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# Affine Super Schur Duality

*To the memory of Goro Shimura*

by

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## Abstract

Schur duality is an equivalence, for $d \leq n$, between the category of finite-dimensional representations over $\mathbb{C}$ of the symmetric group $S_d$ on $d$ letters, and the category of finite-dimensional representations over $\mathbb{C}$ of $\mathrm{GL}(n, \mathbb{C})$ whose irreducible subquotients are subquotients of $\bar{\mathbb{E}}^{\otimes d}$, $\bar{\mathbb{E}} = \mathbb{C}^n$. The latter are called polynomial representations homogeneous of degree $d$. It is based on decomposing $\bar{\mathbb{E}}^{\otimes d}$ as a $\mathbb{C}[S_d] \times \mathrm{GL}(n, \mathbb{C})$-bimodule. It was used by Schur to conclude the semisimplicity of the category of finite-dimensional complex $\mathrm{GL}(n, \mathbb{C})$-modules from the corresponding result for $S_d$ that had been obtained by Young. Here we extend this duality to the affine super case by constructing a functor $\mathcal{F}\colon M \mapsto M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d}$, $\mathbb{E}$ now being the super vector space $\mathbb{C}^{m|n}$, *from* the category of finite-dimensional $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$-modules, or representations of the affine Weyl, or symmetric, group $S_d^a = S_d \ltimes \mathbb{Z}^d$, *to* the category of finite-dimensional representations of the universal enveloping algebra of the affine Lie superalgebra $\mathfrak{U}(\widehat{\mathrm{sl}}(m|n))$ that are $\mathbb{E}^{\otimes d}$-compatible, namely the subquotients of whose restriction to $\mathfrak{U}(\mathrm{sl}(m|n))$ are constituents of $\mathbb{E}^{\otimes d}$. Both categories are not semisimple. When $d < m+n$ the functor defines an equivalence of categories. As an application we conclude that the irreducible finite-dimensional $\mathbb{E}^{\otimes d}$-compatible representations of the affine superalgebra $\widehat{\mathrm{sl}}(m|n)$ are tensor products of evaluation representations at distinct points of $\mathbb{C}^\times$.

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## Contents

1 Introduction 154
2 Superalgebras 158
3 Root systems 162
4 Dynkin diagrams 165
5 Theory of highest weight 168
6 Hook partitions 169

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7 Affine superalgebras 173
8 Generators and relations 175
9 Fundamental representation 178
10 Affine super Schur duality 180
11 Operators are well defined 181
11.1 The relation (S4)(2) 183
12 The relations (S4)(3) 183
13 The relations (S4)(4) 186
14 The functor $\mathcal{F}$ is an equivalence 188
15 Parabolic induction 196
16 Relating representations of $\mathbb{C}[S_d^a]$ and $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n))$ 197
17 Applications: Irreducible representations of $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n))$ 199
References 201

## §1. Introduction

In the beginning the Schur duality [Sch01, Sch27], promoted by Weyl in his book [W53] (see [FH91] and [E11] for modern expositions), was the study of the commuting actions of the symmetric group $S_d$ and the complex general linear group $\mathrm{GL}(n,\mathbb{C})$ on $\mathbb{E}^{\otimes d}$, where $\mathbb{E}$ is the $n$-dimensional Euclidean complex space $\mathbb{C}^n$. It was extended by Drinfel'd [D85] and Jimbo [J86] to the context of the finite-dimensional Iwahori–Hecke algebra $H_d(q^2)$ and the *quantum* algebra $\mathfrak{U}_q(\mathrm{sl}(n))$, on using universal $R$-matrices, which solve the Yang–Baxter equation.

There were two extensions of this duality in the Hecke-quantum case: to the infinite-dimensional *affine quantum* settings by Chari and Pressley [CP96] and to the *super* situation by Moon [Mo03] and by Mitsuhashi [Mi06], who quantum deformed the super Schur duality of (Sergeev [S85], of which they apparently were not aware, nor of the work of each other; see also [CW12]; and of) Berele and Regev [BR87].

We continued this chain of works in [F20] by completing the cube, dealing with the general *affine super quantum* case, relating the commuting actions of the *affine* Iwahori–Hecke algebra $H_d^a(q^2)$, and of the *affine* quantum Lie superalgebra $\mathfrak{U}_q(\widehat{\mathrm{sl}}(m|n,\Pi,p))$ ($\Pi$ is a root system, $p$: parity), using the presentation of the former by Bernstein [F11], and the latter by Yamane [Y99], in terms of generators and relations, acting on the $d$th tensor power of the superspace $\mathbb{E}=\mathbb{C}^{m|n}$. Thus we constructed a functor and showed it is an equivalence of categories of $H_d^a(q^2)$- and $\mathfrak{U}_q(\widehat{\mathrm{sl}}(m|n,\Pi,p))$-modules of finite rank when $d<m+n$.

However, the non-, or pre-quantum, case is interesting in its own right. We study the *affine* extension of the original Schur duality in [F21], which relates the representation theories of the group algebra $\mathbb{C}[S_d^a]$, where $S_d^a = S_d \ltimes \mathbb{Z}^d$ is the affine symmetric group, and of the affine Lie algebra $\mathfrak{U}(\widehat{\mathrm{sl}}(n))$, a duality of which [CP96] is a quantum deformation.

The case of *affine super* Schur duality, which relates the representation theories of the group algebra $\mathbb{C}[S_d^a]$ and of the affine Lie superalgebra $\mathfrak{U}(\widehat{\mathrm{sl}}(m|n,\Pi,p))$, is studied here. In particular, we consider all root systems – a new phenomenon that occurs only in the super case, as the Weyl group does not act transitively on the set of the root data.

The work [F20] is a quantum deformation of the present work, except that we dealt there only with the standard case. We show that the functor that we construct is an equivalence of categories only under the assumption $d < m+n$. Is this a casualty of the method of proof? In the finite-dimensional super situation considered in [S85], the functor is an equivalence in the generality $d < (m+1)(n+1)$. In the classical initial case of Schur, the equivalence is for $d \le n$, thus $m = 0$, $d < n+1$. It is extended to the affine case on the range $d < n$ in [F21], where it is also shown that the functor $\mathcal{F}$ does *not* extend as an equivalence when $d = n$.

As an application of the equivalence of categories we obtain a description of the irreducible finite-dimensional representations of the affine superalgebra $\widehat{\mathrm{sl}}(m|n)$ in terms of evaluation representations, namely that the irreducible $\mathbb{E}^{\otimes d}$-compatible finite-dimensional representations of the affine superalgebra $\widehat{\mathrm{sl}}(m|n)$ are tensor products of evaluation representations at distinct points of $\mathbb{C}^\times$. This is an extension to the affine super case of a result of [F21] in the affine case.

The contents of this article are as follows. We first recall what superalgebras with their even and odd parts, endomorphisms, superdimension, supertrace, supertranspose, and their basic properties are. Then we consider the structure of root systems, where there are even and odd roots in the super case, positive system and fundamental system, and associated decompositions, Weyl group, and Chevalley generators. Next we describe the Dynkin diagrams, where there are white, gray, and black vertices in the super case. We are mainly interested in $\mathrm{gl}(m|n)$ and $\mathrm{sl}(m|n)$, and describe all positive systems for this superalgebra. In contrast to the semisimple Lie algebra, nonsuper, case, there are positive systems for the root system that are not conjugate to each other under the action of the Weyl group. It is possible to pass from one fundamental system to another by means of a sequence of real and odd reflections. Then there is the theory of highest weight, and induced modules for the universal enveloping algebra $\mathfrak{U}(\mathrm{gl}(m|n))$.

To state Sergeev's extension of the Schur duality to the context of the superalgebra $\mathrm{gl}(m|n)$, we recall what partitions $\lambda \vdash d$, and $(m|n)$-hook-partitions and the associated partitions $\lambda^s$, and associated simple $\mathfrak{g}$-modules of highest weight $\lambda^s$ are. We introduce the action $\phi_d$ of $\mathrm{gl}(m|n)$ on $\mathbb{E}^{\otimes d}$ as well as that, $\psi_d$, of $S_d$. The two actions commute. The duality here, in the finite-dimensional super case, asserts a decomposition of $\mathbb{E}^{\otimes d}$, where $\mathbb{E} = \mathbb{C}^{m|n}$, as a $\mathfrak{U}(\mathrm{gl}(m|n)) \otimes \mathbb{C}[S_d]$-module, as a direct sum over $\lambda$ in the set $P_d(m|n)$ of $(m|n)$-hook partitions of $d$, of $L(\lambda^s) \otimes S^\lambda$;

here $L(\lambda^s)$ is the simple $\mathrm{gl}(m|n)$-module of highest weight $\lambda^s$, and $S^\lambda$ is the Specht module of $S_d$ associated with the partition $\lambda$. It can be rephrased as follows:

> If $M$ is a right $(S_d, \psi_d)$-module, define $\mathcal{S}(M) = M \otimes_{\psi_d(\mathbb{C}[S_d])} \mathbb{E}^{\otimes d}$ on objects, with the natural left $(\mathfrak{U}(\mathfrak{g}), \phi_d)$-module structure obtained from that on $\mathbb{E}^{\otimes d}$, and $\mathcal{S}(f) = f \otimes \mathrm{id}_{\mathbb{E}^{\otimes d}}$ on morphisms. If $d < (n+1)(m+1)$ then every partition of $d$ is an $(m|n)$-hook partition, and the functor $M \mapsto \mathcal{S}(M)$ is an equivalence from the category of finite-dimensional $\mathbb{C}[S_d]$-modules to the category of finite-dimensional $\mathbb{E}^{\otimes d}$-compatible $\mathfrak{U}(\mathrm{gl}(m|n))$-modules, namely those that are polynomial of degree $d$.

In particular, $\mathcal{S}$ takes $S^\lambda$ to $S^\lambda \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d} = V^\lambda$ since for any $G$-modules $V$, $W$ we have $V' \otimes_G W = \mathrm{Hom}_G(V, W)$, and $S^\lambda$ is self-dual in characteristic 0 (only!; see [Ja78, Thms. 4.12, 6.7, 8.15; pp. 16, 25, 33]), thus $S^\lambda \otimes_{\mathbb{C}[S_d]} S^\lambda \simeq \mathbb{C}$.

By a finite-dimensional $\mathbb{E}^{\otimes d}$-*compatible* $\mathfrak{U}(\mathfrak{g})$-module, we mean here a $\mathfrak{g}$-module, $\mathfrak{g} = \mathrm{gl}(m|n)$ or $\mathrm{sl}(m|n)$, all of whose subquotients are subquotients of the semisimple module $\mathbb{E}^{\otimes d}$. Below, we say that a $\mathfrak{U}(\hat{\mathfrak{g}})$-module is $\mathbb{E}^{\otimes d}$-*compatible* if its restriction to $\mathfrak{U}(\mathfrak{g})$ is. By a $\mathfrak{g}$- or $\mathcal{L}\mathfrak{g}$-module we mean a module for their universal enveloping algebras $\mathfrak{U}(\mathfrak{g})$ and $\mathfrak{U}(\mathcal{L}\mathfrak{g})$.

To extend this to the affine $\widehat{\mathrm{sl}}(m|n)$, we then introduce an affine Lie superalgebra as a loop algebra augmented with the central elements $c$ and the derivation element $d$. We then describe admissible and affine admissible Lie superalgebras, to describe Yamane's presentation of the affine Lie algebra $\widehat{\mathrm{sl}}(m|n, \Pi, p)$ associated with a datum $(\mathcal{E}, (.\,,.), \Pi, p)$, in terms of generators and relations. There are interesting affine Serre relations in the super case. We need to describe the fundamental representation of $\widehat{\mathrm{sl}}(m|n)$ on the superspace $\mathbb{E} = \mathbb{C}^{m|n}$, to state the main result, Theorem 10.1, which takes the following form. Put $S_d^a = \mathbb{Z} \rtimes S_d$ (superscript $a$ for "affine").

**Theorem 1.1.** *Fix integers $d \geq 0$, $m > n \geq 1$, $m + n > 3$. There exists a functor $\mathcal{F}$ from the category* $\mathrm{Rep}\,\mathbb{C}[S_d^a]$ *of finite-dimensional right $\mathbb{C}[S_d^a]$-modules, to the category* $\mathrm{Rep}(\widehat{\mathrm{sl}}(m|n); d)$ *of finite-dimensional $\mathbb{E}^{\otimes d}$-compatible left $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n, \Pi, p))$-modules, defined as follows. Let $M$ be a right $S_d^a$-module. Define $\mathcal{F}(M)$ to be $\mathcal{S}(M) = M \otimes_{\psi_d(\mathbb{C}[S_d])} \mathbb{E}^{\otimes d}$ as a $\mathfrak{U}_\sigma(\mathrm{sl}(m|n))$-module, thus $\mathfrak{U}_\sigma(\mathrm{sl}(m|n))$ acts on $\mathcal{S}(M)$ via $\phi_d$. Let the remaining generators of $\widehat{\mathrm{sl}}(m|n, \Pi, p)$ act by*

$$(\rho_d(e_0))(\boldsymbol{m} \otimes v) = \sum_{1 \leq j \leq d} \boldsymbol{m} y_j \otimes \rho^{\otimes d}(Y_{j,e}^{(d)})v, \quad Y_{j,e}^{(d)} = (\sigma^{p(\alpha_0)})^{\otimes(j-1)} \otimes e_0 \otimes I^{\otimes(d-j)},$$

$$(\rho_d(f_0))(\boldsymbol{m} \otimes v) = \sum_{1 \leq j \leq d} \boldsymbol{m} y_j^{-1} \otimes \rho^{\otimes d}(Y_{j,f}^{(d)})v, \; Y_{j,f}^{(d)} = (\sigma^{p(\alpha_0)})^{\otimes(j-1)} \otimes f_0 \otimes I^{\otimes(d-j)},$$

*for all* $\boldsymbol{m} \in M$ *and* $v \in \mathbb{E}^{\otimes d}$*. If* $d < m + n$ *then the functor* $\mathcal{F}\colon M \mapsto \mathcal{F}(M)$ *is an equivalence from the category* $\operatorname{Rep}\mathbb{C}[S_d^a]$ *of finite-dimensional* $S_d^a$*-modules onto the category* $\operatorname{Rep}(\widehat{\mathrm{sl}}(m|n); d)$ *of finite-dimensional* $\mathbb{E}^{\otimes d}$*-compatible* $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n))$*-modules.*

We show that our functor is an equivalence only for $d < m + n$. Perhaps this assertion holds for $d < (n+1)(m+1)$, as this is the condition in Theorem 6.2(4), as in [S85]. But our method of proof, which adapts [CP96], shows the surjectivity only for $d < m + n$. In the nonsuper case $n = 0$, it is shown in [F21] that $\mathcal{F}$ is an equivalence when $d < m$, but it is *not* an equivalence when $d = m$ in the affine case, although $\mathcal{S}$ is in the finite-dimensional case.

When $d = 0$ the category on the $S_d$-side is that of finite-dimensional complex vector spaces, and the theorem asserts that there are no nontrivial extensions of $\mathcal{L}\mathfrak{g}$-modules lifted from the trivial $\mathfrak{g}$-module $\mathbb{C}$.

When $d = 1$, an irreducible representation of $\mathbb{C}[S_d \ltimes \mathbb{Z}^d] = \mathbb{C}[\mathbb{Z}] = \mathbb{C}[t, t^{-1}]$ is a $\mathbb{C}$-linear homomorphism $\chi\colon \mathbb{C}[t^{\pm 1}] \to \mathbb{C}$ determined by the value $\chi(t) \in \mathbb{C}^\times$ of $\chi$ at $t$, or at $1 \in \mathbb{Z}$. A finite-dimensional $\mathbb{E}$-compatible irreducible representation of $\mathcal{L}\mathfrak{g} = \mathcal{L} \otimes \mathrm{sl}(n, \mathbb{C})$ (i.e., whose restriction to $\mathrm{sl}(m|n)$ is the standard representation $\rho$ on $\mathbb{E} = \mathbb{C}^{m|n}$) is then of the form $\chi \otimes \rho$, where $\chi\colon \mathcal{L} \to \mathbb{C}$ is a $\mathbb{C}$-linear algebra homomorphism determined by the value $\chi(t) \in \mathbb{C}^\times$ (see Corollary 17.3). On irreducibles the correspondence defined by $\mathcal{F}$ is then $\chi \mapsto \chi \otimes \rho$. Both categories, of finite-dimensional $\mathcal{L}$-modules and of finite-dimensional $\mathbb{E}$-compatible $\mathcal{L}\mathfrak{g}$-modules, are not semisimple.

For the proof we check that the operators that appear in the theorem are well defined. Then we check that the relations stated by Yamane, especially the super Serre relations, are satisfied by our operators. Particularly technical is the verification that the functor $\mathcal{F}$ is an equivalence of categories.

To show that the functor $\mathcal{F}$ – which we have seen is a well-defined functor between the categories specified in the theorem – is an equivalence, one has to show the following:

(a) Every finite-dimensional $\mathbb{E}^{\otimes d}$-compatible $\mathfrak{U}(\widehat{\mathrm{sl}}(m|n, \Pi, p))$-module $W$ is isomorphic to $\mathcal{F}(M) = M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d}$ for some $\mathbb{C}[S_d^a]$-module $M$.

(b) $\mathcal{F}$ is bijective on sets of morphisms.

To prove (a), by the super Schur duality theorem we assume that $W = \mathcal{S}(M)$ for some $\mathbb{C}[S_d]$-module $M$. We then construct the action of the $y_j^{\pm 1}$ on $M$ from the given action of $\rho_d(e_0)$, $\rho_d(f_0)$, $\rho_d(\mathfrak{h})$ on $W$.

As an application, we define induction $M_1 \widetilde{\times} M_2$ of affine Weyl group modules from $\mathbb{C}[S_{d_1}^a] \otimes \mathbb{C}[S_{d_2}^a]$ to $\mathbb{C}[S_{d_1+d_2}^a]$, discuss commutation $\mathcal{F}(M_1 \widetilde{\times} M_2) \simeq$

$\mathcal{F}(M_1) \otimes \mathcal{F}(M_2)$ with the functor $\mathcal{F}$, evaluation representations, implications to universal $\mathbb{C}[S_d^a]$-modules, and then we deduce from Mackey theory that the irreducible finite-dimensional $\mathbb{E}^{\otimes d}$-compatible representations of the affine superalgebra $\widehat{\mathrm{sl}}(m|n)$ are tensor products of evaluation representations at distinct points of $\mathbb{C}^\times$.

It is natural to attempt to state the equivalence of categories in *group*-theoretic terms, rather than in Lie algebra language. Although not touched upon in the present notes, where we work only with affine super*algebras* $\widehat{\mathrm{sl}}(m|n)$, it is tempting to take the hint that the group algebra $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$ is $\mathbb{C}[S_d] \ltimes \mathbb{C}[\mathbb{Z}^d]$ and $\mathbb{C}[\mathbb{Z}^d]$ is the ring $\Gamma(\mathbb{G}_m^d, \mathcal{O})$ of global sections of the torus $\mathbb{G}_m^d = \operatorname{Spec} \mathbb{C}[\mathbb{Z}^d]$, and note that a finite-dimensional $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$-module $M$ can be viewed as the module of global sections $\Gamma(\mathcal{M}, \mathcal{O})$ of an $S_d$-equivariant quasi-coherent sheaf of modules $\mathcal{M} = \pi^* M$ over $\mathbb{G}_m^d = \operatorname{Spec} \mathbb{C}[\mathbb{Z}^d]$, pulled back from a point: $\pi\colon \mathbb{G}_m^d \to \{*\}$.

The role of the affine superalgebra $\widehat{\mathrm{sl}}(m|n)$ has to be replaced by the affine super group $\mathrm{SL}(m|n, \mathcal{L})$, viewed as a functor $A \mapsto \mathrm{SL}(m|n, A[t, t^{-1}])$ on the category of superalgebras $A$ over $\mathbb{C}$. Suitably interpreted, the functor $\mathcal{F}$ may take the (modified to be a limit of subsheaves with finite support) form

$$M = \Gamma(\mathcal{M}, \mathcal{O}) \mapsto \left\{\Gamma((\mathcal{M} \otimes_{\mathbb{G}_m^d} (\mathbb{E}_A^{\otimes d} \otimes \mathbb{G}_m^d)), \mathcal{O})_{\mathbb{C}[S_d]}\right\}_A.$$

Another approach would be to show that a finite-dimensional representation of the super loop algebra $\widehat{\mathrm{sl}}(m|n, \mathbb{C})$ integrates to a compatible family of representations of the super loop group $\mathrm{SL}(m|n, A[t, t^{-1}])$ for all superalgebras $A$ over $\mathbb{C}$. This would permit restating Theorem 1.1 as asserting that the functor $\mathcal{F}$ is, for $d < m + n$, an equivalence of categories between the category of finite-dimensional $\mathbb{C}[S_d^a]$-modules, and the category of compatible families of finite-dimensional representations of $\mathrm{SL}(m|n, A[t, t^{-1}])$, all of whose subquotients as representations of $\mathrm{SL}(m|n, A)$ ($\subset \mathrm{SL}(m|n, A[t, t^{-1}])$ via $A \hookrightarrow A[t, t^{-1}]$) occur in $\mathbb{E}_A^{\otimes d}$, where $\mathbb{E}_A = A^{m|n}$, for $A$ a superalgebra over $\mathbb{C}$. As $\mathbb{E}_A^{\otimes d}$ is semisimple as an $\mathrm{SL}(m|n, A)$-module, "occur in" could be replaced by "are subrepresentations of", or "are subquotients of".

## §2. Superalgebras

We work over the field $\mathbb{C}$ of complex numbers. In the super world, the group $\mathbb{Z}/2$ of two elements, which we denote by $\bar{0}$ and $\bar{1}$, plays a pivotal role. We denote it by $\mathbb{F}_2$, as $\mathbb{Z}/2$ is too long and $\mathbb{Z}_2$ denotes the ring of 2-adic integers. A (vector) *superspace* is a vector space over $\mathbb{C}$ with $\mathbb{F}_2$-gradation: $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $V_{\bar{0}}$ is called the *even* part, and $V_{\bar{1}}$ the *odd* part. Its *dimension* as a superspace, or its *superdimension*, is $\dim_s V = \dim V_{\bar{0}} | \dim V_{\bar{1}}$. As a vector space it is $\dim V = \dim V_{\bar{0}} + \dim V_{\bar{1}}$.

For example, $\mathbb{C}^{m|n}$ denotes the superspace with even part $\mathbb{C}^m$ and odd part $\mathbb{C}^n$; $m$, $n \in \mathbb{Z}_{\geq 0}$ = set of nonnegative integers; $\dim_s \mathbb{C}^{m|n} = m|n$. We often denote $\mathbb{C}^{m|n}$ by $\mathbb{E}$.

A vector $v \in V$ is *homogeneous* if it lies in $V_{\bar{0}}$ or in $V_{\bar{1}}$. It is then called *even* or *odd*, and its *parity* is $p(v) = i$ if $v \in V_i$.

A *subspace* of a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a superspace $W = W_{\bar{0}} \oplus W_{\bar{1}}$ contained in $V$ compatibly, thus $W_{\bar{0}} \subset V_{\bar{0}}$ and $W_{\bar{1}} \subset V_{\bar{1}}$. Writing $p(v)$ for $v \in V$ implies $v$ is homogeneous. Formulae involving such elements are extended below by linearity to all of $V$.

If $V$, $W$ are superspaces, then the space $\mathrm{Hom}(V, W)$ of linear transformations from $V$ to $W$ is a superspace: $T\colon V \to W$ is *even* if $T(V_i) \subset W_i$ ($i \in \mathbb{F}_2$) and *odd* if $T(V_i) \subset W_{i+1}$. Write $\mathrm{End}(V)$ for $\mathrm{Hom}(V, V)$.

The *parity reversing functor* $\Pi$ on the category of superspaces takes $V = V_{\bar{0}} \oplus V_{\bar{1}}$ to $\Pi(V) = \Pi(V)_{\bar{0}} \oplus \Pi(V)_{\bar{1}}$, where $\Pi(V)_i = V_{i+\bar{1}}$, $i \in \mathbb{F}_2$. Then $\Pi^2 = I$, the identity in $\mathrm{End}(V)$.

A *superalgebra* is a superspace $A = A_{\bar{0}} \oplus A_{\bar{1}}$ together with a bilinear multiplication satisfying $A_i A_j \subset A_{i+j}$ ($i$, $j \in \mathbb{F}_2$). A *module* over a superalgebra $A$ is graded: $M = M_{\bar{0}} \oplus M_{\bar{1}}$, with $A_i M_j \subset M_{i+j}$. Also, *subalgebras* and *ideals* of superalgebras are to be understood in the $\mathbb{F}_2$-graded sense. A superalgebra is *simple* if it has no nontrivial ideals.

A *homomorphism* between $A$-modules $M$ and $N$ is a linear map $f\colon M \to N$ with $f(am) = af(m)$ for all $a \in A$, $m \in M$. Such $f$ has *parity* $p(f)$ if $f(M_i) \subset M_{i+p(f)}$, $i \in \mathbb{F}_2$. Note that a homomorphism $f\colon M \to N$ of parity $p(f)$ defines by $f^+(x) = (-1)^{p(f)p(x)} f(x)$ a linear map $f^+\colon M \to N$ of parity $p(f)$ satisfying $f^+(am) = (-1)^{p(a)p(f)} af(m)$ for homogeneous $a \in A$, $m \in M$, and such an $f^+$ defines $f$ by the same formula.

A *Lie superalgebra* is a superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear operation $[.\,,.]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called a *bracket*, with $\mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$, thus $p([x, y]) = p(x) + p(y)$, satisfying, for all homogeneous $x$, $y$, $z \in \mathfrak{g}$,

$$\textit{skew supersymmetry:} \qquad [x, y] = -(-1)^{p(x)p(y)}[y, x],$$
$$\textit{super Jacobi identity:} \quad [x, [y, z]] = [[x, y], z] + (-1)^{p(x)p(y)}[y, [x, z]].$$

A *skew-supersymmetric* (satisfying $(x, y) = (-1)^{p(x)p(y)}(y, x)$) bilinear form $(.\,,.)\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ on a Lie superalgebra $\mathfrak{g}$ is called *invariant* if $([x, y], z) = (x, [y, z])$ for all $x$, $y$, $z \in \mathfrak{g}$.

The even part $\mathfrak{g}_{\bar{0}}$ of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie algebra. In particular, if $\mathfrak{g}_{\bar{1}} = 0$ then $\mathfrak{g} = \mathfrak{g}_{\bar{0}}$ is a Lie algebra. A purely odd Lie superalgebra $\mathfrak{g}$ ($= \mathfrak{g}_{\bar{1}}$, thus $\mathfrak{g}_{\bar{0}} = 0$) is *abelian*: $[\mathfrak{g}, \mathfrak{g}] = 0$, or $[x, y] = 0$ for all $x$, $y$.

