Publ. RIMS Kyoto Univ. **59** (2023), 203–258
DOI 10.4171/PRIMS/59-2-1

Classical and Quantized Maxwell Fields Deduced from Algebraic Many-Photon Theory

by

Alfred Rieckers

Abstract

The deduction starts with the (non-relativistic) one-photon Hilbert space $\mathcal{H}$, equipped with the one-photon Hamiltonian and basic symmetry generators, as the only input information. We recall in the functorially associated boson Fock space the multi-photon dynamics and symmetry transformations, as well as the field operator (as the scaled self-adjoint part of the creation operator) and the q(uasi)-classical states. There is no reference to a presupposed classical Maxwell theory. By abstraction, we go over to the algebraic formulation of the multi-photon theory in terms of a C*-Weyl algebra. Its test function space $E \subset \mathcal{H}$ is constructed as a nuclear Fréchet space, in which – via infrared damping – the dynamics and symmetries are nuclear continuous and their generators bounded. Each w*-closed, singular subspace of the continuous dual E' determines non-Fock coherent states and their mixtures lead to a representation von Neumann algebra with non-trivial center. The symmetry generators restricted to the center can be transformed into the Maxwell form by means of a symplectic transformation and involve the well-known conservation quantities of electrodynamics. This identifies the central part of the represented photon field operator as composed of the two classical canonical electrodynamic field components. We have obtained, therefore, in free space a kind of fusion of the multi-photon theory and the Maxwell theory of transverse electrodynamic fields, where the latter arise as derived quantities. By means of a Bogoliubov transformation one also gets a fusion of the quantized with the classical Maxwell theory, deduced from the photon concept. A sketch of non-relativistic gauging is added in the appendix to gain longitudinal, cohomological, and scalar potentials.

2020 Mathematics Subject Classification: 81T05, 81R15, 22D45, 82B10, 46L55, 7801, 81V80.
Keywords: Boson Fock space, singular coherent states, C*-Weyl algebra, infrared regularization, nuclear Fréchet spaces, GNS-representation, non-trivial center, not universally weakly inner automorphisms, central decompositions, central generators, Maxwell electrodynamics, gauge principle.

Communicated by N. Ozawa. Received October 25, 2020. Revised March 14, 2021.

A. Rieckers: Institut für Theoretische Physik, Universität Tübingen, 72076 Tübingen, Germany;
e-mail: `alfred.rieckers@uni-tuebingen.de`

§1. Introduction

In response to his first paper on light quanta [Einst05P], Einstein met vivid opposition to his new heuristic corpuscle ansatz for electrodynamic radiation traveling free in space. In [Einst09a] he expanded and detailed the intuitive picture as follows:

> Still the picture which seems to me the most natural of all is that the occurrence of the electromagnetic field of light is connected with singular points like the electromagnetic fields in electron theory. One cannot entirely exclude the possibility that in such a theory the total energy of the electromagnetic field can be viewed as being localized in these singularities, just as in the old action at a distance theory. I suppose, say, that each singular point is surrounded by a field of force, which essentially has the character of a plane wave, whose amplitude decreases with increasing distance from the singular point. If many such singular points exist at distances which are small compared to the extension of the field of force of one singular point, the fields of force will overlap and together constitute an undulatory field of force, which differs little from an undulatory field in the sense of the present electromagnetic theory. We need not, of course, especially emphasize that any value should not be attributed to such a picture so long as it does not lead to an exact theory (English translation reproduced from [MehR82]).

In the same year, Einstein again mentioned, in a letter to Sommerfeld, the point-like localization of the energy of the light quanta and assumed that a future theory would be a *fusion* of particle and wave aspects [Pais82, p. 403].

Since this took place long before the invention of the quantum mechanical formalism, one may be inclined not to take Einstein's formulations on singular energy points surrounded by plane waves quite seriously. In fact, the somewhat laconic comment of Pais was that "this fusion now goes under the name of complementarity" [Pais82, p. 404], without giving further comments on Einstein's vision.

Quantum mechanics has, in fact, eliminated the notion of a point-like physical object. But that is totally different from the concept of a classical field, which is still permanently employed in science and engineering and constitutes an objective entity like macroscopic material bodies. Thus Einstein's speculative fusion program is not achieved by the particle–wave duality which removes both the classical particle concept and the classical field concept.

Let us also remark that in the above-reproduced formulations, Einstein requires "many" singular energy points to approach, together with the overlapping force fields, classical radiation fields, and he obviously expected decisively new theoretical developments to settle the light-quantum problem.

The idea that "many" light quanta may approximate a classical electromagnetic field is pursued in quantum optics in terms of the coherent states in Fock space (e.g. [ScullyZub, Vogel, Duncan]). Fock space is commonly constructed, however, in step with the quantization of the classical field. But in order to theoretically formulate a fusion system we intend to set up a *quantum particle system which, under certain conditions, also displays electromagnetic fields.* The electromagnetic fields are in that manner derived quantities, like the collective structures of macroscopic many-particle systems. The fields do not exist without photons, but single photons exist in experiments without classical electromagnetic fields. We must therefore introduce the photon system *without any reference to the classical fields.*

We begin in Section 2 with the introduction of the one-photon Hilbert space $\mathcal{H}$ through specifying the generators of the basic symmetry groups and dynamics, taking into account the transversality of the wave functions. This setup is well founded through experimental non-relativistic quantum optics.

The photonic Fock space is then gained via the *functorial mapping* $\mathcal{H} \mapsto F_+(\mathcal{H})$ giving quite generally the bosonic Fock spaces as a function of the one-particle Hilbert spaces.

We recall in Section 2 some Fock space formalism of the non-relativistically formulated photon theory in a basis-independent manner. The photonic "field operator" $\Phi^\hbar_F(f)$ is directly given (without field quantization) as the scaled sum of the smeared creation and annihilation operators, where the commutation relations of the latter lead to the CCR (canonical commutation relations) for the field operator (and vice versa). This enables the introduction of the Weyl algebra $\mathcal{W}_F(\mathcal{H}, \hbar\sigma)$ in Fock space, where the symplectic form is $\sigma = \mathrm{Im}(\cdot|\cdot)$. The operator algebra $\mathcal{W}_F(\mathcal{H}, \hbar\sigma)$ is viewed as the faithful Fock representation of the abstracted simple C*-Weyl algebra $\mathcal{W}(\mathcal{H}, \hbar\sigma)$.

The Glauber vectors $G(f)$ are indexed by square-integrable one-photon wave functions $f \in \mathcal{H}$. They describe the presence of "many" photons by owning infinitely many n-particle components. They induce via the scalar product method the *coherent states* on $\mathcal{W}_F(\mathcal{H}, \hbar\sigma)$ and also give the abstract coherent states ω_f on $\mathcal{W}(\mathcal{H}, \hbar\sigma)$. In Fock space, the arising *displacement field* is shown, however, not to represent a true classical field since it has a spontaneous transition probability to vanish.

Already the search for coherent states indexed by non-square-integrable functions transcends Fock space formalism. In order to obtain these states as well-defined states on an abstract Weyl algebra, we have to diminish the test function space from $\mathcal{H}$ to a convenient subspace $E \subset \mathcal{H}$, extending therewith the dual space E'. The basic idea of the present paper is to find an E such that $\mathcal{W}(E,\hbar\sigma)$ owns representations $\mathcal{W}_\Pi(E,\hbar\sigma)$, the weak closures of which (the representation von Neumann algebras $\mathcal{M}_\Pi$) give rise to field operators $\Phi_\Pi(f)$ which have a classical (central) component displaying features of the electrodynamic field.

For this purpose, we propose in Section 3 a transversal subspace $E \equiv \mathsf{S}_0^{3\top}$ of the Schwartz space $\mathsf{S}(\mathbb{R}^3,\mathbb{C}^3)$ of rapidly decreasing C^∞-functions which combines infrared regularization with nuclearity in the induced topology τ (and deviates from the E_Δ used in [HonRie15] for Weyl quantization in free space). The advantage is the applicability of the Bochner–Minlos integration on the dual test function space E'. A further advantage is the τ-boundedness of the generators for the free dynamics, space translations and rotations, restricted to this test function space.

The C*-Weyl algebra $\mathcal{W}(E,\hbar\sigma)$ is introduced in Section 3.1. By means of Einstein's one-photon dynamics, the algebraic Heisenberg dynamics is realized by a quasi-free automorphism group $\alpha_\mathbb{R}$, which is "global" in the sense that $\|\alpha_t - \mathrm{id}\| = 2$ for $t \neq 0$, a necessary condition for moving central observables in certain representations.

The generalized coherent states are specified in Section 3.3 by their characteristic functions in a representation-independent manner. In order to gain the *q-classical states*, one forms mixtures of the coherent states over the index space E'. For true radiation states, one needs a subspace of E' consisting solely of norm-unbounded (i.e. singular) functionals.

We demonstrate in Section 3.2 that the w*-closed, singular subspaces of E' are in bijective correspondence with the norm-dense, non-trivial subspaces $E_p \subset E$ arising as the pre-polars of the former. It turns out that E' cannot be decomposed into a *topological* direct sum of $\mathcal{H}$ and a w*-closed, singular subspace. The special Segal subspace $E_p = E_B$ [Segal62], used in [Rie2020] to index in terms of its polar E_B^0 the regular ground states for boson fields with quadratic interaction, is for the present photon system equal to all of E. This implies $E_B^0 = 0$, so that the photons own only the bare vacuum as ground state.

The anti-liminary photonic C*-Weyl algebra $\mathcal{W}(E,\hbar\sigma)$ allows for many non-equivalent representations, which physically have to be selected according to global subsidiary conditions, like temperature or mean particle density. (The represented dynamics depending on those subsidiary conditions is called *effective* in many body physics.) For photons we observe that every class of physical experiment on electrodynamic radiation is restricted to a finite frequency range, thus to a singular subspace $E'_{\mathrm{ess}} \subset E'$ which we assume w*-closed. The set of all q-classical states on $\mathcal{W}(E,\hbar\sigma)$ with Bochner–Minlos measures μ supported on E'_{ess} is shown to constitute a simplex and a stable face, features which point already to a state space of an autonomous classical subtheory. Its states, different from the vacuum, that are all disjoint to all Fock normal states

The GNS-representations over these singular q-classical states with measure μ, evaluated in Section 4, lead to representation von Neumann algebras with non-trivial center, if μ is not a point measure. Since the reference states are regular, even τ-continuous, the representation-dependent quantized field operators exist. They consist of the Fock field additively dressed by μ-mixed, singular, classical fields. The spatial central decomposition of these GNS-representations may be executed, in spite of $\mathcal{W}(E, \hbar\sigma)$ being non-separable. The corresponding sectors are indexed by the sharp singular fields which always are time dependent since $E_B = E$.

A peculiar aspect is that the Fock part of the field operator is split into two components according to the direct decomposition $E = E_p \dot{+} E_e$ with $E_e \subset \overline{E_p}^n$ a (non-unique) complement to $E_p = {}^0E'_{\text{ess}}$. We discuss why E_p indexes the "particle photons" and E_e the so-called "transient photons", and how similar the structures of the represented three-component photon field and the three physical entities in Einstein's picture are.

In Section 5 the dynamics and symmetries are extended to the representation-dependent von Neumann algebra $\mathcal{M}_\mu = \mathcal{L}(F_+(\mathcal{H})) \otimes \mathcal{Z}_\mu$ in W*-tensor product form. Since the symmetries (including the dynamics) act in the same manner on both subalgebras, we have in fact a *fusion* of a quantized particle theory and a classical field theory.

Nevertheless, we show in Section 6 that the classical generators may be brought into the form of the electrodynamic generators in canonical Maxwell theory if we apply a symplectic transformation which distributes, in terms of a new physical constant ϵ_0, the photon energy differently over the two real field components of the complex classical field.

In Section 7 we apply the mentioned symplectic transformation as a Bogoliubov transformation on the algebraic multi-photon theory and arrive in fact at the quantized transverse Maxwell theory, which in this manner is also based on the photon concept and arises in a concise mathematical formulation.

We discuss only briefly the general aspects of the obtained fusion between quantum and classical theory, which is most elegantly formulated in the convex state space approach.

In the appendix we collect some notions for algebraic states and transition probabilities, and supplement the calculations for the angular field momentum and for seminorm estimates. The final Appendix D contains some remarks and conclusions on non-relativistic gauge theory and supplements the longitudinal, cohomological, and scalar potentials, all of these still lacking a photonic foundation in the sense of a fusion theory.

A good deal of our mathematical results are also applicable to more general boson theories of actual interest (e.g. [McCabe, Duncan] and references therein).

We attempt to formulate the mathematical arguments and interpretational remarks in a manner which facilitates a broader readership, not so acquainted with operator algebraic field theory, to get at the main points of the reasoning in this combination of mathematical and theoretical physics.

