthermore, if we write $Y_i^{\mathrm{cpt}} \to Z_{i-1} \to X_{i-1}$ for the Stein factorization of the proper morphism $Y_i^{\mathrm{cpt}} \to X_{i-1}$, then Z_{i-1} is isomorphic to Y_{i-1} over X_{i-1}.*
- (ii) *For each integer i such that $0 \leq i \leq n$, the natural morphism $Y_i \to X_i$ is a connected finite étale covering.*

In particular, Y is a hyperbolic polycurve of relative dimension n over $\mathrm{Nor}(Y/S)$ *and the factorization*

$$Y = Y_n \to Y_{n-1} \to \cdots \to Y_1 \to \mathrm{Nor}(Y/S) = Y_0$$

is a sequence of parametrizing morphisms.

Remark 3.6. Hereafter, if X/S is a hyperbolic polycurve as in Proposition 3.5 and $Y \to X$ is a connected finite étale covering of X, we regard Y as the hyperbolic polycurve over $\mathrm{Nor}(Y/S)$ with the natural sequence of parametrizing morphisms as in Proposition 3.5 unless otherwise specified.

Proposition 3.7 ([10, Prop. 2.4 (i),(ii)]). *Let (m, n) be a pair of integers such that $0 \leq m < n$, S a connected noetherian separated normal scheme over k and X a hyperbolic polycurve of relative dimension n over S. Then the following hold:*

- (i) *For any geometric point $\overline{x}_m \to X_m$ of X_m, the sequence of connected schemes $X \times_{X_m} \overline{x}_m \overset{\mathrm{pr}_1}{\to} X \to X_m$ determines an exact sequence of profinite groups*

$$1 \to \Pi_{X \times_{X_m} \overline{x}_m} \to \Pi_X \to \Pi_{X_m} \to 1.$$

  In particular, we obtain an isomorphism $\Pi_{X \times_{X_m} \overline{x}_m} \overset{\sim}{\to} \Delta_{X/X_m}$ (which is well defined up to Π_X-conjugation).
- (ii) *Let T be a connected noetherian separated normal scheme over S and $T \to X_m$ a morphism over S. Then the natural morphisms $X \times_{X_m} T \overset{\mathrm{pr}_1}{\to} X$ and $X \times_{X_m} T \overset{\mathrm{pr}_2}{\to} T$ determine an outer isomorphism*

$$\Pi_{X \times_{X_m} T} \overset{\sim}{\to} \Pi_X \times_{\Pi_{X_m}} \Pi_T$$

and an isomorphism

$$\Delta_{X \times_{X_m} T/T} \overset{\sim}{\to} \Delta_{X/X_m}$$

(which is well defined up to Π_X-conjugation).

Remark 3.8. Note that, in the notation of Proposition 3.7, for any triplet of integers (i, j, l) such that $0 \leq i < j < l \leq n$, by considering the commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
 & & & & & & \Delta_{X_j/X_i} & & \\
 & & & & & & \downarrow & & \\
1 & \longrightarrow & \Delta_{X_l/X_j} & \longrightarrow & \Pi_{X_l} & \longrightarrow & \Pi_{X_j} & \longrightarrow & 1 \\
 & & \downarrow & & \| & & \downarrow & & \\
1 & \longrightarrow & \Delta_{X_l/X_i} & \longrightarrow & \Pi_{X_l} & \longrightarrow & \Pi_{X_i} & \longrightarrow & 1
\end{array}$$

(cf. Proposition 3.7(i)), we obtain a natural exact sequence of profinite groups

$$1 \to \Delta_{X_l/X_j} \to \Delta_{X_l/X_i} \to \Delta_{X_j/X_i} \to 1.$$

Lemma 3.9. *Let n be a positive integer, S a connected noetherian separated normal scheme over k and X a hyperbolic polycurve of relative dimension n over S. Then the following hold:*

(i) *For any triplet of integers (i, j, l) such that $0 \leq i < j < l \leq n$, the outer homomorphism $\Delta_{X_l/X_i} \to \Delta_{X_j/X_i}$ induced by the outer surjection $\Pi_{X_l} \to \Pi_{X_j}$ (cf. Proposition 3.7(i)) is surjective, and Δ_{X_l/X_i} is the inverse image of $\Delta_{X_j/X_i} \subset \Pi_{X_j}$ by the outer surjection $\Pi_{X_l} \twoheadrightarrow \Pi_{X_j}$.*

(ii) *Let $Y \to X$ be a connected finite étale covering of X. Let us fix a basepoint of Y. Then, for any pair of integers (i, j) such that $0 \leq i < j \leq n$, Π_{Y_i} (cf. Proposition 3.5) naturally coincides with $\mathrm{Im}(\Pi_{Y_j} \hookrightarrow \Pi_{X_j} \twoheadrightarrow \Pi_{X_i})$, and this determines an equality $\Delta_{Y_j/Y_i} = \Delta_{X_j/X_i} \cap \Pi_{Y_j}$.*

(iii) *In the notation of* (ii)*, suppose, moreover, given a pair of integers (i, j) such that $0 \leq i < j \leq n$ and $\Delta_{Y_j/Y_i} = \Delta_{X_j/X_i}$. Then for any pair of integers (l, m) such that $i \leq l < m \leq j$, we obtain an equality $\Delta_{Y_m/Y_l} = \Delta_{X_m/X_l}$.*

Proof. First, we verify assertion (i). It follows immediately that Δ_{X_l/X_i} is the inverse image of $\Delta_{X_j/X_i} \subset \Pi_{X_j}$ by the outer surjection $\Pi_{X_l} \twoheadrightarrow \Pi_{X_j}$. Moreover, it follows from the surjectivity of $\Pi_{X_l} \to \Pi_{X_j}$ that the outer homomorphism

$\Delta_{X_l/X_i} \to \Delta_{X_j/X_i}$ is surjective. This completes the proof of assertion (i). Next, we verify assertion (ii). The commutative diagram of connected schemes

$$\begin{array}{ccc} Y_j & \longrightarrow & X_j \\ \downarrow & & \downarrow \\ Y_i & \longrightarrow & X_i \end{array}$$

determines a commutative diagram of profinite groups

$$\begin{array}{ccc} \Pi_{Y_j} & \longrightarrow & \Pi_{X_j} \\ \downarrow & & \downarrow \\ \Pi_{Y_i} & \longrightarrow & \Pi_{X_i}, \end{array}$$

where the vertical arrows are surjective (cf. Propositions 3.5(i), 3.7(i)) and the horizontal arrows are injective (cf. Proposition 3.5(ii)). Thus, it holds that $\Pi_{Y_i} = \mathrm{Im}(\Pi_{Y_j} \hookrightarrow \Pi_{X_j} \twoheadrightarrow \Pi_{X_i})$. Moreover, it follows immediately that $\Delta_{Y_j/Y_i} \subset \Delta_{X_j/X_i} \cap \Pi_{Y_j}$. On the other hand, it follows from the injectivity of $\Pi_{Y_i} \to \Pi_{X_i}$ that $\Delta_{Y_j/Y_i} \supset \Delta_{X_j/X_i} \cap \Pi_{Y_j}$. This completes the proof of assertion (ii). Finally, we verify assertion (iii). To verify assertion (iii), it suffices to verify that for each integer l such that $i < l < j$, equalities

$$\Delta_{Y_j/Y_l} = \Delta_{X_j/X_l}, \quad \Delta_{Y_l/Y_i} = \Delta_{X_l/X_i}$$

hold. Now it follows from (i) and (ii) that

$$\begin{aligned} \Delta_{Y_j/Y_l} &= \Delta_{X_j/X_l} \cap \Pi_{Y_j} \\ &= \Delta_{X_j/X_l} \cap (\Delta_{X_j/X_i} \cap \Pi_{Y_j}) \\ &= \Delta_{X_j/X_l} \cap \Delta_{Y_j/Y_i} \\ &= \Delta_{X_j/X_l} \cap \Delta_{X_j/X_i} = \Delta_{X_j/X_l}, \\ \Delta_{Y_l/Y_i} &= \mathrm{Im}(\Delta_{Y_j/Y_i} \hookrightarrow \Pi_{Y_j} \twoheadrightarrow \Pi_{Y_l}) \\ &= \mathrm{Im}(\Delta_{Y_j/Y_i} \hookrightarrow \Pi_{Y_j} \twoheadrightarrow \Pi_{Y_l} \hookrightarrow \Pi_{X_l}) \\ &= \mathrm{Im}(\Delta_{X_j/X_i} \hookrightarrow \Pi_{X_j} \twoheadrightarrow \Pi_{X_l}) \\ &= \Delta_{X_l/X_i}. \end{aligned}$$

This completes the proof of assertion (iii). □

Definition 3.10. Let p be a prime number, n a positive integer, S a connected noetherian separated normal scheme over k and X a hyperbolic polycurve of relative dimension n over S. We shall say that X/S satisfies condition $(*)_p$ if for any triplet of integers (i, j, l) such that $0 \leq i < j < l \leq n$, the sequence of profinite groups

$$1 \to \Delta^p_{X_l/X_j} \to \Delta^p_{X_l/X_i} \to \Delta^p_{X_j/X_i} \to 1$$

is exact. We shall say that X/S satisfies condition $(**)_p$ if for any pair of integers (i, j) such that $0 \leq i < j \leq n$, the sequence of profinite groups

$$1 \to \Delta^p_{X_j/X_i} \to \Pi^p_{X_j} \to \Pi^p_{X_i} \to 1$$

is exact.

Remark 3.11. The validity of conditions $(*)_p$ and $(**)_p$ depends on the sequence of parametrizing morphisms (at least by definition). So, precisely, we should say that

$$X = X_n \to X_{n-1} \to \cdots \to X_2 \to X_1 \to S = X_0$$

satisfies condition $(*)_p$ (or $(**)_p$). However, we shall say it as in Definition 3.10 for simplicity. Moreover, if the base scheme S is clear from the context, then we often say more simply that X satisfies condition $(*)_p$ (or $(**)_p$).

Example 3.12. If X is a hyperbolic curve over S, i.e., $n = 1$, then X/S satisfies condition $(*)_p$.

Example 3.13. It is well known that if X/S is a configuration space of a hyperbolic curve over S (cf. [14, Def. 2.1]), then X/S satisfies condition $(*)_p$ (cf. [14, Prop. 2.2]).

Remark 3.14. If X/S satisfies condition $(*)_p$, then $\Delta_{X/S}$ admits various group-theoretic properties (cf., e.g., Proposition 3.16(iii)). However, it is not known whether the validity of condition $(*)_p$ for X/S depends only on the profinite group $\Delta_{X/S}$ or not.

Lemma 3.15. *In the notation of Definition 3.10, X/S satisfies condition $(**)_p$ if and only if X/S satisfies condition $(*)_p$, and $\Delta^p_{X/S} \to \Pi^p_X$ is injective.*

Proof. Note that since the sequences of profinite groups

$$1 \to \Delta_{X_l/X_j} \to \Delta_{X_l/X_i} \to \Delta_{X_j/X_i} \to 1, \quad 1 \to \Delta_{X_j/X_i} \to \Pi_{X_j} \to \Pi_{X_i} \to 1$$

are exact, the two sequences in Definition 3.10 are always right exact. If X/S satisfies condition $(**)_p$, for any triplet of integers (i, j, l) such that $0 \leq i < j <$

$l \leq n$, the composite $\Delta^p_{X_l/X_j} \to \Delta^p_{X_l/X_i} \to \Pi^p_{X_l}$, hence also $\Delta^p_{X_l/X_j} \to \Delta^p_{X_l/X_i}$, is injective. Thus, X/S satisfies condition $(*)_p$. The injectivity of $\Delta^p_{X/S} \to \Pi^p_X$ is trivial. Conversely, suppose that X/S satisfies condition $(*)_p$ and that $\Delta^p_{X/S} \to \Pi^p_X$ is injective. Then for each integer i such that $0 \leq i < n$, $\Delta^p_{X/X_i} \to \Pi^p_X$ is injective. Thus, for any pair of integers (i, j) such that $0 \leq i < j \leq n$, we have a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^p_{X/X_j} & \longrightarrow & \Delta^p_{X/X_i} & \longrightarrow & \Delta^p_{X_j/X_i} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta^p_{X/X_j} & \longrightarrow & \Pi^p_X & \longrightarrow & \Pi^p_{X_j} & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact and $\Delta^p_{X/X_i} \to \Pi^p_X$ is injective. Then $\Delta^p_{X_j/X_i} \to \Pi^p_{X_j}$ is injective. Therefore, we conclude that X/S satisfies condition $(**)_p$. This completes the proof of Lemma 3.15. $\square$

Proposition 3.16. *Let p be a prime number, (m, n) a pair of integers such that $0 \leq m < n$, S a connected noetherian separated normal scheme over k and X a hyperbolic polycurve of relative dimension n over S. Then the following hold:*

(i) *Suppose that X/S satisfies condition $(*)_p$. Then for any geometric point $\overline{x}_m \to X_m$ of X_m, the sequence of connected schemes $X \times_{X_m} \overline{x}_m \stackrel{\mathrm{pr}_1}{\to} X \to X_m$ determines an exact sequence of profinite groups*

$$1 \to \Pi^p_{X \times_{X_m} \overline{x}_m} \to \Pi^p_{X/S} \to \Pi^p_{X_m/S} \to 1.$$

In particular, we obtain an isomorphism $\Pi^p_{X \times_{X_m} \overline{x}_m} \xrightarrow{\sim} \ker(\Pi^p_{X/S} \to \Pi^p_{X_m/S})$ (which is well defined up to $\Pi^p_{X/S}$-conjugation).

(ii) *Suppose that X/S satisfies condition $(*)_p$ (resp. $(**)_p$). Let T be a connected noetherian separated normal scheme over S. Then the hyperbolic polycurve $X \times_S T/T$ satisfies condition $(*)_p$ (resp. $(**)_p$). Moreover, the natural morphisms $X \times_S T \stackrel{\mathrm{pr}_1}{\to} X$ and $X \times_S T \to X_m \times_S T$ determine an outer isomorphism*

$$\Pi^p_{X \times_S T/T} \xrightarrow{\sim} \Pi^p_{X/S} \times_{\Pi^p_{X_m/S}} \Pi^p_{X_m \times_S T/T} \text{ (resp. } \Pi^p_{X \times_S T} \xrightarrow{\sim} \Pi^p_X \times_{\Pi^p_{X_m}} \Pi^p_{X_m \times_S T}\text{)}$$

and an isomorphism

$$\Delta^p_{X \times_S T/X_m \times_S T} \xrightarrow{\sim} \Delta^p_{X/X_m}$$

(which is well defined up to Π^p_{X/X_m}- (resp. Π^p_X-) conjugation).

(iii) *Suppose that X/S satisfies condition $(*)_p$. Then Δ^p_{X/X_m} is nontrivial, topologically finitely generated, slim and torsion-free. In particular, Δ^p_{X/X_m} is infinite.*

(iv) *$\Delta^p_{X_{m+1}/X_m}$ is elastic (cf. [13, Def. 1.1(ii)]), i.e., the following holds: let $N \subset \Delta^p_{X_{m+1}/X_m}$ be a topologically finitely generated closed subgroup of $\Delta^p_{X_{m+1}/X_m}$ that is normal in an open subgroup of $\Delta^p_{X_{m+1}/X_m}$; then N is nontrivial if and only if N is open in $\Delta^p_{X_{m+1}/X_m}$.*

(v) *Suppose that the hyperbolic curve X_{m+1} over X_m is of type (g,r). Then the abelianization of $\Delta^p_{X_{m+1}/X_m}$ is a free $\mathbb{Z}_p$-module of rank $2g + \max\{r-1, 0\}$; $\Delta^p_{X_{m+1}/X_m}$ is a free pro-p group if and only if $r \neq 0$.*

(vi) *For any positive integer N, there exists an open subgroup $H \subset \Delta^p_{X_{m+1}/X_m}$ of $\Delta^p_{X_{m+1}/X_m}$ such that the abelianization of H is a free $\mathbb{Z}_p$-module of rank $\geq N$.*

Proof. (Cf. [10, Prop. 2.4].) First, we verify assertion (i). Let us consider the commutative diagram of profinite groups

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Delta^p_{X/X_m} & & \ker(\Pi^p_{X/S} \twoheadrightarrow \Pi^p_{X_m/S}) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Delta^p_{X/S} & \longrightarrow & \Pi^p_{X/S} & \longrightarrow & \Pi_S & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta^p_{X_m/S} & \longrightarrow & \Pi^p_{X_m/S} & \longrightarrow & \Pi_S & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1. & & & &
\end{array}
$$

Then since the two horizontal sequences and the two vertical sequences of the above diagram are exact (cf. Proposition 3.7(i)), it holds that $\Delta^p_{X/X_m} = \ker(\Pi^p_{X/S} \twoheadrightarrow \Pi^p_{X_m/S})$. Thus, we verify from Proposition 3.7(i) that assertion (i) holds. Next, we verify assertion (ii). Suppose that X/S satisfies condition $(*)_p$. Let $\bar{t} \to X \times_S T$ be a geometric point of $X \times_S T$. Then for any triplet of integers (i, j, l) such that

$1 \leq i < j < l \leq n$, we obtain from Proposition 3.7(ii) that

$$\Delta_{X_j \times_S T/X_i \times_S T} \cong \Pi_{(X_j \times_S T) \times_{(X_i \times_S T)} \bar{t}} = \Pi_{X_j \times_{X_i} \bar{t}} \cong \Delta_{X_j/X_i}.$$

In particular, since X/S satisfies condition $(*)_p$, $X \times_S T/T$ also satisfies condition $(*)_p$. On the other hand, we have the commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^p_{X \times_S T/X_m \times_S T} & \longrightarrow & \Pi^p_{X \times_S T/T} & \longrightarrow & \Pi^p_{X_m \times_S T/T} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta^p_{X/X_m} & \longrightarrow & \Pi^p_{X/S} & \longrightarrow & \Pi^p_{X_m/S} & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact (cf. assertion (i)). Thus, we obtain an outer isomorphism

$$\Pi^p_{X \times_S T/T} \xrightarrow{\sim} \Pi^p_{X/S} \times_{\Pi^p_{X_m/S}} \Pi^p_{X_m \times_S T/T}.$$

If X/S satisfies condition $(**)_p$, then it follows from the commutative diagram of profinite groups

$$\begin{array}{ccc} \Delta^p_{X_j \times_S T/X_i \times_S T} & \longrightarrow & \Pi^p_{X_j \times_S T} \\ \| & & \downarrow \\ \Delta^p_{X_j/X_i} & \longrightarrow & \Pi^p_{X_j}, \end{array}$$

together with the injectivity of $\Delta^p_{X_j/X_i} \hookrightarrow \Pi^p_{X_j}$, that $\Delta^p_{X_j \times_S T/X_i \times_S T} \to \Pi^p_{X_j \times_S T}$ is injective. Thus, it follows from Lemma 3.15 that $X \times_S T/T$ satisfies condition $(**)_p$. On the other hand, we have the commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^p_{X \times_S T/X_m \times_S T} & \longrightarrow & \Pi^p_{X \times_S T} & \longrightarrow & \Pi^p_{X_m \times_S T} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta^p_{X/X_m} & \longrightarrow & \Pi^p_{X} & \longrightarrow & \Pi^p_{X_m} & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact. Thus, we obtain an outer isomorphism

$$\Pi^p_{X \times_S T} \xrightarrow{\sim} \Pi^p_{X} \times_{\Pi^p_{X_m}} \Pi^p_{X_m \times_S T}.$$

This completes the proof of assertion (ii). Next, we verify assertion (iii). Let us observe that it follows from assertion (i) that, to verify assertion (iii), we may assume without loss of generality that $m = n - 1$. On the other hand, if $m = n - 1$, i.e., X is a hyperbolic curve over X_m, assertion (iii) is well known (cf., e.g., [16, Props. 1.1, 1.6], [14, Prop. 1.4]). This completes the proof of assertion (iii). Assertion (iv) follows from [14, Prop. 1.5]. Assertion (v) is well known (cf., e.g., [16, Cor. 1.2]). Finally, we verify assertion (vi). Let $\overline{x} \to X_m$ be a geometric point of X_m. Since $\Delta^p_{X_{m+1}/X_m} \cong \Pi^p_{X_{m+1}\times_{X_m}\overline{x}}$ is an infinite profinite group, there exists an open subgroup $H \subset \Delta^p_{X_{m+1}/X_m}$ of $\Delta^p_{X_{m+1}/X_m}$ such that $d := [\Pi^p_{X_{m+1}\times_{X_m}\overline{x}} : H] \geq N$. Then, if X_{m+1}/X_m is of type (g, r) and H corresponds to a hyperbolic curve of type (g', r'), it follows from Hurwitz's formula (cf., e.g., [9, Chap. IV, Cor. 2.4]) that $2g' - 2 + r' = d(2g - 2 + r)$. Thus, it holds that $\mathrm{rank}_{\mathbb{Z}_p} H^{\mathrm{ab}} = 2g' + \max\{r' - 1, 0\} \geq d(2g + r - 2) \geq d \geq N$. This completes the proof of assertion (vi). □

Lemma 3.17.

(i) *Let G be a profinite group, $H \subset G$ a closed subgroup of G and $V \subset H$ an open subgroup of H. Then there exists an open subgroup $U \subset G$ of G such that $V = H \cap U$.*

(ii) *Let G be a profinite group, $H \subset G$ a closed subgroup of G, $N \subset G$ a normal closed subgroup of G and $V \subset H$ an open subgroup of H such that $V \supset H \cap N$. Then there exists a normal open subgroup $U \subset G$ of G such that $U \supset N$ and $U \cap H \subset V$.*

Proof. Note that if G is a profinite group and H is a closed subgroup (resp. normal closed subgroup) of G, then H is the intersection of all open subgroups (resp. normal open subgroups) of G containing H (cf. [15, Prop. 2.1.4]). First, we verify assertion (i). We have $V = \bigcap_W W = \bigcap_W (W \cap H)$, where W runs over all open subgroups of G containing V. Thus, since $(W \cap H) \setminus V$ is a closed subset of the compact set $H \setminus V$, there are open subgroups $W_1, \ldots, W_n$ of G containing V such that $H \cap \bigcap_{i=1}^n W_i \subset V$. Write $U := \bigcap_{i=1}^n W_i$. Then U is an open subgroup of G. Moreover, since $W_i \supset V$, we obtain $H \cap U = V$. This completes the proof of assertion (i). Similarly, assertion (ii) follows from the fact that $N = \bigcap_W W$, where W runs over all normal open subgroups of G containing N. □

Lemma 3.18 ([1, Prop. 3]). *Let $\Sigma \subset \mathfrak{Primes}$ be a set of prime numbers, G a profinite group and $N \subset G$ a normal closed subgroup of G. If the composite $G \to \mathrm{Aut}(N) \to \mathrm{Aut}(N^\Sigma)$ (where $G \to \mathrm{Aut}(N)$ is the map defined by $g \mapsto (h \mapsto ghg^{-1})$ and $\mathrm{Aut}(N) \to \mathrm{Aut}(N^\Sigma)$ is the natural map) factors through G^Σ, then the kernel of the map $N^\Sigma \to G^\Sigma$ is contained in the center of N^Σ. In particular, if N^Σ is* *center-free, then the map $N^{\Sigma} \to G^{\Sigma}$ is injective. If, for any positive integer n, there are only finitely many open subgroups of index n in N^{Σ} (e.g., the case where N^{Σ} is topologically finitely generated), then the map $G \to \mathrm{Aut}(N^{\Sigma})$ factors through G^{Σ} if and only if the image of G in the profinite group $\mathrm{Aut}(N^{\Sigma})$ is a pro-Σ group.*

Lemma 3.19 ([15, Lem. 4.5.5]). *Let p be a prime number and G a topologically finitely generated pro-p group. Then $\mathrm{Aut}(G)$ has an open pro-p subgroup.*

Proposition 3.20. *Let $\Sigma \subset \mathfrak{Primes}$ be a finite set of prime numbers, S a connected noetherian separated normal scheme over k, X a hyperbolic polycurve over S and $X' \to X$ a connected finite étale covering of X. Then there exists a connected finite étale Galois covering $Y \to X$ of X such that the morphism $Y \to X$ factors through $X' \to X$ and, moreover, for any $p \in \Sigma$, Y satisfies condition $(*)_p$.*

Proof. Write n for the relative dimension of X over S. Then, to verify Proposition 3.20, it follows from Remark 3.8 that it suffices to verify that there exists a connected finite étale Galois covering $Y \to X$ of X such that the morphism $Y \to X$ factors through $X' \to X$ and, moreover, for any $p \in \Sigma$ and for any pair of integers (i, j) such that $0 \leq i < j \leq n$, the homomorphism $\Delta^p_{Y_j/Y_i} \to \Delta^p_{Y_j/Y_0}$ (cf. Proposition 3.5) is injective. Now I claim that the following assertion holds:

> Claim A: Fix an integer m such that $0 \leq m < n$. Suppose given a connected finite étale Galois covering $Y \to X$ of X such that for any $p \in \Sigma$ and any pair of integers (i, j) such that $m < i < j \leq n$, the homomorphism $\Delta^p_{Y_j/Y_i} \to \Delta^p_{Y_j/Y_0}$ is injective. Then there exists a connected finite étale Galois covering $Z \to X$ of X such that the morphism $Z \to X$ factors through $Y \to X$ and, moreover, for any $p \in \Sigma$ and any pair of integers (i, j) such that $m \leq i < j \leq n$, the homomorphism $\Delta^p_{Z_j/Z_i} \to \Delta^p_{Z_j/Z_0}$ (cf. Proposition 3.5) is injective.

Indeed, for each $p \in \Sigma$, we consider the commutative diagram

$$\begin{array}{ccc}
\Delta_{Y_{m+1}/Y_0} & \longrightarrow & \Delta_{Y_m/Y_0} \\
\downarrow & & \downarrow \\
\mathrm{Aut}(\Delta_{Y_{m+1}/Y_m}) & \longrightarrow & \mathrm{Out}(\Delta_{Y_{m+1}/Y_m}) \\
\downarrow & & \downarrow \\
\mathrm{Aut}(\Delta^p_{Y_{m+1}/Y_m}) & \longrightarrow & \mathrm{Out}(\Delta^p_{Y_{m+1}/Y_m}),
\end{array}$$

which is obtained from the exact sequence

$$1 \to \Delta_{Y_{m+1}/Y_m} \to \Delta_{Y_{m+1}/Y_0} \to \Delta_{Y_m/Y_0} \to 1.$$

It follows from Proposition 3.16(iii) and Lemma 3.19 that $\mathrm{Out}(\Delta^p_{Y_{m+1}/Y_m})$ has an open pro-p subgroup. Fix such an open subgroup $H \subset \mathrm{Out}(\Delta^p_{Y_{m+1}/Y_m})$ of $\mathrm{Out}(\Delta^p_{Y_{m+1}/Y_m})$, and write $W_p \subset \Delta_{Y_{m+1}/Y_0}$ for the subgroup obtained by forming the inverse image of $H \subset \mathrm{Out}(\Delta^p_{Y_{m+1}/Y_m})$ by the homomorphism $\Delta_{Y_{m+1}/Y_0} \to \mathrm{Out}(\Delta^p_{Y_{m+1}/Y_m})$. Then W_p is an open subgroup of Δ_{Y_{m+1}/Y_0} containing Δ_{Y_{m+1}/Y_m}, and the image of the composite $W_p \hookrightarrow \Delta_{Y_{m+1}/Y_0} \to \mathrm{Aut}(\Delta^p_{Y_{m+1}/Y_m})$ is pro-p. On the other hand, we have $\Delta_{Y_{m+1}/Y_0} \subset \Delta_{X_{m+1}/X_0} \subset \Pi_{X_{m+1}}$. Moreover, since $\Delta_{Y_{m+1}/Y_m} = \Delta_{X_{m+1}/X_m} \cap \Pi_{Y_{m+1}}$ (cf. Lemma 3.9(ii)), Δ_{Y_{m+1}/Y_m} is a normal closed subgroup of $\Pi_{X_{m+1}}$. Thus, it follows from Lemma 3.17(ii) that there exists a normal open subgroup $V_p \subset \Pi_{X_{m+1}}$ of $\Pi_{X_{m+1}}$ such that $\Delta_{Y_{m+1}/Y_m} \subset V_p \cap \Delta_{Y_{m+1}/Y_0} \subset W_p$. Now let us write $V := \bigcap_{p\in\Sigma} V_p$. Then V is a normal open subgroup of $\Pi_{X_{m+1}}$ containing Δ_{Y_{m+1}/Y_m}. Write $U \subset \Pi_Y$ for the subgroup (which is necessarily normal open in Π_X) obtained by forming the inverse image of $V \subset \Pi_{Y_{m+1}}$ by the composite of the outer injection $\Pi_Y \hookrightarrow \Pi_X$ and the outer surjection $\Pi_X \twoheadrightarrow \Pi_{X_{m+1}}$. Then since $U \subset \Pi_Y$, $U \subset \Pi_X$ corresponds to a connected finite étale Galois covering $Z \to X$ which factors through $Y \to X$. To verify Claim A, it suffices to verify that this covering $Z \to X$ of X satisfies the condition in the statement of Claim A. Note that it follows from Lemma 3.9(ii) that

$$\begin{aligned}
\Pi_{Z_{m+1}} &= \mathrm{Im}(\Pi_Z = U \hookrightarrow \Pi_Y \twoheadrightarrow \Pi_{Y_{m+1}}) = V \cap \Pi_{Y_{m+1}},\\
\Delta_{Z_{m+1}/Z_0} &= \Delta_{Y_{m+1}/Y_0} \cap \Pi_{Z_{m+1}} \subset W_p,\\
\Delta_{Z_{m+1}/Z_m} &= \Delta_{Y_{m+1}/Y_m} \cap \Pi_{Z_{m+1}} = \Delta_{Y_{m+1}/Y_m},\\
\Delta_{Z/Z_{m+1}} &= \Delta_{Y/Y_{m+1}} \cap \Pi_Z = \Delta_{Y/Y_{m+1}}.
\end{aligned}$$

Let $p \in \Sigma$. It suffices to verify that for any pair of integers (i, j) such that $m \le i < j \le n$, the homomorphism $\Delta^p_{Z_j/Z_i} \to \Delta^p_{Z_j/Z_0}$ is injective. If $m < i$, then, since $\Delta_{Z/Z_{m+1}} = \Delta_{Y/Y_{m+1}}$, it follows from Lemma 3.9(iii) that $\Delta_{Z_j/Z_i} = \Delta_{Y_j/Y_i}$. Thus, since the homomorphism $\Delta^p_{Y_j/Y_i} \to \Delta^p_{Y_j/Y_0}$ is injective, $\Delta^p_{Z_j/Z_i} \to \Delta^p_{Z_j/Z_0}$ is also injective. Now suppose that $m = i$. We verify the injectivity of $\Delta^p_{Z_j/Z_i} \to \Delta^p_{Z_j/Z_0}$ by induction on j. If $j = m + 1$, it follows from our choice of $Z \to X$ that the image of the composite

$$\Delta_{Z_{m+1}/Z_0} \hookrightarrow W_p \to \mathrm{Aut}(\Delta^p_{Y_{m+1}/Y_m}) = \mathrm{Aut}(\Delta^p_{Z_{m+1}/Z_m})$$

is a pro-p subgroup. Thus, since $\Delta^p_{Z_{m+1}/Z_m}$ is topologically finitely generated and center-free (cf. Proposition 3.16(iii)), it follows from Lemma 3.18 that $\Delta^p_{Z_{m+1}/Z_m} \to$

$\Delta^p_{Z_{m+1}/Z_0}$ is injective. Now suppose that $m+1 < j \le n$ and that the induction hypothesis is in force. Then we have the commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^p_{Z_j/Z_{j-1}} & \longrightarrow & \Delta^p_{Z_j/Z_m} & \longrightarrow & \Delta^p_{Z_{j-1}/Z_m} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta^p_{Z_j/Z_{j-1}} & \longrightarrow & \Delta^p_{Z_j/Z_0} & \longrightarrow & \Delta^p_{Z_{j-1}/Z_0} & \longrightarrow & 1, \end{array}$$

where, since $j-1 > m$, the two horizontal sequences are exact. Moreover, it follows from the induction hypothesis that $\Delta^p_{Z_{j-1}/Z_m} \to \Delta^p_{Z_{j-1}/Z_0}$ is injective. Thus, $\Delta^p_{Z_j/Z_m} \to \Delta^p_{Z_j/Z_0}$ is also injective. This completes the proof of Claim A.