A *homomorphism* of Lie superalgebras $\mathfrak{g}$, $\mathfrak{g}'$ is an even linear map $f\colon \mathfrak{g} \to \mathfrak{g}'$ respecting the bracket, namely $f([x,y]) = [f(x), f(y)]$ for all $x$, $y \in \mathfrak{g}$.

An associative superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is a Lie superalgebra with the bracket $[x,y] = xy - (-1)^{p(x)p(y)}yx$ (for homogeneous $x$, $y \in A$, and $[.\,,.]$ extended by linearity): it is skew supersymmetric and satisfies the super Jacobi identity.

For example, if $\mathfrak{g}$ is a Lie superalgebra, $\mathrm{End}(\mathfrak{g})$ is an associative superalgebra, hence a Lie superalgebra with the bracket as above. The *adjoint representation* is the map $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ defined by $(\mathrm{ad}(x))(y) = [x,y]$ ($x$, $y \in \mathfrak{g}$). It is a homomorphism of Lie superalgebras, by the super Jacobi identity. The action of $\mathfrak{g}$ on itself is called the *adjoint action*, making $\mathfrak{g}$ into a $\mathfrak{g}$-module.

An endomorphism $D$ of $A$ of parity $j$ ($\in \mathbb{F}_2$), thus $D \in \mathrm{End}(A)_j$, where $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is a superalgebra, is called a *derivation* (of parity $j$) if it satisfies for all homogeneous $x$, $y \in A$,

$$D(xy) = D(x)y + (-1)^{jp(x)}xD(y).$$

The space $\mathrm{Der}(A) = \mathrm{Der}(A)_{\bar{0}} \oplus \mathrm{Der}(A)_{\bar{1}}$ is a subalgebra of the Lie superalgebra $(\mathrm{End}(A), [.\,,.])$.

When $\mathfrak{g}$ is a Lie superalgebra, $\mathrm{ad}(g) \in \mathrm{Der}(\mathfrak{g})$ for all $g \in \mathfrak{g}$, by the super Jacobi identity. These are called *inner derivations*; they form an ideal in $\mathrm{Der}(\mathfrak{g})$.

The restriction $\mathrm{ad}|\mathfrak{g}_{\bar{0}}\colon \mathfrak{g}_{\bar{0}} \to \mathrm{End}(\mathfrak{g}_{\bar{1}})$ of the adjoint map is a homomorphism of Lie algebras, namely $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module under the adjoint action. Thus to a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ we associate a quadruple consisting of

(1) a Lie algebra $\mathfrak{g}_{\bar{0}}$;
(2) a $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ defined by the adjoint action;
(3) a $\mathfrak{g}_{\bar{0}}$-homomorphism $S^2(\mathfrak{g}_{\bar{1}}) \to \mathfrak{g}_{\bar{0}}$ defined by the Lie bracket;
(4) the identity obtained from the super Jacobi identity restricted to $x$, $y$, $z \in \mathfrak{g}_{\bar{1}}$.

Conversely, such a quadruple defines a Lie superalgebra structure on $\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a superspace. The associative superalgebra $\mathrm{End}(V)$ is a Lie superalgebra with the supercommutator defined above, called the *general linear Lie superalgebra*, denoted $\mathrm{gl}(V)$. When $V$ is $\mathbb{E} = \mathbb{C}^{m|n}$, so with the standard basis, write $\mathrm{gl}(m|n)$ for $\mathrm{gl}(V)$.

Choose ordered bases for $V_{\bar{0}}$ and $V_{\bar{1}}$, thus a homogeneous basis for $V$. Parametrize it by the set $I(m|n) = \{\bar{1}, \dots, \overline{m};\ 1, \dots, n\}$ totally ordered by $\bar{1} < \dots < \overline{m} < 0 < 1 < \dots < n$. The size $(m+n) \times (m+n)$ elementary matrices $E_{i,j} = (\delta_{(i,j),(k,\ell)}) \in \mathrm{gl}(m|n)$ ($i$, $j \in I(m|n)$) makes a basis of $\mathrm{End}(V)$, and $\mathrm{gl}(V)$ can be realized as their span, thus $(m+n) \times (m+n)$ complex matrices of the form $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $a$ is an $m \times m$ matrix and $d$ is $n \times n$. The even subalgebra $\mathrm{gl}(V)_{\bar{0}}$

consists of the $g$ with $b = 0 = c$, and the odd $\mathrm{gl}(V)_{\bar{1}}$ of the $g$ with $a = 0$, $d = 0$. Thus $\mathrm{gl}(V)_{\bar{0}} \simeq \mathrm{gl}(m) \oplus \mathrm{gl}(n)$, and $\mathrm{gl}(V)_{\bar{1}}$ is self-dual as a $\mathrm{gl}(V)_{\bar{0}}$-module, and is isomorphic to $(\mathbb{C}^m \otimes \mathbb{C}^{n*}) \oplus (\mathbb{C}^{m*} \otimes \mathbb{C}^n)$, where $\mathbb{C}^{n*}$ signifies the dual space of $\mathbb{C}^n$.

Note that $\mathrm{gl}(V) \to \mathrm{gl}(\Pi V)$, $T \mapsto \Pi T \Pi^{-1}$ is an isomorphism of Lie superalgebras, thus $\mathrm{gl}(m|n) \simeq \mathrm{gl}(n|m)$.

Define the *supertrace* $\mathrm{str}(g)$ of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{gl}(m|n)$ to be $\mathrm{tr}(a) - \mathrm{tr}(d)$, where tr is the trace of a square matrix such as $a$ or $d$. Then $\mathrm{str}([g, g']) = 0$ for all $g$, $g' \in \mathrm{gl}(m|n)$. The *special linear Lie superalgebra* is $\mathrm{sl}(m|n) = \{g \in \mathrm{gl}(m|n);\ \mathrm{str}(g) = 0\}$. This subalgebra of $\mathrm{gl}(m|n)$ satisfies $[\mathrm{gl}(m|n), \mathrm{gl}(m|n)] = \mathrm{sl}(m|n)$; we have $\mathrm{sl}(m|n) \simeq \mathrm{sl}(m|n)$, and when $m \neq n$ and $m + n \geq 2$, $\mathrm{sl}(m|n)$ is simple. Denote the identity matrix in $\mathrm{gl}(\mathbb{E}) = \mathrm{gl}(m|n)$ by $I_{m|n}$. When $m = n$, $\mathrm{sl}(n|n)$ contains a nontrivial center $\mathbb{C}I_{n|n}$, and $\mathrm{sl}(n|n)/\mathbb{C}I_{n|n}$ is simple for $n \geq 2$.

A basis for $\mathfrak{g} = \mathrm{gl}(1|1)$ consists of

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{\bar{1},\bar{1}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{1,1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The element $h = E_{\bar{1},\bar{1}} + E_{1,1} = I_{1|1}$ is central, $[e, f] = h$, and $\{e, f, h\}$ is a basis of $\mathrm{sl}(1|1)$.

Let $\mathbb{I}_{\bar{0}}$ be a set parametrizing an ordered basis of $V_{\bar{0}}$, and $\mathbb{I}_{\bar{1}}$ of $V_{\bar{1}}$. Then $\mathbb{I} = \mathbb{I}_{\bar{0}} \cup \mathbb{I}_{\bar{1}}$ (disjoint union) parametrizes a homogeneous basis of the superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Put $p(i) = j$ for $i \in \mathbb{I}_j$. For example, if $\mathbb{I} = I(m|n)$, then $p(i) = 0$ for $i < 0$ and $p(i) = 1$ for $i > 0$. Choosing a total order on $\mathbb{I}$ we may identify $\mathrm{gl}(V)$ with the space of $|\mathbb{I}| \times |\mathbb{I}|$ matrices. The *supertranspose* of a matrix $A = \sum_{i,j \in \mathbb{I}} a_{i,j} E_{i,j}$ ($a_{i,j} \in \mathbb{C}$) is defined to be ${}^{\mathrm{st}}A = \sum_{i,j \in \mathbb{I}} (-1)^{p(j)(p(i)+p(j))} a_{i,j} E_{j,i}$. For example, if $\mathbb{I} = I(m|n)$ then

$${}^{\mathrm{st}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} {}^t a & {}^t c \\ -{}^t b & {}^t d \end{pmatrix};$$

here ${}^t a$ is the transpose of a matrix $a$.

The *Chevalley automorphism* $\tau\colon \mathrm{gl}(V) \to \mathrm{gl}(V)$ is defined by $\tau(A) = -{}^{\mathrm{st}}A$. It restricts to an automorphism of $\mathrm{sl}(V)$, and its order is 4 when $m$, $n \geq 1$.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a superspace. A bilinear form $B\colon V \times V \to V$ is called *even* if $B(V_i, V_j) = 0$ when $i + j = \bar{1}$, and *odd* if $B(V_i, V_j) = 0$ when $i + j = \bar{0}$. An even bilinear form $B$ is called *supersymmetric* if $B|V_{\bar{0}} \times V_{\bar{0}}$ is symmetric and $B|V_{\bar{1}} \times V_{\bar{1}}$ is skew symmetric. An even bilinear form $B$ is called *skew supersymmetric* if $B|V_{\bar{0}} \times V_{\bar{0}}$ is skew symmetric and $B|V_{\bar{1}} \times V_{\bar{1}}$ is symmetric.

A classification of finite-dimensional complex simple Lie superalgebras was worked out in [K77]. We are interested here only in the case of $\mathrm{sl}(m|n)$, $m > n \geq 1$

(excluding $m|n = 2|1$) and $\mathrm{sl}(m|m)/\mathbb{C}I_{m|m}$ ($m \geq 3$). The remaining cases ($m = n$ or $(m, n) = (2, 1)$) are left to another work.

A *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g} = \mathrm{gl}(m|n)$ or $\mathrm{sl}(m|n)$ is a Cartan algebra of the even $\mathfrak{g}_{\bar{0}}$. Every inner automorphism of $\mathfrak{g}_{\bar{0}}$ extends to one of the Lie superalgebra $\mathfrak{g}$, and Cartan subalgebras of $\mathfrak{g}_{\bar{0}}$ are conjugate under inner automorphisms. Hence the Cartan subalgebras of $\mathfrak{g}$ are conjugate under inner automorphisms.

## §3. Root systems

Let $\mathfrak{h}$ be a Cartan subalgebra of a Lie superalgebra $\mathfrak{g}$. For $\alpha \in \mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ define

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g};\ [h, g] = \alpha(h)g\ \forall\, h \in \mathfrak{h}\}.$$

Such an $\alpha$ is called a *root* if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. The *root system* $\Phi$ is the set of roots. A root $\alpha$ is called *even* if $\mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{0}} \neq 0$ and *odd* if $\mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{1}} \neq 0$. The sets of even and odd roots are denoted $\Phi_{\bar{0}}$ and $\Phi_{\bar{1}}$.

Define the *Weyl group* $W$ of $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ to be the Weyl group of the Lie algebra $\mathfrak{g}_{\bar{0}}$. We continue to work only with $\mathfrak{g} = \mathrm{gl}(m|n)$ and $\mathrm{sl}(m|n)$ ($m > n \geq 1$ but not $(2|1)$), $\mathrm{sl}(n|n)/\mathbb{C}I_{n|n}$ if $m = n \geq 3$, although the following results hold for other ("basic") Lie superalgebras.

(1) There is a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ of $\mathfrak{g}$ wrt $\mathfrak{h}$, and $\mathfrak{g}_0 = \mathfrak{h}$.

(2) $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$. So fix $e_\alpha \in \mathfrak{g}_\alpha$, $e_\alpha \neq 0$.

(3) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for $\alpha$, $\beta \in \Phi \cup \{0\}$ (by the super Jacobi identity).

(4) $\Phi$, $\Phi_{\bar{0}}$, $\Phi_{\bar{1}}$ are invariant under the action of the Weyl group $W$ on $\mathfrak{h}^*$.

(5) There exists a nondegenerate even invariant supersymmetric bilinear form $(.\,,.)$ on $\mathfrak{g}$.

(6) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\alpha = -\beta \in \Phi$.

(7) The restriction of the bilinear form $(.\,,.)$ to $\mathfrak{h} \times \mathfrak{h}$ is nondegenerate and $W$-invariant.

(8) $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha$, where $h_\alpha$ is the *coroot* in $\mathfrak{h}$ determined by $(h_\alpha, h) = \alpha(h)\ \forall\, h \in \mathfrak{h}$.

(9) $\Phi_{\bar{0}} = -\Phi_{\bar{0}}$, $\Phi_{\bar{1}} = -\Phi_{\bar{1}}$, $\Phi = -\Phi$.

(10) Fix $\alpha \in \Phi$. There exists an integer $k \neq \pm 1$ such that $k\alpha \in \Phi$ iff $\alpha$ is an odd root with $(\alpha, \alpha) \neq 0$. In this case $k = \pm 2$.

We shall see these explicitly, and that for each $\alpha \in \Phi$ there is an $i \in \mathbb{F}_2$ with $\mathfrak{g}_\alpha \subset \mathfrak{g}_i$. Then $\Phi$ is the disjoint union of $\Phi_{\bar{0}}$ and $\Phi_{\bar{1}}$, and $\Phi_i = \{\alpha \in \Phi;\ \mathfrak{g}_\alpha \subset \mathfrak{g}_i\}$, $i \in \mathbb{F}_2$.

Note that $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}$ and it is abelian.

A root $\alpha \in \Phi$ is called *isotropic* if $(\alpha, \alpha) = 0$. It is necessarily an odd root. Denote the set of isotropic odd roots by

$$\overline{\Phi}_{\bar{1}} = \{\alpha \in \Phi_{\bar{1}};\ (\alpha, \alpha) = 0\} = \{\alpha \in \Phi_{\bar{1}};\ 2\alpha \notin \Phi\}.$$

The last equality follows from (10). For $\alpha \in \overline{\Phi}_{\bar{1}}$ we have $e_\alpha^2 = \frac{1}{2}[e_\alpha, e_\alpha] = 0$ (by (3)) in the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$, defined in Section 5 below, where it is used. Put $\overline{\Phi}_{\bar{0}} = \{\alpha \in \Phi_{\bar{0}};\ \frac{1}{2}\alpha \notin \Phi\}$.

A nondegenerate supersymmetric bilinear form on $\mathfrak{g} = \mathrm{gl}(m|n)$ is given by

$$(.\,,.) \colon \mathrm{gl}(m|n) \times \mathrm{gl}(m|n) \to \mathbb{C}, \quad (x, y) = \mathrm{str}(xy),$$

where $xy$ indicates matrix multiplication. This form is invariant. On restriction to the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices, one obtains a nondegenerate symmetric bilinear form on $\mathfrak{h}$ satisfying, for $i$, $j \in I(m|n)$,

$$(E_{i,i}, E_{j,j}) = \begin{cases} 1 & \text{if } \bar{1} \le i = j \le \overline{m}, \\ -1 & \text{if } 1 \le i = j \le n, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $\{\delta_i, \varepsilon_j;\ 1 \le i \le m\ , 1 \le j \le n\}$ be the basis of $\mathfrak{h}^*$ dual to the basis $\{E_{i,i}, E_{j,j};\ \bar{1} \le i \le \overline{m},\ 1 \le j \le n\}$ of $\mathfrak{h}$. Using the bilinear form $(.\,,.)$ we can identify $\delta_i$ with $(E_{\bar{i},\bar{i}}, .)$ and $\varepsilon_j$ with $-(E_{j,j}, .)$. We also write $\varepsilon_{\bar{i}}$ for $\delta_i$, $1 \le i \le m$.

The form $(.\,,.)$ on $\mathfrak{h}$ defines a nondegenerate bilinear form on $\mathfrak{h}^*$, denoted also by $(.\,,.)$. For $i$, $j \in I(m|n)$ we have $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}(-1)^{p(\varepsilon_i)}$, where $\delta_{i,j}$ is 1 if $i = j$ and 0 if $i \neq j$.

The root system $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ is given by

$$\Phi_{\bar{0}} = \{\varepsilon_i - \varepsilon_j;\ i \neq j \in I(m|n),\ i, j > 0 \text{ or } i, j < 0\},$$
$$\Phi_{\bar{1}} = \{\pm(\varepsilon_i - \varepsilon_j);\ i, j \in I(m|n),\ i < 0 < j\}.$$

Note that $E_{i,j}$ is a root vector for the root $\varepsilon_i - \varepsilon_j$ for $i \neq j$ in $I(m|n)$. The Weyl group of $\mathrm{gl}(m|n)$ is that of $\mathfrak{g}_{\bar{0}} = \mathrm{gl}(m) \oplus \mathrm{gl}(n)$, isomorphic to the product $S_m \times S_n$ of the symmetric groups on $m$ and $n$ letters.

Let $\Phi$ be a root system for the Lie superalgebra $\mathfrak{g} = \mathrm{sl}(m|n)$ or $\mathrm{gl}(m|n)$, with a fixed Cartan subalgebra $\mathfrak{h}$. Let $E$ be the real vector space spanned by $\Phi$. Then $E \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{h}^*$ for $\mathfrak{g} = \mathrm{sl}(m|n)$. For $\mathfrak{g} = \mathrm{gl}(m|n)$ the space $E \otimes_{\mathbb{R}} \mathbb{C}$ is a subspace of $\mathfrak{h}^*$ of codimension one.

The total ordering $\ge$ on $E$ is taken to be compatible with its real vector space structure: thus $v \ge w$ and $v' \ge w'$ imply $v + v' \ge w + w'$, $-w \ge -v$, and $cv \ge cw$ for $c \in \mathbb{R}_{>0}$.

A *positive system* $\Phi^+$ is a subset of the root system $\Phi$ consisting of the roots $\alpha \in \Phi$ with $\alpha > 0$ for a fixed total ordering of $E$. Given such a $\Phi^+$, define the *fundamental system* $\Pi \subset \Phi^+$ to be the set of $\alpha \in \Phi^+$ that cannot be written as a sum of two roots in $\Phi^+$. The roots in $\Phi^+$ are called *positive* roots. The roots in $\Pi$ are called *simple* roots. Put $\Phi^- = \{\alpha \in \Phi;\ \alpha < 0\}$, $\Phi_i^+ = \Phi^+ \cap \Phi_i$, $\Phi_i^- = \Phi^- \cap \Phi_i$ ($i \in \mathbb{F}_2$). By (9), $\Phi^- = -\Phi^+$, $\Phi_i^- = -\Phi_i^+$ ($i \in \mathbb{F}_2$). Then $\Phi^+ = \Phi_{\bar 0}^+ \cup \Phi_{\bar 1}^+$. Put $\bar\Phi_{\bar 1}^+ = \bar\Phi_{\bar 1} \cap \Phi^+$.

**Lemma 3.1.** *The map "positive system for* $(\mathfrak{g}, \mathfrak{h}) \mapsto$ *fundamental system for* $(\mathfrak{g}, \mathfrak{h})$*", is a bijection between the sets of these systems. The Weyl group of* $\mathfrak{g}$ *acts naturally on these sets.*

*Proof.* Indeed, a positive root that is not simple can be written as a sum of two positive roots. By induction then every positive root is a $\mathbb{Z}_{\geq 0}$-linear combination of simple roots. Hence the positive system is uniquely determined by its fundamental system. By (9), $\Phi = -\Phi$, and $\Phi$ is $W$-invariant. Then the Weyl group $W$ acts naturally on the set of positive systems, hence on the set of fundamental systems by the bijection above. □

A finite-dimensional Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ is called *solvable* if $\mathfrak{g}^{(k)} = 0$ for some $k \geq 1$, where $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(j+1)} = [\mathfrak{g}^{(j)}, \mathfrak{g}^{(j)}]$ for all $j \geq 0$.

Define

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha.$$

These are $\mathrm{ad}(\mathfrak{h})$-stable nilpotent subalgebras of $\mathfrak{g}$. There is a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. The solvable subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is called the *standard Borel subalgebra* of $\mathfrak{g}$ (corresponding to $\Phi^+$). We have $\mathfrak{b} = \mathfrak{b}_{\bar 0} \oplus \mathfrak{b}_{\bar 1}$, where $\mathfrak{b}_i = \mathfrak{b} \cap \mathfrak{g}_i$, $i \in \mathbb{F}_2$.

The Borel subalgebra $\mathfrak{b}$ is not a maximal solvable subalgebra of $\mathfrak{g}$. Indeed, the rank one subalgebra of $\mathfrak{g}$ corresponding to an isotropic simple root is isomorphic to $\mathrm{sl}(1|1) = \{\left(\begin{smallmatrix} a & b \\ c & a \end{smallmatrix}\right)\}$, which is solvable. Hence, enlarging $\mathfrak{b}$ by adding the root space corresponding to a negative isotropic simple root, we obtain a subalgebra that is still solvable.

To a positive system $\Phi^+ = \Phi_{\bar 0}^+ \cup \Phi_{\bar 1}^+$ associate the vectors in $\mathfrak{h}^*$:

$$\rho = \rho_{\bar 0} - \rho_{\bar 1}, \quad \rho_0 = \frac{1}{2} \sum_{\alpha \in \Phi_{\bar 0}^+} \alpha, \quad \rho_1 = \frac{1}{2} \sum_{\beta \in \Phi_{\bar 1}^+} \beta,$$

and $1_{m|n} = (\delta_1 + \cdots + \delta_m) - (\varepsilon_1 + \cdots + \varepsilon_n)$. Then for $\mathfrak{g} = \mathrm{gl}(m|n)$ we have

$$\rho = \sum_{1 \leq i \leq m} (m - i + 1)\delta_i - \sum_{1 \leq j \leq n} (m - i + 1)\delta_i - \sum_{1 \leq j \leq n} j\, \varepsilon_j - \frac{1}{2}(m + n + 1) 1_{m|n}.$$

In the case of $\mathfrak{g} = \mathrm{gl}(m|n)$, the standard Borel subalgebra is the subalgebra of upper triangular matrices in $\mathfrak{g}$. It contains the algebra $\mathfrak{h}$ of diagonal matrices. The *standard positive system* $\Phi^+$ of $\Phi$ is $\{\varepsilon_i - \varepsilon_j;\ i, j \in I(m|n),\ i < j\}$. We also write $\varepsilon_{\bar{i}}$ for $\delta_i$ $(1 \le i \le m)$. The *standard fundamental system* for $\mathrm{gl}(m|n)$ is

$$\big\{\delta_i - \delta_{i+1}, \varepsilon_j - \varepsilon_{j+1}, \delta_m - \varepsilon_1;\ 1 \le i < m,\ 1 \le j < n\big\}.$$

The *standard simple root vectors* are $e_i = E_{i,i+1}$ $(i \in I(m-1|n-1))$ and $e_{\overline{m}} = E_{\overline{m},1}$. The *standard simple coroots* are $h_j = E_{j,j} - E_{j+1,j+1}$ $(j \in I(m-1|n-1))$ and $h_{\overline{m}} = E_{\overline{m},\overline{m}} + E_{1,1}$. Put $f_i = E_{i+1,i}$ for $i \in I(m-1|n-1)$ and $f_{\overline{m}} = E_{1,\overline{m}}$ (where $i+1$ means $\overline{\iota+1}$ for $i = \bar{\iota}$ with $1 \le \iota < m$). Then $\{e_i, h_i, f_i;\ i \in I(m|n-1)\}$ is a set of *Chevalley generators* for $\mathrm{sl}(m|n)$.

We have

$$\begin{aligned} (\delta_i - \delta_{i+1}, \delta_i - \delta_{i+1}) &= 2, \qquad 1 \le i < m, \\ (\delta_m - \varepsilon_1, \delta_m - \varepsilon_1) &= 0, \\ (\varepsilon_j - \varepsilon_{j+1}, \varepsilon_j - \varepsilon_{j+1}) &= -2, \quad 1 \le j < n. \end{aligned}$$

Hence $\delta_m - \varepsilon_1$ is an isotropic simple root.

## §4. Dynkin diagrams

There is a *Dynkin diagram* associated with a fundamental system $\Pi = \{\alpha_1, \ldots, \alpha_k\}$. It consists of vertices labeled by $\alpha_i$, or simply $i$, $1 \le i \le k$, and edges. The vertices are marked

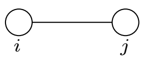
, called *white*, if $(\alpha_i, \alpha_i) \neq 0$ and $p(\alpha_i) = 0$,

, called *gray*, if $(\alpha_i, \alpha_i) = 0$ and $p(\alpha_i) = 1$,

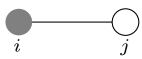
, called *black*, if $(\alpha_i, \alpha_i) \neq 0$ and $p(\alpha_i) = 1$.

We are interested only in Dynkin diagrams whose vertices are white and gray.

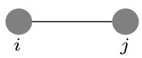
There is an edge between the $i$th and $j$th vertices iff $(\alpha_i, \alpha_j) \neq 0$. There is an edge



if $(\alpha_i, \alpha_i) = (\alpha_j, \alpha_j) = -2(\alpha_i, \alpha_j) \neq 0$,



if $(\alpha_i, \alpha_i) = 0$, $(\alpha_j, \alpha_j) = -2(\alpha_i, \alpha_j) \neq 0$,



if $(\alpha_i, \alpha_i) = 0 = (\alpha_j, \alpha_j)$, $(\alpha_i, \alpha_j) \neq 0$.

The associated *standard Dynkin diagram* is the graph whose vertices are labeled by the roots in the standard fundamental system; see Figure 1, where a white circle denotes an even simple root $\alpha$ (such that $\frac{1}{2}\alpha$ is not a root), and a gray circle denotes an odd isotropic simple root.

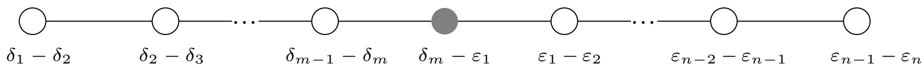



Figure 1. Standard Dynkin diagram for sl$(m|n)$.

To describe all positive systems for gl$(m|n)$, recall again that $\varepsilon_{\bar{i}} = \delta_i$ $(1 \leq i \leq m)$, and suspend the parity of the roots. In this case the root system of gl$(m|n)$ is the same as the root system for gl$(m+n)$, so their positive systems and fundamental systems are described in the same way. So there are $(m+n)!$ such systems. From the standard theory for gl$(m+n)$, a fundamental system for it consists of $(m+n-1)$ roots: $\Pi = (\varepsilon_{i_1} - \varepsilon_{i_2}, \varepsilon_{i_2} - \varepsilon_{i_3}, \ldots, \varepsilon_{i_{m+n_1}} - \varepsilon_{i_{m+n}})$, where $\{i_1, i_2, \ldots, i_{m+n}\}$ is $I(m|n)$. Put $\times$ for a white or gray vertex. Restoring the parity of the simple roots, we get the Dynkin diagram shown in Figure 2.

