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Integrality of v-adic Multiple Zeta Values

by

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Abstract

In this article, we prove the integrality of v-adic multiple zeta values (MZVs). For any index $\mathfrak{s} \in \mathbb{N}^r$ and finite place $v \in A := \mathbb{F}_q[\theta]$, Chang and Mishiba introduced the notion of the v-adic MZVs $\zeta_A(\mathfrak{s})_v$, which is a function field analogue of Furusho's p-adic MZVs. By estimating the v-adic valuation of $\zeta_A(\mathfrak{s})_v$, we show that $\zeta_A(\mathfrak{s})_v$ is a v-adic integer for almost all v. This result can be viewed as a function field analogue of the integrality of p-adic MZVs, which was proved by Akagi–Hirose–Yasuda and Chatzistamatiou.

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§1. Introduction

§1.1. Classical multiple zeta values

Real multiple zeta values, abbreviated as MZVs, are real numbers defined by

$$\zeta(s_1, \ldots, s_r) := \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{R}^{\times},$$

where $\mathfrak{s} := (s_1, \ldots, s_r) \in \mathbb{N}^r$ and $s_1 \geq 2$. Here $\mathrm{dep}(\mathfrak{s}) := r$ is called the *depth*, $\mathrm{wt}(\mathfrak{s}) := \sum_{i=1}^{r} s_i$ is called the *weight* and $\mathrm{ht}(\mathfrak{s}) :=$ the cardinality of $\{i \mid s_i \neq 1\}$ is called the *height*. Interesting properties of MZVs have been established in recent years, but there remain mysteries. For example, although many relations between MZVs have been discovered, the exact $\mathbb{Q}$-linear structure of MZVs is still unclear. We refer the reader to [BGF18, IKZ06, K19, Zh16] for more details.

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Real MZVs come in many variants; we first briefly review the p-adic MZVs introduced by Furusho [F04]. Consider the one-variable multiple polylogarithm

$$\mathrm{Li}_{(s_1,\dots,s_r)}(z) := \sum_{n_1>n_2>\cdots>n_r\geq 1} \frac{z^{n_1}}{n_1^{s_1}\cdots n_r^{s_r}},$$

where $(s_1,\dots,s_r)\in\mathbb{N}^r$ and $s_1\geq 2$. We have

$$\zeta(s_1,\dots,s_r) = \mathrm{Li}_{(s_1,\dots,s_r)}(z)|_{z=1}.$$

We write $\mathrm{Li}_{(s_1,\dots,s_r)}(z)_p$ for the p-adic function defined by the same series on $\mathbb{C}_p$. Then $\mathrm{Li}_{(s_1,\dots,s_r)}(z)_p$ converges on the open unit disk centered at 0. Thus, in the non-archimedean context, it does not make sense to take the limit $z\to 1$. But Furusho [F04] applied Coleman integration [Col82] to p-adically analytically continue $\mathrm{Li}_{(s_1,\dots,s_r)}(z)_p$ to $\mathbb{C}_p\setminus\{1\}$, and then took a certain limit $z\to 1$ to define the p-adic MZV $\zeta(s_1,\dots,s_r)_p$. These p-adic MZVs have features in common with real MZVs; for example, it is shown in [FJ07] that p-adic MZVs satisfy the same regularized double shuffle relations as real MZVs satisfy [IKZ06].

Furusho [F04, Que. 2.26] asked whether his p-adic MZVs are p-adic integers, that is, whether $\zeta(s_1,\dots,s_r)_p\in\mathbb{Z}_p$ for all primes p and $(s_1,\dots,s_r)\in\mathbb{N}^r$. Recently, this question was answered by Akagi–Hirose–Yasuda and Chatzistamatiou.

Theorem 1.1.1 ([AHY, Cha17]). *Every p-adic MZV is a p-adic integer. Moreover, fix an index $(s_1\dots,s_r)\in\mathbb{N}^r$; then for all but finitely many primes p, the p-adic valuation of p-adic MZVs is greater than the weight $\sum_{i=1}^r s_i$.*

Now we describe an application of Theorem 1.1.1, given in [AHY]. Consider the $\mathbb{Q}$-algebra $\mathcal{A} := (\prod_p \mathbb{Z}/p\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$, where p runs over all primes p. Kaneko and Zagier defined *finite multiple zeta values* (abbreviated as FMZVs) by

$$\zeta_{\mathcal{A}}(s_1,\dots,s_r) := (\zeta_{\mathcal{A}}(s_1\dots,s_r)_p)_p \in \mathcal{A},$$

where the p-component $\zeta_{\mathcal{A}}(s_1,\dots,s_r)_p$ is defined by

$$\sum_{p>n_1>n_2>\cdots>n_r\geq 1} \frac{1}{n_1^{s_1}\cdots n_r^{s_r}} \bmod p.$$

Let $w\in\mathbb{N}$, and let $p\in\mathbb{N}$ be a prime number. We set Z_w to be the $\mathbb{Q}$-vector space generated by all real MZVs of weight w, $Z_{w,p}$ to be the $\mathbb{Q}$-vector space generated by all p-adic MZVs of weight w, and $Z_{w,\mathcal{A}}$ to be the $\mathbb{Q}$-vector space generated by all FMZVs of weight w. Then we have the following conjectures which concern the dimensions.

Conjecture 1.1.2. *For each $w \in \mathbb{N}$, the following identities hold:*

(1) *(Zagier)*

$$\frac{1}{1-X^2-X^3} = \sum_{w\geq 0} (\dim_{\mathbb{Q}} Z_w) X^w.$$

(2) *(Furusho–Yamashita [Ya10, Conj. 2])*

$$\frac{1-X^2}{1-X^2-X^3} = \sum_{w\geq 0} (\dim_{\mathbb{Q}} Z_{w,p}) X^w.$$

(3) *(Kaneko–Zagier)*

$$\frac{1-X^2}{1-X^2-X^3} = \sum_{w\geq 0} (\dim_{\mathbb{Q}} Z_{w,\mathcal{A}}) X^w.$$

Akagi–Hirose–Yasuda combine the integrality of p-adic MZVs and the special case of Jarossay's result [J20, Eqn. (1.19)] to give the upper bound of the above conjecture for FMZVs.

Theorem 1.1.3 ([AHY]). *For each integer $w \in \mathbb{N}$, let*

$$\frac{1-X^2}{1-X^2-X^3} = \sum_{w\geq 0} (d_{w,\mathcal{A}}) X^w.$$

Then we have $\dim_{\mathbb{Q}} Z_{w,\mathcal{A}} \leq d_{w,\mathcal{A}}$.

It is natural to ask whether or not the same phenomenon occurs in the characteristic p analogue. The main purpose of this article is to study the function field analogue of Theorem 1.1.1 and to build a suitable framework for future study of the function field analogue of Theorem 1.1.3.

§1.2. MZVs in positive characteristic

The function field analogues of real MZVs were defined by Thakur [T04], generalizing Carlitz zeta values [Ca35]. Let $A := \mathbb{F}_q[\theta]$, $k := \mathbb{F}_q(\theta)$, and define k_∞ to be the completion of k at the infinite place denoted by ∞. For any index $(s_1, \ldots, s_r) \in \mathbb{N}^r$, the ∞-adic MZV is defined by the series

$$\zeta_A(s_1, \ldots, s_r) := \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in k_\infty,$$

where $(a_1, \ldots, a_r) \in A^r$ with a_i monic and $\deg_\theta a_i$ strictly decreasing. In [T09] Thakur showed that $\zeta_A(s_1, \ldots, s_r)$ is non-vanishing for all indices $(s_1, \ldots, s_r) \in \mathbb{N}^r$. For the transcendence problem, Yu [Yu91] proved that all single zeta values $\zeta_A(s_1)$ are transcendental and Chang [C14] proved the transcendence of $\zeta_A(s_1, \ldots, s_r)$

for all indices $(s_1, \ldots, s_r)$. This problem remains open for real MZVs in classical transcendence theory. Some interesting features of real MZVs remain valid for their characteristic p counterpart. For example, Terasoma [Te02] and Goncharov [Gon02] showed that real MZVs are periods of mixed Tate motives, and Anderson–Thakur [AT09] showed that ∞-adic MZVs appear as periods of t-motives.

Inspired by Furusho's definition of p-adic MZVs, Chang and Mishiba considered the Carlitz multiple star polylogarithms (abbreviated as CMSPLs)

$$\mathrm{Li}^{\star}_{(s_1,\ldots,s_r)}(z_1, \ldots, z_r) := \sum_{i_1 \geq \cdots \geq i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}},$$

where $(s_1, \ldots, s_r) \in \mathbb{N}^r$, $L_0 := 1$ and $L_i := (\theta - \theta^q) \cdots (\theta - \theta^{q^i})$ for $i \geq 1$. Chang and Mishiba [CM21, Thm. 5.2.5] proved that ∞-adic MZVs can be written as k-linear combinations of CMSPLs at some precise integral points with explicit coefficients. Let v be a fixed finite place of k and regard $\mathrm{Li}^{\star}_{(s_1,\ldots,s_r)}(z_1, \ldots, z_r)_v$ as a v-adic function defined by the same series on $\mathbb{C}_v^r$ where $\mathbb{C}_v$ is the completion of an algebraic closure of the completion of k at v. Then the series converges on the open unit disk centered at 0. Chang and Mishiba [CM19, Prop. 4.1.1] use the logarithmic interpretation to do the analytic continuation of CMSPLs v-adically. Moreover, they defined v-adic MZVs in [CM21, Def. 6.1.1] by using the same k-linear combinations of CMSPLs, viewed v-adically, and they used Yu's sub-t-module theory [Yu97] to prove that v-adic MZVs satisfy the same k-linear relations as the corresponding ∞-adic MZVs do. This result can be viewed as a positive answer to the function field analogue of Furusho's conjecture [F06, Conj. A], [F07, Sect. 3.1] (see also [CM21, Conj. 1.1.1]), which predict that the p-adic MZVs satisfy the same $\mathbb{Q}$-linear relations that the corresponding real MZVs satisfy. These v-adic MZVs are the main objects of study in this paper.

§1.3. Outline and main result

In Section 2 we fix our notation and setting, then briefly review the terminology of t-modules as in [A86].

In Section 3 we first follow [CM19] closely to review the logarithmic interpretation and v-adic analytic continuation of CMSPLs. Based on the logarithmic interpretation and the functional equation for logarithms of t-modules, it is reasonable to establish certain functional equations for CMSPLs arising from logarithms of t-modules. But the difficulty here is that the logarithmic interpretation for CMSPLs relies on specific evaluation of the logarithm of t-modules at some special points, and this situation may not hold after analytic continuation. We overcome this issue in Section 3. As an application, we study the v-adic valuation of v-adic CMSPLs and prove that the k-vector space generated by the values of v-adic CMSPLs forms an algebra. The latter result is clear for ∞-adic CMSPLs by stuffle relations, but the stuffle relations are still unclear for the v-adic analytic continuation of CMSPLs. Here, we prove it by the functional equation to avoid the use of stuffle relations.

