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Extended Affine Root Supersystems of Types $C(I, J)$ and $BC(1, 1)$

by

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Abstract

In this paper, we complete the characterization of tame irreducible extended affine root supersystems. We give a complete description of tame irreducible extended affine root supersystems of type $X = C(1,1), C(1,2)$, $C(2,2)$ and $BC(1,1)$ and determine isomorphic classes.

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§1. Introduction

The notion of locally finite root supersystems was introduced in [8]; this is a generalization of the two notions of locally finite root systems [3], as well as generalized root systems [4]. More precisely, a symmetric spanning set R of a nontrivial vector space V equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ is called a locally finite root supersystem if

- $0 \in R$,
- for $\alpha \in R$ with $(\alpha,\alpha) \neq 0$ and $\beta \in R$, $2(\beta,\alpha)/(\alpha,\alpha) \in \mathbb{Z}$ and $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in R$,
- the root string property is satisfied for R and
- for $\alpha, \beta \in R$ with $(\alpha,\alpha) = 0$ and $(\alpha,\beta) \neq 0$, $\{\beta-\alpha, \beta+\alpha\} \cap R \neq \emptyset$.

The root system of a basic classical simple Lie superalgebra is an example of a locally finite root supersystem. Irreducible locally finite root supersystems are classified and known as types $\dot{A}(I,J)$, $B(I,J)$, $C(I,J)$, $D(I,J)$ and $BC(I,J)$, together with the root systems of basic classical simple Lie superalgebras; see [8].

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Locally finite root supersystems have a close connection with the super version of affine Lie algebras called affine Lie superalgebras, which was introduced and classified by Van de Leur in 1986 [6]. An affine Lie superalgebra $\mathcal{L}$ is equipped with a nondegenerate invariant supersymmetric bilinear form and has a weight space decomposition with respect to a finite-dimensional abelian subalgebra $\mathcal{H} \subseteq \mathcal{L}_0$ on which the form is nondegenerate. This allows the transfer of the form on $\mathcal{L}$ to a nondegenerate bilinear form $(\cdot,\cdot)$ on the dual space $\mathcal{H}^*$ of $\mathcal{H}$ and the ability to divide nonzero roots of the root system R of $\mathcal{L}$ (with respect to $\mathcal{H}$) into three parts:

- $R_{\rm re}^{\times}$ (nonzero real roots), consisting of those roots α with $(\alpha,\alpha) \neq 0$,
- $R_{\rm im}^{\times}$ (nonzero imaginary roots), consisting of all nonzero roots α with $(\alpha, R) = \{0\}$ and
- $R_{\rm ns}^{\times}$ (nonzero nonsingular roots) consisting of nonzero roots which do not belong to $R_{\rm re}^{\times} \cup R_{\rm im}^{\times}$.

The set of imaginary roots generates a free abelian group $\mathbb{Z}\delta$ of rank 1 and the root system R up to $\mathbb{Z}\delta$ is just a locally finite root supersystem. This motivated us in 2016 to introduce a combinatorial object, called an extended affine root supersystem; see [7]. An extended affine root supersystem R is a specific subset of a vector space and is divided into three parts: $R_{\rm re}$ (real roots), $R_{\rm ns}$ (nonsingular roots) and R^0 (isotropic roots). Up to the group generated by isotropic roots, the extended affine root supersystem R is just a locally finite root supersystem, say $\dot{R}$; see Definition 2.1 for the precise definition. We say R is of type X if $\dot{R}$ is of type X and call it tame if $R^0 \subseteq (R^{\times} - R^{\times})$, where $R^{\times} := (R_{\rm re} \cup R_{\rm ns}) \setminus \{0\}$.

In 2017, another combinatorial object, called an affine generalized reflection root system, was introduced in [2] and the irreducible ones were characterized. Each irreducible affine generalized reflection system is of the form $S^{\times} = S_{\rm re} \cup S_{\rm ns} \setminus \{0\}$ for a complex infinite tame irreducible extended affine root supersystem S satisfying

- $\mathbb{Z}S^{\times} \otimes_{\mathbb{Z}} \mathbb{C}$ is naturally isomorphic to $\text{span}_{\mathbb{C}}\, S^{\times}$,
- if $\alpha, \beta \in S_{\rm ns}^{\times}$ and $\alpha + \beta \in S$, then $\alpha - \beta \notin S$.

There are examples of tame irreducible extended affine root supersystems which do not satisfy the above two conditions; see e.g., Example 3.2 and [2, §1.2.6]. The main goal is finding descriptions of all tame irreducible extended affine root supersystems.

For each irreducible extended affine root supersystem S of type X, there are a locally finite root supersystem $\dot{S}$ of type X and nonempty subsets $S_{\dot{\alpha}}$ $(\dot{\alpha} \in \dot{S})$

of the radical of the form $(\cdot,\cdot)$ on the underlying vector space such that

$$S = \bigcup_{\dot\alpha\in\dot S}(\dot\alpha + S_{\dot\alpha}). \tag{1.1}$$

To get a description of S, we need to know the interactions between the $S_{\dot\alpha}$. In [7], we obtained these interactions for types $X \neq B(0,I)$, $C(2,2)$, $C(1,2)$, $A(\ell,\ell)$ and $BC(1,1)$ which help us to find a description of tame irreducible extended affine root supersystems of the corresponding types. But for type $A(\ell,\ell)$, the interactions are not sufficiently explicit to be investigated directly. One of the difficulties that occurs in finding these interactions for type $A(\ell,\ell)$ is that in contrast with other types, if $\dot\alpha$ is a nonsingular root of $\dot S$, then $S_{\dot\alpha}$ in (1.1), can be not equal to $S_{-\dot\alpha}$. Moreover, for type $A(1,1)$, this phenomenon can even happen for real roots $\dot\alpha \in \dot S$.

Depending on $\ell = 1$ or $\ell \neq 1$, we need different techniques to study type $A(\ell,\ell)$. More precisely, if R is a tame irreducible extended affine root supersystem of type $A(\ell,\ell)$ $(\ell \neq 1)$ in an $\mathbb{F}$-vector space V, we extend V by a 1-dimensional vector space $\mathbb{F}\delta$ and use R to define a new extended affine root supersystem T in the new vector space $V \oplus \mathbb{F}\delta$. Then we describe T instead of R, but T is defined in a way that up to $\mathbb{F}\delta$, it is just R and so we get a description of R by making the quotient on $\mathbb{F}\delta$; see [5] for the details. This technique does not work for $\ell = 1$; see [5, Prop. 2.4(i)].

In this paper, we focus on type $A(1,1) = C(1,1)$. We first give two kinds of examples of extended affine root supersystems (Examples 3.1, 3.2) and then prove that each tame irreducible extended affine root supersystem of type $C(1,1)$ has the expression stated in these examples. Moreover, we complete the study of extended affine root supersystems by giving descriptions of remainder types $C(1,2)$, $C(2,2)$ and $BC(1,1)$. We also determine the isomorphism classes.

§2. Extended affine root supersystems

Throughout this paper, $\mathbb{F}$ is a field of characteristic zero and all vector spaces are defined on $\mathbb{F}$.

Definition 2.1 ([7]). Suppose that V is a nontrivial vector space, S is a subset of V and $(\cdot,\cdot)\colon V\times V \longrightarrow \mathbb{F}$ is a symmetric bilinear form with radical V^0. Set

$$S^0 := S\cap V^0, \qquad S^\times := S\setminus S^0,$$
$$S^\times_{\mathrm{re}} := \{\alpha\in S \mid (\alpha,\alpha)\neq 0\}, \qquad S_{\mathrm{re}} := S^\times_{\mathrm{re}}\cup\{0\},$$
$$S^\times_{\mathrm{ns}} := \{\alpha\in S\setminus S^0 \mid (\alpha,\alpha)= 0\}, \qquad S_{\mathrm{ns}} := S^\times_{\mathrm{ns}}\cup\{0\}.$$

We say $(V,(\cdot,\cdot),S)$ is an *extended affine root supersystem* if the following hold:

(S1) $0\in S$ and S spans V,

(S2) $S=-S$,

(S3) for $\alpha\in S^\times_{re}$ and $\beta\in S$, $2(\alpha,\beta)/(\alpha,\alpha)\in\mathbb{Z}$,

(S4) (root string property) for $\alpha\in S^\times_{re}$ and $\beta\in S$, there are nonnegative integers p,q such that $\{k\in\mathbb{Z}\mid \beta+k\alpha\in S\}=\{-p,\dots,q\}$ and $2(\beta,\alpha)/(\alpha,\alpha)=p-q$,

(S5) for $\alpha\in S_{ns}$ and $\beta\in S$ with $(\alpha,\beta)\neq 0$, $\{\beta-\alpha,\beta+\alpha\}\cap S\neq\emptyset$.

If there is no ambiguity, we say S is an extended affine root supersystem in V. Elements of S^0 are called *isotropic roots,* elements of S_{re} are called *real roots* and elements of S_{ns} are called *nonsingular roots.* The extended affine root supersystem S is called *tame* if $S^0\subseteq S^\times - S^\times$. An extended affine root supersystem S is called *irreducible* if $S_{re}\neq\{0\}$ and $S^\times$ cannot be written as a disjoint union of two nonempty orthogonal subsets. The extended affine root supersystem S is called a *locally finite root supersystem* if the form $(\cdot,\cdot)$ is nondegenerate. A locally finite root supersystem S is called a *finite root supersystem* if S is finite and it is called a *locally finite root system* if $S_{ns}=\{0\}$. We say an extended affine root supersystem $(V,(\cdot,\cdot),R)$ is isomorphic to another extended affine root supersystem $(V',(\cdot,\cdot)',R')$ and write $R\simeq R'$ if there are a linear isomorphism $\varphi\colon V\longrightarrow V'$ and a nonzero scalar r such that $\varphi(R)=R'$ and $r(x,y)=(\varphi(x),\varphi(y))'$.

Remark 2.2. Suppose that $(V,(\cdot,\cdot),S)$ satisfies (S1)–(S3) and (S5); then using [7, Prop. 1.11] and the same argument as in [7, Prop. 2.1], we get that S satisfies (S4) if and only if it satisfies the following:

- $(\overline{V},(\cdot,\cdot)^-,\bar{S})$ is a locally finite root supersystem, in which $\overline{V}$ is the quotient of V over the radical V^0 of the form $(\cdot,\cdot)$ and $(\cdot,\cdot)^-$ is the induced form on $\overline{V}$,
- for $\alpha\in S^\times_{re}$, the reflection
$$r_\alpha\colon V\longrightarrow V,\qquad v\mapsto v-\frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha\quad (v\in V)$$
preserves S.

In what follows, we give the classification of irreducible locally finite root supersystems of [8]. Suppose $\dot{U}$ is a vector space with a basis $\{\eta_1,\eta_2,\eta_3\}$. For $\lambda\in\mathbb{F}\setminus\{0,-1\}$, define the symmetric nondegenerate bilinear form $(\cdot,\cdot)$ on $\dot{U}$ by the linear extension

$$(\eta_1,\eta_1):=\lambda,\quad (\eta_2,\eta_2)=-1-\lambda,\quad (\eta_3,\eta_3):=1,\qquad (\eta_i,\eta_j)=0\quad (1\le i\neq j\le 3). \tag{2.1}$$

Define

$$D(2,1;\lambda)=\{0,\ \pm 2\eta_i,\ \pm\eta_1\pm\eta_2\pm\eta_3 \mid 1\le i\le 3\}. \tag{2.2}$$

Next suppose I and J are two index sets with $I\cup J\neq\emptyset$ and $\dot{U}$ (by abuse of notation) is a vector space with a basis $\{\epsilon_i,\delta_j \mid i\in I,\ j\in J\}$. Define a symmetric bilinear form $(\cdot,\cdot)\colon \dot{U}\times\dot{U}\longrightarrow\mathbb{F}$ with

$$(\epsilon_i,\epsilon_r):=\delta_{i,r},\quad (\delta_j,\delta_s):=-\delta_{j,s}\quad\text{and}\quad(\epsilon_i,\delta_j)=0\quad(i,r\in I,\ j,s\in J).$$

Set[1]

$$\begin{aligned}
\dot{A}(I,I)&:=\pm\{\epsilon_i-\epsilon_r,\delta_i-\delta_r,\epsilon_i-\delta_r-\tfrac{1}{\ell}\textstyle\sum_{k\in I}(\epsilon_k-\delta_k)\mid i,r\in I\}\\
&\qquad(\ell:=|I|\in\mathbb{Z}^{\ge 2}),\\
\dot{A}(I,J)&:=\pm\{\epsilon_i-\epsilon_r,\delta_j-\delta_s,\epsilon_i-\delta_j\mid i,r\in I,\ j,s\in J\}\\
&\qquad(|I|\neq|J|\text{ if } I,\ J \text{ are finite sets}),\\
B(I,J)&:=\pm\{\epsilon_i,\delta_j,\epsilon_i\pm\epsilon_r,\delta_j\pm\delta_s,\epsilon_i\pm\delta_j\mid i,r\in I,\ j,s\in J,\ i\neq r\},\\
C(I,J)&:=\pm\{\epsilon_i\pm\epsilon_r,\delta_j\pm\delta_s,\epsilon_i\pm\delta_j\mid i,r\in I,\ j,s\in J\},\\
D(I,J)&:=\pm\{\epsilon_i\pm\epsilon_r,\delta_j\pm\delta_s,\epsilon_i\pm\delta_j\mid i,r\in I,\ j,s\in J,\ i\neq r\},\\
BC(I,J)&:=\pm\{\epsilon_i,\delta_j,\epsilon_i\pm\epsilon_r,\delta_j\pm\delta_s,\epsilon_i\pm\delta_j\mid i,r\in I,\ j,s\in J\},\\
F(4)&:=\pm\{0,\epsilon,\delta_i\pm\delta_j,\delta_i,\tfrac{1}{2}(\epsilon\pm\delta_1\pm\delta_2\pm\delta_3)\mid 1\le i\neq j\le 3\}\\
&\qquad(I=\{1\},\ J=\{1,2,3\},\ \epsilon:=\sqrt{3}\epsilon_1),\\
G(3)&:=\pm\{0,\nu,2\nu,\epsilon_i-\epsilon_j,2\epsilon_i-\epsilon_j-\epsilon_t,\nu\pm(\epsilon_i-\epsilon_j)\mid\{i,j,t\}=\{1,2,3\}\}\\
&\qquad(I=\{1,2,3\},\ J=\{1\},\ \nu:=\sqrt{2}\delta_1),
\end{aligned}\tag{2.3}$$

in which if I or J is empty, the corresponding indices disappear. We mention that the $\mathbb{F}$-linear spans of all these sets are $\dot{U}$ except for $\dot{A}(I,J)$, so to denote this type, we use $\dot{A}$ instead of A. If X is one of the sets introduced in (2.2) and (2.3), then X is an irreducible locally finite root supersystem, in its linear span, called the irreducible locally finite root supersystem of type X. Up to isomorphism, each irreducible locally finite root supersystem is either an irreducible finite root system or one of the locally finite root supersystems introduced in (2.2) or (2.3); see [3], [4] and [8].