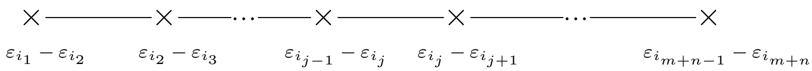



Figure 2. A Dynkin diagram for sl$(m|n)$.

As an example of all gray vertices, consider the case where $m = n$, and the simple roots are $\{\delta_1 - \varepsilon_1, \varepsilon_1 - \delta_2, \delta_2 - \varepsilon_2, \ldots, \delta_{m-1} - \varepsilon_{m-1}, \varepsilon_{m-1} - \delta_m, \delta_m - \varepsilon_m\}$, all odd. The Dynkin diagram is shown in Figure 3.

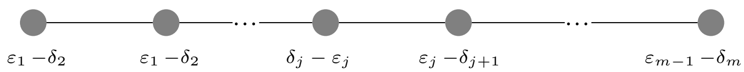



Figure 3. An all gray Dynkin diagram for sl$(m|n)$.

Associate an *$\varepsilon\,\delta$-sequence* to the fundamental system $\Pi = \{\varepsilon_{i_1} - \varepsilon_{i_2}, \ldots, \varepsilon_{i_{m+n-1}} - \varepsilon_{i_{m+n}}\}$ by replacing $\varepsilon_{\bar{i}}$ by $\delta_i$ $(1 \leq i \leq m)$, then erasing the index. We obtain a sequence with $m$ $\delta$'s and $n$ $\varepsilon$'s. The Weyl group shuffles the $\delta$'s and the $\varepsilon$'s, but does not mix them. In particular, the $W$-conjugacy classes of fundamental systems in $\Phi$ are in bijection with the associated $\varepsilon\,\delta$-sequences. So there are $\binom{m+n}{m}$ $W$-conjugacy classes of fundamental systems for gl$(m|n)$. In particular, there are positive systems for $\Phi$ that are not conjugate to each other under the action of the Weyl group, in contrast to the semisimple Lie algebra, nonsuper, case.

For example, there are three $W$-conjugacy classes of fundamental systems for gl$(2|1)$, corresponding to $\delta\,\delta\,\varepsilon$, $\delta\,\varepsilon\,\delta$, $\varepsilon\,\delta\,\delta$ (where $\delta\,\delta\,\varepsilon$ corresponds to $(\delta_1-\delta_2,\delta_2-\varepsilon_1)$ and $(\delta_2-\delta_1,\delta_1-\varepsilon_1)$, $\delta\,\varepsilon\,\delta$ to $(\delta_1-\varepsilon_1,\varepsilon_1-\delta_2)$ and $(\delta_2-\varepsilon_1,\varepsilon_1-\delta_1)$, and $\varepsilon\,\delta\,\delta$ to $(\varepsilon_1-\delta_1,\delta_1-\delta_2)$ and $(\varepsilon_1-\delta_2,\delta_2-\delta_1)$).

The standard Borel algebra of gl$(m|n)$ defines the sequence $\delta^m\,\varepsilon^n=\delta\ldots\delta\;\varepsilon\ldots\varepsilon$ ($\delta$ $m$ times, then $\varepsilon$ $n$ times), and the opposite to it defines $\varepsilon^n\,\delta^m$.

So in contrast to the case of nonsuper, semisimple Lie algebras, the fundamental systems of a root system $\Phi$ are not all $W$-conjugate. This is due to the existence of odd roots in the super case. Recall that a root $\alpha\in\Phi$ is called isotropic if $(\alpha,\alpha)=0$; such a root must be odd. For our superalgebra $\mathfrak{g}$ we have the following lemma.

**Lemma 4.1.** *Let $\Pi$ be a fundamental system of a positive system $\Phi^+$. Let $\alpha$ be an odd simple root. Then $\Phi^+_\alpha=\{-\alpha\}\cup(\Phi^+-\{\alpha\})$ is a positive system whose fundamental system $\Pi_\alpha$ is given by*

$$\big\{\beta\in\Pi;\ (\beta,\alpha)=0,\ \beta\neq\alpha\big\}\cup\big\{\beta+\alpha;\ \beta\in\Pi,\ (\beta,\alpha)\neq 0\big\}\cup\big\{-\alpha\big\}.$$

The process of obtaining $\Pi_\alpha$, $\Phi^+_\alpha$, $\mathfrak{b}^\alpha=\mathfrak{h}\oplus\bigoplus_{\beta\in\Phi^+_\alpha}\mathfrak{g}_\beta$ from $\Pi$, $\Phi^+$, $\mathfrak{b}$ will be called an *odd reflection* wrt $\alpha$, denoted $r_\alpha$. Thus

$$r_\alpha(\Pi)=\Pi_\alpha,\quad r_\alpha(\Phi^+)=\Phi^+_\alpha,\quad r_\alpha(\mathfrak{b})=\mathfrak{b}^\alpha,\quad r_{-\alpha}r_\alpha=1.$$

Real reflections $r_\alpha$ are defined for each even root $\alpha$ (which has to be nonisotropic) as a linear map on $\mathfrak{h}^*$ by

$$r_\alpha(x)=x-2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha,\quad x\in\mathfrak{h}^*,$$

where $(.\,,.)$ is the even nondegenerate supersymmetric bilinear form on $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{h}^*$ of (5) and (7). For an even simple root $\alpha$ we have $\frac{1}{2}\alpha\notin\Phi$, thus $\alpha\in\overline{\Phi}_{\bar{0}}$. Then with $\Phi^+_\alpha$ and $\Pi_\alpha$ as defined in the lemma, and $\mathfrak{b}^\alpha$, we get $r_\alpha(\Pi)=\Pi_\alpha$, $r_\alpha(\Phi^+)=\Phi^+_\alpha$, $r_\alpha(\mathfrak{b})=\mathfrak{b}^\alpha$.

**Proposition 4.2.** *For two fundamental systems $\Pi$ and $\Pi'$ of our superalgebra $\mathfrak{g}$, there exists a sequence of real and odd reflections $r_1,\ldots,r_k$ such that $r_k\ldots r_2r_1(\Pi)=\Pi'$.*

A proof and examples can be found in [CW12].

## §5. Theory of highest weight

To explain the theory of the highest weight, we first record in Proposition 5.1 below the PBW (Poincaré–Birkhoff–Witt) theorem for a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$. A *universal enveloping algebra* of $\mathfrak{g}$ is an associative superalgebra $\mathfrak{U}(\mathfrak{g})$ with a unit, with a homomorphism $\iota\colon \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ of superalgebras, satisfying the following universal property. For every associative superalgebra $A$ and a Lie superalgebra homomorphism $\varphi\colon \mathfrak{g} \to A$, there exists a unique homomorphism of associative superalgebras $\psi\colon \mathfrak{U}(\mathfrak{g}) \to A$ such that $\varphi = \psi \circ \iota$. This implies that representations of $\mathfrak{g}$ and of $\mathfrak{U}(\mathfrak{g})$ coincide. By a standard proof, $\mathfrak{U}(\mathfrak{g})$ exists and is unique up to an isomorphism by the universal property. In fact, one has the following proposition.

**Proposition 5.1.** *Let $\{x_1, \ldots, x_p\}$ be a basis for $\mathfrak{g}_{\bar{0}}$ and $\{y_1, \ldots, y_q\}$ a basis for $\mathfrak{g}_{\bar{1}}$ as vector spaces. Then*

$$\left\{x_1^{r_1} \ldots x_p^{r_p} y_1^{s_1} \ldots y_q^{s_q};\ r_1, \ldots, r_p \in \mathbb{Z}_{\geq 0},\ s_1, \ldots, s_q \in \mathbb{F}_2\right\}$$

*makes a basis for $\mathfrak{U}(\mathfrak{g})$.*

Let $\mathfrak{p}$ be a Lie sub(super)algebra of a finite-dimensional Lie superalgebra $\mathfrak{g}$. Let $V$ be a $\mathfrak{p}$-module. The $\mathfrak{g}$, or $\mathfrak{U}(\mathfrak{g})$, *induced module* is $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} V = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} V$. If $V$ is finite-dimensional then so is $\mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} V$, by the PBW theorem.

Let $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}$ be a finite-dimensional solvable Lie superalgebra with $[\mathfrak{p}_{\bar{1}}, \mathfrak{p}_{\bar{1}}] \subset [\mathfrak{p}_{\bar{0}}, \mathfrak{p}_{\bar{0}}]$. Given $\lambda \in \mathfrak{p}_{\bar{0}}^*$ with $\lambda([\mathfrak{p}_{\bar{0}}, \mathfrak{p}_{\bar{0}}]) = 0$, define a one-dimensional $\mathfrak{p}$-module $\mathbb{C}_\lambda = \mathbb{C} v_\lambda$ by

$$x v_\lambda = \lambda(x) v_\lambda \ \ (x \in \mathfrak{p}_{\bar{0}}), \quad y v_\lambda = 0 \ \ (y \in \mathfrak{p}_{\bar{1}}).$$

Note that $\{\lambda \in \mathfrak{p}_{\bar{0}}^*;\ \lambda([\mathfrak{p}_{\bar{0}}, \mathfrak{p}_{\bar{0}}]) = 0\} \simeq (\mathfrak{p}_{\bar{0}}/[\mathfrak{p}_{\bar{0}}, \mathfrak{p}_{\bar{0}}])^*$.

Under our assumption on $\mathfrak{p}$ ($= \mathfrak{p}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}$, finite-dimensional solvable Lie superalgebra with $[\mathfrak{p}_{\bar{1}}, \mathfrak{p}_{\bar{1}}] \subset [\mathfrak{p}_{\bar{0}}, \mathfrak{p}_{\bar{0}}]$), every finite-dimensional irreducible $\mathfrak{p}$-module is one-dimensional. Any finite-dimensional irreducible $\mathfrak{p}$-module is of the form $\mathbb{C}_\lambda$ for some $\lambda$ in $(\mathfrak{p}_{\bar{0}}/[\mathfrak{p}_{\bar{0}}, \mathfrak{p}_{\bar{0}}])^*$ (see [CW12, Lem. 1.37] for a proof and Example 1.38 there for why the condition $[\mathfrak{p}_{\bar{1}}, \mathfrak{p}_{\bar{1}}] \subset [\mathfrak{p}_{\bar{0}}, \mathfrak{p}_{\bar{0}}]$ is required).

With $\mathfrak{g}$, $\mathfrak{h}$, $\Phi$ as usual, let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ be a Borel subalgebra of $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}$, and $\Phi^+$ the associated positive system. The condition is satisfied for the solvable Lie superalgebra $\mathfrak{b}$ as $\mathfrak{b}_{\bar{1}} = \mathfrak{n}_{\bar{1}}^+$ and

$$[\mathfrak{b}_{\bar{1}}, \mathfrak{b}_{\bar{1}}] = [\mathfrak{n}_{\bar{1}}^+, \mathfrak{n}_{\bar{1}}^+] \subset \mathfrak{n}_{\bar{0}}^+ = [\mathfrak{h}, \mathfrak{n}_{\bar{0}}^+] \subset [\mathfrak{b}_{\bar{0}}, \mathfrak{b}_{\bar{0}}].$$

If $V$ is a finite-dimensional irreducible $\mathfrak{g}$-module, then by the above it contains a one-dimensional $\mathfrak{b}$-module necessarily of the form $\mathbb{C}_\lambda = \mathbb{C} v_\lambda$ for some $\lambda \in \mathfrak{h}^* \simeq (\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}])^*$; thus

$$h v_\lambda = \lambda(h) v_\lambda \ \ (h \in \mathfrak{h}), \quad x v_\lambda = 0 \ \ (x \in \mathfrak{n}^+).$$

Since $V$ is irreducible, by the PBW theorem we get that $V = \mathfrak{U}(\mathfrak{n}^-)v_\lambda$, hence a weight space decomposition

$$V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu, \quad V_\mu = \big\{v \in V;\ hv = \mu(h)v\ \forall\, h \in \mathfrak{h}\big\},$$

where $V_\mu$ is $\{0\}$ unless $\lambda - \mu$ is a $\mathbb{Z}_{\geq 0}$-linear combination of positive roots. The weight $\lambda$ is called the $\mathfrak{b}$-*highest weight* (or extremal weight) of $V$, the space $\mathbb{C}v_\lambda$ the $\mathfrak{b}$-*highest weight space*, and $v_\lambda$ a $\mathfrak{b}$-*highest weight vector* for $V$. When $\mathfrak{b}$ is clear from the context, a reference to it is omitted. In conclusion, every finite-dimensional irreducible $\mathfrak{g}$-module is a $\mathfrak{b}$-highest weight module.

Denote this irreducible highest weight module of weight $\lambda$ by $L(\lambda)$, or $L(\mathfrak{g}, \lambda)$, or $L(\mathfrak{g}, \mathfrak{b}, \lambda)$.

Recall the notation $\Pi_\alpha$, $\mathfrak{b}^\alpha$ associated to an isotropic odd simple root $\alpha$. Denote by $\langle .\,, .\rangle \colon \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ the standard bilinear pairing. Let $h_\alpha$ be the coroot corresponding to $\alpha$, and let $e_\alpha$, $f_\alpha$ be the root vectors of the roots $\alpha$ and $-\alpha$ so that $[e_\alpha, f_\alpha] = h_\alpha$.

We also have [CW12, Lem. 1.40]:

**Lemma 5.2.** *Let $V$ be a simple $\mathfrak{g}$-module, not necessarily finite-dimensional. Let $v$ be a $\mathfrak{b}$-highest weight vector of $V$ of $\mathfrak{b}$-highest weight $\lambda$. Let $\alpha$ be an isotropic odd simple root. Then,*

(1) *if $\langle\lambda, h_\alpha\rangle = 0$, then $V$ is a $\mathfrak{g}$-module of $\mathfrak{b}^\alpha$-highest weight $\lambda$, and $v$ is a $\mathfrak{b}^\alpha$-highest weight vector;*
(2) *if $\langle\lambda, h_\alpha\rangle \neq 0$, then $V$ is a $\mathfrak{g}$-module of $\mathfrak{b}^\alpha$-highest weight $\lambda - \alpha$, and $f_\alpha v$ is a $\mathfrak{b}^\alpha$-highest weight vector.*

## §6. Hook partitions

Once again let $\mathfrak{g}$ be the superalgebra $\mathrm{gl}(m|n)$, and $\mathfrak{h}$ the Cartan subalgebra of diagonal matrices, spanned by the basis elements $\{E_{i,i};\ i \in I(m|n)\}$. Let $\mathfrak{n}^+$ be the subalgebra of strictly upper triangular matrices of $\mathfrak{g}$, and $\mathfrak{n}^-$ the strictly lower ones. Then the fundamental system $\Pi$ of the simple roots of the positive system $\Phi^+$ has the Dynkin diagram whose only nonwhite vertex is gray at the $m$th place. We have the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. The even subalgebra has a compatible triangular decomposition $\mathfrak{g}_{\bar 0} = \mathfrak{n}^-_{\bar 0} \oplus \mathfrak{h} \oplus \mathfrak{n}^+_{\bar 0}$, where $\mathfrak{n}^\pm_{\bar 0} = \mathfrak{g}_{\bar 0} \cap \mathfrak{n}^\pm$. The Borel subalgebras are $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b}_{\bar 0} = \mathfrak{h} \oplus \mathfrak{n}^+_{\bar 0}$.

The Lie superalgebra $\mathfrak{g}$ admits a $\mathbb{Z}$-gradation $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{g}_{\bar 0}$, $\mathfrak{g}_1$ consists of the strictly upper triangular matrices in $\mathfrak{g}_{\bar 1}$, and $\mathfrak{g}_{-1}$ the strictly lower triangular ones; thus $\mathfrak{g}_1 = \mathfrak{g}_{\bar 1} \cap \mathfrak{n}^+$ is generated by the $E_{i,j}$ with $i, j \in I(m|n)$,

$i > 0 > j$ (and $\mathfrak{g}_{-1} = \mathfrak{g}_{\bar{1}} \cap \mathfrak{n}^-$ by $i < 0 < j$). Then $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are abelian Lie superalgebras.

Let $L^0(\lambda)$ be the simple $\mathfrak{g}_{\bar{0}}$-module of highest weight $\lambda \in \mathfrak{h}^*$ relative to the Borel subalgebra $\mathfrak{b}_{\bar{0}}$. It can be extended trivially to $\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_1 = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and induced to a $\mathfrak{g}$-module $K(\lambda) = \mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} L^0(\lambda)$. As a vector space $K(\lambda)$ is $\Lambda(\mathfrak{g}_{-1}) \otimes L^0(\lambda)$ by the PBW theorem. From the embedding $L^0(\lambda) \hookrightarrow L(\lambda)$ of $\mathfrak{g}_{\bar{0}}$-modules, where $L(\lambda)$ is the highest weight irreducible $\mathfrak{g}$-module of highest weight $\lambda$, and Frobenius reciprocity, we obtain a surjective $\mathfrak{g}$-module homomorphism $K(\lambda) \twoheadrightarrow L(\lambda)$, which is unique up to a scalar multiple. The following are equivalent:

(1) $L(\lambda)$ is finite-dimensional;

(2) $L^0(\lambda)$ is finite-dimensional;

(3) $K(\lambda)$ is finite-dimensional.

Indeed, (1) implies (2) since $L^0(\lambda)$ is an irreducible direct summand of $L(\lambda)$ regarded as a $\mathfrak{g}_{\bar{0}}$-module, (2) implies (3) since $K(\lambda) = \Lambda(\mathfrak{g}_{-1}) \otimes L^0(\lambda)$, and (3) implies (1) since $K(\lambda) \twoheadrightarrow L(\lambda)$ is surjective.

Every finite-dimensional simple $\mathfrak{g}$-module is a highest weight module $L(\lambda)$ for some $\lambda \in \mathfrak{h}^*$, and $L(\lambda) \not\simeq L(\mu)$ if $\lambda \neq \mu$.

It follows from the equivalence of (1), (2), (3) above then that the classification of finite-dimensional simple $\mathfrak{g}$-modules is the same as that of the finite-dimensional simple $\mathfrak{g}_{\bar{0}} = \mathrm{gl}(m) \oplus \mathrm{gl}(n)$-modules. A finite-dimensional simple $\mathrm{gl}(m)$-module is uniquely the twist $L_1(\lambda) \otimes \chi$ by a central character $\chi$, that factorizes via the determinant, of a *polynomial* $\mathrm{gl}(m)$-module $L_1(\lambda)$, namely one parametrized by a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$ are nonnegative integers (but $\lambda$, $\chi$ are not uniquely determined by $L_1(\lambda) \otimes \chi$). Thus $L_1(\lambda) \otimes \chi$ is parametrized by $(\lambda_1 + \lambda_0, \lambda_2 + \lambda_0, \ldots, \lambda_m + \lambda_0)$ for some $\lambda_0 \in \mathbb{R}$. Then we have the following proposition.

**Proposition 6.1.** *All pairwise nonisomorphic finite-dimensional simple* $\mathrm{gl}(m|n)$*-modules are* $L(\lambda)$ *for* $\lambda = \sum_{1 \leq i \leq m} \lambda_i \delta_i + \sum_{1 \leq j \leq n} \nu_j \varepsilon_j \in \mathfrak{h}^*$*, with* $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$ *and* $\nu_j - \nu_{j+1} \in \mathbb{Z}_{\geq 0}$ *for all* $i$, $j$.

A sequence $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ of integers $\mu_i \in \mathbb{Z}_{\geq 0}$ with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0$ is called a *partition* of $\mu_1 + \cdots + \mu_m = r$, with $\ell(\mu) = m_0$ parts if $m_0$ ($\leq m$) is the largest integer with $\mu_{m_0} > 0$. These parametrize all the polynomial $\mathrm{gl}(m)$-modules.

A partition $\mu = (\mu_1, \mu_2, \ldots)$ is called an $(m|n)$*-hook partition* if $\mu_{m+1} \leq n$, equivalently if $\mu'_{n+1} \leq m$, where $\mu'$ is the partition conjugate to $\mu$ (thus we write the Young diagram, which has $\mu_1$ boxes in the first row, $\mu_r$ boxes in the $r$th row, all aligned to the left at the fourth quadrant in the plane, so there are $m$ rows if $\ell(\mu) = m$, and $\mu'$ is the transpose Young diagram, thus we write the first row as the first column, second row as the second column, etc., and write $\mu'_i$ for the number of boxes in the $i$th column of $\mu$, thus the $i$th row of $\mu'$).

Thus an $(m|n)$-hook partition is a partition not including $(m+1, n+1)$.

Given an $(m|n)$-hook partition $\mu$, consider the subpartition $\mu^+ = (\mu_{m+1}, \mu_{m+2}, \ldots)$ and its conjugate $\nu = (\mu^+)' = (\nu_1, \ldots, \nu_n)$; this conjugate has at most $n$ parts. Define the weight $\mu^s$ ($s$ for "super") by

$$\begin{aligned}\mu^s &= \mu_1\delta_1 + \cdots + \mu_m\delta_m + \nu_1\,\varepsilon_1 + \cdots + \nu_{n-1}\,\varepsilon_{n-1} + \nu_n\,\varepsilon_n \\ &= (\mu_1, \ldots, \mu_m; \nu_1, \ldots, \nu_n).\end{aligned}$$

Write $P(m|n)$ for the set of $(m|n)$-hook partitions, and $P_d(m|n)$ for the set of $(m|n)$-hook partitions $\mu$ of $d$; thus $\sum_{1\le i\le \ell(\mu)} \mu_i = d$ and $\mu_{m+1} \le n$. Then $P(m|n) = \bigcup_{d\ge 0} P_d(m|n)$, $P_d(m) = P_d(m|0)$ is the set of partitions of $d$ with at most $m$ parts, and $P_d = \bigcup_{m\ge 1} P_d(m)$ is the set of partitions of $d$.

Denote by $L(\lambda^s)$, for $\lambda \in P(m|n)$, the simple $\mathfrak{g}$-module of highest weight $\lambda^s$ with respect to the standard Borel subalgebra. For a partition $\lambda$ of $d$, denote by $S^\lambda$ the Specht module of $S_d$. For example, $S^{(d)}$ is the trivial representation of $S_d$, and $(S^{(1^d)})$ is the sign representation $\mathrm{sgn}_d$ of $S_d$. Representation theory of the symmetric group [Ja78, FH91] establishes that $\{S^\lambda;\ \lambda \in P_d\}$ is a complete list of simple inequivalent $S_d$-modules.

Put $\mathfrak{g} = \mathrm{gl}(m|n)$. Let $V$ be the natural left $\mathfrak{g}$-module $\mathbb{E} = \mathbb{C}^{m|n}$. Then $\mathbb{E}^{\otimes d}$ is a left $\mathfrak{g}$-module by

$$\begin{aligned}&(\phi_d(g))(v_1 \otimes \cdots \otimes v_d) \\ &\quad = (gv_1) \otimes v_2 \otimes \cdots \otimes v_d + (-1)^{p(g)p(v_1)} v_1 \otimes (gv_2) \otimes \cdots \otimes v_d \\ &\qquad + \cdots + (-1)^{p(g)(p(v_1)+\cdots+p(v_{d-1}))} v_1 \otimes v_2 \otimes \cdots \otimes v_{d-1} \otimes (gv_d)\end{aligned}$$

on homogeneous $g \in \mathfrak{g}$ and $v_i \in \mathbb{E}$ for all $i$ $(1 \le i \le d)$, extended by linearity.

The following is the extension of the Schur theorem to the context of the superalgebra $\mathrm{gl}(m|n)$, due to Sergeev [S85], and later to [BR87], and in book form [CW12].

**Theorem 6.2.**

(1) *The formula*

$$\begin{aligned}&(\psi_d((i, i+1)))(v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_d) \\ &\quad = (-1)^{p(v_i)p(v_{i+1})} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_d\end{aligned}$$

*$(1 \le i < d)$, where $(i,j)$ denotes a transposition in $S_d$ and $v_i$, $v_{i+1}$ are homogeneous in $\mathbb{E}$, extends to a left action of the symmetric group $S_d$ on $\mathbb{E}^{\otimes d}$. The actions of $(\mathrm{gl}(m|n), \phi_d)$ and $(S_d, \psi_d)$ on $\mathbb{E}^{\otimes d}$ commute with each other.*

(2) *The images $\phi_d(\mathfrak{U}(\mathfrak{g}))$ and $\psi_d(\mathbb{C}[S_d])$ of $\phi_d$ and $\psi_d$ satisfy the double centralizer property $\phi_d(\mathfrak{U}(\mathfrak{g})) = \mathrm{End}_{\mathbb{C}[S_d]}(\mathbb{E}^{\otimes d})$, $\psi_d(\mathbb{C}[S_d]) = \mathrm{End}_{\mathfrak{U}(\mathfrak{g})}(\mathbb{E}^{\otimes d})$.*

(3) *As a $\mathfrak{U}(\mathrm{gl}(m|n)) \otimes \mathbb{C}[S_d]$-module, one has*

$$(\mathbb{C}^{m|n})^{\otimes d} \simeq \bigoplus_{\lambda \in P_d(m|n)} L(\lambda^s) \otimes S^\lambda.$$

(4) *If $M$ is a right $(S_d, \psi_d)$-module, define $\mathcal{S}(M) = M \otimes_{\psi_d(\mathbb{C}[S_d])} \mathbb{E}^{\otimes d}$, with the natural left $(\mathfrak{U}(\mathfrak{g}), \phi_d)$-module structure obtained from that on $\mathbb{E}^{\otimes d}$. If $d < (n+1)(m+1)$, then every partition of $d$ is an $(m|n)$-hook partition, and the functor $M \mapsto \mathcal{S}(M)$ is an equivalence from the category of finite-dimensional $\mathbb{C}[S_d]$-modules to the category of finite-dimensional $\mathbb{E}^{\otimes d}$-compatible $\mathfrak{U}(\mathrm{gl}(m|n))$-modules, namely those that are polynomial of degree $d$.*

By a finite-dimensional $\mathbb{E}^{\otimes d}$-*compatible* $\mathfrak{U}(\mathrm{gl}(m|n))$-module we mean here a module all of whose subquotients are subquotients of the semisimple module $\mathbb{E}^{\otimes d}$. It is the same as to be "polynomial of degree $d$". In the ordinary, nonsuper case, this notion is discussed in detail in [F21]. We postpone the discussion of this in the super case to a subsequent work.