Now we verify Proposition 3.20. First, let us write $Y \to X$ for the Galois closure of $X' \to X$. Then, if $n-1 > 0$, by applying Claim A, where we take the data "$(m, Y \to X)$" to be $(n-1, Y \to X)$, we obtain a covering $Z \to X$. Next, let us replace $Y \to X$ by $Z \to X$. Then, we apply Claim A again, taking the data "$(m, Y \to X)$" to be $(n-2, Y \to X)$. If $n-2 > 0$, by applying an argument similar to the above argument repeatedly until $m = 0$, we obtain a covering $Z \to X$ which satisfies the condition imposed on "$Y \to X$" in the statement of Proposition 3.20. This completes the proof of Proposition 3.20. □

Proposition 3.21. *Let $\Sigma \subset \mathfrak{Primes}$ be a finite set of prime numbers, S a connected noetherian separated normal scheme over k and X a hyperbolic polycurve of relative dimension n over S. Suppose that for any $p \in \Sigma$, X/S satisfies condition $(*)_p$. Then there exists a connected finite étale Galois covering $T \to S$ of S such that for any $p \in \Sigma$, $X \times_S T/T$ satisfies condition $(**)_p$.*

Proof. For each $p \in \Sigma$, let us consider the sequence $\Pi_S \to \mathrm{Out}(\Delta_{X/S}) \to \mathrm{Out}(\Delta^p_{X/S})$. Then it follows from Proposition 3.16(iii) and Lemma 3.19 that $\mathrm{Out}(\Delta^p_{X/S})$ has an open pro-p subgroup. Fix such an open subgroup $H \subset \mathrm{Out}(\Delta^p_{X/S})$ of $\mathrm{Out}(\Delta^p_{X/S})$, and write $U_p \subset \Pi_S$ for the subgroup obtained by forming the inverse image of $H \subset \mathrm{Out}(\Delta^p_{X/S})$ by the homomorphism $\Pi_S \to \mathrm{Out}(\Delta^p_{X/S})$. Let $U \subset \Pi_S$ be a normal open subgroup of Π_S contained in $\bigcap_{p\in\Sigma} U_p$. Write $T \to S$ for the connected finite étale Galois covering of S corresponding to $U \subset \Pi_S$. Then $X \times_S T \to X$ corresponds to the inverse image of $U \subset \Pi_S$ by the outer homomorphism $\Pi_X \to \Pi_S$ and, moreover, $\Delta_{X\times_S T/T} = \Delta_{X/S}$. Thus, for any $p \in \Sigma$, the image of the homomorphism $\Pi_{X\times_S T} \to \mathrm{Aut}(\Delta^p_{X\times_S T/T}) = \mathrm{Aut}(\Delta^p_{X/S})$ is a pro-p subgroup. Then, since $\Delta^p_{X/S}$ is topologically finitely generated and center-free (cf. Proposition 3.16(iii)), it follows from Lemma 3.18 that $\Delta^p_{X/S} \to \Pi^p_{X\times_S T}$ is injective. On the other hand, $X \times_S T/T$ satisfies condition

$(*)_p$ (cf. Proposition 3.16(ii)). Thus, it follows from Lemma 3.15 that $X \times_S T/T$ satisfies condition $(**)_p$. This completes the proof of Proposition 3.21. □

Lemma 3.22. *Let p be a prime number, n a positive integer, S a connected noetherian separated normal scheme over k and X a hyperbolic polycurve of relative dimension n over S. Then the following hold:*

(i) *Suppose that X/S satisfies condition $(*)_p$. Let $U \subset \Pi^p_{X/S}$ be an open subgroup of $\Pi^p_{X/S}$. Write $Y \to X$ for the connected finite étale covering of X corresponding to U and $S' := \mathrm{Nor}(Y/S)$. Then for each integer i such that $0 \le i \le n$, $\Pi^p_{Y_i/S'}$ (cf. Proposition 3.5) is canonically identified with $\mathrm{Im}(U \hookrightarrow \Pi^p_{X/S} \twoheadrightarrow \Pi^p_{X_i/S})$ and, moreover, for any pair of integers (i,j) such that $0 \le i < j \le n$, $\Delta^p_{Y_j/Y_i} = \Pi^p_{Y_j/S'} \cap \Delta^p_{X_j/X_i}$. In particular, Y/S' satisfies condition $(*)_p$.*

(ii) *Suppose that X/S satisfies condition $(**)_p$. Let $U \subset \Pi^p_X$ be an open subgroup of Π^p_X. Write $Y \to X$ for the connected finite étale covering of X corresponding to U and $S' := \mathrm{Nor}(Y/S)$. Then for each integer i such that $0 \le i \le n$, $\Pi^p_{Y_i}$ (cf. Proposition 3.5) is canonically identified with $\mathrm{Im}(U \hookrightarrow \Pi^p_X \twoheadrightarrow \Pi^p_{X_i})$ and, moreover, for any pair of integers (i,j) such that $0 \le i < j \le n$, $\Delta^p_{Y_j/Y_i} = \Pi^p_{Y_j} \cap \Delta^p_{X_j/X_i}$. In particular, Y/S' satisfies condition $(**)_p$.*

Proof. First, we verify assertion (i). For each integer i such that $0 \le i \le n$, we have a commutative diagram of profinite groups

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Delta_{X/X_i} & \twoheadrightarrow & \Delta^p_{X/X_i} & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \to & \ker(\Delta_{X/S} \twoheadrightarrow \Delta^p_{X/S}) & \to & \Delta_{X/S} & \to & \Delta^p_{X/S} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \ker(\Delta_{X_i/S} \twoheadrightarrow \Delta^p_{X_i/S}) & \to & \Delta_{X_i/S} & \to & \Delta^p_{X_i/S} & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1, & &
\end{array}
$$

where the two horizontal sequences and the two vertical sequences are exact. Thus, the homomorphism $\ker(\Delta_{X/S} \twoheadrightarrow \Delta^p_{X/S}) \to \ker(\Delta_{X_i/S} \twoheadrightarrow \Delta^p_{X_i/S})$ is surjective. On the other hand, since the inverse image of $U \subset \Pi^p_{X/S}$ by the surjection $\Pi_X \twoheadrightarrow \Pi^p_{X/S}$ coincides with $\Pi_Y \subset \Pi_X$, it follows that $\ker(\Delta_{X/S} \twoheadrightarrow \Delta^p_{X/S})$ is contained in Π_Y. Thus, $\ker(\Delta_{X_i/S} \twoheadrightarrow \Delta^p_{X_i/S})$ is contained in $\Pi_{Y_i} = \mathrm{Im}(\Pi_Y \hookrightarrow \Pi_X \twoheadrightarrow \Pi_{X_i})$, hence also in $\Delta_{Y_i/S'} = \Delta_{X_i/S} \cap \Pi_{Y_i}$ (cf. Lemma 3.9(ii)). This implies that $\Delta_{Y_i/S'}$ can be obtained by taking the inverse image of some open subgroup $V \subset \Delta^p_{X_i/S}$ of $\Delta^p_{X_i/S}$ by the surjection $\Delta_{X_i/S} \twoheadrightarrow \Delta^p_{X_i/S}$ and, moreover, $\Delta^p_{Y_i/S'}$ coincides with $V \subset \Delta^p_{X_i/S}$. Thus, we have $\ker(\Delta_{Y_i/S'} \twoheadrightarrow \Delta^p_{Y_i/S'}) = \ker(\Delta_{X_i/S} \twoheadrightarrow \Delta^p_{X_i/S})$, which implies that $\Pi^p_{Y_i/S'} = \Pi_{Y_i}/\ker(\Delta_{Y_i/S'} \twoheadrightarrow \Delta^p_{Y_i/S'})$ coincides with $\mathrm{Im}(\Pi_{Y_i} \hookrightarrow \Pi_{X_i} \twoheadrightarrow \Pi^p_{X_i/S})$ and, moreover, Π_{Y_i} is the inverse image of $\Pi^p_{Y_i/S'} \subset \Pi^p_{X_i/S}$ by the surjection $\Pi_{X_i} \twoheadrightarrow \Pi^p_{X_i/S}$. In particular, we have $U = \Pi^p_{Y/S'} = \mathrm{Im}(\Pi_Y \hookrightarrow \Pi_X \twoheadrightarrow \Pi^p_{X/S})$. Thus, since $\Pi_Y \to \Pi_{Y_i}$ is surjective (cf. Proposition 3.7(i)), by considering the commutative diagram of profinite groups

$$\begin{array}{ccccc} \Pi_Y & \hookrightarrow & \Pi_X & \twoheadrightarrow & \Pi^p_{X/S} \\ \downarrow & & \downarrow & & \downarrow \\ \Pi_{Y_i} & \hookrightarrow & \Pi_{X_i} & \twoheadrightarrow & \Pi^p_{X_i/S}, \end{array}$$

it holds that $\Pi^p_{Y_i/S'} = \mathrm{Im}(U \hookrightarrow \Pi^p_{X/S} \twoheadrightarrow \Pi^p_{X_i/S})$.

Now let (i, j) be a pair of integers such that $0 \le i < j \le n$. Then, since it holds that $\Delta_{Y_j/Y_i} = \Pi_{Y_j} \cap \Delta_{X_j/X_i}$ (cf. Lemma 3.9(ii)), by considering the commutative diagram of profinite groups

$$\begin{array}{ccccccc} \Delta_{Y_j/Y_i} & = & \Pi_{Y_j} \cap \Delta_{X_j/X_i} & \subset & \Delta_{X_j/X_i} & \subset & \Pi_{X_j} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \Pi^p_{Y_j/S'} \cap \Delta^p_{X_j/X_i} & \subset & \Delta^p_{X_j/X_i} & \subset & \Pi^p_{X_j/S}, \end{array}$$

we obtain that Δ_{Y_j/Y_i} is the inverse image of the open subgroup $\Pi^p_{Y_j/S'} \cap \Delta^p_{X_j/X_i} \subset \Delta^p_{X_j/X_i}$ by the surjection $\Delta_{X_j/X_i} \twoheadrightarrow \Delta^p_{X_j/X_i}$. Thus, it holds that $\Delta^p_{Y_j/Y_i} = \Pi^p_{Y_j/S'} \cap \Delta^p_{X_j/X_i}$. In particular, if $0 < i < j \le n$, by considering the commutative diagram of profinite groups

$$\begin{array}{ccc} \Delta^p_{Y_j/Y_i} & \longrightarrow & \Delta^p_{Y_j/Y_0} \\ \| & & \| \\ \Pi^p_{Y_j/S'} \cap \Delta^p_{X_j/X_i} & \hookrightarrow & \Pi^p_{Y_j/S'} \cap \Delta^p_{X_j/X_0}, \end{array}$$

we conclude that $\Delta^p_{Y_j/Y_i} \to \Delta^p_{Y_j/Y_0}$ is injective, i.e., Y/S' satisfies condition $(*)_p$. This completes the proof of assertion (i). Assertion (ii) is proved similarly. $\square$

Lemma 3.23. *Let p be a prime number, (m,n) a pair of integers such that $0 \leq m < n$, S a connected noetherian separated normal scheme over k and X a hyperbolic polycurve of relative dimension n over S satisfying condition $(**)_p$. Then the following hold:*

(i) *The natural surjections $\Pi^p_{X/S} \twoheadrightarrow \Pi^p_X, \Pi^p_{X/S} \twoheadrightarrow \Pi^p_{X_m/S}$ (cf. Proposition 3.16) determine an isomorphism*

$$\Pi^p_{X/S} \xrightarrow{\sim} \Pi^p_X \times_{\Pi^p_{X_m}} \Pi^p_{X_m/S}.$$

(ii) *Let Y be a connected noetherian scheme over X_m. Let us fix a homomorphism $\Pi^p_{Y/S} \to \Pi^p_{X_m/S}$ arising from $Y \to X_m$. Then there exists a natural bijection*

$$\mathrm{Hom}_{\Pi^p_{X_m}}(\Pi^p_Y, \Pi^p_X) \xrightarrow{1:1} \mathrm{Hom}_{\Pi^p_{X_m/S}}(\Pi^p_{Y/S}, \Pi^p_{X/S}).$$

If, moreover, $\Pi^p_{Y/S} \to \Pi^p_{X_m/S}$ is surjective and the image of $\varphi \in \mathrm{Hom}_{\Pi^p_{X_m}}(\Pi^p_Y, \Pi^p_X)$ by the above bijection is $\psi \in \mathrm{Hom}_{\Pi^p_{X_m/S}}(\Pi^p_{Y/S}, \Pi^p_{X/S})$, then there is a natural one-to-one correspondence between the left cosets of $\mathrm{Im}\,\varphi \subset \Pi^p_X$ in Π^p_X and the left cosets of $\mathrm{Im}\,\psi \subset \Pi^p_{X/S}$ in $\Pi^p_{X/S}$. In particular, φ is an open (resp. a surjective) homomorphism if and only if so is ψ.

Proof. Assertion (i) follows from the commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta^p_{X/X_m} & \longrightarrow & \Pi^p_{X/S} & \longrightarrow & \Pi^p_{X_m/S} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta^p_{X/X_m} & \longrightarrow & \Pi^p_X & \longrightarrow & \Pi^p_{X_m} & \longrightarrow & 1,
\end{array}$$

where the horizontal arrows are exact (cf. Lemma 3.15). We verify assertion (ii). Suppose that we are given an element $\varphi \in \mathrm{Hom}_{\Pi^p_{X_m}}(\Pi^p_Y, \Pi^p_X)$. Then a homomorphism $\Pi^p_{Y/S} \to \Pi^p_{X/S}$ over $\Pi^p_{X_m/S}$ is obtained from the commutative diagram

$$\begin{array}{ccccc}
\Pi^p_{Y/S} & \twoheadrightarrow & \Pi^p_Y & \xrightarrow{\varphi} & \Pi^p_X \\
\downarrow & & & \searrow & \downarrow \\
\Pi^p_{X_m/S} & & \twoheadrightarrow & & \Pi^p_{X_m}
\end{array}$$

(cf. assertion (i)). Thus, we obtain a natural map

$$\mathrm{Hom}_{\Pi^p_{X_m}}(\Pi^p_Y, \Pi^p_X) \to \mathrm{Hom}_{\Pi^p_{X_m/S}}(\Pi^p_{Y/S}, \Pi^p_{X/S}).$$

Conversely, since

$$\Pi^p_X = (\Pi^p_{X/S})^p, \quad \Pi^p_Y = (\Pi^p_{Y/S})^p, \quad \Pi^p_{X_m} = (\Pi^p_{X_m/S})^p,$$

we obtain a natural map

$$\mathrm{Hom}_{\Pi^p_{X_m/S}}(\Pi^p_{Y/S}, \Pi^p_{X/S}) \to \mathrm{Hom}_{\Pi^p_{X_m}}(\Pi^p_Y, \Pi^p_X).$$

It follows immediately that these maps are inverse to each other.

Now suppose that $\Pi^p_{Y/S} \to \Pi^p_{X_m/S}$ is surjective. Let $\varphi \in \mathrm{Hom}_{\Pi^p_{X_m}}(\Pi^p_Y, \Pi^p_X)$. Write $\psi \in \mathrm{Hom}_{\Pi^p_{X_m/S}}(\Pi^p_{Y/S}, \Pi^p_{X/S})$ for the image of $\varphi \in \mathrm{Hom}_{\Pi^p_{X_m}}(\Pi^p_Y, \Pi^p_X)$ by the bijection $\mathrm{Hom}_{\Pi^p_{X_m}}(\Pi^p_Y, \Pi^p_X) \overset{1:1}{\to} \mathrm{Hom}_{\Pi^p_{X_m/S}}(\Pi^p_{Y/S}, \Pi^p_{X/S})$. Then, we have a commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \ker(\Pi^p_{Y/S} \twoheadrightarrow \Pi^p_{X_m/S}) & \longrightarrow & \Pi^p_{Y/S} & \longrightarrow & \Pi^p_{X_m/S} & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & \Delta^{(p)}_{Y/X_m} & \longrightarrow & \Pi^p_Y & \longrightarrow & \Pi^p_{X_m} & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow \varphi & & \big\| & & \\
1 & \longrightarrow & \Delta^{(p)}_{X/X_m} & \longrightarrow & \Pi^p_X & \longrightarrow & \Pi^p_{X_m} & \longrightarrow & 1,
\end{array}$$

where the horizontal sequences are exact and, moreover, since the operation of taking the maximal pro-p quotient of a profinite group is right exact, the homomorphism $\ker(\Pi^p_{Y/S} \twoheadrightarrow \Pi^p_{X_m/S}) \to \Delta^{(p)}_{Y/X_m}$ is surjective. Thus, the above diagram induces a one-to-one correspondence between the left cosets of $\mathrm{Im}\,\varphi \subset \Pi^p_X$ in Π^p_X and the left cosets of $\mathrm{Im}(\ker(\Pi^p_{Y/S} \twoheadrightarrow \Pi^p_{X_m/S}) \to \Delta^{(p)}_{X/X_m}) \subset \Delta^{(p)}_{X/X_m}$ in $\Delta^{(p)}_{X/X_m}$. On the other hand, since X/S satisfies condition $(**)_p$, we have $\Delta^{(p)}_{X/X_m} = \Delta^p_{X/X_m}$. Thus, the commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \ker(\Pi^p_{Y/S} \twoheadrightarrow \Pi^p_{X_m/S}) & \longrightarrow & \Pi^p_{Y/S} & \longrightarrow & \Pi^p_{X_m/S} & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow \psi & & \big\| & & \\
1 & \longrightarrow & \Delta^p_{X/X_m} & \longrightarrow & \Pi^p_{X/S} & \longrightarrow & \Pi^p_{X_m/S} & \longrightarrow & 1,
\end{array}$$

where the horizontal sequences are exact, induces a one-to-one correspondence between the left cosets of $\mathrm{Im}(\ker(\Pi^p_{Y/S} \twoheadrightarrow \Pi^p_{X_m/S}) \to \Delta^{(p)}_{X/X_m}) \subset \Delta^{(p)}_{X/X_m}$ in $\Delta^{(p)}_{X/X_m} = \Delta^p_{X/X_m}$ and the left cosets of $\mathrm{Im}\,\psi \subset \Pi^p_{X/S}$ in $\Pi^p_{X/S}$. This completes the proof of assertion (ii). □

Remark 3.24. There are properties similar to Lemma 3.23 if X/S is a hyperbolic polycurve satisfying condition $(*)_p$, i.e., the following hold:

(i) For each integer l such that $0 \leq l \leq m$, the natural surjections $\Pi^p_{X/X_l} \twoheadrightarrow \Pi^p_{X/S}$, $\Pi^p_{X/X_l} \twoheadrightarrow \Pi^p_{X_m/X_l}$ determine an isomorphism

$$\Pi^p_{X/X_l} \xrightarrow{\sim} \Pi^p_{X/S} \times_{\Pi^p_{X_m/S}} \Pi^p_{X_m/X_l}.$$

(ii) Let l be an integer such that $0 \leq l \leq m$ and Y a connected noetherian scheme over X_m. Let us fix a homomorphism $\Pi^p_{Y/X_l} \to \Pi^p_{X_m/X_l}$ arising from $Y \to X_m$. Then there exists a natural bijection

$$\mathrm{Hom}_{\Pi^p_{X_m/S}}(\Pi^p_{Y/S}, \Pi^p_{X/S}) \xrightarrow{1:1} \mathrm{Hom}_{\Pi^p_{X_m/X_l}}(\Pi^p_{Y/X_l}, \Pi^p_{X/X_l}).$$

If, moreover, $\Pi^p_{Y/X_l} \to \Pi^p_{X_m/X_l}$ is surjective and the image of $\varphi \in \mathrm{Hom}_{\Pi^p_{X_m/S}}(\Pi^p_{Y/S}, \Pi^p_{X/S})$ by the above bijection is $\psi \in \mathrm{Hom}_{\Pi^p_{X_m/X_l}}(\Pi^p_{Y/X_l}, \Pi^p_{X/X_l})$, then there is a natural one-to-one correspondence between the left cosets of $\mathrm{Im}\,\varphi \subset \Pi^p_{X/S}$ in $\Pi^p_{X/S}$ and the left cosets of $\mathrm{Im}\,\psi \subset \Pi^p_{X/X_l}$ in Π^p_{X/X_l}. In particular, φ is an open (resp. a surjective) homomorphism if and only if so is ψ.

Definition 3.25. Let p be a prime number and X a variety over k. Then we shall say that X is of *p-LFG-type* if, for any normal variety Y over $\overline{k}$ and any morphism $Y \to X \times_k \overline{k}$ over $\overline{k}$ that is not constant, the image of the outer homomorphism $\Pi^p_Y \to \Pi^p_{X\times_k \overline{k}}$ is infinite.

Remark 3.26. Using an argument similar to the argument in [10, Rem. 2.5.1], it follows that if k'/k is a field extension of k, then X is of p-LFG-type if and only if $X \times_k k'$ is of p-LFG-type. On the other hand, it follows immediately from the definition that if X is of p-LFG-type, then X is of LFG-type (cf. [10, Def. 2.5]).

Lemma 3.27. *Let p be a prime number and X, Y varieties over k. Suppose that X is of p-LFG-type. Then the following hold:*

(i) *Suppose that there exists a quasi-finite morphism $Y \to X$ over k. Then Y is of p-LFG-type.*

(ii) *Let $f : X \to Y$ be a morphism over k. Suppose that $\Delta^p_{f/k}$ is finite. Then f is quasi-finite. If, moreover, f is surjective, then Y is of p-LFG-type.*

(iii) *Let* $f : X \to Y$ *be a morphism over* k. *Suppose that* $\Pi^p_{X\times_k\overline{k}} \to \Pi^p_X$ *is injective and* $\Delta^{(p)}_f$ *is finite. Then* f *is quasi-finite. If, moreover,* f *is surjective, then* Y *is of* p*-LFG-type.*

Proof. (Cf. [10, Lem. 2.6].) First, we verify assertion (i). It follows from the fact that if $Y \to X$ is quasi-finite then so is the morphism $Y \times_k \overline{k} \to X \times_k \overline{k}$, determined by $Y \to X$, that to verify assertion (i), we may assume without loss of generality that $k = \overline{k}$. Let Z be a normal variety over k and $Z \to Y$ a nonconstant morphism over k. Then since Y is quasi-finite over X, it follows that the composite $Z \to Y \to X$ is nonconstant. In particular, since X is of p-LFG-type, the image of the composite $\Pi^p_Z \to \Pi^p_Y \to \Pi^p_X$, hence also $\Pi^p_Z \to \Pi^p_Y$, is infinite. This completes the proof of assertion (i). Next, we verify assertion (ii). Note that we have equalities

$$\Delta^p_{f/k} = \ker(\Delta^p_{X/k} \to \Delta^p_{Y/k}) = \ker(\Pi^p_{X\times_k\overline{k}} \to \Pi^p_{Y\times_k\overline{k}}) = \Delta^p_{X\times_k\overline{k}\to Y\times_k\overline{k}/\overline{k}}.$$

Thus, it follows from the fact that if the morphism $Y \times_k \overline{k} \to X \times_k \overline{k}$ determined by f is quasi-finite then so is f (cf. [2, Prop. 1.9.4]), together with the fact that if f is surjective then so is the morphism $Y \times_k \overline{k} \to X \times_k \overline{k}$ determined by f, that to verify assertion (ii), we may assume without loss of generality that $k = \overline{k}$. Let $\overline{y} \to Y$ be a k-valued geometric point of Y and F a connected component (which is necessarily a normal variety over k) of the normalization of the geometric fiber of f at $\overline{y}$. Then, since the composite of the outer homomorphism $\Pi^p_F \to \Pi^p_X$ induced by the natural morphism $F \to X$ and $\Pi^p_X \to \Pi^p_Y$ factors through $\Pi^p_{\overline{y}} = \{1\}$, $\Pi^p_F \to \Pi^p_X$ factors through $\Delta^p_{f/k} \subset \Pi^p_X$. In particular, since $\Delta^p_{f/k}$ is finite, the image of $\Pi^p_F \to \Pi^p_X$ is finite. Thus, since X is of p-LFG-type, it follows that F is finite over k. This implies that f is quasi-finite.

Now suppose that f is surjective. Let Z be a normal variety over k and $Z \to Y$ a nonconstant morphism over k. Then since f is a quasi-finite surjection, and $Z \to Y$ is nonconstant, there exists a connected component C (which is necessarily a normal variety over k) of the normalization of $Z \times_Y X$ such that the natural morphism $C \to X$ over k is nonconstant. Thus, since X is of p-LFG-type, the image of $\Pi^p_C \to \Pi^p_X$, hence also that of $\Pi^p_C \to \Pi^p_X \to \Pi^p_Y$, is infinite. In particular, since the composite $C \to X \to Y$ factors through $Z \to Y$, it follows that the image of $\Pi^p_Z \to \Pi^p_Y$ is infinite, which implies that Y is of p-LFG-type. This completes the proof of assertion (ii). Finally, we verify assertion (iii). Let us observe that, since if $\Pi^p_{X\times_k\overline{k}} = \Delta^p_{X/k} \to \Pi^p_X$ is injective then $\Delta^{(p)}_f \supset \ker(\Pi^p_{X\times_k\overline{k}} \to \Pi^p_{Y\times_k\overline{k}})$, it follows from an argument similar to the argument used at the beginning of the proof of assertion (ii) that to verify assertion (iii), we may assume without loss of generality that $k = \overline{k}$. But then assertion (iii) is the same as assertion (ii), which has already been verified. This completes the proof of assertion (iii). □

Proposition 3.28. *Let p be a prime number. Then every hyperbolic polycurve over k satisfying condition $(*)_p$ is of p-LFG-type.*

Proof. (Cf. [10, Prop. 2.7].) First, let us observe that it follows that, to verify Proposition 3.28, we may assume without loss of generality that $k = \overline{k}$. Let X be either $\mathrm{Spec}\, k$ or a hyperbolic polycurve over k satisfying condition $(*)_p$. Write $n := \dim(X)$. We verify that X is of p-LFG-type by induction on n. If $n = 0$, i.e., $X = \mathrm{Spec}\, k$, then X is clearly of p-LFG-type. Now suppose that $n \geq 1$ and that the induction hypothesis is in force. Let Y be a normal variety over k and $Y \to X$ a nonconstant morphism over k. To verify Proposition 3.28, it suffices to verify that the image of $\Pi_Y^p \to \Pi_X^p$ is infinite.

Now suppose that the composite $Y \to X \to X_{n-1}$ is nonconstant. It follows from the induction hypothesis that X_{n-1} is of p-LFG-type. Thus, the image of the composite $\Pi_Y^p \to \Pi_X^p \twoheadrightarrow \Pi_{X_{n-1}}^p$, hence also that of $\Pi_Y^p \to \Pi_X^p$, is infinite.