In Section 4 we first follow [CM21] closely to construct v-adic MZVs via v-adic CMSPLs. Then we explain how to arrive at our main result via the properties of v-adic CMSPLs obtained in Section 3. Our main result, stated as Theorem 4.2.1 later, is the following:

Theorem 1.3.1. *Let $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and q_v be the cardinality of the residue field A/vA. If we set*

$$B_{w,v} := \min_{n \geq 0}\{q_v^n - n \cdot w\},$$

then we have

$$\operatorname{ord}_v(\zeta_A(\mathfrak{s})_v) \geq B_{\operatorname{wt}(\mathfrak{s}),v} - \frac{\operatorname{wt}(\mathfrak{s}) - \operatorname{dep}(\mathfrak{s}) - \operatorname{ht}(\mathfrak{s})}{q_v - 1}.$$

In particular, if we set A_v to be the valuation ring inside the completion of k at v, then we have

$$\zeta_A(\mathfrak{s})_v \in A_v \quad \text{if } q_v \geq \operatorname{wt}(\mathfrak{s}).$$

We will provide an example which shows that certain conditions for the integrality of v-adic MZVs are necessary.

In Section 5 we formulate a new framework, the *adelic* MZVs, based on our integrality result. We develop some properties of adelic MZVs and we hope that they may be helpful for the later study of the function field analogue of Theorem 1.1.3.

§2. Preliminaries

§2.1. Notation

$\mathbb{F}_q$:= a finite field with q elements, for q a power of a prime number p.

A := $\mathbb{F}_q[\theta]$, the polynomial ring in the variable θ over $\mathbb{F}_q$.

v := a monic, irreducible polynomial in A.

ϵ_v := $\deg_\theta(v)$, the degree of v with respect to θ.

q_v := q^{ϵ_v}, the cardinality of the residue field A/vA.

k := $\mathbb{F}_q(\theta)$, the fraction field of A.

$|\cdot|_v$:= the normalized absolute value on k such that $|v|_v = q_v^{-1}$.

$|\cdot|_\infty$:= the normalized absolute value on k such that $|\theta|_\infty = q$.

$$\begin{aligned}
\operatorname{ord}_v(\cdot) &:= \text{the associated valuation of } |\cdot|_v.\\
\operatorname{ord}_\infty(\cdot) &:= \text{the associated valuation of } |\cdot|_\infty.\\
k_v &:= \text{the completion of } k \text{ with respect to } |\cdot|_v.\\
k_\infty &:= \text{the completion of } k \text{ with respect to } |\cdot|_\infty.\\
A_v &:= \text{the valuation ring inside } k_v.\\
A_\infty &:= \text{the valuation ring inside } k_\infty.\\
\mathbb{C}_v &:= \text{the completion of an algebraic closure of } k_v.\\
\mathbb{C}_\infty &:= \text{the completion of an algebraic closure of } k_\infty.\\
\bar{k} &:= \text{a fixed algebraic closure of } k \text{ with fixed embeddings into } \mathbb{C}_v \text{ and } \mathbb{C}_\infty\\
&\quad\ \text{respectively.}\\
\operatorname{wt}(\mathfrak{s}) &:= s_1 + \cdots + s_r, \text{ the weight of } \mathfrak{s} := (s_1, \ldots, s_r) \in \mathbb{N}^r.\\
\operatorname{dep}(\mathfrak{s}) &:= r, \text{ the depth of } \mathfrak{s} := (s_1, \ldots, s_r) \in \mathbb{N}^r.
\end{aligned}$$

§2.2. Basic setting

In this section we briefly recall some basic objects which are fundamental for the arithmetic of A. First, we set $D_0 := 1$, $L_0 := 1$ and for $i \in \mathbb{N}$, we set

$$[i] := \theta^{q^i} - \theta \in A, \quad D_i := [i][i-1]^q \cdots [1]^{q^{i-1}}, \quad \text{and} \quad L_i := (-1)^i [i][i-1] \cdots [1].$$

Next, we recall the Carlitz factorials. For each non-negative integer n, we write

$$n = \sum_{j \geq 0} n_j q^j, \quad 0 \leq n_j \leq q-1.$$

The Carlitz factorial is defined by

$$\Gamma_{n+1} := \prod_j D_j^{n_j} \in A.$$

Given an index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, we further define $\Gamma_{\mathfrak{s}} := \Gamma_{s_1} \cdots \Gamma_{s_r}$. For these objects, we have the following basic but useful proposition on its v-adic valuation.

Proposition 2.2.1. *The following assertions hold:*

(1) *Let $i \in \mathbb{Z}_{\geq 0}$. If we write $i = \alpha \cdot \epsilon_v + \beta$, $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, and $0 \leq \beta < \epsilon_v$, then*

$$\operatorname{ord}_v(D_i) = q^\beta \cdot \frac{q_v^\alpha - 1}{q_v - 1} \quad \textit{and} \quad \operatorname{ord}_v(L_i) = \alpha.$$

(2) *Let $s \in \mathbb{N}_{\geq 2}$ and $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$. Then $\operatorname{ord}_v(\Gamma_1) = 0$,*

$$\operatorname{ord}_v(\Gamma_s) \leq \frac{s-2}{q_v - 1} \quad \textit{and} \quad \operatorname{ord}_v(\Gamma_{\mathfrak{s}}) \leq \frac{\operatorname{wt}(\mathfrak{s}) - \operatorname{dep}(\mathfrak{s}) - \operatorname{ht}(\mathfrak{s})}{q_v - 1}.$$

Proof. The case $i = 0$ is clear, so we may assume that $i \geq 1$. Since $[i]$ is the product of all monic irreducible polynomials f in A with $\deg(f)$ divides i [Go96, Prop. 3.1.6], we have that $\operatorname{ord}_v([i]) = 1$ if and only if ϵ_v divides i and $\operatorname{ord}_v([i]) = 0$ otherwise. Then direct calculation shows that

$$\operatorname{ord}_v(D_i) = \sum_{j=1}^{i} q^{i-j} \cdot \operatorname{ord}_v([j]) = \sum_{j'=1}^{\alpha} q^{i-j'\cdot\epsilon_v} \cdot \operatorname{ord}_v([j' \cdot \epsilon_v]) = q^{\beta} \cdot \frac{q_v^{\alpha} - 1}{q_v - 1}$$

and

$$\operatorname{ord}_v(L_i) = \sum_{j=1}^{\alpha} \operatorname{ord}_v([j \cdot \epsilon_v]) = \alpha.$$

The first assertion now follows. To prove the second part, we write $n := s - 1 = \sum_{j\geq 0} n_j q^j$ and $j = \alpha_j \cdot \epsilon_v + \beta_j$ for each non-negative integer j, where $\alpha_j, \beta_j \in \mathbb{Z}_{\geq 0}$ and $0 \leq \beta_j < \epsilon_v$. Then

$$\begin{aligned}\operatorname{ord}_v(\Gamma_s) = \sum_{j\geq 0} n_j \cdot \operatorname{ord}_v(D_j) &= \sum_{j\geq 0} n_j \cdot q^{\beta_j} \cdot \frac{q_v^{\alpha_j} - 1}{q_v - 1} \\ &= \frac{1}{q_v - 1}\left(\sum_{j\geq 0} n_j q^j - \sum_{j\geq 0} n_j q^{\beta_j}\right) \\ &\leq \frac{1}{q_v - 1}\left(n - \sum_{j\geq 0} n_j\right) \leq \frac{n-1}{q_v - 1} = \frac{s-2}{q_v - 1}.\end{aligned}$$

Finally,

$$\begin{aligned}\operatorname{ord}_v(\Gamma_{\mathfrak{s}}) &= \sum_{j=1}^{r} \operatorname{ord}_v(\Gamma_{s_j}) \\ &\leq \left(\sum_{j=1}^{r} \frac{s_i - 1}{q_v - 1}\right) - \frac{\operatorname{ht}(\mathfrak{s})}{q_v - 1} = \frac{\operatorname{wt}(\mathfrak{s}) - \operatorname{dep}(\mathfrak{s}) - \operatorname{ht}(\mathfrak{s})}{q_v - 1}.\end{aligned}$$ □

§2.3. Anderson's t-modules

In this section we quickly review the theory of t-modules introduced by Anderson [A86]. Let τ be the Frobenius qth power operator

$$\tau := (x \mapsto x^q) \colon \mathbb{C}_v \to \mathbb{C}_v.$$

This τ-action naturally extends to matrices by componentwise action. Let $\mathbb{C}_v[\tau]$ be the non-commutative polynomial ring generated by τ equipped with the relation

$$\tau\alpha = \alpha^q \tau \quad \text{for } \alpha \in \mathbb{C}_v.$$

For any d-dimensional additive algebraic group $\mathbb{G}_a^d$ defined over $\mathbb{C}_v$, one may identify the ring of $\mathbb{F}_q$-linear endomorphisms of $\mathbb{G}_a^d$ with $\mathrm{Mat}_d(\mathbb{C}_v[\tau])$. We also define the partial differential operator

$$\partial := \left(\sum_{i\geq 0}\alpha_i\tau^i \mapsto \alpha_0\right)\colon \mathrm{Mat}_d(\mathbb{C}_v[\tau]) \to \mathrm{Mat}_d(\mathbb{C}_v).$$

Now we are ready to give a precise definition of t-modules.

Definition 2.3.1. Let $d \in \mathbb{N}$. A d-dimensional t-module is a pair $G = (\mathbb{G}_a^d, \rho)$, where

$$\rho\colon \mathbb{F}_q[t] \to \mathrm{Mat}_d(\mathbb{C}_v[\tau]),$$
$$a \mapsto \rho_a$$

is an $\mathbb{F}_q$-linear ring homomorphism so that $\partial\rho_t - \theta I_d$ is a nilpotent matrix.