When $m > n$ we have $\mathrm{gl}(m|n) = \mathrm{sl}(m|n) \oplus \mathfrak{z}$, where $\mathfrak{z} = \mathbb{C}I_{m|n}$ is the center of $\mathrm{gl}(m|n)$, and $I_{m|n}$ is the identity matrix in $\mathrm{gl}(m|n)$. The data of a $\mathrm{gl}(m|n)$-module $\Pi$ with a given central character $\chi$ is equivalent to that of an $\mathrm{sl}(m|n)$-module $\pi$ and the character $\chi$ of the center $\mathfrak{z}$. Indeed, given $\Pi$, $\pi$ is the restriction of $\Pi$ to $\mathrm{sl}(m|n)$, in particular it is irreducible if $\Pi$ is; given $\pi$ we put $\Pi(s,z) = \chi(z)\pi(s)$ $(s \in \mathrm{sl}(m|n)$, $z \in \mathfrak{z})$. As $z \in \mathbb{C}I_{m|n}$ acts on $\mathbb{E}^{\otimes d}$ as multiplication by $z$, we may replace $\mathrm{gl}(m|n)$ by $\mathrm{sl}(m|n)$ in (3) and (4) of the theorem. From this perspective, "$\mathbb{E}^{\otimes d}$-compatible" is a better term than "polynomial of degree $d$", which makes no sense for $\mathrm{sl}(m|n)$ once the term "polynomial" is fully explained (see [F21]).

We refer to a $\mathfrak{U}(\mathfrak{g}) \otimes \mathbb{C}[S_d]$-module also as a $\mathfrak{g} \times S_d$-module, $\mathfrak{g} = \mathrm{gl}(m|n)$ or $\mathrm{sl}(m|n)$.

When $d < (n+1)(m+1)$, every partition $\lambda \vdash d$ lies in $P_d(m|n)$, namely it is an $(m|n)$-hook partition of $d$, since $(m+1)(n+1) > d = \sum_i \lambda_i \ge \sum_{1 \le i \le m+1} \lambda_i \ge (m+1)\lambda_{m+1}$ implies $\lambda_{m+1} \le n$. So in this case, every simple $S_d$-module appears in the duality decomposition (3), as stated in (4).

When $n = 0$ the theorem reduces to the usual (nonsuper) Schur duality.

If $d = 2$, then $(\mathbb{C}^m)^{\otimes 2} = S^2(\mathbb{C}^m) \oplus \Lambda^2(\mathbb{C}^m)$. The modules on the right are irreducible of highest weights $2\delta_1$ and $\delta_1 + \delta_2$; see [FH91].

## §7. Affine superalgebras

Our aim is to develop an affine analogue of super Schur duality. Recall that a Lie superalgebra is an $\mathbb{F}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, together with a bilinear operation $[.\,,.]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that for homogeneous $x$, $y$, $z$ in $\mathfrak{g}$ it satisfies skew supersymmetry,

$$[x, y] = -(-1)^{p(x)p(y)}[y, x] \in \mathfrak{g}_{p(x)+p(y)},$$

and the super Jacobi identity,

$$[x, [y, z]] = [[x, y], z] + (-1)^{p(x)p(y)}[y, [x, z]].$$

Here $p\colon \mathfrak{g}_{\bar{0}} \cup \mathfrak{g}_{\bar{1}} \to \mathbb{F}_2$ is the parity function, which takes a homogeneous element $x \in \mathfrak{g}_i$ to $i$. A bilinear form $(.\,,.)\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is *invariant* if it satisfies $(x, y) = -(-1)^{p(x)p(y)}(y, x)$ (skew supersymmetry) and $([x, y], z) = (x, [y, z])$. It is *nondegenerate* if its restriction to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}} \subset \mathfrak{g}$ is (thus for $h \in \mathfrak{h}$, $(\mathfrak{h}, h) = 0$ iff $h = 0$). We also define the *adjoint action* $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by $(\mathrm{ad}(x))(y) = [x, y]$ ($x$, $y \in \mathfrak{g}$). A $c \in \mathfrak{g}$ is called *central* if $\mathrm{ad}(c) = 0$, thus $[c, x] = 0$ for all $x \in \mathfrak{g}$.

Let $\mathcal{L} = \mathbb{C}[t, t^{-1}]$ be the algebra of polynomials in $t$ and $t^{-1}$ over $\mathbb{C}$. Consider a finite-dimensional (over $\mathbb{C}$) Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$. The *loop superalgebra* of $\mathfrak{g}$ is $\mathcal{L}(\mathfrak{g}) = \mathcal{L}\mathfrak{g} = \mathcal{L} \otimes \mathfrak{g} = \mathcal{L}(\mathfrak{g})_{\bar{0}} \oplus \mathcal{L}(\mathfrak{g})_{\bar{1}}$, $\mathcal{L}(\mathfrak{g})_i = \mathcal{L} \otimes \mathfrak{g}_i$, with the Lie superalgebra structure defined by $[P \otimes x, Q \otimes y] = PQ \otimes [x, y]$, $P$, $Q \in \mathcal{L}$, $x$, $y \in \mathfrak{g}$. In particular, the parity is $\bar{0}$ on $\mathcal{L}$. Note that $\mathcal{L}(\mathfrak{g})$ can be viewed as the Lie superalgebra of polynomial maps from the unit circle to $\mathfrak{g}$, whence the name *loop* superalgebra of $\mathfrak{g}$.

A skew-supersymmetric invariant nondegenerate bilinear form $(.\,,.)\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ exists, and it is unique up to a scalar multiple when $\mathfrak{g}$ is simple, as we now assume. Using it, define $(.\,,.)_t\colon \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to \mathcal{L}$ by

$$(P \otimes x, Q \otimes y)_t = PQ(x, y) \in \mathcal{L}, \quad P, Q \in \mathcal{L},\ x, y \in \mathfrak{g}.$$

Define linear maps $\frac{d}{dt}\colon \mathcal{L}(\mathfrak{g}) \to \mathcal{L}(\mathfrak{g})$ and $\mathrm{Res}\colon \mathcal{L} \to \mathbb{C}$ by

$$\frac{d}{dt}(t^n \otimes x) = nt^{n-1} \otimes x, \quad \mathrm{Res}(t^n) = \delta_{n+1,0} \quad n \in \mathbb{Z},\ x \in \mathfrak{g}.$$

Res is the unique functional on $\mathcal{L}$ satisfying $\mathrm{Res}\, t^{-1} = 1$ and $\mathrm{Res}\, \frac{dP}{dt} = 0$ for all $P \in \mathcal{L}$. A more natural presentation of the residue would be to view $f \in \mathcal{L}\mathfrak{g}$ as a morphism $f\colon \mathbb{C}^\times \to \mathfrak{g}$, where $\mathbb{C}^\times$ is the multiplicative group $\mathbb{C} - \{0\}$ of $\mathbb{C}$. The differential $df$ of $f$ is a 1-form with values in $\mathfrak{g}$. Then $(df, g)_t$ is a 1-form on $\mathbb{C}^\times$, whose residue at 0 is denoted by $\mathrm{Res}_0((df, g)_t)$ ($= \mathrm{Res}((\frac{df}{dt}, g)_t)$). In particular, $\mathrm{Res}_0(dP) = 0$ for all $P \in \mathcal{L}$.

Define a bilinear map $\nu\colon \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to \mathbb{C}$ by $\nu(f,g) = \operatorname{Res}_0((df,g)_t)$.

**Lemma 7.1.** *The map $\nu$ is a 2-supercocycle on $\mathcal{L}(\mathfrak{g})$. Namely, for all $f$, $g$, $h \in \mathcal{L}(\mathfrak{g})$, we have skew supersymmetry $\nu(f,g) = -(-1)^{p(g)p(f)}\nu(g,f)$, and*

$$\nu([f,g],h) + \iota_1\nu([g,h],f) + \iota_2\nu([h,f],g) = 0,$$

*where $\iota_1 = (-1)^{p(x)(p(y)+p(z))}$, and $\iota_2 = (-1)^{p(z)(p(x)+p(y))}$.*

*Proof.* For $P$, $Q \in \mathcal{L}$ and $x$, $y \in \mathfrak{g}$ we have

$$\begin{aligned}\nu(P\otimes x, Q\otimes y) + (-1)^{p(y)p(x)}\nu(Q\otimes y, P\otimes x) &= (x,y)\operatorname{Res}_0(dPQ + PdQ)\\ &= (x,y)\operatorname{Res}_0(d(PQ))\\ &= 0.\end{aligned}$$

For $P$, $Q$, $R \in \mathcal{L}$ and $x$, $y$, $z \in \mathfrak{g}$ we have

$$\begin{aligned}&\nu([P\otimes x, Q\otimes y], R\otimes z) + \iota_1\nu([Q\otimes y, R\otimes z], P\otimes x)\\ &\qquad + \iota_2\nu([R\otimes z, P\otimes x], Q\otimes y)\\ &\quad = ([x,y],z)\operatorname{Res}_0(d(PQ)R) + \iota_1([y,z],x)\operatorname{Res}_0(d(QR)P)\\ &\qquad + \iota_2([z,x],y)\operatorname{Res}_0((RP)Q)\\ &\quad = ([x,y],z)\operatorname{Res}_0(d(PQ)R + d(QR)P + d(RP)Q)\\ &\quad = ([x,y],z)\operatorname{Res}_0(d(PQR))\\ &\quad = 0.\end{aligned}$$

The passage from the second to the third row follows from

$$\iota_1([y,z],x) = (x,[y,z]) = ([x,y],z) \quad \text{and} \quad ([z,x],y) = (z,[x,y]) = \iota_2([x,y],z),$$

as $p([x,y]) = p(x) + p(y)$. □

The *pre-affine Lie superalgebra* is $\tilde{\mathfrak{g}} = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c$, where $c$ is a formal central element of parity 0, and the Lie superalgebra structure is defined by

$$[f,g] = [f,g]_{\mathcal{L}(\mathfrak{g})} + \nu(f,g)c,$$

and $[f,g]_{\mathcal{L}(\mathfrak{g})}$ is the bracket in $\mathcal{L}(\mathfrak{g})$.

The skew supersymmetry for $\tilde{\mathfrak{g}}$ is a consequence of the skew supersymmetry of $[.\,,.]_{\mathcal{L}(\mathfrak{g})}$, namely of $[.\,,.]$ on $\mathfrak{g}$, and the skew supersymmetry of $\nu$ (first equality of the lemma). The super Jacobi identity for $\tilde{\mathfrak{g}}$ is a consequence of this identity for $\mathcal{L}(\mathfrak{g})$ (which follows from that for $\mathfrak{g}$), and the last equality of the lemma.

To be able to have linearly independent simple roots, one adds the derivation element $d$, of parity 0, and defines the *affine Lie superalgebra* to be

$$\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}d = \hat{\mathfrak{g}}_{\bar{0}} \oplus \hat{\mathfrak{g}}_{\bar{1}}, \quad \hat{\mathfrak{g}}_{\bar{0}} = \mathcal{L} \otimes \mathfrak{g}_{\bar{0}} \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \hat{\mathfrak{g}}_{\bar{1}} = \mathcal{L} \otimes \mathfrak{g}_{\bar{1}},$$

where $d$ acts by

$$[d, P \otimes x] = t\frac{dP}{dt} \otimes x, \quad [d, c] = 0.$$

Thus we obtain a Lie superalgebra with $(a_1, a_2, b_1, b_2 \in \mathbb{C};\ m, n \in \mathbb{Z})$,

$$\begin{aligned}&[t^m \otimes x + a_1 c + b_1 d, t^n \otimes y + a_2 c + b_2 d]\\ &\quad = t^{m+n} \otimes [x, y] + m\delta_{m,-n}(x, y)c + b_1 n t^n \otimes y - b_2 m t^m \otimes x.\end{aligned}$$

## §8. Generators and relations

An equivalent definition as an abstract symmetrizable Kac–Moody Lie superalgebra, analogous to [K90, Sect. 1.3], is as follows. We follow Kac [K77] and Yamane [Y99].

Let $\mathcal{E}$ be a finite-dimensional $\mathbb{C}$-vector space. Let $(\,.\,,.\,)$ denote a nondegenerate symmetric bilinear form on $\mathcal{E}$. Let $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_{\mathbf{n}}\}$ be a linearly independent subset of $\mathcal{E}$. Put ($\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ and)

$$P = \mathbb{Z}\alpha_0 \oplus \cdots \oplus \mathbb{Z}\alpha_{\mathbf{n}}, \quad P_+ = \mathbb{Z}_{\geq 0}\alpha_0 \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\alpha_{\mathbf{n}}, \quad P_- = -P_+.$$

An element $\alpha_i \in \Pi$ is called a *simple root* and $P$ the *root lattice*. A function $p\colon \Pi \to \mathbb{F}_2$ extends uniquely to a group homomorphism $p\colon P \to \mathbb{F}_2$, called *parity*. Put $\mathfrak{h} = \mathcal{E}^*$ for the dual space of $\mathcal{E}$. Identify $\nu \in \mathcal{E}$ with $h_\nu \in \mathfrak{h}$ by $\mu(h_\nu) = (\mu, \nu)$ for all $\mu \in \mathcal{E}$. A *datum* is a quadruple $(\mathcal{E}, (\,.\,,.\,), \Pi, p)$ as above. We abbreviate it to $(\mathcal{E}, \Pi, p)$. We *associate to a datum a Lie superalgebra* $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}(\mathcal{E}, \Pi, p)$ generated by *generators* $h \in \mathfrak{h}$, $e_i$, $f_i$ $(0 \leq i \leq k)$, and *relations* $[h, h'] = 0$ $(h, h' \in \mathfrak{h})$,

$$[h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i, \quad [e_i, f_j] = \delta_{i,j}h_{\alpha_i},$$

and *parities*

$$p(e_i) = p(\alpha_i) = p(f_i), \quad p(h) = 0 \ \ (h \in \mathfrak{h}).$$

The superalgebra $\widetilde{\mathcal{G}}$ has a triangular decomposition $\widetilde{\mathcal{G}} = \tilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_-$, where $\tilde{\mathfrak{n}}_+$ (resp. $\tilde{\mathfrak{n}}_-$) is the free superalgebra generated by the $e_i$ (resp. $f_i$).

An ideal $r'$ of the Lie superalgebra $\widetilde{\mathcal{G}}$ is called *admissible* if $r' \cap \mathfrak{h} = \{0\}$, and then we say the quotient $\mathcal{G}' = \widetilde{\mathcal{G}}/r'$ is admissible. For a *fixed* datum $(\mathcal{E}, \Pi, p)$, the associated admissible Lie superalgebras make a partially ordered set $I = I(\mathcal{E}, \Pi, p)$. We say $\mathcal{G}' = \widetilde{\mathcal{G}}/r' > \mathcal{G}'' = \widetilde{\mathcal{G}}/r''$ if $r' \subset r''$. Then $\widetilde{\mathcal{G}}$ is the unique top element of $I$. Note that $\mathcal{G}' > \mathcal{G}''$ iff there is a surjection $\psi = \psi(\mathcal{G}', \mathcal{G}'')\colon \mathcal{G}' \twoheadrightarrow \mathcal{G}''$ with

$(h, e_i, f_i) \mapsto (h, e_i, f_i)$. Denote by $\mathcal{G} = \mathcal{G}(\mathcal{E}, \Pi, p)$ the unique bottom element of $I(\mathcal{E}, \Pi, p)$, the object of study of this work. It is the *affine Lie superalgebra*.

For $\mathcal{G}' = \mathcal{G}'(\mathcal{E}, \Pi, p)$ and $\alpha \in \mathcal{E}$, denote $\mathcal{G}'_\alpha = \{x \in \mathcal{G}';\ [h, x] = \alpha(h)x\ \forall\, h \in \mathfrak{h}\}$ and $\Phi' = \Phi(\mathcal{G}') = \{\alpha \in \mathcal{E} - \{0\};\ \mathcal{G}'_\alpha \neq \{0\}\}$. The linear space $\mathcal{G}'_0$ is equal to $\mathfrak{h}$ for all $\mathcal{G}'$. It is named the *Cartan algebra* of $\mathcal{G}'$, and $\mathfrak{h} \subset \mathfrak{h}_{\bar{0}}$ is even. By the defining relations, $\Phi' \subset P_+ \cup P_- - \{0\}$. Put $\Phi = \Phi(\mathcal{E}, \Pi, p)$ for $\Phi(\mathcal{G})$, where $\mathcal{G}$ is the minimal element of $I$. Note that $\mathcal{G}' > \mathcal{G}''$ implies $\Phi(\mathcal{G}'') \subset \Phi(\mathcal{G}')$.

For $B \subset I$, put $\mathcal{G}_B = \widetilde{\mathcal{G}} / \bigcap_{\mathcal{G}' \in B} \ker \psi(\widetilde{\mathcal{G}}, \mathcal{G}')$. It is an admissible Lie superalgebra $\mathcal{G}_B \in I$. Then $\mathcal{G}_B > \mathcal{G}''$ for all $\mathcal{G}'' \in B$, and $\Phi(\mathcal{G}_B) = \bigcup_{\mathcal{G}'' \in B} \Phi(\mathcal{G}'')$. For $\alpha \in \mathcal{E}$, if $\dim \mathcal{G}''_\alpha$ does not depend on $\mathcal{G}'' \in B$ then $\dim \mathcal{G}_{B,\alpha} = \dim \mathcal{G}''_\alpha$ for all $\mathcal{G}'' \in B$.

For $\alpha,\ \beta \in P_+$, write $\beta < \alpha$ if $\alpha - \beta \in P_+ - \{0\}$. Fix a datum $(\mathcal{E}, \Pi, p)$. Fix $\mathcal{G}' \in I(\mathcal{E}, \Pi, p)$. As argued in the proof of [K77, Prop. 9.11], as in [Y99, p. 327] we have the following proposition.

**Proposition 8.1.**

(1) *Let $\rho \in \mathcal{E}$ be such that $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$ for all $\alpha_i \in \Pi$. If $\alpha \in P_+$ satisfies $(\alpha, \alpha) \neq 2(\rho, \alpha)$, and $\dim \mathcal{G}'_\beta = \dim \mathcal{G}_\beta$ for all $\beta \in P_+$ with $\beta < \alpha$, then $\dim \mathcal{G}'_\alpha = \dim \mathcal{G}_\alpha$.*

(2) *Fix $\alpha_i \in \Pi$. Then,*

- (i) $\dim \mathcal{G}'_{k\alpha_i}$ *equals* 1 *if $k = 1$, or if $k = 2$ and $(\alpha_i, \alpha_i) \neq 0$ and $p(\alpha_i) = 1$; it equals* 0 *if $k = 2$ and $p(\alpha_i) = 0$, or $k \geq 3$;*
- (ii) *when $p(\alpha_i) = 1$ and $(\alpha_i, \alpha_i) = 0$ we have $\dim \mathcal{G}'_{2\alpha_i} = 0$ iff $[e_i, e_i] = 0$;*
- (iii) *statements* (i) *and* (ii) *also hold with $\alpha_i$ replaced by $-\alpha_i$;*
- (iv) $\dim \mathcal{G}'_\beta$ *is* 0 *for $\beta \in P - P_+ \cup P_-$.*

(3) *Fix $a_i \in \mathbb{C}^\times$ for all $i$, $1 \leq i \leq N = 1 + \mathbf{n} = |\Pi|$. There exists a unique automorphism $\phi(a_1, \ldots, a_N)$ of $\mathcal{G}'$ with $(h, e_i, f_i) \mapsto (h, a_i e_i, a_i^{-1} f_i)$. An automorphism $\phi$ of $\mathcal{G}'$ satisfies $\phi|\mathfrak{h} = 1_{\mathfrak{h}}$ iff $\phi = \phi(a_1, \ldots, a_N)$ for some $a_i \in \mathbb{C}^\times$ for all $i$, $1 \leq i \leq N$. These $a_i$ are uniquely determined by $\phi$.*

If $\phi\colon \mathcal{G}' \to \mathcal{G}''$ and $\varphi\colon \mathcal{G}' \to \mathcal{G}''$ are homomorphisms, where $\mathcal{G}' = \mathcal{G}'(\mathcal{E}, \Pi, p)$ and $\mathcal{G}'' = \mathcal{G}''(\mathcal{E}', \Pi', p')$, we write $\phi \equiv \varphi$ if $\varphi = \phi \circ (a_1, \ldots, a_N)$ for some $a_i \in \mathbb{C}^\times$ $(1 \leq i \leq N)$. Then $\equiv$ is an equivalence relation. If $\phi$, $\varphi$ are isomorphisms and $\phi(\mathfrak{h}) = \mathfrak{h} = \varphi(\mathfrak{h})$, then $\phi \equiv \varphi$ iff $\varphi = \phi(b_1, \ldots, b_N) \circ \phi$ for some $b_i \in \mathbb{C}^\times$, $1 \leq i \leq N$.

The Dynkin diagram associated with a datum $(\mathcal{E}, \Pi, p)$ is described in general in [Y99, Sect. 1.3], but we need it only for type $(\mathrm{AA})^{(1)}$, where only white and gray vertices occur, no black ones, and no twisting. Thus we shall be interested only in a datum with Dynkin diagram as shown in Figure 4.

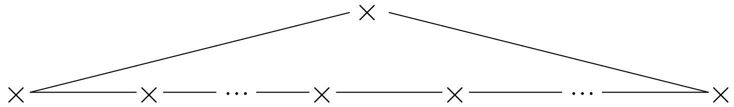



Figure 4. A type (AA) Dynkin diagram, for $\widehat{\mathrm{sl}}(m|n)$.

Fix a datum $(\mathcal{E}, \Pi = \{\alpha_0, \ldots, \alpha_{\mathbf{n}}\}, p)$ whose Dynkin diagram is (AA); the $i$th vertex is labeled by the $i$th root $\alpha_i$. Let $\mathcal{E}^{\mathrm{ex}}$ (ex for extended) be an $(\mathbf{n}+2)$-dimensional $\mathbb{C}$-vector space, with a nondegenerate symmetric bilinear form $(.\,,.)$, and a basis $\{\varepsilon_1, \ldots, \varepsilon_{\mathbf{n}+1}, \delta\}$, such that

$$(\varepsilon_i, \varepsilon_j) = \delta_{i,j}(-1)^{p(\varepsilon_i)}, \quad (\varepsilon_i, \delta) = (\delta, \delta) = 0.$$

It is common to add to the basis another vector $\Lambda_0$ with $(\delta, \Lambda_0) = 1$, $(\Lambda_0, \Lambda_0) = 0$, $(\varepsilon_i, \Lambda_0) = 0$, but we do not need it. Write $(\mathrm{AA})^g$ for (AA) ($g$ for good) if $\sum_{1 \le i \le \mathbf{n}+1} (-1)^{p(\varepsilon_i)} \neq 0$, and $(\mathrm{AA})^b$ ($b$ for bad) if this sum is 0. Put $\mathcal{E} = \mathcal{E}^{\mathrm{ex}}$ if $(\mathrm{AA})^b$, and $\mathcal{E} = \{x \in \mathcal{E}^{\mathrm{ex}};\ (x, \theta) = 0\}$ if $(\mathrm{AA})^g$, where $\theta = \sum_{1 \le i \le \mathbf{n}+1} (-1)^{p(\varepsilon_i)} \varepsilon_i$. Note that $(.\,,.)$ restricts to a nondegenerate symmetric bilinear form on $\mathcal{E}$. Assume there is a simple odd root. The vectors $\varepsilon_1, \ldots, \varepsilon_{\mathbf{n}+1}, \delta$ (and $\Lambda_0$) are called the *fundamental* elements of $(\mathcal{E}, \Pi, p)$. In [Y99], $\mathbf{n}+1$ is denoted by $N$ (in our case of $(\mathrm{AA})^g$, thus $m \neq n$), and it is equal to our $m+n$.

The Kac–Moody Lie superalgebra $\mathcal{G}(\mathcal{E}, \Pi, p)$ is called an affine Lie superalgebra of type (AA); we denote it also by $\widehat{\mathrm{sl}}(m|n, \Pi, p)$.

Note that $\widehat{\mathrm{sl}}(m|n) = A(m-1, n-1)$ $(m \neq n)$ is $(\mathrm{AA})^g$, $\widehat{\mathrm{sl}}(m|m)/\mathbb{C} \cdot I_{2m} = A(m-1|m-1)$. Note that $A(m-1|m-1)$ and $\widehat{\mathrm{sl}}(m|m)$ are not Kac–Moody Lie superalgebras, since their simple roots are linearly dependent, and $\mathrm{gl}(m|m)$ is a Kac–Moody Lie superalgebra. Define $A(m-1|m-1)^{\mathfrak{h}}$ as follows. Let $\widehat{\mathrm{sl}}(m|m)^{\mathfrak{h}}$ be the subalgebra $\widehat{\mathrm{sl}}(m|m) \oplus \mathbb{C}E_{1,1}$ of $\widehat{\mathrm{gl}}(m|m)$ ($E_{1,1}$ is the $2m \times 2m$ matrix with all entries $a_{i,j}$ equal to 0 except $a_{1,1} = 1$). Then $A(m-1|m-1)^{\mathfrak{h}}$ is defined to be the quotient $\widehat{\mathrm{sl}}(m|m)^{\mathfrak{h}}/(\bigoplus_{k \neq 0} \mathbb{C}I_{2m} \otimes t^k)$. This is a Kac–Moody Lie superalgebra, while $A(m-1|m-1)$ is not, since its simple roots are linearly dependent.

By [Y99, Prop. 3.1.1], $\dim \mathcal{G}_\alpha = 1$ for all $\alpha \in \Phi(\mathcal{E}, \Pi, p) - \mathbb{Z}\delta$, $\mathcal{G} = \mathcal{G}(\mathcal{E}, \Pi, p)$.

Let $\mathcal{G}' = \mathcal{G}'(\mathcal{E}, \Pi, p)$ be an admissible Lie superalgebra with respect to $(\mathcal{E}, \Pi, p)$, namely $\mathcal{G}' > \mathcal{G} = \mathcal{G}(\mathcal{E}, \Pi, p)$. As in [Y99, Def. 3.1.2], say $\mathcal{G}'$ is *affine admissible* if

$$\Phi(\mathcal{G}'(\mathcal{E}, \Pi, p)) = \Phi(\mathcal{E}, \Pi, p), \quad \dim \mathcal{G}'_\alpha = 1 \quad \forall\, \alpha \in \Phi(\mathcal{E}, \Pi, p) - \mathbb{Z}\delta.$$

Let $\mathrm{AI} = \mathrm{AI}(\mathcal{E}, \Pi, p)$ be the set of affine-admissible Lie superalgebras with respect to $(\mathcal{E}, \Pi, p)$. Let $\mathcal{G}_{\mathrm{AI}} = \mathcal{G}_{\mathrm{AI}}(\mathcal{E}, \Pi, p)$ be the admissible Lie superalgebra $\bigcup_{\mathcal{G}' \in \mathrm{AI}} \mathcal{G}'$. It is the unique maximal affine-admissible Lie superalgebra in $\mathrm{AI}(\mathcal{E}, \Pi, p)$: it is in AI and $\mathcal{G}_{\mathrm{AI}} \geq \mathcal{G}'$ for all $\mathcal{G}' \in \mathrm{AI}$. It satisfies $\dim \mathcal{G}_{\mathrm{AI},\alpha} = \dim \mathcal{G}_\alpha$ for all $\alpha \in (P_+ \cup P_- - \mathbb{Z}\delta) \cup \{0\}$.