In the sequel, if either I or J is a finite set, we may replace it by its cardinality in each type, e.g., we may denote $B(I,J)$ by $B(|I|,|J|)$ if I and J are finite sets. We should point out that our notation has a minor difference compared with the notation in the literature; more precisely, $D(1,n)$ for $n\in\mathbb{Z}^{\ge 1}$ and $\dot{A}(m,n)$ for

[1] We denote the cardinal number of a set A by $|A|$.

$m, n \in \mathbb{Z}^{\geq 1}$ in our sense are denoted by $C(n+1)$ and $A(m-1, n-1)$ respectively in the literature.

The real part of an irreducible locally finite root supersystem $(V, (\cdot,\cdot), S)$ is a locally finite root system; in fact, we have

$$S_{\rm re} = \bigcup_{i=1}^{n} S^i,$$

where $n \in \{1,2,3\}$, each S^i is an irreducible locally finite root system and $(S^i, S^j) = \{0\}$ for $i \neq j$. Moreover, we have

$$S^i = \{0\} \cup S^i_{\rm sh} \cup S^i_{\rm lg} \cup S^i_{\rm ex},$$

where

$$\begin{aligned} S^i_{\rm sh} &:= \big\{\alpha \in S^i \setminus \{0\} \mid (\alpha,\alpha) \leq (\beta,\beta)\ \forall\, \beta \in S^i \setminus \{0\}\big\},\\ S^i_{\rm ex} &:= 2S^i_{\rm sh} \cap S^i,\\ S^i_{\rm lg} &= S^i \setminus (\{0\} \cup S^i_{\rm ex} \cup S^i_{\rm sh}), \end{aligned}$$

which are called the sets of short, extra-long and long roots of S^i, respectively. We set

$$(S_{\rm re})_* := \bigcup_{i=1}^{n} S^i_* \quad (* = {\rm sh}, {\rm lg}, {\rm ex}). \tag{2.4}$$

Next assume $\bar{\ }: V \longrightarrow \overline{V} := V/V^0$ is the canonical projection map. Then the form induces a form on $\overline{V}$ and $\bar{R}$ is an irreducible locally finite root supersystem in $\overline{V}$; see [7, Prop. 1.11]. We say R is of *type* X if $\bar{R}$ is of type X.

Pick a subset $\Pi = \{\upsilon_i \mid i \in I\} \subseteq R$ such that $\overline{\Pi} = \{\bar{\upsilon}_i \mid i \in I\}$ is a basis for $\overline{V}$ and set

$$\dot{V} := {\rm span}_{\mathbb{F}} \Pi.$$

Then we have

$$V = \dot{V} \oplus V^0$$

and that

$$\dot{R} := \{\dot{\alpha} \in \dot{V} \mid \dot{\alpha} + \sigma \in R \text{ for some } \sigma \in V^0\}$$

is an irreducible locally finite root supersystem in $\dot{V}$ isomorphic to $\bar{R}$. We mention that

$$\Pi \subseteq R \cap \dot{R}. \tag{2.5}$$

Setting

$$S_{\dot{\alpha}} := \big\{\sigma \in V^0 \mid \dot{\alpha} + \sigma \in R\big\} \quad (\dot{\alpha} \in \dot{R}),$$

we have

$$R = \bigcup_{\dot\alpha\in\dot R}(\dot\alpha + S_{\dot\alpha}). \tag{2.6}$$

To get a description of R, one needs to know the interaction between the $S_{\dot\alpha}$. Although, these interactions depend on the type of R and the choice of Π, we have the following four general facts:

Fact 1. For $\dot\alpha \in \dot R^\times_{\mathrm{re}}$ and $\dot\beta \in \dot R$, we have

$$S_{\dot\beta} - \frac{2(\dot\alpha,\dot\beta)}{(\dot\alpha,\dot\alpha)}S_{\dot\alpha} \subseteq S_{r_{\dot\alpha}(\dot\beta)} = S_{\dot\beta - \frac{2(\dot\beta,\dot\alpha)}{(\dot\alpha,\dot\alpha)}\dot\alpha}.$$

This follows from the fact that

$$r_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in R \quad (\alpha \in R^\times_{\mathrm{re}},\ \beta\in R).$$

Fact 2. If $\dot\alpha \in \dot R^\times_{\mathrm{re}}$ and $0 \in S_{\dot\alpha}$, we have $S_{\dot\beta} = S_{r_{\dot\alpha}(\dot\beta)}$ for each $\dot\beta \in \dot R$. This follows from Fact 1.

Fact 3. Suppose $\dot\alpha,\dot\beta \in \dot R^\times$ with $\dot\alpha + \dot\beta \notin \dot R$ and $(\dot\alpha,\dot\beta) \neq 0$; then $\dot\alpha - \dot\beta \in \dot R$ and $S_{\dot\alpha} - S_{\dot\beta} \subseteq S_{\dot\alpha-\dot\beta}$. For $\sigma \in S_{\dot\alpha}$, $\tau \in S_{\dot\beta}$ and $\alpha := \dot\alpha + \sigma$, $\beta := \dot\beta + \tau \in R$, since $(\dot\alpha,\dot\beta) \neq 0$ and $\dot\alpha + \dot\beta \notin \dot R$, we have using (S5) and (S4) that $\alpha - \beta \in R$ and $\dot\alpha - \dot\beta \in \dot R$. So $\sigma - \tau \in S_{\dot\alpha-\dot\beta}$. This means that $S_{\dot\alpha} - S_{\dot\beta} \subseteq S_{\dot\alpha-\dot\beta}$.

Fact 4. For $\dot\alpha \in \dot R$, $S_{-\dot\alpha} = -S_{\dot\alpha}$. This follows easily from (S2).

Using these facts together with some technical points, some descriptions of all tame irreducible extended affine root supersystems except for types $\dot A(1,1) \simeq C(1,1)$, $B(0,I)$, $C(2,J) \simeq C(J,2)$ and $BC(1,1)$, are given in [7]. In this work, we deal with these remainder types. Regarding type $C(I,J)$ with $(|I|,|J|) \neq (1,1)$, we give, in general, a description of type $C(I,J)$ for two nonempty sets I and J with $|I| \geq 1$ and $|J| \geq 2$.

Locally finite root supersystems $B(I,0)$, $B(0,I) = BC(0,I) \simeq BC(I,0)$, $C(I,0) \simeq C(0,I) = D(0,I)$ and $D(I,0)$ are locally finite root systems known as types B_I, BC_I, C_I and D_I respectively. Moreover, $BC_I \simeq B(0,I)$ appears as the root system of some Lie superalgebra with nonzero odd part; namely, $B(0,n)$ is the root system of basic classical simple Lie superalgebra $B(0,n)$.

The following theorem is proved as in [1, (2.18), Prop. 2.23 & Thm. 3.1]:

Theorem 1. *Suppose that $(\dot{U},(\cdot,\cdot),\dot{R})$ is an irreducible locally finite root supersystem of type $B(0,I)=BC_I$ for a nonempty set I and U is a vector space. Let L (if $|I|\geq 2$), S and E be nonempty subsets of U satisfying*

$$
\begin{aligned}
&0\in S,\ \ S-2S\subseteq S,\ \ E-2E\subseteq E \quad \textit{and} \quad 0\in L,\ \ L-2L\subseteq L \ \ (\textit{if } |I|\geq 2),\\
&L+L\subseteq L \ \ (\textit{if } |I|\geq 3) \quad \textit{and} \quad \operatorname{span}_{\mathbb{F}} S=U,\\
(\dagger)\quad &S+L\subseteq S,\ \ L+2S\subseteq L \ \ (\textit{if } |I|\geq 2),\\
&S+E\subseteq S,\ \ E+4S\subseteq E \ \ (\textit{if } |I|=1),\\
&L+E\subseteq L,\ \ E+2L\subseteq E \ \ (\textit{if } |I|\geq 2).
\end{aligned}
$$

Extend $(\cdot,\cdot)$ to a form on $\dot{U}\oplus U$ such that U is the radical of this new form and set

$$
R=\dot{R}(S,L,E):=(S-S)\cup(\dot{R}_{\mathrm{sh}}+S)\cup(\dot{R}_{\mathrm{lg}}+L)\cup(\dot{R}_{\mathrm{ex}}+E), \tag{2.7}
$$

where if $|I|=1$, the part $\dot{R}_{\mathrm{lg}}+L$ disappears. Then $(\dot{U}\oplus U,(\cdot,\cdot),R)$ is a tame irreducible extended affine root supersystem of type $B(0,I)$ in $\dot{U}\oplus U$. Conversely, each tame irreducible extended affine root supersystem of type $BC_I=B(0,I)$ has an expression as in (2.7). Moreover, if V is a vector space with subspaces S', L', E' satisfying the same conditions as in ($\dagger$), $\dot{R}(S,L,E)$ is isomorphic to $\dot{R}(S',L',E')$ if and only if there are $\tau'\in L'$ (if $|I|\geq 2$), $\sigma'\in S'$ and a linear isomorphism $\psi\colon U\longrightarrow V$ with

$$
\psi(L)=L'+\tau' \quad (\textit{if } |I|\geq 2), \quad \psi(S)=S'+\sigma' \quad \textit{and} \quad \psi(E)=E'+2\sigma'.
$$

§3. Type $C(1,1)$

Suppose that $(\dot{U},(\cdot,\cdot),\dot{R})$ is a finite root supersystem of type $C(1,1)$ and U is a vector space. We know from (2.3) that $\dot{R}_{\mathrm{re}}$ is the direct sum of two irreducible finite root systems of type A_1, say $\dot{R}_{\mathrm{re}}=\dot{R}_1\oplus\dot{R}_2$ with

$$
\dot{R}_1=\{0,\pm 2\epsilon\}, \quad \dot{R}_2=\{0,\pm 2\delta\}, \quad \dot{R}_{\mathrm{ns}}=\{\pm\epsilon\pm\delta\}
$$

and

$$
(\epsilon,\delta)=0 \quad \text{and} \quad (\epsilon,\epsilon)=1=-(\delta,\delta).
$$

Set

$$
\dot{R}_1^{+}:=\{2\epsilon\}.
$$

Extend the form on $\dot{U}$ to a symmetric bilinear form on $\dot{U}\oplus U$ such that U is the radical of this new form.

Example 3.1. Suppose that K is a subgroup of U and E, F and T are nonempty subsets of K such that

- $K = E \cup F$ and $\operatorname{span}_{\mathbb{F}} K = U$,
- $0 \in F$, $F - 2F \subseteq F$ and $E - 2E \subseteq -E$,
- $(E - E) \cup (F - F) \subseteq T = -T \subseteq K$.

Then we claim that

$$U = \operatorname{span}_{\mathbb{F}} T$$

and that

$$\begin{aligned} R = \dot{R}(T, E, F, K) &:= T \cup \pm(\dot{R}_1^+ + E) \cup (\dot{R}_2^\times + F) \cup (\dot{R}_{\mathrm{ns}}^\times + K) \\ &= T \cup \pm(2\epsilon + E) \cup (\pm 2\delta + F) \cup (\pm\epsilon \pm \delta + K) \end{aligned}$$

is a tame irreducible extended affine root supersystem of type $C(1,1)$ in $\dot{U} \oplus U$.

We first note that $R^0 = T \subseteq K = K - K = (\epsilon+\delta+K)-(\epsilon+\delta+K) \subseteq R^\times - R^\times$. So, using Remark 2.2, we get that R is a tame irreducible extended affine root supersystem if we verify property (S5) of an extended affine root supersystem for R. To this end, suppose that $\alpha \in R_{\mathrm{ns}}^\times = \pm\epsilon \pm \delta + K$ and $\beta \in R$ with $(\alpha, \beta) \neq 0$. Since K is a group and $K = E \cup F$, if $\beta \in R_{\mathrm{re}}$, it is trivial that either $\alpha + \beta \in R$ or $\alpha - \beta \in R$. So we assume $\alpha, \beta \in R_{\mathrm{ns}}$. Since $(\alpha, \beta) \neq 0$, we have $\alpha = t\epsilon + t'\delta + \tau_1$ and $\beta = r\epsilon + r'\delta + \tau_2$ for some $\tau_1, \tau_2 \in K$ and $t, t', r, r' = \pm 1$ with $tt' = -rr'$.

Since $\alpha + \beta = \alpha - (-\beta)$ and $R = -R$, replacing α with $-\alpha$ and β with $-\beta$ if necessary, we may assume $\alpha = \epsilon + r\delta + \tau_1$ and $\beta = \epsilon - r\delta + \tau_2$ for some $r = \pm 1$ and $\tau_1, \tau_2 \in K$. Note that

$$\text{if } \tau_1, \tau_1 - \tau_2 \in E \Longrightarrow \tau_1 + \tau_2 = (\tau_2 - \tau_1) + 2\tau_1 \in -E + 2E \subseteq E,$$

which in turn gives that

$$\tau_1 + \tau_2 \notin E \xrightarrow{K = E \cup F} \tau_1 \in F \text{ or } \tau_1 - \tau_2 \in F.$$

So we have the following cases:

- $\tau_1 + \tau_2 \in E$. Then $\alpha + \beta = 2\epsilon + (\tau_1 + \tau_2) \in R$ and we are done.
- $\tau_1 + \tau_2 \notin E$ and $\tau_1 - \tau_2 \in F$. Then $\alpha - \beta = 2r\delta + (\tau_1 - \tau_2) \in R$ as we desired.
- $\tau_1 + \tau_2 \notin E$ and $\tau_1 \in F$. As $\tau_1 + \tau_2 \in K + K \subseteq K = E \cup F$, we get $\tau_1 + \tau_2 \in F$. So we have
$$\tau_1 - \tau_2 = -(\tau_1 + \tau_2) + 2\tau_1 \in F - 2F \subseteq F,$$
and we get $\alpha - \beta \in R$ as in the previous case.