§2. Photons in Fock space

§2.1. The photon observables

Inversely to the historical developments, we start from the point of view that experiments in a fixed inertial system have discovered a mass-0 quantum particle with spin 1, called a *photon*, and the collective structures of these bosonic particles have not been clarified yet. As the input for the theoretical analysis, we take solely the one-photon data, expressed in quantum mechanical terms, and proceed then to the traditional multi-photon theory in Fock space (without any reference to classical electrodynamics). The transition to the algebraic treatment of the multi-photon theory is guided and motivated by physical needs.

The complex one-particle Hilbert space of that photon must carry an irreducible representation of the rotation group SO(3) connected with spin 1, realizable by three-component wave functions $f \in L^2(\mathbb{R}^3, \mathbb{C}^3, d^3x)$. For the energy operator one finds the formula $B = \hbar c\sqrt{-\Delta}$. This is a self-adjoint strictly positive operator which acts on wave packets as $B \int_{\mathbb{R}^3} \exp\{ik \cdot x\} \hat{f}(k)\, d^3k = \int_{\mathbb{R}^3} \hbar c|k| \exp\{ik \cdot x\} \hat{f}(k)\, d^3k$ and displays an absolutely continuous spectrum (being the operator formulation of Einstein's historical photon energy for plane waves). The photonic momentum in direction $n \in \mathbb{R}^3$ (measurable by Compton scattering) evaluates as $n \cdot \mathbf{p} = p_n = -i\hbar\, n \cdot \nabla$. By means of group theory, the angular momentum about the axis n has then the form $\mathsf{L}_\mathsf{n} = -i\hbar n \cdot (\mathbf{I} - (x \times \nabla))$, where the x-independent part $n \cdot \mathbf{I} := n_1\ell_1 + n_2\ell_2 + n_3\ell_3$ refers to a basis (ℓ_ν) of the Lie algebra for SO(3) represented by 3×3-matrices (see Appendix B). Polarizability (used e.g. in two-photon EPR experiments) shows transversality and implies $\nabla \cdot f = 0$.

The mentioned observables identify the elementary particle "photon" and its one-particle Hilbert space $\mathcal{H} = P^\top L^2(\mathbb{R}^3, \mathbb{C}^3, d^3x)$, where $P^\top$ denotes the orthogonal projection onto the (closed) kernel of the div operator. (For a group-theoretical classification of one-particle spaces – a cornerstone for quantum field theory – cf. e.g. [Araki] and references therein.) The self-adjoint operators B, p_n, L_n commute with $P^\top$ and are considered self-adjoint operators on $\mathcal{H}$ (see also Section 3.1).

Established treatment leads then to the many-boson theory in the symmetrized Fock space $F_+(\mathcal{H}) := \bigoplus_{m=0}^{\infty} P_+(\bigotimes_m \mathcal{H})$, with P_+ the symmetrization operator and with the vector $\Omega_0 := 1 \oplus 0 \oplus 0 \oplus 0 \oplus \cdots \in P_+(\bigotimes_0 \mathcal{H}) \equiv \mathbb{C} \subset F_+(\mathcal{H})$ representing the *bare or Fock vacuum.*

One-particle observables K (for the moment dimensionless) are extended to the m-particle space by introducing $K_m := K \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \cdots \otimes K$, for $m \in \mathbb{N}$, an essentially self-adjoint operator on $P_+(\bigotimes_m \mathcal{H})$ (e.g. [ReedSim1, Thm. VIII.33]). Defining $K_0 := 0$, one forms $\bigoplus_{m=0}^{\infty} K_m|_{P_+}$, which is essentially self-adjoint on $F_+(\mathcal{H})$, with self-adjoint closure $d\Gamma(K)$. Especially for $K = \mathbf{1}$, we obtain the unbounded Fock particle number operator $N_F := d\Gamma(\mathbf{1})$ which makes explicit the particle character of the mass-0 particle. Conventionally, one expresses the $d\Gamma(K)$ in terms of the creation and annihilation operators, which for all $f, g \in \mathcal{H}$ satisfy the $\hbar$-independent commutation relations

$$[a_F(f), a_F^*(g)] \subset (f|g)\mathbf{1}_{F_+}, \quad [a_F(f), a_F(g)] \subset 0 \supset [a_F^*(f), a_F^*(g)], \tag{2.1}$$

and act on the vacuum as

$$a_F(f)\Omega_0 = 0 \equiv 0 \oplus 0 \oplus 0 \oplus \cdots, \quad a_F^*(f)\Omega_0 = f \equiv 0 \oplus f \oplus 0 \oplus 0 \oplus \cdots. \tag{2.2}$$

Proposition 2.1. (a) *Let $\{e_n\}_{\mathbb{N}}$ be an orthonormal basis in the domain of the self-adjoint K on $\mathcal{H}$. Then the so-called "second quantization" of K is given by*

$$d\Gamma(K) = \sum_{n \in \mathbb{N}} a_F^*(Ke_n) a_F(e_n) = \sum_{n,m \in \mathbb{N}} a_F^*(e_m)(e_m|Ke_n) a_F(e_n), \tag{2.3}$$

where the infinite sum is the strong resolvent limit of the closed finite-sum operators.

(b) *One gets* $\exp\{is\, d\Gamma(K)\} a_F^{(*)}(f) \exp\{-is\, d\Gamma(K)\} = a_F^{(*)}(\exp\{isK\}f)$*, which reduces the many-particle transformations generated by* $d\Gamma(K)$ *to the unitary single-particle transformations* $\exp\{i\mathbb{R}K\}$ *on* $\mathcal{H}$.

(c) *The mapping $K = K^* \mapsto d\Gamma(K)$ is $\mathbb{R}$-linear, and $d\Gamma(K)$ is positive, if K is so.*

(d) *If the self-adjoint K_1 and K_2 commute on $\mathcal{H}$, then $d\Gamma(K_1)$ and $d\Gamma(K_2)$ commute on $F_+(\mathcal{H})$ (expressed by commuting spectral projections).*

(e) *If on $\mathcal{H}$ we have $K_n \to K$ in the strong resolvent sense for self-adjoint operators, then $d\Gamma(K_n) \to d\Gamma(K)$ in the strong resolvent sense on $F_+(\mathcal{H})$.*

For a given $U \in \mathcal{L}(\mathcal{H})$, the m-fold tensor product $\bigotimes_m U$ is a bounded operator on $P_+(\bigotimes_m \mathcal{H})$. Provided $U \in \mathcal{L}_1(\mathcal{H})$ (meaning $\|U\| \leq 1$), we may introduce $\Gamma(U) := \bigoplus_{m=0}^{\infty} \bigotimes_m U|_{P_+}$, with $\bigotimes_0 U|_{P_+} := 1 \in \mathbb{C}$. Thus $\Gamma(U)$ is in $\mathcal{L}_1(F_+(\mathcal{H}))$.

For $U, V \in \mathcal{L}_1(\mathcal{H})$, it holds that $\Gamma(U^*) = \Gamma(U)^*$, $\Gamma(UV) = \Gamma(U)\Gamma(V)$, $U \mapsto \Gamma(U)$ is strong–strong continuous and $\Gamma(\exp\{i\mathbb{R}K\}) = \exp\{i\mathbb{R}d\Gamma(K)\}$ for all self-adjoint K on $\mathcal{H}$.

The particle number operator N_F generates the first-kind gauge transformations $\Gamma(\exp\{is\}\mathbf{1}) = \exp\{isN_F\}$, $s \in \mathbb{R}/[2\pi]$ on $F_+(\mathcal{H})$ and leaves the vacuum vector invariant, as do all $\Gamma(U)$.

§2.2. Fock fields and q-classical states

The self-adjoint operator, given as the sum of two unbounded closed operators,

$$\Phi_F^\hbar(f) := \frac{\sqrt{\hbar}}{\sqrt{2}}(\overline{a_F(f) + a_F^*(f)}) \quad \forall\, f \in \mathcal{H}, \tag{2.4}$$

is called a *field operator*. It constitutes an operator-valued functional $f \mapsto \Phi_F^\hbar(f)$ and satisfies the $\hbar$-dependent CCR

$$[\Phi_F^\hbar(f), \Phi_F^\hbar(g)] \subset i\hbar\sigma(f,g)\mathbf{1}_{F_+} \quad \forall\, f, g \in \mathcal{H}, \tag{2.5}$$

on a dense domain in $F_+(\mathcal{H})$, if one sets $\sigma(f,g) := \mathrm{Im}(f|g)$. The unitary operators $W_F^\hbar(f) := \exp\{i\Phi_F^\hbar(f)\} \in \mathcal{L}(F_+(\mathcal{H}))$ represent the *abstract algebraic Weyl relations*

$$W(f)W(g) = \exp\{-\tfrac{i}{2}\hbar\sigma(f,g)\}W(f+g), \quad W(f)^* = W(-f), \quad \forall\, f, g \in \mathcal{H}. \tag{2.6}$$

For algebraic Weyl theory it is useful to restrict the wave function arguments to a (complex) norm-dense subspace E of the one-particle Hilbert space $\mathcal{H}$ which we describe for photons in Section 3; E must be invariant under the basic symmetry transformations $\exp\{i\mathbb{R}K\}$, $K \in \{B, p_n, \mathsf{L_n}\}$.

Definition 2.2. We restrict the algebraic Weyl relations of equation (2.6) to the "test functions" $f \in E$. We introduce the abstract *-algebra $\Delta(E, \hbar\sigma) := \mathrm{LH}\{W(f) \mid f \in E\}$. From the Fock representation we have a C*-norm, i.e. a Banach-algebra norm with $\|A^*A\| = \|A\|^2$ for all $A \in \Delta(E, \hbar\sigma)$ (where in some sense this norm is unique). The completion in norm leads to the C*-Weyl algebra

$$\mathcal{W}(E, \hbar\sigma) := \overline{\Delta(E, \hbar\sigma)}^{\|\cdot\|}. \tag{2.7}$$

One knows that $\mathcal{W}(E, \hbar\sigma)$ is simple and that the (faithful) Fock-represented Weyl algebra $\mathcal{W}_F(E, \hbar\sigma) = \Pi_F(\mathcal{W}(E, \hbar\sigma))$ is irreducible and does not contain a non-trivial compact operator. This indicates that $\mathcal{W}(E, \hbar\sigma)$ is anti-liminary (see, e.g., [Peders79]) and owns infinitely many non-equivalent irreducible representations.

Notational Remark 2.3. The introduction of the Planck constant $\hbar$ in equation (2.4) is necessary if the field operator $\Phi_F^\hbar(f)$ arises from canonical quantization. It is, however, common in dealing with Fock space operators, where equation (2.1) is considered to be the basic commutation relation, to set $\hbar = 1$. We join this usage if not stated otherwise, and denote $W_F^{\hbar=1}(f) \equiv W_F(f)$. Thus $W_F(f)$ is the Fock representative $\Pi_F(W(f))$ of the abstract $W(f) \in \mathcal{W}(E, \sigma)$ for $f \in E$.

The (vector norm) closure of $\mathcal{W}_F(\mathcal{H}, \sigma)\Omega_0$ contains all $a_F^*(f_1) \dots a_F^*(f_n)\Omega_0$ and equals $F_+(\mathcal{H})$, so $\mathcal{W}_F(\mathcal{H}, \sigma)$ owns Ω_0 as a cyclic vector. The same can be said for $\mathcal{W}_F(E, \sigma)$, because $f \mapsto W_F(f)$ is strong–strong continuous.

Since we know $\exp\{i\mathbb{R}d\Gamma(K)\}\Phi_F(f)\exp\{-i\mathbb{R}d\Gamma(K)\} = \Phi_F(\exp\{i\mathbb{R}K\}f)$, it is appropriate to define the abstract symmetry transformations in terms of the so-called "quasi-free" automorphisms

$$(2.8) \qquad \gamma_s^K(W(f)) := W(\exp\{isK\}f) \quad \forall\, s \in \mathbb{R},\ \forall\, f \in E.$$

This defines γ_s^K by linear extension on $\Delta(E, \hbar\sigma)$, and by norm-continuous extension on $\mathcal{W}(E, \sigma)$. Via the Baker–Hausdorff formula we obtain

$$(2.9) \qquad W_F(f) = \exp\{-\tfrac{1}{4}\|f\|^2\}\exp\{\tfrac{i}{\sqrt{2}}a_F^*(f)\}\exp\{\tfrac{i}{\sqrt{2}}a_F(f)\},$$

which is valid in application to analytic vectors for $a_F(f)$ (on which the power series in $a_F(f)$ exists), and obtain

$$(2.10) \qquad (\Omega_0|W_F(f)\Omega_0)_{F_+} = \exp\{-\tfrac{1}{4}\|f\|^2\} =: \langle\omega_0; W(f)\rangle \quad \forall\, f \in E.$$

This is the *characteristic function* of the vacuum viewed as a state ω_0 on the abstract Weyl algebra$\mathcal{W}(E, \sigma)$. Quite generally, a state ω on $\mathcal{W}(E, \sigma)$ is defined by its expectation values $\langle\omega; A\rangle$, $A \in \mathcal{W}_F(E, \sigma)$ (a point of view already expressed in the seminal book of von Neumann [vNeum]). According to their statistical–physical meaning, these expectation value functionals have to act linearly on $\mathcal{W}(E, \sigma)$ and must satisfy $\langle\omega; \mathbf{1}\rangle = 1$, as well as $\langle\omega; A^*A\rangle \geq 0$. Then they are automatically continuous with respect to the algebraic norm.