[Y99, Prop. 3.1.3] shows that if $(\delta, \rho) \neq 0$ then $\mathcal{G}_{\mathrm{AI}} = \mathcal{G}$, and gives examples where the conclusion fails if $(\delta, \rho) = 0$. In fact, this condition holds for $(\mathrm{AA})^g$, thus affine $\mathrm{sl}(m|n)$, $m \neq n$, which we denote by $\widehat{\mathrm{sl}}(m|n)$, and fails for $(\mathrm{AA})^b$, where $\mathcal{G}_{\mathrm{AI}}$ is described in [Y99, Thm. 3.5.1]. The main result of [Y99, Thm. 4.1.1] of use for us describes $\widehat{\mathrm{sl}}(m|n, \Pi, p) = \mathcal{G}_{\mathrm{AI}}(\mathcal{E}, \Pi, p)$ in terms of generators and relations. Recall that $\Pi = \{\alpha_0, \dots, \alpha_{\mathbf{n}}\}$, $\mathbf{n} + 1 = N = m + n$,

$$\mathcal{E} = \mathrm{Span}\big\{\varepsilon_i - \varepsilon_j\ , (i \neq j \in I(m|n)),\ \delta\big\} \subset \mathcal{E}^{\mathrm{ex}} = \mathrm{Span}\big\{\varepsilon_j\ (j \in I(m|n)),\ \delta\big\}.$$

We use [Y99, Thm. 4.1.1] only for $m > n \geq 1$, $m + n > 3$. The cases $m = n$ and $(m, n) = (2, 1)$ are left to another work.

**Theorem 8.2.** *Let $(\mathcal{E}, \Pi, p)$ be of affine (AA) type. The affine Lie superalgebra $\widehat{\mathrm{sl}}(m|n, \Pi, p) = \mathcal{G}_{\mathrm{AI}}(\mathcal{E}, \Pi, p)$ can alternatively be defined by generators $h \in \mathfrak{h}$, $e_i$, $f_i$ for all $i$, $0 \leq i \leq \mathbf{n}$, parities $p(h) = 0$, $p(e_i) = p(\alpha_i) = p(f_i)$ for all $i$, $0 \leq i \leq \mathbf{n}$, and "affine Serre" relations*

(S1) $[h, h'] = 0$ *for* $h, h' \in \mathfrak{h}$;

(S2) $[h, e_i] = \alpha_i(h) e_i$, $[h, f_i] = -\alpha_i(h) f_i$;

(S3) $[e_i, f_j] = \delta_{i,j} h_{\alpha_i}$;

(S4)(1) $[e_i, e_j] = 0$ *if* $i \neq j$ *and* $(\alpha_i, \alpha_j) = 0$;

(S4)(2) $[e_i, e_i] = 0$ *if* $(\alpha_i, \alpha_i) = 0$ *and then* $p(\alpha_i) = \bar{1}$;

(S4)(3) $[e_i, [e_i, [\dots [e_i, e_j] \dots]]] = 0$, *$e_i$ appears* $1 - \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$ *times if* $(\alpha_i, \alpha_i) \neq 0$, *then* $p(\alpha_i) = \bar{0}$, *and the* $i$*th vertex is white*;

(S4)(4) $[[[e_i, e_j], e_k], e_j] = 0$ *if* $(\alpha_i, \alpha_j) = -(\alpha_j, \alpha_k) \neq 0 = (\alpha_j, \alpha_j)$, *so the* $j$*th vertex is gray*;

(S5)($a$) $1 \leq a \leq 4$. *The same relations as* (S4)($a$) *with* $f_r$ *in place of* $e_r$.

Note that for $\widehat{\mathrm{sl}}(m|n)$, (S4)(3) becomes $[e_i, [e_i, e_j]] = 0$ if $(\alpha_i, \alpha_i) = -2(\alpha_i, \alpha_j) = \pm 2, = (\alpha_j, \alpha_j)$ if it is nonzero. Then vertex $i$ is white, and the adjacent vertex $j$ is gray if $(\alpha_j, \alpha_j)$ is zero, and white if not.

## §9. Fundamental representation

Our aim is to relate the finite-dimensional representations of the affine Lie algebra $\widehat{\mathrm{sl}}(m|n)$ with those of the affine symmetric group $S_d^a = \mathbb{Z}^d \rtimes S_d$, the semidirect product of the symmetric group $S_d$ and the lattice $\mathbb{Z}^d$, where $S_d$ acts on $\mathbb{Z}^d$ by permutations. Denote by $y_i = (0, \dots, 0, 1, 0, \dots, 0)$ (1 in the $i$th place, $1 \le i \le d$) the standard $d$ generators of $\mathbb{Z}^d$ as a free abelian group.

Before stating the relation, recall that the Dynkin diagram of our $\widehat{\mathrm{sl}}(m|n) = \widehat{\mathrm{sl}}(m|n; \Pi, p)$ is described by Figure 4, where the vertices are labeled by the roots $\alpha_j = \varepsilon_{i_j} - \varepsilon_{i_{j+1}}$ in a fundamental system ($i_j \in I(m|n)$, $1 \le j < m+n$, $i_j \ne i_{j'}$ if $j \ne j'$), and $\alpha_0 = \delta - (\varepsilon_{i_1} - \varepsilon_{i_{m+n}})$, where $(\delta, \varepsilon_j) = 0 = (\delta, \delta)$ ($j \in I(m|n)$). The root vector corresponding to the root $\alpha_j = \varepsilon_{i_j} - \varepsilon_{i_{j+1}}$ is $e_j = E_{i_j, i_{j+1}}$. The corresponding coroot is $h_{\alpha_j} = E_{i_j, i_j} - (-1)^{p(\alpha_j)} E_{i_{j+1}, i_{j+1}}$, where $p(\alpha_j) = \bar{0}$ if $\bar{1} \le i_j,\ i_{j+1} \le \overline{m}$ or $1 \le i_j,\ i_{j+1} \le n$, and then the vertex labeled $j$, associated with $\alpha_j$, is white, and $p(\alpha_j) = \bar{1}$ otherwise, and then the associated vertex is gray. In particular, $(\alpha_j, \alpha_{j+1}) = \pm 1$, and $p(\alpha_j) = \bar{1}$ precisely when $p(\varepsilon_{i_j}) + p(\varepsilon_{i_{j+1}}) = \bar{1}$, thus $p(\alpha_j) = p(\varepsilon_{i_j}) + p(\varepsilon_{i_{j+1}})$. The $\varepsilon_i$ satisfy $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}(-1)^{p(\varepsilon_i)}$, and we added a vector $\delta$ with $(\varepsilon_i, \delta) = (\delta, \delta) = 0$ for all $i$. The root vector $e_0$ corresponding to $\alpha_0$ is the transpose ${}^tE_{i_1, i_{m+n}} = E_{i_{m+n}, i_1}$ of $E_{i_1, i_{m+n}}$, and the corresponding coroot is $h_{\alpha_0} = E_{i_{m+n}, i_{m+n}} - (-1)^{p(\alpha_0)} E_{i_1, i_1}$.

The superspace $V = \mathbb{E}$, where $\mathbb{E} = \mathbb{C}^{m|n}$, has a natural structure of an $\widehat{\mathrm{sl}}(m|n)$-module, called the *fundamental representation*, denoted $\rho$. Recall that $\mathbb{E} = \mathbb{E}_{\bar{0}} \oplus \mathbb{E}_{\bar{1}}$ is a superspace, thus it is $\mathbb{F}_2$-graded, $\mathbb{E}_{\bar{0}} = \bigoplus_i \mathbb{C} u_i$ ($\bar{1} \le i \le \overline{m}$), $\mathbb{E}_{\bar{1}} = \bigoplus_j \mathbb{C} u_j$ ($1 \le j \le n$), with basis $(u_{\bar{1}}, \dots, u_n)$, and there is a parity function $p\colon \mathbb{E}_{\bar{0}} \cup \mathbb{E}_{\bar{1}} \to \mathbb{F}_2$, with $p$ being $\bar{0}$ on $\mathbb{E}_{\bar{0}}$ and $\bar{1}$ on $\mathbb{E}_{\bar{1}}$. Define a $\mathbb{C}$-linear operator $\rho(\sigma)$ on $\mathbb{E}$ by $\rho(\sigma) u_i = (-1)^{p(u_i)} u_i$ ($i \in I(m|n)$), thus $\rho(\sigma) = \mathrm{diag}(I_m, -I_n)$, in the basis ($u_i$; $i \in I(m|n)$). Put $u_{\bar{0}} = 0 = u_{n+1}$ (where $\bar{0} < \bar{1} < \cdots < \overline{m} < 0 < 1 < \cdots < n < n+1$). In the basis $(u_{\bar{1}}, \dots, u_n)$, $\alpha_0 = \delta - (\varepsilon_{i_1} - \varepsilon_{i_{m+n}})$ has root vector $e_0$ and

$$\rho(e_0) = \rho(e_{\alpha_0}) = E_{i_{m+n}, i_1}, \quad \rho(f_0) = \rho(f_{\alpha_0}) = E_{i_1, i_{m+n}},$$
$$\rho(h_0) = \rho(h_{\alpha_0}) = E_{i_{m+n}, i_{m+n}} - (-1)^{p(\alpha_0)} E_{i_1, i_1},$$

$\alpha_j = \varepsilon_{i_j} - \varepsilon_{i_{j+1}}$ has root vector $e_j$, $1 \le j < m+n$, and

$$\rho(e_j) = \rho(e_{\alpha_j}) = E_{i_j, i_{j+1}}, \quad \rho(f_j) = \rho(f_{\alpha_j}) = E_{i_{j+1}, i_j},$$
$$\rho(h_j) = \rho(h_{\alpha_j}) = E_{i_j, i_j} - (-1)^{p(\alpha_j)} E_{i_{j+1}, i_{j+1}},$$

so

$$\rho(e_j) u_i = \delta_{i, i_{j+1}} u_{i_j}, \quad \rho(f_j) u_i = \delta_{i, i_j} u_{i_{j+1}},$$

and

$$\rho(h_j) u_i = \begin{cases} u_{i_j} & \text{if } i = i_j, \\ -u_{j+1} & \text{if } i = i_{j+1}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $[\rho(e_i), \rho(f_j)] = \delta_{i,j} \rho(h_{\alpha_i})$. Note that $\varepsilon_i \in \mathcal{E} = \mathfrak{h}^*$ and $u_i \in \mathbb{E}$.

## §10. Affine super Schur duality

The universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of the superalgebra $\mathfrak{g}$ has the structure of a Hopf superalgebra, whose comultiplication $\Delta$, counit $\varepsilon$, antipode $S$ are

$$\Delta(e_i) = e_i \otimes 1 + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + 1 \otimes f_i,$$
$$\varepsilon(e_i) = 0 = \varepsilon(f_i), \quad S(e_i) = -e_i, \quad S(f_i) = -f_i.$$

One can introduce a Hopf algebra structure by replacing $\mathfrak{U}(\mathfrak{g})$ by $\mathfrak{U}_\sigma(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g}) \rtimes \langle \sigma \rangle$, where $\sigma$ is the involution on $\mathfrak{g}$ given by $\sigma(e_i) = (-1)^{p(\alpha_i)} e_i$, $\sigma(f_i) = (-1)^{p(\alpha_i)} f_i$. Then $\mathfrak{U}_\sigma(\mathfrak{g})$ is a Hopf algebra with comultiplication $\Delta_\sigma$, counit $\varepsilon_\sigma$, antipode $S_\sigma$ given by

$$\Delta_\sigma(\sigma) = \sigma \otimes \sigma, \quad \Delta_\sigma(e_i) = e_i \otimes 1 + \sigma^{p(\alpha_i)} \otimes e_i, \quad \Delta_\sigma(f_i) = f_i \otimes 1 + \sigma^{p(\alpha_i)} \otimes f_i,$$
$$\varepsilon_\sigma(\sigma) = 1, \quad \varepsilon_\sigma(e_i) = 0 = \varepsilon_\sigma(f_i),$$
$$S_\sigma(\sigma) = \sigma, \quad S_\sigma(e_i) = -\sigma^{p(\alpha_i)} e_i, \quad S_\sigma(f_i) = -\sigma^{p(\alpha_i)} f_i.$$

The representation $(\rho, \mathbb{E})$ extends to a representation $\rho_d$ of $\mathfrak{U}_\sigma = \mathfrak{U}_\sigma(\mathfrak{g})$ on $\mathbb{E}^{\otimes d}$ via the map

$$\Delta^{(k)} = (\Delta_\sigma \otimes I^{\otimes(k-1)})\Delta^{(k-1)} \colon \mathfrak{U}_\sigma \to \mathfrak{U}_\sigma^{\otimes(k+1)},$$

where $\Delta^{(1)} = \Delta_\sigma \colon \mathfrak{U}_\sigma \to \mathfrak{U}_\sigma^{\otimes 2}$. Thus we put

$$\rho_d(x) = \rho^{\otimes d} \circ \Delta^{(d-1)}(x), \quad x \in \mathfrak{U}_\sigma = \mathfrak{U}_\sigma(\mathrm{gl}(m|n)),$$

Explicitly, $\rho_d(\sigma) = \rho(\sigma)^{\otimes d}$,

$$\rho_d(e_i) = \sum_{1 \le k \le d} \rho(\sigma^{p(\alpha_i)})^{\otimes(k-1)} \otimes \rho(e_i) \otimes I^{\otimes(d-k)},$$
$$\rho_d(f_i) = \sum_{1 \le k \le d} \rho(\sigma^{p(\alpha_i)})^{\otimes(k-1)} \otimes \rho(f_i) \otimes I^{\otimes(d-k)}.$$

We can now state our main result, an affine extension of the super Schur duality of Sergeev [S85, CW12]. Recall that $S_d^a = \mathbb{Z}^d \rtimes S_d$. Put $\widehat{\mathrm{sl}}(m|n) = \widehat{\mathrm{sl}}(m|n, \Pi, p)$.

**Theorem 10.1.** *Fix integers $d \ge 0$, $m > n \ge 1$, $(m,n) \ne (2,1)$. There exists a functor $\mathcal{F}$ from the category* $\mathrm{Rep}\,\mathbb{C}[S_d^a]$ *of finite-dimensional right* $\mathbb{C}[S_d^a]$*-modules, to the category* $\mathrm{Rep}(\widehat{\mathrm{sl}}(m|n); d)$ *of finite-dimensional left* $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n, \Pi, p))$*-modules whose restriction to* $\mathrm{sl}(m|n)$ *is* $\mathbb{E}^{\otimes d}$*-compatible, defined as follows. Let $M$ be a right $S_d^a$-module. Define $\mathcal{F}(M)$ to be $\mathcal{S}(M) = M \otimes_{\psi_d(\mathbb{C}[S_d])} \mathbb{E}^{\otimes d}$ as a* $\mathfrak{U}_\sigma(\mathrm{sl}(m|n))$*-module.*

*Let the remaining generators of* $\widehat{\mathrm{sl}}(m|n,\Pi,p)$ *act by*

$$(\rho_d(e_0))(\boldsymbol{m}\otimes v)=\sum_{1\le j\le d}\boldsymbol{m}y_j\otimes\rho^{\otimes d}(Y^{(d)}_{j,e})v,\quad Y^{(d)}_{j,e}=(\sigma^{p(\alpha_0)})^{\otimes(j-1)}\otimes e_0\otimes I^{\otimes(d-j)},$$

$$(\rho_d(f_0))(\boldsymbol{m}\otimes v)=\sum_{1\le j\le d}\boldsymbol{m}y_j^{-1}\otimes\rho^{\otimes d}(Y^{(d)}_{j,f})v,\ Y^{(d)}_{j,f}=(\sigma^{p(\alpha_0)})^{\otimes(j-1)}\otimes f_0\otimes I^{\otimes(d-j)},$$

*for all* $\boldsymbol{m}\in M$ *and* $v\in\mathbb{E}^{\otimes d}$*. If* $d<m+n$ *then the functor* $\mathcal{F}\colon M\mapsto\mathcal{F}(M)$ *is an equivalence from the category* $\mathrm{Rep}\,\mathbb{C}[S^a_d]$ *of finite-dimensional* $S^a_d$*-modules, onto the category* $\mathrm{Rep}(\widehat{\mathrm{sl}}(m|n);d)$ *of finite-dimensional* $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n))$*-modules whose restriction to* $\mathrm{sl}(m|n)$ *is* $\mathbb{E}^{\otimes d}$*-compatible.*

The vector $\boldsymbol{m}\in M$ is unrelated to the integer $m=\dim\mathbb{E}_{\bar{0}}$.

We show that our functor is an equivalence only for $d<m+n$. Perhaps this assertion holds for other $d<(n+1)(m+1)$, as this is the condition in Theorem 6.2(4), as in [S85]. But our method of proof, which adapts [CP96], shows the surjectivity only for $d<m+n$. In the ordinary case of $n=0$, it is shown in [F21] that $\mathcal{F}$ is an equivalence when $d<m$, but it is *not* an equivalence when $d=m$ in the affine case, although $\mathcal{S}$ is in the finite-dimensional case. Determination of the upper bound of $d$ for which the theorem holds is left for another work.

In the trivial case $d=0$, $\mathbb{C}[S^a_d]=\mathbb{C}$ and $\mathbb{E}^{\otimes d}=\mathbb{C}$; the category on the $S_d$-side is that of finite-dimensional complex vector spaces, and the theorem asserts that there are no nontrivial extensions of $\mathcal{L}\mathfrak{g}$-modules lifted from the trivial $\mathfrak{g}$-module $\mathbb{C}$.

When $d=1$ an irreducible representation of $\mathbb{C}[S_d\ltimes\mathbb{Z}^d]=\mathbb{C}[\mathbb{Z}]=\mathbb{C}[t,t^{-1}]$ is a $\mathbb{C}$-linear homomorphism $\chi\colon\mathbb{C}[t^{\pm1}]\to\mathbb{C}$ determined by the value $\chi(t)\in\mathbb{C}^\times$ of $\chi$ at $t$, or at $1\in\mathbb{Z}$. An $\mathbb{E}$-compatible irreducible representation of $\mathcal{L}\mathfrak{g}=\mathcal{L}\otimes\mathrm{sl}(n,\mathbb{C})$ (i.e., whose restriction to $\mathrm{sl}(m|n)$ is the standard representation $\rho$ on $\mathbb{E}=\mathbb{C}^{m|n}$) is then of the form $\chi\otimes\rho$, where $\chi\colon\mathcal{L}\to\mathbb{C}$ is a $\mathbb{C}$-linear algebra homomorphism determined by the value $\chi(t)\in\mathbb{C}^\times$, by Corollary 17.3. On irreducibles the correspondence defined by $\mathcal{F}$ is then $\chi\mapsto\chi\otimes\rho$. Both categories, of finite-dimensional $\mathcal{L}$-modules, and of finite-dimensional $\mathbb{E}$-compatible $\mathcal{L}\mathfrak{g}$-modules, are not semisimple.

## §11. Operators are well defined

The first task on the way to the proof of the theorem is to check that the operators $\rho_d(e_0)$ and $\rho_d(f_0)$ are well defined. Then we need to check they satisfy the relations that define $\mathfrak{g}$. Only the new relations, those involving the new generators $e_0$ and $f_0$, need to be checked. Then we need to verify that the functor is an equivalence of categories.

**Proposition 11.1.** *The operators are well defined.*

*Proof.* First we verify that for all $s \in S_d$,

$$(\rho_d(f_0))(\boldsymbol{m}s \otimes v) = (\rho_d(f_0))(\boldsymbol{m} \otimes sv)$$

for all $\boldsymbol{m} \in M$ and $v \in \mathbb{E}^{\otimes d}$; namely, as operators on $\mathcal{S}(M) = M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d}$ we have

$$\sum_{1 \le j \le d} sy_j \otimes \rho^{\otimes d}(Y_{j,f}^{(d)}) = \sum_{1 \le j \le d} y_j \otimes \rho^{\otimes d}(Y_{j,f}^{(d)})s,$$

where we recall that $Y_{j,f}^{(d)} = (\sigma^{p(\alpha_0)})^{\otimes(j-1)} \otimes f_0 \otimes I^{\otimes(d-j)}$. It suffices to show this for a set of generators of the symmetric group $S_d$, so we take $s$ to be a transposition $s_i = (i, i+1)$, $1 \le i < d$. The terms with $j \ne i$, $i+1$ on both sides are equal to one another, as $s_i$ commutes with $y_j$ and with $\rho^{\otimes d}(Y_{j,f}^{(d)})$. It remains to show

$$s_i y_i \otimes Y_{i,f}^{(d)} + s_i y_{i+1} \otimes Y_{i+1,f}^{(d)} = y_i \otimes Y_{i,f}^{(d)} s_i + y_{i+1} \otimes Y_{i+1,f}^{(d)} s_i.$$

Only the $i$th and $(i+1)$th factors in $Y$ are affected by $s_i$, so to simplify the notation we assume that $s = (12)$ and $d = 2$, and we are to show that

$$(\rho_d(f_0))(\boldsymbol{m}s \otimes v) = A + B,$$
$$A = \boldsymbol{m}sy_1 \otimes (\rho(f_0) \otimes I)v, \quad B = \boldsymbol{m}sy_2 \otimes (\rho(\sigma^{p(\alpha_0)}) \otimes \rho(f_0))v$$

equals

$$(\rho_d(f_0))(\boldsymbol{m} \otimes s \cdot v) = C + D,$$
$$C = \boldsymbol{m}y_1 \otimes (\rho(f_0) \otimes I)s \cdot v, \quad D = \boldsymbol{m}y_2 \otimes (\rho(\sigma^{p(\alpha_0)}) \otimes \rho(f_0))s \cdot v.$$

Here $s \cdot v$ stands for $\psi_d(s)v$, $v$ is $v_1 \otimes v_2$ where $v_i$ are homogeneous (in $\mathbb{E}_\iota$), $\rho(\sigma)v_i = (-1)^{p(v_i)}v_i$, $(\psi_2(s))(v_1 \otimes v_2) = (-1)^{p(v_1)p(v_2)}v_2 \otimes v_1$. Then $A$ is $\boldsymbol{m}y_2$ times (i.e., $\otimes$)

$$\begin{aligned}\psi(s)(\rho(f_0)v_1 \otimes v_2) &= (-1)^{p(\rho(f_0)v_1)p(v_2)}v_2 \otimes \rho(f_0)v_1 \\ &= (-1)^{(p(\alpha_0)+p(\alpha_1))p(v_2)}v_2 \otimes \rho(f_0)v_1,\end{aligned}$$

and $D$ is $\boldsymbol{m}y_2$ times

$$(\rho(\sigma^{p(\alpha_0)}) \otimes \rho(f_0))(-1)^{p(v_1)p(v_2)}v_2 \otimes v_1 = (-1)^{p(v_1)p(v_2)}(-1)^{p(\alpha_0)p(v_2)}v_2 \otimes \rho(f_0)v_1,$$

thus $A = D$, and $B$ is $\boldsymbol{m}y_1$ times

$$\psi(s)(-1)^{p(\alpha_0)p(v_1)}v_1 \otimes \rho(f_0)v_2 = (-1)^{p(\alpha_0)p(v_1)}(-1)^{p(\rho(f_0)v_2)p(v_1)}\rho(f_0)v_2 \otimes v_1$$

and $C$ is $\boldsymbol{m}y_1$ times

$$(\rho(f_0) \otimes (-1)^{p(v_1)p(v_2)}I)v_2 \otimes v_1 = (-1)^{p(v_1)p(v_2)}\rho(f_0)v_2 \otimes v_1,$$

so $C = D$ as $p(\rho(f_0)v_2) = p(\alpha_0) + p(v_2)$.

The same computation holds with $e_0$ replacing $f_0$ (and $y$ replaced by $y^{-1}$). □

**Remark 11.2.** Our proof is patterned on that of [CP96]. Our Section 11 corresponds to the first paragraph in the proof of [CP96, Thm. 4.2]; our verification of the relations (which are more complex in the present case) in Sections 11.1, 12, 13 reminds one of [CP96, midpage 305 to midpage 306]; and our Section 14 of [CP96, Sects. 4.3–4.6]. In particular, [CP96, lemma in Sect. 4.3] is given there without a proof; our version of it is proven as Proposition 14.1 below.

## §11.1. The relation (S4)(2)

The relation (S4)(2) needs to be checked only for the vertex in the affine Dynkin diagram of $\widehat{\mathrm{sl}}(m|n)$, that is not in the diagram of $\mathrm{sl}(m|n)$, namely that indexed by $\alpha_0$. This relation asserts that if $p(\alpha_0)=\bar{1}$, then $[e_0,e_0]=e_0^2-(-1)^{p(\alpha_0)p(\alpha_0)}e_0^2=2e_0^2$ is 0. Thus we need to check that $\rho_d(e_0^2)=0$ if $p(\alpha_0)=\bar{1}$. We then compute

$$\rho_d(e_0)^2(\boldsymbol{m}\otimes v)=\sum_{1\le i,j\le d}\boldsymbol{m}y_j^{-1}y_i^{-1}\otimes(\rho(\sigma)^{\otimes(i-1)}\otimes\rho(e_0)\otimes I^{\otimes(d-i)})\\ \times(\rho(\sigma)^{\otimes(j-1)}\otimes\rho(e_0)\otimes I^{\otimes(d-j)})v.$$

First we consider the summands associated with $i<j$: thus $\boldsymbol{m}y_j^{-1}y_i^{-1}\otimes$

$$I^{\otimes(i-1)}\otimes\rho(e_0)\rho(\sigma)\otimes\rho(\sigma)^{\otimes(j-i-1)}\otimes\rho(e_0)\otimes I^{\otimes(d-j)}$$

plus

$$I^{\otimes(i-1)}\otimes\rho(\sigma)\rho(e_0)\otimes\rho(\sigma)^{\otimes(j-i-1)}\otimes\rho(e_0)\otimes I^{\otimes(d-j)}$$

is 0 since $\rho(\sigma)\rho(e_0)=-\rho(e_0)\rho(\sigma)$. Then for the summands labeled by $i=j$ we have $\rho(e_0)^2=0$ since $\rho(e_0)\in\mathrm{End}_{\bar{1}}\mathbb{E}$, $\dim\mathbb{E}=m+n$, $\dim_s\mathbb{E}=m|n$, so $\rho(e_0)$ is nilpotent of order 2. The relation (S5)(2) is verified in the same way, with $f_0$ replacing $e_0$.