Finally, we show that $U = \operatorname{span}_{\mathbb{F}} T$. To this end, we fix $\sigma \in E$. We have

$$U = \operatorname{span}_{\mathbb{F}} K = \operatorname{span}_{\mathbb{F}}(E \cup F) \subseteq \operatorname{span}_{\mathbb{F}}((E - \sigma) \cup F \cup \{\sigma\}) \subseteq U.$$

This means that

$$U = \operatorname{span}_{\mathbb{F}}((E-\sigma)\cup F\cup\{\sigma\}) \subseteq \operatorname{span}_{\mathbb{F}}(T\cup\{\sigma\}) \subseteq U;$$

i.e.,

$$U = \operatorname{span}_{\mathbb{F}}(T\cup\{\sigma\}).$$

Since $\sigma \in E \subseteq K$ and K is a group, we have $-\sigma \in K = E\cup F$. If $-\sigma \in E$, we have $-2\sigma = -\sigma-\sigma \in E-\sigma \subseteq T$. Also, if $-\sigma \in F$, we have $-\sigma \in T$ as $F \subseteq T$. Therefore, in both cases, we have $\sigma \in \operatorname{span}_{\mathbb{F}} T$ and so $U = \operatorname{span}_{\mathbb{F}} T$.

Theorem 2. *Suppose that U and V are vector spaces and suppose that subspaces T, K, E, F of U and subspaces T', K', E', F' of V satisfy the same conditions as those stated in Example 3.1. Then $\dot{R}(T,E,F,K)$ is isomorphic to $\dot{R}(T',E',F',K')$ if and only if there exist $\tau_1,\tau_2 \in K'$ with $\frac{1}{2}(\tau_1+\tau_2)\in K'$ and a linear isomorphism $\psi\colon U \longrightarrow V$ such that one of the following occurs:*

(i) $\psi(T)=T'$, $\psi(E)=E'-\tau_1$, $\psi(F)=F'-\tau_2$ *and* $\psi(K)=K'$;

(ii) $\psi(T)=T'$, $\psi(E)=-E'-\tau_1$, $\psi(F)=F'-\tau_2$ *and* $\psi(K)=K'$;

(iii) $\psi(T)=T'$, $\psi(E)=F'-\tau_1$, $\psi(F)=E'-\tau_2$ *and* $\psi(K)=K'$.

Proof. We mention that as $0\in F$,

(3.1) in cases (i), (ii), we get $\tau_2\in F'$ and in case (iii), we have $\tau_2 \in E'$.

(i) Suppose that the conditions of (i) occur and define

$$\begin{aligned}\varphi\colon \dot{U}\oplus U &\longrightarrow \dot{U}\oplus V,\\ 2\epsilon+\sigma &\mapsto 2\epsilon+\tau_1+\psi(\sigma),\\ 2\delta+\sigma &\mapsto 2\delta+\tau_2+\psi(\sigma)\quad (\sigma\in U).\end{aligned}$$

Then for $t_1,t_2\in\{\pm1\}$, we have

$$\begin{aligned}\varphi(t_1\epsilon+t_2\delta+K) &= t_1\epsilon+t_2\delta+\frac{t_1}{2}\tau_1+\frac{t_2}{2}\tau_2+K'\\ &= t_1\epsilon+t_2\delta+t_1\frac{\tau_1+\tau_2}{2}+\frac{t_2-t_1}{2}\tau_2+K'\\ &\overset{K'\colon\text{group}}{=} t_1\epsilon+t_2\delta+K'.\end{aligned}$$

We also have

$$\begin{aligned}\varphi(\pm(2\epsilon+E)) &= \pm(2\epsilon+\tau_1+E'-\tau_1)=\pm(2\epsilon+E'),\\ \varphi(2\delta+F) &= 2\delta+\tau_2+F'-\tau_2 = 2\delta+F',\\ \varphi(-2\delta+F) &= -2\delta-\tau_2+F'-\tau_2 \overset{(3.1)}{=} -2\delta+\underbrace{F'-2\tau_2}_{\in F'-2F'} = -2\delta+F'.\end{aligned}$$

Moreover, denoting the bilinear forms defined on $\dot{U}\oplus U$ and $\dot{U}\oplus V$ respectively by $(\cdot,\cdot)$ and $(\cdot,\cdot)'$, we have

$$(\varphi(u),\varphi(v))' = (u,v) \quad (u,v\in \dot{U}\oplus U).$$

These, together with the fact that $\varphi(R^0)=\varphi(T)=\psi(T)=T'=(R')^0$ imply that φ defines an isomorphism from $\dot{R}(T,E,F,K)$ to $\dot{R}(T',E',F',K')$.

(ii) Suppose that the conditions of (ii) occur and define

$$\begin{aligned}\varphi\colon \dot{U}\oplus U &\longrightarrow \dot{U}\oplus V,\\ 2\epsilon+\sigma &\mapsto -2\epsilon+\tau_1+\psi(\sigma),\\ 2\delta+\sigma &\mapsto 2\delta+\tau_2+\psi(\sigma) \quad (\sigma\in U).\end{aligned}$$

Then

$$\begin{aligned}\varphi(\pm(2\epsilon+E)) &= \pm(-2\epsilon+\tau_1-E'-\tau_1)=\mp(2\epsilon+E'),\\ \varphi(2\delta+F) &= 2\delta+\tau_2+F'-\tau_2=2\delta+F',\\ \varphi(-2\delta+F) &= -2\delta-\tau_2+F'-\tau_2 \overset{(3.1)}{=} -2\delta+\underbrace{F'-2\tau_2}_{\in F'-2F'}=-2\delta+F',\end{aligned}$$

and for $t_1,t_2\in\{\pm1\}$, we have

$$\begin{aligned}\varphi(t_1\epsilon+t_2\delta+K) &= -t_1\epsilon+t_2\delta+\frac{-t_1}{2}\tau_1+\frac{t_2}{2}\tau_2+K'\\ &= -t_1\epsilon+t_2\delta-t_1\frac{\tau_1+\tau_2}{2}+\frac{t_2+t_1}{2}\tau_2+K'\\ &\overset{K'\colon \text{group}}{=} -t_1\epsilon+t_2\delta+K'.\end{aligned}$$

We also have

$$\varphi(R^0)=(R')^0 \quad\text{and}\quad (\varphi(u),\varphi(v))'=(u,v) \quad (u,v\in\dot{U}\oplus U).$$

So φ defines an isomorphism from $\dot{R}(T,E,F,K)$ to $\dot{R}(T',E',F',K')$.

(iii) Suppose that the conditions of (iii) occur and define

$$\begin{aligned}\varphi\colon \dot{U}\oplus U &\longrightarrow \dot{U}\oplus V,\\ 2\epsilon+\sigma &\mapsto 2\delta+\tau_1+\psi(\sigma),\\ 2\delta+\sigma &\mapsto 2\epsilon+\tau_2+\psi(\sigma) \quad (\sigma\in U).\end{aligned}$$

Then as in the previous cases, for $t_1,t_2\in\{\pm1\}$, we have

$$\varphi(t_1\epsilon+t_2\delta+K)=t_1\delta+t_2\epsilon+\frac{t_1}{2}\tau_1+\frac{t_2}{2}\tau_2+K'=t_1\delta+t_2\epsilon+K',$$

and

$$\begin{aligned}\varphi(\pm(2\epsilon+E)) &= \pm(2\delta+\tau_1+F'-\tau_1) = \pm(2\delta+F') = \pm 2\delta + F',\\ \varphi(2\delta+F) &= 2\epsilon+\tau_2+E'-\tau_2 = 2\epsilon+E',\\ \varphi(-2\delta+F) &= -2\epsilon-\tau_2+E'-\tau_2 \overset{(3.1)}{=} -2\epsilon + \underbrace{E'-2\tau_2}_{\in E'-2E'} = -2\epsilon - E'.\end{aligned}$$

Moreover, we have

$$\varphi(R^0) = (R')^0 \quad \text{and} \quad (\varphi(u),\varphi(v))' = -(u,v) \quad (u,v \in \dot{U}\oplus U),$$

i.e., φ is an isomorphism from $\dot{R}(T,E,F,K)$ to $\dot{R}(T',E',F',K')$.

Conversely, assume $R := \dot{R}(T,E,F,K)$ and $R' := \dot{R}(T',E',F',K')$ are isomorphic. Denote the bilinear forms on the underlying vector spaces $\dot{U}\oplus U$ and $\dot{U}\oplus V$ respectively by $(\cdot,\cdot)$ and $(\cdot,\cdot)'$. So there is a linear isomorphism $\varphi\colon \dot{U}\oplus U \longrightarrow \dot{U}\oplus V$ and a nonzero scalar r such that

$$\varphi(R) = R' \quad \text{and} \quad (\varphi(u),\varphi(v))' = r(u,v)$$

for $u,v \in \dot{U}\oplus U$. Since $\varphi(R^0) = (R')^0$, we have

$$\varphi(T) = T'. \tag{3.2}$$

Moreover, there are linear transformations

$$\zeta\colon \dot{U} \longrightarrow \dot{U}, \quad \eta\colon \dot{U} \longrightarrow V, \quad \psi\colon U \longrightarrow V$$

such that

$$\varphi(\dot{\alpha}+\sigma) = \zeta(\dot{\alpha}) + \eta(\dot{\alpha}) + \psi(\sigma) \quad (\dot{\alpha}\in\dot{U},\ \sigma\in U),$$

ζ is an isomorphism from $\dot{R}$ to $\dot{R}$ and ψ is a linear isomorphism. Since

$$-1 = (2\delta,2\delta) = r(\zeta(2\delta),\zeta(2\delta))',$$

we get either $r=1$ or $r=-1$. In the former case, we have

$$\zeta(\{\pm\epsilon\}) = \{\pm\epsilon\} \quad \text{and} \quad \zeta(\{\pm\delta\}) = \{\pm\delta\},$$

and in the latter case, we have

$$\zeta(\{\pm\epsilon\}) = \{\pm\delta\} \quad \text{and} \quad \zeta(\{\pm\delta\}) = \{\pm\epsilon\}.$$

If $r=1$, we have

$$\begin{aligned}\zeta(2\epsilon)+\eta(2\epsilon)+\psi(E) &= \varphi(2\epsilon+E) \in \pm(2\epsilon+E'),\\ \zeta(2\delta)+\eta(2\delta)+\psi(F) &= \varphi(2\delta+F) \in \{\pm 2\delta\}+F',\end{aligned}$$

and also

$$\zeta(\epsilon+\delta)+\eta(\epsilon+\delta)+\psi(K)=\varphi(\epsilon+\delta+K)\in\pm\epsilon\pm\delta+K'.$$

So setting

$$\tau_1 := \eta(2\epsilon) \quad \text{and} \quad \tau_2 := \eta(2\delta),$$

one of the following occurs:

- $\psi(E)=E'-\tau_1$, $\psi(F)=F'-\tau_2$ and $\psi(K)=K'-\frac{1}{2}(\tau_1+\tau_2)$: Since $0\in F\cap K$, these imply that $\tau_2\in F'\subseteq K'$ and $\frac{1}{2}(\tau_1+\tau_2)\in K'$; in particular, as K' is a group, we have $\psi(K)=K'$ and $\tau_1=(\tau_1+\tau_2)-\tau_2\in K'$. So by (3.2), the conditions stated in (i) are fulfilled.
- $\psi(E)=-E'-\tau_1$, $\psi(F)=F'-\tau_2$ and $\psi(K)=K'-\frac{1}{2}(\tau_1+\tau_2)$: As in the previous case, $\tau_1,\tau_2,\frac{1}{2}(\tau_1+\tau_2)\in K'$ and $\psi(K)=K'$. So the conditions stated in (ii) are satisfied; see also (3.2).