The *exponential function* of G is an $\mathbb{F}_q$-linear power series of the form

$$\exp_G := I_d + \sum_{i\geq 1} Q_i\tau^i, Q_i \in \mathrm{Mat}_d(\mathbb{C}_v).$$

It is the unique power series satisfying the property that

$$\exp_G \circ \partial\rho_a = \rho_a \circ \exp_G \quad \text{for all } a \in \mathbb{F}_q[t].$$

The *logarithm* of G, denoted by $\log_G$, is defined to be the formal inverse of $\exp_G$. It is an $\mathbb{F}_q$-linear power series of the form

$$\log_G := I_d + \sum_{i\geq 1} P_i\tau^i, \quad P_i \in \mathrm{Mat}_d(\mathbb{C}_v),$$

with the property that

$$\log_G \circ \rho_a = \partial\rho_a \circ \log_G \quad \text{for all } a \in \mathbb{F}_q[t].$$

The logarithm $\log_G$ is one of the most important objects in our study as it has a deep connection to CMSPLs and MZVs (see [AT90, CM19, CM21]).

§3. v-adic CMSPL

In this section we will follow [CM19] closely to construct a specific t-module $G_{\mathfrak{s},\mathbf{u}}$ and an explicit special point $\mathbf{v}_{\mathfrak{s},\mathbf{u}}$ to define the v-adic CMSPL. Then we study the properties of the k-vector space generated by certain values of v-adic CMSPLs with the same weight.

§3.1. Formulation through iterated extension of Carlitz tensor powers

Throughout this section we fix $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and $\mathbf{u} = (u_1, \ldots, u_r) \in (\bar{k}^\times)^r$. We define the $\mathfrak{s}$th Carlitz multiple polylogarithm (CMPL) as (see [C14])

$$\mathrm{Li}_{\mathfrak{s}}(z_1, \ldots, z_r) := \sum_{i_1 > \cdots > i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \in k[\![z_1, \ldots, z_r]\!].$$

We also define the $\mathfrak{s}$th CMSPL as (see [CM19])

$$\mathrm{Li}^\star_{\mathfrak{s}}(z_1, \ldots, z_r) := \sum_{i_1 \geq \cdots \geq i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \in k[\![z_1, \ldots, z_r]\!].$$

We write $\mathrm{Li}_{\mathfrak{s}}(z_1, \ldots, z_r)_v$ and $\mathrm{Li}^\star_{\mathfrak{s}}(z_1, \ldots, z_r)_v$ when we consider the v-adic convergence of these two infinite series.

For $1 \leq \ell \leq r$, we set $d_\ell := s_\ell + \cdots + s_r$ and $d := d_1 + \cdots + d_r$. Let B be a $d \times d$-matrix of the form

$$\left(\begin{array}{c|c|c} B[11] & \cdots & B[1r] \\ \hline \vdots & & \vdots \\ \hline B[r1] & \cdots & B[rr] \end{array}\right),$$

where $B[\ell m]$ is a $d_\ell \times d_m$-matrix for each ℓ and m. We call $B[\ell m]$ the (ℓ, m)th block submatrix of B.

For $1 \leq \ell \leq m \leq r$, we set

$$N_\ell := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \in \mathrm{Mat}_{d_\ell}(\bar{k}),$$

$$N := \begin{pmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_r \end{pmatrix} \in \mathrm{Mat}_d(\bar{k}),$$

$$E[\ell m] := \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{Mat}_{d_\ell \times d_m}(\bar{k}) \quad (\text{if } \ell = m),$$

$$E[\ell m] := \begin{pmatrix} 0 & & \cdots\cdots 0 \\ \vdots & & \ddots \quad \vdots \\ 0 & & \ddots \vdots \\ (-1)^{m-\ell}\prod_{e=\ell}^{m-1} u_e & 0 & \cdots 0 \end{pmatrix} \in \mathrm{Mat}_{d_\ell \times d_m}(\bar{k}) \quad (\text{if } \ell < m),$$

$$E := \begin{pmatrix} E[11] & E[12] & \cdots & E[1r] \\ & E[22] & \ddots & \vdots \\ & & \ddots & E[r-1,r] \\ & & & E[rr] \end{pmatrix} \in \mathrm{Mat}_d(\bar{k}).$$

Also, we define $\mathbb{1}_\ell := (\delta_{\ell,j})_{1\le j\le d}$ to be the vector in $\mathrm{Mat}_{1\times d}(\bar{k})$ where $\delta_{\ell,j} = 1$ if $\ell = j$ and $\delta_{\ell,j} = 0$ otherwise. Finally, we define the t-module $G_{\mathfrak{s},\mathbf{u}} := (\mathbb{G}_a^d, \rho)$ by

$$\rho_t = \theta I_d + N + E\tau \in \mathrm{Mat}_d(\bar{k}[\tau]). \tag{3.1.1}$$

Note that $G_{\mathfrak{s},\mathbf{u}}$ depends only on $u_1, \ldots, u_{r-1}$.

Let

$$\mathbf{v}_{\mathfrak{s},\mathbf{u}} := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^{r-1}u_1\cdots u_r \\ 0 \\ \vdots \\ 0 \\ (-1)^{r-2}u_2\cdots u_r \\ \vdots \\ 0 \\ \vdots \\ 0 \\ u_r \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\\ 0\end{matrix}}\right\} d_1 \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\\ 0\end{matrix}}\right\} d_2 \\ \vdots \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\\ 0\end{matrix}}\right\} d_r \end{matrix} \in G_{\mathfrak{s},\mathbf{u}}(\bar{k}). \tag{3.1.2}$$

If we denote $|M|_v := \max\{|M_{ij}|_v\}$ for each matrix $M = (M_{ij})$ with entries in $\mathbb{C}_v$, then $|\mathbf{v}_{\mathfrak{s},\mathbf{u}}|_v < 1$ when $|u_m|_v \le 1$ for each $1 \le m < r$ and $|u_r|_v < 1$. In this case, $\log_{G_{\mathfrak{s},\mathbf{u}}}(\mathbf{v}_{\mathfrak{s},\mathbf{u}})$ converges v-adically. Furthermore, we have the following theorem.

Theorem 3.1.3 ([CM19, Thm. 3.3.3]). *Let $u_1, \dots, u_r \in \bar{k}^\times$ with $|u_m|_v \le 1$ for each $1 \le m < r$ and $|u_r|_v < 1$. Let $G_{\mathfrak{s},\mathbf{u}}$ and $\mathbf{v}_{\mathfrak{s},\mathbf{u}}$ be as above. Then we have*

$$\log_{G_{\mathfrak{s},\mathbf{u}}}(\mathbf{v}_{\mathfrak{s},\mathbf{u}}) = \begin{pmatrix} * \\ \vdots \\ * \\ (-1)^{r-1}\operatorname{Li}^\star_{(s_r,\dots,s_1)}(u_r,\dots,u_1)_v \\ * \\ \vdots \\ * \\ (-1)^{r-2}\operatorname{Li}^\star_{(s_r,\dots,s_2)}(u_r,\dots,u_2)_v \\ \vdots \\ * \\ \vdots \\ * \\ \operatorname{Li}^\star_{s_r}(u_r)_v \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} * \\ \vdots \\ * \\ * \end{matrix}}\right\} d_1 \\ \left.\vphantom{\begin{matrix} * \\ \vdots \\ * \\ * \end{matrix}}\right\} d_2 \\ \vdots \\ \left.\vphantom{\begin{matrix} * \\ \vdots \\ * \\ * \end{matrix}}\right\} d_r \end{matrix} \in \operatorname{Lie} G_{\mathfrak{s},\mathbf{u}}(\mathbb{C}_v).$$

In [CM19], the logarithmic interpretation was used to analytically continue CMSPLs v-adically. In what follows, we will recall their result and give the formulation of v-adic CMSPLs via the logarithm of the t-module $G_{\mathfrak{s},\mathbf{u}}$.

We let $\mathcal{O}_{\mathbb{C}_v}$ be the valuation ring inside $\mathbb{C}_v$ and $\mathfrak{m}_v$ be the maximal ideal of $\mathcal{O}_{\mathbb{C}_v}$. We further denote by $v(t) \coloneqq v(\theta)|_{\theta=t} \in \mathbb{F}_q[t]$.

Proposition 3.1.4 ([CM19, Prop. 4.1.1]). *Let $u_1, \dots, u_r \in \bar{k}^\times$ with $|u_i|_v \le 1$ for each $1 \le i \le r$. Let $G_{\mathfrak{s},\mathbf{u}}$ be the t-module defined in (3.1.1) and $\mathbf{v}_{\mathfrak{s},\mathbf{u}} \in G_{\mathfrak{s},\mathbf{u}}(\bar{k})$ be defined in (3.1.2). Let $\ell \ge 1$ be an integer such that each image of u_i in $\mathcal{O}_{\mathbb{C}_v}/\mathfrak{m}_v \cong \overline{\mathbb{F}_q}$ is contained in $\mathbb{F}_{q^\ell}$. We further set*

$$a(t) \coloneqq (v(t)^{d_1\ell} - 1)(v(t)^{d_2\ell} - 1)\cdots(v(t)^{d_r\ell} - 1). \tag{3.1.5}$$

Then

$$\rho_a(\mathbf{v}_{\mathfrak{s},\mathbf{u}}) \in G_{\mathfrak{s},\mathbf{u}}(\mathfrak{m}_v).$$

In particular, $\log_{G_{\mathfrak{s},\mathbf{u}}}(\rho_a(\mathbf{v}_{\mathfrak{s},\mathbf{u}}))$ converges in $\operatorname{Lie} G_{\mathfrak{s},\mathbf{u}}(\mathbb{C}_v)$.

From Proposition 3.1.4, we can make the following formulation.

Definition 3.1.6. Let

$$\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r, \quad \mathbf{u} = (u_1, \dots, u_r) \in \bar{k}^\times \quad \text{with } |u_i|_v \le 1.$$

We define

$$\tilde{\mathfrak{s}} = (s_r, \dots, s_1), \quad \tilde{\mathbf{u}} = (u_r, \dots, u_1), \quad \tilde{d}_\ell \coloneqq s_{r+1-\ell} + \cdots + s_1, \quad 1 \le \ell \le r.$$

Consider the t-module $G_{\tilde{\mathfrak{s}},\tilde{\mathbf{u}}}$ defined in (3.1.1), the special point $\mathbf{v}_{\tilde{\mathfrak{s}},\tilde{\mathbf{u}}}$ defined in (3.1.2), and any polynomial $a(t) \in \mathbb{F}_q[t]$ so that

$$a(t) \neq 0 \quad \text{and} \quad \rho_a(\mathbf{v}_{\tilde{\mathfrak{s}},\tilde{\mathbf{u}}}) \in G_{\tilde{\mathfrak{s}},\tilde{\mathbf{u}}}(\mathfrak{m}_v).$$

Then we define $\mathrm{Li}^{\star}_{\mathfrak{s}}(u_1,\ldots,u_r)_v$ to be the value

$$\frac{(-1)^{r-1}}{a(\theta)} \times \text{the } \tilde{d}_1\text{th coordinate of } \log_{G_{\tilde{\mathfrak{s}},\tilde{\mathbf{u}}}}(\rho_a(\mathbf{v}_{\tilde{\mathfrak{s}},\tilde{\mathbf{u}}})). \tag{3.1.7}$$

Remark 3.1.8. The proposition above guarantees the existence of such an $a(t)$. Moreover, the definition above is independent of the choice of $a(t)$ (see [CM19, Rem. 4.1.3]).