Next, suppose that the composite $Y \to X \to X_{n-1}$ is constant. Write $\overline{x} \to X_{n-1}$ for the k-valued geometric point of X_{n-1} through which the constant morphism $Y \to X \to X_{n-1}$ factors. Then the composite $Y \to X \to X_{n-1}$ determines a nonconstant morphism $Y \to X \times_{X_{n-1}} \overline{x}$ over k. Since $X \times_{X_{n-1}} \overline{x}$ is a hyperbolic curve over $\overline{x}$, it follows that the morphism $Y \to X \times_{X_{n-1}} \overline{x}$ is dominant. Thus, it follows from Lemma 2.3 that the outer homomorphism $\Pi_Y \to \Pi_{X \times_{X_{n-1}} \overline{x}}$, hence also $\Pi_Y^p \to \Pi_{X \times_{X_{n-1}} \overline{x}}^p$, is open. Now let us observe that $\Pi_{X \times_{X_{n-1}} \overline{x}}^p \xrightarrow{\sim} \Delta_{X/X_{n-1}}^p$ (cf. Proposition 3.7(i)) is infinite (cf. Proposition 3.16(iii)). Thus, since X/S satisfies condition $(*)_p$, the image of the composite $\Pi_Y^p \to \Pi_{X \times_{X_{n-1}} \overline{x}}^p \xrightarrow{\sim} \Delta_{X/X_{n-1}}^p \hookrightarrow \Pi_X^p$ is infinite. This completes the proof of Proposition 3.28. □

Lemma 3.29 (Cf. [16, Lem. 1.10]). *Let p be a prime number, (g_0, r_0) a pair of nonnegative integers and X a hyperbolic curve (resp. a nonproper hyperbolic curve) over k. Then there exists a normal open subgroup $U \subset \Pi_{X \times_k \overline{k}}$ of $\Pi_{X \times_k \overline{k}}$ such that $\Pi_{X \times_k \overline{k}}/U$ is a p-group, and that if we write (g, r) for the type of the hyperbolic curve corresponding to $U \subset \Pi_{X \times_k \overline{k}}$, then $g \geq g_0$ (resp. $g \geq g_0$, $r \geq r_0$).*

Lemma 3.30. *Let p be a prime number, (g_0, r_0) a pair of nonnegative integers, S a connected noetherian separated normal scheme over k and X a hyperbolic curve (resp. a nonproper hyperbolic curve) over S. Then there exists a connected finite étale Galois covering $Y \to X$ of X such that if we write $S' := \mathrm{Nor}(Y/S)$ and (g, r) for the type of the hyperbolic curve Y/S', then $g \geq g_0$ (resp. $g \geq g_0$, $r \geq r_0$), and that $\Pi_{Y/S'}^p \to \Pi_{X/S}^p$ is injective.*

Proof. Let $\overline{s} \to S$ be a geometric point of S. Then it follows from Lemma 3.29 that there exists a normal open subgroup $V \subset \Delta_{X/S}$ of $\Delta_{X/S} \cong \Pi_{X \times_S \overline{s}}$ (cf. Proposition 3.7(i)) such that $\Delta_{X/S}/V$ is a p-group and that the pair of integers (g', r') corresponding to V satisfies $g' \geq g_0$ (resp. $g' \geq g_0$, $r' \geq r_0$). On the other hand, let us observe that $\ker(\Delta_{X/S} \twoheadrightarrow \Delta^p_{X/S}) \subset V \subset \Delta_{X/S} \subset \Pi_X$. Thus, since $\ker(\Delta_{X/S} \twoheadrightarrow \Delta^p_{X/S}) \subset \Pi_X$ is a normal closed subgroup of Π_X, it follows from Lemma 3.17(ii) that there exists a normal open subgroup $U \subset \Pi_X$ such that $\ker(\Delta_{X/S} \twoheadrightarrow \Delta^p_{X/S}) \subset U \cap \Delta_{X/S} \subset V$. Write $Y \to X$ for the connected finite étale Galois covering of X corresponding to $U \subset \Pi_X$, $S' := \mathrm{Nor}(Y/S)$ and (g, r) for the type of Y/S'. Then, since $U \cap \Delta_{X/S} = \Delta_{Y/S'} \subset V$, we obtain $g \geq g' \geq g_0$ (resp. $g \geq g' \geq g_0$, $r \geq r' \geq r_0$). Moreover, since $\Delta_{Y/S'} \supset \ker(\Delta_{X/S} \twoheadrightarrow \Delta^p_{X/S})$, the homomorphism $\Delta^p_{Y/S'} \to \Delta^p_{X/S}$ is injective. Thus, we have

$$\ker(\Delta_{Y/S'} \twoheadrightarrow \Delta^p_{Y/S'}) = \ker(\Delta_{X/S} \twoheadrightarrow \Delta^p_{X/S}) \cap \Delta_{Y/S'} = \ker(\Delta_{X/S} \twoheadrightarrow \Delta^p_{X/S}).$$

This implies that $\Pi^p_{Y/S'} \to \Pi^p_{X/S}$ is injective. This completes the proof of Lemma 3.30. □

Lemma 3.31. *Let p be a prime number, S a connected noetherian separated normal scheme over k, X a hyperbolic curve over S, R a strictly henselian discrete valuation ring over S, K the field of fractions of R and $\operatorname{Spec} K \to X$ a morphism over S. Then it holds that the morphism $\operatorname{Spec} K \to X$ factors through the open immersion $\operatorname{Spec} K \hookrightarrow \operatorname{Spec} R$ if and only if the image of the outer homomorphism $\Pi_{\operatorname{Spec} K} \to \Pi^p_{X/S}$ induced by the morphism $\operatorname{Spec} K \to X$ is trivial.*

Proof. (Cf. [10, Lem. 2.8].) Since $\Pi_{\operatorname{Spec} R} = \{1\}$ (cf., e.g., [6, Thm. (18.5.11)]), necessity is immediate. We verify sufficiency. Note that we have

$$\Pi^p_{X \times_S \operatorname{Spec} R/\operatorname{Spec} R} \xrightarrow{\sim} \Pi^p_{X/S} \times_{\Pi_S} \Pi_{\operatorname{Spec} R} = \ker(\Pi^p_{X/S} \twoheadrightarrow \Pi_S)$$

(cf. Proposition 3.16(ii)). In particular, $\Pi^p_{X \times_S \operatorname{Spec} R/\operatorname{Spec} R} \to \Pi^p_{X/S}$ is injective. Thus, the image of $\Pi_{\operatorname{Spec} K} \to \Pi^p_{X \times_S \operatorname{Spec} R/\operatorname{Spec} R}$ is trivial. This implies that, to verify sufficiency, we may assume without loss of generality that $S = \operatorname{Spec} R$.

Next, let us write $\widehat{R}$ for the completion of R and $\widehat{K}$ for the field of fractions of $\widehat{R}$. Then, since $\Pi^p_{X \times_R \widehat{R}/\widehat{R}} \cong \Pi^p_{X/R} \times_{\Pi_R} \Pi_{\widehat{R}} = \Pi^p_{X/R}$ (cf. Proposition 3.16(ii)), it follows that if the image of $\Pi_{\operatorname{Spec} K} \to \Pi^p_{X/R}$ is trivial, then so is $\Pi_{\operatorname{Spec} \widehat{K}} \to \Pi^p_{X \times_R \widehat{R}/\widehat{R}}$. Thus, if we verify Lemma 3.31 in the case where R is complete, then the morphism $\operatorname{Spec} \widehat{K} \to X \times_R \widehat{R}$ factors through $\operatorname{Spec} \widehat{K} \hookrightarrow \operatorname{Spec} \widehat{R}$. Then, it follows from the commutative diagram of schemes

$$\begin{array}{ccccc} \operatorname{Spec}\widehat{R} & \longrightarrow & X \times_R \widehat{R} & \xrightarrow{\mathrm{pr}_1} & X \\ \downarrow & & & & \downarrow \\ \operatorname{Spec} R & & \longrightarrow & & X^{\mathrm{cpt}} \end{array}$$

that the image of the morphism $\operatorname{Spec} R \to X^{\mathrm{cpt}}$ is contained in X. Thus, it follows from the valuative criterion of properness (cf., e.g., [9, Chap. II, Thm. 4.7]) applied to the morphism $X^{\mathrm{cpt}} \to S$ that the given morphism $\operatorname{Spec} K \to X$ factors through $\operatorname{Spec} K \hookrightarrow \operatorname{Spec} R$. This implies that, to verify sufficiency, we may assume without loss of generality that R is complete.

Now, to verify sufficiency, assume that the given morphism $\operatorname{Spec} K \to X$ does not factor through $S = \operatorname{Spec} R$. Then it follows from Lemma 3.30 that there exists a connected finite étale covering $Y \to X$ of X such that if we write (g, r) for the type of the hyperbolic curve Y/S (note that it follows easily from the fact that Π_S is trivial, together with Proposition 3.5(ii), that $\mathrm{Nor}(Y/S) \xrightarrow{\sim} S$), then $r \geq 2$, and that $\Pi^p_{Y/S} \to \Pi^p_{X/S}$ is injective. For each cusp c of the hyperbolic curve X over R, let c' be a cusp of the hyperbolic curve Y over R which lies over c. Write X^{cpt}_c (resp. $Y^{\mathrm{cpt}}_{c'}$) for the spectrum of the ring obtained by completing X^{cpt} (resp. Y^{cpt}) along c (resp. c'), and $X_c := X \times_{X^{\mathrm{cpt}}} X^{\mathrm{cpt}}_c$, $Y_{c'} := Y \times_{Y^{\mathrm{cpt}}} Y^{\mathrm{cpt}}_{c'}$. Let $\overline{y} \to S$ be a geometric point of S. Then we have an exact sequence

$$0 \to \widehat{\mathbb{Z}}(1) \to \bigoplus_r \widehat{\mathbb{Z}}(1) \to (\Pi_{Y_{\overline{y}}})^{\mathrm{ab}}$$

(cf. [16, (1–5)]), where the homomorphism $\widehat{\mathbb{Z}}(1) \to \bigoplus_r \widehat{\mathbb{Z}}(1)$ is the diagonal embedding, and $\Pi_{Y_{c'}}$ is one of the direct summands $\widehat{\mathbb{Z}}(1)$ of $\bigoplus_r \widehat{\mathbb{Z}}(1)$. Thus, since $r \geq 2$, the morphism $\Pi^p_{Y_{c'}} \to (\Pi_{Y_{\overline{y}}})^{\mathrm{ab},p}$, hence also $\Pi^p_{Y_{c'}} \to \Pi^p_{Y_{\overline{y}}} = \Delta^p_{Y/S} \hookrightarrow \Pi^p_{Y/S}$, is injective. Next, let $h \in \ker(\Pi^p_{X_c} \to \Pi^p_{X/S})$. Then, since $\Pi^p_{Y_{c'}}$ is an open subgroup of $\Pi^p_{X_c}$, there exists a positive integer n such that $h^n \in \Pi^p_{Y_{c'}}$. Thus, since $\Pi^p_{X_c} \cong \mathbb{Z}_p$ is torsion-free, it follows from our choice of $Y \to X$ that $h = 1$. This implies that $\Pi^p_{X_c} \to \Pi^p_{X/S}$ is injective. On the other hand, it follows from the valuative criterion of properness applied to the morphism $X^{\mathrm{cpt}} \to S$ that the morphism $\operatorname{Spec} K \to X^{\mathrm{cpt}}$ factors through $\operatorname{Spec} K \hookrightarrow S = \operatorname{Spec} R$. Thus, since the given morphism $\operatorname{Spec} K \to X$ does not factor through $\operatorname{Spec} K \hookrightarrow S = \operatorname{Spec} R$, $\operatorname{Spec} K \to X$ factors through the natural morphism $X_c \to X$ associated to a suitable cusp c of X. Thus, since the image of the natural outer homomorphism $\Pi_{\operatorname{Spec} K} \to \Pi^p_{X/S}$ is trivial, it follows that the image of $\Pi_{\operatorname{Spec} K} \to \Pi_{X_c}$ is contained in $\ker(\Pi_{X_c} \twoheadrightarrow \Pi^p_{X_c})$. Note that $\ker(\Pi_{X_c} \twoheadrightarrow \Pi^p_{X_c})$ is the intersection of all open subgroups $U \subset \Pi_{X_c}$ such that Π_{X_c}/U is a p-group. Such an open subgroup of U contains $\mathrm{Im}(\Pi_{\mathrm{Spec}\,K} \to \Pi_{X_c}) \subset U$ and, moreover, the pull-back of the étale covering of X_c corresponding to U on $\mathrm{Spec}\,K$ is a disjoint union of copies of $\mathrm{Spec}\,K$. Now let us consider the diagram of affine schemes

$$\begin{array}{ccc} \mathrm{Spec}\,K & \longrightarrow & X_c \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spec}\,R & \longrightarrow & X_c^{\mathrm{cpt}}. \end{array}$$

The diagram obtained by taking global sections of the structure sheaves of the (affine) schemes in the above diagram is

$$\begin{array}{ccc} R\left[\dfrac{1}{\pi}\right] & \longleftarrow & R[[T]]\left[\dfrac{1}{T}\right] \\ \big\uparrow & & \big\uparrow \\ R & \longleftarrow & R[[T]], \end{array}$$

where π is a uniformizing parameter of R. Write $a \in R$ for the image of $T \in R[[T]]$ by the ring homomorphism $R[[T]] \to R$. Then a is contained in the maximal ideal of R. On the other hand, the covering of X_c corresponding to U is the spectrum of $R[[T^{1/p^m}]]$ for a suitable nonnegative integer m. Note that if U runs over all open subgroups as above, then m runs over all nonnegative integers. Thus, we conclude that for each nonnegative integer m there exists $b \in K^\times$ such that $a = b^{p^m}$. However, by considering the valuation, it follows that there is no such element $a \in R$. This completes the proof of sufficiency, hence also that of Lemma 3.31. $\square$

Lemma 3.32. *Let p be a prime number, k' a finite extension field of k, S a normal variety over k and Y, Z normal varieties over k', X a hyperbolic polycurve over S satisfying condition $(*)_p$, $Z \to Y$ a morphism over k', $Y \to S$ a morphism over k and $f : Z \to X$ a morphism over S. Suppose that the following conditions are satisfied:*

(1) *$Z \to Y$ is dominant and generically geometrically irreducible. (Thus, it follows from Lemma 2.6 that the natural outer homomorphism $\Pi_Z \to \Pi_Y$, hence also $\Pi^p_{Z/S} \to \Pi^p_{Y/S}$, is surjective.)*

(2) $\Delta^p_{Z\to Y/S} \subset \Delta^p_{Z\to X/S}$. *(Thus, it follows from the surjectivity of* $\Pi^p_{Z/S} \to \Pi^p_{Y/S}$ *that the natural outer homomorphism* $\Pi^p_{Z/S} \to \Pi^p_{X/S}$ *induced by* f *determines an outer homomorphism* $\Pi^p_{Y/S} \to \Pi^p_{X/S}$.*)*

Then the morphism $f : Z \to X$ *admits a unique factorization* $Z \to Y \to X$ *such that the morphism* $Y \to X$ *is an* S*-morphism.*

Proof. (Cf. [10, Lem. 2.9].) First, let us observe that the asserted uniqueness of the factorization under consideration follows from the fact that the morphism $Z \to Y$ is dominant. Next, we verify that, to verify Lemma 3.32, it suffices to verify Lemma 3.32 in the case where X is a hyperbolic curve over S. To verify this, assume that Lemma 3.32 holds if X is a hyperbolic curve over S. We verify Lemma 3.32 by induction on the relative dimension n of the hyperbolic polycurve X/S. We have already assumed that Lemma 3.32 in the case $n = 1$ holds. Now suppose that $n \geq 2$ and that the induction hypothesis is in force. Then since $\Delta^p_{Z\to Y/S} \subset \Delta^p_{Z\to X/S} \subset \Delta^p_{Z\to X_1/S}$, it follows from the case $n = 1$ that the morphism $Z \to X_1$ admits a unique factorization $Z \to Y \to X_1$ such that $Y \to X_1$ is an S-morphism. On the other hand, since X/S satisfies condition $(*)_p$, it follows that $\Delta^p_{X/X_1} \to \Delta^p_{X/S}$ is injective. Thus, since $\ker(\Delta_{Z/X_1} \to \Delta^p_{Y/X_1}) \subset \ker(\Delta_{Z/X_1} \to \Delta^p_{Y/X_1} \to \Delta^p_{X/S})$, we obtain $\Delta^p_{Z\to Y/X_1} \subset \Delta^p_{Z\to X/X_1}$. By the induction hypothesis, since X/X_1 satisfies condition $(*)_p$, the morphism $f : Z \to X$ admits a unique factorization $Z \to Y \to X$ such that $Y \to X$ is an X_1-morphism (hence an S-morphism).

Now let us assume that X/S is a hyperbolic curve. Moreover, let us assume that $k = k'$ until Claim F below. Write $\Gamma_0 \subset X \times_S Y$ for the scheme-theoretic image of the natural morphism $Z \to X \times_S Y$ over S and $\Gamma := \mathrm{Nor}(Z/\Gamma_0)$. Then Γ_0 is an integral variety over k and the natural morphism $Z \to \Gamma_0$ is dominant. Moreover, it follows from Lemma 2.9 that Γ is a normal variety over k, the resulting morphism $Z \to \Gamma$ is dominant and generically geometrically irreducible and the natural morphism $\Gamma \to \Gamma_0$ is finite and surjective.

Next, I claim that the following assertion holds:

> Claim A: Let $\overline{y} \to Y$ be a geometric point of Y. Then the image of the morphism $Z \times_Y \overline{y} \to X \times_S \overline{y}$ determined by f consists of finitely many closed points of $X \times_S \overline{y}$.

Indeed, let $F \to Z \times_Y \overline{y}$ be a connected component (which is necessarily a normal variety over $\overline{y}$) of the normalization of the reduced scheme associated to $Z \times_Y \overline{y}$. Then, since the composite of natural morphisms $F \to Z \times_Y \overline{y} \to Z \to Y$ factors through the geometric point $\overline{y} \to Y$, we obtain $\mathrm{Im}(\Pi^p_F \to \Pi^p_{Z/S}) \subset \Delta^p_{Z\to Y/S}$. Thus, it follows from condition (2) that the image of the outer homomorphism $\Pi^p_F \to \Pi^p_{X/S}$ is trivial. On the other hand, the composite of natural morphisms

$F \to Z \times_Y \overline{y} \overset{\mathrm{pr}_1}{\to} Z \to X$ factors through the projection $X \times_S \overline{y} \overset{\mathrm{pr}_1}{\to} X$. Thus, since the outer homomorphism $\Pi^p_{X \times_S \overline{y}} \to \Pi^p_{X/S}$ induced by $X \times_S \overline{y} \overset{\mathrm{pr}_1}{\to} X$ is injective (cf. Proposition 3.16(i)), it follows that the image of the outer homomorphism $\Pi^p_F \to \Pi^p_{X \times_S \overline{y}}$ induced by the morphism $F \to X \times_S \overline{y}$ is trivial. In particular, since $X \times_S \overline{y}$ is a hyperbolic curve over $\overline{y}$, hence of p-LFG-type (cf. Proposition 3.28), and the morphism $F \to X \times_S \overline{y}$ is a morphism between varieties over $\overline{y}$, it follows that the image of $F \to X \times_S \overline{y}$ consists of a closed point of $X \times_S \overline{y}$. Thus, the image of the morphism $Z \times_Y \overline{y} \to X \times_S \overline{y}$ consists of finitely many closed points of $X \times_S \overline{y}$. This completes the proof of Claim A.

Next, I claim that the following holds:

> Claim B: The composite $\Gamma \to \Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$, hence also the composite $\Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$, is dominant and induces an isomorphism between the respective function fields.

Indeed, since $Z \to Y$ is dominant and generically geometrically irreducible (cf. condition (1)) and factors through $\Gamma \to Y$, it follows from [5, Prop. (4.5.9)] that $\Gamma \to Y$ is dominant and generically geometrically irreducible. Since k, hence also the respective function fields of Γ and Y, is of characteristic zero, to verify Claim B, it suffices to verify that $\Gamma \to Y$ is generically quasi-finite. To verify that $\Gamma \to Y$ is generically quasi-finite, let $\overline{\eta}_Y \to Y$ be a geometric point of Y whose image is the generic point of Y. Then since the operation of forming the scheme-theoretic image commutes with base-change by a flat morphism, $\Gamma_0 \times_Y \overline{\eta}_Y$ is naturally isomorphic to the scheme-theoretic image of the natural morphism $Z \times_Y \overline{\eta}_Y \to X \times_S \overline{\eta}_Y$. Thus, since the image of the morphism $Z \times_Y \overline{\eta}_Y \to X \times_S \overline{\eta}_Y$ consists of finitely many closed points of $X \times_S \overline{\eta}_Y$ (cf. Claim A), it follows that the composite $\Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$, hence (by the finiteness of $\Gamma \to \Gamma_0$) also the morphism $\Gamma \to Y$, is generically quasi-finite. This completes the proof of Claim B.

Next, I claim that the following assertion holds:

> Claim C: $\Delta^p_{\Gamma \to Y/S} \subset \Delta^p_{\Gamma \to X/S}$.

Indeed, it follows from Lemma 2.6 that $\Pi_Z \to \Pi_\Gamma$, hence also $\Pi^p_{Z/S} \to \Pi^p_{\Gamma/S}$, is surjective. Thus, since $\Delta^p_{\Gamma \to Y/S}$ (resp. $\Delta^p_{\Gamma \to X/S}$) is the image of the subgroup $\Delta^p_{Z \to Y/S}$ (resp. $\Delta^p_{Z \to X/S}$) of $\Pi^p_{Z/S}$ by the surjection $\Pi^p_{Z/S} \twoheadrightarrow \Pi^p_{\Gamma/S}$, it follows from condition (2) that $\Delta^p_{\Gamma \to Y/S} \subset \Delta^p_{\Gamma \to X/S}$. This completes the proof of Claim C.

Next, I claim that the following assertion holds:

> Claim D: Let $\overline{y} \to Y$ be a geometric point of Y. Then the image of the morphism $\Gamma \times_Y \overline{y} \to X \times_S \overline{y}$ determined by $\Gamma \to \Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_1}{\to} X$ consists of finitely many closed points of $X \times_S \overline{y}$.

Indeed, we obtain a proof of Claim D by replacing "Z" in the proof of Claim A with Γ (cf. Claim C). This completes the proof of Claim D.

Next, I claim that the following assertion holds:

Claim E: The composite $\Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$ is an open immersion.

Indeed, let $\overline{y} \to Y$ be a geometric point of Y. Then it follows from Claim D that the image of the composite $\Gamma \times_Y \overline{y} \to \Gamma_0 \times_Y \overline{y} \hookrightarrow X \times_S \overline{y}$ consists of finitely many closed points of $X \times_S \overline{y}$. Thus, since $\Gamma \to \Gamma_0$ is surjective, and the morphism $\Gamma_0 \times_Y \overline{y} \hookrightarrow X \times_S \overline{y}$ is a closed immersion, we conclude that $\Gamma_0 \times_Y \overline{y}$ is quasi-finite over $\overline{y}$. In particular, $\Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$ is quasi-finite. Thus, it follows from Claim B, together with [4, Cor. (4.4.9)], that the composite $\Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$ is an open immersion. This completes the proof of Claim E.

Next, I claim that the following assertion holds:

Claim F: If X is proper over S, then $f : Z \to X$ admits a factorization $Z \to Y \to X$ such that $Y \to X$ is an S-morphism.

Indeed, if X is proper over S, then the composite $\Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$ is proper. Thus, it follows from Claim E that the composite $\Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$ is an isomorphism over S. In particular, $f : Z \to X$ admits a factorization $Z \to Y \to X$ such that $Y \to X$ is an S-morphism. This completes the proof of Claim F.

Next, I claim that the following assertion holds (note that in Claims G and H below, we do not assume that $k = k'$):

Claim G: If the genus of the hyperbolic curve X over S is ≥ 2, then f admits a factorization $Z \to Y \to X$ such that $Y \to X$ is an S-morphism.

Indeed, let us consider the commutative diagram of schemes

$$\begin{array}{ccccc}
Z & \longrightarrow & X \times_k k' & \overset{\mathrm{pr}_1}{\longrightarrow} & X \\
\downarrow & & \downarrow & & \downarrow \\
Y & \longrightarrow & S \times_k k' & \overset{\mathrm{pr}_1}{\longrightarrow} & S \\
 & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow \\
 & & \operatorname{Spec} k' & \longrightarrow & \operatorname{Spec} k,
\end{array}$$

where $X \times_k k'$ is a hyperbolic curve of genus ≥ 2 over $S \times_k k'$. Since $\Pi_{S \times_k k'} \to \Pi_S$ is injective, it follows that $\Delta_{Z/S} = \Delta_{Z/S \times_k k'}$, $\Delta_{Y/S} = \Delta_{Y/S \times_k k'}$. Thus, we obtain $\Pi^p_{Z/S} = \Pi^p_{Z/S \times_k k'}$, $\Pi^p_{Y/S} = \Pi^p_{Y/S \times_k k'}$. This implies that $\Delta^p_{Z \to Y/S} = \Delta^p_{Z \to Y/S \times_k k'}$.

Moreover, since $\Pi^p_{X\times_k k'/S\times_k k'}$ is the inverse image of the open subgroup $G_{k'} \subset G_k$ by the composite $\Pi^p_{X/S} \to \Pi_S \to G_k$, $\Pi^p_{Z/S} = \Pi^p_{Z/S\times_k k'} \to \Pi^p_{X/S}$ factors through $\Pi^p_{X\times_k k'/S\times_k k'}$. Thus, we conclude that $\Delta^p_{Z\to X/S} = \Delta^p_{Z\to X\times_k k'/S\times_k k'}$. In particular, to verify Claim G, we may assume without loss of generality that $k = k'$.

Now, since the genus of X/S is ≥ 2, X^{cpt} is a proper hyperbolic curve over S. Thus, since $\Delta^p_{Z\to Y/S} \subset \Delta^p_{Z\to X/S} \subset \Delta^p_{Z\to X^{\mathrm{cpt}}/S}$, by applying Claim F, where we take the data "(S, Y, Z, X)" to be $(S, Y, Z, X^{\mathrm{cpt}})$, we conclude that the morphism $Z \to X^{\mathrm{cpt}}$ over S factors as a composite $Z \to Y \to X^{\mathrm{cpt}}$, where $Y \to X^{\mathrm{cpt}}$ is an S-morphism. This implies that, to verify Claim G, it suffices to verify that $Y \to X^{\mathrm{cpt}}$ factors through $X \subset X^{\mathrm{cpt}}$. Note that since $Z \to Y$ is dominant by condition (1), it follows that the image of the generic point of Y by the morphism $Y \to X^{\mathrm{cpt}}$ is contained in $X \subset X^{\mathrm{cpt}}$. Let $y \in Y$ be a point of Y that is not the generic point of Y and R_0 a discrete valuation ring dominating $\mathcal{O}_{Y,y}$ (cf., e.g., [3, Prop. (7.1.7)]). Write R for the strict henselization of R_0. Then R is a strict henselian discrete valuation ring and, moreover, the image of the closed point of $\operatorname{Spec} R$ by the composite $\operatorname{Spec} R \to \operatorname{Spec} R_0 \to \operatorname{Spec} \mathcal{O}_{Y,y} \to Y$ is y. On the other hand, since the composite $\eta_R \to \operatorname{Spec} R \to Y$, where we write η_R for the spectrum of the quotient field of R, factors through $\Gamma \to Y$ (cf. Claim B), it follows from the fact that $\Pi_{\operatorname{Spec} R} = \{1\}$ that the image of the composite $\Pi_{\eta_R} \to \Pi_{\operatorname{Spec} R} \to \Pi^p_{Y/S}$, hence also the composite $\Pi_{\eta_R} \to \Pi^p_{\Gamma/S} \to \Pi^p_{X/S}$ (cf. Claim C), is trivial. Thus, it follows from Lemma 3.31 that $\eta_R \to \Gamma \to X$ factors through $\eta_R \hookrightarrow \operatorname{Spec} R$. In particular, the composite $\operatorname{Spec} R \to Y \to X^{\mathrm{cpt}}$ factors through $X \subset X^{\mathrm{cpt}}$. This implies that the image of $y \in Y$ by the morphism $Y \to X^{\mathrm{cpt}}$ is contained in $X \subset X^{\mathrm{cpt}}$. Thus, the morphism $Y \to X^{\mathrm{cpt}}$ factors through $X \subset X^{\mathrm{cpt}}$. This completes the proof of Claim G.

Finally, I claim that the following assertion holds:

Claim H: f admits a factorization $Z \to Y \to X$ such that $Y \to X$ is an S-morphism.