If $r=-1$, we have

$$\zeta(2\epsilon)+\eta(2\epsilon)+\psi(E)=\varphi(2\epsilon+E)\in\pm2\delta+F',$$
$$\zeta(2\delta)+\eta(2\delta)+\psi(F)=\varphi(2\delta+F)\in\pm(2\epsilon+E')$$

and

$$\zeta(\epsilon+\delta)+\eta(\epsilon+\delta)+\psi(K)=\varphi(\epsilon+\delta+K)\in\pm\epsilon\pm\delta+K'.$$

So setting

$$\tau_1 := \eta(2\epsilon) \quad \text{and} \quad \tau_2 := \eta(2\delta),$$

one of the following occurs:

- $\psi(E)=F'-\tau_1$, $\psi(F)=E'-\tau_2$ and $\psi(K)=K'-\frac{1}{2}(\tau_1+\tau_2)$: These together with the fact that $0\in F\cap K$ imply that $\tau_2\in E'\subseteq K'$ and $\frac{1}{2}(\tau_1+\tau_2)\in K'$. Since K' is a group, we have $\psi(K)=K'$ and $\tau_1=(\tau_1+\tau_2)-\tau_2\in K$; i.e., recalling (3.2), we get the conditions stated in (iii).
- $\psi(E)=F'-\tau_1$, $\psi(F)=-E'-\tau_2$ and $\psi(K)=K'-\frac{1}{2}(\tau_1+\tau_2)$: Since $0\in F$ and K is a group, we get $-\tau_2\in E'\subseteq K'$, $\frac{1}{2}(\tau_1+\tau_2)\in K'$ and $\psi(K)=K'$. So for $\tau_2' := -\tau_2\in E'$, we have $\psi(F)=\psi(-F)=E'+\tau_2=E'-\tau_2'$. Moreover, since K' is a group, we have $\frac{1}{2}(\tau_1+\tau_2')=\frac{1}{2}(\tau_1-\tau_2)=\frac{1}{2}(\tau_1+\tau_2)-\tau_2\in K'$. In fact, we have $\tau_1,\tau_2',\frac{1}{2}(\tau_1+\tau_2')\in K'$,

$$\psi(E)=F'-\tau_1,\quad \psi(F)=E'-\tau_2',\quad \psi(K)=K' \quad\text{and}\quad \psi(T)=T'.$$

In other words, again, we get the conditions stated in (iii). This completes the proof. □

Example 3.2. For a quadruple (U, G, T, τ) in which G is a subgroup of the vector space U, $T \subseteq U$ and $\tau \in U$ with $U = \mathrm{span}_{\mathbb{F}}(T \cup \{\tau\})$ and $G \subseteq T = -T \subseteq G \cup (G \pm 2\tau)$, set

$$R = \dot{R}(T, G, \tau) := T \cup (\dot{R}^{\times}_{\mathrm{re}} + G) \cup (\dot{R}^{\times}_{\mathrm{ns}} + G \pm \tau) \subseteq U.$$

It is easily seen that $\dot{R}(T, G, \tau)$ is a tame irreducible extended affine root supersystem of type $C(1, 1)$ in $\dot{U} \oplus U$. Furthermore, we claim that if $U \neq \mathrm{span}_{\mathbb{F}} T$, then $T = G$. In fact, if $T \neq G$, then since $G \subseteq T \subseteq G \cup (G \pm 2\tau)$, there are $r = \pm 1$ and $g \in G$ such that $g + 2r\tau \in T$, so $2\tau \in T - rg \subseteq \mathrm{span}_{\mathbb{F}} T$. Therefore, $U = \mathrm{span}_{\mathbb{F}}(T \cup \{\tau\}) = \mathrm{span}_{\mathbb{F}} T$. We moreover note that if $\tau \in G$, then $T = G$ and $R = \dot{R} + G$. In particular, if $\alpha, \beta \in R_{\mathrm{ns}}$ with $(\alpha, \beta) \neq 0$, then $\alpha + \beta, \alpha - \beta \in R$. This phenomenon does not happen for affine reflection root systems [2, §1.2.6].

Theorem 3. *Suppose that quadruples (U, G, T, τ) and (V, G', T', τ') satisfy the same conditions as stated in Example 3.2. Then $\dot{R}(T, G, \tau) \simeq \dot{R}(T', G', \tau')$ if and only if there exist $\sigma_1, \sigma_2 \in V$ and a linear isomorphism $\psi \colon U \longrightarrow V$ such that*

- $2\sigma_1, 2\sigma_2 \in G'$,
- $\psi(T) = T'$, $\psi(G) = G'$ *and* $\psi(\tau) \in G' \pm \tau' + \sigma_1 + \sigma_2$.

Proof. Denote the bilinear forms defined on $\dot{U} \oplus U$ and $\dot{U} \oplus V$ respectively by $(\cdot, \cdot)$ and $(\cdot, \cdot)'$ and suppose σ_1, σ_2 and ψ are as in the statements. Define

$$\varphi \colon \dot{U} \oplus U \longrightarrow \dot{U} \oplus V,$$
$$\epsilon \mapsto \epsilon + \sigma_1, \quad \delta \mapsto \delta + \sigma_2, \quad \sigma \mapsto \psi(\sigma) \quad (\sigma \in U).$$

Then for $t_1, t_2 = \pm 1$, we have

$$\varphi(2t_1\epsilon + G) = 2t_1\epsilon + G' + 2t_1\sigma_1 = 2t_1\epsilon + G',$$
$$\varphi(2t_2\delta + G) = 2t_2\delta + G' + 2t_2\sigma_2 = 2t_2\delta + G'$$

and as G' is a group with $2\sigma_1, 2\sigma_2 \in G'$, we get

$$\varphi(t_1\epsilon + t_2\delta \pm \tau + G) \in t_1\epsilon + t_2\delta + t_1\sigma_1 + t_2\sigma_2 \pm \tau' + \sigma_1 + \sigma_2 + G' = t_1\epsilon + t_2\delta \pm \tau' + G'.$$

Also, we have

$$(\varphi(u), \varphi(u'))' = (u, u') \quad (u, u' \in \dot{U} \oplus U).$$

These together with the fact that $\varphi(T) = \psi(T) = T'$ imply that φ defines an isomorphism from $\dot{R}(T, G, \tau)$ to $\dot{R}(T', G', \tau')$.

Conversely, assume $R := \dot{R}(T, G, \tau)$ and $R' := \dot{R}(T', G', \tau')$ are isomorphic. So there are a nonzero scalar r and a linear isomorphism $\varphi \colon \dot{U} \oplus U \longrightarrow \dot{U} \oplus V$

such that

$$\varphi(R) = R' \quad \text{and} \quad (\varphi(u), \varphi(u'))' = r(u,u') \quad (u,u' \in \dot{U} \oplus U).$$

This gives that there are linear transformations

$$\zeta\colon \dot{U} \longrightarrow \dot{U}, \quad \eta\colon \dot{U} \longrightarrow V \quad \text{and} \quad \psi\colon U \longrightarrow V$$

such that

$$\varphi(\dot{\alpha}+\sigma) = \zeta(\dot{\alpha}) + \eta(\dot{\alpha}) + \psi(\sigma) \quad (\dot{\alpha} \in \dot{U},\ \sigma \in U).$$

In particular, ζ defines an isomorphism of $\dot{R}$ and

$$\psi(T) = \psi(R^0) = \varphi(R^0) = (R')^0 = T'.$$

Set

$$\sigma_1 := -\eta(\epsilon) \quad \text{and} \quad \sigma_2 := -\eta(\delta);$$

then we have

$$\zeta(2\epsilon) - 2\sigma_1 = \varphi(2\epsilon) \in R' \quad \text{and} \quad \zeta(2\delta) - 2\sigma_2 = \varphi(2\delta) \in R'.$$

So

$$2\sigma_1, 2\sigma_2 \in G'.$$

Moreover, we have

$$\zeta(2\epsilon) + \eta(2\epsilon) + \psi(G) = \varphi(2\epsilon + G) \subseteq \dot{R}_{\mathrm{re}}^{\times} + G'.$$

Since $\zeta(2\epsilon) \in \{\pm 2\epsilon, \pm 2\delta\}$, it follows that

$$\psi(G) = G'.$$

Finally, we have

$$\zeta(\epsilon) + \zeta(\delta) - \sigma_1 - \sigma_2 + \psi(\tau) = \varphi(\epsilon + \delta + \tau) \in \pm\epsilon \pm \delta \pm \tau' + G',$$

which gives

$$\psi(\tau) \in G' \pm \tau' + \sigma_1 + \sigma_2.$$

This fulfills the conditions stated in the statement. □

Theorem 4. *Each tame irreducible extended affine root supersystem* $(V, (\cdot,\cdot), R)$ *of type* $C(1,1)$ *has an expression as* R *in Examples 3.1, 3.2. Moreover, if* $\operatorname{span}_{\mathbb{F}} R^0 \neq V^0$*, then for* $\alpha, \beta \in R_{\mathrm{ns}}$ *with* $(\alpha,\beta) \neq 0$*, one and only one of* $\alpha+\beta$ *and* $\alpha-\beta$ *is an element of* R*.*

Proof. Suppose $(V,(\cdot,\cdot),R)$ is a tame irreducible extended affine root supersystem of type $C(1,1)$. As in (2.5) and (2.6), for $\dot{R}=\{0,\pm 2\zeta,\pm 2\eta,\pm\zeta\pm\eta\}$, which is an irreducible finite root supersystem of type $C(1,1)$ with $(\zeta,\eta)=0$, $(\zeta,\zeta)=-(\eta,\eta)=1$ and $\Pi=\{\zeta-\eta,2\eta\}$, there is a class $\{S_{\dot\alpha}\}_{\dot\alpha\in\dot R}$ of nonempty subsets of V^0 such that

$$R=\bigcup_{\dot\alpha\in\dot R}(\dot\alpha+S_{\dot\alpha})\text{ with }0\in S_{\dot\alpha}\quad(\dot\alpha\in\Pi).$$

Since $0\in S_{\zeta-\eta}\cap S_{2\eta}$, Fact 2 implies that

$$S_{2\eta}=S_{-2\eta},\quad S_{\zeta-\eta}=S_{\zeta+\eta},\quad S_{-\zeta-\eta}=S_{-\zeta+\eta}.$$

Also as $R=-R$, we have

$$S_{2\zeta}=-S_{-2\zeta}\quad\text{and}\quad S_{\zeta\pm\eta}=-S_{-\zeta\pm\eta}.$$

Set

$$E_1:=S_{2\zeta},\quad F:=S_{\pm 2\eta}\quad\text{and}\quad K:=S_{\zeta\pm\eta}.$$

So

$$R^\times=\pm(2\zeta+E_1)\cup(\pm 2\eta+F)\cup\pm(\zeta\pm\eta+K)\tag{3.3}$$

with

$$0\in F=-F\quad\text{and}\quad 0\in K.\tag{3.4}$$

We continue the proof in the following steps.

Step 1. We have the following:

(a) $E_1-2E_1\subseteq -E_1$, $F-2F\subseteq F$, $E_1-K\subseteq K$ and $K-F\subseteq K$.

(b) $K=E_1\cup F$. Moreover,

$$K\text{ is a subgroup of }V^0\iff K=-K\iff E_1\setminus(E_1\cap -E_1)\subseteq F;$$

in particular, if $E_1=-E_1$, K is a subgroup of V^0.

Reason: (a) Fact 1 implies that $E_1-2E_1\subseteq -E_1$ and $F-2F\subseteq F=-F$. Next assume $x\in K$, $y\in E_1$ and $z\in F$. Since $(2\zeta+y,\zeta+\eta+x),(-2\eta-z,\zeta+\eta+x)\neq 0$ while $2\zeta+(\zeta+\eta),-2\eta-(\zeta+\eta)\notin\dot R$, from (S5), we have

$$\begin{aligned}\zeta-\eta+y-x&=(2\zeta+y)-(\zeta+\eta+x)\in R,\\ \zeta-\eta-z+x&=(-2\eta-z)+(\zeta+\eta+x)\in R,\end{aligned}$$

which in turn implies that

$$E_1-K\subseteq K\quad\text{and}\quad K-F\subseteq K.$$

(b) Since $0 \in K$ and $F = -F$, (a) implies that $E_1 \cup F \subseteq K$. Now suppose $x \in K$; then $\zeta + \eta + x$, $\zeta - \eta \in R$. So (S5) implies that either $2\zeta + x \in R$ or $2\eta + x \in R$. Therefore, either $x \in E_1$ or $x \in F$; in other words, $K = E_1 \cup F$.

Next suppose K is a subgroup of V^0; then $K = -K$. Conversely, suppose $K = -K$, so

$$K = -K \xrightarrow{E_1 - K \subseteq K} K - E_1 \subseteq K \xrightarrow{K - F \subseteq K \;\&\; K = E_1 \cup F} K - K \subseteq K,$$

i.e., K is a subgroup of V^0.

To complete the proof of this step, we need to show that $K = -K$ if and only if $E_1 \setminus -E_1 \subseteq F$. First assume $K = -K$. Since $-E_1 \subseteq -K = K = E_1 \cup F$ and $F = -F$, if $x \in E_1 \setminus -E_1$, then $-x \in K \setminus E_1$, so $-x \in F$; i.e., $E_1 \setminus (E_1 \cap -E_1) \subseteq F$. Conversely, suppose $E_1 \setminus (E_1 \cap -E_1) \subseteq F$; then as $F = -F$, we have

$$-K = -E_1 \cup -F \subseteq (E_1 \cap -E_1) \cup -F \subseteq E_1 \cup -F = E_1 \cup F = K$$

as we desired.

Step 2. Let $\sigma \in E_1 \subseteq K$ and set $E := E_1 - \sigma \subseteq E_1 - K \overset{\text{Step 1(a)}}{\subseteq} K$. Then

(a) $0 \in F \cap E$, $2E - E \subseteq E$, $2F - F \subseteq F$.

(b) Set $\epsilon := \zeta + \frac{\sigma}{2}$ and $L := K - \frac{\sigma}{2}$; then

$$R^\times = (\pm 2\epsilon + E) \cup (\pm 2\eta + F) \cup (\pm\epsilon \pm \eta + L).$$

(c) $-L = L = (E + \frac{\sigma}{2}) \cup (F - \frac{\sigma}{2})$.

(d) $E \cup F \subseteq (E - E) \cup (F - F) \subseteq R^0 \subseteq K - K$ and $V^0 = \mathrm{span}_{\mathbb{F}}(R^0 \cup \{\sigma\})$.

(e) Suppose $\sigma \notin \mathrm{span}_{\mathbb{F}} R^0$. Then for $\alpha, \beta \in R_{\mathrm{ns}}$ with $(\alpha, \beta) \neq 0$, one and only one of $\alpha + \beta$ and $\alpha - \beta$ is a root; in particular, we get the last assertion of the theorem.