Remark 3.1.9. If $\mathbf{u} = (u_1,\ldots,u_r) \in \bar{k}^{\times}$ with $|u_1|_v < 1$ and $|u_i|_v \leq 1$ for $2 \leq i \leq r$, then the definition above coincides with the original power series expansion, namely

$$\mathrm{Li}^{\star}_{\mathfrak{s}}(u_1,\ldots,u_r)_v = \sum_{i_1 \geq \cdots \geq i_r \geq 0} \frac{u_1^{q^{i_1}} \cdots u_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \in \mathbb{C}_v.$$

§3.2. The k-vector space spanned by v-adic CMSPLs

The aim of this subsection is to study the properties of the k-vector space spanned by $\mathrm{Li}^{\star}_{\mathfrak{s}}(\mathbf{u})_v$ with the same weight. More precisely, we get a lower bound of the v-adic valuation of $\mathrm{Li}^{\star}_{\mathfrak{s}}(\mathbf{u})_v$ in terms of $\mathrm{wt}(\mathfrak{s})$ and ϵ_v. In addition, we prove a functional equation of v-adic CMSPLs arising from the logarithm of the t-module. As an application, we will prove that the k-vector space spanned by $\mathrm{Li}^{\star}_{\mathfrak{s}}(\mathbf{u})_v$ forms an algebra.

Definition 3.2.1. Let $w \in \mathbb{Z}_{\geq 0}$, $S_{0,v} := \{1\}$, and

$$S_{w,v} := \left\{\mathrm{Li}^{\star}_{\mathfrak{s}}(\mathbf{u})_v \mid r \in \mathbb{N},\ \mathfrak{s} \in \mathbb{N}^r,\ \mathrm{wt}(\mathfrak{s}) = w, \mathbf{u} \in A^r\right\} \quad \text{for } w > 0.$$

We define $\mathcal{L}_{w,v}$ to be the k-vector space generated by $S_{w,v}$.

To study the properties of $\mathcal{L}_{w,v}$, we need some further information on $\mathrm{Li}^{\star}_{\mathfrak{s}}(\mathbf{u})_v$. For the convenience of our later use, we adopt the following setting: for any $x \in \mathbb{C}_v$, we define

$$x^{(i)} := \tau^i(x) = x^{q^i}.$$

For any $M = (M_{ij}) \in \mathrm{Mat}_{m\times n}(\mathbb{C}_v)$, we define $M^{(\ell)} = (M_{ij}^{(\ell)})$. Furthermore, given any two square matrices M_1, M_2 with the same size, we define

$$\mathrm{ad}(M_1)^0(M_2) := M_2$$

and for non-negative integers j,

$$\mathrm{ad}(M_1)^{j+1}(M_2) := M_1(\mathrm{ad}(M_1)^j(M_2)) - (\mathrm{ad}(M_1)^j(M_2))M_1.$$

Proposition 3.2.2. *Let*

$$\mathfrak{s} := (s_1, \dots, s_r) \in \mathbb{N}^r, \quad \mathbf{u} := (u_1, \dots, u_r) \in (\bar{k}^\times)^r,$$

and let $G_{\mathfrak{s},\mathbf{u}}$ be the t-module defined in (3.1.1). Consider

$$\log_{G_{\mathfrak{s},\mathbf{u}}} := \sum_{i\geq 0} P_i \tau^i, \quad P_0 = I_d,$$

and put

$$d_1 th \text{ row of } P_i := (Y_1^{\langle i\rangle}, \dots, Y_r^{\langle i\rangle}), \quad \text{where } Y_m^{\langle i\rangle} \in \bar{k}^{d_m} \text{ for } 1 \leq m \leq r.$$

If we set

$$Y_m^{\langle i\rangle} = (y_{m,1}^{\langle i\rangle}, \dots, y_{m,d_m}^{\langle i\rangle}),$$

then for $1 \leq j \leq d_1$ we have

$$y_{1,j}^{\langle i\rangle} = \frac{(-[i])^{d_1-j}}{L_i^{d_1}}, \tag{3.2.3}$$

and for $m \geq 2$, $1 \leq j \leq d_m$ we have

$$y_{m,j}^{\langle i\rangle} = (-1)^{m-1}(-[i])^{d_m-j} \sum_{0\leq i_1\leq\cdots\leq i_{m-1}<i} \frac{u_1^{(i_1)}\cdots u_{m-1}^{(i_{m-1})}}{L_{i_1}^{s_1}\cdots L_{i_{m-1}}^{s_{m-1}} L_i^{d_m}}. \tag{3.2.4}$$

Remark 3.2.5. This proposition is an improvement of [CM19, Prop. 3.2.1], using the strategy of [AT90]. More precisely, Chang and Mishiba proved in [CM19, Prop. 3.2.1] that

$$y_{1,d_1}^{\langle i\rangle} = \frac{1}{L_i^{d_1}},$$

and for $m \geq 2$,

$$y_{m,d_m}^{\langle i\rangle} = (-1)^{m-1} \sum_{0\leq i_1\leq\cdots\leq i_{m-1}<i} \frac{u_1^{(i_1)}\cdots u_{m-1}^{(i_{m-1})}}{L_{i_1}^{s_1}\cdots L_{i_{m-1}}^{s_{m-1}} L_i^{d_m}}.$$

Remark 3.2.6. For the depth 1 case, i.e., $\mathfrak{s} = s$, $\mathbf{u} = u$, our t-module $G_{\mathfrak{s},\mathbf{u}}$ is the sth tensor power of the Carlitz module and our result coincides with [Pp, Cor. 4.1.5].

Proof of Proposition 3.2.2. Consider the functional equation

$$\log_{G_{\mathfrak{s},\mathbf{u}}} \circ \rho_t = \partial\rho_t \circ \log_{G_{\mathfrak{s},\mathbf{u}}}.$$

We have

$$\left(\sum_{i\geq 0} P_i\tau^i\right)\cdot(\theta I_d + N + E\tau) = (\theta I_d + N)\cdot\left(\sum_{i\geq 0} P_i\tau^i\right).$$

By comparison with the coefficient matrix of τ^{i+1}, we obtain

$$P_{i+1}(\theta^{q^{i+1}}\cdot I_d + N) + P_iE^{(i)} = (\theta\cdot I_d + N)\cdot P_{i+1}.$$

Therefore,

$$[i+1]P_{i+1} + P_{i+1}N - NP_{i+1} = -P_iE^{(i)}$$

and thus

$$P_{i+1} - \frac{\mathrm{ad}(N)^1(P_{i+1})}{[i+1]} = -\frac{P_iE^{(i)}}{[i+1]}.$$

By applying $\frac{\mathrm{ad}(N)^j(P_{i+1})}{[i+1]^j}$ and using a telescopic sum, we may use the fact

$$\mathrm{ad}(N)^{2d_1-1}(P_{i+1}) = 0$$

to obtain

$$P_{i+1} = -\sum_{j=0}^{2d_1-2}\frac{\mathrm{ad}(N)^j(P_iE^{(i)})}{[i+1]^{j+1}}.$$

Now we consider the vector $\mathbb{1}_{d_1} = (\delta_{d_1,\ell})_{1\leq\ell\leq d}$. Note that $\mathbb{1}_{d_1}N = 0$ and $\mathrm{ad}(N)^j(P_iE^{(i)})$ can be expressed as

$$\mathrm{ad}(N)^j(P_iE^{(i)}) = NB + (-1)^jP_iE^{(i)}N^j$$

for some $B \in \mathrm{Mat}_d(\bar{k})$. Hence

$$\mathbb{1}_{d_1}P_{i+1} = -\sum_{j=0}^{2d_1-2}\frac{\mathbb{1}_{d_1}\,\mathrm{ad}(N)^j(P_iE^{(i)})}{[i+1]^{j+1}} = \sum_{j=0}^{2d_1-2}\frac{\mathbb{1}_{d_1}P_iE^{(i)}N^j}{(-[i+1])^{j+1}}. \tag{3.2.7}$$

For each i, j, we have

$$E^{(i)}N^j = \begin{pmatrix} E[11]^{(i)} & E[12]^{(i)} & \cdots & E[1r]^{(i)} \\ & E[22]^{(i)} & \ddots & \vdots \\ & & \ddots & E[r-1,r]^{(i)} \\ & & & E[rr]^{(i)} \end{pmatrix}\begin{pmatrix} N_1^j & & & \\ & N_2^j & & \\ & & \ddots & \\ & & & N_r^j \end{pmatrix}$$

$$= \begin{pmatrix} E[11]^{(i)}N_1^j & E[12]^{(i)}N_2^j & \cdots & E[1r]^{(i)}N_r^j \\ & E[22]^{(i)}N_2^j & \ddots & \vdots \\ & & \ddots & E[r-1,r]^{(i)}N_r^j \\ & & & E[rr]^{(i)}N_r^j \end{pmatrix}.$$

For $1 \le \ell \le m \le r$, $0 \le j \le d_m - 1$, we have

$$E[\ell m]^{(i)} N_m^j = \begin{pmatrix} 0 \cdots \cdots & & 0 & & \cdots \cdots 0 \\ \vdots & & \vdots & & \vdots \\ \vdots & & 0 & & \vdots \\ 0 \cdots & 0 & (-1)^{m-\ell} \prod_{e=\ell}^{m-1} u_e^{(i)} & 0 & \cdots 0 \end{pmatrix}$$

where the only non-zero element appears in the $(d_m, j+1)$th entry. If we denote

$$Y_m^{\langle i \rangle} = (y_{m,1}^{\langle i \rangle}, \dots, y_{m,d_m}^{\langle i \rangle}),$$

then by comparing with both sides of (3.2.7), we obtain

$$Y_1^{\langle i+1 \rangle} = \left(\frac{y_{1,d_1}^{\langle i \rangle}}{-[i+1]}, \frac{y_{1,d_1}^{\langle i \rangle}}{(-[i+1])^2}, \dots, \frac{y_{1,d_1}^{\langle i \rangle}}{(-[i+1])^{d_1}} \right)$$

and for $m \ge 2$,

$$Y_m^{\langle i+1 \rangle} = \left(\frac{y_{m,d_m}^{\langle i \rangle} + \sum_{n=1}^{m-1} y_{n,d_n}^{\langle i \rangle} (-1)^{m-n} \prod_{e=n}^{m-1} u_e^{(i)}}{-[i+1]}, \right.$$
$$\left. \dots, \frac{y_{m,d_m}^{\langle i \rangle} + \sum_{n=1}^{m-1} y_{n,d_n}^{\langle i \rangle} (-1)^{m-n} \prod_{e=n}^{m-1} u_e^{(i)}}{(-[i+1])^{d_m}} \right).$$