The great advantage of bosonic theory in terms of the Weyl algebra is that for determining a state ω it suffices to already know $C_\omega(f) := \langle\omega; W(f)\rangle$ for all $f \in E$, i.e. the *characteristic function* (because it gives all expectations on $\mathcal{W}(E, \sigma)$ by linear and norm-continuous extension). Applying the Weyl relations one obtains the *twisted positive-definiteness* condition

$$(2.11) \qquad \sum_{i,j=1}^{n} \overline{z_i} z_j \exp\{\tfrac{i}{2}\hbar\sigma(f_i, f_j)\}C_\omega(f_j - f_i) \geq 0, \quad n \in \mathbb{N},\ f_i \in E,\ z_j \in \mathbb{C}.$$

Formula (2.10) shows that ω_0 is a C^∞-state, for which the characteristic function is infinitely differentiable. The C^2-property already implies $\Delta_F(E,\sigma)\Omega_0 \subset \operatorname{dom}\Phi_F(f)$ for all $f \in E$. On account of this, we obtain on a dense domain in the Fock representation, comprising $\Delta_F(E,\sigma)\Omega_0$, for the symmetry generators in the Heisenberg picture,

$$\begin{aligned}\frac{d}{ids} W_F(\exp\{isK\}f)|_{s=0} &= [d\Gamma(K), W_F(\exp\{isK\}f)]|_{s=0} \\ &= \Phi_F(iKf)W_F(f) \quad \forall f \in \operatorname{dom} K.\end{aligned} \tag{2.12}$$

The *displaced vacua* or *coherent states* are defined in terms of the *Glauber vectors*

$$W_F(\tfrac{\sqrt{2}}{i}f)\Omega_0 = \exp\{-\tfrac{1}{2}\|f\|^2\} \bigoplus_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \Big(\bigotimes_m f \Big) \tag{2.13}$$

$$= D_F(f)\Omega_0 =: G(f), \quad D_F(f) := W_F(\tfrac{\sqrt{2}}{i}f), \tag{2.14}$$

where $D_F(f)$ is called the *displacement operator*. For $f = 0$ one gets $G(0) = \Omega_0$.

Proposition 2.4. *The following results are derivable from the series expansion equation* (2.13) *(see the textbooks on quantum optics for the one-mode case and [HonRie15, Sect. 18.5] for the general case):*

(a) $a_F(f)G(g) = (f|g)G(g)$ *for all* $f, g \in \mathcal{H}$.

(b) $(G(g)|d\Gamma(K)G(g))_{F_+} = (g|Kg)$ *for all* $K = K^*$*, for all* $g \in \operatorname{dom} K$.

(c) $(G(f)|G(g))_{F_+} = \exp\{(f|g) - \frac{1}{2}\|f\|^2 - \frac{1}{2}\|g\|^2\} \neq 0$ *for all* $f, g \in \mathcal{H}$.

(d) *The mapping* $\mathcal{H} \ni f \mapsto G(f) \in F_+(\mathcal{H})$ *is norm–norm continuous (but is not linear).*

(e) *The vectors* $G(f)$*,* $f \in \mathcal{H}$ *are total in* $F_+(\mathcal{H})$.

(f) *We have*

$$W_F(f)G(g) = \exp\{i2^{-1/2}\operatorname{Re}(f|g)\}G(g + i2^{-1/2}f) \quad \forall f, g \in \mathcal{H}. \tag{2.15}$$

(g) *The linear hull of* $G(g)$ *equals* $\Delta_F(E,\sigma)\Omega_0$ *and is a core for* $\Phi_F(g)$ *for all* $g \in \mathcal{H}$.

For $f, g \in \mathcal{H}$, one obtains from equations (2.6) and (2.14), and differentiation,

$$\begin{aligned} D_F(f)W_F(g)D_F(f)^* &= W_F(g)\exp\{i\sqrt{2}\operatorname{Re}(f|g)\}, \\ D_F(f)\Phi_F(g)D_F(f)^* &= \Phi_F(g) + \sqrt{2}\operatorname{Re}(f|g)\mathbf{1}_{F_+}. \end{aligned} \tag{2.16}$$

The additive "displacement" $\sqrt{2}\operatorname{Re}(g|f)$ in equation (2.16) has the shape of a real-valued – and apparently "classical" – smeared field, arising by a kind of dequantization from the quantized field operator. Quantized boson theory seems to provide per se classical field expressions which are norm bounded as required by the strong–strong continuity of $f \mapsto W_F(f)$. It seems that the Glauber vectors, respectively the associated *coherent states*, have a special affinity to classical fields. We obtain the characteristic functions of the *coherent states* as

$$\begin{aligned} \langle\omega_g; W(f)\rangle &:= (G(g)|W_F(f)G(g))_{F_+} = \exp\{-\tfrac{1}{4}\|f\|^2\}\exp\{i\sqrt{2}\,\mathrm{Re}(g|f)\} \\ &= C_0(f)\exp\{i\sqrt{2}\,\mathrm{Re}(g|f)\} =: C_g(f) \quad \forall\, g,f \in E \end{aligned} \tag{2.17}$$

(use Baker–Hausdorff and Proposition 2.4(a)).

Also from Proposition 2.4(a), we deduce the strong cluster relation

$$\begin{aligned} &(G(g)|a_F^*(f_1)\cdots a_F^*(f_m)a_F(f_{m+1})\cdots a_F(f_{m+n})G(g))_{F_+} \\ &\quad = (g|f_1)\cdots(g|f_m)\overline{(g|f_{m+1})}\cdots\overline{(g|f_{m+n})} \end{aligned} \tag{2.18}$$

for all $1 \leq m,n < \infty$. This covers *all-order optical coherence* given by values $m = n \in \mathbb{N}$. The (smeared) *coherence function* equals here the complex field expectation $(G(g)|a_F^*(f)G(g))_{F_+} = (g|f)$. In classical coherence theory, that factorization signifies the presence of a sharp, non-fluctuating complex field, belonging to the real field $\sqrt{2}\,\mathrm{Re}(g|f) = (G(g)|\Phi_F(f)G(g))_{F_+}$ (whereas for the quantized field $\Phi_F(f)$ the vacuum fluctuations have to be taken into account). Note that here the clustering does not mean disorder for the state, rather it expresses a robustness against perturbations and thus an extreme form of ODLRO (= off-diagonal long-range order [Sewell02]), so to speak.

Definition 2.5. We call a state ω_ϱ on $\mathcal{W}_F(E,\sigma)$, representable by a density operator $\varrho \in \mathcal{T}_1^+(F_+)$, a *Fock normal state* and write $\mathcal{S}_{Fn}$ for the convex set of all Fock normal states.

If we consider *mixtures* of the coherent states ω_g, we arrive at the so-called "classical states". Since this naming has led to severe misunderstandings, both in the applied and fundamental physical literature (e.g. [McCabe, Duncan] and references therein), we found it necessary to distinguish these states from the states occurring in a classical field theory and call the first ones "q-classical" (where the "q" may be read as "quasi", cf. [CohTann], or as "quantum", cf. [HonRie15]).

Definition 2.6. A state $\omega \in \mathcal{S}_{Fn}$ is "q-classical" if its ϱ has the shape (see [DaviesOp])

$$\varrho \equiv \varrho_\mu = \int_{\mathcal{H}} |G(h))(G(h)|\, d\mu(h) \tag{2.19}$$

for some $\mu \in M_p(\mathcal{H})$, a probability measure on the Borel sets $\mathsf{B}(\mathcal{H})$ induced by the strong vector topology on $\mathcal{H}$. The integral is performed in the trace-norm topology on $\mathcal{T}_1^+(F_+)$ and produces in fact a density operator on Fock space.

The set of all Fock-normal q-classical states, comprising the coherent states, is denoted by $\mathcal{S}_{\mathrm{Fcl}} \subset \mathcal{S}_{Fn}$.

Observation 2.7. The characteristic function of an $\omega_\varrho \in \mathcal{S}_{\mathrm{Fcl}}$ has the form

$$C_\varrho(f) \equiv C_\mu(f) = \int_{\mathcal{H}} \exp\{-\tfrac{1}{4}\|f\|^2\} \exp\{i\sqrt{2}\,\mathrm{Re}(h|f)\}\, d\mu(h) \tag{2.20}$$

for all $f \in \mathcal{H}$ and some $\mu \in M_p(\mathcal{H})$.

This follows since the integrand is continuous and bounded as a function of $h \in \mathcal{H}$. Thus we can evaluate $\mathrm{tr}[\varrho W_F(f)]$ by exchanging the order of tracing and integration and we arrive via (2.17) at equation (2.20).

From Proposition 2.4(c) it follows, however, that there is a non-vanishing probability amplitude $(G(g)|G(0))_{F_+} = \exp\{-\frac{1}{2}\|g\|^2\}$ for going over to the vacuum and, therewith, for the spontaneous vanishing of the displacement field $\sqrt{2}\,\mathrm{Re}(g|f)$.

This is to be considered a severe obstruction to the interpretation of the displacement field as a proper classical radiation field, notwithstanding the facts that a coherent state involves infinitely many photons and exhibits a strong internal ordering. A classical field surely should constitute a stable, objective physical entity.

This counterargument dissolves if we are able to include in our photon theory "singular coherent states ω_g" in which g has reached an infinite norm, that is, in which $f \mapsto (g|f)$ has been replaced by an unbounded complex-linear form $f \mapsto L(f)$. That limiting procedure is even required by the physical meaning of a radiation field per se, which typically is not square integrable. According to Proposition 2.4(b) it makes the $(G(g)|d\Gamma(K)G(g))_{F_+}$ singular, especially the photon number for $K = \mathbf{1}$. Such an extension transcends principally Fock space formalism and leads straightforwardly to algebraic quantum field theory. We have to look for further representations of an adapted abstract Weyl algebra.

§3. The C*-algebraic many-photon theory

§3.1. The test function space and C*-Weyl algebra

One knows that the Fock representation is not sufficient for all applications and that one has, to obtain the physically desirable representations, to reduce the test function space for the smeared Weyl operators from a Hilbert space $\mathcal{H}$ to an appropriate locally convex (LC-) space E, norm dense in $\mathcal{H}$. According to the

Wightman axioms [StrWigh], one could be tempted to suggest for the photon theory in non-relativistic formulation the Schwartz space of rapidly decreasing C^∞-functions $\mathsf{S}(\mathbb{R}^3, \mathbb{C}^3)$. The problem is that already the free photon dynamics $v_{\mathbb{R}} = \exp\{i\mathbb{R}B\}$ does not leave this test function space invariant and requires an *infrared regularization*. This severe complication cannot be avoided if one wants a photon algebra fitting the dynamics.

In [BogShir] the ultraviolet regularization is achieved by using only those test functions for smearing the time-ordered, or retarded, n-point functions of the S-matrix expansion, which vanish to a certain order at $(x, t) = 0$, that is at the coinciding space-time arguments.

For the infrared regularization we have to require a vanishing of the Fourier-transformed test functions $\hat{f} \in \hat{\mathsf{S}}(\mathbb{R}^3, \mathbb{C}^3)$ at $k = 0$, and this to infinite order. In detail, we want to equip the three-component Schwartz space $\mathsf{S}(\mathbb{R}^3, \mathbb{C}^3) \equiv \mathsf{S}^3$ with the locally convex direct sum topology τ^3 (e.g. [Schaef66]) of the three one-component spaces $\mathsf{S}(\mathbb{R}^3, \mathbb{C}) \equiv \mathsf{S}^1_\nu$, $1 \le \nu \le 3$. We denote in accordance with [Schaef66] $\mathsf{S}^3 = \bigoplus_\nu \mathsf{S}^1_\nu$ and write $\pi_\nu \colon \mathsf{S}^3 \to \mathsf{S}^1_\nu$ for the projections. (The inductive topology τ^3 is the finest topology, so that the injection maps $\pi_\nu^{-1} \colon \mathsf{S}^1_\nu \to \mathsf{S}^3$ are continuous.) By [Schaef66, I, 2.1], the π_ν are continuous too.

The usual LC-topology τ^1 of S^1 is defined by the following system of seminorms (e.g. [ReedSim1]). We use the notation $\mathbb{R}^3 \ni x = (x_1, x_2, x_3) \equiv (x_\nu)$, $x^2 = \sum_{\nu=1}^3 x_\nu^2$, and $|x| = \sqrt{x^2}$. We define in $\mathsf{A} := \mathbb{N}_0^3 \ni \alpha$ the componentwise sum, write $|\alpha| = \sum_{\nu=1}^3 \alpha_\nu$, and set $\alpha \ge \alpha'$ if $\alpha_\nu \ge \alpha'_\nu$ for $1 \le \nu \le 3$. Denoting $x^\alpha = \prod_{\nu=1}^3 x_\nu^{\alpha_\nu}$ and $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_3^{\alpha_3}$, this gives $x^\alpha x^{\alpha'} = x^{\alpha+\alpha'}$ and $D^\alpha \circ D^{\alpha'} = D^{\alpha+\alpha'}$.