Indeed, it follows from an argument similar to the argument used at the beginning of the proof of Claim G that to verify Claim H, we may assume without loss of generality that $k = k'$. Then, it follows from Lemma 3.30 that there exists a connected finite étale Galois covering $X' \to X$ of X such that if we write $S' := \mathrm{Nor}(X'/S)$, then the genus of X'/S' is ≥ 2 and, moreover, $\Pi^p_{X'/S'} \to \Pi^p_{X/S}$ is injective. Write $Y' \to Y$ for the connected finite étale Galois covering of Y corresponding to the inverse image of $\Pi^p_{X'/S'} \subset \Pi^p_{X/S}$ by $\Pi^p_{Y/S} \to \Pi^p_{X/S}$ (cf. condition (2)), and $Z' := Z \times_Y Y' \stackrel{\mathrm{pr}_1}{\to} Z$ for the connected (cf. condition (1)) finite étale Galois covering of Z corresponding to $Y' \to Y$. Then, since the image of $\Delta^p_{Z'\to Y'/S'} \subset \Pi^p_{Z'/S'}$ by the composite $\Pi^p_{Z'/S'} \to \Pi^p_{Y'/S'} \to \Pi^p_{Y/S} \to \Pi^p_{X/S}$ is trivial, it follows from the injectivity of $\Pi^p_{X'/S'} \to \Pi^p_{X/S}$ that the image of $\Delta^p_{Z'\to Y'/S'} \subset \Pi^p_{Z'/S'}$ by $\Pi^p_{Z'/S'} \to \Pi^p_{X'/S'}$ is trivial. Thus, we conclude that $\Delta^p_{Z'\to Y'/S'} \subset \Delta^p_{Z'\to X'/S'}$. On the other hand, the image of $\Pi_{S'} \subset \Pi_S$ (resp. $\Pi_{Y'} \subset \Pi_Y$) by the surjection $\Pi_S \twoheadrightarrow G_k$ (resp. $\Pi_Y \twoheadrightarrow G_k$) is an open subgroup of G_k, which corresponds to some finite field extension k' (resp. k''). Then, $(S', Y', Z', X', k', k'')$ satisfies conditions (1), (2) for "(S, Y, Z, X, k, k')" in the statement of Lemma 3.32. Thus, since X' is a hyperbolic curve over S' of genus ≥ 2, it follows from Claim G that the natural morphism $Z' \to X'$ over S' factors as a composite $Z' \to Y' \to X'$, where $Y' \to X'$ is an S'-morphism. In particular, the natural morphism $Z' \to X$ over S admits a unique factorization $Z' \to Y' \to X$, where $Y' \to X$ is an S-morphism. Moreover, in light of this uniqueness, the factorization $Z' \to Y' \to X'$ is compatible with the natural actions of $\mathrm{Gal}(Z'/Z) \cong \mathrm{Gal}(Y'/Y)$. This compatibility with Galois actions thus implies that we obtain a factorization $Z \to Y \to X$ such that $Y \to X$ is an S-morphism. This completes the proof of Claim H, hence also of Lemma 3.32. $\square$

Corollary 3.33. *Let p be a prime number, k' a finite extension field of k, S a normal variety over k and Y, Z normal varieties over k', X a hyperbolic polycurve over S satisfying condition $(**)_p$, $Z \to Y$ a morphism over k', $Y \to S$ a morphism over k and $f : Z \to X$ a morphism over S. Suppose that the following conditions are satisfied:*

(1) *$Z \to Y$ is dominant and generically geometrically irreducible.*

(2) $\Delta^{(p)}_{Z/Y} \subset \Delta^{(p)}_{Z/X}$.

Then the morphism $f : Z \to X$ admits a unique factorization $Z \to Y \to X$, where the morphism $Y \to X$ is an S-morphism.

Proof. It follows from Lemma 3.23(ii) that the outer homomorphism $\Pi^p_Y \to \Pi^p_X$ over Π^p_S (cf. condition (2)) determines an outer homomorphism $\Pi^p_{Y/S} \to \Pi^p_{X/S}$ such that the composite $\Pi^p_{Z/S} \to \Pi^p_{Y/S} \to \Pi^p_{X/S}$ coincides with the outer homomorphism $\Pi^p_{Z/S} \to \Pi^p_{X/S}$ induced by f. Thus, we obtain $\Delta^p_{Z\to Y/S} \subset \Delta^p_{Z\to X/S}$. This implies that it follows from Lemma 3.32 that the morphism $f : Z \to X$ admits a unique factorization $Z \to Y \to X$. This completes the proof of Corollary 3.33. $\square$

Lemma 3.34. *Let p be a prime number, S, Y normal varieties over k, $Y \to S$ a morphism, X a hyperbolic polycurve over S satisfying condition $(*)_p$ (resp. $(**)_p$) and $\phi : \Pi^p_{Y/S} \to \Pi^p_{X/S}$ (resp. $\phi : \Pi^p_Y \to \Pi^p_X$) a homomorphism. Write $\eta \to Y$ for the generic point of Y. Then the following conditions are equivalent:*

(1) *The homomorphism ϕ arises from a morphism $Y \to X$ over S.*

(2) *There exists a morphism $\eta \to X$ over S such that the outer homomorphism $\Pi^p_{\eta/S} \to \Pi^p_{X/S}$ (resp. $\Pi^p_\eta \to \Pi^p_X$) induced by this morphism $\eta \to X$ coincides*

with the composite of the outer surjection (cf. Lemma 2.2) $\Pi^p_{\eta/S} \twoheadrightarrow \Pi^p_{Y/S}$ *(resp.* $\Pi^p_\eta \twoheadrightarrow \Pi^p_Y$*) induced by* $\eta \to Y$ *and the outer homomorphism determined by* ϕ.

Proof. (Cf. [10, Lem. 2.10].) The implication (1) $\Rightarrow$ (2) is immediate. We verify the implication (2) $\Rightarrow$ (1). Suppose that condition (2) is satisfied. Let $U \subset Y$ be a nonempty open subscheme of Y such that the morphism $\eta \to X$ in condition (2) extends to a morphism $U \to X$ over S. Then it follows from Lemma 2.2 that the outer homomorphism $\Pi_\eta \to \Pi_U$, hence also $\Pi^p_{\eta/S} \to \Pi^p_{U/S}$ (resp. $\Pi^p_\eta \to \Pi^p_U$), is surjective. Thus, it follows that the outer homomorphism $\Pi^p_{U/S} \to \Pi^p_{X/S}$ (resp. $\Pi^p_U \to \Pi^p_X$) coincides with the composite of the outer surjection $\Pi^p_{U/S} \twoheadrightarrow \Pi^p_{Y/S}$ and the outer homomorphism determined by ϕ. By applying Lemma 3.32 (resp. Corollary 3.33), where we take the data "(k, k', S, Y, Z, X)" to be (k, k, S, Y, U, X), we conclude that the homomorphism $U \to X$ factors through $Y \to X$. This completes the proof of Lemma 3.34. □

Lemma 3.35. *Let* p *be a prime number,* X *a hyperbolic curve over* k*,* Y *a normal variety over* k *and* $f : Y \to X$ *a morphism over* k*. Write* $\phi_f : \Pi_Y \to \Pi_X$*,* $\phi^p_{f/k} : \Pi^p_{Y/k} \to \Pi^p_{X/k}$*,* $\phi^p_f : \Pi^p_Y \to \Pi^p_X$ *for the outer homomorphisms induced by* f*. Consider the following conditions:*

(1) *f is surjective, smooth and generically geometrically connected.*

(2) *ϕ_f is surjective and the kernel Δ_f of ϕ_f is topologically finitely generated.*

(2)′ *$\phi^p_{f/k}$ is surjective and the kernel $\Delta^p_{f/k}$ of $\phi^p_{f/k}$ is topologically finitely generated.*

(2)″ *ϕ^p_f is surjective and the kernel $\Delta^{(p)}_f$ of ϕ^p_f is topologically finitely generated.*

(3) *f is surjective and generically geometrically connected.*

(4) *Let C be a hyperbolic curve over k and $C \to X$ a morphism over k. Then if f factors through $C \to X$, then $C \to X$ is an isomorphism.*

Then we have implications and an equivalence: (1) $\Rightarrow$ (2) $\Rightarrow$ (2)′ $\Rightarrow$ (3) $\Leftrightarrow$ (4), (2)′ $\Rightarrow$ (2)″*. Moreover, if* $\Pi^p_{X \times_k \bar{k}} \to \Pi^p_X$ *and* $\Pi^p_{Y \times_k \bar{k}} \to \Pi^p_Y$ *are injective, then we have an equivalence* (2)′ $\Leftrightarrow$ (2)″.

Proof. (Cf. [10, Lem. 2.11].) The implication (1) $\Rightarrow$ (2) and the equivalence (3) $\Leftrightarrow$ (4) are proved in [10, Lem. 2.11]. First, we verify the implication (2) $\Rightarrow$ (2)′. Suppose that condition (2) is satisfied. Then, the surjectivity of $\phi^p_{f/k}$ is immediate. Moreover, since there is a surjection $\Delta_f \to \Delta^p_{f/k}$ (cf. Lemma 2.18), $\Delta^p_{f/k}$ is topologically finitely generated. This completes the proof of the implication (2) $\Rightarrow$ (2)′. The implication (2)′ $\Rightarrow$ (2)″ may be proved similarly. Next, we verify the implication (2)′ $\Rightarrow$ (4). Suppose that condition (2)′ is satisfied. Let C be a hyperbolic curve over k and $C \to X$ a morphism over k. Then, if $C \times_k \overline{k} \to X \times_k \overline{k}$ is an isomorphism, then so is $C \to X$ (cf. [2, Cor. 1.8.4]). On the other hand, by considering the commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi^p_{Y\times_k\overline{k}} & \longrightarrow & \Pi^p_{Y/k} & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \phi^p_{f/k} & & \| & & \\ 1 & \longrightarrow & \Pi^p_{X\times_k\overline{k}} & \longrightarrow & \Pi^p_{X/k} & \longrightarrow & G_k & \longrightarrow & 1, \end{array}$$

we obtain that $\Pi^p_{Y\times_k\overline{k}/\overline{k}} = \Pi^p_{Y\times_k\overline{k}} \to \Pi^p_{X\times_k\overline{k}} = \Pi^p_{X\times_k\overline{k}/\overline{k}}$ is surjective and, moreover, $\Delta^p_{f/k} = \ker(\Pi^p_{Y\times_k\overline{k}} \to \Pi^p_{X\times_k\overline{k}})$. Thus, to verify that condition (4) is satisfied, we may assume without loss of generality that $k = \overline{k}$. Suppose that f factors through $C \to X$. Then, since X is a hyperbolic curve over k, it follows from Proposition 3.16(iii) that $\Delta^p_{X/k} = \Pi^p_X$ is infinite. Thus, since C is a hyperbolic curve over k, the surjectivity of $\phi^p_{f/k}$ implies that f, hence also $Y \to C$, is dominant. In particular, it follows from Lemma 2.3 that the induced outer homomorphism $\Pi_Y \to \Pi_C$, hence also $\Pi^p_Y \to \Pi^p_C$, is open. Moreover, since $\phi^p_{f/k}$ is surjective, $\Pi^p_C \to \Pi^p_X$ is surjective. On the other hand, since the kernel of $\phi^p_{f/k}$ is topologically finitely generated, it follows from the openness of $\Pi^p_Y \to \Pi^p_C$ that $\Delta^p_{C\to X/k}$ admits an open subgroup which is topologically finitely generated. Thus, $\Delta^p_{C\to X/k}$ is topologically finitely generated. Now the surjectivity of $\Pi^p_C \to \Pi^p_X$ implies that $\Pi^p_C/\Delta^p_{C\to X/k} \xrightarrow{\sim} \Pi^p_X$. Thus, since Π^p_X is infinite, $\Delta^p_{C\to X/k} \subset \Pi^p_C = \Delta^p_{C/k}$ is not open in $\Delta^p_{C/k}$. This implies that $\Delta^p_{C\to X/k}$ is trivial (cf. Proposition 3.16(iv)). Thus, we conclude that $\Delta^p_{C/k} = \Pi^p_C \to \Pi^p_X = \Delta^p_{X/k}$ is an outer isomorphism. Write (g_X, r_X), (g_C, r_C) for the type of X/k, C/k, respectively. Then, since $\Delta^p_{C/k} \cong \Delta^p_{X/k}$, it follows from Proposition 3.16(v) that $2g_X + \max\{r_X - 1, 0\} = 2g_C + \max\{r_C - 1, 0\}$ and, moreover, that $r_X = 0$ if and only if $r_C = 0$. On the other hand, since $C \to X$ determines the surjection $C^{\mathrm{cpt}} \to X^{\mathrm{cpt}}$, we have $r_C \geq r_X$. Moreover, it follows from Hurwitz' formula (cf., e.g., [9, Chap. IV, Cor. 2.4]) that $g_C \geq g_X$. Thus, it follows immediately that $g_C = g_X$, $r_C = r_X$. Moreover, since $C^{\mathrm{cpt}} \to X^{\mathrm{cpt}}$ determines the bijection between the points of $C^{\mathrm{cpt}} \setminus C$ and the points of $X^{\mathrm{cpt}} \setminus X$, $C^{\mathrm{cpt}} \to X^{\mathrm{cpt}}$ is totally ramified over $C^{\mathrm{cpt}} \setminus C$. Thus, if we write n for the degree of $C^{\mathrm{cpt}} \to X^{\mathrm{cpt}}$ and e_P for the ramification index at $P \in C^{\mathrm{cpt}}$, then it follows from Hurwitz' formula that

$$2g_C - 2 = n(2g_X - 2) + \sum_{P\in C^{\mathrm{cpt}}} (e_P - 1) = n(2g_X - 2) + r_C(n-1) + \sum_{P\in C} (e_P - 1).$$

This implies that $n = 1$, and that for any $P \in C$, $e_P = 1$. Thus, we conclude that $C \to X$ is an isomorphism. This completes the proof of the implication $(2)' \Rightarrow (4)$.

Finally, we verify the implication $(2)'' \Rightarrow (2)'$, assuming that $\Pi^p_{X\times_k\overline{k}} \to \Pi^p_X$ and $\Pi^p_{Y\times_k\overline{k}} \to \Pi^p_Y$ are injective. Suppose that condition $(2)''$ is satisfied. Then, the two commutative diagrams of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi^p_{Y\times_k\overline{k}} & \longrightarrow & \Pi^p_Y & \longrightarrow & G^p_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \phi^p_f} & & \| & & \\ 1 & \longrightarrow & \Pi^p_{X\times_k\overline{k}} & \longrightarrow & \Pi^p_X & \longrightarrow & G^p_k & \longrightarrow & 1 \end{array}$$

and

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi^p_{Y\times_k\overline{k}} & \longrightarrow & \Pi^p_{Y/k} & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \phi^p_{f/k}} & & \| & & \\ 1 & \longrightarrow & \Pi^p_{X\times_k\overline{k}} & \longrightarrow & \Pi^p_{X/k} & \longrightarrow & G_k & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact, imply that condition $(2)'$ is satisfied. This completes the proof of the implication $(2)'' \Rightarrow (2)'$, hence also of Lemma 3.35. $\square$

Lemma 3.36. *In the notation of Lemma 3.35, suppose, moreover, that Y is of p-LFG-type. Then the following hold:*

(i) *Consider the following conditions:*

(1) *f is an isomorphism.*

(2) *ϕ_f is an outer isomorphism.*

(3) *ϕ_f is surjective and the kernel Δ_f of ϕ_f is finite.*

(4) *$\phi^p_{f/k}$ is an outer isomorphism.*

(5) *$\phi^p_{f/k}$ is surjective and the kernel $\Delta^p_{f/k}$ of $\phi^p_{f/k}$ is finite.*

(6) *ϕ^p_f is an outer isomorphism.*

(7) *ϕ^p_f is surjective and the kernel $\Delta^{(p)}_f$ of ϕ^p_f is finite.*

Then we have implications and equivalences: $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Rightarrow (6) \Rightarrow (7)$. *Moreover, if $\Pi^p_{X\times_k\overline{k}} \to \Pi^p_X$ and $\Pi^p_{Y\times_k\overline{k}} \to \Pi^p_Y$ are injective, then the above conditions are all equivalent.*

(ii) *The following conditions are equivalent:*

(1) f *is a finite étale covering and the degree of the Galois closure of* $Y \times_k \overline{k} \to X \times_k \overline{k}$ *determined by* f *is a power of* p.

(2) $\phi^p_{f/k}$ *is an outer open injection.*

(3) $\phi^p_{f/k}$ *is open and the kernel* $\Delta^p_{f/k}$ *of* $\phi^p_{f/k}$ *is finite.*

(iii) *Suppose that* $\Pi^p_{X\times_k\overline{k}} \to \Pi^p_X$ *and* $\Pi^p_{Y\times_k\overline{k}} \to \Pi^p_Y$ *are injective. Then the following conditions are equivalent:*

(1) f *is a finite étale covering and the degree of the Galois closure of* f *is a power of* p.

(2) ϕ^p_f *is an outer open injection.*

(3) ϕ^p_f *is open and the kernel* $\Delta^{(p)}_f$ *of* ϕ^p_f *is finite.*

Proof. (Cf. [10, Lem. 2.12].) First, we verify assertion (i). The implication (3) ⇒ (5) follows from Lemma 2.18 and, moreover, the implications (1) ⇒ (2) ⇒ (3), (2) ⇒ (4) ⇒ (5) and (2) ⇒ (6) ⇒ (7) are immediate. Now we verify the implication (5) ⇒ (1). Suppose that condition (5) is satisfied. Then it follows from the implication (2)′ ⇒ (3) of Lemma 3.35 that f is surjective and generically geometrically connected. On the other hand, it follows from Lemma 3.27(ii) that f is quasi-finite. Thus, it follows from [4, Cor. (4.4.9)] that f is an open immersion, hence an isomorphism. This completes the proof of the implication (5) ⇒ (1). Similarly, the implication (7) ⇒ (1) (assuming that $\Pi^p_{X\times_k\overline{k}} \to \Pi^p_X$ and $\Pi^p_{Y\times_k\overline{k}} \to \Pi^p_Y$ are injective) follows from the implication (2)″ ⇒ (3) of Lemma 3.35, together with Lemma 3.27(iii).

Next, we verify assertion (ii). First, we verify the implication (1) ⇒ (2). Suppose that condition (1) is satisfied. Then, since the open subgroup $\Pi_Y \subset \Pi_X$ corresponding to f contains $\ker(\Delta_{X/k} \twoheadrightarrow \Delta^p_{X/k})$, it follows that $\Pi^p_{Y/k}$ is the image of $\Pi_Y \subset \Pi_X$ by the surjection $\Pi_X \twoheadrightarrow \Pi^p_{X/k}$. This completes the implication (1) ⇒ (2). The implication (2) ⇒ (3) is immediate. Thus, it remains to verify the implication (3) ⇒ (1). To verify this implication, suppose that condition (3) is satisfied. Write $X' \to X$ for the connected finite étale covering corresponding to the open subgroup $\mathrm{Im}(\phi^p_{f/k}) \subset \Pi^p_{X/k}$ of $\Pi^p_{X/k}$. Then the degree of the Galois closure of $X' \times_k \overline{k} \to X \times_k \overline{k}$ determined by $X' \to X$ is a power of p. Moreover, since $\Pi_{X'}$ is the inverse image of $\mathrm{Im}(\phi^p_{f/k}) \subset \Pi^p_{X/k}$ by the surjection $\Pi_X \twoheadrightarrow \Pi^p_{X/k}$, it follows that $\phi_f : \Pi_Y \to \Pi_X$ factors through $\Pi_{X'} \hookrightarrow \Pi_X$. Thus, $Y \to X$ factors through $X' \to X$. On the other hand, it follows from our choice of $X' \to X$ that $\Pi^p_{Y/k} \to \Pi^p_{X'/k}$ is surjective and, moreover, $\Delta^p_{Y\to X'/k} = \Delta^p_{f/k}$. Thus, it follows from the implication (5) ⇒ (1) of (i) that the morphism $Y \to X'$ is an isomorphism.

This completes the proof of the implication (3) $\Rightarrow$ (1), hence also of assertion (ii). Similarly, assertion (iii) follows from the implication (7) $\Rightarrow$ (1) of (i). This completes the proof of Lemma 3.36. □

Lemma 3.37. *In the notation of Lemma 3.35, suppose, moreover, that Y is a hyperbolic curve over k. Then the following hold:*

(i) *Consider the following conditions:*

  (1) *f is an isomorphism.*
  (2) *ϕ_f is an outer isomorphism.*
  (3) *ϕ_f is surjective and the kernel Δ_f of ϕ_f is topologically finitely generated.*
  (4) *$\phi^p_{f/k}$ is an outer isomorphism.*
  (5) *$\phi^p_{f/k}$ is surjective and the kernel $\Delta^p_{f/k}$ of $\phi^p_{f/k}$ is topologically finitely generated.*
  (6) *ϕ^p_f is an outer isomorphism.*
  (7) *ϕ^p_f is surjective and the kernel $\Delta^{(p)}_f$ of ϕ^p_f is topologically finitely generated.*

  Then we have implications and equivalences: (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7). *Moreover, if $\Pi^p_{X\times_k\overline{k}} \to \Pi^p_X$ and $\Pi^p_{Y\times_k\overline{k}} \to \Pi^p_Y$ are injective, then the above conditions are all equivalent.*

(ii) *The following conditions are equivalent.*

  (1) *f is a finite étale covering and the degree of the Galois closure of $Y\times_k\overline{k} \to X\times_k\overline{k}$ determined by f is a power of p.*
  (2) *$\phi^p_{f/k}$ is an outer open injection.*
  (3) *$\phi^p_{f/k}$ is open and the kernel $\Delta^p_{f/k}$ of $\phi^p_{f/k}$ is topologically finitely generated.*

(iii) *Suppose that $\Pi^p_{X\times_k\overline{k}} \to \Pi^p_X$ and $\Pi^p_{Y\times_k\overline{k}} \to \Pi^p_Y$ are injective. Then the following conditions are equivalent:*

  (1) *f is a finite étale covering and the degree of the Galois closure of f is a power of p.*
  (2) *ϕ^p_f is an outer open injection.*
  (3) *ϕ^p_f is open and the kernel $\Delta^{(p)}_f$ of ϕ^p_f is topologically finitely generated.*

Proof. (Cf. [10, Lem. 2.13].) If we verify assertion (i), then assertions (ii) and (iii) follow from an argument similar to the argument used in the proof of Lemma 3.36.

Thus, it remains to verify assertion (i). Since Y is of p-LFG-type (cf. Proposition 3.28), the implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) follow from Lemma 3.36. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (5) follow from Lemma 2.18. The implications (2) $\Rightarrow$ (6) $\Rightarrow$ (7) are immediate. Now we verify the implication (5) $\Rightarrow$ (4). Suppose that condition (5) is satisfied. Let us observe that it follows from the commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi^p_{Y\times_k\overline{k}} & \longrightarrow & \Pi^p_{Y/k} & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \phi^p_{f/k} & & \| & & \\ 1 & \longrightarrow & \Pi^p_{X\times_k\overline{k}} & \longrightarrow & \Pi^p_{X/k} & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

that, to verify that condition (4) is satisfied, we may assume without loss of generality that $k = \overline{k}$. Then, it follows from Proposition 3.16(iii), together with the surjectivity of $\phi^p_{f/k}$, that the image of $\phi^p_{f/k}$ is infinite, i.e., $\Delta^p_{f/k}$ is not open in $\Pi^p_Y = \Delta^p_{Y/k}$. Thus, it follows from Proposition 3.16(iv) that $\Delta^p_{f/k}$ is trivial. This completes the proof of the implication (5) $\Rightarrow$ (4). Finally, we verify the implication (7) $\Rightarrow$ (5), assuming that $\Pi^p_{X\times_k\overline{k}} \to \Pi^p_X$ and $\Pi^p_{Y\times_k\overline{k}} \to \Pi^p_Y$ are injective. It follows from an argument similar to the argument used in the proof of the implication $(2)'' \Rightarrow (2)'$ of Lemma 3.35 that, to verify the implication (7) $\Rightarrow$ (5), we may assume without loss of generality that $k = \overline{k}$. But then condition (7) is the same as condition (5). This completes the proof of the implication (7) $\Rightarrow$ (5), hence also of Lemma 3.37. $\square$

Lemma 3.38. *Suppose that $k = \overline{k}$. Let p be a prime number, n a positive integer, X a hyperbolic polycurve over k satisfying condition $(*)_p$, F a normal variety over k of dimension $\geq n$ and $F \to X$ a quasi-finite morphism over k. (Thus, it holds that $n \leq \dim(F) \leq \dim(X)$.) Write $\Pi^p_{F\to X/k} := \Pi^p_{F/k}/\Delta^p_{F\to X/k}$. (Note that $\Pi^p_{F\to X/k}$ is canonically identified with the image of $\Pi^p_{F/k} \to \Pi^p_{X/k}$, and that since $k = \overline{k}$, it holds that $\Pi^p_{F/k} = \Pi^p_F, \Pi^p_{X/k} = \Pi^p_X$.) Then there exists a sequence of normal closed subgroups of $\Pi^p_{F\to X/k}$,*

$$1 = H_0 \subset H_1 \subset \cdots \subset H_{n-1} \subset H_n = \Pi^p_{F\to X/k},$$

such that, for each integer i such that $0 < i \leq n$, the closed subgroup H_i is topologically finitely generated and the quotient H_i/H_{i-1} is infinite.

Proof. (Cf. [10, Lem. 2.14].) Write $d := \dim(X)$. For each integer j such that $0 \leq j \leq d$, write $F[j] \to X_j$ for the normalization in F of the scheme-theoretic image of the composite $F \to X \to X_j$. Then we obtain a commutative diagram of normal varieties over k,

$$\begin{array}{ccccccccc} F & \longrightarrow & F[d] & \longrightarrow & \cdots & \longrightarrow & F[1] & \longrightarrow & \operatorname{Spec} k = F[0] \\ \downarrow & & \downarrow & & & & \downarrow & & \| \\ X & = & X_d & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & \operatorname{Spec} k = X_0, \end{array}$$

where the horizontal arrows are dominant and generically geometrically connected and the vertical arrows (except for the morphism $F \to X$) are finite (cf. Lemma 2.9), which implies that $\dim F[i] \leq i$, $0 \leq \dim(F[i+1]) - \dim(F[i]) \leq 1$. Now since $\dim(F) \geq n$, there exists a uniquely determined subset $\{D_0, \ldots, D_{n-1}\} \subset \{0, \ldots, d-1\}$ of cardinality n such that, for each integer i such that $0 \leq i < n$, the normal variety $F[D_i + 1]$ is of dimension $i + 1$, but the normal variety $F[D_i]$ is of dimension i. Write $F[D_n] := F$. Next, since k is of characteristic zero, and the horizontal arrows in the above commutative diagram are dominant and generically geometrically connected, one verifies easily that, for each integer i such that $0 \leq i \leq n$, there exists a nonempty open subscheme $U[D_i] \subset F[D_i]$ of $F[D_i]$ such that, for each integer i such that $1 \leq i \leq n$, the image of the open subscheme $U[D_i] \subset F[D_i]$ by the morphism $F[D_{i-1}] \to F[D_{i-1}]$ is contained in $U[D_{i-1}] \subset F[D_{i-1}]$ and, moreover, the resulting morphism $U[D_i] \to U[D_{i-1}]$ is surjective, smooth and geometrically connected. Thus, we obtain a commutative diagram of normal varieties over k,

$$\begin{array}{ccccccccc} U[D_n] & \longrightarrow & U[D_{n-1}] & \longrightarrow & \cdots & \longrightarrow & U[D_1] & \longrightarrow & \operatorname{Spec} k = U[D_0] \\ \downarrow & & \downarrow & & & & \downarrow & & \| \\ F[D_n] & \longrightarrow & F[D_{n-1}] & \longrightarrow & \cdots & \longrightarrow & F[D_1] & \longrightarrow & \operatorname{Spec} k = F[D_0], \end{array}$$

where the vertical arrows are open immersions and the upper horizontal arrows are surjective, smooth and geometrically connected.