Reason: (a) Since $2\eta \in R$ and $\sigma \in E_1$, we have $0 \in E \cap F$. Using Step 1(a), we have

$$E_1 - 2E_1 \subseteq -E_1 \quad \text{and} \quad F - 2F \subseteq F,$$

so

$$E - 2E = E_1 - \sigma - 2(E_1 - \sigma) = E_1 - 2E_1 + \sigma \subseteq -E_1 + \sigma = -E.$$

Since $0 \in E \cap F$, these imply that $E = -E$ and $F = -F$ and consequently

$$2E - E \subseteq E = -E \quad \text{and} \quad 2F - F \subseteq F = -F.$$

(b) We first show that $K - \sigma = -K$. In fact, we have, using Step 1(a), that

$$K - \sigma \subseteq K - E_1 \subseteq -K.$$

Also using (3.3), if $\tau \in -K$, we have $-\zeta+\eta+\tau \in R$ and as $\sigma \in E_1$, we have $2\zeta+\sigma \in R$. But $(-\zeta+\eta+\tau, 2\zeta+\sigma) \neq 0$ and $(-\zeta+\eta+\tau)-(2\zeta+\sigma) \notin R$, so we get from (S5) that $\zeta+\eta+\sigma+\tau \in R$, which in turn implies that $\sigma+\tau \in K$. In other words, $\tau \in K-\sigma$; that is

$$-K \subseteq K-\sigma.$$

Therefore, we have

$$L = K-\frac{\sigma}{2} = K-\sigma+\frac{\sigma}{2} = -K+\frac{\sigma}{2} = -\Big(K-\frac{\sigma}{2}\Big) = -L.$$

Use (3.3) and recall that $\epsilon = \zeta+\frac{\sigma}{2}$ to get

$$\begin{aligned} R^\times &= \pm\,(2\zeta+E_1)\cup(\pm 2\eta+F)\cup\pm(\zeta\pm\eta+K)\\ &= \pm\,\Big(2\Big(\zeta+\frac{\sigma}{2}\Big)-\sigma+E_1\Big)\cup(\pm 2\eta+F)\cup\pm\Big(\zeta\pm\eta+\frac{\sigma}{2}-\frac{\sigma}{2}+K\Big)\\ &= \pm\,(2\epsilon+E)\cup(\pm 2\eta+F)\cup\pm(\epsilon\pm\eta+L). \end{aligned}$$

But we have already seen that $E=-E$ and $L=-L$, so we have

$$R^\times = (\pm 2\epsilon+E)\cup(\pm 2\eta+F)\cup(\pm\epsilon\pm\eta+L).$$

(c) We know from Step 1(b) that $K = E_1\cup F$. Therefore, we have

$$\begin{aligned} L = K-\frac{\sigma}{2} = \Big(E_1-\frac{\sigma}{2}\Big)\cup\Big(F-\frac{\sigma}{2}\Big) &= \Big(E+\sigma-\frac{\sigma}{2}\Big)\cup\Big(F-\frac{\sigma}{2}\Big)\\ &= \Big(E+\frac{\sigma}{2}\Big)\cup\Big(F-\frac{\sigma}{2}\Big). \end{aligned}$$

(d) Assume $x, y \in E$, since by part (b), $2\epsilon+x, 2\epsilon+y \in R$, the root string property implies that $x-y \in R^0$. Similarly, $F-F \subseteq R^0$. So as $0 \in E\cap F$, we have

$$E\cup F \subseteq (E-E)\cup(F-F) \subseteq R^0. \tag{3.5}$$

Also as R is tame, we have $R^0 \subseteq R^\times - R^\times$. So we get

$$R^0 \subseteq (E-E)\cup(F-F)\cup(K-K) \subseteq K-K.$$

For the last assertion of (d), suppose that $v \in V^0 \subseteq V = \mathrm{span}_{\mathbb{F}} R$. Since $\mathrm{span}_{\mathbb{F}} R \subseteq \mathrm{span}_{\mathbb{F}}(R^\times \cup (R^\times - R^\times))$, (3.3) implies that

$$v \in \mathrm{span}_{\mathbb{F}}(E_1\cup F\cup K) \overset{\text{Step 1(b)}}{=} \mathrm{span}_{\mathbb{F}}(E_1\cup F) \overset{E=E_1-\sigma}{=} \mathrm{span}_{\mathbb{F}}(E\cup F\cup\{\sigma\}).$$

Therefore, we have

$$V^0 \subseteq \mathrm{span}_{\mathbb{F}}(E\cup F\cup\{\sigma\}) \subseteq V^0.$$

So

$$V^0 = \mathrm{span}_{\mathbb{F}}(E \cup F \cup \{\sigma\}) \overset{(3.5)}{\subseteq} \mathrm{span}_{\mathbb{F}}(R^0 \cup \{\sigma\}) \subseteq V^0$$

as we desired.

(e) Let $\sigma \notin \mathrm{span}_{\mathbb{F}} R^0$ and $\alpha, \beta \in R_{\mathrm{ns}}$ with $(\alpha, \beta) \neq 0$. Then by (S5), either $\alpha+\beta \in R$ or $\alpha-\beta \in R$. To the contrary, assume both $\alpha+\beta$ and $\alpha-\beta$ are elements of R. Using parts (b) and (c), without loss of generality, we assume $\alpha = \epsilon+\eta+\tau+r\frac{\sigma}{2}$ and $\beta = \epsilon - \eta + \tau' + s\frac{\sigma}{2}$, where $\tau, \tau' \in E \cup F$ and $r, s \in \{\pm 1\}$. Therefore,

$$\alpha + \beta = 2\epsilon + (\tau + \tau') + (r + s)\frac{\sigma}{2}, \quad \alpha - \beta = 2\eta + (\tau - \tau') + (r - s)\frac{\sigma}{2} \in R.$$

So by (3.5), $(\tau + \tau') + (r + s)\frac{\sigma}{2} \in E \subseteq R^0$ and $(\tau - \tau') + (r - s)\frac{\sigma}{2} \in F \subseteq R^0$. Since $\tau + \tau', \tau - \tau' \in \mathrm{span}_{\mathbb{F}} R^0$, it follows that $\sigma \in \mathrm{span}_{\mathbb{F}} R^0$, a contradiction.

Step 3. If $E_1 \setminus (E_1 \cap -E_1) \subseteq F$, then R has an expression as in Example 3.1. *Reason:* If $E_1 \setminus (E_1 \cap -E_1) \subseteq F$, then by Step 1(b), K is a group. Moreover, picking $\sigma \in E_1 \subseteq K$, by Step 2(d),

$$(E_1 - E_1) \cup (F - F) = ((E_1 - \sigma) - (E_1 - \sigma)) \cup (F - F) \subseteq R^0 = -R^0 \subseteq K$$

and so again using Step 2(d), together with the fact that $\sigma \in E_1 \subseteq K$, we get

$$V^0 = \mathrm{span}_{\mathbb{F}}(R^0 \cup \{\sigma\}) \subseteq \mathrm{span}_{\mathbb{F}} K \subseteq V^0.$$

Also by (3.3), we have

$$R^\times = \pm(2\zeta + E_1) \cup (\pm 2\eta + F) \cup (\pm\zeta \pm \eta + K).$$

These together with Step 1 and (3.4) give that R has an expression as in Example 3.1.

Step 4. If $E_1 \setminus (E_1 \cap -E_1) \not\subseteq F$, then R has an expression as in Example 3.2. *Reason:* Fix $\sigma \in E_1 \setminus (-E_1 \cup F)$ and set $E := E_1 - \sigma$. For

$$\epsilon := \zeta + \frac{\sigma}{2} \quad \text{and} \quad \delta := \eta,$$

Step 2 implies that

$$E = -E \subseteq K \cap -K \quad \text{and} \quad R^\times = (\pm 2\epsilon + E) \cup (\pm 2\delta + F) \cup (\pm\epsilon \pm \delta + L), \tag{3.6}$$

where $L := K - \frac{\sigma}{2} = -L$. We complete our argument in this step through the following stages:

Stage 1. $2E \subseteq F$ and $2F \subseteq E$: Suppose that $x \in E = -E \subseteq -K \cap K$. Then

$$\alpha := \epsilon + \delta + x + \frac{\sigma}{2} \in \epsilon + \delta - K + \frac{\sigma}{2} = \epsilon + \delta + L \subseteq R,$$
$$\beta := \epsilon - \delta - x + \frac{\sigma}{2} \in \epsilon - \delta - K + \frac{\sigma}{2} = \epsilon - \delta + L \subseteq R.$$

Since $(\alpha, \beta) \neq 0$, by (S5), we have either $\alpha+\beta = 2\epsilon+\sigma \in R$ or $\alpha-\beta = 2\delta+2x \in R$. If $2\epsilon + \sigma \in R$, we get $\sigma \in E = E_1 - \sigma$ which in turn implies that $2\sigma \in E_1$. So

$$0 = 2\sigma - 2\sigma \in 2E_1 - E_1 \subseteq E_1.$$

Therefore, $-E_1 = 0 - E_1 \subseteq 2E_1 - E_1 \subseteq E_1$, which implies that $\sigma \in E_1 = -E_1$, a contradiction. So $\alpha - \beta = 2\delta + 2x \in R$, that is, $2x \in F$ by (3.6) as we expected.

Next assume $y \in F = -F \subseteq K \cap -K$. Then as above $\alpha := \epsilon + \delta + y - \frac{\sigma}{2}$, $\beta := -\epsilon+\delta-y-\frac{\sigma}{2} \in R$ with $(\alpha, \beta) \neq 0$, which in turn implies that either $\alpha+\beta \in R$ or $\alpha - \beta \in R$. If $\alpha + \beta = 2\delta - \sigma \in R$, we get $\sigma \in F$, which is a contradiction. So $\alpha - \beta = 2\epsilon + 2y \in R$, which in turn implies that $2y \in E$; see (3.6). So $2F \subseteq E$.

Stage 2. E is a subgroup of V^0: To the contrary, assume there are $x, y \in E = -E \subseteq K \cap -K$ such that $x - y \notin E$. Since

$$\alpha := \epsilon + \delta + \underbrace{x + \frac{\sigma}{2}}_{\in -K+\frac{\sigma}{2}=L}, \qquad \beta := -\epsilon + \delta + \underbrace{y + \frac{\sigma}{2}}_{\in -K+\frac{\sigma}{2}=L} \overset{(3.6)}{\in} R$$

and $(\alpha, \beta) \neq 0$, (S5) implies that either $\alpha - \beta \in R$ or $\alpha + \beta \in R$. Since $x - y \notin E$, we have $\alpha - \beta = 2\epsilon + (x - y) \notin R$, so $\alpha + \beta = 2\delta + (x + y) + \sigma \in R$. Therefore, $(x + y) + \sigma \in F$ and so by Stage 1, $2x + 2y + 2\sigma \in E$. This implies that

$$2\sigma = (2x + 2y + 2\sigma) - 2x - 2y \in E - 2E - 2E \overset{\text{Step 2(a)}}{\subseteq} E.$$

But this implies that $-2\sigma \in -E = E = E_1 - \sigma$, that is, $-\sigma \in E_1$, a contradiction as $\sigma \in E_1 \setminus (-E_1 \cup F)$.

Stage 3. F is a subgroup of V^0: To the contrary, assume there are $x, y \in F$ with $x - y \notin F$. Using the same argument as in Stage 2, for $\alpha := \epsilon + \delta + x - \frac{\sigma}{2}$, $\beta := \epsilon - \delta + y - \frac{\sigma}{2} \in R$ as $x - y \notin F$, we have $\alpha + \beta = 2\epsilon + (x + y) - \sigma \in R$. This implies that $(x + y) - \sigma \in E$ and so by Stage 1, $2x + 2y - 2\sigma \in F$. Thus, we have

$$-2\sigma = (2x + 2y - 2\sigma) - 2x - 2y \in F - 2F - 2F \subseteq F = -F.$$

Therefore, Step 1 gives

$$-\sigma = \sigma - 2\sigma \in E_1 - F \subseteq K - F \subseteq K = E_1 \cup F.$$

So either $-\sigma \in F$ or $-\sigma \in E_1$, i.e., either $\sigma \in F$ or $-\sigma \in E_1$, which both result in a contradiction as $\sigma \in E_1 \setminus (-E_1 \cup F)$.

Stage 4. $E=F$: We first show that $F\subseteq E$. To the contrary, assume there is $y\in F\setminus E$ and fix $x\in E\subseteq K\cap -K$. Then since $F\subseteq K$ and $L=-L=K-\frac{\sigma}{2}$ (see Steps 1, 2), we have

$$\alpha:=\epsilon+\delta+\underbrace{y-\frac{\sigma}{2}}_{\in K-\frac{\sigma}{2}=L},\qquad \beta:=\epsilon-\delta+\underbrace{x+\frac{\sigma}{2}}_{\in -K+\frac{\sigma}{2}=L}\in R.$$

But $(\alpha,\beta)\neq 0$, so by (S5), either $\alpha+\beta\in R$ or $\alpha-\beta\in R$. If $2\epsilon+(x+y)=\alpha+\beta\in R$, we have $x+y\in E$, which together with Stage 2 implies that $y\in E$, a contradiction. Also, if $2\delta+(y-x)-\sigma=\alpha-\beta\in R$, we have $y-x-\sigma\in F=-F$. Therefore, as F is a group and $y\in F$, we have $x+\sigma\in F$ and so we get, using Stage 1, that $2x+2\sigma\in E$, which implies that $2\sigma\in E$ as E is a group. But we have already seen in Stage 2 that $2\sigma\in E$ implies that $\sigma\in -E_1$, a contradiction.

We next show that $E\subseteq F$. To the contrary, assume there is $y\in E\setminus F$ and fix $x\in F$. Then as above, since

$$\alpha:=\epsilon+\delta+\underbrace{y+\frac{\sigma}{2}}_{\in -K+\frac{\sigma}{2}=L},\qquad \beta:=-\epsilon+\delta+\underbrace{x-\frac{\sigma}{2}}_{\in K-\frac{\sigma}{2}=L}\in R$$

and $(\alpha,\beta)\neq 0$, we have either $\alpha+\beta\in R$ or $\alpha-\beta\in R$. If $2\delta+(y+x)=\alpha+\beta\in R$, we have $y+x\in F$. Since F is a group, this implies that $y\in F$, a contradiction. Also, if $2\epsilon+(y-x)+\sigma=\alpha-\beta\in R$, we get $y-x+\sigma\in E$. Therefore, we have $-x+\sigma\in E$ and so by Stage 1, we have $-2x+2\sigma\in F$. But F is a group, so we get $-2\sigma\in F$. This gives a contradiction as we saw in Stage 3.