For all $\alpha, \beta \in \mathsf{A}$ a seminorm on S^1 is defined by

$$\|\varphi\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^3} |x^\alpha D^\beta \varphi(x)|, \quad \varphi \in \mathsf{S}^1. \tag{3.1}$$

In this manner, (S^1, τ^1) is a (separable, complete) Fréchet space. A function $\varphi \colon \mathbb{R}^3 \to \mathbb{C}$ is in S^1 if and only if $\|\varphi\|_{\alpha,\beta} < \infty$ for all $\alpha, \beta \in \mathsf{A}$ or, equivalently, if and only if it is C^∞ and decreases with all its derivatives faster than any negative power for $|x| \to \infty$. By transforming S^1 into its so-called N-representation it is shown to be a nuclear space [Hida80, ReedSim1]. According to [Schaef66, III, 7.4], S^3 is a nuclear Fréchet space too.

Defining the Fourier transform $\hat{\varphi}(k) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \exp\{-ix \cdot k\} \varphi(x)\, d^3x$ for the components, we obtain the k-space representation $\hat{\mathsf{S}}^1$ of S^1 which consists of the same set of rapidly decreasing C^∞-functions and is equipped with the same topology as S^1 (see [ReedSim2]).

Since $\pi\colon \mathsf{S}^3 \xrightarrow{\text{onto}} \bigoplus_\nu \mathsf{S}^1_\nu$ is by construction a homeomorphism, we may denote $\mathsf{S}^3 \ni f(x) = (\varphi_1(x), \dots, \varphi_3(x)) \equiv (\varphi_\nu(x))$ with $\varphi_\nu := \pi_\nu f$. The net $(f_i)_I \subset \mathsf{S}^3$ converges in τ^3 to $f \in \mathsf{S}^3$, written $f_i \to f$, if and only if $\varphi_{\nu,i} \to \varphi_\nu$ for $1 \le \nu \le 3$. The componentwise Fourier transform $\hat{f}$ is given by $\hat{f}(k) = (\hat{\varphi}_\nu(k))$ and defines $\hat{\mathsf{S}}^3$.

Definition 3.1. We set $\hat{\mathsf{S}}^1_0 := \{\hat{\varphi} \in \hat{\mathsf{S}}^1 \mid (D^\beta \hat{\varphi})(0) = 0 \;\forall\, \beta \in \mathsf{A}\}$ and continue to denote by τ^1 its topology induced from $\hat{\mathsf{S}}^1$. We write $\hat{\mathsf{S}}^3_0$ for the locally convex direct sum of the $\hat{\mathsf{S}}^1_0$ with the inductive topology τ^3.

Finally, we define our transversal photonic test function space in the Fourier representation as $\hat{E} := \hat{\mathsf{S}}^{3\top}_0 = \{P^\top \hat{f} \mid \hat{f} \in \hat{\mathsf{S}}^3_0\}$, with $(P^\top \hat{f}) := -\frac{k}{|k|} \times (\frac{k}{|k|} \times \hat{f})$ (where we use mute variables in multiplication operators).

Observation 3.2. We observe that $\varphi \in \hat{\mathsf{S}}^1_0$ implies $|\varphi| \in \hat{\mathsf{S}}^1_0$. If $\hat{\varphi}$ is in $\hat{\mathsf{S}}^1_0$ then $|k|^p \hat{\varphi}$ exists (as $\lim_{k\to 0}$) for all $p \in \mathbb{Z}$. This is because $|k|^p \hat{\varphi}(k) \le (k^2)^p |\hat{\varphi}(k)|$ for $p < 0$ and $0 < |k| < 1$, which tends to 0 for $k \to 0$, because of the l'Hospital rule. By equation (C.1) we know then that for $\hat{f} \in \hat{\mathsf{S}}^3_0$, also $|k|^p \hat{f} \in \hat{\mathsf{S}}^3_0$ for $p \in \mathbb{Z}$, and that, therefore, the $\mathcal{H}$-norms $\|\, |k|^p \hat{f}\| =: \|f\|_p$ exist.

This shows that the present $(\hat{E}, \tau) = (\hat{\mathsf{S}}^{3\top}_0, \tau)$ is a (not topological) subspace of the Fourier-transformed $E_\Delta + iE_\Delta$ in [HonRie15, I, equation (10.2.1)], where the latter is topologized by the seminorms $\|f\|_p$, $p \in 2\mathbb{Z}$, and arose from smearing fields in rather general cavities, but is not nuclear.

Theorem 3.3. (a) *The test function space $\hat{E} = P^\top \hat{\mathsf{S}}^3_0$ is a well-defined subspace of $\hat{\mathsf{S}}^3_0 \subset \hat{\mathsf{S}}^3$ and constitutes with the topology τ, induced from $(\hat{\mathsf{S}}^3_0, \tau^3)$, a nuclear Fréchet space (not locally compact, being infinite-dimensional).*

Via inverse Fourier transformation we obtain the same topological assertions in the position space representation for $E = P^\top \mathsf{S}^3_0 \subset \mathsf{S}^3_0 \subset \mathsf{S}^3$ (where here $P^\top = (-\Delta)^{-1}\operatorname{curl}^2$ and the subscript "0" is defined by the behavior of the Fourier-transformed functions).

(b) *The topology τ is strictly stronger than the norm topology on E, and E is a norm-dense strict subspace of $\mathcal{H} := P^\top L^2(\mathbb{R}^3, \mathbb{C}^3, d^3x)$ so that the scalar product $(\cdot|\cdot)$ of $\mathcal{H}$ is jointly τ-continuous.*

(c) *After embedding (without notation), E and $\mathcal{H}$ are w^*-dense in the dual space E' of τ-continuous, $\mathbb{C}$-linear functionals on E, and $E \subset \mathcal{H} \subset E'$ constitutes a Gelfand triple.*

Proof. (a) Clearly, $\hat{\mathsf{S}}^3_0$ is a $\mathbb{C}$-linear subspace of $\hat{\mathsf{S}}^3$ and is then nuclear (see [Schaef66, III, 7.4]). It is also τ^3-closed, since in $\hat{\mathsf{S}}^1$ the τ^1 convergence $\hat{\varphi}_i \to \hat{\varphi}$ implies for all $\beta \in \mathsf{A}$ that $\sup_k |D^\beta \hat{\varphi}_i(k) - D^\beta \hat{\varphi}(k)| \to 0$, which gives $D^\beta \hat{\varphi}(0) = 0$ for that smooth function, if the $\hat{\varphi}_i$ are in $\hat{\mathsf{S}}^1_0$.

By Observation 3.2 we have $\frac{k_\nu}{|k|}\varphi_{\nu'} \in \hat{\mathsf{S}}^1_0$ for $\hat{\varphi}_{\nu'} \in \hat{\mathsf{S}}^1_0$, a necessary and sufficient condition for $P^\top \hat{f} \in \hat{\mathsf{S}}^3_0$ if $\hat{f} \in \hat{\mathsf{S}}^3_0$.

(b) Since $\hat{\mathsf{S}}^3_0$ contains the functions of compact support $C^\infty_c(\mathbb{R}^3\backslash 0)$, it is norm dense in $L^2(\mathbb{R}^3, \mathbb{C}^3, d^3x)$ and thus E is norm dense in $\mathcal{H} := P^\top L^2(\mathbb{R}^3, \mathbb{C}^3, d^3x)$.

(c) The topology τ is clearly stronger than norm, and, therefore, $\overline{E}^\tau$ is strictly contained in $\mathcal{H}$, and the scalar product $(\cdot|\cdot)$ of $\mathcal{H}$ is jointly τ-continuous. It holds that $\{g \in E \mid (g|f) = 0 \;\forall\, f \in E\}$ equals 0 since E is norm dense in $\mathcal{H}$, and this in turn implies the w*-denseness of E in E' by means of [Conway85, Cor. IV, 3.1.4]. This gives our Gelfand triple. □

The proof of the following theorem is given in Appendix C, where we use that $K\colon E \to E$ is by definition *τ-continuous* if for any $\alpha, \beta \in \mathsf{A}$ there are $c > 0$, $\alpha_n, \beta_n \in \mathsf{A}$, so that $\|Kf\|_{\alpha,\beta} \le c\sum_{n=1}^N \|f\|_{\alpha_n,\beta_n}$ for an $N \in \mathbb{N}$ and all $f \in E$ (e.g. [Schaef66, Chap. III]).

Theorem 3.4. (a) *The groups* $\exp\{i\mathbb{R}K\}$*, with* $K \in \{B, p_n, \mathsf{L}_\mathsf{n}\}$*, act τ-continuously on E, and depend pointwise τ-continuously on $s \in \mathbb{R}$.*

(b) *The named generators K are τ-continuous too, as is also $P^\top$. The K are unbounded self-adjoint on $\mathcal{H}$ and own E as a common core.*

Notational Remark 3.5. We do not perform consequent bookkeeping for the physical dimensions and mostly replace $\hbar$ by 1. But for $K \in \{B, p_n, \mathsf{L}_\mathsf{n}\}$, equipped with the correct physical dimensions, the quotients $Ks/\hbar$ are dimensionless if s obtains the dimensions sec, cm, 1, respectively.

That means that, in any case, $\hbar$ drops out from the symmetry generators $K/\hbar$, which we denote by the same symbols in both the x- and k-representations.

Proposition 3.6. *Let $\gamma^K_s, s \in \mathbb{R}$ denote the quasi-free automorphism on $\mathcal{W}(E, \sigma)$ that is induced by* $\exp\{isK\}$*, where K is self-adjoint on $\mathcal{H}$, which leads to* $\exp\{isK\}\colon E \to E$.

For each such non-trivial γ^K_s, $s \in \mathbb{R}$, it holds that $\|\gamma^K_s - \mathrm{id}\| = 2$, where id *is the identity automorphism on $\mathcal{W}(E, \sigma)$ and the norm refers to the Banach space of bounded linear transformations $\mathcal{L}(\mathcal{W}(E, \sigma))$.*

Therefore, γ^K_s is not "universally weakly inner" but acts "globally" on the photonic field algebra so that it may move central observables in von Neumann algebras of GNS-representations.

Proof. Relation $\sup\{\|(\gamma^K_s - \mathrm{id})(A)\| \mid \|A\| \le 1\}$ can be calculated directly by inserting for A all finite sums $\sum_k c_k W(f_k)$ with the $c_k \in \mathbb{C}$ satisfying $\sum_k |c_k| \le 1$, a norm-dense subset of the unit ball by construction of the Weyl algebra. The supremum 2 is reached for all $A = W(f)$ with $\exp\{isK\}f \neq f$, since $\|W(g) - W(f)\| = 2$ for $g \neq f$.

From [KadR83, Exe. 10.5.74] (see also [KadR67]) we infer that for each automorphism α it holds that $\|\alpha - \mathrm{id}\| \leq 2$, and that for $\|\alpha - \mathrm{id}\| < 2$, α is *universally weakly inner* as described in Appendix A.1. □

Let us mention in passing that the photon field dynamics $\alpha_{\mathbb{R}} = \gamma^B_{\mathbb{R}}$ exhibits the strong ergodic condition of asymptotic abelianness (via the absolute continuous spectrum of B), and the $\gamma^K_{\mathbb{R}}$, with $\ker K = 0$, the weaker condition of $\mathbb{R}$-centrality (cf. [Rie2020, App. A]).

Remark 3.7. Without giving details, let us remark that the choice of the τ-closed subspace $\hat{\mathsf{S}}^3_0 \subset \hat{\mathsf{S}}^3$ for an infrared regularization leads to a direct topological decomposition $\hat{\mathsf{S}}^3 = \hat{\mathsf{S}}^3_0 \dot{+} \hat{\mathsf{S}}^3_c$ with dual decomposition $(\hat{\mathsf{S}}^3)' = (\hat{\mathsf{S}}^3_0)' \dot{+} (\hat{\mathsf{S}}^3_0)^0$, involving the polar $(\hat{\mathsf{S}}^3_0)^0$ which consists of all tempered distributions supported on $k = 0$.

According to [BogShir, HeppReg], renormalization would mean in the present context the extension of $L \in (\hat{\mathsf{S}}^3_0)'$ to an $L_{\mathrm{ren}} \in (\hat{\mathsf{S}}^3)'$, which would require the adaption of infinitely many renormalization constants (all the more for the n-point functions of the quantized field, given as distributions over the tensorized copies of $\hat{\mathsf{S}}^3_0$). But the radiation fields under consideration are in $P^\top(\hat{\mathsf{S}}^3_0)'$ and cannot be extended to $P^\top(\hat{\mathsf{S}}^3_0)^0$ since the latter is not defined. Therefore, there is no infrared renormalization problem for the present transversal fields.