Now, for each integer i such that $0 \leq i \leq n$, let us write

$$H_i := \operatorname{Im}(\Delta^p_{U[D_n] \to U[D_{n-i}]/k} \hookrightarrow \Pi^p_{U[D_n]} \twoheadrightarrow \Pi^p_F \twoheadrightarrow \Pi^p_{F \to X/k}).$$

Let us observe that since $\Delta^p_{U[D_n] \to U[D_{n-i}]/k}$ is a normal subgroup of $\Pi^p_{U[D_n]}$, H_i is a normal subgroup of $\Pi^p_{F \to X/k}$. We verify that the sequence of normal subgroups of $\Pi^p_{F \to X/k}$,

$$1 = H_0 \subset H_1 \subset \cdots \subset H_{n-1} \subset H_n = \Pi^p_{F \to X/k},$$

satisfies the condition in the statement of Lemma 3.38. Fix an integer i such that $0 < i \leq n$. First, by applying Lemma 2.11, where we take the data "(X, S, U)" to be $(U[D_n], U[D_{n-1}], U[D_n])$, it follows that $\Delta_{U[D_n]/U[D_{n-1}]} \subset \Pi_{U[D_n]}$ is topologically finitely generated. On the other hand, since $\Pi_{U[D_n]} \to \Pi_{U[D_{n-i}]}$ is surjective (cf. Lemma 2.6), it follows from Lemma 2.18 that $\Delta^p_{U[D_n]\to U[D_{n-i}]/k} \subset \Pi^p_{U[D_n]}$ is the image of $\Delta_{U[D_n]/U[D_{n-i}]} \subset \Pi_{U[D_n]}$ by $\Pi_{U[D_n]} \twoheadrightarrow \Pi^p_{U[D_n]}$. Thus, H_i is the image of $\Delta_{U[D_n]/U[D_{n-i}]} \subset \Pi_{U[D_n]}$ by the composite $\Pi_{U[D_n]} \twoheadrightarrow \Pi^p_{U[D_n]} \twoheadrightarrow \Pi^p_F \twoheadrightarrow \Pi^p_{F\to X/k}$. In particular, H_i is topologically finitely generated. Thus, it remains to verify that the quotient H_i/H_{i-1} is infinite. Write Ω for an algebraic closure of the function field of $U[D_{n-i}]$, $\overline{a} = \operatorname{Spec}\Omega \to U[D_{n-i}]$ for the generic geometric point of $U[D_{n-i}]$ determined by Ω, and $U_{D_{n-i+1}/D_{n-i}} := U[D_{n-i+1}] \times_{U[D_{n-i}]} \overline{a}$, which is a smooth variety over Ω of dimension 1 (resp. $\dim(F) - n + 1$) if $i \neq 1$ (resp. $i = 1$). Then, since the morphism $U[D_{n-i+1}] \to U[D_{n-i}]$ is surjective, smooth, geometrically connected (hence geometrically integral), it follows from our choice of the geometric point $\overline{a} \to U[D_{n-i}]$ that $(U[D_{n-i}], U[D_{n-i+1}], U[D_{n-i+1}], \overline{a} \to U[D_{n-i}])$ satisfies conditions (1), (2), (3), (4) for "$(S, X, Y, \overline{s} \to S)$" of Lemma 2.10. Thus, the sequence of profinite groups

$$\Pi_{U_{D_{n-i+1}/D_{n-i}}} \longrightarrow \Pi_{U[D_{n-i+1}]} \longrightarrow \Pi_{U[D_{n-i}]} \longrightarrow 1$$

is exact, which determines a surjection $\Pi_{U_{D_{n-i+1}/D_{n-i}}} \twoheadrightarrow \Delta_{U[D_{n-i+1}]/U[D_{n-i}]}$. On the other hand, the exact sequence of profinite groups

$$1 \to \Delta_{U[D_n]/U[D_{n-i+1}]} \to \Delta_{U[D_n]/U[D_{n-i}]} \to \Delta_{U[D_{n-i+1}]/U[D_{n-i}]} \to 1$$

determines an isomorphism

$$\Delta_{U[D_n]/U[D_{n-i}]}/\Delta_{U[D_n]/U[D_{n-i+1}]} \xrightarrow{\sim} \Delta_{U[D_{n-i+1}]/U[D_{n-i}]}.$$

Thus, we obtain a sequence of profinite groups

$$\Pi_{U_{D_{n-i+1}/D_{n-i}}} \twoheadrightarrow \Delta_{U[D_n]/U[D_{n-i}]}/\Delta_{U[D_n]/U[D_{n-i+1}]} \twoheadrightarrow H_i/H_{i-1}.$$

On the other hand, since H_i (resp. H_{i-1}) is the image of $\Delta^p_{U[D_n]\to U[D_{n-i}]/k}$ (resp. $\Delta^p_{U[D_n]\to U[D_{n-i+1}]/k}$) $\subset \Pi^p_{U[D_n]}$ by the composite $\Pi^p_{U[D_n]} \twoheadrightarrow \Pi^p_F \twoheadrightarrow \Pi^p_{F\to X/k} = \Pi^p_F/\Delta^p_{F\to X/k}$ and, moreover, the subgroups $\Delta^p_{U[D_n]\to X/k}$ and $\Delta^p_{U[D_n]\to U[D_{n-i+1}]/k}$ of $\Pi^p_{U[D_n]}$ are contained in the kernel of the composite $\Pi^p_{U[D_n]} \twoheadrightarrow \Pi^p_F \to \Pi^p_X \twoheadrightarrow \Pi^p_{X_{D_{n-i+1}}}$ (where we write $X_{D_n} := X$), it follows that there is a natural homomorphism $H_i/H_{i-1} \to \Pi^p_{X_{D_{n-i+1}}}$. Thus, the composite of natural morphisms

$$U_{D_{n-i+1}/D_{n-i}} = U[D_{n-i+1}] \times_{U[D_{n-i}]} \overline{a} \overset{\mathrm{pr}_1}{\to} U[D_{n-i+1}] \to X_{D_{n-i+1}}$$

determines a sequence of profinite groups

$$\Pi^p_{U_{D_{n-i+1}/D_{n-i}}} \to H_i/H_{i-1} \to \Pi^p_{X_{D_{n-i+1}}}.$$

On the other hand, since the natural morphism $F[D_{n-i+1}] \to X_{D_{n-i+1}}$, hence also $U[D_{n-i+1}] \hookrightarrow F[D_{n-i+1}] \to X_{D_{n-i+1}}$, is quasi-finite, it follows that

$$\begin{aligned} U_{D_{n-i+1}/D_{n-i}} &= U[D_{n-i+1}] \times_{U[D_{n-i}]} \overline{a} \\ &= (U[D_{n-i+1}] \times_k \Omega) \times_{(U[D_{n-i}]\times_k \Omega)} \overline{a} \to X_{D_{n-i+1}} \times_k \Omega \end{aligned}$$

is quasi-finite, hence nonconstant. Moreover, since $X_{D_{n-i+1}} \times_k \Omega$ is a hyperbolic polycurve over Ω satisfying condition $(*)_p$, it follows from Proposition 3.28 that $X_{D_{n-i+1}} \times_k \Omega$ is of p-LFG-type. This implies that the image of the composite $\Pi^p_{U_{D_{n-i+1}/D_{n-i}}} \to \Pi^p_{X_{D_{n-i+1}} \times_k \Omega} \xrightarrow{\sim} \Pi^p_{X_{D_{n-i+1}}}$, hence also the image of $H_i/H_{i-1} \to \Pi^p_{X_{D_{n-i+1}}}$, is infinite. Thus, we conclude that H_i/H_{i-1} is infinite. This completes the proof of Lemma 3.38. □

§4. Pro-p Grothendieck conjecture for hyperbolic polycurves

In the present Section 4, we consider the pro-p version of the Grothendieck conjecture for hyperbolic polycurves. Let k be a field of characteristic zero, $\overline{k}$ an algebraic closure of k and $G_k := \mathrm{Gal}(\overline{k}/k)$.

Definition 4.1 (Cf. [11, Def. 15.4(i)]). Let p be a prime number. Then we shall say that k is *sub-p-adic* if k is isomorphic to a subfield of a finitely generated extension of $\mathbb{Q}_p$.

Proposition 4.2. *Let p be a prime number, X a hyperbolic polycurve over k satisfying condition $(*)_p$ and Y a geometrically integral variety over k. Then the following hold:*

(i) *Write $\mathrm{Hom}^{\mathrm{dom}}_k(Y,X) \subset \mathrm{Hom}_k(Y,X)$ for the subset of dominant morphisms from Y to X over k and $\mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi^p_{Y/k},\Pi^p_{X/k}) \subset \mathrm{Hom}_{G_k}(\Pi^p_{Y/k},\Pi^p_{X/k})$ for the subset of open homomorphisms from $\Pi^p_{Y/k}$ to $\Pi^p_{X/k}$ over G_k. Then the natural map*

$$\mathrm{Hom}^{\mathrm{dom}}_k(Y,X) \to \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi^p_{Y/k},\Pi^p_{X/k})/\,\mathrm{Inn}(\Delta^p_{X/k})$$

(cf. Lemma 2.3) is injective.

(ii) *Suppose that k is sub-p-adic. Then the natural map*

$$\mathrm{Hom}_k(Y,X) \to \mathrm{Hom}_{G_k}(\Pi^p_{Y/k},\Pi^p_{X/k})/\,\mathrm{Inn}(\Delta^p_{X/k})$$

is injective.

Proof. (Cf. [10, Prop. 3.2].) Write $n := \dim(X)$. First, we verify assertion (i). I claim that the following assertion holds:

Claim A: If $n = 1$, then assertion (i) holds.

Indeed, since k is of characteristic zero, it follows that Y contains a dense open subscheme which is smooth over k. Thus, by replacing Y by such an open subscheme, we may assume without loss of generality that Y is smooth over k. Then, if k is sub-p-adic, Claim A follows from [11, Thm. A]. Now we verify Claim A for an arbitrary k. Let $f, g \in \mathrm{Hom}_k^{\mathrm{dom}}(Y, X)$ be elements of $\mathrm{Hom}_k^{\mathrm{dom}}(Y, X)$ that map to the same element by the above map. Then there exist a subfield k' of k which is finitely generated over $\mathbb{Q}$, a hyperbolic curve X' over k', a smooth variety Y' over k' and $f', g' \in \mathrm{Hom}_{k'}^{\mathrm{dom}}(Y', X')$ such that the base-change of f', g' to k is f, g, respectively. Then, since k' is finitely generated over $\mathbb{Q}$ (hence sub-p-adic), the map $\mathrm{Hom}_{k'}^{\mathrm{dom}}(Y', X') \to \mathrm{Hom}_{G_{k'}}^{\mathrm{open}}(\Pi^p_{Y'/k'}, \Pi^p_{X'/k'})/\operatorname{Inn}(\Delta^p_{X'/k'})$ is injective. Moreover, since $\Delta^p_{X'/k'}$ is slim (cf. Proposition 3.16(iii)), it follows from Lemma 2.20 that

$$\mathrm{Hom}_{G_{k'}}^{\mathrm{open}}(\Pi^p_{Y'/k'}, \Pi^p_{X'/k'})/\operatorname{Inn}(\Delta^p_{X'/k'}) \to \mathrm{Hom}(\Delta^p_{Y'/k'}, \Delta^p_{X'/k'})/\operatorname{Inn}(\Delta^p_{X'/k'})$$

is injective. Then, since $f', g' \in \mathrm{Hom}_{k'}^{\mathrm{dom}}(Y', X')$ map to the same element in

$$\mathrm{Hom}(\Delta^p_{Y'/k'}, \Delta^p_{X'/k'})/\operatorname{Inn}(\Delta^p_{X'/k'}) = \mathrm{Hom}(\Delta^p_{Y/k}, \Delta^p_{X/k})/\operatorname{Inn}(\Delta^p_{X/k}),$$

it follows that $f' = g'$, which implies that $f = g$. This completes the proof of Claim A.

Next, we verify assertion (i) by induction on n. If $n = 1$, then assertion (i) is the same as Claim A. Now suppose that $n \geq 2$ and that the induction hypothesis is in force. Let $f, g \in \mathrm{Hom}_k^{\mathrm{dom}}(Y, X)$ be elements of $\mathrm{Hom}_k^{\mathrm{dom}}(Y, X)$ that map to the same element by the above map. Write f_{n-1}, g_{n-1} for the composites of $X \to X_{n-1}$ and f, g, respectively. Then f_{n-1}, g_{n-1} induce the same $\Delta^p_{X_{n-1}/k}$-conjugacy class of homomorphisms $\Pi^p_{Y/k} \to \Pi^p_{X_{n-1}/k}$. Thus, it follows from the induction hypothesis that $f_{n-1} = g_{n-1}$. Let $\overline{\eta} \to X_{n-1}$ be a generic geometric point of X_{n-1}. Let $C \subset Y \times_{X_{n-1}} \overline{\eta}$ (where we take $Y \to X_{n-1}$ to be $f_{n-1} = g_{n-1}$) be an irreducible component of $Y \times_{X_{n-1}} \overline{\eta}$ with the reduced induced structure. Write $f', g' : Y \times_{X_{n-1}} \overline{\eta} \to X \times_{X_{n-1}} \overline{\eta}$ for the base-change of f, g, respectively. Now let us fix a basepoint of C and consider the diagram of profinite groups

$$\begin{array}{ccccc}
\Pi^p_{C/\overline{\eta}} & \rightrightarrows & \Pi^p_{X\times_{X_{n-1}}\overline{\eta}/\overline{\eta}} & \longrightarrow & \Pi_{\overline{\eta}} = \{1\} \\
\downarrow & & \downarrow & & \downarrow \\
\Pi^p_{Y/k} & \rightrightarrows & \Pi^p_{X/k} & \longrightarrow & \Pi^p_{X_{n-1}/k}
\end{array}$$

induced by the diagram of schemes

$$\begin{array}{ccccccc} C & \hookrightarrow & Y \times_{X_{n-1}} \overline{\eta} & \overset{f'}{\underset{g'}{\rightrightarrows}} & X \times_{X_{n-1}} \overline{\eta} & \xrightarrow{\mathrm{pr}_1} & \overline{\eta} \\ & \searrow & \downarrow & & \downarrow & & \downarrow \\ & & Y & \overset{f}{\underset{g}{\rightrightarrows}} & X & \longrightarrow & X_{n-1}. \end{array}$$

Then, since $X \times_{X_{n-1}} \overline{\eta}$ is a hyperbolic curve over $\overline{\eta}$, it follows from Proposition 3.16(ii) that

$$\Pi^p_{X \times_{X_{n-1}} \overline{\eta}/\overline{\eta}} \xrightarrow{\sim} \Pi^p_{X/X_{n-1}} \times_{\Pi_{X_{n-1}}} \Pi_{\overline{\eta}} = \Delta^p_{X/X_{n-1}} \subset \Delta^p_{X/k} \subset \Pi^p_{X/k}.$$

Thus, $\phi_{f'}, \phi_{g'} : \Pi^p_{C/\overline{\eta}} \to \Pi^p_{X \times_{X_{n-1}} \overline{\eta}/\overline{\eta}}$ induced by the dominant morphisms f', g', respectively, are determined by $\phi_f, \phi_g : \Pi^p_{Y/k} \to \Pi^p_{X/k}$ induced by f, g, respectively. On the other hand, since ϕ_f is a $\Delta^p_{X/k}$-conjugate of ϕ_g, we can choose an element $a \in \Delta^p_{X/k}$ such that $\phi_g = a\phi_f a^{-1}$. Write $\psi : \Pi^p_{X/k} \to \Pi^p_{X_{n-1}/k}$ for the right-hand lower horizontal arrow of the above diagram of profinite groups. Then the composites $\psi \circ \phi_f$ and $\psi \circ \phi_g = \psi(a) \cdot (\psi \circ \phi_f) \cdot \psi(a)^{-1}$ are induced by $f_{n-1} = g_{n-1}$, hence $\psi \circ \phi_f = \psi \circ \phi_g$. Thus, since $\psi(a) \in \Delta^p_{X_{n-1}/k}$, it follows that $\psi(a) \in Z_{\Delta^p_{X_{n-1}/k}}(\mathrm{Im}(\psi \circ \phi_f) \cap \Delta^p_{X_{n-1}/k})$. On the other hand, since ϕ_f is open and ψ is surjective (cf. Proposition 3.7(i)), $\mathrm{Im}(\psi \circ \phi_f) \cap \Delta^p_{X_{n-1}/k} \subset \Delta^p_{X_{n-1}/k}$ is an open subgroup of $\Delta^p_{X_{n-1}/k}$, which implies that $Z_{\Delta^p_{X_{n-1}/k}}(\mathrm{Im}(\psi \circ \phi_f) \cap \Delta^p_{X_{n-1}/k}) = \{1\}$ (cf. Proposition 3.16(iii)). Thus, it follows that $a \in \ker \psi = \Delta^p_{X/X_{n-1}}$, i.e., that ϕ_f is a $\Delta^p_{X/X_{n-1}}$-conjugate of ϕ_g, which implies that $\phi_{f'}$ is a $\Delta^p_{X/X_{n-1}}$-conjugate of $\phi_{g'}$. In particular, by applying Claim A, where we take the data "$(\mathrm{Spec}\, k, X, Y)$" to be $(\overline{\eta}, X \times_{X_{n-1}} \overline{\eta}, C)$, we obtain that $f' = g'$. Since the morphism $C \to Y$ is schematically dense, we conclude that $f = g$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Write $\eta \to Y$ for the generic point of Y. Note that the hyperbolic polycurve $X \times_k \eta/\eta$ satisfies condition $(*)_p$ (cf. Proposition 3.16(ii)). Fix a homomorphism $\Pi_\eta \to \Pi^p_{Y/k}$ arising from the natural morphism $\eta \to Y$. Then we have a natural $\Pi^p_{X/k}$-conjugacy class of isomorphisms $\Delta^p_{X \times_k \eta/\eta} \xrightarrow{\sim} \Delta^p_{X/k}$ (cf. Proposition 3.16(ii)), a natural outer isomorphism $\Pi^p_{X \times_k \eta/\eta} \xrightarrow{\sim} \Pi^p_{X/k} \times_{G_k} \Pi_\eta$ (cf. Proposition 3.16(ii)) and a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_k(Y, X) & \longrightarrow & \mathrm{Hom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k}) / \mathrm{Inn}(\Delta^p_{X/k}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_\eta(\eta, X \times_k \eta) & \longrightarrow & \mathrm{Hom}_{\Pi_\eta}(\Pi_\eta, \Pi^p_{X \times_k \eta/\eta}) / \mathrm{Inn}(\Delta^p_{X \times_k \eta/\eta}). \end{array}$$

Now, since $\eta \to Y$ is schematically dense, the left-hand vertical arrow of the above diagram is injective. Thus, since the function field of Y is finitely generated over the sub-p-adic field k (hence the function field of Y itself is sub-p-adic), by replacing k by the function field of Y and Y by $\mathrm{Spec}\, k$, to verify assertion (ii), we may assume without loss of generality that $Y = \mathrm{Spec}\, k$. Now we verify assertion (ii) by induction on n. If $n = 1$, then assertion (ii) follows from [11, Thm. C]. Now suppose that $n \geq 2$ and that the induction hypothesis is in force. Let $f, g \in \mathrm{Hom}_k(\mathrm{Spec}\, k, X)$ be elements of $\mathrm{Hom}_k(\mathrm{Spec}\, k, X)$ that determine the same element of $\mathrm{Hom}_{G_k}(G_k, \Pi^p_{X/k})/\mathrm{Inn}(\Delta^p_{X/k})$. Then, it follows from the induction hypothesis that the composite of $X \to X_{n-1}$ and f coincides with that of g. Write $x \in X_{n-1}$ for the image of the morphism $\mathrm{Spec}\, k \to X_{n-1}$ determined by f (i.e., by g), and $\phi_f, \phi_g : G_k \to \Pi^p_{X/k}$ for the homomorphism induced by f, g, respectively. Choose an element $a \in \Delta^p_{X/k}$ such that $\phi_g = a\phi_f a^{-1}$. Then it follows immediately that $k(x) = k$. Moreover, since $X_x := X \times_{X_{n-1}} x$ is a hyperbolic curve over $\mathrm{Spec}\, k(x)$, it follows from Proposition 3.16(ii) that $\Delta^p_{X_x/k(x)} \xrightarrow{\sim} \Delta^p_{X/X_{n-1}} \subset \Delta^p_{X/k}$, which implies that $\Pi^p_{X_x/k(x)} \to \Pi^p_{X/k}$ is injective. Thus, if we write $\psi : \Pi^p_{X/k} \to \Pi^p_{X_{n-1}/k}$ for the outer homomorphism induced by $X \to X_{n-1}$, then it follows from an argument similar to the argument used in the proof of assertion (i) that it suffices to show that $\psi(a) = 1$. Now, the section $\psi \circ \phi_f = \psi \circ \phi_g$ induced by $\mathrm{Spec}\, k \to X_{n-1}$, together with the action of $\Pi^p_{X_{n-1}/k}$ on $\Delta^p_{X_{n-1}/k}$ by conjugation, determines an action of G_k on $\Delta^p_{X_{n-1}/k}$. Then, it follows from the easily verified fact that $\psi(a) \in (\Delta^p_{X_{n-1}/k})^{G_k}$ that to verify assertion (ii), it suffices to verify that the following assertion holds:

> Claim B: Suppose that k is sub-p-adic. Let X be a hyperbolic polycurve over k satisfying condition $(*)_p$ and $\mathrm{Spec}\, k \to X$ a k-rational point. Then, on the group action of G_k on $\Delta^p_{X/k}$ determined by the section of $\Pi^p_{X/k} \twoheadrightarrow G_k$ induced by $\mathrm{Spec}\, k \to X$, we have $(\Delta^p_{X/k})^{G_k} = \{1\}$.

Indeed, let us observe that it follows from induction on the dimension of X that, to verify Claim B, we may assume without loss of generality that X is a hyperbolic curve over k. Now assume that $(\Delta^p_{X/k})^{G_k} \neq \{1\}$. Let us choose an element $a \in (\Delta^p_{X/k})^{G_k} \setminus \{1\}$. Then there exists a characteristic open subgroup $V \subset \Delta^p_{X/k}$ of $\Delta^p_{X/k}$ such that $a \notin V$ (cf. Proposition 3.16(iii), [15, Prop. 2.5.1(b)]). Write $U := V \cdot \langle a \rangle$. Then U is an open subgroup of $\Delta^p_{X/k}$. Moreover, since $V \subset \Delta^p_{X/k}$ is normal, it follows that $[U, U] \subset V$. In particular, $a \in U \setminus [U, U]$, which implies that $(U^{\mathrm{ab}})^{G_k} \neq \{1\}$. Write W for the (necessarily open) subgroup of $\Pi^p_{X/k}$ generated by U and the image of the section $G_k \twoheadrightarrow \Pi^p_{X/k}$ induced by the given k-rational point $\mathrm{Spec}\, k \to X$. Then $W \subset \Pi^p_{X/k}$ corresponds to a hyperbolic curve X' over k. Thus, to verify Claim B, by replacing X/k by X'/k, it suffices to verify that

$(\Delta^{p,\mathrm{ab}}_{X/k})^{G_k}$ is trivial. Moreover, by replacing k by its finite extension if necessary, we may assume that $S(\overline{k}) = S(k)$, where we write $S := X^{\mathrm{cpt}} \setminus X$. Write (g, r) for the type of the hyperbolic curve X/k, J for the Jacobian variety of X^{cpt} and T_pJ for the p-adic Tate module of J. Then, if $r = 0$, we have a canonical isomorphism $\Delta^{p,\mathrm{ab}}_{X/k} \cong T_pJ$ (cf. [16, (1–3)]). If $r > 0$, then we have the exact sequence

$$0 \to \mathbb{Z}_p(1) \to \bigoplus_{x \in S(k)} \mathbb{Z}_p(1) \to \Delta^{p,\mathrm{ab}}_{X/k} \to T_pJ \to 0$$

(cf. [16, (1–5)]). Thus, to verify Claim B, it suffices to verify that $(\mathbb{Z}_p(1))^{G_k}$ and $(T_pJ)^{G_k}$ are trivial. First, we verify that $(\mathbb{Z}_p(1))^{G_k}$ is trivial. Since k is sub-p-adic, there exists an injection $k \hookrightarrow K$, where K is a finitely generated field extension of $\mathbb{Q}_p$. Then, the action of G_k on $\mathbb{Z}_p(1)$ determines a character $\chi : G_k \to \mathbb{Z}_p^\times$. Now let us consider the commutative diagram of profinite groups

$$\begin{array}{ccc} G_K & \longrightarrow & G_{\mathbb{Q}_p} \\ \downarrow & & \downarrow \\ G_k & \xrightarrow[\chi]{} & \mathbb{Z}_p^\times. \end{array}$$

Then, since $G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ is surjective and $G_K \to G_{\mathbb{Q}_p}$ is open, the image of $G_K \to \mathbb{Z}_p^\times$, hence also that of χ, is nontrivial. Thus, we conclude that $(\mathbb{Z}_p(1))^{G_k}$ is trivial. Next, we verify that $(T_pJ)^{G_k}$ is trivial. It follows from the sequence $G_K \to G_k \to \mathrm{Aut}(T_pJ)$ that, to verify that $(T_pJ)^{G_k}$ is trivial, we may assume without loss of generality that k is finitely generated over $\mathbb{Q}_p$. Then there exist a normal domain R with quotient field k which is finitely generated over $\mathbb{Q}_p$ and an abelian scheme $\mathcal{A}$ over R such that $J \to \operatorname{Spec} k$ is the base-change of $\mathcal{A} \to \operatorname{Spec} R$ by the morphism $\operatorname{Spec} k \to \operatorname{Spec} R$. Let x be a closed point of $\operatorname{Spec} R$. Then, by considering the action $G_{k(x)} \to \mathrm{Aut}(T_p\mathcal{A}_x) \xleftarrow{\sim} \mathrm{Aut}(T_pJ)$, we conclude that to verify that $(T_pJ)^{G_k}$ is trivial, it suffices to verify that, for each finite extension k of $\mathbb{Q}_p$ and abelian variety A over k, the module $(T_pA)^{G_k}$ of G_k-invariants is trivial. Now let us observe that

$$(T_pA)^{G_k} = \varprojlim_n A[p^n](\overline{k})^{G_k} = \varprojlim_n A[p^n](k) = \varprojlim_n A(k)[p^n].$$

On the other hand, since $A(k)$ is a compact abelian p-adic Lie group, it follows that $A(k)$ is isomorphic, as a topological group, to the direct sum of $\mathbb{Z}_p^m$ for a suitable nonnegative integer m and a finite abelian group. Thus, we conclude that $\varprojlim_n A(k)[p^n]$ is trivial. This completes the proof of Claim B, hence also of Proposition 4.2. $\square$

Corollary 4.3. *Let p be a prime number, X a hyperbolic polycurve over k satisfying condition $(**)_p$ and Y a geometrically integral variety over k. Then the following hold:*

(i) *Write $\mathrm{Hom}_k^{\mathrm{dom}}(Y, X) \subset \mathrm{Hom}_k(Y, X)$ for the subset of dominant morphisms from Y to X over k and $\mathrm{Hom}_{G_k^p}^{\mathrm{open}}(\Pi_Y^p, \Pi_X^p) \subset \mathrm{Hom}_{G_k^p}(\Pi_Y^p, \Pi_X^p)$ for the subset of open homomorphisms from Π_Y^p to Π_X^p over G_k^p. Then the natural map*

$$\mathrm{Hom}_k^{\mathrm{dom}}(Y, X) \to \mathrm{Hom}_{G_k^p}^{\mathrm{open}}(\Pi_Y^p, \Pi_X^p)/\,\mathrm{Inn}(\Delta_{X/k}^p)$$

(cf. Lemma 2.3) is injective.

(ii) *Suppose that k is sub-p-adic. Then the natural map*

$$\mathrm{Hom}_k(Y, X) \to \mathrm{Hom}_{G_k^p}(\Pi_Y^p, \Pi_X^p)/\,\mathrm{Inn}(\Delta_{X/k}^p)$$

is injective.