Consequently, we get that

$$y_{m,j}^{\langle i \rangle} = (-[i])^{d_m - j} y_{m,d_m}^{\langle i \rangle}.$$

The desired result now follows immediately from the relation $L_{i+1} = -[i+1]L_i$ and [CM19, Prop. 3.2.1]. □

For convenience, we define $\mathrm{Li}_{\varnothing}^{\star} = 0$. Finally, we consider the Carlitz difference operator $\triangle_1$, which acts on $f \in k[\![z_1, \dots, z_r]\!]$ by

$$(\triangle_1 f)(z_1, \dots, z_r) := f(\theta z_1, z_2, \dots, z_r) - \theta f(z_1, \dots, z_r).$$

Lemma 3.2.8. *Let $\triangle_1^j := \triangle_1 \circ \cdots \circ \triangle_1$ be the j-fold composition of the Carlitz difference operator. Then we have*

$$(\triangle_1^j f)(z_1, \dots, z_r) = \sum_{\ell=0}^{j} (-1)^\ell \binom{j}{\ell} \theta^\ell f(\theta^{j-\ell} z_1, z_2, \dots, z_r).$$

Proof. The lemma is essentially an application of the binomial theorem. We introduce two operators S_θ and I_θ. Here S_θ is the operator defined by

$$(S_\theta f)(z_1, \dots, z_r) := f(\theta z_1, z_2, \dots, z_r).$$

We let I_θ be the multiplication operator defined by

$$(I_\theta f)(z_1,\dots,z_r) := \theta f(z_1,\dots,z_r).$$

Note that $\triangle_1 = S_\theta - I_\theta$ and $S_\theta \circ I_\theta = I_\theta \circ S_\theta$. Thus, we have

$$\triangle_1^j = (S_\theta - I_\theta)^j = \sum_{\ell=0}^{j} (-1)^\ell \binom{j}{\ell} S_\theta^{j-\ell} \circ I_\theta^\ell.$$

Consequently, we obtain the result. □

A simple application of Lemma 3.2.8 comes from the observation that

$$\triangle_1 \, z^{q^i} = [i] z^{q^i}.$$

More generally, if we have an $\mathbb{F}_q$-linear power series,

$$f(z_1,\dots,z_r) = \sum_{i=0}^{\infty} c_i z_1^{q^i} \in k[\![z_1,\dots,z_r]\!],$$

where $c_i \in k[\![z_2,\dots,z_r]\!]$. Then we have

$$(\triangle_1^j \, f)(z_1,\dots,z_r) = \sum_{i=0}^{\infty} c_i [i]^j z_1^{q^i}. \tag{3.2.9}$$

Surprisingly, this phenomenon also occurs in the calculation of the v-adic CMSPLs. To formulate the statement, we adopt the following notation. Let $\mathfrak{s} = (s_1,\dots,s_r) \in \mathbb{N}^r$. Then we define

$$\mathfrak{s}_{(i)} = (s_1 + \cdots + s_i, s_{i+1}, \dots, s_r) \text{ for } 1 \le i \le r \quad \text{and} \quad \mathfrak{s}_{(i)} = \varnothing \text{ for } i > r \text{ or } i < 0.$$

Theorem 3.2.10. *Let*

$$\mathfrak{s} = (s_1,\dots,s_r) \in \mathbb{N}^r, \quad \mathbf{u} = (u_1,\dots,u_r) \in (\bar{k}^\times)^r \ \text{ with } |u_i|_v \le 1.$$

We define

$$\tilde{\mathfrak{s}} = (s_r,\dots,s_1), \quad \tilde{\mathbf{u}} = (u_r,\dots,u_1), \quad \tilde{d}_\ell := s_{r+1-\ell} + \cdots + s_1, \ \ 1 \le \ell \le r.$$

Consider the t-module $G_{\tilde{\mathfrak{s}},\tilde{\mathbf{u}}}$ defined in (3.1.1), the special point $\mathbf{v}_{\tilde{\mathfrak{s}},\tilde{\mathbf{u}}}$ defined in (3.1.2), and any polynomial $a(t) \in \mathbb{F}_q[t]$ so that

$$\rho_a(\mathbf{v}_{\tilde{\mathfrak{s}},\tilde{\mathbf{u}}}) \in G_{\tilde{\mathfrak{s}},\tilde{\mathbf{u}}}(\mathfrak{m}_v).$$

We further set

$$(V_1,\dots,V_r)^{\mathrm{tr}} = \rho_a(\mathbf{v}_{\tilde{\mathfrak{s}},\tilde{\mathbf{u}}}), \quad \text{where } V_m \in \bar{k}^{\tilde{d}_m}.$$

If we set

$$V_m \coloneqq (V_{m,1}, \ldots, V_{m,\tilde{d}_m})^{\mathrm{tr}},$$

then we have the following identity

$$\begin{aligned}\mathrm{Li}^{\star}_{\mathfrak{s}}(\mathbf{u})_v = \frac{(-1)^{r-1}}{a(\theta)} \sum_{m=1}^{r} \sum_{j=0}^{\tilde{d}_m-1} \sum_{\ell=0}^{j} (-1)^{j+\ell+m-1} \binom{j}{\ell} \theta^{\ell} \\ \times \big\{ \mathrm{Li}^{\star}_{\mathfrak{s}_{(r+1-m)}}(\theta^{j-\ell} V_{m,\tilde{d}_m-j}, u_{r+2-m}, \ldots, u_r)_v \\ - \mathrm{Li}^{\star}_{\mathfrak{s}_{(r+2-m)}}(\theta^{j-\ell} V_{m,\tilde{d}_m-j} u_{r+2-m}, u_{r+3-m}, \ldots, u_r)_v \big\}.\end{aligned}$$

Proof. Let $\log_{G_{\tilde{\mathfrak{s}},\tilde{\mathbf{u}}}} = \sum_{i\geq 0} \widetilde{P}_i \tau^i$ and

$$\tilde{d}_1\text{th row of } \widetilde{P}_i \coloneqq (\widetilde{Y}_1^{\langle i\rangle}, \ldots, \widetilde{Y}_r^{\langle i\rangle}), \quad \text{where } \widetilde{Y}_m^{\langle i\rangle} \in \bar{k}^{\tilde{d}_m} \text{ for } 1 \leq m \leq r.$$

Then by Definition 3.1.6 we have

$$\begin{aligned}\mathrm{Li}^{\star}_{\mathfrak{s}}(\mathbf{u})_v &= \frac{(-1)^{r-1}}{a(\theta)} \sum_{i\geq 0} (\widetilde{Y}_1^{\langle i\rangle}, \ldots, \widetilde{Y}_r^{\langle i\rangle}) \cdot (V_1^{(i)}, \ldots, V_r^{(i)})^{\mathrm{tr}} \\ &= \sum_{i\geq 0} \big\{ (\widetilde{Y}_1^{\langle i\rangle}) \cdot (V_1^{(i)})^{\mathrm{tr}} + \cdots + (\widetilde{Y}_r^{\langle i\rangle}) \cdot (V_r^{(i)})^{\mathrm{tr}} \big\}.\end{aligned}$$

For $m = 1$, we have

$$\begin{aligned}\sum_{i\geq 0} (\widetilde{Y}_1^{\langle i\rangle}) \cdot (V_1^{(i)})^{\mathrm{tr}} &= \sum_{i\geq 0} \sum_{j=1}^{\tilde{d}_1} \frac{(-1)^{\tilde{d}_1-j} [i]^{\tilde{d}_1-j} V_{1,j}^{q^i}}{L_i^{\tilde{d}_1}} \\ &= \sum_{j=0}^{\tilde{d}_1-1} \sum_{i\geq 0} \frac{(-1)^j [i]^j V_{1,\tilde{d}_1-j}^{q^i}}{L_i^{\tilde{d}_1}} \\ &= \sum_{j=0}^{\tilde{d}_1-1} (-1)^j (\triangle_1^j \, \mathrm{Li}^{\star}_{\tilde{d}_1})(V_{1,\tilde{d}_1-j}) \\ &= \sum_{j=0}^{\tilde{d}_1-1} \sum_{\ell=0}^{j} (-1)^{j+\ell} \binom{j}{\ell} \theta^{\ell} \, \mathrm{Li}^{\star}_{\tilde{d}_1}(\theta^{j-\ell} V_{1,\tilde{d}_1-j}) \\ &= \sum_{j=0}^{\tilde{d}_1-1} \sum_{\ell=0}^{j} (-1)^{j+\ell} \binom{j}{\ell} \theta^{\ell} \big\{ \mathrm{Li}^{\star}_{\mathfrak{s}_{(r)}}(\theta^{j-\ell} V_{1,\tilde{d}_1-j}) \\ &\qquad - \mathrm{Li}^{\star}_{\mathfrak{s}_{(r+1)}}(\theta^{j-\ell} V_{1,\tilde{d}_1-j}) \big\}.\end{aligned}$$

The first equality comes from (3.2.3). The third equality follows by combining (3.2.9) and the definition of $\mathrm{Li}^{\star}$. The fourth equality follows by Lemma 3.2.8.