§3.2. Singular classical field spaces

We are going to analyze GNS-representations over general *q-classical states* on $\mathcal{W}(E, \hbar\sigma)$ and *again set* $\hbar = 1$, if not stated otherwise. These states are defined through the integral decompositions of their characteristic functions indicating mixtures of coherent states, now over the extended integration space $E' \supset \mathcal{H}$. According to the Bochner–Minlos theorem (e.g. [Hida80]), each τ-continuous, normalized, positive-definite function $P\colon E \to \mathbb{C}, P(0) = 1$, on the nuclear Fréchet space E, that is, $P \in \mathcal{P}_\tau(E)$, has a Fourier decomposition

$$P(f) = \int_{E'} \exp\{i\sqrt{2}\,\mathrm{Re}\, L(f)\}\, d\mu(L), \quad f \in E, \tag{3.2}$$

in terms of a unique probability measure $\mu \in M_p(E')$. The pertinent σ-algebra $\Sigma(E', E)$ is defined as the smallest σ-algebra which contains all cylinder sets $U(f_1, \dots, f_n; \Lambda) := \{L \in E' \mid (L(f_1), \dots, L(f_n)) \in \Lambda\}$, where $n \in \mathbb{N}$, Λ is a Borel set in $\mathbb{C}^n$, and $f_i \in E$. The cylinder sets are w*-Borel sets in E' and $\Sigma(E', E) \subset \mathsf{B}(\mathsf{E}')$.

Conversely, each integral of the form equation (3.2) produces a $P \in \mathcal{P}_\tau(E)$.

If Y is a subspace of E' then we apply the above construction to obtain the σ-algebra $\Sigma(Y,E)$, via the cylinder sets $U_Y(f_1,\dots,f_n;\Lambda) := \{L \in Y \mid (L(f_1),\dots, L(f_n)) \in \Lambda\}$. If $Y = \mathcal{H}$, then $\Sigma(\mathcal{H},E)$ equals the Borel sets of $\mathcal{H}$ in the strong topology on $\mathcal{H}$ (which follows from [DaviesOp, Thm. 11.2]).

Definition 3.8. Denote by E'_s the set of all "singular fields" given by the norm-unbounded functionals in E' (being no subspace of E'). For a linear subspace $Y \subset E'$ we write $Y \subset E'_s$ if $Y \neq 0$ and if for all $0 \neq L \in Y$ it holds that $L \in E'_s$.

The norm-bounded functionals from $\mathcal{H} \subset E'$ are called "bounded fields".

In order to work with the singular fields, detached from the bounded ones, the first idea could be to employ $E'/\mathcal{H} \ni [L] = L + \mathcal{H}$, defined as the quotient of LC-spaces. But, since $\mathcal{H}$ is not closed in the w*-topology, the quotient topology of $E'/\mathcal{H}$ is not Hausdorff, and would even prevent point measures (cf. [Schaef66, I, 2.3]). Instead, we generalize a method in [Rie2020] which enables the construction of quite general singular w*-closed, linear spaces now, and is basic for our present approach.

Theorem 3.9. (a) *It holds that "$Y \subset E'_s$ and Y is w*-closed" if and only if its pre-polar ${}^0Y := \{f \in E \mid L(f) = 0 \;\forall\, L \in Y\}$ is a norm-dense strict subspace X of E.*

Stated otherwise, the τ-continuous polars X^0 of all norm-dense strict subspaces X of E exhaust all w-closed subspaces $Y \subset E'_s$.*

(b) *If we have a family $\{X_i \mid i \in I\}$ of norm-dense strict subspaces $X_i \subset E$ then it holds for $\bigcap_{i\in I} X_i =: X_p$ that for the Hilbert space orthogonal complements we have $X_p^\perp = \bigvee_{i\in I} \widehat{X}_i^\perp = 0$, and the true subspace X_p is norm dense in E. The polar $X_p^0 =: Y_p$ equals $\overline{\mathrm{LH}\bigcup_{i\in I} X_i^0}^{w*} \subset E'_s$.*

(c) *Given a closed subspace $Y \subset E'_s$ with ${}^0Y = X_p \subset E$, we fix a (non-unique) direct complement X_e such that $E = X_p \dot{+} X_e$. This provides a unique decomposition $f = f_p + f_e$ for all $f \in E$. Denoting the continuous dual of X_e by X'_e, we find the linear homeomorphism in the respective w*-topologies*

$$X'_e \cong X_p^0 \quad \textit{for all choices of a complement } X_e \textit{ of } X_p. \tag{3.3}$$

(d) *A linear subspace $Y \subset E'$ is contained in E'_s if and only if the quotient map $q\colon Y \to E'/\mathcal{H}$ is a linear isomorphism.*

In particular, if a linear subspace E'_c is a direct complement of $\mathcal{H}$ in E', i.e. $E' = \mathcal{H} \dot{+} E'_c$, then E'_c is linear isomorphic to $E'/\mathcal{H}$. Thus $E'_c \subset E'_s$, but E'_c is not w-closed.*

Proof. (a) Assume $Y \subset E'_s$ and Y is w*-closed. If ${}^0Y$ were not norm dense in E, and thus not norm dense in $\mathcal{H}$, then there would exist a $0 \neq g \in \mathcal{H}$ which is orthogonal to ${}^0Y$ and, therefore, contained in $({}^0Y)^0 = Y$, so that the latter would not be contained in E'_s. If ${}^0Y$ were E, then Y would be 0.

If X is a norm-dense strict subspace of E then it holds for each norm continuous $L \in X^0 \subset E'$ that L vanishes on a norm-dense set and equals 0. Thus X^0 is contained in E'_s. Since $X \neq E$ it follows that $Y \neq 0$.

(b) $X \subset E$ is norm dense in E if and only if it is norm dense in $\mathcal{H}$, and that is equivalent to $X^\perp = 0$.

The continuous polar for the norm-dense linear subspace $\bigcap_{i\in I} X_i \subset E$ is certainly a w*-closed, linear space in E'_s which contains all X_i^0 and is the smallest of these linear spaces.

(c) If $L_e \in {X_e}'$ then there exists for it a unique extension $L \in E'$ which vanishes on X_p and vice versa. The w*-convergence of $(L_i)_I \subset X_p^0$ implies the same for their restrictions $(L_{ei})_I$ on X_e and vice versa.

(d) Quite generally, $q\colon Y \to E'/\mathcal{H}$ is a linear map. If different L_1, L_2 are in $Y \subset E'_s$ and $q(L_1) = q(L_2)$ then $L_1 - L_2 \in \mathcal{H}$ would not be in E'_s, which would contradict the subspace property of Y. Thus q is injective.

If Y is a linear subspace of E' and $q\colon Y \to E'/\mathcal{H}$ is injective, then each difference $q^{-1}[L_1] - q^{-1}[L_2]$, for different L_1, L_2 in Y, must be in E'_s and thus $Y \subset E'_s$.

Clearly E'_c is a subspace of E'_s, but is not w*-closed by Observation 3.10(b) below. □

Observation 3.10. We supplement some topological features connected with singular fields.

(a) Fix a direct algebraic decomposition $E' = \mathcal{H} \dot{+} E'_c$, $E'_c \subset E'_s$, and consider the projection $\mathsf{P}_{\mathcal{H}}\colon E' \to \mathcal{H}$ picking in each direct decomposition $L = g + L_c \in E'$ the unique component $g \in \mathcal{H} \subset E'$. Then $\mathsf{P}_{\mathcal{H}}$ is not continuous in the w*-topology.

This is because there are sequences $\{L_n\} = \{g_n + L_c\} \subset E'$ with $\{g_n\} \subset \mathcal{H}$ converging to $0 \neq L_s \in E'_c$, since $\mathcal{H}$ is w*-dense in E'. Then $\lim \mathsf{P}_{\mathcal{H}}(L_n) = \lim g_n = L_s \neq 0$ but $\mathsf{P}_{\mathcal{H}}(\lim L_n) = \mathsf{P}_{\mathcal{H}}(L_s + L_c) = 0$.

(b) Because of (a), $\mathsf{P}_c\colon \mathbf{1} \dot{-} \mathsf{P}_{\mathcal{H}}$ is w*-discontinuous also, and $\mathsf{P}_c E' = E'_c$ is not w*-closed. This implies that there exists a sequence $\{L_n\} \subset E'_c$ which converges to a $g \in \mathcal{H}$ in the w*-topology, since otherwise a discontinuity of P_c could not arise. But then all sequences $\{L_n + f\} \subset E'_s$, $f \in \mathcal{H}$, converge to any prescribed $g + f =: h \in \mathcal{H}$. We conclude that E'_s is w*-dense in E'. This means that $\mathcal{H}$ and E'_s are intimately interwoven, disjoint sets in the w*-topology.

(c) From (a) it follows that $E' = \mathcal{H} \dotplus E'_c$ is not a topological direct sum (in the sense of [Schaef66, p. 21]) and the algebraic linear isomorphism $q\colon E'_c \to E'/\mathcal{H}$ is not continuous (according to [Schaef66, p. 22]). Also, compare [Conway85, Prop. 5.22] and the subsequent discussion.

(d) Similarly one may show that $E = X_p \dotplus X_e$, is not a topological direct sum in the norm topology and the projections $P_p\colon E \to X_p$ and $P_e\colon E \to X_e$ are discontinuous. (Apply, e.g., P_p to (g_n) of Lemma 3.11.) Our examples show that possibly $\dim X_e < \infty$ (a possibility emphasized by [Schaef66, II, Ex. 8 and IV, Ex. 12]).

For later use we need the following fact.

Lemma 3.11. *If L is in a w^*-closed, singular subspace Y with pre-polar ${}^0Y = X_p$ then according to Theorem 3.9 we have $E = X_p \dotplus X_e$. Given any $f_e \in X_e$ we can choose a sequence $\{g_n\} \subset X_p$ such that $g_n \to -f_e$, since X_p is norm dense in E. Then $L(g_n + f_e) \to L(f_e)$ in the stationary sense, but $\|g_n + f_e\| \to 0$.*

To construct *examples* for w*-closed, linear subspaces $Y \subset E'_s$ we generalize a strategy of [Rie2020].

Definition 3.12. Let a self-adjoint K on $\mathcal{H}$ with $\ker K = 0$ be given and write $w_s := \exp\{isK\}$ for all $s \in \mathbb{R}$. Denote by $E_K \subset E$ the union of all finite linear combinations $\sum_n c_n(w_s f_n - f_n)$, $c_n \in \mathbb{C}$, $s \in \mathbb{R}$, $f_n \in E$.

Due to the self-adjointness of K, we have $\mathcal{H} = \overline{\operatorname{ran} K}^n \oplus \ker K$. If $g \in \ker K$ it holds that $(g|w_s f - f) = (w_{-s} g - g|f) = 0$ for all $f \in E$ and all $s \in \mathbb{R}$ and it follows that $\overline{E_K}^n$ is $(\ker K)^\perp$. Thus $\overline{E_K}^n = \mathcal{H}$. If E_K is a true subset of E then E_K^0 is a non-trivial, w*-closed subspace of E'_s, which is time invariant if $[K, B] = 0$.

If we have a family $\{K_i \mid i \in I\}$ of such self-adjoint operators, then $E_p := \bigcap_i E_{K_i}$ is still n-dense in E according to Theorem 3.9(c)(b), and time invariant. Then E_p^0 provides a time-invariant, linear, w*-closed space of singular fields.

To introduce our *main examples* we choose as singular fields transversal plane waves $\mathbb{R}^3 \ni x \mapsto e_k(x) = a_k \exp\{ik \cdot x\}/(2\pi)^{3/2}$, with $\mathbb{R}^3 \ni k \neq 0$. The term e_k is the inverse Fourier transform of $\hat{e}_k = a_k \delta_k \in \hat{E}'_s$. Here $0 \neq a_k \in \mathbb{C}^3$, with $k \cdot a_k = 0$, determines the intensity and kind of polarization. If $\operatorname{Arg}(a_2) - \operatorname{Arg}(a_1)$ equals 0 or π, the wave is linearly polarized in the 1-2-plane provided k points along the 3-axis. We admit arbitrary such $0 \neq a_k \in \mathbb{C}^3$, i.e. elliptically polarized plane waves, but not partially polarized waves which would mean mixed states. Set $\kappa := k/|k|$ and $a_\kappa^0 := a_k/|a_k|$ so that the polarizer projection is $P_\kappa = a_\kappa^0 \otimes a_\kappa^0$ and $P_\kappa a_k \delta_k = a_k \delta_k$ (where in optics one works with the real part of the $e_k(x)$, taken as the **E**-field; e.g. [KleinF]).

To illustrate the essential features it is sufficient to assume the entire photon system is polarized by the fixed P_κ, that is, we vary in the considered $e_k(x)$ only the frequencies $\omega_k = c|k|$ and intensities $|a|$, but have, of course many other $L \in E'$ with $P_\kappa L = L$. Correspondingly, it is sufficient to restrict the test functions to the form $\hat{f}(l) = (a^0_{\kappa,\nu}\hat{\varphi}(l))$ where $\hat{\varphi} \in \hat{\mathsf{S}}^1_0$. Then we obtain in particular $\hat{e}_k(\hat{f}) = |a_k|\hat{\varphi}(k) \in \mathbb{C}$, illustrating that the now restricted $\hat{f}$ also test the intensity.