Proof. This follows from Proposition 4.2, together with Lemma 3.23(ii). □

Theorem 4.4. *Let p be a prime number, k a sub-p-adic field, X a hyperbolic curve over k (resp. a hyperbolic curve over k satisfying condition $(**)_p$), Y a normal variety over k and $\phi : \Pi_{Y/k}^p \to \Pi_{X/k}^p$ (resp. $\phi : \Pi_Y^p \to \Pi_X^p$) an open homomorphism over G_k (resp. G_k^p). Then ϕ arises from a uniquely determined dominant morphism $Y \to X$ over k.*

Proof. (Cf. [10, Thm. 3.3].) First, let us observe that, if X/k satisfies condition $(**)_p$, then it follows from Lemma 3.23(ii) that the homomorphism $\Pi_Y^p \to \Pi_X^p$ canonically determines $\Pi_{Y/k}^p \to \Pi_{X/k}^p$. Thus, in light of Proposition 4.2(i) and Corollary 4.3(i), to verify Theorem 4.4, it suffices to verify that an open homomorphism $\phi : \Pi_{Y/k}^p \to \Pi_{X/k}^p$ over G_k arises from a dominant morphism $Y \to X$ over k. Now, let us observe that there exists a dense open subscheme U of Y which is smooth over k. Then, it follows from [11, Thm. A] that the composite $\Pi_{U/k}^p \to \Pi_{Y/k}^p \xrightarrow{\phi} \Pi_{X/k}^p$ arises from a uniquely determined morphism $U \to X$ over k. Write $\eta \to U$ for the generic point of U. Then, since $\Pi_{\eta/k}^p \to \Pi_{U/k}^p \to \Pi_{Y/k}^p \xrightarrow{\phi} \Pi_{X/k}^p$ is induced by $\eta \to U \to X$, it follows from Lemma 3.34 that ϕ arises from a morphism $Y \to X$ over k. Moreover, since ϕ is open, it follows that the morphism $Y \to X$ is dominant. This completes the proof of Theorem 4.4. □

Lemma 4.5. *Let p be a prime number, n a positive integer, S, Y normal varieties over k, X a hyperbolic polycurve of relative dimension n over S and $\phi : \Pi_Y^p \to \Pi_X^p$ an open homomorphism over G_k^p. Suppose that the composite $\Pi_Y^p \xrightarrow{\phi} \Pi_X^p \twoheadrightarrow \Pi_S^p$ arises from a morphism $Y \to S$ over k. Write $S' \subset S$ for the scheme-theoretic* *image of the morphism* $Y \to S$*,* $Z := \mathrm{Nor}(Y/S')$ *and* $\eta \to Z$ *for the generic point of* Z*. Then the following hold:*

(i) *The morphism* $Y \to Z$ *over* k *is dominant and generically geometrically connected. In particular,* $Y_\eta := Y \times_Z \eta$ *is a (nonempty) normal variety over* η*.*

(ii) *There exist nonempty open subschemes* $U_Y \subset Y$*,* $U_Z \subset Z$ *of* Y*,* Z*, respectively, such that the image of* $U_Y \subset Y$ *by the natural morphism* $Y \to Z$ *is contained in* $U_Z \subset Z$ *and, moreover, the resulting morphism* $U_Y \to U_Z$ *is surjective, smooth and geometrically connected.*

(iii) *Write* $N \subset \Pi^p_Y$ *for the normal closed subgroup of* Π^p_Y *obtained by forming the image of the normal closed subgroup* $\Delta^{(p)}_{U_Y/U_Z} \subset \Pi^p_{U_Y}$ *of* $\Pi^p_{U_Y}$ *by* $\Pi^p_{U_Y} \to \Pi^p_Y$*. Then the image of the composite* $\Delta^{(p)}_{Y_\eta/\eta} \hookrightarrow \Pi^p_{Y_\eta} \to \Pi^p_Y$*, hence also the composite* $\Pi_{Y_{\overline{\eta}}} \xrightarrow{\sim} \Delta_{Y_\eta/\eta} \twoheadrightarrow \Delta^{(p)}_{Y_\eta/\eta} \hookrightarrow \Pi^p_{Y_\eta} \to \Pi^p_Y$*, coincides with* $N \subset \Pi^p_Y$*.*

(iv) *The image of* $N \subset \Pi^p_Y$ *by the composite* $\Pi^p_Y \to \Pi^p_X \to \Pi^p_S$ *is trivial. In particular, we obtain a natural* Π^p_X*-conjugacy class of homomorphisms* $N \to \Delta^{(p)}_{X/S}$*.*

(v) *If, moreover,* $\dim(Y) > \dim(S)$*,* Y *is of* p*-LFG-type and* $\Pi^p_{Y \times_k \overline{k}} \to \Pi^p_Y$ *is injective, then* N *is infinite.*

(vi) *If moreover,* $\dim(Y) > \dim(S)$ *and* Y *is a hyperbolic polycurve over* k *satisfying condition* $(**)_p$*, then there exists a sequence of normal closed subgroups of* N*,*

$$\{1\} = H_0 \subset H_1 \subset \cdots \subset H_{\dim(Y)-\dim(S)-1} \subset H_{\dim(Y)-\dim(S)} = N,$$

such that, for each integer i *such that* $1 \leq i \leq \dim(Y) - \dim(S)$*, the closed subgroup* H_i *is topologically finitely generated and the quotient* H_i/H_{i-1} *is infinite.*

(vii) *If, moreover,* $n = 1$*,* k *is sub-*p*-adic,* X/S *satisfies condition* $(**)_p$ *and the image of* $N \to \Delta^{(p)}_{X/S}$ *of* (iv) *is nontrivial, then* ϕ *arises from a morphism* $Y \to X$ *over* S*.*

Proof. (Cf. [10, Lems. 3.4, 3.5].) Assertion (i) follows from Lemma 2.9. Assertion (ii) follows from assertion (i) and the fact that k is of characteristic zero. Next, we verify assertion (iii). Let $\overline{\eta} \to U_Z$ be a generic geometric point of U_Z. Write $Y_{\overline{\eta}} := Y \times_Z \overline{\eta}$ and $(U_Y)_{\overline{\eta}} := U_Y \times_{U_Z} \overline{\eta}$. Then it follows from Lemma 2.10, together with the right exactness of the operation of taking the maximal pro-p quotient, that we obtain a surjection $\Pi_{(U_Y)_{\overline{\eta}}} \to \Delta^{(p)}_{U_Y/U_Z}$. Thus, N is the image of the composite $\Pi_{(U_Y)_{\overline{\eta}}} \twoheadrightarrow \Delta^{(p)}_{U_Y/U_Z} \hookrightarrow \Pi^p_{U_Y} \to \Pi^p_Y$, which coincides with the composite $\Pi_{(U_Y)_{\overline{\eta}}} \to \Pi_{Y_{\overline{\eta}}} \xrightarrow{\sim} \Delta_{Y_\eta/\eta} \to \Delta^{(p)}_{Y_\eta/\eta} \hookrightarrow \Pi^p_{Y_\eta} \to \Pi^p_Y$. On the other hand, it follows from Lemma 2.2 that the homomorphism $\Pi_{(U_Y)_{\overline{\eta}}} \to \Pi_{Y_{\overline{\eta}}}$ is surjective. Moreover, it follows from the surjectivity of $\Pi_{Y_\eta} \twoheadrightarrow \Pi_\eta$, together with the right exactness of the operation of taking the maximal pro-p quotient, that $\Delta_{Y_\eta/\eta} \to \Delta^{(p)}_{Y_\eta/\eta}$ is surjective. Thus, N is the image of the composite $\Delta^{(p)}_{Y_\eta/\eta} \hookrightarrow \Pi^p_{Y_\eta} \to \Pi^p_Y$. This completes the proof of assertion (iii). Assertion (iv) follows from assertion (iii), together with the fact that the composite $Y_\eta \hookrightarrow Y \to S$ factors through $\eta \to S$. Next, we verify assertion (v). It follows from our choice of (U_Y, U_Z) that the geometric fiber F of $U_Y \to U_Z$ at a $\overline{k}$-valued geometric point of U_Z is a smooth variety over $\overline{k}$ of dimension $\geq \dim(Y) - \dim(S) > 0$. In particular, the natural morphism $F \to Y \times_k \overline{k}$ over $\overline{k}$ is nonconstant. Thus, since Y is of p-LFG-type, the image of $\Pi^p_F \to \Pi^p_{Y\times_k \overline{k}}$, hence also that of $\Pi^p_F \to \Pi^p_Y$, is infinite. On the other hand, it follows from our choice of F that $\Pi^p_F \to \Pi^p_Y$ factors through the composite $\Delta^{(p)}_{U_Y/U_Z} \hookrightarrow \Pi^p_{U_Y} \to \Pi^p_Y$. Thus, we conclude that N is infinite. This completes the proof of assertion (v). Next, we verify assertion (vi). The morphism $Y_{\overline{\eta}} = Y \times_Z \overline{\eta} \stackrel{\mathrm{pr}_1}{\to} Y$ factors through a natural closed immersion $Y_{\overline{\eta}} \hookrightarrow Y \times_k \overline{\eta}$. Then, since $Y_{\overline{\eta}}$ is a normal variety over $\overline{\eta}$ of dimension $\geq \dim(Y) - \dim(S)$ and, moreover, $Y \times_k \overline{\eta}$ is a hyperbolic polycurve over $\overline{\eta}$ satisfying condition $(*)_p$ (cf. Proposition 3.16(ii)), it follows from Lemma 3.38 that the image of $\Pi^p_{Y_{\overline{\eta}}} \to \Pi^p_{Y\times_k \overline{\eta}}$ admits a sequence of closed subgroups as in the statement of assertion (vi). On the other hand, any homomorphism $\Pi^p_{Y\times_k\overline{\eta}} \to \Pi^p_Y$ induced by $Y \times_k \overline{\eta} \to Y$ determines an isomorphism $\Pi^p_{Y\times_k\overline{\eta}} \stackrel{\sim}{\to} \Delta^p_{Y/k}$ (cf. Lemma 2.5, Proposition 3.16(ii)). Thus, the image of $\Pi^p_{Y_{\overline{\eta}}} \to \Pi^p_{Y\times_k\overline{\eta}}$ is isomorphic to that of $\Pi^p_{Y_{\overline{\eta}}} \to \Pi^p_Y$, which coincides with N (cf. assertion (iii)). This completes the proof of assertion (vi). Finally, we verify assertion (vii). Note that since X/S satisfies condition $(**)_p$, we have $\Delta^{(p)}_{X/S} = \Delta^p_{X/S}$. It follows from assertion (iii) that the image of $\Delta^{(p)}_{Y_\eta/\eta} \subset \Pi^p_{Y_\eta}$ by the composite $\Pi^p_{Y_\eta} \to \Pi^p_Y \stackrel{\phi}{\to} \Pi^p_X$ coincides with the image of $N \to \Delta^p_{X/S}$, which is assumed to be nontrivial. On the other hand, it follows from Lemma 2.2 that $\Pi_{Y_\eta} \to \Pi_Y$, hence also $\Pi^p_{Y_\eta} \to \Pi^p_Y$, is surjective. Thus, since $\Delta^{(p)}_{Y_\eta/\eta} \subset \Pi^p_{Y_\eta}$ is a normal subgroup of $\Pi^p_{Y_\eta}$, in light of the openness of ϕ, it follows that $\mathrm{Im}(\Pi^p_{Y_\eta} \to \Pi^p_X) \cap \Delta^p_{X/S}$ is an open subgroup of $\Delta^p_{X/S}$ and, moreover, $\mathrm{Im}(\Delta^{(p)}_{Y_\eta/\eta} \to \Delta^p_{X/S})$ is a normal subgroup of $\mathrm{Im}(\Pi^p_{Y_\eta} \to \Pi^p_X) \cap \Delta^p_{X/S}$. On the other hand, it follows from Lemmas 2.5 and 2.7 that $\Delta_{Y_\eta/\eta}$, hence also $\Delta^{(p)}_{Y_\eta/\eta}$, is topologically finitely generated. Thus, we conclude that $\mathrm{Im}(\Delta^{(p)}_{Y_\eta/\eta} \to \Delta^p_{X/S})$ is an open subgroup of $\Delta^p_{X/S}$ (cf. Proposition 3.16(iv)). Write $X_\eta := X \times_S \eta$. Let us fix an isomorphism $\Pi^p_{X_\eta} \stackrel{\sim}{\to} \Pi^p_X \times_{\Pi^p_S} \Pi^p_\eta$ (cf. Proposition 3.16(ii)) over Π^p_η arising from morphisms $X_\eta \stackrel{\mathrm{pr}_1}{\to} X$, $X_\eta \stackrel{\mathrm{pr}_2}{\to} \eta$ over S and a homomorphism $\Pi^p_{Y_\eta} \to \Pi^p_Y \times_{\Pi^p_Z} \Pi^p_\eta$ over Π^p_η arising from morphisms $Y_\eta \stackrel{\mathrm{pr}_1}{\to} Y$, $Y_\eta \stackrel{\mathrm{pr}_2}{\to} \eta$ over Z.

Then ϕ determines a homomorphism

$$\phi_\eta : \Pi^p_{Y_\eta} \to \Pi^p_Y \times_{\Pi^p_Z} \Pi^p_\eta \to \Pi^p_X \times_{\Pi^p_S} \Pi^p_\eta \overset{\sim}{\leftarrow} \Pi^p_{X_\eta}$$

over Π^p_η. On the other hand, we have $\Delta^{(p)}_{X_\eta/\eta} = \Delta^p_{X_\eta/\eta} \overset{\sim}{\to} \Delta^p_{X/S}$ (cf. Proposition 3.16(ii)). Thus, it follows from the openness of $\Delta^{(p)}_{Y_\eta/\eta} \to \Delta^p_{X/S}$, together with the commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^{(p)}_{Y_\eta/\eta} & \longrightarrow & \Pi^p_{Y_\eta} & \longrightarrow & \Pi^p_\eta & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \phi_\eta} & & \| & & \\ 1 & \longrightarrow & \Delta^{(p)}_{X_\eta/\eta} & \longrightarrow & \Pi^p_{X_\eta} & \longrightarrow & \Pi^p_\eta & \longrightarrow & 1, \end{array}$$

that the image of ϕ_η is a closed subgroup of $\Pi^p_{X_\eta}$ of finite index, hence open. Thus, since X_η is a hyperbolic curve over η satisfying condition $(**)_p$ and, moreover, η is the spectrum of a sub-p-adic field, it follows from Theorem 4.4 that ϕ_η arises from a morphism $Y_\eta \to X_\eta$ over η. Write $\xi \to Y_\eta$ for the generic point of $Y_\eta = Y \times_Z \eta \subset Y$. Let us consider $\Pi^p_\xi \to \Pi^p_X$ induced by the morphism $\xi \to Y_\eta \to X_\eta \to X$ over S. Then we obtain a commutative diagram of profinite groups

$$\begin{array}{ccccc} \Pi^p_\xi & \longrightarrow & \Pi^p_Y \times_{\Pi^p_Z} \Pi^p_\eta & \longrightarrow & \Pi^p_X \times_{\Pi^p_S} \Pi^p_\eta \\ & & \downarrow & & \downarrow \\ & & \Pi^p_Y & \xrightarrow[\phi]{} & \Pi^p_X. \end{array}$$

Thus, $\Pi^p_\xi \to \Pi^p_X$ coincides with the composite of ϕ and $\Pi^p_\xi \to \Pi^p_Y$ arising from $\xi \to Y$, which implies that ϕ arises from a morphism $Y \to X$ over S (cf. Lemma 3.34). This completes the proof of assertion (vii). □

Definition 4.6. Let p be a prime number, X, Y normal varieties over k and $\phi : \Pi^p_Y \to \Pi^p_X$ a homomorphism over G^p_k.

(i) We shall say that ϕ is *nondegenerate* if ϕ is open and, moreover, for any open subscheme $U \subset Y$ of Y, any normal variety Z over k such that $\dim(Z) < \dim(X)$ and any smooth, geometrically connected, surjective morphism $U \to Z$ over k, the composite $\Pi^p_U \to \Pi^p_Y \overset{\phi}{\to} \Pi^p_X$ does not factor through $\Pi^p_U \to \Pi^p_Z$.

(ii) Suppose that X is a hyperbolic polycurve of relative dimension n over k. Then we shall say that the homomorphism ϕ is *poly-nondegenerate* if there exists a sequence of parametrizing morphisms

$$X = X_n \to X_{n-1} \to \cdots \to X_1 \to \operatorname{Spec} k = X_0$$

such that X/k satisfies condition $(**)_p$ with respect to this sequence and that for each integer i such that $0 \le i \le n$, the composite $\Pi_Y^p \to \Pi_X^p \twoheadrightarrow \Pi_{X_i}^p$ is nondegenerate.

Theorem 4.7. *Let p be a prime number, k a sub-p-adic field, X a hyperbolic polycurve over k satisfying condition $(**)_p$ and Y a normal variety over k. Write $\operatorname{Hom}_k^{\mathrm{dom}}(Y,X) \subset \operatorname{Hom}_k(Y,X)$ for the subset of dominant morphisms from Y to X over k and $\operatorname{Hom}_{G_k^p}^{\mathrm{PND}}(\Pi_Y^p,\Pi_X^p) \subset \operatorname{Hom}_{G_k^p}(\Pi_Y^p,\Pi_X^p)$ for the subset of poly-nondegenerate homomorphisms from Π_Y^p to Π_X^p over G_k^p. Then the natural map*

$$\operatorname{Hom}_k^{\mathrm{dom}}(Y,X) \to \operatorname{Hom}_{G_k^p}(\Pi_Y^p,\Pi_X^p)/\operatorname{Inn}(\Delta_{X/k}^p)$$

determines a bijection

$$\operatorname{Hom}_k^{\mathrm{dom}}(Y,X) \xrightarrow{1:1} \operatorname{Hom}_{G_k^p}^{\mathrm{PND}}(\Pi_Y^p,\Pi_X^p)/\operatorname{Inn}(\Delta_{X/k}^p).$$

Proof. (Cf. [10, Thm. 3.7].) First, I claim that the following assertion holds:

Claim A: Any homomorphism $\phi_f : \Pi_Y^p \to \Pi_X^p$ over G_k^p that arises from a dominant morphism $f : Y \to X$ over k is poly-nondegenerate.

Indeed, suppose that there exist an integer i, an open subscheme $U \subset Y$ of Y, a normal variety Z over k, a smooth, geometrically connected, surjective morphism $U \to Z$ over k and a sequence of parametrizing morphisms

$$X = X_n \to X_{n-1} \to \cdots \to X_1 \to \operatorname{Spec} k = X_0$$

such that $0 \le i \le n$, X/k satisfies condition $(**)_p$ with respect to this sequence and, moreover, the composite $\Pi_U^p \to \Pi_Y^p \xrightarrow{\phi_f} \Pi_X^p \twoheadrightarrow \Pi_{X_i}^p$ factors through $\Pi_U^p \to \Pi_Z^p$. Then, by applying Corollary 3.33, where we take the data "(k,k',S,Y,Z,X,f)" to be $(k,k,\operatorname{Spec} k, Z, U, X_i, U \hookrightarrow Y \xrightarrow{f} X \to X_i)$, we conclude that the composite $U \hookrightarrow Y \xrightarrow{f} X \to X_i$ factors through $U \to Z$. In particular, since f is dominant, it holds that $\dim(Z) \ge \dim(X_i)$. This completes the proof of Claim A.

It follows from Claim A that we have a natural map

$$\operatorname{Hom}_k^{\mathrm{dom}}(Y,X) \to \operatorname{Hom}_{G_k^p}^{\mathrm{PND}}(\Pi_Y^p,\Pi_X^p)/\operatorname{Inn}(\Delta_{X/k}^p).$$

Moreover, it follows from Corollary 4.3(i) that this natural map is injective. Thus, to verify Theorem 4.7, it suffices to verify the surjectivity of the above map. Let $\phi \in \operatorname{Hom}_{G_k^p}^{\mathrm{PND}}(\Pi_Y^p,\Pi_X^p)$ be a poly-nondegenerate homomorphism over G_k^p and

$$X = X_n \to X_{n-1} \to \cdots \to X_1 \to \operatorname{Spec} k = X_0,$$

a sequence of parametrizing morphisms as in Definition 4.6(ii). Now I claim that the following assertion holds:

Claim B: Suppose that there exists a morphism $f : Y \to X$ over k from which ϕ arises. Then f is dominant.

Indeed, assume that f is not dominant. Write $X' \subset X$ for the scheme-theoretic image of f and $S := \mathrm{Nor}(Y/X')$. Then since the natural morphism $Y \to S$ is dominant and generically geometrically irreducible (cf. Lemma 2.9) and k is of characteristic zero, there exist open subschemes $U_Y \subset Y$, $U_S \subset S$ of Y, S, respectively, such that the image of $U_Y \subset Y$ by the morphism $Y \to S$ is contained in $U_S \subset S$ and, moreover, the resulting morphism $U_Y \to U_S$ is surjective, smooth and geometrically connected. On the other hand, since f is not dominant, it follows that X', hence also U_S, is of dimension $< \dim(X)$. However, since $\Pi^p_{U_Y} \to \Pi^p_X$ factors through $\Pi^p_{U_Y} \to \Pi^p_{U_S}$ and ϕ is poly-nondegenerate, we obtain a contradiction. This completes the proof of Claim B.

It follows from the discussion preceding Claim B that, to verify Theorem 4.7, it suffices to verify that the following assertion holds:

Claim C: For each integer i such that $0 \le i < n$, if the composite $\Pi^p_Y \to \Pi^p_X \twoheadrightarrow \Pi^p_{X_i}$ arises from a dominant morphism $Y \to X_i$ over k, then the composite $\Pi^p_Y \to \Pi^p_X \twoheadrightarrow \Pi^p_{X_{i+1}}$ arises from a dominant morphism $Y \to X_{i+1}$ over k.

To verify Claim C, let us write $Z := \mathrm{Nor}(Y/X_i)$, $\eta \to Z$ for the generic point of Z and $Y_\eta := Y \times_Z \eta$. Now I claim that the following assertion holds:

Claim C.1: The image of any homomorphism that belongs to the $\Pi^p_{X_{i+1}}$-conjugacy class of homomorphisms $N \to \Delta^{(p)}_{X_{i+1}/X_i}$ of Lemma 4.5(iv), where we take the data "(S, Y, X)" to be (X_i, Y, X_{i+1}), is nontrivial.

Indeed, assume that the image of $N \to \Delta^{(p)}_{X_{i+1}/X_i}$ is trivial. Let $U_Y \subset Y$, $U_Z \subset Z$ be open subschemes of Y, Z, respectively, as in Lemma 4.5(ii). Then it follows from Lemma 4.5(iii) that the image of $\Delta^{(p)}_{U_Y/U_Z} \subset \Pi^p_{U_Y}$ by the composite $\Pi^p_{U_Y} \xrightarrow{\phi} \Pi^p_X \twoheadrightarrow \Pi^p_{X_{i+1}}$ is trivial. Thus, it follows that the composite $\Pi^p_{U_Y} \to \Pi^p_Y \xrightarrow{\phi} \Pi^p_X \twoheadrightarrow \Pi^p_{X_{i+1}}$ factors through $\Pi^p_{U_Y} \to \Pi^p_{U_Z}$. On the other hand, since $\dim(U_Z) = \dim(Z) = i < i + 1 = \dim(X_{i+1})$, and ϕ is poly-nondegenerate, we obtain a contradiction. This completes the proof of Claim C.1.

It follows from Claim C.1, together with Lemma 4.5(vii), that the composite $\Pi^p_Y \to \Pi^p_X \twoheadrightarrow \Pi^p_{X_{i+1}}$ arises from a morphism $Y \to X_{i+1}$ over k. Moreover, it follows from Claim B that this morphism is dominant. This completes the proof of Claim C, hence also of Theorem 4.7. $\square$

Remark 4.8. It follows from Theorem 4.7, together with the proof of Claim A in Theorem 4.7, that a poly-nondegenerate homomorphism satisfies the condition in Definition 4.6(ii) with respect to an arbitrary sequence of parametrizing morphisms of X/S with respect to which X/S satisfies condition $(**)_p$.

Theorem 4.9. *Let p be a prime number, k a sub-p-adic field, Y, S normal varieties over k, X a hyperbolic curve over S satisfying condition $(**)_p$ and $\phi : \Pi_Y^p \to \Pi_X^p$ a homomorphism over G_k^p. Suppose that the following conditions are satisfied:*

(1) *The composite $\Pi_Y^p \xrightarrow{\phi} \Pi_X^p \to \Pi_S^p$ arises from a morphism $Y \to S$ over k.*
(2) *ϕ is open and its kernel is finite.*
(3) *Y is of p-LFG-type and, moreover, $\Pi_{Y \times_k \overline{k}}^p \to \Pi_Y^p$ is injective.*
(4) *$\dim(X)$ $(= \dim(S) + 1) \leq \dim(Y)$.*

Then ϕ arises from a quasi-finite dominant morphism $Y \to X$ over S. In particular, $\dim(X) = \dim(Y)$.

Proof. (Cf. [10, Thm 3.8].) It follows from conditions (3), (4), together with Lemma 4.5(v), that the closed subgroup $N \subset \Pi_Y^p$ defined in Lemma 4.5(iii) is infinite. Thus, it follows from condition (2) that the image of $N \subset \Pi_Y^p$ by ϕ is nontrivial. This implies that ϕ arises from a morphism $Y \to X$ over S (cf. Lemma 4.5(vii)). Moreover, it follows from conditions (2), (3), together with Lemma 3.27(iii), that $Y \to X$ is quasi-finite, hence dominant (cf. condition (4)). This completes the proof of Theorem 4.9. □

Definition 4.10. Let p be a prime number, n a positive integer and $\mathcal{C}$ a condition on a connected noetherian separated normal scheme S over k, a hyperbolic polycurve X over S and a sequence of parametrizing morphisms

$$X = X_m \to X_{m-1} \to \cdots \to X_1 \to S = X_0$$

which satisfies the following conditions:

(1) If X/S satisfies condition $\mathcal{C}$ and, moreover, $m \geq 2$, then X/X_1 satisfies condition $\mathcal{C}$.
(2) If X/S satisfies condition $\mathcal{C}$, then, for any connected noetherian separated normal scheme T over k and any morphism $T \to S$, $X \times_S T/T$ satisfies condition $\mathcal{C}$.
(3) If X/S satisfies condition $\mathcal{C}$, then, for any open subgroup $U \subset \Pi_X^p$ of Π_X^p, the hyperbolic polycurve corresponding to U satisfies condition $\mathcal{C}$.

Then we shall say that the assertion $(\dagger_n)_p^{\mathcal{C}}$ holds if, for any hyperbolic polycurve X of relative dimension n over $\overline{k}$ satisfying conditions $(**)_p$ and $\mathcal{C}$, Π_X^p does not admit a sequence of closed subgroups of Π_X^p,

$$\{1\} = H_0 \subset H_1 \subset \cdots \subset H_n \subset H_{n+1} = \Pi_X^p,$$

such that, for each integer i such that $0 \leq i \leq n$, the closed subgroup H_i is topologically finitely generated and normal in H_{i+1} and the quotient H_{i+1}/H_i is infinite.

Example 4.11. Suppose that $\mathcal{C}$ is one of the following:

- X/S is an arbitrary hyperbolic polycurve.
- X/S is a hyperbolic polycurve such that $X \to S$ is proper.
- X/S is a hyperbolic polycurve such that, for each integer i such that $1 \leq i \leq m$ (where we write m for the relative dimension of X/S), if we write (g_i, r_i) for the type of the hyperbolic curve X_i/X_{i-1}, then $r_i > 0$.

Then $\mathcal{C}$ satisfies conditions (1), (2), (3) in Definition 4.10.

Lemma 4.12. *For an arbitrary condition $\mathcal{C}$ as in Definition 4.10, the assertion $(\dagger_1)_p^{\mathcal{C}}$ holds.*

Proof. (Cf. [10, Lem. 3.10].) This follows from Proposition 3.16(iv). □

Theorem 4.13. *Let p be a prime number, n a positive integer, k a sub-p-adic field, $\mathcal{C}$ a condition as in Definition 4.10, S a normal variety over k, X a hyperbolic polycurve of relative dimension n over S satisfying condition $(**)_p$, Y a hyperbolic polycurve over k satisfying condition $(**)_p$ and $\phi : \Pi_Y^p \to \Pi_X^p$ a homomorphism over G_k^p. Suppose that the following conditions are satisfied:*

(1) *The composite $\Pi_Y^p \xrightarrow{\phi} \Pi_X^p \to \Pi_S^p$ arises from a morphism $Y \to S$ over k.*
(2) *ϕ is an open injection.*
(3) $\dim(X)$ $(= \dim(S) + n) \leq \dim(Y)$.
(4) *If $n \geq 2$, then X/X_1 satisfies condition $\mathcal{C}$.*
(5) *For each integer i such that $0 < i < n$, the assertion $(\dagger_i)_p^{\mathcal{C}}$ holds.*

Then ϕ arises from a quasi-finite dominant morphism $Y \to X$ over S. In particular, $\dim(X) = \dim(Y)$.

Proof. (Cf. [10, Thm. 3.11].) Fix a surjection $\Pi_X^p \twoheadrightarrow \Pi_{X_1}^p$ over G_k^p arising from the morphism $X \to X_1$ over k. First, I claim that the following assertion holds:

> Claim A: If $n \geq 2$, then the composite $\Pi_Y^p \xrightarrow{\phi} \Pi_X^p \to \Pi_{X_1}^p$ arises from a morphism $Y \to X_1$ over S.

Indeed, write $S' \subset S$ for the scheme-theoretic image of the morphism $Y \to S$, $Z := \mathrm{Nor}(Y/S')$, $\eta \to Z$ for the generic point of Z and $Y_\eta := Y \times_Z \eta$. Then, by applying Lemma 4.5(vii), where we take the data "(S, Y, X, ϕ)" to be $(S, Y, X_1, \Pi_Y^p \stackrel{\phi}{\to} \Pi_X^p \twoheadrightarrow \Pi_{X_1}^p)$, it suffices to verify that the image of the closed subgroup $N \subset \Pi_Y^p$ defined in Lemma 4.5(iii) by the homomorphism $\Pi_Y^p \to \Pi_{X_1}^p$ is nontrivial. To verify this, assume that the image of $N \subset \Pi_Y^p$ by $\Pi_Y^p \to \Pi_{X_1}^p$ is trivial, i.e., that the image of $N \subset \Pi_Y^p$ by ϕ is contained in $\Delta_{X/X_1}^{(p)} = \Delta_{X/X_1}^p \subset \Pi_X^p$. Then, since $N \subset \Pi_Y^p$ is normal in Π_Y^p and ϕ is open, it follows that the image $\phi(N)$ is normal in the open subgroup $\mathrm{Im}\,\phi \subset \Pi_X^p$ of Π_X^p. On the other hand, it follows from Lemma 4.5(vi) that there exists a sequence of normal closed subgroups of N,

$$\{1\} = H_0 \subset H_1 \subset \cdots \subset H_{\dim(Y)-\dim(S)} = N,$$

such that, for each integer i such that $1 \le i \le \dim(Y) - \dim(S)$, the closed subgroup H_i is topologically finitely generated, and that the quotient H_i/H_{i-1} is infinite. Write $U := \mathrm{Im}\,\phi \cap \Delta_{X/X_1}^p \subset \Delta_{X/X_1}^p$ and, for each integer i such that $0 \le i \le \dim(Y) - \dim(S)$, $H_i^U := \phi(H_i) \subset \Delta_{X/X_1}^p$. Then, since $\mathrm{Im}\,\phi \subset \Pi_X^p$ is open in Π_X^p, U is an open subgroup of Δ_{X/X_1}^p. Moreover, since ϕ is injective, the following hold:

- $H_{\dim(Y)-\dim(S)}^U$ is a normal closed subgroup of $U =: H_{\dim(Y)-\dim(S)+1}^U$.
- For each integer i such that $1 \le i \le \dim(Y) - \dim(S) + 1$, H_i^U is topologically finitely generated.
- For each integer i such that $1 \le i \le \dim(Y) - \dim(S)$, H_i^U is normal in $H_{\dim(Y)-\dim(S)}^U$ and, moreover, the quotient H_i^U/H_{i-1}^U is infinite.

Now suppose that $H_{\dim(Y)-\dim(S)+1}^U/H_{\dim(Y)-\dim(S)}^U$ is finite. Since $H_{\dim(Y)-\dim(S)}^U$ is an open subgroup of Δ_{X/X_1}^p, then it follows from Proposition 3.16(ii) and Lemma 3.22(ii), together with conditions (2) and (3) in Definition 4.10, that $H_{\dim(Y)-\dim(S)}^U$ may be regarded as the maximal pro-p quotient of the fundamental group of a hyperbolic polycurve of dimension $n-1$ over $\overline{k}$ satisfying conditions $(**)_p$ and $\mathcal{C}$. Thus, since we have assumed that the assertion $(\dagger_{n-1})_p^{\mathcal{C}}$ holds, for each integer i such that $1 \le i \le n$, by taking the "H_i" in Definition 4.10 to be $H_{\dim(Y)-\dim(S)-n+i}^U$, we obtain a contradiction. Next, suppose that $H_{\dim(Y)-\dim(S)+1}^U/H_{\dim(Y)-\dim(S)}^U$ is infinite. Then, for each integer i such that $1 \le i \le n$, by taking the "H_i" in Definition 4.10 to be $H_{\dim(Y)-\dim(S)-n+1+i}^U$, we obtain a contradiction. This completes the proof of Claim A.