Stage 5. R has the expression of Example 3.2: By (3.6), Stage 4 and Step 2(c), we have

$$R=R^0\cup(\dot{R}_{\rm re}^{\times}+E)\cup\Big(\dot{R}_{\rm ns}^{\times}+E\pm\frac{\sigma}{2}\Big).$$

Since $2\epsilon+E\subseteq R$ (in particular, $2\epsilon\in R$), the root string property implies that $E\subseteq R^0$; also as R is tame and E is a group, we get

$$E\subseteq R^0=-R^0=R^0\cap(R^\times-R^\times)\subseteq(E-E)\cup\Big(\Big(E\pm\frac{\sigma}{2}\Big)-\Big(E\pm\frac{\sigma}{2}\Big)\Big)\subseteq E\cup(E\pm\sigma).$$

Furthermore, we know from Step 2(d) that $\mathrm{span}_{\mathbb{F}}(R^0\cup\{\sigma\})=V^0$. So R has the expression as in Example 3.2. This completes the proof. □

§4. Type $C(I,J)$ ($|I|\geq 1$, $|J|\geq 2$)

From [7, Thm. 2.2], we know a description of tame irreducible extended affine root supersystems of type $C(J,I)\simeq C(I,J)$ when $|J|>2$ and here we want to find a description for tame irreducible extended affine root supersystems of types $C(1,2) \simeq C(2,1)$ and $C(2,2)$. As arguments giving descriptions of types $C(2,2)$ and $C(1,2) \simeq C(2,1)$ are similar and also work for the general type $C(I,J)$ ($|I| \geq 1$, $|J| \geq 2$), we give a description of the general case and, moreover, we determine the isomorphic classes.

Convention 4.1. Throughout this section, we suppose I and J are index sets with $|I| \geq 1$ and $|J| \geq 2$. We always assume $1 \in I$ and $1, 2 \in J$. Moreover, if $|I| \geq 2$, we assume 2 is also an element of I.

Suppose that $(\dot{U}, (\cdot,\cdot), \dot{R})$ is a locally finite root supersystem of type $C(I,J)$. Without loss of generality, we assume $\dot{R}$ has the expression as in (2.3), that is,

$$\dot{R} = \big\{\pm\epsilon_i \pm \epsilon_j, \pm\delta_p \pm \delta_q, \pm\epsilon_i \pm \delta_p \mid i,j \in I,\ p,q \in J\big\}$$

with $(\epsilon_i, \delta_p) = 0$, $(\epsilon_i, \epsilon_j) = \delta_{i,j}$, $(\delta_p, \delta_q) = -\delta_{p,q}$ ($i,j \in I$, $p,q \in J$) and $\dot{R}_{\rm re} = \dot{R}^1 \cup \dot{R}^2$, where

$$\dot{R}^1 := \big\{\pm\epsilon_i \pm \epsilon_j \mid i,j \in I\big\} \quad \text{and} \quad \dot{R}^2 := \big\{\pm\delta_p \pm \delta_q \mid p,q \in J\big\}.$$

Next assume U is a vector space. Let L_1, L_2 and F be subsets of U satisfying

$$(\ddagger) \qquad \begin{gathered} 0 \in L_1, \quad L_i - 2L_i \subseteq L_i, \quad L_i + F \subseteq F \quad (i = 1,2), \\ L_1 + 2F \subseteq L_1 \ (\text{if } |I| \geq 2), \quad L_2 + 2F \subseteq L_2 \quad \text{and} \quad F = L_1 \cup L_2. \end{gathered}$$

Extend $(\cdot,\cdot)$ to a form on $\dot{U} \oplus U$ such that U is the radical of this new form. Recall (2.4) and set

$$(4.1) \qquad \begin{aligned} R &= \dot{R}(F, L_1, L_2) \\ &:= F \cup (((\dot{R}_{\rm re})_{\rm sh} \cup \dot{R}_{\rm ns}) + F) \cup (\dot{R}^1_{\rm lg} + L_1) \cup (\dot{R}^2_{\rm lg} + L_2) \quad \text{if } |I| > 1 \end{aligned}$$

and

$$(4.2) \qquad \begin{aligned} R &= \dot{R}(F, L_1, L_2) \\ &:= F \cup ((\dot{R}^2_{\rm sh} \cup \dot{R}_{\rm ns}) + F) \cup (\dot{R}^1_{\rm sh} + L_1) \cup (\dot{R}^2_{\rm lg} + L_2) \quad \text{if } |I| = 1. \end{aligned}$$

Then using Remark 2.2, it is readily seen that R is a tame irreducible extended affine root supersystem of type $C(I,J)$.

Theorem 5. *Suppose that $(V, (\cdot,\cdot), R)$ is a tame irreducible extended affine root supersystem of type $C(I,J)$. Then R has an expression as $\dot{R}(F, L_1, L_2)$; see (4.1) and (4.2).*

Proof. Assume $(V, (\cdot,\cdot), R)$ is a tame irreducible extended affine root supersystem of type $C(I,J)$. Keeping the same notation as in the text and recalling (2.5) and

(2.6), we may assume

$$\Pi = \{2\epsilon_1, \epsilon_i - \epsilon_1, \delta_p - \epsilon_1 \mid 1 \neq i \in I,\ p \in J\}$$

and

$$R = \bigcup_{\dot\alpha \in \dot R} (\dot\alpha + S_{\dot\alpha})$$

for some nonempty subsets $S_{\dot\alpha}$ of V^0.

Step 1. Recall that $\dot R^1 = \{\pm\epsilon_i \pm \epsilon_j \mid i,j \in I\}$. For $\dot\alpha, \dot\beta \in (\dot R^1)^\times$ with $(\dot\alpha,\dot\alpha) = (\dot\beta,\dot\beta)$, we have $0 \in S_{\dot\alpha} = S_{\dot\beta}$: We know that $\dot R^1$ is an irreducible locally finite root system of type C_I and $B := \{2\epsilon_1, \epsilon_i - \epsilon_1 \mid 1 \neq i \in I\}$ is a reflectable base of R in the sense that $(\dot R^1)^\times = W_B B$ in which W_B is the group generated by the reflections r_α $(\alpha \in B)$. Since $B \subseteq \Pi \subseteq R \cap \dot R$ and $(\dot R^1)^\times = W_B B$, Fact 2 implies that $0 \in S_{\dot\alpha}$ for all $\dot\alpha \in (\dot R^1)^\times$. Since $(\dot R^1)^\times = W_B B$, for $\dot\alpha, \dot\beta \in (\dot R^1)^\times$ with $(\dot\alpha,\dot\alpha) = (\dot\beta,\dot\beta)$, there is $w \in W_B$ with $w(\dot\alpha) = \dot\beta$, so again using Fact 2, we get $S_{\dot\alpha} = S_{\dot\beta}$.

Step 2. For $p \neq q \in J$, we have $0 \in S_{\pm\delta_p \pm\delta_q}$: Using Fact 2, we get

$$S_{\epsilon_1 - \delta_p} = S_{r_{2\epsilon_1}(\epsilon_1 - \delta_p)} = S_{-\epsilon_1 - \delta_p}. \tag{4.3}$$

Set $\dot\alpha := -\epsilon_1 - \delta_p$ and $\dot\beta := \epsilon_1 - \delta_q$. Using Facts 3, 4, we have $S_{-\epsilon_1-\delta_p} + S_{\epsilon_1 - \delta_q} \subseteq S_{-\delta_p - \delta_q}$. But $\epsilon_1 - \delta_p, \epsilon_1 - \delta_q \in \Pi \subseteq R \cap \dot R$ which in turn, together with (4.3), implies that

$$0 \in S_{\epsilon_1-\delta_p} \cap S_{\epsilon_1-\delta_q} = S_{-\epsilon_1-\delta_p} \cap S_{\epsilon_1-\delta_q}. \tag{4.4}$$

Then we get

$$0 \in S_{-\epsilon_1-\delta_p} + S_{\epsilon_1-\delta_q} \subseteq S_{-\delta_p-\delta_q}.$$

This, together with Fact 1 and the fact that $r_{-\delta_p-\delta_q}(\epsilon_1 - \delta_p) = \epsilon_1 + \delta_q$, implies that $S_{\epsilon_1+\delta_q} = S_{\epsilon_1-\delta_p}$. So $0 \in S_{\epsilon_1+\delta_q}$ by (4.4). Changing the roles of p and q, we get $0 \in S_{\epsilon_1+\delta_p}$; in fact, we have using (4.4) that

$$0 \in S_{\epsilon_1\pm\delta_p} \cap S_{\epsilon_1\pm\delta_q}. \tag{4.5}$$

Since for $t_1, t_2 \in \{1,-1\}$, $(\epsilon_1 + t_1\delta_p) + (\epsilon_1 + t_2\delta_q) \notin \dot R$, we get using Fact 3 that $S_{\epsilon_1+t_1\delta_p} - S_{\epsilon_1+t_2\delta_q} \subseteq S_{t_1\delta_p - t_2\delta_q}$; in particular, (4.5) implies that $0 \in S_{\pm\delta_p\pm\delta_q}$.

Step 3. For $i \in I$ and $p \in J$, we have

$$0 \in F := S_{\delta_1-\epsilon_1} = S_{\pm\epsilon_i\pm\delta_p}:$$

We first note that $\delta_1 - \epsilon_1 \in \Pi \subseteq \dot R \cap R$, so

$$0 \in S_{\delta_1-\epsilon_1} = F.$$

Next suppose $p \neq 1$; since $r_{\delta_p \pm \delta_1}(\epsilon_i + \delta_p) = \epsilon_i \mp \delta_1$, we get $S_{\epsilon_i+\delta_p} = S_{\epsilon_i \mp \delta_1}$ by Fact 2 and Step 2; i.e.,

$$S_{\epsilon_i+\delta_p} = S_{\epsilon_i+\delta_1} = S_{\epsilon_i-\delta_1}.$$

So Fact 2 implies that

$$S_{-\epsilon_i+\delta_1} = S_{r_{2\epsilon_i}(\epsilon_i+\delta_1)} = S_{\epsilon_i+\delta_1} = S_{\epsilon_i-\delta_1} = S_{r_{2\epsilon_i}(\epsilon_i-\delta_1)} = S_{-\epsilon_i-\delta_1}$$

and

$$\begin{aligned} S_{-\epsilon_i+\delta_p} = S_{r_{2\epsilon_i}(\epsilon_i+\delta_p)} = S_{\epsilon_i+\delta_p} = S_{\epsilon_i-\delta_1} &= S_{r_{\delta_p-\delta_1}(\epsilon_i-\delta_1)} = S_{\epsilon_i-\delta_p} \\ &= S_{r_{2\epsilon_i}(\epsilon_i-\delta_p)} = S_{-\epsilon_i-\delta_p}. \end{aligned}$$

This completes the proof.

Step 4. Recall F from Step 3. We have $F - F \subseteq F$ and for $i \neq j \in I$ and $p \neq q \in J$,

$$F = S_{\pm\epsilon_i \pm\epsilon_j} = S_{\pm\delta_p \pm\delta_q}:$$

Suppose $t, t_1, t_2 \in \{\pm 1\}$ and set $\dot\alpha := t\epsilon_i + t_1\epsilon_j$ and $\dot\beta := t\epsilon_i + t_2\delta_p$. Since $(\dot\alpha, \dot\beta) \neq 0$ and $\dot\alpha + \dot\beta \notin \dot R$, Fact 3 implies that

$$S_{t\epsilon_i+t_1\epsilon_j} - F = S_{t\epsilon_i+t_1\epsilon_j} - S_{t\epsilon_i+t_2\delta_p} \subseteq S_{t_1\epsilon_j - t_2\delta_p} = F.$$

Since $0 \in F$ by Step 3, this means that

$$S_{\pm\epsilon_i\pm\epsilon_j} \subseteq F.$$

One also knows that $(\epsilon_i + \delta_p) + (\epsilon_i + \delta_p) \notin R$, so again using Fact 3, we have

$$F - F = S_{\epsilon_i+\delta_p} - S_{\epsilon_i+\delta_p} \subseteq S_{\epsilon_i-\epsilon_j} \subseteq F;$$

but $0 \in F$, so these, all together with Step 1, imply that $F - F \subseteq F = S_{\epsilon_i-\epsilon_j} = S_{\pm\epsilon_i\pm\epsilon_j}$. The same argument implies that $F = S_{\pm\delta_p\pm\delta_q}$.

Step 5. For $i \in I$ and $p \in J$, we have

$$L_1 := S_{2\epsilon_1} = S_{\pm 2\epsilon_i} \quad \text{and} \quad L_2 := S_{2\delta_1} = S_{\pm 2\delta_p}:$$

It follows from Fact 2, Steps 1, 4 and the fact that for $p \neq 1$,

$$r_{\delta_1 \mp \delta_p}(2\delta_1) = \pm 2\delta_p \quad \text{and} \quad r_{\delta_2+\delta_1}(2\delta_2) = -2\delta_1.$$

Step 6. We have $0 \in L_1$ and $L_i - 2L_i \subseteq L_i$ for $i = 1, 2$; in particular, $L_i = -L_i$: From Step 1, we see $0 \in L_1$. For the last assertion, suppose $\dot\alpha = \dot\beta := 2\epsilon_1$. Then Fact 1 implies that $L_1 - 2L_1 \subseteq L_1$. Similarly, we get $L_2 - 2L_2 \subseteq L_2$.

Step 7. We have $L_2 + 2F \subseteq L_2$ and if $|I| \geq 2$, $L_1 + 2F \subseteq L_1$: Set $\dot\beta = 2\delta_1$ and $\dot\alpha := \delta_1 + \delta_2$ and use Fact 1 to get

$$S_{2\delta_1} - 2S_{\delta_1+\delta_2} = S_{\dot\beta} - 2S_{\dot\alpha} \subseteq S_{-2\delta_2}.$$

By Step 4 and Fact 4, we have $S_{\dot\alpha} = F = -F$ and by Step 5, we have $L_2 = S_{2\delta_1} = S_{-2\delta_2}$, so we get $L_2 + 2F \subseteq L_2$. If $|I| \geq 2$, setting $\dot\beta := 2\epsilon_1$ and $\dot\alpha := \epsilon_1 + \epsilon_2$ and using the same argument as above, we get $L_1 + 2F \subseteq L_1$ as we desired.