## §12. The relations (S4)(3)

Next we check the relation(s) (S4)(3). It states, for our $\hat{\mathfrak{g}}$,

$$[e_i,[e_i,e_j]]=\begin{cases}0 & \text{if } (\alpha_i,\alpha_i)=-2(\alpha_i,\alpha_j)=\pm 2,\\ (\alpha_j,\alpha_j) & \text{if } (\alpha_j,\alpha_j)\neq 0.\end{cases}$$

Here the vertices $i$ and $j$ are adjacent, the $i$-vertex is white, and so is the $j$th, unless $(\alpha_j,\alpha_j)=0$ when the $j$-vertex is gray. It suffices to check that $\rho_d$ preserves this relation only for the new vertex in the Dynkin diagram of $\hat{\mathfrak{g}}$, which does not appear in the diagram for $\mathfrak{g}$. This vertex is (labeled by the root) $\alpha_0$, so we require that $e_i$ or $e_j$ is $e_0$, and then the other is $e_{\bar{1}}$ or $e_{\mathbf{n}}$, as $(\alpha_i,\alpha_j)\neq 0$.

If $j = 0$ then $i = \bar{1}$ or $\mathbf{n}$, and the relation is

$$[e_{\bar{1}}, [e_{\bar{1}}, e_0]] = \begin{cases} 0 & \text{if } (e_{\bar{1}}, e_{\bar{1}}) = -2(e_{\bar{1}}, e_0) = \pm 2, \\ (e_0, e_0) & \text{if} \neq 0, \end{cases}$$

$$[e_{\mathbf{n}}, [e_{\mathbf{n}}, e_0]] = \begin{cases} 0 & \text{if } (\alpha_{\mathbf{n}}, \alpha_{\mathbf{n}}) = -2(\alpha_{\mathbf{n}}, \alpha_0) = \pm 2, \\ (e_0, e_0) & \text{if} \neq 0. \end{cases}$$

If $i = 0$ then $j = \bar{1}$ or $\mathbf{n}$, and the relation is

$$[e_0, [e_0, e_{\bar{1}}]] = \begin{cases} 0 & \text{if } (e_0, e_0) = -2(e_0, e_{\bar{1}}) = \pm 2, \\ (e_{\bar{1}}, e_{\bar{1}}) & \text{if} \neq 0, \end{cases}$$

$$[e_0, [e_0, e_{\mathbf{n}}]] = \begin{cases} 0 & \text{if } (\alpha_0, \alpha_0) = -2(\alpha_0, \alpha_{\mathbf{n}}) = \pm 2, \\ (e_{\mathbf{n}}, e_{\mathbf{n}}) & \text{if} \neq 0. \end{cases}$$

The first of these triality relations, since $p(\alpha_{\bar{1}}) = 0$, becomes

$$0 = [e_{\bar{1}}, [e_{\bar{1}}, e_0]] = [e_{\bar{1}}, e_{\bar{1}} e_0 - e_0 e_{\bar{1}}] = e_{\bar{1}}^2 e_0 - 2 e_{\bar{1}} e_0 e_{\bar{1}} + e_0 e_{\bar{1}}^2.$$

We then need to show the vanishing of

$$\rho_d([e_{\bar{1}}, [e_{\bar{1}}, e_0]])(\boldsymbol{m} \otimes v) = \sum_{1 \le j \le d} \boldsymbol{m} y_j^{-1} \otimes [\rho_d(e_{\bar{1}}), [\rho_d(e_{\bar{1}}), \rho^{\otimes d}(Y_{j,e}^{(d)})]] v.$$

It suffices to show the vanishing of $\rho^{\otimes d}$ of

$$[\Delta^{(d-1)}(e_{\bar{1}}), [\Delta^{(d-1)}(e_{\bar{1}}), Y_{j,e}^{(d)}]].$$

When $d = 1$ this leads to $\rho([e_{\bar{1}}, [e_{\bar{1}}, e_0]]) = [\rho(e_{\bar{1}}), [\rho(e_{\bar{1}}), \rho(e_0)]]$. From the relations on $\alpha_{\bar{1}}$, $\alpha_0$ we may assume $\alpha_{\bar{1}} = \varepsilon_i - \varepsilon_j$, $\alpha_0 = \varepsilon_k - \varepsilon_i$ ($\bar{1} \le i, j, k \le \overline{m}$ or $1 \le i, j, k \le n$) if $(\alpha_{\bar{1}}, \alpha_{\bar{1}}) = (\alpha_0, \alpha_0) = -2(\alpha_{\bar{1}}, \alpha_0)$, hence $\rho(e_{\bar{1}}) = E_{i,j}$ satisfies $\rho(e_{\bar{1}})^2 = 0$, and $\rho(e_0) = E_{k,i}$, $k \neq j$, so $\rho(e_{\bar{1}})\rho(e_0) = E_{i,j} E_{k,i}$ is 0. When $p(\alpha_0) = 1$ we have $\bar{1} \le i, j \le \overline{m}$ and $1 \le k \le n$ or $\bar{1} \le k \le \overline{m}$ and $1 \le i, j \le n$, and the same conclusion is obtained.

When $d = 2$, we are led to $\rho^{\otimes d}$ of

$$\begin{aligned} &[\Delta(e_{\bar{1}}), [\Delta(e_{\bar{1}}), e_0 \otimes 1 + \sigma^{p(\alpha_0)} \otimes e_0]] = A + B, \\ &A = [e_{\bar{1}} \otimes 1 + 1 \otimes e_{\bar{1}}, [e_{\bar{1}} \otimes 1 + 1 \otimes e_{\bar{1}}, \sigma^{p(\alpha_0)} \otimes e_0]], \\ &B = [e_{\bar{1}} \otimes 1 + 1 \otimes e_{\bar{1}}, [e_{\bar{1}} \otimes 1 + 1 \otimes e_{\bar{1}}, e_0 \otimes 1]]. \end{aligned}$$

Using $\rho(e_{\bar{1}}) = E_{i,j}$, $p(e_{\bar{1}}) = 0$, $\rho(e_0) = E_{k,i}$, $\rho(e_{\bar{1}})\rho(e_0) = E_{i,j} E_{k,i} = 0$, $\rho(e_{\bar{1}})^2 = 0$, and putting $\rho(\sigma) = J$, we see that the inner $[.,.]$ in $\rho^{\otimes 2}(A)$ is

$$E_{i,j} J^{p(\alpha_0)} \otimes E_{k,i} - J^{p(\alpha_0)} E_{i,j} \otimes E_{k,i} - J^{p(\alpha_0)} \otimes E_{k,i} E_{i,j}.$$

If $p(\alpha_0) = \bar{0}$ then the first two terms cancel each other, so $\rho^{\otimes 2}(A)$ becomes

$$-E_{i,j} \otimes E_{k,j} + E_{i,j} \otimes E_{k,j} = 0.$$

If $p(\alpha_0) = \bar{1}$ we get the same conclusion since $E_{i,j}J = JE_{i,j}$, as $\bar{1} \le i, j \le \overline{m}$ or $1 \le i, j \le n$.

The inner bracket in $\rho^{\otimes 2}(B)$ is $\rho(e_0) \otimes \rho(e_{\bar{1}}) - \rho(e_0)\rho(e_{\bar{1}}) \otimes 1 - \rho(e_0) \otimes \rho(e_{\bar{1}}) = -\rho(e_0)\rho(e_{\bar{1}}) \otimes 1$, hence $\rho^{\otimes 2}(B)$ is $-\rho(e_0)\rho(e_{\bar{1}}) \otimes \rho(e_{\bar{1}}) + \rho(e_0)\rho(e_{\bar{1}}) \otimes \rho(e_{\bar{1}}) = 0$.

For $d \ge 3$, to verify that $0 = \rho_d([e_{\bar{1}}, [e_{\bar{1}}, e_0]]) = [\rho_d(e_{\bar{1}}), [\rho_d(e_{\bar{1}}), \rho_d(e_0)]]$, where we recall that

$$\rho_d(e_{\bar{1}}) = \sum_{1 \le s \le d} \rho(\sigma^{p(\alpha_i)})^{\otimes(s-1)} \otimes \rho(e_i) \otimes I^{\otimes(d-s)} \quad (\bar{1} \le i < n)$$

and

$$(\rho_d(e_0))(\boldsymbol{m} \otimes v) = \sum_{1 \le j \le d} \boldsymbol{m} y_j^{-1} \otimes \rho^{\otimes d}(Y_{j,e}^{(d)})v, \quad Y_{j,e}^{(d)} = (\sigma^{p(\alpha_0)})^{\otimes(j-1)} \otimes e_0 \otimes I^{\otimes(d-j)},$$

it suffices to show that after applying $\rho^{\otimes d}$, which we omit to simplify the notation, the sum $\sum_{1 \le s,t \le d} a(s,t,j)$ is mapped to 0 for each $j$, where (here $p(\alpha_{\bar{1}}) = 0$)

$$\begin{aligned} a(s,t,j) = &[I^{\otimes(s-1)} \otimes e_{\bar{1}} \otimes I^{\otimes(d-s)}, \\ &[I^{\otimes(t-1)} \otimes e_{\bar{1}} \otimes I^{\otimes(d-t)}, (\sigma^{p(\alpha_0)})^{\otimes(j-1)} \otimes e_0 \otimes I^{\otimes(d-j)}]]. \end{aligned}$$

Fix $j$. The term $s = t = j$ is zero since this case reduces to that of $d = 1$, as the components at all other positions commute. So ($\rho^{\otimes 3}$ of) $a(j,j,j)$ is 0.

Fix $j' \ne j$. If $s$, $t$ range over the set $\{j, j'\}$, the corresponding part of the sum reduces to the case of $d = 2$, for the same reason. In particular, the sum of the terms $a(j',j,j)$, $a(j,j',j)$, $a(j,j,j')$ is zero.

Now fix $j'$ and $j''$ such that $|\{j, j', j''\}| = 3$. It remains to show that $a(j',j'',j) + a(j'',j',j)$ is 0 for all triples $\{j, j', j''\}$. As the components at all other positions commute, it suffices to consider the case where $d = 3$. There are three cases: $j = 1$, 2, 3. Consider $j = 1$. We have ($(s,t) = (2,3)$, $(3,2)$) $\rho^{\otimes 3}$ of

$$[I \otimes e_{\bar{1}} \otimes I, [I \otimes I \otimes e_{\bar{1}}, e_0 \otimes I \otimes I]] + [I \otimes I \otimes e_{\bar{1}}, [I \otimes e_{\bar{1}} \otimes I, e_0 \otimes I \otimes I]] = 0.$$

When $j = 2$ we have $\rho^{\otimes 3}$ of

$$[e_{\bar{1}} \otimes I \otimes I, [I \otimes I \otimes e_{\bar{1}}, \sigma^{p(e_0)} \otimes e_0 \otimes I]] = 0 \quad (s = 1,\ t = 3)$$

plus the term for $(s,t) = (3,1)$, $\rho^{\otimes 3}$ of

$$[I \otimes I \otimes e_{\bar{1}}, [e_{\bar{1}} \otimes I \otimes I, \sigma^{p(e_0)} \otimes e_0 \otimes I]].$$

Here the inner bracket has the form $(\rho(e_{\bar{1}})J - J\rho(e_{\bar{1}})) \otimes \rho(e_0) \otimes I$ since $p(e_{\bar{1}}) = 0$, which vanishes since $J = \operatorname{diag}(I_m, -I_n)$ and $\rho(e_{\bar{1}}) = \operatorname{diag}(A, B)$ with $A$ of size $m$, $B$ of size $n$.

Finally, when $j = 3$, we get $\rho^{\otimes 3}$ of ($s = 1$, $t = 2$; use $p(e_{\bar{1}}) = 0$)

$$[e_{\bar{1}} \otimes I \otimes I, [I \otimes e_{\bar{1}} \otimes I, \sigma^{p(e_0)} \otimes \sigma^{p(e_0)} \otimes e_0]] = 0,$$

since the inner bracket is $J^{p(e_0)} \otimes (\rho(e_{\bar{1}})J^{p(e_0)} - J^{p(e_0)}\rho(e_{\bar{1}})) \otimes e_0 = 0$, and (for $s = 2$, $t = 1$) we get $\rho^{\otimes 3}$ of

$$[I \otimes e_{\bar{1}} \otimes I, [e_{\bar{1}} \otimes I \otimes I, \sigma^{p(e_0)} \otimes \sigma^{p(e_0)} \otimes e_0]] = 0,$$

since the first component in the inner bracket is $\rho(e_{\bar{1}})J^{p(e_0)} - J^{p(e_0)}\rho(e_{\bar{1}}) = 0$.

The verification of the relation $\rho_d([e_{\mathbf{n}}, [e_{\mathbf{n}}, e_0]]) = 0$ is obtained by simply replacing $e_{\bar{1}}$ by $e_{\mathbf{n}}$ in the computation above.

The remaining pair of triality relations $\rho_d([e_0, [e_0, e_i]]) = 0$ for $i = \bar{1}$ or $n$ is verified similarly. We note that only the standard case, where $p(e_0) = \bar{1}$, is discussed in [F20], but as we show here, the same computations apply to all data $(\mathcal{E}, \Pi, p)$.

This completes the verification that the relations (S4)(3) are preserved under $\rho_d$.

The relations (S5)(3), in which $e_i$ are replaced by $f_i$, are verified by analogous computations.

## §13. The relations (S4)(4)

The relations are $[[[e_i, e_j], e_k], e_j] = 0$ if $p(\alpha_j) = 1$, thus $(\alpha_j, \alpha_j) = 0$, and $(\alpha_i, \alpha_j) = -(\alpha_j, \alpha_k) \neq 0$. The new, affine, cases, are those that involve $\alpha_0$. Only these relations need to be verified. So these relations are

$$(*) \qquad [[[e_{\bar{1}}, e_0], e_{\mathbf{n}}], e_0] = 0, \quad p(e_0) = 1, \quad (\alpha_{\bar{1}}, \alpha_0) = -(\alpha_0, \alpha_{\mathbf{n}}) \neq 0$$

corresponding to the three consecutive vertices ($i = \bar{1}$, $j = 0$, $k = \mathbf{n}$), where the vertex $j = 0$ is gray; and the relation obtained on interchanging $e_{\bar{1}}$ and $e_{\mathbf{n}}$,

$$(**) \qquad [[[e_0, e_{\bar{1}}], e_{\bar{2}}], e_{\bar{1}}] = 0, \quad p(e_{\bar{1}}) = 1, \quad (\alpha_0, \alpha_{\bar{1}}) = -(\alpha_{\bar{1}}, \alpha_{\bar{2}}) \neq 0$$

corresponding to the three consecutive vertices ($i = 0$, $j = \bar{1}$, $k = \bar{2}$), where the vertex $j = \bar{1}$ is gray; and the relation obtained on interchanging $e_{\bar{2}}$ and $e_0$, and replacing $(0, \bar{1}, \bar{2})$ with $(0, \mathbf{n}, \mathbf{n} - 1)$ and $(\mathbf{n} - 1, \mathbf{n}, 0)$: this is the case where $j = \mathbf{n}$ (is gray), $\{i, k\} = \{0, \mathbf{n} - 1\}$.

Consider the case where $p(\alpha_0) = p(\alpha_{\bar{1}}) = p(\alpha_{\mathbf{n}}) = 1$ of $(*)$, the most extreme case unique to the super situation. Then $\alpha_0 = \delta_i - \varepsilon_j$ and $\rho(e_0) = E_{i,j}$. We may take

(1) $\alpha_{\bar{1}} = \varepsilon_k - \delta_i$, then $\alpha_{\mathbf{n}} = \varepsilon_j - \delta_\ell$, or $\alpha_{\bar{1}} = \delta_i - \varepsilon_k$, then $\alpha_{\mathbf{n}} = \delta_\ell - \varepsilon_j$; or

(2) $\alpha_{\bar{1}} = \varepsilon_j - \delta_k$, then $\alpha_{\mathbf{n}} = \varepsilon_\ell - \delta_i$, or $\alpha_{\bar{1}} = \delta_k - \varepsilon_j$, then $\alpha_{\mathbf{n}} = \delta_i - \varepsilon_\ell$.

Our task is then to show that

$$[[[\rho_d(e_{\bar{1}}), \rho_d(e_0)], \rho_d(e_{\mathbf{n}})], \rho_d(e_0)] = 0.$$

Since

$$(\rho_d(e))(\boldsymbol{m} \otimes v) = \sum_{1 \le j \le d} \boldsymbol{m} y_j^{-1} \otimes \rho^{\otimes d}(Y_{j,e}^{(d)}), \quad Y_{j,e}^{(d)} = \sigma^{\otimes(j-1)} \otimes e \otimes I^{\otimes(d-j)}$$

for each of our $e = e_{\bar{1}}$, $e_0$, $e_{\mathbf{n}}$ (as $\sigma^{p(e)} = \sigma$, since $p(e) = \bar{1}$), we need to consider the sum of terms of the form (we write $J$ for $\rho(\sigma)$, $E_i$ for $\rho(e_i)$ ($i = 0, \bar{1}, \mathbf{n}$), $1 \le j_i \le d$)

$$\begin{aligned} a(j_1, j_2, j_3, j_4) = [[[&J^{\otimes(j_1-1)} \otimes E_1 \otimes I^{\otimes(d-j_1)}, J^{\otimes(j_2-1)} \otimes E_0 \otimes I^{\otimes(d-j_2)}], \\ &J^{\otimes(j_3-1)} \otimes E_{\mathbf{n}} \otimes I^{\otimes(d-j_3)}], J^{\otimes(j_4-1)} \otimes E_0 \otimes I^{\otimes(d-j_4)}] \end{aligned}$$

applied to $\boldsymbol{m} \otimes v$.

To keep track of the computations, the procedure will be to fix $(j_2, j_4)$, and consider the sum of the terms $a$ for all possibilities for $j_1$, $j_3$. In all cases the sum is zero. There are too many cases to record all computations here, but the technique is as in the previous section, of a triple bracket. We describe a few cases. Consider then the case of (1) above. If all $j_i$ are equal to the same $j$, then we may assume that $d = 1$, as the other components in the tensor product commute. In this case we are reduced to the computation of $[[[E_1, E_0], E_{\mathbf{n}}], E_0]$, where $E_1 = E_{k,i}$, $E_{\mathbf{n}} = E_{j,\ell}$, $E_0 = E_{i,j}$. This bracket becomes $[E_{k,i}E_{i,j}E_{j,\ell}, E_{i,j}] = 0$, since $j$, $k$, $\ell$ are all different.

Next we consider the case of $j_2 = j_4 = j$, and $\{\bar{j}_1, j_3\} \subset \{j, j'\}$. Then we may work with $d = 2$, so $j = 1$ or 2. When $j = 1$, $j_1 = 1$, $j_3 = 2$, the term is

$$[[[E_1 \otimes I, E_0 \otimes I], J \otimes E_{\mathbf{n}}], E_0 \otimes 1].$$

The first, innermost bracket, is $E_1 E_0$ with $p = 0$, as $E_0 E_1 = 0$. The second bracket is $E_1 E_0 J - J E_1 E_0 = 0$, since $p(e_0 e_1) = 0$ so $E_1 E_0$ commutes with $J$.

When $j = 1$, $j_1 = 2$ (any $j_3 \in \{1, 2\}$), the inner bracket is

$$[J \otimes E_1, E_0 \otimes 1] = (JE_0 + E_0 J) \otimes E_0,$$

and this is zero since $p(e_0) = 1$ (thus $JE_0 = -E_0 J$).

When $j = 2$ and $j_1 = 1$, the inner bracket

$$[E_1 \otimes 1, J \otimes E_0] = (E_1 J + J E_1) \otimes E_0$$

is zero as $p(e_1) = 1$. When $j = 2 = j_1$, $j_3 = 2$, the term is

$$[[[J \otimes E_1, J \otimes E_0], E_n \otimes I], J \otimes E_0].$$

The innermost bracket is $I \otimes E_1 E_0$, as $E_0 E_1 = 0$. Since $p(e_1 e_0) = 0$, the second bracket is 0, as $I \otimes E_1 E_0$ commutes with $E_{\mathbf{n}} \otimes I$.

If $j_2 = j_4 = j$ and $j_1$, $j_3 \neq j$, then we may work with $d = 3$. Thus if $j = 1$, $(j_1, j_3)$ is $(2,3)$ or $(3,2)$. If $j = 2$, $(j_1, j_3)$ is $(1,3)$ or $(3,1)$. If $j = 3$, $(j_1, j_3)$ is $(1,2)$ or $(2,1)$.

If $j_2 \neq j_4 = j$ and $j_1$, $j_3 \in \{j_2, j_4\}$, we may assume $d = 2$, and then $(j_2, j_4) = (1,2)$ and $(j_1, j_3) = (1,2)$ and $(2,1)$, or $(j_2, j_4) = (2,1)$ and $(j_1, j_3) = (1,2)$ and $(2,1)$. If $j_1, j_3 \in \{j_2, j_4, j'\}$ but not both in $\{j_2, j_4\}$, then we may assume $d = 3$. In this case, one of $j_1$, $j_3$ is in $\{j_2, j_4\}$, the other is not, or $j_1 = j_3 = j'$, a case we consider next.

If $j_2 \neq j_4 = j$ and $j_1$, $j_3 \notin \{j_2, j_4\}$, we work with $d = 3$ if $j_1 = j_3$, and with $d = 4$ if not. In particular, it suffices always to work with $d \leq 4$, and in each case the computation is reduced to an elementary matrix multiplication, that can easily be verified.

These considerations verify (S4)(4). The verification of the (S5) cases, where the generators $e$ are replaced by the generators $f$, is analogous.

## §14. The functor $\mathcal{F}$ is an equivalence

The super Schur duality (Theorem 6.2) asserts the existence of an equivalence of categories when $d < (m+1)(n+1)$. The proof described below, which is an adaptation of that of [CP96] in the affine quantum nonsuper case, of [F20] in the affine quantum super case, and of [F21] in the affine case, seems to hold only under the restriction $d < m + n$. So we assume this in the present section, and ask whether the result, that our functor $\mathcal{F}$ is an equivalence, extends to bigger $d < (m+1)(n+1)$. In [F21] it is shown that the affine extension of Schur's duality holds for $d < n$ but not for $d = n$ when $\mathfrak{g} = \mathrm{sl}_n(\mathbb{C})$, although Schur's duality holds for $d \leq n$. Recall that $\mathbb{E} = \mathbb{C}^{m|n}$.

To show that the functor $\mathcal{F}$ – which we have seen is a well-defined functor between the categories specified in the theorem – is an equivalence, one has to show the following:

(a) Every finite-dimensional $\mathbb{E}^{\otimes d}$-compatible $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n, \Pi, p))$-module $W$, i.e., each of its irreducible constituents when restricted to $\mathfrak{U}_\sigma(\mathrm{sl}(m|n, \Pi, p))$ is a constituent of $\mathbb{E}^{\otimes d}$, is isomorphic to $\mathcal{F}(M) = M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d}$ for some $\mathbb{C}[S_d^a]$-module $M$. We write $\otimes_{\mathbb{C}[S_d]}$ for $\otimes_{\psi_d(\mathbb{C}[S_d])}$.

(b) $\mathcal{F}$ is bijective on sets of morphisms.

To prove (a), by the super Schur duality Theorem 6.2 we assume that $W = \mathcal{S}(M)$ for some $\mathbb{C}[S_d]$-module $M$. We shall construct the action of the $y_j^{\pm 1}$ on $M$ from the given action of $\rho_d(e_0)$, $\rho_d(f_0)$, $\rho_d(\mathfrak{h})$ on $W$.

Put $\Pi_0 = \Pi - \{\alpha_0\} = \{\alpha_1 = \varepsilon_{j_1} - \varepsilon_{j_2}, \dots, \alpha_{\mathbf{n}} = \varepsilon_{j_{m+n-1}} - \varepsilon_{j_{m+n}};\ j_i \in I(m|n)\} \subset \mathcal{E} = \mathfrak{h}^*$.

Recall that $\{u_i;\ i \in I(m|n)\}$ denotes the standard basis of $\mathbb{E}$, with $p(u_i) = \bar{0}$ if $\bar{1} \le i \le \overline{m}$, $p(u_i) = \bar{1}$ if $1 \le i \le n$. We also write $v_i$ for $u_{\bar{i}}$, $1 \le i \le m$, and $v_{i+m}$ for $u_i$, $1 \le i \le n$. Further, $\mathbf{n} + 1 = N = m + n$. Write $\mathfrak{U}_\sigma(\mathrm{sl}(m|n))$ for $\mathfrak{U}_\sigma(\mathrm{sl}(m|n, \Pi_0, p))$.

**Proposition 14.1.**

(a) *Let $M$ be a finite-dimensional $\mathbb{C}[S_d]$-module. Fix $v \in \mathbb{E}^{\otimes d}$ such that $\mathbb{E}^{\otimes d} = \rho_d(\mathfrak{U}_\sigma(\mathrm{sl}(m|n))) \cdot v$. Then the map $M \to \mathcal{S}(M)$, $\boldsymbol{m} \mapsto \boldsymbol{m} \otimes v$ is injective.*

(b) *Suppose $v = u_{i_1} \otimes \cdots \otimes u_{i_d} \in \mathbb{E}^{\otimes d}$, where $i_1, \dots, i_d \in I(m|n)$ are distinct. Then $\mathbb{E}^{\otimes d} = \rho_d(\mathfrak{U}_\sigma(\mathrm{sl}(m|n))) \cdot v$. In particular, $v$ satisfies the condition stated in* (a).

*Proof.* Choose an isomorphism $\mathbb{E}^{\otimes d} = \bigoplus_\lambda L(\lambda^s) \otimes S^\lambda$, where $\lambda \in P_d(m|n)$. As we assume $d < m + n < (m+1)(n+1)$, every partition $\lambda$ of $d$ is an $(m|n)$-hook partition of $d$, so the sum ranges over $P_d(m+n)$. The length $\ell(\lambda^s)$ of $\lambda^s$ is $< m + n$. Here $S^\lambda$ is the $\lambda$-Specht representation of $S_d$ and $L(\lambda^s)$ is the $\mathrm{sl}(m|n)$-module parametrized by $\lambda^s$. The vector $v = \sum_\lambda x_\lambda$ spans $\mathbb{E}^{\otimes d}$ under the action of $\rho_d(\mathfrak{U}_\sigma(\mathfrak{g}))$, in particular $\rho_d(\mathfrak{U}_\sigma(\mathfrak{g})) \cdot x_\lambda = S^\lambda \otimes L(\lambda^s) = \mathrm{Hom}_{\mathbb{C}}(L(\lambda^s)^\vee, S^\lambda)$. As $\dim L(\lambda^s) \ge \dim S^\lambda$, we may assume $x_\lambda \colon L(\lambda^s)^\vee \to S^\lambda$ is onto, for each $\lambda$.