For $2 \leq m \leq r$, we have

$$
\begin{aligned}
\sum_{i\geq 0}(\widetilde{Y}_m^{\langle i\rangle})\cdot(V_m^{(i)})^{\mathrm{tr}}
&= \sum_{i\geq 0}\sum_{j=1}^{\tilde{d}_m}(-1)^{m-1+\tilde{d}_m-j}\sum_{\substack{0\leq i_1\leq\cdots\\ \leq i_{m-1}<i}}\frac{[i]^{\tilde{d}_m-j}u_r^{(i_1)}\cdots u_{r+2-m}^{(i_{m-1})}V_{m,j}^{(i)}}{L_{i_1}^{s_r}\cdots L_{i_{m-1}}^{s_{r+2-m}}L_i^{\tilde{d}_m}}\\
&= \sum_{i\geq 0}\sum_{j=1}^{\tilde{d}_m}(-1)^{m-1+\tilde{d}_m-j}\Bigg(\sum_{\substack{0\leq i_1\leq\cdots\\ \leq i_{m-1}\leq i}}-\sum_{\substack{0\leq i_1\leq\cdots\\ \leq i_{m-1}=i}}\Bigg)\frac{[i]^{\tilde{d}_m-j}u_r^{(i_1)}\cdots u_{r+2-m}^{(i_{m-1})}V_{m,j}^{(i)}}{L_{i_1}^{s_r}\cdots L_{i_{m-1}}^{s_{r+2-m}}L_i^{\tilde{d}_m}}\\
&= \sum_{j=0}^{\tilde{d}_m-1}(-1)^{m+j-1}\sum_{i\geq 0}\Bigg(\sum_{\substack{0\leq i_1\leq\cdots\\ \leq i_{m-1}\leq i}}-\sum_{\substack{0\leq i_1\leq\cdots\\ \leq i_{m-1}=i}}\Bigg)\frac{[i]^{j}u_r^{(i_1)}\cdots u_{r+2-m}^{(i_{m-1})}V_{m,\tilde{d}_m-j}^{(i)}}{L_{i_1}^{s_r}\cdots L_{i_{m-1}}^{s_{r+2-m}}L_i^{\tilde{d}_m}}\\
&= \sum_{j=0}^{\tilde{d}_m-1}(-1)^{m+j-1}\big\{(\triangle_1^j\,\mathrm{Li}^\star_{\mathfrak{s}_{(r+1-m)}})(V_{m,d_m-j},u_{r+2-m},\ldots,u_r)\\
&\qquad\qquad -(\triangle_1^j\,\mathrm{Li}^\star_{\mathfrak{s}_{(r+2-m)}})(V_{m,d_m-j}u_{r+2-m},u_{r+3-m},\ldots,u_r)\big\}\\
&= \sum_{j=0}^{\tilde{d}_m-1}\sum_{\ell=0}^{j}(-1)^{j+\ell+m-1}\binom{j}{\ell}\theta^\ell\big\{\mathrm{Li}^\star_{\mathfrak{s}_{(r+1-m)}}(\theta^{j-\ell}V_{m,\tilde{d}_m-j},u_{r+2-m},\ldots,u_r)_v\\
&\qquad\qquad -\mathrm{Li}^\star_{\mathfrak{s}_{(r+2-m)}}(\theta^{j-\ell}V_{m,\tilde{d}_m-j}u_{r+2-m},u_{r+3-m},\ldots,u_r)_v\big\}.
\end{aligned}
$$

The first equality comes from (3.2.4). The third equality follows by combining (3.2.9) and the definition of $\mathrm{Li}^\star$. The fourth equality follows by Lemma 3.2.8. The desired result now follows by combining the above two cases. □

Remark 3.2.11. In order to state the following corollary of Theorem 3.2.10, let $S^\star_{0,v} := \{1\}$ and $S^\star_{w,v} := \{\mathrm{Li}^\star_{\mathfrak{s}}(\mathbf{u})_v \mid r\in\mathbb{N},\ \mathfrak{s}\in\mathbb{N}^r,\ \mathrm{wt}(\mathfrak{s})=w,\ \mathbf{u}\in vA\times A^{r-1}\}$ for $w\in\mathbb{N}$. It is important to point out that $\mathrm{Li}^\star_{\mathfrak{s}}(\mathbf{u})_v\in S^\star_{w,v}$ and $\mathrm{Li}^\star_{\mathfrak{s}'}(\mathbf{u}')_v\in S^\star_{w',v}$ satisfy stuffle relations analogous to the characteristic zero case. Indeed, every $\mathrm{Li}^\star_{\mathfrak{s}}(\mathbf{u})_v\in S^\star_{w,v}$ has a power series expansion, namely

$$\mathrm{Li}^\star_{\mathfrak{s}}(\mathbf{u})_v=\sum_{i_1\geq\cdots\geq i_r\geq 0}\frac{u_1^{q^{i_1}}\cdots u_r^{q^{i_r}}}{L_{i_1}^{s_1}\cdots L_{i_r}^{s_r}}\in k_v.$$

Then the inclusion–exclusion principle on the set

$$\{i_1\geq\cdots\geq i_r\geq 0\}$$

shows that CMSPLs can be written as $\mathbb{F}_q$-linear combinations of CMPLs. For example,

$$\{i_1 \geq i_2 \geq 0\} = \{i_1 > i_2 \geq 0\} \cup \{i_1 = i_2 \geq 0\}.$$

It follows that

$$\mathrm{Li}^{\star}_{(s_1,s_2)}(u_1,u_2)_v = \mathrm{Li}_{(s_1,s_2)}(u_1,u_2) + \mathrm{Li}_{(s_1+s_2)}(u_1 \cdot u_2).$$

Then [C14, Sect. 5.2] and [CM19, Prop. 5.2.3] now provide the stuffle relations for CMSPLs. For example, for $\mathfrak{s} = s_1 \in \mathbb{N}$, $\mathfrak{s}' = s_2 \in \mathbb{N}$, $\mathbf{u} = u_1 \in vA$, $\mathbf{u}' = u_2 \in vA$, we have

$$\mathrm{Li}^{\star}_{s_1}(u_1)_v \cdot \mathrm{Li}^{\star}_{s_2}(u_2)_v = \mathrm{Li}^{\star}_{(s_1,s_2)}(u_1,u_2)_v + \mathrm{Li}^{\star}_{(s_2,s_1)}(u_2,u_1)_v - \mathrm{Li}^{\star}_{(s_1+s_2)}(u_1u_2)_v.$$

On the other hand, due to the lack of a power series expansion, it does not seem clear from Definition 3.1.6 whether elements in $S_{w,v} \setminus S^{\star}_{w,v}$ satisfy the same stuffle relations.

Now we are ready to state a corollary for Theorem 3.2.10.

Corollary 3.2.12. *Let $\mathcal{L}_{w,v}$ be the k-vector space generated by $S_{w,v}$ (see Definition 3.2.1). Then we have the following:*

(1) *As a k-vector space, $\mathcal{L}_{w,v}$ is generated by $S^{\star}_{w,v}$. Moreover, let*

$$\mathcal{L}_v = \sum_{w \in \mathbb{Z}_{\geq 0}} \mathcal{L}_{w,v}.$$

Then $\mathcal{L}_v$ forms a k-algebra.

(2) *Let*

$$B_{w,v} \coloneqq \min_{n \geq 0}\{q_v^n - n \cdot w\}.$$

Then

$$\mathrm{ord}_v(\mathrm{Li}^{\star}_{\mathfrak{s}}(\mathbf{u})_v) \geq B_{w,v} \quad \text{for every } \mathrm{Li}^{\star}_{\mathfrak{s}}(\mathbf{u})_v \in S_{w,v}.$$

In particular,

$$\mathrm{Li}^{\star}_{\mathfrak{s}}(\mathbf{u})_v \in A_v \quad \text{if } q_v \geq \mathrm{wt}(\mathfrak{s}).$$

Proof. Indeed, by Proposition 3.1.4, we may always pick $a(t) \in \mathbb{F}_q[t]$ as in (3.1.5). Then Theorem 3.2.10 implies that every $\mathrm{Li}^{\star}_{\mathfrak{s}}(\mathbf{u})_v \in S_{w,v}$ is just a k-linear combination of elements in $S^{\star}_{w,v}$. Thus, $\mathcal{L}_{w,v}$ is generated by $S^{\star}_{w,v}$. Consequently, the following inclusion holds by Remark 3.2.11:

$$\mathcal{L}_{w_1,v} \cdot \mathcal{L}_{w_2,v} \subset \mathcal{L}_{w_1+w_2,v}.$$

The first assertion now follows immediately. For the second part, one observes that

$$\operatorname{ord}_v(\operatorname{Li}^\star_{\mathfrak{s}}(\mathbf{u})_v) \geq B_{w,v} \quad \text{for every } \operatorname{Li}^\star_{\mathfrak{s}}(\mathbf{u})_v \in S^\star_{w,v}$$

since

$$\operatorname{ord}_v\Big(\frac{u_1^{q^{i_1}} \dots u_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}}\Big) \geq q^{i_1} - \sum_{j=1}^r s_j \operatorname{ord}_v(L_{i_j}) \geq q^{i_1} - \operatorname{wt}(\mathfrak{s})\Big\lfloor \frac{i_1}{\epsilon_v} \Big\rfloor$$
$$\geq q_v^\alpha - \operatorname{wt}(\mathfrak{s})\alpha \geq B_{w,v}.$$

Here, the first inequality comes from the fact that $\mathbf{u} \in vA \times A^{r-1}$. The second inequality follows by combining Proposition 2.2.1 and the inequality $i_1 \geq \cdots \geq i_r$. The third inequality comes from the fact that if we write $i_1 = \alpha \cdot \epsilon_v + \beta$, where $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ with $0 \leq \beta < \epsilon_v$, then

$$q^{i_1} - \operatorname{wt}(\mathfrak{s})\Big\lfloor \frac{i_1}{\epsilon_v} \Big\rfloor \geq q_v^\alpha - \operatorname{wt}(\mathfrak{s})\alpha.$$

On the other hand, every $\operatorname{Li}^\star_{\mathfrak{s}}(\mathbf{u}) \in S_{w,v}$ is just a k-linear combination of elements in $S^\star_{w,v}$. Moreover, every coefficient lies in $k \cap A_v$ since $\operatorname{ord}_v(a(\theta)) = 0$ by our choice from (3.1.5). Hence,

$$\operatorname{ord}_v(\operatorname{Li}^\star_{\mathfrak{s}}(\mathbf{u})_v) \geq B_{w,v} \quad \text{for every } \operatorname{Li}^\star_{\mathfrak{s}}(\mathbf{u})_v \in S_{w,v}.$$

Finally, if $q_v \geq \operatorname{wt}(\mathfrak{s})$ then we claim that for $w = \operatorname{wt}(\mathfrak{s})$ we have $B_{w,v} \geq 0$. To see this, consider the function

$$f_{w,v}(x) := q_v^x - x \cdot w.$$

Then we have

$$f_{w,v}(2) = q_v^2 - 2w \geq q_v^2 - 2q_v \geq 0.$$

Moreover, we also have

$$f'(x) = \ln q_v \cdot q_v^x - w > \tfrac{1}{2} \cdot q_v^x - w \geq q_v^{x-1} - w \geq 0 \quad \text{for } x \geq 2.$$

Thus $f_{w,v}(x)$ is monotonically increasing on $x \geq 2$ and hence

$$B_{w,v} \geq \min\{1, q_v - w\} \geq 0.$$

In other words,

$$\operatorname{Li}^\star_{\mathfrak{s}}(\mathbf{u})_v \in A_v \quad \text{for } q_v \geq \operatorname{wt}(\mathfrak{s}).$$