Step 8. $L_i + F \subseteq F$ $(i = 1, 2)$ and $F = L_1 \cup L_2$: Contemplating Step 3 and using Fact 1 by taking $\dot\alpha := 2\delta_1$ and $\dot\beta := \epsilon_1 - \delta_1$, we have $F + L_1 \subseteq F$. Similarly, we have $F + L_2 \subseteq F$; in particular

$$L_i \subseteq F \quad (i = 1, 2)$$

as $0 \in F$. For the last assertion, suppose $\sigma \in F$. We have $\alpha := \epsilon_1 + \delta_1 + \sigma$, $\beta := \epsilon_1 - \delta_1 \in R$; see Step 3. Since $(\alpha, \beta) \neq 0$, by (S5) we have either $\alpha + \beta \in R$ or $\alpha - \beta \in R$. This implies that either $\sigma \in S_{2\epsilon_1}$ or $\sigma \in S_{2\delta_1}$. Therefore, $F \subseteq L_1 \cup L_2$ and so we are done.

Summarizing our result, there are an abelian group $F \subseteq V^0$ and subspaces $L_1, L_2 \subseteq F$ with

$$\begin{gathered} 0 \in L_1, \quad L_i - 2L_i \subseteq L_i, \quad L_i + F \subseteq F \quad (i = 1, 2), \\ F = L_1 \cup L_2, \quad L_1 + 2F \subseteq L_1 \text{ if } |I| \geq 2 \quad \text{and} \quad L_2 + 2F \subseteq L_2 \end{gathered}$$

such that

$$R = R^0 \cup \big(((\dot R_{\mathrm{re}})_{\mathrm{sh}} \cup \dot R_{\mathrm{ns}}) + F\big) \cup (\dot R^1_{\mathrm{lg}} + L_1) \cup (\dot R^2_{\mathrm{lg}} + L_2)$$

if $|I| > 1$ and

$$R = R^0 \cup ((\dot R^2_{\mathrm{sh}} \cup \dot R_{\mathrm{ns}}) + F) \cup (\dot R^1_{\mathrm{sh}} + L_1) \cup (\dot R^2_{\mathrm{lg}} + L_2)$$

if $|I| = 1$, where

$$\dot R^1 = \big\{\pm\epsilon_i \pm \epsilon_j \mid i, j \in I\big\} \quad \text{and} \quad \dot R^2 = \big\{\pm\delta_p \pm \delta_q \mid p, q \in J\big\}.$$

Since R is tame, we have $R^0 \subseteq R^\times - R^\times$, so we get $R^0 \subseteq F - F \subseteq F$. On the other hand, for $\dot\alpha := \delta_1 + \delta_2$ and $\sigma \in F$, we have $\dot\alpha + \sigma, \dot\alpha \in R$ and so the root string property implies that $\sigma \in R$; i.e., $F \subseteq R^0$. Therefore, we have $R^0 = F$. So

$$R = \dot R(F, L_1, L_2).$$

This completes the proof. $\square$

Theorem 6. *Assume U and V are vector spaces and $F, L_1, L_2 \subseteq U$, as well as $F', L'_1, L'_2 \subseteq V$, satisfy the same conditions as in (‡). Then $\dot{R}(F, L_1, L_2)$ and $\dot{R}(F', L'_1, L'_2)$ are isomorphic extended affine root supersystems in $\dot{U} \oplus U$ and $\dot{U} \oplus V$ respectively if and only if there are $\tau_1, \tau_2 \in F'$ with $\frac{1}{2}(\tau_1 + \tau_2) \in F'$ and a linear isomorphism $\psi\colon U \longrightarrow V$ such that if $|I| \neq |J|$, then*

$$\psi(F) = F' \quad \textit{and} \quad \psi(L_i) = L'_i - \tau_i \quad (i = 1, 2)$$

and if $|I| = |J|$, then

$$\psi(F) = F' \quad \textit{and} \quad \psi(L_i) = L'_j - \tau_j \quad (\{i, j\} = \{1, 2\} \textit{ or } 1 \leq i = j \leq 2).$$

Proof. Suppose that $\dot{R}(F, L_1, L_2)$ and $\dot{R}(F', L'_1, L'_2)$ are isomorphic. We denote corresponding bilinear forms on the underlying vector spaces respectively by $(\cdot, \cdot)$ and $(\cdot, \cdot)'$. Then there are a nonzero scalar r and a linear isomorphism $\varphi\colon \dot{U} \oplus U \longrightarrow \dot{U} \oplus V$ such that

$$\varphi(R) = R' \quad \text{and} \quad (u, v) = r(\varphi(u), \varphi(v))' \quad (u, v \in \dot{U} \oplus U);$$

in particular, $\varphi(U) = V$ (equivalently, $\varphi^{-1}(V) = U$), so there are linear maps

$$\zeta\colon \dot{U} \longrightarrow \dot{U}, \quad \eta\colon \dot{U} \longrightarrow V \quad \text{and} \quad \psi\colon U \longrightarrow V \tag{4.6}$$

such that ζ and ψ are linear isomorphisms and

$$\varphi(\dot{u} + \sigma) = \zeta(\dot{u}) + \eta(\dot{u}) + \psi(\sigma) \quad (\dot{u} \in \dot{U},\ \sigma \in U).$$

In fact, ζ defines an isomorphism from $\dot{R}$ to $\dot{R}$. Since for $p, q \in J$ with $p \neq q$, $\delta_p - \delta_q \in R \cap \dot{R}$, we have

$$-2 = (\delta_p - \delta_q, \delta_p - \delta_q) = r(\zeta(\delta_p - \delta_q), \zeta(\delta_p - \delta_q))',$$

we get either $r = 1$ or $r = -1$. In the former case,

$$\zeta(\{\pm\epsilon_i \mid i \in I\}) = \{\pm\epsilon_i \mid i \in I\} \quad \text{and} \quad \zeta(\{\pm\delta_p \mid p \in J\}) = \{\pm\delta_p \mid p \in J\},$$

and in the latter case,

$$\zeta(\{\pm\epsilon_i \mid i \in I\}) = \{\pm\delta_p \mid p \in J\} \quad \text{and} \quad \zeta(\{\pm\delta_p \mid p \in J\}) = \{\pm\epsilon_i \mid i \in I\};$$

in particular, in the latter case,

$$|I| = |J|.$$

Case 1. $r = 1$: In this case,

$$\begin{aligned}\varphi(\{\pm\delta_p \pm \delta_q \mid p \neq q \in J\} + F) &= \{\pm\delta_p \pm \delta_q \mid p \neq q \in J\} + F',\\ \varphi(\{\pm\epsilon_i \pm \delta_p \mid i \in I, p \in J\} + F) &= \{\pm\epsilon_i \pm \delta_q \mid i \in I, p \in J\} + F',\\ \varphi(\{\pm 2\epsilon_i \mid i \in I\} + L_1) &= \{\pm 2\epsilon_i \mid i \in I\} + L_1',\\ \varphi(\{\pm 2\delta_p \mid p \in J\} + L_2) &= \{\pm 2\delta_p \mid p \in J\} + L_2'.\end{aligned} \tag{4.7}$$

For $\gamma \coloneqq \eta(\delta_1 - \delta_2)$, we have $\zeta(\delta_1 - \delta_2) + \gamma = \varphi(\delta_1 - \delta_2)$, so we get using (4.7) that $\gamma \in F'$ and

$$\zeta(\delta_1 - \delta_2) + F' = \varphi(\delta_1 - \delta_2 + F) = \zeta(\delta_1 - \delta_2) + \gamma + \psi(F).$$

This implies that

$$\psi(F) = F' - \gamma = F' \tag{4.8}$$

as F' is a group. Also, setting

$$\tau_1 \coloneqq \eta(2\epsilon_1) \quad \text{and} \quad \tau_2 \coloneqq \eta(2\delta_1),$$

we have

$$\varphi(2\epsilon_1 + L_1) = \zeta(2\epsilon_1) + \eta(2\epsilon_1) + \psi(L_1) \quad \text{and} \quad \varphi(2\delta_1 + L_2) = \zeta(2\delta_1) + \eta(2\delta_1) + \psi(L_2).$$

This together with (4.7) implies that

$$\psi(L_i) = L_i' - \tau_i \quad (i = 1, 2).$$

We know that

$$L_1' \cup L_2' = F' \overset{(4.8)}{=} \psi(F) = \psi(L_1 \cup L_2) = (L_1' - \tau_1) \cup (L_2' - \tau_2).$$

But F' is a group, so this implies that

$$\tau_1, \tau_2 \in F'.$$

Again using (4.7), we have

$$\begin{aligned}\zeta(\epsilon_1 + \delta_1) + F' = \varphi(\epsilon_1 + \delta_1 + F) &= \zeta(\epsilon_1 + \delta_1) + \eta(\epsilon_1 + \delta_1) + \psi(F)\\ &\overset{(4.8)}{=} \zeta(\epsilon_1 + \delta_1) + \frac{1}{2}(\tau_1 + \tau_2) + F',\end{aligned}$$

so we get

$$\frac{1}{2}(\tau_1 + \tau_2) \in F'.$$

Case 2. $r=-1$: In this case, $|I|=|J|$ and

$$\begin{aligned}\varphi(\{\pm\delta_p\pm\delta_q \mid p\neq q\in J\}+F)&=\{\pm\epsilon_i\pm\epsilon_j \mid i,j\in I, i\neq j\}+F',\\ \varphi(\{\pm\epsilon_i\pm\delta_p \mid i\in I, p\in J\}+F)&=\{\pm\epsilon_i\pm\delta_p \mid i\in I, p\in J\}+F',\\ \varphi(\{\pm 2\epsilon_i \mid i\in I\}+L_1)&=\{\pm 2\delta_p \mid p\in J\}+L_2',\\ \varphi(\{\pm 2\delta_p \mid p\in J\}+L_2)&=\{\pm 2\epsilon_i \mid i\in I\}+L_1'.\end{aligned} \tag{4.9}$$

Recall (4.6) and set

$$\tau_2 := \eta(2\epsilon_1) \quad \text{and} \quad \tau_1 := \eta(2\delta_1);$$

then we have

$$\varphi(2\epsilon_1+L_1)=\zeta(2\epsilon_1)+\tau_2+\psi(L_1) \quad \text{and} \quad \varphi(2\delta_1+L_1)=\zeta(2\delta_1)+\tau_1+\psi(L_2).$$

This together with (4.9) implies that

$$\psi(L_i)=L_j'-\tau_j \quad (\{i,j\}=\{1,2\}).$$

Also as in the previous case, we have

$$\psi(F)=F' \quad \text{and} \quad \tau_1,\tau_2\in F' \text{ with } \frac{1}{2}(\tau_1+\tau_2)\in F'.$$

Conversely, suppose that there is a linear isomorphism $\psi\colon U\longrightarrow V$, as well as $\tau_1,\tau_2\in F'$ with $\frac{1}{2}(\tau_1+\tau_2)\in F'$, such that

$$\psi(F)=F' \quad \text{and} \quad \psi(L_i)=L_i'-\tau_i \quad (i=1,2).$$

Define

$$\begin{aligned}\varphi\colon \dot{U}\oplus U &\longrightarrow \dot{U}\oplus V,\\ 2\epsilon_i+\sigma &\longrightarrow 2\epsilon_i+\tau_1+\psi(\sigma) \quad (i\in I, \sigma\in U),\\ 2\delta_p+\sigma &\longrightarrow 2\delta_p+\tau_2+\psi(\sigma) \quad (p\in J, \sigma\in U).\end{aligned}$$

Then as for $i=1,2$, $L_i'\subseteq F'$ and $L_i'-2L_i'\subseteq L_i'$, for $j\in I$ and $p\in J$ we have

$$\begin{aligned}\varphi(2\epsilon_j+L_1)&=2\epsilon_j+\tau_1+L_1'-\tau_1=2\epsilon_j+L_1',\\ \varphi(-2\epsilon_j+L_1)&=-2\epsilon_j-\tau_1+L_1'-\tau_1=-2\epsilon_j+L_1'-2\tau_1=-2\epsilon_j+L_1',\\ \varphi(2\delta_p+L_2)&=2\delta_p+\tau_2+L_2'-\tau_2=2\delta_p+L_2',\\ \varphi(-2\delta_p+L_2)&=-2\delta_p-\tau_2+L_2'-\tau_2=-2\delta_p+L_2'-2\tau_2=-2\delta_p+L_2'.\end{aligned}$$

Also, as F' is a group, for $t_1,t_2\in\{\pm1\}$, for $i\neq j\in I$ and $p\neq q\in J$ we have

$$\begin{aligned}\varphi(t_1\epsilon_i+t_2\delta_p+F)&=t_1\epsilon_i+\frac{t_1}{2}\tau_1+t_2\delta_p+\frac{t_2}{2}\tau_2+\psi(F)\\&=t_1\epsilon_i+t_2\delta_p+F'+\frac{t_1}{2}(\tau_1+\tau_2)+\underbrace{\Big(\frac{t_2}{2}-\frac{t_1}{2}\Big)}_{\in\{0,\pm1\}}\tau_2\\&=t_1\epsilon_i+t_2\delta_p+F';\end{aligned}$$

similarly, we get $\varphi(t_1\epsilon_i+t_2\epsilon_j+F)\cup\varphi(t_1\delta_p+t_2\delta_q+F)\subseteq R$. Moreover,

$$(u,v)=(\varphi(u),\varphi(v))'\quad(u,v\in\dot U\oplus U).$$

This means that φ is an isomorphism from $\dot R(F,L_1,L_2)$ to $\dot R(F',L_1',L_2')$.