The dual dynamics is given by $(v'_t e_k)(x) = a_k \exp\{-i(\omega_k t - k\cdot x)\}$, which in the k-representation is written as $v'_t\hat{e}_k = \exp\{-i\omega_k t\}\hat{e}_k \in \mathbb{C}\hat{e}_k =: \widehat{Y}_k \subset \hat{E}'_s$. So the one-dimensional, singular, w*-closed subspace $\widehat{Y}_k$ is time invariant.

The pre-polar is ${}^0\widehat{Y}_k = \widehat{X}_{k,p} = \{a^0_k\hat{\varphi} \in \hat{E} \mid \varphi(l) = 0 \text{ for } l = k\}$ and is a true, norm-dense subspace of the now restricted $\hat{E}$, by Theorem 3.9. A direct complement $\widehat{X}_{k,e}$ of $\widehat{X}_{k,p}$ with respect to $\hat{E}$ is, e.g., $\mathbb{C}\hat{f}_{k,\lambda}$, with $\lambda > 0$, where $\hat{f}_{k,\lambda}(l) := a_k\hat{\varphi}_{k,\lambda}(l) = a_k \exp\{-\lambda(k-l)^2/|l|\}$ for all $l \in \mathbb{R}^3$. The associated direct decomposition $\hat{f} = \hat{f}_p \dotplus \hat{f}_e$ is for any $\hat{f} \in \hat{E}$ given by $\hat{f}(l) = (\hat{f}(l) - \hat{f}(k)\hat{\varphi}_{k,\lambda}(l)) \dotplus \hat{f}(k)\hat{\varphi}_{k,\lambda}(l)$. If we let λ increase then $\hat{f}_{k,\lambda}$ and all other elements in $\widehat{X}_{k,e}$ concentrate more and more around $l = k$, whereas the elements in $X_{k,e}$ are more and more *spatially extended*. Notice that the $\hat{f}_{k,\lambda}/\|\hat{f}_{k,\lambda}\|$ approximate thereby $\hat{e}_k$.

If $\mathcal{K} \subset \mathbb{R}^3\backslash 0$ is an arbitrary subset, then for $\bigcap_{k\in\mathcal{K}} \widehat{X}_{k,p} =: \widehat{X}_p$ we still have $\widehat{X}_p^\perp = \bigvee_{k\in\mathcal{K}} \widehat{X}^\perp_{k,p} = 0$, and the true subspace $\widehat{X}_p$ is norm dense in $\hat{E}$. We arrive at the time-invariant, w*-closed, singular subspace $\widehat{Y} := \widehat{X}^0_p = \overline{\mathrm{LH}\bigcup_{k\in\mathcal{K}} \widehat{Y}_k}^{w^*}$.

§3.3. Generalized q-classical photon states

In the following we take the point of view that we consider a class of photon systems for which we know a w*-closed space $Y \subset E'_s$ which should comprise all singular classical fields which may possibly arise. For example, the experimental possibilities certainly restrict the range of frequencies for the (always singular) radiation fields.

The *space* Y *being fixed*, we term it $Y \equiv E'_{\mathrm{ess}}$ and denote its pre-polar ${}^0E'_{\mathrm{ess}}$ by E_p. We have thus $E = E_p \dotplus E_e$ in the sense of Proposition 3.9(c).

Since according to equation (2.2) each f constitutes in the Fock representation a one-particle wave function, the splitting of E induces two different types of photons. As indicated in the main example and further discussed below, in E_p there are the genuine *particle bosons*, and in E_e are bosons of somewhat diffuse character, since E_e is not unique. We term them *transient or extended photons* since a typical physical feature is their wider spatial extension. The relation $E'_{\mathrm{ess}} = E'_e$, especially, demonstrates the close relation between classical fields and transient photons.

We consider only q-classical states $\mathcal{S}_{\mathrm{cl}}$ with a τ-continuous characteristic function, which makes them into a subset of the folium $\mathcal{F}_\tau \subset \mathcal{S}$ of "τ-continuous states" (cf. [HonRie15, I, Thm. 18.2-3], where also the τ-strong continuity of $E \ni f \mapsto W_\varphi(f)$ is demonstrated for all $\varphi \in \mathcal{F}_\tau$).

Definition 3.13. We denote the state space $\mathcal{S}(\mathcal{W}(E,\sigma))$ simply by $\mathcal{S}$ and the classical state space $\mathcal{S}(\mathcal{W}(E,0))$ by $\mathcal{S}^0$, and choose an E'_{ess}.

(a) Any state $\omega \in \mathcal{S}$ is termed "q-classical", written $\omega \in \mathcal{S}_{\mathrm{cl}}$, if its characteristic function has the form

$$C_\omega(f) \equiv C_\mu(f) = \int_{E'} \exp\{-\tfrac{1}{4}\|f\|^2\} \exp\{i\sqrt{2}\,\mathrm{Re}\,L(f)\}\, d\mu(L) \tag{3.4}$$

for all $f \in E$ and some Bochner–Minlos measure $\mu \in M_p(E')$ and is then written $\omega \equiv \omega_\mu$. (Recall that the support $\operatorname{supp}\mu$ is the usual smallest closed set, the complement of which has zero measure, since E' is metrizable in the w*-topology.)

(b) A state $\omega \equiv \omega_\mu \in \mathcal{S}_{\mathrm{cl}}$ is called "Fock q-classical", denoted $\omega \in \mathcal{S}_{\mathrm{Fcl}}$, if $\operatorname{supp}\mu \subset \mathcal{H}$.

(c) If μ equals a point measure δ_L with $L \in E'$ then $\omega_{\delta_L} \equiv \omega_L$ is called a "coherent state". If $L \in E'_s$, then ω_L is also called a "singular coherent state", otherwise a "Fock coherent state". In the latter case we write $\omega_L = \omega_g$, if $L(f) = (g|f)$, $g \in \mathcal{H}$ (cf. equation (2.17)).

(d) An $\omega \equiv \omega_\mu \in \mathcal{S}_{\mathrm{cl}}$ is called "essential (q-classical)", denoted $\omega \in \mathcal{S}_{\mathrm{ess}}$, if $\operatorname{supp}\mu \subset E'_{\mathrm{ess}}$.

If a plane wave with $e_k = L \in E'_s$ (which is not a one-photon wave function) is included in E'_{ess} then it gives rise in the present setup to a well-defined multi-photon state $\omega_L \in \mathcal{S}_{\mathrm{ess}}$.

Theorem 3.14. (a) *$\mathcal{S}_{\mathrm{cl}}$ is a simplex contained in $\mathcal{F}_\tau$.*

The extremal boundary $\partial_e \mathcal{S}_{\mathrm{cl}}$ is given by $\{\omega_L \mid L \in E'\}$, with the characteristic functions

$$C_{\delta_L}(f) \equiv C_L(f) = \exp\{-\tfrac{1}{4}\|f\|^2\} \exp\{i\sqrt{2}\,\mathrm{Re}\,L(f)\}, \quad L \in E'. \tag{3.5}$$

It holds that $\partial_e \mathcal{S}_{\mathrm{cl}} \subset \partial_e \mathcal{S}$.

Moreover, $\partial_e \mathcal{S}_{\mathrm{cl}}$ is the disjoint union of the two subsets $\partial_e \mathcal{S}_{\mathrm{Fcl}}$ and $\partial_e \mathcal{S}_{\mathrm{scl}}$, each of which is dense in $\partial_e \mathcal{S}_{\mathrm{cl}}$ in the w-topology on $\mathcal{S}$.*

(b) *The present $\mathcal{S}_{\mathrm{Fcl}}$ coincides with that of Definition 2.6.*

This means that the Bochner–Minlos measure $\mu \in M_p(E')$ reduced to $\mathcal{H} \subset E'$ is a Borel measure in the strong topology on $\mathcal{H}$ and that the boundary *integrals converge not only in the w*-topology, but even in the norm topology on $\mathcal{S}$, where the latter is equivalent to the trace-norm topology for the corresponding density operators on $F_+(\mathcal{H})$.*

Thus $\mathcal{S}_{\mathrm{Fcl}}$ is a simplex of Fock-normal states, and $\partial_e\mathcal{S}_{\mathrm{Fcl}}$ is given by $\{\omega_L \mid L \in \mathcal{H}\}$, the only pure states in $\mathcal{S}_{\mathrm{Fcl}}$.

The space $\mathcal{S}_{\mathrm{Fcl}}$ is a face of $\mathcal{S}_{\mathrm{cl}}$, but not a face of $\mathcal{S}$.

(c) *$\mathcal{S}_{\mathrm{ess}}$ is a simplex, and $\partial_e\mathcal{S}_{\mathrm{ess}}$ is given by $\{\omega_L \mid L \in E'_{\mathrm{ess}}\}$.*

The space $\mathcal{S}_{\mathrm{ess}}$ is even a face of $\mathcal{S}$ which is stable in the sense that any orthogonal decomposition measure μ' of $\omega_\mu \in \mathcal{S}_{\mathrm{ess}}$ is supported on $\mathcal{S}_{\mathrm{ess}}$.

In other words,all possible decompositions of the quantum state $\omega_\mu \in \mathcal{S}_{\mathrm{ess}}$ are induced by the decompositions of the corresponding mixed, classical, singular field states on the center.

Proof. (a) In equation (3.4) the vacuum part may be extracted from the integral and is τ-continuous since τ is stronger than norm. The remaining integral is a τ-continuous positive-definite function by Bochner–Minlos. The product of both is twisted positive definite and defines a state in $\mathcal{F}_\tau$.

By construction, $\mathcal{S}_{\mathrm{cl}}$ is affinely isomorphic to the simplex $M_p(E')$. Then $\partial_e\mathcal{S}_{\mathrm{cl}}$ is in bijection with the point measures δ_L, $L \in E'$ which exist, since the measurable space $(E', \Sigma(E', E))$ is Hausdorff in the w*-topology. That all ω_L are pure follows, e.g., from their irreducible GNS-representations, which are seen in Theorem 4.1.

If L_n converges to L in the w*-topology on E' then $C_{L_n} \to C_L$ in the pointwise Δ-topology, and $\omega_{L_n} \to \omega_L$ in the w*-topology on $\mathcal{S}$. Besides the w*-denseness of $\mathcal{H}$ in E', we have from Observation 3.10 (b) that also E'_s is w*-dense in E', where E' indexes all ω_L.

(b) Since our $\mathcal{H}$ is separable, its strong Borel σ-algebra coincides with $\Sigma(\mathcal{H}, E)$ according to [DaviesOp, Thm. 11.2].

Since for a present $\omega \in \mathcal{S}_{\mathrm{Fcl}}$ the characteristic function C_ω is the same as in equation (2.20), it is Fock normal and has – as a density operator ϱ_ω – the integral decomposition in the trace norm, given in [DaviesOp, p. 120]. This can be applied to all $A \in \mathcal{L}(F_+(\mathcal{H}))$ and gives the decomposition in the state norm, since the trace class operators $\mathcal{T}(\mathcal{H})$ provide the pre-dual of $\mathcal{L}(F_+(\mathcal{H}))$.

The simplex property of $\mathcal{S}_{\mathrm{Fcl}}$ and its boundary follow again from the affine isomorphy to a measure space, here to $M_p(\mathcal{H})$. That the $\omega_g, g \in \mathcal{H}$ are the only pure states in $\mathcal{S}_{\mathrm{Fcl}}$ is the content of [DaviesOp, Chap. 8, Lem. 2.1].

If $\mathcal{S}_{\mathrm{Fcl}} \ni \omega \equiv \omega_\mu = \lambda\omega_1 + (1-\lambda)\omega_2$ with $0 < \lambda < 1$ and $\omega_i \in \mathcal{S}_{\mathrm{cl}}$, then neither μ_1 nor μ_2 can have a partially singular support. Otherwise, there would contribute a non-negligible $Y \subset \operatorname{supp}\mu \subset E'_s$ which could not be compensated via the convex state composition.

Since for different $g_1, g_2 \in \mathcal{H}$ the q-classical $\frac{1}{2}(|G(g_1))(G(g_1)|+|G(g_2))(G(g_2)|)$ has extremal decompositions into non-coherent one-dimensional projections, it follows that $\mathcal{S}_{\mathrm{Fcl}}$ is no face of $\mathcal{S}$.

(c) By construction $\mathcal{S}_{\mathrm{ess}}$ is affinely isomorphic to the simplex $M_p(E'_{\mathrm{ess}})$ and $\partial_e \mathcal{S}_{\mathrm{cl}}$ is in bijection with the point measures δ_L, $L \in E'_{\mathrm{ess}}$. Observe that $E'_e = E'_{\mathrm{ess}}$.