Hence we complete the proof. □

We provide an explicit example for Theorem 3.2.10 as follows:

Example 3.2.13. Consider $r = 1$, $v = \theta$, $u \in A$ and $s \in \mathbb{N}$. In this case, we have $a(t) = t^s - 1$ and thus

$$\rho_a(\mathbf{v}_{\tilde{\mathfrak{s}},\tilde{\mathbf{u}}}) = \left(\binom{s}{1}\theta u, \binom{s}{2}\theta^2 u, \dots, \binom{s}{s-1}\theta^{s-1}u, \theta^s + (u^q - u)\right)^{\mathrm{tr}}.$$

By Theorem 3.2.10, we obtain

$$\mathrm{Li}_s^\star(u)_v = \frac{1}{\theta^s - 1}\left\{\mathrm{Li}_s^\star(\theta^s u + u^q - u)_v + \sum_{j=1}^{s-1}\sum_{k=0}^{j}(-1)^{j+k}\binom{j}{k}\theta^k \,\mathrm{Li}_s^\star(\binom{s}{j}\theta^{s-k}u)_v\right\}.$$

In particular, if $s = p^\ell$ for some $\ell \in \mathbb{Z}_{\geq 0}$, then

$$\mathrm{Li}_s^\star(u)_v = \frac{1}{\theta^s - 1}\,\mathrm{Li}_s^\star(\theta^s u + u^q - u)_v.$$

Remark 3.2.14. If we replace $u_1, \dots, u_r$ by r independent variables $z_1, \dots, z_r$, then by considering the t-module G defined over $A[z_1, \dots, z_r]$, the formula in Theorem 3.2.10 is still valid in the formal power series ring $k[\![z_1, \dots, z_r]\!]$ (cf. [CM19, Rem. 3.3.6]).

Remark 3.2.15. For the depth 1 case, i.e., $\mathfrak{s} = s$ and $\mathbf{u} = u$, our t-module $G_{\mathfrak{s},\mathbf{u}}$ is the sth tensor power of the Carlitz module and the formula in Theorem 3.2.10 coincides with [T04, Rem. 7.6.2].

§4. v-adic MZVs

The aim of this section is to provide a criterion for the integrality of v-adic MZVs. We first recall the formulation for v-adic MZVs via v-adic CMSPLs. Then we estimate the v-adic valuation of v-adic MZVs by using Corollary 3.2.12. As a consequence, we provide an explicit lower bound for the v-adic valuation and a precise criterion for the integrality of v-adic MZVs.

§4.1. Formulation through v-adic CMSPLs

To introduce the formula of v-adic MZVs via v-adic CMSPLs, we need to review the Anderson–Thakur polynomials [AT90]. Let t be a variable independent of θ. We set $F_0 := 1$, $F_i := \prod_{j=1}^{i}(t^{q^i} - \theta^{q^i})$. Then the Anderson–Thakur polynomials $H_n \in A[t]$ are defined by the following generating function:

$$\left(1 - \sum_{i=0}^{\infty}\frac{F_i}{D_i|_{\theta=t}}x^{q^i}\right)^{-1} = \sum_{n=0}^{\infty}\frac{H_n}{\Gamma_{n+1}|_{\theta=t}}x^n.$$

For each $1 \leq i \leq r$, we express the Anderson–Thakur polynomial $H_{s_i-1}(t) \in A[t]$ as

$$H_{s_i-1}(t) = \sum_{j=0}^{m_i} u_{ij} t^j,$$

where $u_{ij} \in A$ and $u_{im_i} \neq 0$. Now we define

$$\mathbf{J}_{\mathfrak{s}} := \{0, 1, \ldots, m_1\} \times \cdots \times \{0, 1, \ldots, m_r\}.$$

Given $\mathbf{j} = (j_1, \ldots, j_r) \in \mathbf{J}_{\mathfrak{s}}$, we put $\mathbf{u_j} := (u_{1j_1}, \ldots, u_{rj_r}) \in A^r$ and $a_{\mathbf{j}} := t^{j_1+\cdots+j_r} \in A[t]$. To state the formulation, we further introduce the following definition.

Definition 4.1.1 (Cf. [CM21, Def. 5.2.1]). Let $\mathfrak{s} := (s_1, \ldots, s_r) \in \mathbb{N}^r$ and $\mathbf{u} \in A^r$ with $r > 1$ and let $S := \{0, 1\}$. For any $\mathbf{w} := (\mathbf{w}_1, \ldots, \mathbf{w}_{r-1}) \in S^{r-1}$, we define

$$\mathbf{w}(\mathfrak{s}) := (s_1 \lambda(\mathbf{w}_1) s_2 \lambda(\mathbf{w}_2) \cdots \lambda(\mathbf{w}_{r-1}) s_r),$$

where $\lambda(0) =$ "," (comma) and $\lambda(1) =$ "+" (addition). We also define

$$\mathbf{w}^{\times}(\mathbf{u}) := (u_1 \mu(\mathbf{w}_1) u_2 \mu(\mathbf{w}_2) \cdots \mu(\mathbf{w}_{r-1}) u_r),$$

where $\mu(0) =$ "," (comma) and $\mu(1) =$ " $\times$ " (multiplication).

Now we are ready to give the formulation of v-adic MZVs via v-adic CMSPLs.

Definition 4.1.2. For any index $\mathfrak{s} := (s_1, \ldots, s_r) \in \mathbb{N}^r$, let the notation be the same as above. We renumber the set

$$\left\{((-1)^{r-1} a_{\mathbf{j}}(\theta), \mathbf{w}(\mathfrak{s}), \mathbf{w}^{\times}(\mathbf{u_j})) \mid \mathbf{j} \in \mathbf{J}_{\mathfrak{s}},\ \mathbf{w} \in S^{r-1}\right\} = \left\{(b_\ell, \mathfrak{s}_\ell, \mathbf{u}_\ell)\right\}.$$

Then we define the v-adic MZV $\zeta_A(\mathfrak{s})_v$ to be

$$\zeta_A(\mathfrak{s})_v := \frac{1}{\Gamma_{\mathfrak{s}}} \sum_\ell b_\ell \cdot (-1)^{\operatorname{dep}(\mathfrak{s}_\ell)-1} \operatorname{Li}^{\star}_{\mathfrak{s}_\ell}(\mathbf{u}_\ell)_v \in k_v.$$

Remark 4.1.3. This definition, given in [CM21, Def. 6.1.1], is inspired by Furusho's p-adic MZVs (see [F04]) and the logarithmic interpretation of ∞-adic MZVs (see [AT90, CM21]). Note that we have the following identity [CM21, Thm. 5.2.5]:

$$\zeta_A(\mathfrak{s}) = \frac{1}{\Gamma_{\mathfrak{s}}} \sum_\ell b_\ell \cdot (-1)^{\operatorname{dep}(\mathfrak{s}_\ell)-1} \operatorname{Li}^{\star}_{\mathfrak{s}_\ell}(\mathbf{u}_\ell) \in k_\infty.$$

§4.2. Main result and examples

In this subsection we prove the explicit lower bound for the v-adic valuation and the precise criterion for the integrality of v-adic MZVs.

Theorem 4.2.1. *Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$. If we set*

$$B_{w,v} := \min_{n \geq 0}\{q_v^n - n \cdot w\},$$

then we have

$$\operatorname{ord}_v(\zeta_A(\mathfrak{s})_v) \geq B_{\mathrm{wt}(\mathfrak{s}),v} - \frac{\mathrm{wt}(\mathfrak{s}) - \mathrm{dep}(\mathfrak{s}) - \mathrm{ht}(\mathfrak{s})}{q_v - 1}.$$

In particular,

$$\zeta_A(\mathfrak{s})_v \in A_v \quad \text{if } q_v \geq \mathrm{wt}(\mathfrak{s}).$$

Remark 4.2.2. We will provide a non-integral example when the restriction in the theorem is omitted. The example was found by using the computer algebra system SageMath. The author is grateful to Yoshinori Mishiba for providing the example.

Proof of Theorem 4.2.1. Since $\mathrm{wt}(s_\ell) = \mathrm{wt}(s)$ for all ℓ by definition, we can use Corollary 3.2.12 to obtain

$$\operatorname{ord}_v(\mathrm{Li}^\star_{\mathfrak{s}_\ell}(\mathbf{u}_\ell)_v) \geq B_{\mathrm{wt}(\mathfrak{s}),v} \quad \text{for all } \ell.$$

On the other hand, b_ℓ is just a power of θ and hence $\operatorname{ord}_v(b_\ell) \geq 0$. Finally, by Proposition 2.2.1,

$$\operatorname{ord}_v(\Gamma_\mathfrak{s}) \leq \frac{\mathrm{wt}(\mathfrak{s}) - \mathrm{dep}(\mathfrak{s}) - \mathrm{ht}(\mathfrak{s})}{q_v - 1}.$$

We can conclude that

$$\begin{aligned}\operatorname{ord}_v(\zeta_A(\mathfrak{s})_v) &= \operatorname{ord}_v\left(\frac{1}{\Gamma_\mathfrak{s}}\sum_\ell b_\ell \cdot (-1)^{\mathrm{dep}(s_\ell)-1}\,\mathrm{Li}^\star_{\mathfrak{s}_\ell}(\mathbf{u}_\ell)_v\right)\\ &\geq \min_\ell\{\operatorname{ord}_v(b_\ell \cdot (-1)^{\mathrm{dep}(s_\ell)-1}\,\mathrm{Li}^\star_{\mathfrak{s}_\ell}(\mathbf{u}_\ell)\} - \frac{\mathrm{wt}(\mathfrak{s}) - \mathrm{dep}(\mathfrak{s}) - \mathrm{ht}(\mathfrak{s})}{q_v - 1}\\ &\geq B_{\mathrm{wt}(\mathfrak{s}),v} - \frac{\mathrm{wt}(\mathfrak{s}) - \mathrm{dep}(\mathfrak{s}) - \mathrm{ht}(\mathfrak{s})}{q_v - 1}.\end{aligned}$$

In particular, if $q_v \geq \mathrm{wt}(\mathfrak{s})$, then we have

$$\frac{\mathrm{wt}(\mathfrak{s}) - \mathrm{dep}(\mathfrak{s}) - \mathrm{ht}(\mathfrak{s})}{q_v - 1} < 1.$$

On the other hand, the condition $q_v \geq \mathrm{wt}(\mathfrak{s})$ guarantees that $B_{\mathrm{wt}(\mathfrak{s}),v} \geq 0$. As a result, if $q_v \geq \mathrm{wt}(\mathfrak{s})$, then

$$\mathrm{ord}_v(\zeta_A(\mathfrak{s})_v) \geq B_{\mathrm{wt}(\mathfrak{s}),v} - \frac{\mathrm{wt}(\mathfrak{s}) - \mathrm{dep}(\mathfrak{s})}{q_v - 1} > -1.$$

The desired result now follows from the fact that $\mathrm{ord}(\zeta_A(\mathfrak{s})_v) \in \mathbb{Z}$. □

Remark 4.2.3. There is another way to formulate Theorem 4.2.1. Let $w \in \mathbb{N}$ and

$$v^{[w]} := \mathrm{Span}_{A_v}\left\{\frac{v^{q^n}}{L_n^w} \,\middle|\, n \geq 0\right\} \subset k_v.$$

Then according to our proof of Corollary 3.2.12 and Theorem 4.2.1, for each $r > 0$, $\mathfrak{s} \in \mathbb{N}^r$ we have

$$\zeta_A(\mathfrak{s})_v \in \frac{1}{\Gamma_{\mathfrak{s}}} v^{[\mathrm{wt}(\mathfrak{s})]}.$$

In particular, if $q_v := q^{\deg v} > \mathrm{wt}(\mathfrak{s})$, then

$$\zeta_A(\mathfrak{s})_v \in \frac{1}{\Gamma_{\mathfrak{s}}} v^{[\mathrm{wt}(\mathfrak{s})]} \subset (vA_v)^{B_{\mathrm{wt}(\mathfrak{s}),v}}.$$

Now we give a non-integral example.