To see why the dimension of the GL-representation ($L(\lambda^s)$ or $V^\lambda$) is no less than the dimension of the corresponding representation ($S^\lambda$) of the symmetric group, when $d \le n$, I follow a message from Vera Serganova. First take the case $d = n$, and the usual, nonsuper case. Consider the diagonal subgroup of GL. Take the subspace $M$ of $\mathbb{E}^{\otimes d}$ on which $\mathrm{diag}(x_1, \dots, x_n)$ acts by the character $x_1 \cdots x_n$. This subspace is the regular representation of $S_d$ (when $d = n$). Then we get $\dim(M \cap V^\lambda) = \dim S^\lambda$. (Indeed, $M \cap V^\lambda = c_\lambda M$, where $c_\lambda$ is the Young projector, basically by definition. But $\dim c_\lambda X$ is the multiplicity of $S^\lambda$ in $X$ for a representation $X$ of $S_d$. In our case $X = M$ is regular, hence the multiplicity of any irreducible representation is equal to its dimension.) This gives the inequality, since $\dim V^\lambda \ge \dim(M \cap V^\lambda) = \dim S^\lambda$. This proof works perfectly well in the super case. Now when $n > d$, the dimension of $V^\lambda$ only grows. This is also clear in the super case, where $V^\lambda$ is replaced by $L(\lambda^s)$. It is important to note that in the super case we mean the usual dimension, $\dim L(\lambda^s)_0 + \dim L(\lambda^s)_1$. For the superdimension the inequality is wrong. For example, all irreducible representations of gl(1|1) have superdimension $\leq 2$. But for the (1|1)-hook partition $\lambda = (2,1,1,\dots,1)$ of size $d$, the corresponding symmetric group module $S^\lambda$ has dimension $d-1$, $>2$ for $d \geq 4$. In the nonsuper case the dimension inequality follows also from the dimension formula of [FH91, Ex. 6.4*, p. 78].

Now since $\mathbb{C}[S_d]$ is semisimple, by the Maschke theorem the finite-dimensional $\mathbb{C}[S_d]$-module $M$ is completely reducible. Thus $M = \bigoplus_{\mu\vdash d} M_\mu$, where $M_\mu$ are the $\mu$-isotypical components of $M$. Hence $M_\mu \simeq S^\mu \otimes A_\mu$, where $A_\mu = \mathrm{Hom}_{\mathbb{C}[S_d]}(S^\mu, M)$ is a vector space. Since $S^\mu \simeq (S^\mu)'$ is self dual, $M_\mu \simeq \mathrm{Hom}_{\mathbb{C}}(S^\mu, A_\mu)$.

We next use the fact that $S^\lambda$ is self-dual, and Schur's lemma: $V' \otimes_G W \simeq \mathrm{Hom}_G(V, W)$ is $\mathbb{C}$ if the irreducible $G$-modules $V$, $W$ are isomorphic, 0 if not; $G = S_d$. Consider the map

$$\begin{aligned} M \times \mathbb{E}^{\otimes d} \to M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d} &= \Big(\bigoplus_\mu \mathrm{Hom}_{\mathbb{C}}(S^\mu, A_\mu)\Big) \otimes_{\mathbb{C}[S_d]} \Big(\bigoplus_\lambda S^\lambda \otimes L(\lambda^s)\Big) \\ &= \bigoplus_\lambda A_\lambda \otimes L(\lambda^s) \\ &= \bigoplus_\lambda \mathrm{Hom}_{\mathbb{C}}(L(\lambda^s)^\vee, A_\lambda), \end{aligned}$$

$$\begin{aligned} ((f_\lambda \colon S^\lambda \to A_\lambda), (x_\lambda \in S^\lambda \otimes L(\lambda^s))) &\mapsto (f_\lambda(x_\lambda) \in A_\lambda \otimes L(\lambda^s) \\ &= \mathrm{Hom}_{\mathbb{C}}(L(\lambda^s)^\vee, A_\lambda)), \end{aligned}$$

where $f_\lambda(x_\lambda)$ is $f_\lambda \circ x_\lambda \colon L(\lambda^s)^\vee \to S^\lambda \to A_\lambda$.

The injectivity of the map $M \mapsto M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d}$, $\boldsymbol{m} \mapsto \boldsymbol{m} \otimes v$, means that $f_\lambda \circ x_\lambda = 0$ implies $f_\lambda = 0$ for all $\lambda$. This holds when $x_\lambda \colon L(\lambda^s)^\vee \to S^\lambda$ is surjective, as assumed; (a) follows.

It does not suffice to assume that $x_\lambda \neq 0$ for all $\lambda$ as assumed in [CP96, Lem. 4.3(a)]: if for example $x_\lambda = a \otimes b$, $a \in S^\lambda$, $b \in L(\lambda^s)$, $\dim S^\lambda \geq 2$ and $A_\lambda \neq 0$, there are nonzero $f_\lambda \colon S^\lambda \to A_\lambda$ that send $a$ to 0. For this reason we write a proof of the proposition.

Claim (b) is elementary. We need to show that under the action of the universal enveloping algebra $\rho_d(\mathfrak{U}_\sigma(\mathrm{sl}(m|n)))$, each basis vector $\varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_d}$ of $\mathbb{E}^{\otimes d}$ can be obtained from $\varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_d}$ with distinct $i_1, \dots, i_d \in \{1, \dots, m+n\}$. To simplify the notation it suffices to show this for $d = 2$. Then $m + n > 2$ by our standing assumption $d < m + n$. Thus it suffices to show that $\rho_d(\mathfrak{U}_\sigma(\mathfrak{g}))$ takes $\varepsilon_1 \otimes \varepsilon_2$ to $\varepsilon_k \otimes \varepsilon_\ell$ for any $k$, $\ell$ between 1 and $m+n$. Recall that $\rho_d(g)$ acts as $\Delta_d(g)v = \sum_{1\leq j\leq d} (-1)^{p(g)(p(v_1)+\cdots+p(v_{j-1}))} (I^{\otimes(j-1)} \otimes g \otimes I^{\otimes(d-j)})v$ on

$v = v_1 \otimes \cdots \otimes v_d$. Write $g = (a, b, c)$ for the $(m+n) \times (m+n)$-matrix with first column $a$, second column $b$, $k$th column $c$, $k > 2$; the other columns are not written, to save on notation; it suffices to take $m + n = 3$ and $k = 3$. Then $\Delta_d(\varepsilon_k, 0, *)$ takes $\varepsilon_1 \otimes \varepsilon_2$ to $\varepsilon_k \otimes \varepsilon_2$, where $*$ means any column; $\Delta_d(*, \varepsilon_\ell, 0)$ takes $\varepsilon_k \otimes \varepsilon_2$ to $\varepsilon_k \otimes \varepsilon_\ell$ $(k \neq 2)$; $\Delta_d(0, *, \varepsilon_2)$ takes $\varepsilon_k \otimes \varepsilon_1$ to $\varepsilon_2 \otimes \varepsilon_1$; $\Delta_d(\varepsilon_\ell, 0, *)$ takes $\varepsilon_2 \otimes \varepsilon_1$ to $\varepsilon_2 \otimes \varepsilon_\ell$; $\Delta_d(0, \varepsilon_k, *)$ takes $\varepsilon_1 \otimes \varepsilon_2$ to $\varepsilon_1 \otimes \varepsilon_k$, of course up to a sign. □

**Proposition 14.2.** *For $j$, $1 \leq j \leq d$, put $a(j) = v_2 \otimes \cdots \otimes v_j$, $b(j) = v_{j+1} \otimes \cdots \otimes v_d$,*

$$v^{(j)} = a(j) \otimes v_{m+n} \otimes b(j), \quad w^{(j)} = a(j) \otimes v_1 \otimes b(j).$$

*In particular,*

$$\begin{aligned} v^{(1)} &= v_{m+n} \otimes v_2 \otimes \cdots \otimes v_d, & v^{(d)} &= v_2 \otimes \cdots \otimes v_d \otimes v_{m+n}, \\ w^{(1)} &= v_1 \otimes v_2 \otimes \cdots \otimes v_d, & w^{(d)} &= v_2 \otimes \cdots \otimes v_d \otimes v_1. \end{aligned}$$

*Then there exists $\alpha_{j,f} \in \operatorname{End}_{\mathbb{C}} M$ with*

$$(\rho_d(f_0))(\boldsymbol{m} \otimes v^{(j)}) = \alpha_{j,f}(\boldsymbol{m}) \otimes \rho^{\otimes d}(Y_{j,f}^{(d)}) v^{(j)}$$

*and $\alpha_{j,e} \in \operatorname{End}_{\mathbb{C}} M$ with*

$$(\rho_d(e_0))(\boldsymbol{m} \otimes w^{(j)}) = \alpha_{j,e}(\boldsymbol{m}) \otimes \rho^{\otimes d}(Y_{j,e}^{(d)}) w^{(j)}$$

*for all $j, 1 \leq j \leq d$. We have $\rho^{\otimes d}(Y_{j,f}^{(d)}) v^{(j)} = \pm w^{(j)}$ and $\rho^{\otimes d}(Y_{j,e}^{(d)}) w^{(j)} = \pm v^{(j)}$.*

*Proof.* For $\tau$ in the symmetric group $S_d$ on $d$ letters, put

$$w_\tau^{(j)} = (v_{\tau(2)} \otimes \cdots \otimes v_{\tau(j)}) \otimes v_{\tau(1)} \otimes (v_{\tau(j+1)} \otimes \cdots \otimes v_{\tau(d)}).$$

The set $\{w_\tau^{(j)};\ \tau \in S_d\}$ spans the subspace of $\mathbb{E}^{\otimes d}$ of weight $\lambda_d = \varepsilon_{j_1} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_d}$. Now $\rho_d(f_0)$ adds $\varepsilon_{j_1} - \varepsilon_{j_{m+n}}$ to the weight, hence it takes $\varepsilon_{j_{m+n}}$ to $\varepsilon_{j_1}$. Thus for every $\boldsymbol{m} \in M$ we have

$$(\rho_d(f_0))(\boldsymbol{m} \otimes v^{(j)}) = \sum_{\tau \in S_d} \boldsymbol{m}_\tau \otimes w_\tau^{(j)}$$

for some $\boldsymbol{m}_\tau \in M$. But $w_\tau^{(j)}$ is a nonzero scalar multiple of $h \cdot w^{(j)}$ for some $h \in \mathbb{C}[S_d]$, $h = h(\tau)$. Hence $(\rho_d(f_0))(\boldsymbol{m} \otimes v^{(j)})$ equals $\boldsymbol{m}' \otimes w^{(j)}$ for some $\boldsymbol{m}' \in M$. Then there exists $\alpha_{j,f} \in \operatorname{End}_{\mathbb{C}} M$ with $\boldsymbol{m}' = \alpha_{j,f}(\boldsymbol{m})$ for all $\boldsymbol{m} \in M$ by Proposition 14.1. The existence of $\alpha_{j,e} \in \operatorname{End}_{\mathbb{C}} M$ is proven analogously. □

**Proposition 14.3.** *For all $m \in M$ and $v \in \mathbb{E}^{\otimes d}$ we have*

$$(\rho_d(e_0))(\boldsymbol{m} \otimes v) = \sum_{1 \le j \le d} \alpha_{j,e}(\boldsymbol{m}) \otimes \rho^{\otimes d}(Y_{j,e}^{(d)})v,$$
$$(\rho_d(f_0))(\boldsymbol{m} \otimes v) = \sum_{1 \le j \le d} \alpha_{j,f}(\boldsymbol{m}) \otimes \rho^{\otimes d}(Y_{j,f}^{(d)})v.$$

*Proof.* Suppose $v = u_{i_1} \otimes \cdots \otimes u_{i_d}$. Then $(\rho_d(f_0))(\boldsymbol{m} \otimes v)$ will be 0 if no $i_j$ is $j_{m+n}$, as then $-\varepsilon_{j_{m+n}} + \varepsilon_{j_1} + \varepsilon_{i_1} + \cdots + \varepsilon_{i_d}$ cannot be a weight of $\mathbb{E}^{\otimes d}$. So we may assume some component of $v$ is $u_{j_{m+n}}$.

Let $r \ge 0$, $s \ge 1$, $r + s \le d$, $1 \le j_1 < j_2 < \cdots < j_r \le d$, $1 \le j'_1 < j'_2 < \cdots < j'_s \le d$, and assume $\{j_1, \ldots, j_r\} \cap \{j'_1, \ldots, j'_s\} = \emptyset$. Write $j = (j_1, \ldots, j_r)$, $j' = (j'_1, \ldots, j'_s)$. Let $\mathbb{E}^{(j,j')}$ be the subspace of $\mathbb{E}^{\otimes d}$ spanned by the vectors that have $v_1$ in positions $j_1, \ldots, j_r$; $v_{m+n}$ in positions $j'_1, \ldots, j'_s$; and vectors from $\{v_2, \ldots, v_{m+n-1}\}$ in the remaining positions. We prove the proposition when $v$ is in $\mathbb{E}^{(j,j')}$ for all $j$, $j'$ in two steps:

(i) for $s = 1$, by induction on $r$,

(ii) for all $r$, by induction on $s$.

By Proposition 14.1, applied to the subalgebra of $\mathfrak{U}_\sigma$ generated by the $e_i$, $f_i$, $h_{\alpha_i}^{\pm 1}$ for $i \in \{2, \ldots, m+n-2\}$, to prove our proposition for all $v \in \mathbb{E}^{(j,j')}$ it suffices to prove it for one $0 \ne v \in \mathbb{E}^{(j,j')}$ whose components have no vector from $\{v_2, \ldots, v_{m+n-1}\}$ twice. Such vectors exist since $1 \le d + 1 - r - s \le d \le m + n - 1$. Here we used the condition $d < m + n$.

*Proof of step* (i). Here $s = 1$. The case of $r = 0$ follows from Proposition 14.2: take

$$v = a(j'_1) \otimes v_{m+n} \otimes b(j'_1), \quad w = a(j'_1) \otimes v_1 \otimes b(j'_1)$$

(recall that $a(j) = v_2 \otimes \cdots \otimes v_j$, $b(j) = v_{j+1} \otimes \cdots \otimes v_d$). As $Y_{j,f}^{(d)} = (\sigma^{p(\alpha_0)})^{\otimes(j-1)} \otimes f_0 \otimes I^{\otimes(d-j)}$ and $\rho(f_0) = E_{1,m+n}$, we have $\rho^{\otimes d}(Y_{j'_1,f}^{(d)})v = w$ times a sign, and $\rho^{\otimes d}(Y_{j,f}^{(d)})v = 0$ for all $j \ne j'_1$, hence we have $(\rho_d(f_0))(\boldsymbol{m} \otimes v) = \sum_{1 \le j \le d} \alpha_{j,f}(\boldsymbol{m}) \otimes \rho^{\otimes d}(Y_{j,f}^{(d)})v$, where $\alpha_{j,f}(\boldsymbol{m})$ is the sign times $\boldsymbol{m}$.

Assume step (i) holds for $r-1$. Put $\tilde{j} = (j_2, \ldots, j_r)$. Define $v' \in \mathbb{E}^{(\tilde{j},j')}$ to be a pure tensor with $v_2$ in the $j_1$ position, and distinct vectors from $\{v_3, \ldots, v_{m+n-1}\}$ in the remaining positions. Then $v = \rho_d(e_1)v'$. Indeed, recall that $\rho_d(e_1) = \sum_k (J^{p(\alpha_1)})^{\otimes(k-1)} \otimes \rho(e_1) \otimes 1^{\otimes(d-k)}$, that $\rho(e_1)v_j = \delta_{2,j}v_1$, and that $v'$ has $v_2$ only at position $j_1$ (and $v_1$ only at positions $j_2, \ldots, j_r$), so only $k = j_1$ survives in the sum over $k$ that defines $\rho_d(e_1)$, and $(\rho_d(e_1))v' = v$.

Define $v''$ by replacing $v_{m+n}$ in position $j' = j'_1$ in $v'$ by $v_1$, and $v'''$ by replacing $v_2$ in position $j_1$ in $v''$ by $v_1$. Now $r(v') = r - 1$, so we can apply the induction on $r$ (in the third equality below, and (S3) in the second):

$$\begin{aligned}(\rho_d(f_0))(\boldsymbol{m}\otimes v) &= \rho_d(f_0)\rho_d(e_1)(\boldsymbol{m}\otimes v') \\ &= \rho_d(e_1)\rho_d(f_0)(\boldsymbol{m}\otimes v') \\ &= \rho_d(e_1)\sum_{1\le \ell\le d}\alpha_{\ell,f}(\boldsymbol{m})\otimes \rho^{\otimes d}(Y_{\ell,f}^{(d)})v'.\end{aligned}$$

Recall again that $Y_{\ell,f}^{(d)}$ is $(\sigma^{p(\alpha_0)})^{\otimes(\ell-1)}\otimes f_0\otimes I^{\otimes(d-\ell)}$, and $\rho(f_0)=E_{1,m+n}$, and $v_{m+n}$ occurs only at position $j_1'$ in $v'$. Then only $\ell=j_1'$ survives in the sum, which becomes a multiple of $v''$, by a sign $\iota$. Since $v_2$ occurs in $v''$ only in position $j_1$, in the sum defining $\rho_d(e_1)$ only the summand indexed by $k=j_1$ survives when acting on $v''$, and it is $(J^{p(\alpha_1)})^{\otimes(j_r-1)}\otimes\rho(e_1)\otimes 1^{\otimes(d-j_r)}$. So $\rho_d(e_1)$ maps $v''$ to $v'''$. We obtain $\alpha_{j_1',f}(\boldsymbol{m})$ times $\iota v'''=\rho^{\otimes d}(Y_{j_1',f}^{(d)})v$. For other $j$ we have $0=\rho^{\otimes d}(Y_{j,f}^{(d)})v$. So we end up with $\sum_j \alpha_{j,f}(\boldsymbol{m})\otimes\rho^{\otimes d}(Y_{j,f}^{(d)})v$, completing step (i).

*Proof of step* (ii). Assume the proposition holds for all $v\in\mathbb{E}^{(j,j')}$ with fewer than $s$ components $v_{m+n}$. As in step (i), it suffices to prove the claim for one element $v\neq 0$ in $\mathbb{E}^{(j,j')}$ that has distinct entries from $\{v_2,\dots,v_{m+n-1}\}$ in the remaining positions. Fix such a $v$. Let $v'$ be the tensor obtained from $v$ on replacing $v_{m+n}$ in positions $j_{s-1}'$ and $j_s'$ by $v_{m+n-1}$. We claim that

$$\rho_d(f_{m+n-1})^2 v' = 2v.$$

To see this, recall that $\rho(f_{m+n-1})=E_{m+n,m+n-1}$, and $p(\alpha_{m+n-1})=0$, and

$$\rho_d(f_{m+n-1})=\sum_{1\le k\le d}(J^{p(\alpha_{m+n-1})})^{\otimes(k-1)}\otimes\rho(f_{m+n-1})\otimes I^{\otimes(d-k)}.$$

So in $\rho_d(f_{m+n-1})^2v'$ the sum over $k$ in each $\rho_d(f_{m+n-1})$ reduces to $k=j_{s-1}'$, $j_s'$, and all factors in positions $\neq j_{s-1}'$, $j_s'$ in each summand, commute. At these two positions the components of $v'$ are $v_{m+n-1}\otimes v_{m+n-1}$ and those of $\rho_d(f_{m+n-1})^2$ are

$$\begin{aligned}&(\rho(f_{m+n-1})\otimes I+J^{p(\alpha_{m+n-1})}\otimes\rho(f_{m+n-1}))\\ &\qquad\times(\rho(f_{m+n-1})\otimes I+J^{p(\alpha_{m+n-1})}\otimes\rho(f_{m+n-1}))\\ &\quad=\rho(f_{m+n-1})J^{p(\alpha_{m+n-1})}\otimes\rho(f_{m+n-1})+J^{p(\alpha_{m+n-1})}\rho(f_{m+n-1})\otimes\rho(f_{m+n-1})\end{aligned}$$

as $\rho(f_{m+n-1})^2=0$. So $\rho_d(f_{m+n-1})^2v'$ equals

$$I^{\otimes(j_{s-1}'-1)}\otimes\rho(f_{m+n-1})\otimes I^{\otimes(j_s'-1-j_{s-1}')}\otimes(\rho(f_{m+n-1})+\rho(f_{m+n-1}))\otimes I^{\otimes(d-j_s')}v'.$$

Now $\rho(f_{m+n-1})v_{m+n-1}=v_{m+n}$, so in conclusion $v=\frac{1}{2}\rho_d(f_{m+n-1})^2v'$, as claimed.

To continue we use the equality (S5)(3),

$$\rho_d(f_0)\rho_d(f_{m+n-1})^2 = 2\rho_d(f_{m+n-1})\rho_d(f_0)\rho_d(f_{m+n-1}) - \rho_d(f_{m+n-1})^2\rho_d(f_0),$$

in the second equality below:

$$(\rho_d(f_0))(\boldsymbol{m}\otimes v) = \tfrac{1}{2}\rho_d(f_0)\rho_d(f_{m+n-1})^2(\boldsymbol{m}\otimes v') = A + B,$$
$$A = \rho_d(f_{m+n-1})\rho_d(f_0)\rho_d(f_{m+n-1})(\boldsymbol{m}\otimes v'),$$
$$B = -\tfrac{1}{2}\rho_d(f_{m+n-1})^2\rho_d(f_0)(\boldsymbol{m}\otimes v').$$

To find $B$, we write by induction

$$(\rho_d(f_0))(\boldsymbol{m}\otimes v') = \sum_{1\le k\le s-2} \alpha_{j'_k,f}(\boldsymbol{m})\otimes \rho^{\otimes d}(Y^{(d)}_{j'_k,f})v',$$
$$Y^{(d)}_{j,f} = (\sigma^{p(\alpha_0)})^{\otimes(j-1)}\otimes f_0\otimes I^{\otimes(d-j)},$$

as $v_{m+n}$ occurs only at the $s-2<s$ positions $j'_1,\dots,j'_{s-2}$ in $v'$. Recall that $\rho(f_0)=E_{1,m+n}$. Note that $\rho_d(f_{m+n-1})$ changes the factors ($v_{m+n-1}$ to $v_{m+n}$) of $v'$ only at the positions $j'_{s-1}$, $j'_s$. Applying $\rho_d(f_{m+n-1})$ to $(\rho_d(f_0))(\boldsymbol{m}\otimes v')$ would send the part $v_{m+n-1}\otimes v_{m+n-1}$ at the positions $j'_{s-1}$ and $j'_s$ to $v_{m+n}\otimes v_{m+n-1}$ (from the summand of $\rho_d(f_{m+n-1})$ with $(j'_{s-1},j'_s)$ parts $\rho(f_{m+n-1})\otimes I$), plus $v_{m+n-1}\otimes v_{m+n}$ (from the summand of $\rho_d(f_{m+n-1})$ with $(j'_{s-1},j'_s)$ parts $I\otimes\rho(f_{m+n-1})$). Applying $\rho_d(f_{m+n-1})$ again we obtain

$$v_{m+n}\otimes v_{m+n} + v_{m+n}\otimes v_{m+n} = 2v_{m+n}\otimes v_{m+n}.$$

Now $\rho^{\otimes d}(Y^{(d)}_{j'_k,f})$ acts on the two factors $v_{m+n}\otimes v_{m+n}$ of $v$ at the positions $(j'_{s-1},j'_s)$ trivially, and also on $v'$. So in summary,

$$B = -\sum_{1\le k\le s-2}\alpha_{j'_k}(\boldsymbol{m})\otimes\rho^{\otimes d}(Y^{(d)}_{j'_k,f})v.$$

To compute $A$, let $v''$ (resp. $v'''$) be obtained from $v'$ on replacing the vector $v_{m+n-1}$ at the $j'_{s-1}$ (resp. $j'_s$) position by $v_{m+n}$. Observe that

$$(\rho_d(f_{m+n-1}))(\boldsymbol{m}\otimes v') = \boldsymbol{m}\otimes v'' + \boldsymbol{m}\otimes v'''.$$

(Applying $\rho_d(f_{m+n-1})$ again we recover the result from the start of the proof: $(\rho_d(f_{m+n-1})^2)(\boldsymbol{m}\otimes v') = 2(\boldsymbol{m}\otimes v)$.) As $s(v'') = s-1 = s(v''') < s$, by induction we get

$$\rho_d(f_0)\rho_d(f_{m+n-1})(\boldsymbol{m}\otimes v') = \sum_{k\ne s}\alpha_{j'_k,f}(\boldsymbol{m})\otimes\rho^{\otimes d}(Y^{(d)}_{j'_k,f})v'' + \sum_{k\ne s-1}\alpha_{j'_k,f}(\boldsymbol{m})\otimes\rho^{\otimes d}(Y^{(d)}_{j'_k,f})v'''.$$

Now we apply $\rho_d(f_{m+n-1})$. As $v''$ has $v_{m+n-1}$ only at the $j'_s$ position, we get

$$\rho_d(f_{m+n-1})\sum_{k\neq s}\alpha_{j'_k,f}(\boldsymbol{m})\otimes\rho^{\otimes d}(Y^{(d)}_{j'_k,f})v''=\sum_{k\leq s-1}\alpha_{j'_k,f}(\boldsymbol{m})\otimes\rho^{\otimes d}(Y^{(d)}_{j'_k,f})v.$$

Denote this by $A_1$. As $v'''$ has $v_{m+n-1}$ only at the $j'_{s-1}$ position,

$$\rho_d(f_{m+n-1})\sum_{k\neq s-1}\alpha_{j'_k,f}(\boldsymbol{m})\otimes\rho^{\otimes d}(Y^{(d)}_{j'_k,f})v'''=A_2+A_3,$$
$$A_2=\alpha_{j'_s,f}(\boldsymbol{m})\otimes\rho^{\otimes d}(Y^{(d)}_{j'_s,f})v,\quad A_3=\sum_{k\leq s-2}\alpha_{j'_k,f}(\boldsymbol{m})\otimes\rho^{\otimes d}(Y^{(d)}_{j'_k,f})v.$$

Then $A=A_1+A_2+A_3$, and $B+A$ is

$$\begin{aligned}(\rho_d(f_0))(\boldsymbol{m}\otimes v)&=-\sum_{1\leq k\leq s-2}\alpha_{j'_k,f}(\boldsymbol{m})\otimes\rho^{\otimes d}(Y^{(d)}_{j'_k,f})v\\&\quad+\sum_{k\leq s-2}\alpha_{j'_k,f}(\boldsymbol{m})\otimes\rho^{\otimes d}(Y^{(d)}_{j'_k,f})v\\&\quad+\alpha_{j'_s,f}(\boldsymbol{m})\otimes\rho^{\otimes d}(Y^{(d)}_{j'_s,f})v+\sum_{1\leq k\leq s-1}\alpha_{j'_k,f}(\boldsymbol{m})\otimes\rho^{\otimes d}(Y^{(d)}_{j'_k,f})v\\&=\sum_{1\leq k\leq s}\alpha_{j'_k,f}(\boldsymbol{m})\otimes\rho^{\otimes d}(Y^{(d)}_{j'_k,f})v.\end{aligned}$$

□

**Proposition 14.4.** *Setting* $\boldsymbol{m}y_j=\alpha_{j,e}(\boldsymbol{m})$, $\boldsymbol{m}y_j^{-1}=\alpha_{j,f}(\boldsymbol{m})$ *defines a right* $\mathbb{C}[S^a_d]$*-module structure on* $M$*, extending its* $\mathbb{C}[S_d]$*-module structure.*

*Proof.* We have to check the following relations:

(i) $y_jy_j^{-1}=1=y_j^{-1}y_j$;

(ii) $y_jy_k=y_ky_j$;

(iii) $y_{j+1}=s_jy_js_j$, where $s_j=(j,j+1)\in S_d$.