By applying Claim A and using condition (1) in Definition 4.10 inductively, to verify Theorem 4.13, we may assume without loss of generality that X is a hyperbolic curve over S. Then it follows from Proposition 3.28 and Theorem 4.9 that ϕ arises from a quasi-finite dominant morphism $Y \to X$ over S. ☐

Corollary 4.14. *Let p be a prime number, k a sub-p-adic field, S a normal variety over k, X a hyperbolic polycurve of relative dimension 2 over S satisfying condition $(**)_p$, Y a hyperbolic polycurve over k satisfying condition $(**)_p$ and $\phi : \Pi_Y^p \to \Pi_X^p$ a homomorphism over G_k^p. Suppose that the following conditions are satisfied:*

(1) *The composite $\Pi_Y^p \xrightarrow{\phi} \Pi_X^p \to \Pi_S^p$ arises from a morphism $Y \to S$ over k.*
(2) *ϕ is an open injection.*
(3) *$\dim(X)$ $(= \dim(S) + 2) \leq \dim(Y)$.*

Then ϕ arises from a quasi-finite dominant morphism $Y \to X$ over S. In particular, $\dim(X) = \dim(Y)$.

Proof. (Cf. [10, Cor. 3.12].) This follows from Theorem 4.13, together with Lemma 4.12. ☐

Lemma 4.15 ([10, Lem. 3.13]). *Let G_1, G_2 be profinite groups, $H_1 \subset G_1$, $H_2 \subset G_2$ closed subgroups of G_1, G_2, respectively, and $\phi : G_1 \to G_2$ a homomorphism. Suppose that $\phi(H_1) \subset H_2$. Then the homomorphism $H_1 \to H_2$ induced by ϕ is surjective if and only if the following condition is satisfied: for any open subgroup $U \subset G_2$ of G_2 and any normal open subgroup $N \subset U$ of U, if the composite $H_2 \cap U \hookrightarrow U \twoheadrightarrow U/N$ is surjective, then the composite $H_1 \cap \phi^{-1}(U) \hookrightarrow \phi^{-1}(U) \xrightarrow{\phi} U \twoheadrightarrow U/N$ is surjective.*

Theorem 4.16. *Let p be a prime number, k a sub-p-adic field, X a hyperbolic polycurve of dimension 2 over k satisfying condition $(**)_p$, Y a normal variety over k and $\phi : \Pi_Y^p \to \Pi_X^p$ an open homomorphism over G_k^p. Suppose that $\Pi^p_{Y \times_k \overline{k}} \to \Pi_Y^p$ is injective and, moreover, that the kernel of ϕ is topologically finitely generated. Then ϕ arises from a uniquely determined dominant morphism $Y \to X$ over k. In particular, $\dim(Y) \geq 2$.*

Proof. (Cf. [10, Thm. 3.14].) First, by replacing X by the connected finite étale covering of X corresponding to $\phi(\Pi_Y^p) \subset \Pi_X^p$, to verify Theorem 4.16, we may assume without loss of generality that ϕ is surjective. (Note that it follows from Lemma 3.22(ii) that X satisfies condition $(**)_p$ even if we replace X as above.) Then since ϕ and $\Pi_X^p \to \Pi_{X_1}^p$ are surjective (cf. Proposition 3.16(i)) and their kernels are topologically finitely generated (cf. Proposition 3.16(iii)), the composite $\Pi_Y^p \xrightarrow{\phi} \Pi_X^p \twoheadrightarrow \Pi_{X_1}^p$ is surjective and its kernel is topologically finitely generated. Thus, since X_1 is a hyperbolic curve over k satisfying condition $(**)_p$, it follows from Theorem 4.4, together with the implication $(2)'' \Rightarrow (3)$ of Lemma 3.35, that $\Pi_Y^p \to \Pi_{X_1}^p$ arises from a uniquely determined morphism $Y \to X_1$ over k which is surjective and generically geometrically connected. Write $\eta \to X_1$ for the generic point of X_1, $Y_\eta := Y \times_{X_1} \eta$ and $X_\eta := X \times_{X_1} \eta$. (Thus, Y_η is a normal variety over η.) Now I claim that the following assertion holds:

> Claim A: Any homomorphism that belongs to the Π_X^p-conjugacy class of homomorphisms $N \to \Delta_{X/X_1}^p$ of Lemma 4.5(iv), where we take the data "(S, Y, X)" to be (X_1, Y, X), is surjective.

Let us observe that $N \to \Delta_{X/X_1}^p$ is surjective if and only if $\Delta_{Y_\eta/\eta} \to \Delta_{X/X_1}^p$ is surjective (cf. Lemma 4.5(iii)). Thus, it follows from Lemma 4.15 that, to verify Claim A, it suffices to verify that the following assertion holds:

> Claim A.1: Let $U \subset \Pi_X^p$ be an open subgroup of Π_X^p and $V \subset U$ a normal open subgroup of U. Write $X' \to X$ for the connected finite étale covering of X corresponding to $U \subset \Pi_X^p$, $X'' \to X'$ for the connected finite étale Galois covering of X' corresponding to $V \subset U = \Pi_{X'}^p$, $Y' \to Y$ for the connected finite étale covering of Y corresponding to $\phi^{-1}(U) \subset \Pi_Y^p$, $Y'' \to Y'$ for the connected finite étale Galois covering of Y' corresponding to $\phi^{-1}(V) \subset \phi^{-1}(U) = \Pi_{Y'}^p$, $Y'_\eta := Y' \times_{X_1} \eta$ $(= Y' \times_Y Y_\eta)$ and $Y''_\eta := Y'' \times_{X_1} \eta$ $(= Y'' \times_Y Y_\eta)$ (note that it follows from Lemma 2.2 that $Y_\eta \to Y$ induces an outer surjection $\Pi_{Y_\eta} \to \Pi_Y$, which implies that Y'_η and Y''_η are connected). Suppose that the composite $\Delta_{X/X_1}^p \cap \Pi_{X'}^p \hookrightarrow \Pi_{X'}^p \twoheadrightarrow \Pi_{X'}^p/\Pi_{X''}^p = U/V$ is surjective. Then the composite $\Delta_{Y_\eta/\eta} \cap \Pi_{Y'_\eta} \hookrightarrow \Pi_{Y'_\eta} \twoheadrightarrow \Pi_{Y'_\eta}/\Pi_{Y''_\eta}$ is surjective.

Indeed, it follows from Proposition 3.5 and Lemma 3.22(ii) that the sequence of schemes $X' \to X'_1 := \mathrm{Nor}(X'/X_1) \to X'_0 := \mathrm{Nor}(X'/\operatorname{Spec} k)$ determines a structure of hyperbolic polycurve of dimension 2 on X' which satisfies condition $(**)_p$ and, moreover, the natural morphisms $X'_1 \to X_1$, $\eta' \to \eta$, where we write $\eta' \to X'_1$ for the generic point of X'_1, are connected finite étale coverings. In particular, it follows from Lemma 3.9(ii) that the natural inclusions $\Pi_{X'}^p \hookrightarrow \Pi_X^p$, $\Pi_{Y'_\eta} \hookrightarrow \Pi_{Y_\eta}$ determine equalities

$$\Delta_{X/X_1}^p \cap \Pi_{X'}^p = \Delta_{X'/X'_1}^p, \quad \Delta_{Y_\eta/\eta} \cap \Pi_{Y'_\eta} = \Delta_{Y'_\eta/\eta'}.$$

Thus, to verify Claim A.1, by replacing X by X', it suffices to verify that for any covering $X'' \to X$ corresponding to a normal open subgroup of Π_X^p, if $\Delta_{X/X_1}^p \to \Pi_X^p/\Pi_{X''}^p$ is surjective, then $\Delta_{Y_\eta/\eta} \to \Pi_{Y_\eta}/\Pi_{Y''_\eta}$ is surjective. Moreover, since $\Pi_{X''}$ is the inverse image of $\Pi_{X''}^p \subset \Pi_X^p$ by the surjection $\Pi_X \twoheadrightarrow \Pi_X^p$, it follows that the natural homomorphism $\Pi_X/\Pi_{X''} \to \Pi_X^p/\Pi_{X''}^p$ is an isomorphism. Thus, to verify Claim A.1, it suffices to verify that if $\Delta_{X/X_1} \to \Pi_X/\Pi_{X''}$ is surjective, then $\Delta_{Y_\eta/\eta} \to \Pi_{Y_\eta}/\Pi_{Y''_\eta}$ is surjective. Let $\overline{\eta} \to X_1$ be a generic geometric point of X_1. Then it follows from the natural isomorphism $\Pi_{X \times_{X_1} \overline{\eta}} \xrightarrow{\sim} \Delta_{X/X_1}$ (resp. $\Pi_{Y_\eta \times_\eta \overline{\eta}} = \Pi_{Y \times_{X_1} \overline{\eta}} \xrightarrow{\sim} \Delta_{Y_\eta/\eta}$) that $\Pi_{X \times_{X_1} \overline{\eta}} \to \Pi_X/\Pi_{X''}$ (resp. $\Pi_{Y_\eta \times_{X_1} \overline{\eta}} \to \Pi_{Y_\eta}/\Pi_{Y''_\eta}$) is surjective if and only if $X'' \times_X (X \times_{X_1} \overline{\eta}) = X'' \times_{X_1} \overline{\eta}$ (resp. $Y''_\eta \times_{Y_\eta} (Y \times_{X_1} \overline{\eta}) = Y''_\eta \times_{X_1} \overline{\eta}$) is connected. Thus, we conclude that to verify Claim A.1, it suffices to verify that if $X'' \times_{X_1} \overline{\eta}$ is connected, then $Y''_\eta \times_\eta \overline{\eta}$ is connected. To verify this, assume that $X'' \times_{X_1} \overline{\eta}$ is connected, i.e., $X'' \to X_1$ is generically geometrically connected. Then, since the composite $X'' \to X \to X_1$ is smooth and surjective, it follows from the implication (1) $\Rightarrow$ (2)$''$ of Lemma 3.35 that the composite $\Pi^p_{X''} \hookrightarrow \Pi^p_X \twoheadrightarrow \Pi^p_{X_1}$ is surjective and its kernel is topologically finitely generated. On the other hand, we have assumed that ϕ is surjective and $\ker \phi$ is topologically finitely generated. Thus, it holds that the composite $\Pi^p_{Y''} \twoheadrightarrow \Pi^p_{X''} \hookrightarrow \Pi^p_X \twoheadrightarrow \Pi^p_{X_1}$ is surjective and its kernel is topologically finitely generated. In particular, it follows from the implication (2)$''$ $\Rightarrow$ (3) of Lemma 3.35 that the morphism $Y'' \to X_1$ is generically geometrically connected, which implies that $Y'' \times_{X_1} \overline{\eta}$ is connected. This completes the proof of Claim A.1, hence also of Claim A.

It follows from Claim A, together with Proposition 3.16(iii) and Lemma 4.5(vii), that ϕ arises from a morphism $Y \to X$ over k. Moreover, it follows from Corollary 4.3(i) that $Y \to X$ is unique. On the other hand, it follows from Claim A, together with Lemma 4.5(iii), that $\Pi_{Y \times_Z \overline{\eta}} \to \Pi^p_{X \times_{X_1} \overline{\eta}}$ (where $Z := \mathrm{Nor}(Y/X_1)$ and $\overline{\eta} \to Z$ is a generic geometric point of Z) is surjective. Thus, since $X \times_{X_1} \overline{\eta}$ is a hyperbolic curve over $\overline{\eta}$, it follows from Proposition 3.16(iii) that the morphism $Y \times_Z \overline{\eta} \to X \times_{X_1} \overline{\eta}$, hence also $Y \times_{X_1} \overline{\eta} \to X \times_{X_1} \overline{\eta}$, is dominant. This implies that $Y \to X$ is dominant. This completes the proof of Theorem 4.16. □

Theorem 4.17. *Let p be a prime number, k a sub-p-adic field, X a hyperbolic polycurve of dimension 2 over k satisfying condition $(*)_p$, Y a normal variety over k and $\phi : \Pi^p_{Y/k} \to \Pi^p_{X/k}$ an open homomorphism over G_k. Suppose that the kernel of ϕ is topologically finitely generated. Then ϕ arises from a uniquely determined dominant morphism $Y \to X$ over k. In particular, $\dim(Y) \geq 2$.*

Proof. There exists a finite Galois extension k_1 of k such that $X \times_k k_1/k_1$ satisfies condition $(**)_p$ (cf. Proposition 3.21). We can choose a finite extension k_2 of k_1 such that $Y \times_k k_2$ has a k_2-rational point. Then the section of $\Pi_{Y \times_k k_2} \twoheadrightarrow G_{k_2}$ induced by a k_2-rational point determines a homomorphism $G_{k_2} \to \mathrm{Aut}(\Pi^p_{Y \times_k \overline{k}})$. On the other hand, it follows from Lemma 2.7 that $\Pi_{Y \times_k \overline{k}}$, hence also $\Pi^p_{Y \times_k \overline{k}}$, is topologically finitely generated. Thus, $\mathrm{Aut}(\Pi^p_{Y \times_k \overline{k}})$ has an open pro-p subgroup U (cf. Lemma 3.19). Let us choose a finite Galois extension k' of k such that $G_{k'} \subset G_k$

is contained in the inverse image of $U \subset \mathrm{Aut}(\Pi^p_{Y\times_k\overline{k}})$ by $G_{k_2} \to \mathrm{Aut}(\Pi^p_{Y\times_k\overline{k}})$. Then k' is a finite extension of k_2. Moreover, since the image of the composite $G_{k'} \hookrightarrow G_{k_2} \to \mathrm{Aut}(\Pi^p_{Y\times_k\overline{k}})$ is pro-p, $G_{k'} \hookrightarrow G_{k_2} \to \mathrm{Aut}(\Pi^p_{Y\times_k\overline{k}})$ factors through the surjection $G_{k'} \twoheadrightarrow G^p_{k'}$. Thus we obtain a homomorphism $G^p_{k'} \to \mathrm{Aut}(\Pi^p_{Y\times_k\overline{k}})$, which determines a semidirect product $\Pi^p_{Y\times_k\overline{k}} \rtimes G^p_{k'}$. Then by construction, we obtain a surjection $\Pi_{Y\times_k\overline{k}} \rtimes G_{k'} \twoheadrightarrow \Pi^p_{Y\times_k\overline{k}} \rtimes G^p_{k'}$. Now $\Pi_{Y\times_k\overline{k}} \rtimes G_{k'} \cong \Pi_{Y\times_k k'}$ (cf. Lemma 2.5) and, moreover, the image of $\Pi_{Y\times_k\overline{k}} \subset \Pi_{Y\times_k k'}$ by $\Pi_{Y\times_k k'} \twoheadrightarrow \Pi^p_{Y\times_k\overline{k}} \rtimes G^p_{k'}$ is $\Pi^p_{Y\times_k\overline{k}}$. Since $\Pi^p_{Y\times_k\overline{k}} \rtimes G^p_{k'}$ is pro-p, the composite $\Pi_{Y\times_k\overline{k}} \subset \Pi_{Y\times_k k'} \twoheadrightarrow \Pi^p_{Y\times_k\overline{k}} \rtimes G^p_{k'}$ determines a sequence $\Pi^p_{Y\times_k\overline{k}} \to \Pi^p_{Y\times_k k'} \twoheadrightarrow \Pi^p_{Y\times_k\overline{k}} \rtimes G^p_{k'}$. In particular, since $\Pi^p_{Y\times_k\overline{k}} \to \Pi^p_{Y\times_k\overline{k}} \rtimes G^p_{k'}$ is injective, we conclude that $\Pi^p_{Y\times_k\overline{k}} \to \Pi^p_{Y\times_k k'}$ is injective.

Now $\Pi^p_{X\times_k k'/k'}$, $\Pi^p_{Y\times_k k'/k'}$ are the inverse images of the normal open subgroup $G_{k'} \subset G_k$ by the surjections $\Pi^p_{X/k} \twoheadrightarrow G_k$, $\Pi^p_{Y/k} \twoheadrightarrow G_k$, respectively. Thus, if we write $\phi' : \Pi^p_{Y\times_k k'/k'} \to \Pi^p_{X\times_k k'/k'}$ for the open homomorphism over $G_{k'}$ determined by ϕ, then $\ker\phi' = \ker\phi$. Write $\widetilde{\phi}' : \Pi^p_{Y\times_k k'} \to \Pi^p_{X\times_k k'}$ for the open homomorphism over $G^p_{k'}$ determined by ϕ'. Then since $\Pi^p_{Y\times_k\overline{k}} \to \Pi^p_{Y\times_k k'}$ is injective, we have $\ker\phi' = \ker\widetilde{\phi}'$. On the other hand, since $X \times_k k_1/k_1$ satisfies condition $(**)_p$, it follows from Proposition 3.16(ii) that $X \times_k k'/k'$ satisfies condition $(**)_p$. Thus, $\widetilde{\phi}'$ arises from a dominant morphism $Y \times_k k' \to X \times_k k'$ over k' (cf. Theorem 4.16). Since the image of $\widetilde{\phi}'$ by the map of Lemma 3.23(ii) is ϕ', this implies that ϕ' arises from the above dominant morphism $Y \times_k k' \to X \times_k k'$, which is compatible with the natural actions of $\mathrm{Gal}(k'/k)$ (cf. Proposition 4.2(i)). Thus, by descending the morphism, we obtain a dominant morphism $Y \to X$ over k. Since $\Delta^p_{X/k}$ is slim (cf. Proposition 3.16(iii)), it follows from Lemma 2.20 that $\Pi^p_{Y/k} \to \Pi^p_{X/k}$ induced by the morphism $Y \to X$ belongs to the same $\Delta^p_{X/k}$-conjugacy class determined by ϕ, which implies that ϕ arises from a dominant morphism $Y \to X$. Moreover, it follows from Proposition 4.2(i) that $Y \to X$ is unique. This completes the proof of Theorem 4.17. □

Corollary 4.18. *Let p be a prime number, k a sub-p-adic field, X a hyperbolic polycurve of dimension 3 over k satisfying condition $(**)_p$, Y a normal variety over k and $\phi : \Pi^p_Y \to \Pi^p_X$ a homomorphism over G^p_k. Suppose that the following conditions are satisfied:*

(1) *ϕ is open and its kernel is finite.*

(2) *Y is of p-LFG-type and, moreover, $\Pi^p_{Y\times_k\overline{k}} \to \Pi^p_Y$ is injective.*

(3) *$3 \leq \dim(Y)$.*

Then ϕ arises from a uniquely determined quasi-finite dominant morphism $Y \to X$ over k. In particular, $\dim(Y) = 3$.

Proof. (Cf. [10, Cor. 3.15].) It follows from condition (1), together with Proposition 3.16(iii), that the kernel of the composite $\Pi_Y^p \xrightarrow{\phi} \Pi_X^p \twoheadrightarrow \Pi_{X_2}^p$ is topologically finitely generated. Thus, it follows from Theorem 4.16 that the composite $\Pi_Y^p \xrightarrow{\phi} \Pi_X^p \twoheadrightarrow \Pi_{X_2}^p$ arises from a dominant morphism $Y \to X_2$ over k. In particular, it follows from Theorem 4.9 that ϕ arises from a quasi-finite dominant morphism $Y \to X$ over k. Moreover, it follows from Corollary 4.3(i) that $Y \to X$ is unique. This completes the proof of Corollary 4.18. □

Corollary 4.19. *Let p be a prime number, k a sub-p-adic field, X a hyperbolic polycurve of dimension* 3 *over k satisfying condition $(*)_p$, Y a normal variety over k and $\phi : \Pi_{Y/k}^p \to \Pi_{X/k}^p$ a homomorphism over G_k. Suppose that the following conditions are satisfied:*

(1) *ϕ is open and its kernel is finite.*

(2) *Y is of p-LFG-type.*

(3) $3 \leq \dim(Y)$.

Then ϕ arises from a uniquely determined quasi-finite dominant morphism $Y \to X$ over k. In particular, $\dim(Y) = 3$.

Proof. This follows from Corollary 4.18, together with an argument similar to the argument used in the proof of Theorem 4.17. □

Corollary 4.20. *Let p be a prime number, k a sub-p-adic field, X a hyperbolic polycurve of dimension* 4 *over k satisfying condition $(**)_p$, Y a hyperbolic polycurve over k satisfying condition $(**)_p$ and $\phi : \Pi_Y^p \to \Pi_X^p$ a homomorphism over G_k^p. Suppose that the following conditions are satisfied:*

(1) *ϕ is an open injection (resp. isomorphism).*

(2) $4 \leq \dim(Y)$.

Then ϕ arises from a uniquely determined finite étale covering (resp. isomorphism) $Y \to X$ over k. In particular, $\dim(Y) = 4$.

Proof. (Cf. [10, Cor. 3.16].) First, by replacing X by the connected finite étale covering of X corresponding to $\phi(\Pi_Y^p) \subset \Pi_X^p$ (cf. Proposition 3.5, Lemma 3.22(ii)), to verify Corollary 4.20, we may assume without loss of generality that ϕ is an isomorphism. Then it follows from Proposition 3.16(iii) that the kernel of the composite $\Pi_Y^p \xrightarrow{\phi} \Pi_X^p \twoheadrightarrow \Pi_{X_2}^p$ is topologically finitely generated. Thus, it follows from Theorem 4.16 that the composite $\Pi_Y^p \xrightarrow{\phi} \Pi_X^p \twoheadrightarrow \Pi_{X_2}^p$ arises from a dominant morphism $Y \to X_2$ over k. In particular, it follows from Corollary 4.14 that ϕ arises from a quasi-finite dominant morphism $Y \to X$ over k, which implies that

$4 = \dim(X) = \dim(Y)$. By applying an argument similar to the above argument to ϕ^{-1}, we obtain a quasi-finite dominant morphism $X \to Y$ over k. Then it follows from Corollary 4.3(i) that the two morphisms $Y \to X$ and $X \to Y$ are inverse to each other. In particular, this morphism $Y \to X$ is an isomorphism. Moreover, it follows from Corollary 4.3(i) that $Y \to X$ is unique. This completes the proof of Corollary 4.20. □

Corollary 4.21. *Let p be a prime number, k a sub-p-adic field, X a hyperbolic polycurve of dimension 4 over k satisfying condition $(*)_p$, Y a hyperbolic polycurve over k satisfying condition $(*)_p$ and $\phi : \Pi^p_{Y/k} \to \Pi^p_{X/k}$ a homomorphism over G_k. Suppose that the following conditions are satisfied:*

(1) *ϕ is an open injection (resp. isomorphism).*

(2) $4 \leq \dim(Y)$.

Then ϕ arises from a uniquely determined finite étale covering (resp. isomorphism) $Y \to X$ over k. In particular, $\dim(Y) = 4$.

Proof. This follows from Corollary 4.20, together with an argument similar to the argument used in the proof of Theorem 4.17. □

Corollary 4.22. *Let p be a prime number, n_X, n_Y positive integers, k a sub-p-adic field and X, Y hyperbolic polycurves of dimension n_X, n_Y over k satisfying condition $(*)_p$, respectively. Suppose that either $n_X \leq 4$ or $n_Y \leq 4$. Then the natural maps*

$$\mathrm{Isom}_k(Y, X) \to \mathrm{Isom}_{G_k}(\Pi_Y, \Pi_X)/\,\mathrm{Inn}(\Delta_{X/k}) \to \mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k})/\,\mathrm{Inn}(\Delta^p_{X/k})$$

are bijective.

Proof. The bijectivity of the map

$$\mathrm{Isom}_k(Y, X) \to \mathrm{Isom}_{G_k}(\Pi_Y, \Pi_X)/\,\mathrm{Inn}(\Delta_{X/k})$$

is proved in [10, Cor. 3.18], and the injectivity of the map

$$\mathrm{Isom}_k(Y, X) \to \mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k})/\,\mathrm{Inn}(\Delta^p_{X/k})$$

follows from Proposition 4.2(i). Thus, it remains to verify the surjectivity of the latter map. Let $\phi : \Pi^p_{Y/k} \xrightarrow{\sim} \Pi^p_{X/k}$ be an isomorphism over G_k. Then, by replacing (X, Y, ϕ) by (Y, X, ϕ^{-1}) if necessary, we may assume without loss of generality that $n_X \leq n_Y$. In particular, $n_X \leq 4$. Thus, it follows from Proposition 3.28, Theorems 4.4, 4.17 and Corollaries 4.19, 4.21 that ϕ arises from a uniquely determined quasi-finite dominant morphism $Y \to X$ over k. In particular, we obtain that $n_X = n_Y \leq 4$. Thus, by applying an argument similar to the above argument to ϕ^{-1}, we obtain a quasi-finite dominant morphism $X \to Y$ over k. Then it follows from Corollary 4.3(i) that the two morphisms $Y \to X$ and $X \to Y$ are inverse to each other. Thus, $Y \to X$ is an isomorphism. This completes the proof of Corollary 4.22. □

Remark 4.23. It seems that the assertion $(\dagger_n)_p^{\mathcal{C}}$ holds for every positive integer n. However, it is not known whether there exists an integer $n > 1$ such that the assertion $(\dagger_n)_p^{\mathcal{C}}$ can be proven (with a sufficiently general condition $\mathcal{C}$). If one proves that the assertion $(\dagger_n)_p^{\mathcal{C}}$ holds for every positive integer n, then, by applying an argument similar to the argument applied in the proof of Corollary 4.22, except that instead of applying Theorems 4.4, 4.17 and Corollaries 4.19, 4.21, one applies Theorem 4.13, we can prove the assertion obtained by replacing the assumption "either $n_X \leq 4$ or $n_Y \leq 4$" of Corollary 4.22 by "X, Y satisfy condition $\mathcal{C}$".

Proposition 4.24 ([10, Prop. 3.19]). *Let k_X, k_Y be finitely generated extension fields of $\mathbb{Q}$. Then the following hold:*

(i) *Let $H \subset G_{k_X}$ be a closed subgroup of G_{k_X}. Suppose that H is topologically finitely generated and normal in an open subgroup of G_{k_X}. Then H is trivial.*

(ii) *The natural map $\mathrm{Isom}(\overline{k_X}/k_X, \overline{k_Y}/k_Y) \to \mathrm{Isom}(G_{k_Y}, G_{k_X})$ is bijective.*

Corollary 4.25. *Let p be a prime number, k_X, k_Y fields of characteristic zero, n a positive integer, X a hyperbolic polycurve of dimension n over k_X satisfying condition $(*)_p$, Y a normal variety over k_Y and $\phi : \Pi^p_{Y/k_Y} \to \Pi^p_{X/k_X}$ an open homomorphism. Suppose that one of the following conditions (1), (2), (3), (4) is satisfied:*

(1) $n = 1$.

(2) *The following conditions are satisfied:*

  (2-i) $n = 2$.

  (2-ii) *The kernel of ϕ is topologically finitely generated.*

(3) *The following conditions are satisfied:*

  (3-i) $n = 3$.

  (3-ii) *The kernel of ϕ is finite.*

  (3-iii) *Y is of p-LFG-type.*

  (3-iv) $3 \leq \dim(Y)$.

(4) *The following conditions are satisfied:*

(4-i) $n = 4$.
(4-ii) ϕ *is injective.*
(4-iii) Y *is a hyperbolic polycurve over* k_Y *satisfying condition* $(*)_p$.
(4-iv) $4 \leq \dim(Y)$.

Then the following hold:

(i) *Suppose that both* k_X, k_Y *are finitely generated over* $\mathbb{Q}$. *Then the open homomorphism* ϕ *lies over an open homomorphism* $G_{k_Y} \to G_{k_X}$.

(ii) *In situation* (i), *suppose that the homomorphism* $G_{k_Y} \to G_{k_X}$ *obtained in* (i) *is injective. Then* ϕ *arises from a dominant morphism* $Y \to X$.

(iii) *Suppose that both* k_X, k_Y *are sub-p-adic and, moreover, that the open homomorphism* ϕ *lies over an open homomorphism* $G_{k_Y} \to G_{k_X}$ *that arises from a homomorphism* $k_X \hookrightarrow k_Y$ *of fields. Then* ϕ *arises from a dominant morphism* $Y \to X$.