According to Theorem 4.2 below, $\mathcal{M}_\mu$ is spatially decomposable into irreducible, mutually disjoint representations. This entails that $\mathcal{M}'_\mu = \mathcal{Z}_\mu$ is maximal abelian and by the Tomita theorem (e.g. [Takes79]) each orthogonal decomposition measure μ' of ω_μ is associated with a sub–von Neumann algebra $\mathcal{N}_{\mu'} \subset \mathcal{Z}_\mu$. Thus μ' is a coarsening of the central decomposition and supported on $\mathcal{S}_{\mathrm{ess}}$. So $\mathcal{S}_{\mathrm{ess}}$ is a stable face of $\mathcal{S}$. □

§4. Effective photon fields over q-classical states

§4.1. Represented photon algebras

Theorem 4.1. (a) *For any q-classical ω_μ, $\mu \in M_p(E')$, the GNS-triple is*

$$
\begin{aligned}
&(W_\mu(f), \mathcal{H}_\mu, \Omega_\mu) = (W_F(f) \otimes W^0_\mu(f), P_\mu[F_+(\mathcal{H}) \otimes L^2(E', \mu)], \Omega_0 \otimes 1_\mu),\\
&P_\mu[F_+(\mathcal{H}) \otimes L^2(E', \mu)] := \overline{\mathrm{LH}\{W_\mu(f)\Omega_\mu \mid f \in E\}},\\
&W^0_\mu(f) := \textit{multiplication by } L \mapsto W^0(f)[L] := \mathrm{e}^{i\sqrt{2}\,\mathrm{Re}\,L(f)} \textit{ on } L^2(E', \mu).
\end{aligned}
$$

(b) *If $\mu = \delta_L$, $L \in E'$, then $(W_\mu(f), \mathcal{H}_\mu, \Omega_\mu) \equiv (W_L(f), \mathcal{H}_L, \Omega_L)$ is (unitary equivalent to) $(\mathrm{e}^{i\sqrt{2}\,\mathrm{Re}\,L(f)} W_F(f), F_+(\mathcal{H}), \Omega_0)$ and is irreducible (but disjoint to the Fock representation for $L \in E'_s$).*

(c) *For a $\omega_\mu \in \mathcal{S}'_{\mathrm{ess}}$, with $\mu \in M_p(E'_{\mathrm{ess}})$, the GNS-triple specializes to*

$$(W_\mu(f), \mathcal{H}_\mu, \Omega_\mu) = (W_F(f) \otimes W^0_\mu(f), F_+(\mathcal{H}) \otimes L^2(E'_{\mathrm{ess}}, \mu), \Omega_0 \otimes 1_\mu). \tag{4.1}$$

(d) *Let $\mu \in M_p(E'_{\mathrm{ess}})$ and $\mathcal{M}_\mu := \mathcal{W}_\mu(E, \hbar\sigma)''$. Then $\mathcal{M}_\mu$, its commutant $\mathcal{M}'_\mu$, and its center $\mathcal{Z}_\mu := \mathcal{M}_\mu \cap \mathcal{M}'_\mu$ are given by*

$$\mathcal{M}_\mu = \mathcal{L}(F_+(\mathcal{H})) \otimes L^\infty(E'_{\mathrm{ess}}, \mu), \tag{4.2}$$

$$\mathcal{M}'_\mu = \mathbf{1}_F \otimes L^\infty(E'_{\mathrm{ess}}, \mu) = \mathcal{Z}_\mu, \tag{4.3}$$

where "$\otimes$" between von Neumann algebras includes the weak closure.

Proof. (a) That $(\Omega_\mu | W_\mu(f)\Omega_\mu) = C_\mu(f)$ is immediate, and cyclicity holds by construction.

(b) Formed by arbitrary finite sets $\{c_n\} \subset \mathbb{C}$, $\{\sum_n c_n \Pi_L(\mathcal{W}(f_n))\Omega_0\}$ equals the sets $\{c'_n \Pi_F(\mathcal{W}(f_n))\Omega_0\}$, with $c_n = c'_n \mathrm{e}^{i\sqrt{2}\,\mathrm{Re}\,L(-f)}$, $c'_n \in \mathbb{C}$, which are dense in

$F_+(\mathcal{H})$. The map $f \mapsto W_L(f)$ is not norm-weak continuous if L is singular and in that case $\mathcal{S}_n(\Pi_L) \cap \mathcal{S}_n(\Pi_F) = \emptyset$ by Theorem 4.2 below.

(c) Denote $(g_n + f_e)$ from Lemma 3.11 by (f_n). Then for each $L \in E'_{\text{ess}}$ we have $W_F(f_n) \otimes \exp\{i\sqrt{2}\,\mathrm{Re}\,L(f_n)\} \to \mathbf{1}_F \otimes \exp\{i\sqrt{2}\,\mathrm{Re}\,L(f_e)\} =: Z(f)$ if $n \to \infty$, since $E \ni g \mapsto W_F(g)$ is norm-strong continuous so that $Z(f) \in \mathcal{M}_\mu$.

Multiplying $Z(f)$ by $W_F(f) \otimes \exp\{i\sqrt{2}\,\mathrm{Re}\,L(-f)\}$ leads to $W_F(f) \otimes \mathbf{1}_\mu$ for arbitrary $f \in E$. Hence $\mathcal{M}_\mu$ contains all elements $W_F(f') \otimes \exp\{i\sqrt{2}\,\mathrm{Re}\,L(f)\}$ with $f', f \in E$. According to the Stone–Weierstrass theorem [KadR83], the characters on E'_{ess}, given by $Z(f)$, $f \in E$, generate algebraically a norm-dense subalgebra of $C(E'_{\text{ess}})$ and generate, by application to Ω_μ, linear extension, and vector-norm closure all of $\Omega_0 \otimes L^2(E'_{\text{ess}}, \mu)$, whereas the $W_F(f) \otimes \mathbf{1}_\mu$ generate, by application to Ω_μ, linear extension, and vector-norm closure all of $F_+(\mathcal{H}) \otimes 1$. This shows that $\mathcal{H}_\mu = F_+(\mathcal{H}) \otimes L^2(E'_{\text{ess}}, \mu)$.

Since the commutant is the tensor product of the commutants of the separate factors [Takes79], we arrive at equation (4.3). □

In contradiction with the correspondence limit $\hbar \to 0$, which deforms the entire quantum algebra into a classical one, the foregoing investigation reveals the phase space functions $L^\infty(E'_{\text{ess}}, \mu)$ as the central part of an effective quantum field algebra for fixed $\hbar > 0$. Its pure states are the restrictions of the quantum states $\{\omega_L \mid L \in E'_{\text{ess}}\}$ to the center. Each ω_L has in general a spontaneous transition probability to pass into some of its "weak quantum perturbations" from the folium $\mathcal{F}_L = \mathcal{S}_n(\Pi_L)$, but should not spontaneously go over to a different $\omega_{L'}$. It is interesting that we can confirm this by directly calculating the algebraic transition probabilities between states on $\mathcal{W}(E, \sigma)$.

Theorem 4.2. *Employ the setup of Theorem 4.1 and the notions in Appendix A.*

(a) *Consider two different singular coherent states ω_L and $\omega_{L'}$ on $\mathcal{W}(E, \hbar\sigma) \equiv \mathcal{A}$.*
Then for all state tuples $(\omega, \varphi) \in \mathcal{S}_n(\Pi_L) \times \mathcal{S}_n(\Pi_{L'})$, we obtain $T_\mathcal{A}(\omega, \varphi) = 0$. This implies that all states in $\mathcal{S}_n(\Pi_L)$ are macroscopically different (= disjoint) from the states in $\mathcal{S}_n(\Pi_{L'})$.
Particularly for $L' = 0$ this demonstrates that no vacuum fluctuation – that is a state in $\mathcal{S}_n(\Pi_F)$ – can achieve the spontaneous arise of a true classical field associated with $\mathcal{S}_n(\Pi_L)$.

(b) *We conclude that the state decomposition of an essential q-classical state $\omega \equiv \omega_\mu$,*

$$\omega = \int_{E'_{\text{ess}}} \omega_L \, d\mu(L) = \int_{\mathcal{S}} \varphi \, d\mu_\omega(\varphi), \tag{4.4}$$

implied by equation (3.4)*, is not only extremal on* $\mathcal{S}$ *but also a parametrization of the central measure* μ_ω *on* $\mathcal{S}$*.*

(c) *It follows by the separability of* E'_{ess} *that each* $\omega_\mu \in \mathcal{S}_{\mathrm{ess}}$ *is* spatially decomposable *in the sense that*

$$(4.5)\qquad (\Pi_\mu, \mathcal{H}_\mu, \Omega_\mu, \mathcal{M}_\mu) \overset{U}{=} \int_{E'_{\mathrm{ess}}}^{\oplus} (\Pi_L, \mathcal{H}_L, \Omega_L, \mathcal{M}_L)\, d\mu(L),$$

such that $\mathcal{Z}_\mu$ *is unitary equivalent to the diagonal algebra* $\int_{E'_{\mathrm{ess}}}^{\oplus} \mathbb{C}_L \mathbf{1}_L \, d\mu(L)$*.*

Proof. (a) Let us begin with the pure essentially singular classical states $\omega_L, L \in E'_{\mathrm{ess}}$. Then there exists a sequence $(g_n) \subset E$ with $g_n \xrightarrow{w*} L \in E'_{\mathrm{ess}}$, entailing $\|g_n\| \to \infty$. This also gives $C_{g_n}(f) \to C_L(f)$ for all $f \in E$, and, by the affine Δ-to-weak* homeomorphy between characteristic functions and states, $\omega_{g_n} \xrightarrow{w*} \omega_L$ in $\mathcal{S}$. We can now try to apply Proposition A.1. First we have $T_{\mathcal{A}}(\omega_{g_n}, \omega_f) = |(G(g_n)|G(f))_{F_+}|^2$, because the vectors $G(g_n)$, $G(f)$ represent ω_{g_n}, ω_f both in the Fock representation, in the commutant of which there are only scalar unitaries.

Then, according to Proposition 2.4(c), we obtain

$$(4.6)\qquad (G(g_n)|G(f))_{F_+} = \exp\{(g_n|f) - \tfrac{1}{2}\|g_n\|^2 - \tfrac{1}{2}\|f\|^2\} \to 0 \quad \forall f \in E,$$

$$(4.7)\qquad \text{implying that } T_{\mathcal{A}}(\omega_{g_n}, \omega_f) \to T_{\mathcal{A}}(\omega_L, \omega_f) = 0 \;\forall f \in E,\, \forall L \in E'_{\mathrm{ess}}.$$

In fact, since $T_{\mathcal{A}}(\omega, \omega')$ is jointly, weak* lower semicontinuous the limit of falling values is assumed [ReedSim1, I, Suppl. IV.5].

If also $f_m \xrightarrow{w*} L' \in E'_{\mathrm{ess}}$, with $L' \neq L$, then $L' - L \in E'_{\mathrm{ess}}$ and is approximated by $g_n - f_m$ with $\|g_n - f_m\| \to \infty$. Then $|\exp\{(g_n|f_m) - \frac{1}{2}\|g_n\|^2 - \frac{1}{2}\|f_m\|^2\}|$ is dominated by $\exp\{\frac{-1}{2}\|g_n - f_m\|^2\} \to 0$ and $T_{\mathcal{A}}(\omega_{g_n}, \omega_{f_m}) \to T(\omega_L, \omega_{L'}) = 0$ for all $L, L' \in E'_{\mathrm{ess}}$. We observe that

$$(G(g_n)|W_F(h)G(f_m))_{F_+} = (G(g_n)|\exp\{i2^{-1/2}\operatorname{Re}(h|f_m)\}G(f_m + i2^{-1/2}h))_{F_+}$$

(apply equation (2.15)), which tends to 0 by the preceding argument. Then also $T(\omega_L, \omega^C_{L'}) = 0$ if $\omega^C_{L'}$ symbolizes a finite convex combination of states, approached by normalized vectors $W_F(h_i)G(f_m)/\|W_F(h_i)G(f_m)\|_{F_+}$, taking into account the separate convexity of $T_{\mathcal{A}}(\omega, \varphi)$. Since $\Delta(E, \sigma)$ is norm dense in $\mathcal{W}(E, \sigma)$ we approximate any $\varphi \in \mathcal{S}_n(\Pi_{L'}, \mathcal{H}_{L'})$ by the described Weyl perturbations in the state norm, in which $\varphi \mapsto T_{\mathcal{A}}(\omega, \varphi)$ is continuous. Apply the same procedure to the other argument ω in $T_{\mathcal{A}}(\omega, \varphi)$. That this implies the disjointness of ω and φ follows from Proposition A.2.

(b) If we consider two disjoint subsets E'_1 and E'_2 of E'_{ess}, with non-vanishing μ-measure each, then $\int_{E'_1}^{\oplus} \omega_L \, d\mu(L)/\mu(E'_1)$ and $\int_{E'_2}^{\oplus} \omega_L \, d\mu(L)/\mu(E'_2)$ give two disjoint states by the reasoning of (a). It characterizes the central decomposition (e.g. [BratRob1]), written as the second integral in equation (4.4) by means of the w*–w*-continuous map $L \mapsto \omega_L$ to transpose the measure.