Finally, assume $|I|=|J|$ and that there are a linear isomorphism $\psi\colon U\longrightarrow V$ and $\tau_1,\tau_2\in F'$ with $\frac{1}{2}(\tau_1+\tau_2)\in F'$ such that

$$\psi(F)=F'\quad\text{and}\quad\psi(L_i)=L_j'-\tau_j\quad(\{i,j\}=\{1,2\}).$$

Without loss of generality, we assume $I=J$ and define

$$\begin{aligned}\varphi\colon\dot U\oplus U&\longrightarrow\dot U\oplus V\\2\epsilon_i+\sigma&\longrightarrow2\delta_i+\tau_2+\psi(\sigma)\quad(i\in I,\sigma\in U),\\2\delta_i+\sigma&\longrightarrow2\epsilon_i+\tau_1+\psi(\sigma)\quad(i\in I,\sigma\in U).\end{aligned}$$

So as in the previous case, we get that φ is an isomorphism from $\dot R(F,L_1,L_2)$ to $\dot R(F',L_1',L_2')$. □

§5. Type $BC(1,1)$

Suppose that $(\dot U,(\cdot,\cdot),\dot R)$ is an irreducible finite root supersystem of type $BC(1,1)$. Without loss of generality, we assume $(\dot U,(\cdot,\cdot),\dot R)$ is as in (2.3), i.e.,

$$\dot R=\{\pm\epsilon,\pm\delta,\pm2\epsilon,\pm2\delta,\pm\epsilon\pm\delta\}$$

with

$$(\epsilon,\epsilon)=-(\delta,\delta)=1\quad\text{and}\quad(\epsilon,\delta)=0.$$

Suppose that U is a vector space and S, F, E_1 and E_2 are subsets of U satisfying

$$(\sharp)\qquad\begin{gathered}0\in S,\quad S-2S\subseteq S,\quad F+2S\subseteq F,\\E_i+S\subseteq S,\quad E_i-2E_i\subseteq E_i,\quad E_i+4S\subseteq E_i\quad(i=1,2),\\F\text{ is a subgroup of }U\text{ and }F=E_1\cup E_2.\end{gathered}$$

Note that we have

$$S = -S \quad \text{and} \quad E_i = -E_i \subseteq S \quad (i = 1, 2) \quad (\text{in particular } F \subseteq S). \tag{5.1}$$

Set

$$\dot{R}(S, F, E_1, E_2) := (S - S) \cup (\{\pm\epsilon, \pm\delta\} + S) \cup (\{\pm\epsilon \pm \delta\} + F) \\ \cup (\{\pm 2\epsilon\} + E_1) \cup (\{\pm 2\delta\} + E_2),$$

and extend the form $(\cdot, \cdot)$ to $\dot{U} \oplus U$ such that U is the radical of this new form. Then one can easily check that $(\dot{U} \oplus U, (\cdot, \cdot), R)$ is a tame irreducible extended affine root supersystem of type $BC(1, 1)$ in $\dot{U} \oplus U$.

Theorem 7. *Each tame irreducible extended affine root supersystem of type $BC(1, 1)$ is of the form $\dot{R}(S, F, E_1, E_2)$.*

Proof. Assume $(V, (\cdot, \cdot), R)$ is a tame irreducible extended affine root supersystem of type $BC(1, 1)$. As in (2.6), for $\dot{R} = \{\pm\epsilon, \pm\delta, \pm 2\epsilon, \pm 2\delta, \pm\epsilon\pm\delta\}$ and $\Pi = \{\epsilon - \delta, \delta\}$, we have

$$R = \bigcup_{\dot{\alpha} \in \dot{R}} (\dot{\alpha} + S_{\dot{\alpha}})$$

for some nonempty subsets $S_{\dot{\alpha}}$ of V^0. As $\Pi \subseteq R$, we have in particular that

$$0 \in S_{\epsilon-\delta} \cap S_{\delta}. \tag{5.2}$$

Since $(\epsilon - \delta) - \delta \notin \dot{R}$ and $(\epsilon - \delta) + \epsilon \notin \dot{R}$, for $\sigma \in S_{\epsilon-\delta}$, $\tau \in S_{\delta}$ and $\gamma \in S_{\epsilon}$ (equivalently, $-\gamma \in -S_{-\epsilon}$ by Fact 4), we have

$$(\epsilon - \delta + \sigma) - (\delta + \tau) \notin R \quad \text{and} \quad (\epsilon - \delta + \sigma) + (\epsilon + \gamma) \notin R$$

and so by (S5),

$$\epsilon + (\sigma + \tau) = (\epsilon - \delta + \sigma) + (\delta + \tau) \in R \quad \text{and} \quad (-\delta + \sigma - \tau) = (\epsilon - \delta + \sigma) - (\epsilon + \gamma) \in R.$$

This gives that

$$S_{\epsilon-\delta} + S_{\delta} \subseteq S_{\epsilon} \quad \text{and} \quad S_{\epsilon-\delta} + S_{-\epsilon} \subseteq S_{-\delta}; \tag{5.3}$$

in particular, we get using (5.2) that

$$0 \in S_{\delta} \subseteq S_{\epsilon} \quad \text{and} \quad -S_{\epsilon} = S_{-\epsilon} \subseteq S_{-\delta} = -S_{\delta}.$$

So we have

$$0 = S_{\delta} = S_{\epsilon}. \tag{5.4}$$

Therefore, Fact 2 implies that

$$F := S_{\epsilon+\delta} = S_{r_\delta(\epsilon-\delta)} = S_{\epsilon-\delta} = S_{r_\epsilon(\epsilon-\delta)} = S_{-\epsilon-\delta} = S_{r_\delta(-\epsilon+\delta)} = S_{-\epsilon+\delta},$$

and

$$\begin{aligned} S_{-\epsilon} &= S_{r_\epsilon(\epsilon)} = S_\epsilon, & S_{-\delta} &= S_{r_\delta(\delta)} = S_\delta, \\ S_{-2\epsilon} &= S_{r_\epsilon(2\epsilon)} = S_{2\epsilon}, & S_{-2\delta} &= S_{r_\delta(2\delta)} = S_{2\delta}. \end{aligned}$$

So (5.2) and (5.4) imply that

$$0 \in F = S_{\pm\epsilon\pm\delta} \quad \text{and} \quad 0 \in S := S_\delta = S_{-\delta} = S_\epsilon = S_{-\epsilon}. \tag{5.5}$$

Set $E_1 := S_{2\epsilon} = S_{-2\epsilon}$ and $E_2 := S_{2\delta} = S_{-2\delta}$. So (5.5) implies that

$$R^\times = (\{\pm\epsilon, \pm\delta\} + S) \cup (\{\pm\epsilon \pm \delta\} + F) \cup (\{\pm 2\epsilon\} + E_1) \cup (\{\pm 2\delta\} + E_2). \tag{5.6}$$

Using Fact 1, we have

$$\begin{aligned} &S - 2S \subseteq S, \quad F + 2S \subseteq F, \quad F + E_i \subseteq F \quad (\text{in particular, } E_i \subseteq F), \\ &S + E_i \subseteq S, \quad E_i - 2E_i \subseteq E_i \quad \text{and} \quad E_i + 4S \subseteq E_i \quad (i = 1, 2). \end{aligned} \tag{5.7}$$

Moreover, for $\sigma \in F$, we have $\epsilon + \delta + \sigma, \epsilon - \delta \in R$, so by (S5), either $2\epsilon + \sigma \in R$ or $2\delta + \sigma \in R$. Thus, we get either $\sigma \in E_1$ or $\sigma \in E_2$, in other words, $F = E_1 \cup E_2$. In particular, as $F + E_i \subseteq F$ and $E_i = -E_i$ $(i = 1, 2)$, we get that F is a group.

To complete the proof, we just need to show that $R^0 = S - S$. Since R is tame, we have $R^0 \subseteq R^\times - R^\times$. By using (5.6), (5.7) and (5.1), we have $R^0 \subseteq S - S$. Also, since for $\sigma, \tau \in S$, $\epsilon + \sigma, \epsilon + \tau \in R$, the root string property implies that $\sigma - \tau \in R^0$, that is, $S - S \subseteq R^0$. So $R^0 = S - S$. This completes the proof. □

Proposition 8. *Suppose that U and V are vector spaces and S, F, E_1, E_2 and S', F', E_1', E_2' are subspaces of U and V respectively, satisfying the same conditions as in (♯). Then $\dot{R}(S, F, E_1, E_2)$ is isomorphic to $\dot{R}(S', F', E_1', E_2')$ if and only if there are $\sigma \in S'$, $\tau \in F' \cap (S' + S')$ and a linear isomorphism $\psi\colon U \longrightarrow V$ such that one of the following occurs:*

(i) $\psi(S) = S' + \sigma$, $\psi(F) = F'$, $\psi(E_1) = E_1' + 2\sigma$, $\psi(E_2) = E_2' + 2\sigma - 2\tau$,

(ii) $\psi(S) = S' + \sigma$, $\psi(F) = F'$, $\psi(E_1) = E_2' + 2\sigma$, $\psi(E_2) = E_1' + 2\sigma - 2\tau$.

Proof. Denote the forms on $\dot{U} \oplus U$ and $\dot{U} \oplus V$ respectively by $(\cdot,\cdot)$ and $(\cdot,\cdot)'$. Suppose that the conditions of (i) are fulfilled and define

$$\begin{aligned} &\varphi\colon \dot{U} \oplus U \longrightarrow \dot{U} \oplus V \\ &\epsilon \mapsto \epsilon - \sigma, \quad \delta \mapsto \delta - \tau + \sigma, \quad \gamma \mapsto \psi(\gamma) \quad (\gamma \in U). \end{aligned}$$

Then since $S'+F'=S'-(E'_1\cup E'_2)\subseteq S'$ and $S'-2S'\subseteq S'$, we have

$$\varphi(\pm\epsilon+S)=\pm(\epsilon-\sigma)+S'+\sigma=\pm\epsilon+S'$$

and

$$\varphi(\pm\delta+S)=\pm(\delta-\tau+\sigma)+S'+\sigma=\pm\delta+S'.$$

Also, as $F'=E'_1\cup E'_2\subseteq S'$, $F'+2S'\subseteq F'$ and $E'_i+4S'\subseteq E'_i$ $(i=1,2)$, for $t_1,t_2\in\{\pm1\}$ we have

$$\begin{aligned}\varphi(t_1\epsilon+t_2\delta+F)&=t_1\epsilon+t_2\delta-t_1\sigma+t_2(-\tau+\sigma)+F'=t_1\epsilon+t_2\delta+F',\\ \varphi(\pm2\epsilon+E_1)&=\pm(2\epsilon-2\sigma)+E'_1+2\sigma=\pm2\epsilon+E'_1,\\ \varphi(\pm2\delta+E_2)&=\pm(2\delta-2\tau+2\sigma)+E'_2+2\sigma-2\tau=\pm2\delta+E'_2.\end{aligned}$$

These together with the fact that $(\varphi(u),\varphi(u'))'=(u,u')$ $(u,u'\in\dot{U}\oplus U)$ give that φ is an isomorphism from $\dot{R}(S,F,E_1,E_2)$ to $\dot{R}(S',F',E'_1,E'_2)$.

Next assume (ii) is fulfilled and define

$$\begin{gathered}\varphi\colon\dot{U}\oplus U\longrightarrow\dot{U}\oplus V,\\ \epsilon\mapsto\delta-\sigma,\quad\delta\mapsto\epsilon-\tau+\sigma,\quad\gamma\mapsto\psi(\gamma)\quad(\gamma\in U).\end{gathered}$$

A similar argument to above implies that φ is an isomorphism from $\dot{R}(S,F,E_1,E_2)$ to $\dot{R}(S',F',E'_1,E'_2)$.

Conversely, suppose that $\dot{R}(S,F,E_1,E_2)$ is isomorphic to $\dot{R}(S',F',E'_1,E'_2)$. So there are a nonzero scalar r and a linear isomorphism $\varphi\colon\dot{U}\oplus U\longrightarrow\dot{U}\oplus V$ such that

$$\varphi(R)=R'\quad\text{and}\quad(\varphi(u),\varphi(u'))'=r(u,u')\quad(u,u'\in\dot{U}\oplus U).$$

Therefore, there are linear transformations

$$\zeta\colon\dot{U}\longrightarrow\dot{U},\quad\eta\colon\dot{U}\longrightarrow V\quad\text{and}\quad\psi\colon U\longrightarrow V$$

such that

$$\varphi(\dot{u}+\gamma)=\zeta(\dot{u})+\eta(\dot{u})+\psi(\gamma)\quad(\dot{u}\in\dot{U},\ \gamma\in U).$$

Set

$$\sigma:=-\eta(\epsilon)\in S'\quad\text{and}\quad\tau:=-\underbrace{\eta(\epsilon)}_{\in S'}-\underbrace{\eta(\delta)}_{\in S'}=-\eta(\epsilon+\delta)\in F'\cap(S'+S').$$

Then

$$\zeta(\epsilon)+\eta(\epsilon)+\psi(S)=\varphi(\epsilon+S)\in\{\pm\epsilon,\pm\delta\}+S'.$$

So

$$\psi(S)=S'+\sigma.$$

We also have

$$\zeta(\epsilon+\delta)+\eta(\epsilon+\delta)+\psi(F)=\varphi(\epsilon+\delta+F)\in\{\pm\epsilon\pm\delta\}+F'.$$

Therefore, as F' is a group, we have

$$\psi(F)=F'+\tau=F'.$$

Next we recall that for $u,u'\in\dot{U}\oplus U$, we have $(\varphi(u),\varphi(u'))'=r(u,u')$. So it follows that $r=\pm1$. We first suppose $r=1$; then

$$\zeta(2\epsilon)+\eta(2\epsilon)+\psi(E_1)=\varphi(2\epsilon+E_1)\subseteq\pm2\epsilon+E_1'.$$

So we get $\psi(E_1)=E_1'+2\sigma$. We also have

$$\zeta(2\delta)+\eta(2\delta)+\psi(E_2)=\varphi(2\delta+E_2)\subseteq\pm2\delta+E_2',$$

which gives $\psi(E_2)=E_2'+2\tau-2\sigma$. These altogether imply that condition (i) is satisfied. Next assume $r=-1$. Then

$$\zeta(2\epsilon)+\eta(2\epsilon)+\psi(E_1)=\varphi(2\epsilon+E_1)\subseteq\pm2\delta+E_2'.$$

So we get $\psi(E_1)=E_2'+2\sigma$. Moreover, we have

$$\zeta(2\delta)+\eta(2\delta)+\psi(E_2)=\varphi(2\delta+E_2)\subseteq\pm2\epsilon+E_1',$$

which in turn implies that $\psi(E_2)=E_1'+2\tau-2\sigma$. This completes the proof. □

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