Proof. (Cf. [10, Cor. 3.20].) First, we verify assertion (i). It follows from Lemma 2.7 and Proposition 3.7(i) that $\Pi_{Y_{\overline{k_Y}}} \xrightarrow{\sim} \Delta_{Y/k_Y}$, hence also the image of the composite $\Delta_{Y/k_Y} \twoheadrightarrow \Delta^p_{Y/k_Y} \hookrightarrow \Pi^p_{Y/k_Y} \xrightarrow{\phi} \Pi^p_{X/k_X} \to G_{k_X}$, is topologically finitely generated. Moreover, the image of the composite $\Delta_{Y/k_Y} \twoheadrightarrow \Delta^p_{Y/k_Y} \hookrightarrow \Pi^p_{Y/k_Y} \xrightarrow{\phi} \Pi^p_{X/k_X} \to G_{k_X}$ is normal in the image of $\Pi^p_{Y/k_Y} \xrightarrow{\phi} \Pi^p_{X/k_X} \twoheadrightarrow G_{k_X}$, which is an open subgroup of G_{k_X}. Thus it follows from Proposition 4.24(i) that the image of the composite $\Delta_{Y/k_Y} \twoheadrightarrow \Delta^p_{Y/k_Y} \to G_{k_X}$ is trivial. In particular, the composite $\Pi^p_{Y/k_Y} \xrightarrow{\phi} \Pi^p_{X/k_X} \twoheadrightarrow G_{k_X}$ factors through $\Pi^p_{Y/k_Y} \twoheadrightarrow G_{k_Y}$. Then ϕ lies over a resulting homomorphism $G_{k_Y} \to G_{k_X}$. Moreover, since ϕ is open and the outer homomorphism $\Pi^p_{X/k_X} \twoheadrightarrow G_{k_X}$ is surjective, we conclude that $G_{k_Y} \to G_{k_X}$ is open. This completes the proof of assertion (i). Next, we verify assertion (ii). Let us observe that, by replacing X by the connected finite étale covering of X corresponding to $\phi(\Pi^p_{Y/k_Y}) \subset \Pi^p_{X/k_X}$ (cf. Proposition 3.5, Lemma 3.22(i)), to verify assertion (ii), we may assume without loss of generality that ϕ, hence also the injection $G_{k_Y} \to G_{k_X}$, is surjective. Then it follows from Proposition 4.24(ii) that the isomorphism $G_{k_Y} \xrightarrow{\sim} G_{k_X}$ arises from an isomorphism $\overline{k_X} \xrightarrow{\sim} \overline{k_Y}$ that determines an isomorphism $k_X \xrightarrow{\sim} k_Y$. In particular, by replacing $(X \times_{k_X} k_Y, k_Y, \overline{k_Y})$ by $(X, k_X, \overline{k_X})$, we may assume without loss of generality that $(k_X, \overline{k_X}) = (k_Y, \overline{k_Y})$, and that the homomorphism $G_{k_Y} \to G_{k_X}$ of (i) is the identity automorphism of G_{k_X}. Then it follows from Theorems 4.4, 4.17 and Corollaries 4.19, 4.21 that ϕ arises from a dominant morphism $Y \to X$. This completes the proof of assertion (ii). Finally, we verify assertion (iii). Since

$\Pi^p_{X\times_{k_X}k_Y/k_Y} \xrightarrow{\sim} \Pi^p_{X/k_X} \times_{G_{k_X}} G_{k_Y}$ (cf. Proposition 3.16(ii)), it follows that ϕ determines a homomorphism $\Pi^p_{Y/k_Y} \to \Pi^p_{X\times_{k_X}k_Y/k_Y}$, which arises from a dominant morphism $Y \to X \times_{k_X} k_Y$ (cf. Theorems 4.4, 4.17, Corollaries 4.19, 4.21). Thus, ϕ arises from the composite $Y \to X \times_{k_X} k_Y \stackrel{\mathrm{pr}_1}{\to} X$. This completes the proof of assertion (iii). □

§5. Finiteness of the set of outer isomorphisms between geometrically pro-p étale fundamental groups of hyperbolic polycurves

In the present Section 5, we discuss the finiteness of a certain set of outer isomorphisms between the pro-p étale fundamental groups of hyperbolic polycurves. Let k be a field of characteristic zero, $\overline{k}$ an algebraic closure of k and $G_k := \mathrm{Gal}(\overline{k}/k)$.

Lemma 5.1 ([10, Lem. 4.1]). *Let G be a profinite group, $H \subset G$ an open subgroup of G, A a group and $A \to \mathrm{Aut}(G)$ a homomorphism. Write $A_H \subset A$ for the subgroup of A consisting of $a \in A$ such that the automorphism of G obtained by forming the image of a in $\mathrm{Aut}(G)$ preserves $H \subset G$. Suppose that G is topologically finitely generated. Then A_H is of finite index in A.*

Lemma 5.2. *Let p be a prime number, n a positive integer, S a connected noetherian separated normal scheme over k and X a hyperbolic polycurve of relative dimension n over S satisfying condition $(*)_p$. Then there exists an open subgroup $H \subset \Delta^p_{X/S}$ of $\Delta^p_{X/S}$ such that, if we write $H_i := H \cap \Delta^p_{X/X_i}$ for each integer i such that $0 \le i \le n$, then, for each integer i such that $0 < i < n$, it holds that*

$$\mathrm{rank}_{\mathbb{Z}_p}((H_i/H_{i+1})^{\mathrm{ab}}) < \mathrm{rank}_{\mathbb{Z}_p}((H_{i-1}/H_i)^{\mathrm{ab}}).$$

Proof. (Cf. [10, Lem. 4.2(i)].) We verify Lemma 5.2 by induction on n. If $n = 1$, then Lemma 5.2 is immediate. Now suppose that $n \ge 2$, and that the induction hypothesis is in force. Then it follows from the induction hypothesis that there exists an open subgroup $U \subset \Delta^p_{X/X_1}$ of Δ^p_{X/X_1} such that, if we write $U_i := U \cap \Delta^p_{X/X_i}$ for each integer i such that $1 \le i \le n$, then, for each integer i such that $1 < i < n$, it holds that

$$\mathrm{rank}_{\mathbb{Z}_p}((U_i/U_{i+1})^{\mathrm{ab}}) < \mathrm{rank}_{\mathbb{Z}_p}((U_{i-1}/U_i)^{\mathrm{ab}}).$$

Now it follows from Lemma 3.17(i) that there exists an open subgroup $V \subset \Delta^p_{X/S}$ of $\Delta^p_{X/S}$ such that $U = V \cap \Delta^p_{X/X_1}$. Write W for the image of $V \subset \Delta^p_{X/S}$ by the surjection $\Delta^p_{X/S} \twoheadrightarrow \Delta^p_{X_1/S}$. Then since W is an open subgroup of $\Delta^p_{X_1/S}$, there exists an open subgroup $Q \subset W$ of W such that

$$\mathrm{rank}_{\mathbb{Z}_p}(Q^{\mathrm{ab}}) > \mathrm{rank}_{\mathbb{Z}_p}((U_1/U_2)^{\mathrm{ab}})$$

(cf. Proposition 3.16(vi)). Write H for the inverse image of $Q \subset W$ by the surjection $V \twoheadrightarrow W$. Then H is an open subgroup of V, hence also of $\Delta^p_{X/S}$. Moreover, since $U = V \cap \Delta^p_{X/X_1} \subset H \subset V$, we have $H \cap \Delta^p_{X/X_1} = U$. Thus, if we write $H_i := H \cap \Delta^p_{X/X_i}$ for each integer i such that $0 \le i \le n$, then, for each integer i such that $1 \le i \le n$, it holds that $H_i = U_i$. Moreover, since $H_0 = H$ and $H_1 = U$, it follows from the exact sequence $1 \to U \to H \to Q \to 1$ that we have an isomorphism $H_0/H_1 \xrightarrow{\sim} Q$. In particular, for each integer i such that $1 \le i \le n-1$, it holds that

$$\mathrm{rank}_{\mathbb{Z}_p}((H_i/H_{i+1})^{\mathrm{ab}}) < \mathrm{rank}_{\mathbb{Z}_p}((H_{i-1}/H_i)^{\mathrm{ab}}).$$

This completes the proof of Lemma 5.2. □

Lemma 5.3. *Let p be a prime number, n a positive integer and X, Y hyperbolic polycurves of dimension n over k satisfying condition $(*)_p$. Then the following hold:*

(i) *Let $\phi : \Delta^p_{Y/k} \xrightarrow{\sim} \Delta^p_{X/k}$ be an isomorphism from $\Delta^p_{Y/k}$ to $\Delta^p_{X/k}$. Suppose that there exists an open subgroup $H \subset \Delta^p_{Y/k}$ of $\Delta^p_{Y/k}$ such that, if we write $H_i := H \cap \Delta^p_{Y/Y_i}$, $H'_i := \phi(H) \cap \Delta^p_{X/X_i}$ for each integer i such that $0 \le i \le n$, then, for any integers i, j such that $0 \le i < j < n$, it holds that*

$$\mathrm{rank}_{\mathbb{Z}_p}((H_i/H_{i+1})^{\mathrm{ab}}) > \mathrm{rank}_{\mathbb{Z}_p}((H'_j/H'_{j+1})^{\mathrm{ab}}),$$
$$\mathrm{rank}_{\mathbb{Z}_p}((H'_i/H'_{i+1})^{\mathrm{ab}}) > \mathrm{rank}_{\mathbb{Z}_p}((H_j/H_{j+1})^{\mathrm{ab}}).$$

Then, for each integer i such that $0 \le i \le n$, it holds that $\phi(\Delta^p_{Y/Y_i}) = \Delta^p_{X/X_i}$.

(ii) *Let $\psi : \Pi^p_{Y/k} \xrightarrow{\sim} \Pi^p_{X/k}$ be an isomorphism from $\Pi^p_{Y/k}$ to $\Pi^p_{X/k}$ over G_k. Suppose that k is sub-p-adic, and that for each integer i such that $0 \le i \le n$, it holds that $\psi(\Delta^p_{Y/Y_i}) = \Delta^p_{X/X_i}$ (e.g., the case where $\psi|_{\Delta^p_{Y/k}}$ satisfies the condition appearing in the statement of assertion* (i)*). Then ψ arises from an isomorphism $Y \xrightarrow{\sim} X$ over k.*

Proof. (Cf. [10, Lem. 4.2(ii),(iii)].) First, we verify assertion (i) by induction on n. If $n = 1$, then assertion (i) is immediate. Now suppose that $n \ge 2$ and that the induction hypothesis is in force. To verify assertion (i), I claim that the following assertion holds:

Claim A: $\phi(H_{n-1}) = H'_{n-1}$.

Indeed, there exists a unique integer m such that $0 \le m < n$, and the image of the composite $H_{n-1} \hookrightarrow H \xrightarrow{\phi} \phi(H) \twoheadrightarrow \phi(H)/H'_{m+1}$ is nontrivial, but the image of the composite $H_{n-1} \hookrightarrow H \xrightarrow{\phi} \phi(H) \twoheadrightarrow \phi(H)/H'_m$ is trivial. Then the composite $H_{n-1} \hookrightarrow H \xrightarrow{\phi} \phi(H) \twoheadrightarrow \phi(H)/H'_{m+1}$ determines a nontrivial homomorphism

$H_{n-1} \to H'_m/H'_{m+1}$. Now since $H \overset{\phi}{\to} \phi(H) \twoheadrightarrow \phi(H)/H'_{m+1}$ is surjective, and $H_{n-1} \subset H$ is normal in H, it follows that the image of $H_{n-1} \to H'_m/H'_{m+1}$ is normal in H'_m/H'_{m+1}. On the other hand, it follows from the commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H'_{m+1} & \longrightarrow & H'_m & \longrightarrow & H'_m/H'_{m+1} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta^p_{X/X_{m+1}} & \longrightarrow & \Delta^p_{X/X_m} & \longrightarrow & \Delta^p_{X_{m+1}/X_m} & \longrightarrow & 1 \end{array}$$

that the image of $H'_m/H'_{m+1} \hookrightarrow \Delta^p_{X_{m+1}/X_m}$ is open. Thus, H'_m/H'_{m+1} may be regarded as the maximal pro-p quotient of the fundamental group of a hyperbolic curve over an algebraically closed field, which implies that H'_m/H'_{m+1} is elastic (cf. Proposition 3.16(iv)). In particular, since H_{n-1} is topologically finitely generated, the image of $H_{n-1} \to H'_m/H'_{m+1}$ is open, which implies that $\mathrm{rank}_{\mathbb{Z}_p}((H'_m/H'_{m+1})^{\mathrm{ab}}) \leq \mathrm{rank}_{\mathbb{Z}_p}(H^{\mathrm{ab}}_{n-1})$. Thus, it follows from our assumption that $m = n-1$, i.e., $\phi(H_{n-1}) \subset H'_{n-1}$. Moreover, by applying an argument similar to the above argument to ϕ^{-1}, we conclude that $\phi(H_{n-1}) = H'_{n-1}$. This completes the proof of Claim A. Next, I claim that the following assertion holds:

Claim B: $\phi(\Delta^p_{Y/Y_{n-1}}) = \Delta^p_{X/X_{n-1}}$.

Indeed, if we write N for the intersection of all $\Delta^p_{Y/k}$-conjugates of H_{n-1}, then N is a normal subgroup of $\Delta^p_{Y/k}$. Moreover, since $\Delta^p_{Y/Y_{n-1}}$ is topologically finitely generated (cf. Proposition 3.16(iii)) and normal in $\Delta^p_{Y/k}$, and $H_{n-1} \subset \Delta^p_{Y/Y_{n-1}}$ is open in $\Delta^p_{Y/Y_{n-1}}$, N is a finite intersection of open subgroups of $\Delta^p_{Y/Y_{n-1}}$ of the form $gH_{n-1}g^{-1}$ (where $g \in \Delta^p_{Y/k}$), hence N is also open. Thus, $\Delta^p_{Y/Y_{n-1}}/N \subset \Delta^p_{Y/k}/N$ is a finite subgroup of $\Delta^p_{Y/k}/N$. In particular, since $\Delta^p_{Y_{n-1}/k}$ is torsion-free (cf. Proposition 3.16(iii)), $\Delta^p_{Y/Y_{n-1}}/N \subset \Delta^p_{Y/k}/N$ is the unique maximal torsion subgroup of $\Delta^p_{Y/k}/N$. On the other hand, it follows from Claim A that the image of $N \subset \Delta^p_{Y/k}$ by the isomorphism ϕ is the intersection of all $\Delta^p_{X/k}$-conjugates of H'_{n-1}. Thus, it follows from an argument similar to the above argument that $\Delta^p_{X/X_{n-1}}/\phi(N) \subset \Delta^p_{X/k}/\phi(N)$ is the unique maximal torsion subgroup of $\Delta^p_{X/k}/\phi(N)$. In particular, the image of $\Delta^p_{Y/Y_{n-1}}/N \subset \Delta^p_{Y/k}/N$ by the isomorphism $\Delta^p_{Y/k}/N \overset{\sim}{\to} \Delta^p_{X/k}/\phi(N)$ determined by ϕ is $\Delta^p_{X/X_{n-1}}/\phi(N) \subset \Delta^p_{X/k}/\phi(N)$. Thus, we conclude that $\phi(\Delta^p_{Y/Y_{n-1}}) = \Delta^p_{X/X_{n-1}}$. This completes the proof of Claim B.

It follows from Claim B that ϕ determines an isomorphism $\overline{\phi} : \Delta^p_{Y_{n-1}/k} \overset{\sim}{\to} \Delta^p_{X_{n-1}/k}$. Write $\overline{H}$ for the image of $H \subset \Delta^p_{Y/k}$ by the surjection $\Delta^p_{Y/k} \twoheadrightarrow \Delta^p_{Y_{n-1}/k}$.

For each integer i such that $0 \le i < n$, write, moreover,

$$\overline{H}_i := \overline{H} \cap \Delta^p_{Y_{n-1}/Y_i},$$
$$\overline{H}'_i := \overline{\phi}(\overline{H}) \cap \Delta^p_{X_{n-1}/X_i}.$$

Then, since the inverse image of $\overline{H}_i \subset \Delta^p_{Y_{n-1}/k}$ by the surjection $\Delta^p_{Y/k} \twoheadrightarrow \Delta^p_{Y_{n-1}/k}$ is $H_i \Delta^p_{Y/Y_{n-1}}$, for each integer i such that $0 \le i < n-1$, it holds that $H_i/H_{i+1} \cong \overline{H}_i/\overline{H}_{i+1}$. Similarly, for each integer i such that $0 \le i < n-1$, it holds that $H'_i/H'_{i+1} \cong \overline{H}'_i/\overline{H}'_{i+1}$. Thus, it follows from the induction hypothesis that for each integer i such that $0 \le i < n$, $\overline{\phi}(\Delta^p_{Y_{n-1}/Y_i}) = \Delta^p_{X_{n-1}/X_i}$. On the other hand, for each integer i such that $0 \le i \le n-1$, the image of $\Delta^p_{Y/Y_i} \subset \Delta^p_{Y/k}$ by the surjection $\Delta^p_{Y/k} \twoheadrightarrow \Delta^p_{Y_{n-1}/k}$ is $\Delta^p_{Y_{n-1}/Y_i}$. Thus, since $\overline{\phi}(\Delta^p_{Y_{n-1}/Y_i}) = \Delta^p_{X_{n-1}/X_i}$, the image of $\phi(\Delta^p_{Y/Y_i}) \subset \Delta^p_{X/k}$ by the surjection $\Delta^p_{X/k} \twoheadrightarrow \Delta^p_{X_{n-1}/k}$ is $\Delta^p_{X_{n-1}/X_i}$. In particular, $\phi(\Delta^p_{Y/Y_i})$ is contained in the inverse image of $\Delta^p_{X_{n-1}/X_i}$ by the surjection $\Delta^p_{X/k} \twoheadrightarrow \Delta^p_{X_{n-1}/k}$, which coincides with Δ^p_{X/X_i}. Now, by applying an argument similar to the above argument to ϕ^{-1}, we conclude that $\phi(\Delta^p_{Y/Y_i}) = \Delta^p_{X/X_i}$. This completes the proof of assertion (i).

Finally, we verify assertion (ii). It follows from Proposition 3.16(i) that, for each integer i such that $0 \le i \le n$, ψ induces an isomorphism $\psi_i : \Pi^p_{Y_i/k} \xrightarrow{\sim} \Pi^p_{X_i/k}$ over G_k. By induction on i, to verify assertion (ii), it suffices to verify that the following assertion holds:

Claim C: For each integer i such that $0 \le i < n$, if the isomorphism ψ_i arises from an isomorphism $f_i : Y_i \xrightarrow{\sim} X_i$ over k, then ψ_{i+1} arises from an isomorphism $Y_{i+1} \xrightarrow{\sim} X_{i+1}$ over k.

Indeed, write $\eta \to Y_i$ for the generic point of Y_i, $(Y_{i+1})_\eta := Y_{i+1} \times_{Y_i} \eta$, and $(X_{i+1})_\eta := X_{i+1} \times_{X_i} \eta$ (where $\eta \to X_i$ is the composite $\eta \to Y_i \xrightarrow{f_i} X_i$). Then it follows from Proposition 3.16(ii) that

$$\Pi^p_{(Y_{i+1})_\eta/\eta} \xrightarrow{\sim} \Pi^p_{Y_{i+1}/Y_i} \times_{\Pi^p_{Y_i/Y_i}} \Pi^p_{Y_i \times_{Y_i} \eta/\eta} = \Pi^p_{Y_{i+1}/Y_i} \times_{\Pi_{Y_i}} \Pi_\eta.$$

Moreover, it follows from Remark 3.24(i) that $\Pi^p_{Y_{i+1}/Y_i} \xrightarrow{\sim} \Pi^p_{Y_{i+1}/k} \times_{\Pi^p_{Y_i/k}} \Pi_{Y_i}$, which implies that $\Pi^p_{(Y_{i+1})_\eta/\eta} \xrightarrow{\sim} \Pi^p_{Y_{i+1}/k} \times_{\Pi^p_{Y_i/k}} \Pi_\eta$. Similarly, it holds that $\Pi^p_{(X_{i+1})_\eta/\eta} \xrightarrow{\sim} \Pi^p_{X_{i+1}/k} \times_{\Pi^p_{X_i/k}} \Pi_\eta$. Thus, ψ_{i+1} determines an isomorphism $\Pi^p_{(Y_{i+1})_\eta/\eta} \xrightarrow{\sim} \Pi^p_{(X_{i+1})_\eta/\eta}$ over Π_η. Now it follows from Theorem 4.4 that the isomorphism $\Pi^p_{(Y_{i+1})_\eta/\eta} \xrightarrow{\sim} \Pi^p_{(X_{i+1})_\eta/\eta}$ arises from a dominant morphism $(Y_{i+1})_\eta \to (X_{i+1})_\eta$ over η, which is actually an isomorphism (cf. Lemma 3.37(i)). Write $\xi \to (Y_{i+1})_\eta$ for the generic point of $(Y_{i+1})_\eta \subset Y_{i+1}$. Then it follows from the commutative diagram of profinite groups

$$\begin{array}{ccccc} \Pi_{\xi} & \longrightarrow & \Pi^{p}_{(Y_{i+1})_{\eta}/\eta} & \xrightarrow{\sim} & \Pi^{p}_{(X_{i+1})_{\eta}/\eta} \\ \downarrow & & \downarrow & & \downarrow \\ \Pi^{p}_{\xi/k} & \longrightarrow & \Pi^{p}_{Y_{i+1}/k} & \underset{\psi_{i+1}}{\longrightarrow} & \Pi^{p}_{X_{i+1}/k}, \end{array}$$

together with Lemma 3.34, that ψ_{i+1} arises from a morphism $Y_{i+1} \to X_{i+1}$ over k. Moreover, by applying an argument similar to the above argument to ψ_{i+1}^{-1}, we conclude that ψ_{i+1}^{-1} arises from a morphism $X_{i+1} \to Y_{i+1}$ over k. Then it follows from Proposition 4.2(i) that the two morphisms $Y_{i+1} \to X_{i+1}$ and $X_{i+1} \to Y_{i+1}$ are inverse to each other. Thus, $Y_{i+1} \xrightarrow{\sim} X_{i+1}$ is an isomorphism. This completes the proof of Claim C, hence also of assertion (ii). □

Theorem 5.4. *Let p be a prime number, n a positive integer, k a sub-p-adic field and X, Y hyperbolic polycurves of dimension n over k satisfying condition $(*)_p$. For each integer i such that $1 \leq i \leq n$, write (g_i, r_i) for the type of the hyperbolic curve X_i/X_{i-1}, and (g'_i, r'_i) for the type of the hyperbolic curve Y_i/Y_{i-1}. Suppose that, for any integers i, j such that $0 \leq i < j < n$,*

$$2g_i + \max\{r_i - 1, 0\} > 2g'_j + \max\{r'_j - 1, 0\},$$
$$2g'_i + \max\{r'_i - 1, 0\} > 2g_j + \max\{r_j - 1, 0\}.$$

Then the natural map

$$\mathrm{Isom}_k(Y, X) \to \mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k}) / \mathrm{Inn}(\Delta^p_{X/k})$$

is bijective, i.e., every isomorphism $\Pi^p_{Y/k} \xrightarrow{\sim} \Pi^p_{X/k}$ over G_k arises from a uniquely determined isomorphism $Y \to X$ over k.

Proof. (Cf. [10, Thm. 4.3].) The injectivity of the map in question holds from Proposition 4.2(i). The surjectivity of the map in question follows from Lemma 5.3 (where we take "H" to be $\Delta^p_{Y/k}$), together with Proposition 3.16(v). □

Proposition 5.5 ([10, Prop. 4.5]). *Let S, Y be integral varieties over k, $Y \to S$ a dominant morphism over k and X a hyperbolic polycurve over S. Then the set* $\mathrm{Hom}^{\mathrm{dom}}_S(Y, X)$ *of dominant morphisms from Y to X over S is finite.*

Theorem 5.6. *Let p be a prime number, k a sub-p-adic field and X, Y hyperbolic polycurves over k. Suppose that at least one of X/k, Y/k satisfies condition $(*)_p$. Then the set*

$$\mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k}) / \mathrm{Inn}(\Delta^p_{X/k})$$

is finite.

Proof. (Cf. [10, Thm. 4.4].) If $\mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k}) = \emptyset$, then Theorem 5.6 is immediate. Thus, to verify Theorem 5.6, we may assume without loss of generality that $\mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k}) \neq \emptyset$. Then any element of $\mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k})$ determines a bijection between $\mathrm{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k})/\mathrm{Inn}(\Delta^p_{X/k})$ and $\mathrm{Aut}_{G_k}(\Pi^p_{X/k})/\mathrm{Inn}(\Delta^p_{X/k})$. Thus, to verify Theorem 5.6, we may assume without loss of generality that $X = Y$ and X/k satisfies condition $(*)_p$. Let $H \subset \Delta^p_{X/k}$ be an open subgroup of $\Delta^p_{X/k}$ that satisfies the condition appearing in the statement of Lemma 5.2. Then, by applying Lemma 5.1, where we take the data "(G, H, A)" to be $(\Delta^p_{X/k}, H, \mathrm{Aut}_{G_k}(\Pi^p_{X/k}))$ (note that there exists a natural homomorphism $\mathrm{Aut}_{G_k}(\Pi^p_{X/k}) \to \mathrm{Aut}(\Delta^p_{X/k})$), we conclude that there exists a subgroup $A \subset \mathrm{Aut}_{G_k}(\Pi^p_{X/k})$ of $\mathrm{Aut}_{G_k}(\Pi^p_{X/k})$ of finite index such that each $\phi \in A$ preserves $H \subset \Delta^p_{X/k}$. Then it follows from Lemma 5.3(ii) that every element of A arises from an automorphism of X over k, i.e., the image of the composite

$$A \hookrightarrow \mathrm{Aut}_{G_k}(\Pi^p_{X/k}) \twoheadrightarrow \mathrm{Aut}_{G_k}(\Pi^p_{X/k})/\mathrm{Inn}(\Delta^p_{X/k})$$

is contained in the image of the natural injection $\mathrm{Aut}_k(X) \hookrightarrow \mathrm{Aut}_{G_k}(\Pi^p_{X/k})/\mathrm{Inn}(\Delta^p_{X/k})$ (cf. Proposition 4.2(i)). On the other hand, $\mathrm{Aut}_k(X)$, hence also the image of the composite

$$A \hookrightarrow \mathrm{Aut}_{G_k}(\Pi^p_{X/k}) \twoheadrightarrow \mathrm{Aut}_{G_k}(\Pi^p_{X/k})/\mathrm{Inn}(\Delta^p_{X/k})$$

is finite (cf. Proposition 5.5). Thus, it follows from our choice of A that $\mathrm{Aut}_{G_k}(\Pi^p_{X/k})/\mathrm{Inn}(\Delta^p_{X/k})$ is finite. This completes the proof of Theorem 5.6. ☐

Corollary 5.7. *Let p be a prime number, k_X, k_Y finite extensions of $\mathbb{Q}$ and X, Y hyperbolic polycurves over k_X, k_Y, respectively. Suppose that at least one of X/k_X, Y/k_Y satisfies condition $(*)_p$. Then the set*

$$\mathrm{Isom}(\Pi^p_{Y/k_Y}, \Pi^p_{X/k_X})/\mathrm{Inn}(\Pi^p_{X/k_X})$$

is finite.

Proof. (Cf. [10, Cor. 4.6].) It follows from an argument similar to the argument used at the beginning of the proof of Theorem 5.6 that to verify Corollary 5.7, we may assume without loss of generality that $X = Y$, and X satisfies condition $(*)_p$. Then, for each $\phi \in \mathrm{Aut}(\Pi^p_{X/k_X})$, the image of the composite $\Delta^p_{X/k_X} \hookrightarrow \Pi^p_{X/k_X} \stackrel{\phi}{\to} \Pi^p_{X/k_X} \twoheadrightarrow G_{k_X}$ is a topologically finitely generated normal closed subgroup of G_{k_X}, hence trivial (cf. Proposition 4.24(i)). Thus, since $\Pi^p_{X/k_X}/\Delta^p_{X/k_X} \xrightarrow{\sim} G_{k_X}$, there exists a unique homomorphism $G_{k_X} \to G_{k_X}$ such that

$$\begin{array}{ccc} \Pi^p_{X/k_X} & \xrightarrow{\phi} & \Pi^p_{X/k_X} \\ \downarrow & & \downarrow \\ G_{k_X} & \longrightarrow & G_{k_X} \end{array}$$

is commutative. Moreover, by applying an argument similar to the above argument to ϕ^{-1}, we conclude that the homomorphism $G_{k_X} \to G_{k_X}$ is an isomorphism. Thus, we have a natural exact sequence

$$1 \to \mathrm{Aut}_{G_{k_X}}(\Pi^p_{X/k_X}) \to \mathrm{Aut}(\Pi^p_{X/k_X}) \to \mathrm{Aut}(G_{k_X}).$$

Write $N \subset \mathrm{Out}(\Pi^p_{X/k_X})$ for the image of $\mathrm{Aut}_{G_{k_X}}(\Pi^p_{X/k_X}) \subset \mathrm{Aut}(\Pi^p_{X/k_X})$ by $\mathrm{Aut}(\Pi^p_{X/k_X}) \twoheadrightarrow \mathrm{Out}(\Pi^p_{X/k_X})$. Then since $\Pi^p_{X/k_X} \to G_{k_X}$ is surjective, the sequence

$$1 \to N \to \mathrm{Out}(\Pi^p_{X/k_X}) \to \mathrm{Out}(G_{k_X})$$

induced by the above exact sequence is exact. Thus, to verify Corollary 5.7, it suffices to verify that N and $\mathrm{Out}(G_{k_X})$ are finite. Now since $\mathrm{Aut}_{G_{k_X}}(\Pi^p_{X/k_X})/\mathrm{Inn}(\Delta^p_{X/k_X})$ is finite (cf. Theorem 5.6), it follows that N is finite. Finally, we verify the finiteness of $\mathrm{Out}(G_{k_X})$. It follows from Proposition 4.24(ii) that the natural map

$$\mathrm{Isom}(\overline{k_X}/k_X, \overline{k_X}/k_X) \ni \varphi \mapsto (G_{k_X} \ni \sigma \mapsto \varphi\sigma\varphi^{-1} \in G_{k_X}) \in \mathrm{Aut}(G_{k_X})$$

is bijective. Let $f, g \in \mathrm{Aut}(G_{k_X})$. Then, if we write $\varphi_f, \varphi_g \in \mathrm{Isom}(\overline{k_X}/k_X, \overline{k_X}/k_X)$ for the element of $\mathrm{Isom}(\overline{k_X}/k_X, \overline{k_X}/k_X)$ corresponding to f, g, respectively, then one verifies easily that f and g are G_{k_X}-conjugate if and only if $\varphi_f|_{k_X} = \varphi_g|_{k_X}$. Thus, it holds that $\mathrm{Out}(G_{k_X}) \cong \mathrm{Aut}(k_X)$, which implies that $\mathrm{Out}(G_{k_X})$ is finite. This completes the proof of Corollary 5.7. □

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