To prove (i) and (ii), we compute both sides of the equality

$$(\rho_d([e_0,f_0]))(\boldsymbol{m}\otimes v)=\rho_d(h_{\alpha_0})(\boldsymbol{m}\otimes v).$$

For (i) we take $v$ with $v_{m+n}$ in the $j$th position and $v_{m+n-(d-1)},\dots,v_{m+n-1}$ in the remaining positions, in any order.

For (ii) take $v$ to be a tensor with $v_1$ in the $j$th place, $v_{m+n}$ in the $k$th position, and distinct vectors from $\{v_2,\dots,v_{m+n-1}\}$ in the other positions.

For (iii), take $v=v_{i_1}\otimes\cdots\otimes v_{i_d}\in\mathbb{E}^{\otimes d}$ with $i_j=2$, $i_{j+1}=1$, and the remaining $i_k$ are distinct from $\{3,\dots,m+n-1\}$. This is possible since $d\leq m+n-1$. (Once again we use the condition $d<m+n$.) So $v$ has $v_2$ at position $j$, $v_1$ at position

$j+1$. The vector $v'$ is obtained from $v$ on replacing $v_1$ at position $j+1$ by $v_{m+n}$. The vector $v''$ is obtained from $v'$ on replacing $v_2$ at position $j$ by $v_{m+n}$ and $v_{m+n}$ at position $j+1$ by $v_2$. The vector $v'''$ is obtained from $v$ on replacing $v_2$ at position $j$ by $v_1$ and $v_1$ at position $j+1$ by $v_2$.

Now looking at the indices $(i,j) = (2, m+n)$ only, we have $s(v_{m+n} \otimes v_2) = v_2 \otimes v_{m+n}$ and $s(v_2 \otimes v_1) = v_1 \otimes v_2$, so $sv = v'''$ and $sv'' = v'$. Then

$$\begin{aligned} \boldsymbol{m} \cdot s_j y_j s_j \otimes v = \boldsymbol{m} \cdot s_j y_j \otimes v''' &= (\rho_d(f_0))(\boldsymbol{m} \cdot s_j \otimes v'') \\ &= (\rho_d(f_0))(\boldsymbol{m} \otimes s_j v'') \\ &= (\rho_d(f_0))(\boldsymbol{m} \otimes v') = \boldsymbol{m} y_{j+1} \otimes v. \end{aligned}$$

Since $v$ has distinct components, Proposition 14.1 implies that $\boldsymbol{m} \cdot y_{j+1} = \boldsymbol{m} \cdot s_j y_j s_j$ for all $\boldsymbol{m} \in M$.

This completes the proof that $W \simeq \mathcal{F}(M)$ as a $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n, \Pi, p))$-module. $\square$

To show that $\mathcal{F}$ is an equivalence we still need to show that it is bijective on sets of morphisms. Injectivity of $\mathcal{F}$ follows from that of $\mathcal{S}$. For surjectivity, let $F\colon \mathcal{F}(M) \to \mathcal{F}(M')$ be a homomorphism of $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n, \Pi, p))$-modules. By Proposition 14.1, $F = \mathcal{S}(f)$ for some homomorphism $f\colon M \to M'$ of $\mathbb{C}[S_d]$-modules. Since $F$ commutes with the action of $\rho(f_0)$ we have $(\rho(f_0)F)(\boldsymbol{m} \otimes v) = (F\rho(f_0))(\boldsymbol{m} \otimes v)$, i.e.,

$$\sum_{1 \le j \le d} f(\boldsymbol{m}) \cdot y_j \otimes \rho^{\otimes d}(Y^{(d)}_{j,f})v = \sum_{1 \le j \le d} f(\boldsymbol{m} y_j) \otimes \rho^{\otimes d}(Y^{(d)}_{j,f})v$$

for all $\boldsymbol{m} \in M$ and $v \in \mathbb{E}^{\otimes d}$. Choosing $v$ suitably we deduce that $f(\boldsymbol{m} y_j) = f(\boldsymbol{m}) y_j$ for all $j$ $(1 \le j \le d)$. This completes the proof of Theorem 10.1.

## §15. Parabolic induction

Let $S_{d_i}$ be the symmetric group on $d_i$ letters, $i = 1, 2$. There is a natural embedding of $S_{d_1} \times S_{d_2}$ in $S_{d_1+d_2}$, given by viewing $S_{d_1+d_2}$ as the group of permutations of the letters $t_1, \dots, t_{d_1+d_2}$, $S_{d_1}$ as the symmetric group of $t_1, \dots, t_{d_1}$, and $S_{d_2}$ of $t_{d_1+1}, \dots, t_{d_1+d_2}$. This naturally extends to an embedding of group algebras, $\mathbb{C}[S_{d_1}] \otimes \mathbb{C}[S_{d_2}] \hookrightarrow \mathbb{C}[S_{d_1+d_2}]$, and also to an embedding of affine symmetric groups $\phi_{d_1,d_2}\colon S^a_{d_1} \times S^a_{d_2} \hookrightarrow S^a_{d_1+d_2}$, and their group algebras $\phi_{d_1,d_2}\colon \mathbb{C}[S^a_{d_1}] \otimes \mathbb{C}[S^a_{d_2}] \hookrightarrow \mathbb{C}[S^a_{d_1+d_2}]$. Here $S^a_d = \mathbb{Z}^d \rtimes S_d$, $S_d$ acts on $\mathbb{Z}^d$ by permutations, $S_d$ is generated by $s_i = (i, i+1)$ $(1 \le i < d)$, $\mathbb{Z}^d$ by $y_j = (0, \dots, 0, 1, 0, \dots, 0)$ (1 at the $j$th place); the embedding maps $s_i \in S_{d_1}$ (or $s_i \otimes 1$) to $s_i$, and $y_j \in S^a_{d_1}$ (or $y_j \otimes 1$ in $\mathbb{C}[S^a_{d_1}] \otimes \mathbb{C}[S^a_{d_2}]$) to $y_j$, and $s_i \in S_{d_2}$ $(= 1 \otimes s_i \in 1 \otimes \mathbb{C}[S^a_{d_2}])$ to $s_{d_1+i}$, $y_j \in S^a_{d_2}$ to $y_{d_1+j}$.

Let $M_i$ be a finite-dimensional right $\mathbb{C}[S^a_{d_i}]$-module. Their outer tensor product, $M_1 \otimes_{\mathbb{C}} M_2$, is a $\mathbb{C}[S^a_{d_1}] \otimes_{\mathbb{C}} \mathbb{C}[S^a_{d_2}]$-module. The induced $\mathbb{C}[S^a_{d_1+d_2}]$-module $M_1 \widetilde{\times} M_2$ is defined by

$$M_1 \widetilde{\times} M_2 = \mathrm{ind}^{\mathbb{C}[S^a_{d_1+d_2}]}_{\mathbb{C}[S^a_{d_1}]\otimes\mathbb{C}[S^a_{d_2}]}(M_1 \otimes M_2) = (M_1 \otimes M_2) \otimes_{\mathbb{C}[S^a_{d_1}]\otimes\mathbb{C}[S^a_{d_2}]} \mathbb{C}[S^a_{d_1+d_2}].$$

This induction is associative up to a canonical isomorphism.

For finite-dimensional $\mathbb{C}[S_{d_i}]$-modules $M_i$ we define $M_1 \times M_2$ analogously for the finite groups $S_{d_i}$ and their group algebras, with the superscript $a$ removed.

If $M$ is a $\mathbb{C}[S^a_d]$-module, by $M|\mathbb{C}[S_d]$ we mean $M$ regarded as a $\mathbb{C}[S_d]$-module by restriction.

**Proposition 15.1.** *Let $M_i$ be a finite-dimensional $\mathbb{C}[S^a_{d_i}]$-module, $i = 1, 2$. Then there is a natural isomorphism $M_1 \widetilde{\times} M_2|\mathbb{C}[S_{d_1+d_2}] \simeq M_1|\mathbb{C}[S_{d_1}] \times M_2|\mathbb{C}[S_{d_2}]$.*

*Proof.* The natural map from the left to the right sides,

$$(m_1 \otimes m_2) \otimes h \mapsto (m_1 \otimes m_2) \otimes h \quad (m_i \in M_i,\ h \in \mathbb{C}[S_{d_1+d_2}]),$$

is a well-defined surjective homomorphism of $\mathbb{C}[S_{d_1+d_2}]$-modules. Note that $\mathbb{C}[S_d] \hookrightarrow \mathbb{C}[S^a_d]$ and $\mathbb{C}[S_d] \otimes_{\mathbb{C}} \mathbb{C}[y_1^{\pm1}, \dots, y_d^{\pm1}] \to \mathbb{C}[S^a_d]$ is an isomorphism of $\mathbb{C}$-vector spaces. Hence the rank of $\mathbb{C}[S^a_{d_1+d_2}]$ as a $\mathbb{C}[S^a_{d_1}] \otimes \mathbb{C}[S^a_{d_2}]$-module is equal to the rank of $\mathbb{C}[S_{d_1+d_2}]$ as a $\mathbb{C}[S_{d_1}] \otimes \mathbb{C}[S_{d_2}]$-module. It follows that $\dim_{\mathbb{C}} M_1 \widetilde{\times} M_2 = \dim_{\mathbb{C}} M_1 \times M_2$. □

Let $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{C}^{\times d}$, and define the evaluation map $\varepsilon_{\mathbf{a}} \colon \mathbb{C}[S_d \ltimes \mathbb{Z}^d] \to \mathbb{C}[S_d]$ by $\sigma_i \mapsto \sigma_i$ $(1 \le i < d)$, $y_j \mapsto a_j$ $(1 \le j \le d)$. Let $I_{\mathbf{a}}$ be the ideal generated by $y_j - a_j$ $(1 \le j \le d)$ in the algebra $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$, and $M_{\mathbf{a}}$ the quotient of $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$ by $I_{\mathbf{a}}$. Then $M_{\mathbf{a}}$ is a finite-dimensional $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$-module. As a $\mathbb{C}[S_d]$-module it is isomorphic to the right regular representation. Thus $M_{\mathbf{a}}$ is the pullback of the right regular representation $\mathbb{C}[S_d]$ via the evaluation map $\varepsilon_{\mathbf{a}}$.

Some representations of $\mathbb{C}[S^a_d]$ can be lifted from those of $\mathbb{C}[S_d]$.

**Proposition 15.2.** *For each $z \in \mathbb{C}^\times$ there is a unique homomorphism $\mathrm{ev}_z \colon \mathbb{C}[S^a_d] \to \mathbb{C}[S_d]$ that is the identity on $\mathbb{C}[S_d] \hookrightarrow \mathbb{C}[S_d]$, and it maps $y_1$ to $z$. Hence $\mathrm{ev}_z(y_j) = z$ for all $j$, $1 \le j \le d$.*

## §16. Relating representations of $\mathbb{C}[S^a_d]$ and $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n))$

The functor $\mathcal{F}$ is a functor of $\mathbb{C}$-linear categories. It commutes with induction. Recall that we write $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n, \Pi, p))$ for $U_{\mathrm{AI}}(\mathcal{E}, \Pi, p)$ for simplicity.

**Proposition 16.1.** *Let $M_i$ be a finite-dimensional $\mathbb{C}[S^a_{d_i}]$-module $(i = 1, 2)$. Then there is a natural isomorphism $\mathcal{F}(M_1 \widetilde{\times} M_2) \simeq \mathcal{F}(M_1) \otimes \mathcal{F}(M_2)$ of $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n, \Pi, p))$-modules.*

*Proof.* Let $\phi\colon B \to A$ be a homomorphism of associative algebras with a unit over a field, $M$ a right $B$-module, $W$ a left $A$-module, and $W|B$ is $W$ regarded as a left $B$-module via $\phi$. Then there is a natural isomorphism of vector spaces: $\mathrm{ind}^A_B(M) \otimes W \simeq M \otimes_B W|B$. This form of Frobenius reciprocity is given by $(\boldsymbol{m} \otimes a) \otimes w \mapsto \boldsymbol{m} \otimes aw$ $(\boldsymbol{m} \in M,\ a \in A,\ w \in W)$.

Take $A = \mathbb{C}[S_{d_1+d_2}]$, $B = \mathbb{C}[S_{d_1}] \otimes \mathbb{C}[S_{d_2}]$, $\phi = \phi(d_1, d_2)$, $M = M_1 \otimes M_2$, $W = \mathbb{E}^{\otimes(d_1+d_2)}$, where $\mathbb{E} = \mathbb{C}^{m|n} = \mathbb{E}_{\bar{0}} \oplus \mathbb{E}_{\bar{1}} = \mathbb{C}^m \oplus \mathbb{C}^n$ (of dimension $m|n$) being the natural representation of $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n, \Pi, p))$. Note that $W \simeq \mathbb{E}^{\otimes d_1} \otimes \mathbb{E}^{\otimes d_2}$ as an $\mathbb{C}[S_{d_1}] \otimes \mathbb{C}[S_{d_2}]$-module. We get a natural isomorphism of vector spaces

$$\mathcal{F}(M_1 \widetilde{\times} M_2) \to (M_1 \otimes M_2) \otimes_{\mathbb{C}[S_{d_1}] \otimes \mathbb{C}[S_{d_2}]} (\mathbb{E}^{\otimes d_1} \otimes \mathbb{E}^{\otimes d_2}).$$

The right-hand side is isomorphic to $\mathcal{F}(M_1) \otimes \mathcal{F}(M_2)$ as a vector space. It remains to check that the resulting isomorphism $\mathcal{F}(M_1 \widetilde{\times} M_2) \to \mathcal{F}(M_1) \otimes \mathcal{F}(M_2)$ of $\mathbb{C}$-vector spaces commutes with the action of $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n, \Pi, p))$. □

Consider the fundamental $\mathfrak{U}_\sigma(\mathrm{sl}(m|n, \Pi, p))$-module $\mathbb{E}$. For $a \in \mathbb{C}^\times$ we view $\mathbb{E}$ as an $\mathcal{L}(\widehat{\mathrm{sl}}(m|n, \Pi, p))$-module $\mathbb{E}(a)$, on which $t$ acts as multiplication by $a$. In other words, $\mathbb{E}(a)$ is the $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n, \Pi, p))$-module which is $\mathbb{E}$ as a $\mathfrak{U}_\sigma(\mathrm{sl}(m|n, \Pi, p))$-module, the central element $c$ and the derivation $d$ act as 0, and $t$ acts as multiplication by $a$.

Using the equivalence $\mathcal{F}$ we now relate the universal $\mathbb{C}[S^a_d]$-module $M_{\mathbf{a}}$ ($\mathbf{a} \in \mathbb{C}^{\times d}$) and the $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n, \Pi, p))$-modules $\mathbb{E}(a_i)$, $a_i \in \mathbb{C}^\times$.

**Proposition 16.2.** *Let $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{C}^{\times d}$, $d \geq 1$, $m$, $n \geq 2$. Then there exists a natural isomorphism $\mathcal{F}(M_{\mathbf{a}}) \simeq \mathbb{E}(a_1) \otimes \cdots \otimes \mathbb{E}(a_d)$.*

*Proof.* As a $\mathbb{C}[S_d]$-module, $M_a$ is the right regular representation. Hence the map $\mathbb{E}^{\otimes d} \to \mathcal{S}(M_{\mathbf{a}})$, $v \mapsto 1 \otimes v$ is an isomorphism of $\mathfrak{U}_\sigma(\mathrm{sl}(m|n, \Pi, p))$-modules,

$$(\rho_d(e_0))(1 \otimes v) = \sum_{1 \leq j \leq d} 1 \cdot y_j \otimes \rho^{\otimes d}(Y^{(d)}_{j,e})v = 1 \otimes \Bigg(\sum_{1 \leq j \leq d} a_j \otimes \rho^{\otimes d}(Y^{(d)}_{j,e})\Bigg)v.$$

Also $\rho_d(e_0) = \sum_{1 \leq j \leq d} \rho(\sigma^{p(\alpha_0)})^{\otimes(j-1)} \otimes \rho(e_0) \otimes I^{\otimes(d-j)}$ acts on $\mathbb{E}(a_1) \otimes \cdots \otimes \mathbb{E}(a_d)$ as

$$\sum_{1 \leq j \leq d} \rho(\sigma^{p(\alpha_0)})^{\otimes(j-1)} \otimes a_j \rho(e_0) \otimes I^{\otimes(d-j)} = \rho^{\otimes d}\Bigg(\sum_{1 \leq j \leq d} a_j Y^{(d)}_{j,e}\Bigg).$$

The map $\mathbb{E}^{\otimes d} \to \mathcal{S}(M_{\mathbf{a}})$ commutes with the action of $\rho(f_0)$, $\rho(e_0)$. □

## §17. Applications: Irreducible representations of $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n))$

The irreducible representations of $S_d \ltimes \mathbb{Z}^d$ can be described by Mackey theory; see e.g., [Sr77, Sect. 8.2], as follows.

Let $G = H \ltimes A$ be a group, where $A$ is a normal commutative subgroup and $H$ a finite subgroup, acting on $A$. Let $\chi\colon A \to \mathbb{C}^\times$ be a character (multiplicative function). Denote the stabilizer of $\chi$ in $G$ by $A_\chi = \{g \in G;\ \chi(gag^{-1}) = \chi(a)$ for all $a \in A\}$. This stabilizer is a subgroup of $G = H \ltimes A$, and it contains $A$, hence it is of the form $A_\chi = H' \ltimes A$ for some subgroup $H'$ of $H$. Then $\chi$ extends to $\chi'\colon H' \ltimes A \to \mathbb{C}^\times$ by $\chi'(ha) = \chi(a)$. Let $\rho$ be an irreducible representation of $H'$. Define $\rho'$ to be the composition of $\rho$ followed by the natural projection $H' \ltimes A \twoheadrightarrow H'$. Mackey theory asserts the following.

**Proposition 17.1.** *The induced representation* $\mathrm{Ind}(\chi' \otimes \rho';\ H' \ltimes A, G)$ *is irreducible. It uniquely determines the datum* $(H', \chi, \rho)$*. Each irreducible representation of* $G$ *has this form.*

We use this with $A = \mathbb{Z}^d$, $H = S_d$. A character $\chi$ of $\mathbb{Z}^d$ is a $d$-tuple $\mathbf{a} = (p_1^{d_1}, \dots, p_k^{d_k}) \in \mathbb{C}^{\times d}$, where $p_i^{d_i} = (p_i, \dots, p_i) \in \mathbb{C}^{\times d_i}$. The stabilizer has the form $H' \ltimes \mathbb{Z}^d$ with $H' = S_{d_1} \times \cdots \times S_{d_k}$. So an irreducible representation of $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$ is determined by $(d_1, \dots, d_k)$, $d_i \geq 1$, $d_1 + \cdots + d_k = d$, distinct $a_i \in \mathbb{C}^\times$, and irreducible representations $\rho_i$ of $S_{d_i}$, $1 \leq i \leq k$.

Let us express this using evaluation maps. Define the group algebra homomorphism $\varepsilon_{d.a}\colon \mathbb{C}[S_d^a] \to \mathbb{C}[S_d]$ that maps each $\sigma \in S_d$ to itself, and $y_j$ to $a$ for all $j$, $1 \leq j \leq d$. Then $\varepsilon_{d.a} = \varepsilon_{\mathbf{a}}$ with $\mathbf{a} = (a, \dots, a) \in \mathbb{C}^{\times d}$. If $M$ is an irreducible $\mathbb{C}[S_d]$-module, pulling $M$ back by $\varepsilon_{d.a}$ gives an irreducible $\mathbb{C}[S_d^a]$-module $M_{\mathbf{a}} = M_{d.a} := \varepsilon_{d.a}^* M$. When $\mathbf{a} = (p_1^{d_1}, \dots, p_k^{d_k})$, $p_i^{d_i} = (p_i, \dots, p_i) \in \mathbb{C}^{\times d_i}$, and $M_i$ are $\mathbb{C}[S_{d_i}]$-modules, we write

$$\begin{aligned}(M_1 \times \cdots \times M_k)_{\mathbf{a}} &= \varepsilon_{\mathbf{a}}^*(M_1 \times \cdots \times M_k)\\ &= \varepsilon_{d_1.p_1}^* M_1 \widetilde{\times} \cdots \widetilde{\times}\, \varepsilon_{d_k.p_k}^* M_k\\ &= M_{1,d_1.p_1} \widetilde{\times} \cdots \widetilde{\times} M_{k,d_k.p_k}.\end{aligned}$$

In summary we deduce the following from Mackey theory.

**Proposition 17.2.** *Every finite-dimensional irreducible* $\mathbb{C}[S_d^a]$*-module is isomorphic to a product* $M_{1,d_1.p_1} \widetilde{\times} \cdots \widetilde{\times} M_{k,d_k.p_k}$ *of* $M_{d_i.p_i}$*,* $d = d_1 + \cdots + d_k$*, distinct* $p_i$*.*

The theorem permits translating this result to the context of $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n))$, as follows.

As above, for each $a \in \mathbb{C}^\times$ there is a Lie algebra homomorphism

$$\mathrm{ev}_a\colon \mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n)) \to \mathfrak{U}_\sigma(\mathrm{sl}(m|n)),$$

defined by $\mathrm{ev}_a(P(t)\otimes x) = P(a)\otimes x$. If $W$ is an irreducible $\mathfrak{U}_\sigma(\mathrm{sl}(m|n))$-module, its pullback by $\mathrm{ev}_a$ is an irreducible $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n))$-module $W_a$.

Applying the functor $\mathcal{F}$, for $a \in \mathbb{C}^\times$ and a $\mathbb{C}[S_d]$-module $M$ we obtain

$$\begin{aligned}\mathcal{F}(M_{d.a}) = \mathcal{F}(\varepsilon^*_{d.a}\,M) &= (\varepsilon^*_{d.a}\,M)\otimes_{\mathbb{C}[S_d]}\mathbb{E}^{\otimes d}\\ &= \mathrm{ev}^*_a(M\otimes_{\mathbb{C}[S_d]}\mathbb{E}^{\otimes d})\\ &= \mathrm{ev}^*_a(\mathcal{S}(M)) = \mathcal{S}(M)_a.\end{aligned}$$

In general,

$$\begin{aligned}\mathcal{F}(M_{1,d_1.p_1}\,\widetilde{\times}\cdots\widetilde{\times} M_{k,d_k.p_k}) &= \mathcal{F}((M_1\times\cdots\times M_k)_{\mathbf{a}})\\ &= \varepsilon^*_{\mathbf{a}}(M_1\times\cdots\times M_k)\otimes_{\mathbb{C}[S_d]}\mathbb{E}^{\otimes d}\\ &= \mathrm{ev}^*_{p_1}(M_1\otimes_{\mathbb{C}[S_{d_1}]}\mathbb{E}^{\otimes d_1})\otimes\cdots\\ &\quad\ \otimes\mathrm{ev}^*_{p_k}(M_k\otimes_{\mathbb{C}[S_{d_k}]}\mathbb{E}^{\otimes d_k})\\ &= \mathrm{ev}^*_{p_1}(\mathcal{S}(M_1))\otimes\cdots\otimes\mathrm{ev}^*_{p_k}(\mathcal{S}(M_k)).\end{aligned}$$

From Theorem 10.1 we then conclude the following corollary.

**Corollary 17.3.** *Every finite-dimensional irreducible $\mathbb{E}^{\otimes d}$-compatible representation of the universal enveloping algebra of the affine superalgebra $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n))$ is a tensor product of evaluation representations $W_{p_i}$ at distinct points $p_i$. Here $W_{p_i} = \mathrm{ev}^*_{p_i}(\mathcal{S}(M_i))$, where $M_i$ is an irreducible $\mathbb{C}[S_{d_i}]$-module, $d = d_1+\cdots+d_k$.*

Recall that by an $\mathbb{E}^{\otimes d}$-*compatible* finite-dimensional irreducible representation of the affine superalgebra we mean that the subquotients of its restriction to the superalgebra are subrepresentations of $\mathbb{E}^{\otimes d}$, $\mathbb{E} = \mathbb{C}^{m|n}$.

**Corollary 17.4.**

(a) *Every finite-dimensional irreducible $\mathbb{C}[S^a_d]$-module is isomorphic to a quotient of some $M_{\mathbf{a}}$, $\mathbf{a} = (a_1,\dots,a_d)\in\mathbb{C}^{\times d}$.*

(b) *For all $\mathbf{a}\in\mathbb{C}^{\times d}$, $M_{\mathbf{a}}$ is isomorphic as a $\mathbb{C}[S_d]$-module to the right regular representation.*

(c) *$M_{\mathbf{a}}$ is reducible as a $\mathbb{C}[S^a_d]$-module iff $a_j = a_k$ for some $j\neq k$.*

**Corollary 17.5.** *Let $1\le d< m+n$.*

(a) *Every finite-dimensional $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n,\Pi,p))$-module that occurs as a subquotient of $\mathbb{E}^{\otimes d}$ as a $\mathfrak{U}_\sigma(\mathrm{sl}(m|n,\Pi,p))$-module is isomorphic to a quotient of $\mathbb{E}(b_1)\otimes\cdots\otimes\mathbb{E}(b_d)$ for some $b_1,\dots,b_d\in\mathbb{C}^\times$.*

(b) *Let* $b_1, \dots, b_d \in \mathbb{C}^\times$. *Then* $\mathbb{E}(b_1)\otimes\cdots\otimes\mathbb{E}(b_d)$ *is reducible as a* $\mathfrak{U}_\sigma(\widehat{\mathrm{sl}}(m|n,\Pi,p))$*-module iff* $b_j = b_k$ *for some* $j$, $k$ *with* $j \neq k$.

*Proof.* This follows from the preceding proposition, for the group algebra $\mathbb{C}[S_d^a]$ of the affine symmetric group and the fact that $\mathcal{F}$ is an equivalence of categories. □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