Example 4.2.4. Consider $q = 2$, $v = \theta$, and $\mathfrak{s} = (4, 1)$. Then

$$\begin{aligned} \Gamma_{(4,1)} &= \Gamma_4 \cdot \Gamma_1 = \theta^2 + \theta, & H_{s_1-1}(t) &= H_3(t) = t^2 + t \\ H_{s_2-1}(t) &= H_0(t) = 1, & \mathbf{J}_{\mathfrak{s}} &= \{(0,0), (1,0), (2,0)\} \end{aligned}$$

and

$$\begin{aligned} (b_1, \mathbf{s}_1, \mathbf{u}_1) &= (1, (4,1), (0,1)), & (b_2, \mathbf{s}_2, \mathbf{u}_2) &= (1, (5), (0)), \\ (b_3, \mathbf{s}_3, \mathbf{u}_3) &= (\theta, (4,1), (1,1)), & (b_4, \mathbf{s}_4, \mathbf{u}_4) &= (\theta, (5), (1)), \\ (b_5, \mathbf{s}_5, \mathbf{u}_5) &= (\theta^2, (4,1), (1,1)), & (b_6, \mathbf{s}_6, \mathbf{u}_6) &= (\theta^2, (5), (1)). \end{aligned}$$

Thus,

$$\zeta_A(4,1)_\theta = \mathrm{Li}^{\star}_{(4,1)}(1,1)_\theta + \mathrm{Li}^{\star}_{(5)}(1)_\theta. \tag{4.2.5}$$

Here we use the fact that $1 = -1$ since $q = 2$. Note that

$$\mathrm{ord}_\theta(d_1\text{th coordinate of } P_i\tau^i(\mathbf{v})) \geq 2^i - 5i.$$

Thus, in the computation of θ-adic CMSPLs of the right-hand side of (4.2.5), we can just compute the first four terms of $\log_G(\rho_a(\mathbf{v}))$. Consequently, we obtain

$$\zeta_A(4,1)_\theta = \theta^{-3} + \theta^2 + O(\theta^7) \notin A_\theta.$$

§5. Adelic MZVs and finite MZVs

In this section we formulate the adelic MZVs over function fields and investigate their properties. We apply [CM21, Thm. 6.1.1] to see that the $\bar{k}$-linear space spanned by ∞-adic MZVs is isomorphic to the $\bar{k}$-linear space spanned by adelic MZVs. Thus, the dimension formula which was conjectured by Todd [To18] also fits into our adelic framework. Finally, we discuss potential connections between adelic MZVs and finite MZVs.

§5.1. Formulation of adelic MZVs

We recall the ∞-adic MZVs over function fields which were initially studied by Thakur [T04]. Then we give our formulation of the adelic MZVs by using our integrality result Theorem 4.2.1.

Definition 5.1.1. Let $\mathfrak{s} := (s_1, \ldots, s_r) \in \mathbb{N}^r$. Then

(1) for the infinite place, we define

$$\zeta_A(\mathfrak{s})_\infty := \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in k_\infty,$$

where $a_1, \ldots, a_r$ runs over all monic polynomials in A with $0 \leq |a_r|_\infty < \cdots < |a_1|_\infty$;

(2) for the adelic setting, we define

$$\zeta_{\mathbb{A}_k}(\mathfrak{s}) := (\zeta_A(\mathfrak{s})_v)_{v \in M_k} \in \mathbb{A}_k,$$

where M_k is the set of all places of k and $\mathbb{A}_k$ is the adele ring of k.

Remark 5.1.2. Theorem 4.2.1 implies that $\zeta_{\mathbb{A}_k}(\mathfrak{s})$ is well defined.

§5.2. Dimension conjecture

In what follows, we recall a result from [CM21, Thm. 6.4.1] which asserts that the v-adic MZVs satisfy the same $\bar{k}$-linear relations that their corresponding ∞-adic MZVs satisfy. Then we can obtain the $\bar{k}$-vector space isomorphism from the above discussion.

Theorem 5.2.1 ([CM21, Thm. 6.4.1]). *Let v be a finite place of k and fix an embedding $\bar{k} \hookrightarrow \mathbb{C}_v$. Let w be a positive integer and let $\overline{\mathcal{Z}}_w$ be the $\bar{k}$-vector space spanned by all ∞-adic MZVs of weight w, and let $\overline{\mathcal{Z}}_{w,v}$ be the $\bar{k}$-vector space spanned by all v-adic MZVs of weight w. Then we have a well-defined surjective $\bar{k}$-linear map*

$$\overline{\mathcal{Z}}_w \twoheadrightarrow \overline{\mathcal{Z}}_{w,v}$$

given by

$$\zeta_A(\mathfrak{s})_\infty \mapsto \zeta_A(\mathfrak{s})_v$$

and the kernel contains the one-dimensional vector space $\bar{k} \cdot \zeta_A(w)$ *when* w *is divisible by* $q-1$*.*

As an application of Theorems 4.2.1 and 5.2.1, we have the following corollary:

Corollary 5.2.2. *Let* $\overline{\mathcal{Z}}_{w,\mathbb{A}_k}$ *be the* $\bar{k}$*-vector space spanned by all adelic MZVs of weight* w*. Then the map*

$$\overline{\mathcal{Z}}_w \to \overline{\mathcal{Z}}_{w,\mathbb{A}_k}$$

given by

$$\zeta_A(\mathfrak{s})_\infty \mapsto (\zeta_A(\mathfrak{s})_v)_{v\in M_k}$$

is a well-defined $\bar{k}$*-linear isomorphism. In particular,*

$$\overline{\mathcal{Z}}_w \cong \overline{\mathcal{Z}}_{w,\mathbb{A}_k}$$

as a k*-vector space.*

Proof. Indeed, this map is a well-defined surjective k-linear map by Theorem 5.2.1 and the injectivity comes from the ∞-adic coordinate. □

Todd [To18] discovers some linear relations among the same weight ∞-adic MZVs, and makes the following conjecture.

Conjecture 5.2.3 (Todd's dimension conjecture). *Let* $w \in \mathbb{N}$ *and* $\mathcal{Z}_w$ *be the* k*-vector space generated by all* ∞*-adic MZVs of weight* w*. Then we have*

$$\dim_k \mathcal{Z}_w = \begin{cases} 2^{w-1} & \text{if } 1 \le w < q, \\ 2^{w-1} - 1 & \text{if } w = q, \\ \sum_{i=1}^{q} \dim_k \mathcal{Z}_{w-i} & \text{if } w > q. \end{cases}$$

In order to see the connection between Todd's dimension conjecture and the dimension of our adelic MZVs space, we recall the following result from [C14].

Theorem 5.2.4 ([C14, Thm. 2.2.1]). *If the given* ∞*-adic MZVs* $Z_1, \ldots, Z_m$ *are linearly independent over* k*, then*

$$1, Z_1, \ldots, Z_m$$

are linearly independent over $\bar{k}$*.*

As a consequence, studying the $\bar{k}$-linear relations among ∞-adic MZVs is equivalent to studying k-linear relations among ∞-adic MZVs. In other words, the dimension of the k-vector space $\mathcal{Z}_w$ and the dimension of the $\bar{k}$-vector space $\overline{\mathcal{Z}}_w$ must be the same. We summarize this discussion as the following theorem.

Theorem 5.2.5. *We adopt the same notation. Then*

$$\dim_k \mathcal{Z}_w = \dim_{\bar{k}} \overline{\mathcal{Z}}_w = \dim_{\bar{k}} \overline{\mathcal{Z}}_{w,\mathbb{A}_k}.$$

§5.3. Finite MZVs in positive characteristic

Let $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$. Consider the k-algebra $\mathcal{A}_k := (\prod_v A/vA) \otimes_A k$, where v runs over all finite places v. Following Kaneko and Zagier, finite MZVs in positive characteristic are defined by

$$\zeta_{\mathcal{A}_k}(\mathfrak{s}) := (\zeta_{\mathcal{A}_k}(\mathfrak{s})_v)_v \in \mathcal{A}_k,$$

where the v-component $\zeta_{\mathcal{A}_k}(\mathfrak{s})_v$ is defined by

$$\sum_{\deg_\theta v > \deg_\theta a_1 > \cdots > \deg_\theta a_r \geq 0} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \bmod v.$$

Several properties of these values have already been established. For example, relations between FMZVs and finite CMPLs were developed in [CM17], several identities among FMZVs were worked out in [Shi18], and non-vanishing properties were studied in [ANDTR19] and [PP18].

Inspired by Conjecture 1.1.2 and Theorem 1.1.3, we raise the following question:

Question 5.3.1. *Let $\mathcal{Z}_{w,\mathcal{A}_k}$ be the k-vector space spanned by all finite MZVs of weight w. Do we have a well-defined surjective k-linear map from $\mathcal{Z}_{w,\mathbb{A}_k}$ into $\mathcal{Z}_{w,\mathcal{A}_k}$? In particular, is it true that*

$$\dim_k \mathcal{Z}_{w,\mathcal{A}_k} \leq \dim_k \mathcal{Z}_{w,\mathbb{A}_k}?$$

If the answer to the above question is positive, then how should one formulate the dimension conjecture for $\mathcal{Z}_{w,\mathcal{A}_k}$ from Conjecture 5.2.3?

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