(c) If $\{f_n \mid n \in \mathbb{N}\}$ is τ-dense in E then $\{W_\varphi(f_n) \mid n \in \mathbb{N}\}$ is weakly dense in $\overline{\mathcal{W}_\varphi(E,\sigma)} = \mathcal{M}_\varphi$, because of the continuity of $f \mapsto W_\varphi(f)$, $\varphi \in \mathcal{F}_\tau$. Thus $\{W_\varphi(f_n)\Omega_\varphi \mid n \in \mathbb{N}\}$ is dense in $\mathcal{H}_\varphi$ for all $\varphi \in \mathcal{F}_\tau$. This gives a family of integrable Hilbert spaces and von Neumann algebras. Since μ_ω is supported on $\mathcal{F}_\tau$ and is an orthogonal measure, the argument that $\int_{\mathcal{S}}^{\oplus} \Omega_\varphi \, d\mu_\omega(\varphi)$ is a cyclic vector for the integral representation runs as for the case of separable C*-algebras (see [Takes79], and for Weyl algebras [HonRie15, III, Prop. 48.2-15]). This makes ω_μ spatially decomposable in the sense of [HonRie15, III], meaning the validity of equation (4.5). □

By Theorem 4.2(b) we have another argument to deduce the GNS-representation over an $\omega_\mu \in \mathcal{S}_{\text{ess}}$. The decisive limit is here $E \ni g_n \to L \in E'_{\text{ess}}$, an apparently more constructive manner to reach the singular classical fields, whereas in Theorem 4.1 the singular classical fields are presumed, and the alternative limit, there, serves to separate them off from the quantum degrees of freedom.

We have arrived at a "sector decomposition" of the effective field algebra which we understand merely as a decomposition into irreducible mutually disjoint representations of the abstract field algebra which is not always to consider as forming "superselection rules" (cf. [Earman]).

§4.2. The dressed photon fields

By differentiating $\mathbb{R} \ni s \mapsto W_\mu(sf)$ of equation (4.1) we obtain the field operators $\Phi_\mu(f) = \frac{1}{\sqrt{2}}(a_\mu^*(f) + a_\mu(f))$ over $\omega_\mu \in \mathcal{S}_{\text{scl}}$ on a dense domain in $\mathcal{H}_\mu$ in the form

$$\begin{aligned} \Phi_\mu(f) &= \Phi_F(f) \otimes \mathbf{1}_\mu + \mathbf{1}_{F_+} \otimes \Phi_\mu^0(f), \quad \text{with } \Phi_\mu^0(f) := \sqrt{2}\,\mathrm{Re}\, L(f), \\ a_\mu^*(f) &= a_F^*(f) \otimes \mathbf{1}_\mu + \mathbf{1}_{F_+} \otimes L(f), \quad a_\mu(f) = a_F(f) \otimes \mathbf{1}_\mu + \mathbf{1}_{F_+} \otimes \overline{L(f)}, \end{aligned} \tag{4.8}$$

where $L \in E'_{\text{ess}}$ is the mute variable for the multiplication operators on $L^2(E'_{\text{ess}}, \mu)$. Dropping the unit operators we write

$$\Phi_\mu(f) = \Phi_\mu(P_p f) + \Phi_\mu(P_e f) = \Phi_F(f_p) + \Phi_F(f_e) + \Phi_\mu^0(f_e). \tag{4.9}$$

Let us recall that quite generally a restrictive linear subsidiary condition on the test functions, under which the corresponding smeared field is measured, implements a *quality* on the quantized field. If, e.g., the f are all supported on a volume $V \subset \mathbb{R}^3$, then we speak of a "field localized in V". In an analogous sense we call $\Phi_F(f_p)$, $f_p \in E_p$ the "particle photon field" and $\Phi_F(f_e)$, $f_e \in E_p$ the "transient photon field".

By the "strong resolvent continuity" of $E \ni f \mapsto \Phi_F(f)$, induced by the strong continuity of the map $f \mapsto W_F(f)$, we obtain – using $\{f_n\}$ from Lemma 3.11 –

$$\lim_n \Phi_\mu(f_n) = \Phi^0_\mu(f_e), \quad \{f_n\} = \{g_n + f_e\},\ g_n \to -f_e. \tag{4.10}$$

We may interpret that convergence of the represented photon field to the alone standing classical field part $\Phi^0_\mu(f_e)$ as the existence of coarsened measurement procedures which neglect the bosonic particle aspects and lead to merely observing the classical field. Mathematically, the limiting behavior of the particle test functions $g_n = P_p g_n$ may clear away, not only the particle part, but also the transient test functions, and this is enabled by the denseness of E_p in E and by the discontinuity of P_p. Physically, these measurement methods are so coarse that they do not even detect the extended transient photons.

If one measures, e.g., the laser light in a glass-fiber cable, one is usually interested only in the quality of the information-carrying classical field part which, for itself, already displays fluctuations, the classical noise (cf. also our formula (7.4)). More precise, technically relevant measurements also exhibit the quantum noise. In the extreme limit, one must shrink the cable to a high-quality cavity (e.g. [ScullyZub, Trifonov]) in order to measure n exemplars of localized particle photons. In the case that these constitute all of the photonic system, $E_e = 0$ and there are neither transient photons nor classical fields.

Altogether, the decomposition equation (4.9) of the effective photon field operator, deduced from microscopic algebraic quantum theory, already displays a similarity to Einstein's vision, if we relate $\Phi_F(f_p)$ with the "singular points" and $\Phi^0(f_e)$ with the "undulatory fields of force". The spatial extension of f_p – and thus of all particles describable by $\Phi_F(f_p)$ – is almost point-like displayed in the nearly instantaneous act of atomic absorption, and the photonic energy is correspondingly concentrated. (An indivisible energy point will be treated in a forthcoming investigation.)

The name "undulatory fields of force" is only intelligible if one understands "fields of force". The latter are to mediate between the singular point particles and the "undulatory fields of force", as do our transient photons $\Phi_F(f_e)$ between the quantum particles and classical waves. Einstein's notions appeal to the influence of the photons on charged matter, whereas we presently deal with free photons only. We supplement, however, some remarks in Appendix D on how the gauge principle discloses the capacity to interact with charged fermions as an intrinsic feature of photons. This amplifies the kinship of the transient photons to the "fields of force". Moreover, the increasingly overlapping "fields of force" approaching an electrodynamic wave are not completely foreign to the high-intensity limit of the transient photon amplitudes.

The big difference between a transient one-photon wave function f_e and a classical "field of force" lies, of course, in the principal statistical character of the former. It seems, however, not clear whether Einstein assigned the spatially decreasing "fields of force" the same classical stability as the "undulatory fields of force". The first-named rather appear to be a diffuse concept, similarly to the transient photon wave functions which depend on the choice of the complementary subspace $E_e \subset E$. Let us only emphasize that just the necessity of such interpolative quantities between particles and classical fields demonstrates that classical field properties are not one-photon properties, neither in Einstein's formulation nor in the quantized theory, where one-photon wave functions are basically different from electromagnetic fields.

The intuitive picture must further be substantiated by elaborating the behavior of the effective field operators under time and symmetry transformations.

§5. Dynamics and symmetries

§5.1. Time-invariant classical fields and photon states

We venture to draw further physical conclusions from the topological properties of (E, τ) and its τ-continuous dual E' of classical fields. Because of their physical significance we present the following results as a theorem.

Theorem 5.1. *Consider a general quasi-free bosonic field system, described in terms of a C^*-Weyl algebra $\mathcal{W}(E, \sigma)$ with (E, τ) an infinite-dimensional locally convex test function space. The automorphic Heisenberg dynamics is induced from the τ-continuous unitary test function dynamics $v_{\mathbb{R}} = \exp\{i\mathbb{R}B\}$ with $B > 0$.*

We introduce the subspace $E_B \subset E$ according to Definition 3.12 and know that $\overline{E_B}^{n} = \mathcal{H}$.

(a) *Denoting by $E'_{\mathrm{inv}} \subset E'$ the set of all time-invariant classical fields we have $E_B^0 = E'_{\mathrm{inv}} \subset E'_s$, where the polar formation is performed under τ-continuity.*
That is, all *time-invariant dual fields are given by the polar of E_B and are singular (if not vanishing).*

(b) *It holds for the present photonic case that $E_B = E$ and thus $E_B^0 = E'_{\mathrm{inv}} = 0$. There is no time-invariant coherent photon state besides ω_0.*

(c) *The only (pure or mixed) regular photonic ground state is the bare vacuum ω_0.*

Proof. (a) $L(v_t f - f) = 0$ for all $t \in \mathbb{R}$, for all $f \in E \Leftrightarrow L \in E'_{\mathrm{inv}}$. Thus E'_{inv} equals E_B^0 and is singular since E_B is dense in E, according to Theorem 3.9(a).

(b) If $f \in E$ then for $t \neq 0$ we have $f = (v_t - \mathbf{1})g$ with $g = (v_t - \mathbf{1})^{-1} f \in E$ since $k \mapsto \exp\{it|k| - 1\}^{-1} f(k)$ is C^∞ at $k = 0$ too, with vanishing derivatives there, via the l'Hospital rule, giving $g \in E$. Thus $f \in E_B \subset E = E_B$.

(c) According to [Rie2020, Thm. 4.2], all regular bosonic ground states are q-classical if $B > 0$, with integral decomposition over E_B^0. Now apply (b). □

General quasi-free dynamical symmetry transformations combine automorphisms and anti-automorphisms and are induced from symplectic transformations $w \in \mathrm{symp}(E, \sigma)$ (where $\mathrm{symp}(E, \sigma)$ consists of all real-linear bijections on E which leave σ invariant).

Definition 5.2. Consider the photonic Heisenberg dynamics $(\mathcal{W}(E, \hbar\sigma), \alpha_{\mathbb{R}})$ induced from the dynamical generator $B > 0$ on E.

(a) The general "dynamical symmetry group" G^α is defined as the abstraction of the transformation group $\{w \in \mathrm{symp}(E, \sigma) \mid [v_{\mathbb{R}}, w] = 0,\ w \text{ is } \tau\text{-continuous}\}$. The abstract group elements are identified with those acting on E and we write $w \in G^\alpha$.
(b) The dual actions are $(w'L)(f) := L(wf)$ for all $L \in E'$ and all $f \in E$, and the associated quasi-free automorphisms are written γ_w, $w \in G^\alpha$.
(c) The corresponding state transformations, given by duality as affine bijections, are denoted $\nu_w \colon \mathcal{S} \to \mathcal{S}$, $\nu_w \omega := \omega \circ \gamma_w$ for all $w \in G^\alpha$
(d) If K is self-adjoint on $\mathcal{H}$ we write $\exp\{i\mathbb{R}K\} \subset G_c^\alpha$ if $\exp\{i\mathbb{R}K\} \subset G^\alpha$ and K is τ-bounded.

Notice that $w \mapsto w'$ and $w \mapsto \gamma_w$ represent G^α anti-homomorphically, introduced so for notational simplicity. We apply these definitions, however, only to commutative subgroups of G^α.

Proposition 5.3. (a) *G^α is a group.*
(b) *Each $w \in G^\alpha$, extends uniquely to a unitary on $\mathcal{H}$.*

Proof. (a) For $w_i \in G^\alpha$, one shows the τ-continuity of the operator $w_1 w_2$ by iteration, and from [ReedSim1, Thm. 5.6] follows the τ-continuity of w_i^{-1} (where the second assertion uses (E, τ) being a Fréchet space).

(b) In [Weinl69] there is proven the following assertion: Suppose that the unitary group $U_{\mathbb{R}}$ on a complex Hilbert space $\mathcal{H}$ has a strictly positive generator and that S is a bounded real-linear transformation on $\mathcal{H}$ which commutes with $U_{\mathbb{R}}$. Then S is complex linear. A partial generalization, covering our case, is demonstrated in the proof of [HonRie15, I, Thm. 9.1-2]: If S is an only densely defined, possibly unbounded, real-linear symplectic transformation, its commuting with the strictly positive generated $U_{\mathbb{R}}$ makes it also complex linear. But a complex-linear symplectic transformation on E is uniquely extensible to a unitary on $\mathcal{H}$. $\square$

The following features are readily verified.

Observation 5.4. (a) Since $w(v_t f - f) = v_t w f - w f$, each $w \in G^\alpha$ leaves E_B invariant. Thus, by duality, we have also $E_B^0 \circ G^\alpha = G^{\alpha\prime} E_B^0 = E_B^0$.

(b) E'_{ess} is invariant under w', $w \in G^\alpha$, if and only if E_p is w-invariant, and then E_e is so, and $[w, P_p] = [w, P_e] = 0$. For, if $f \in E_p$ then also $L(w_s f) = w'_s L(f) = 0$ for all $L \in E'_{\mathrm{ess}}$ and thus $w_s f \in E_p$. Since $P_e = \mathbf{1} \dot{-} P_p$, also E_e is w-invariant. The implication $\Leftarrow$ goes similarly.

(c) Clearly $C_{\nu_w \omega}(f) = \langle \omega; \gamma_w W(f) \rangle = C_\omega(wf)$ for all $w \in G^\alpha$ and all $\omega \in \mathcal{S}$.

(d) If $\omega \in \mathcal{S}_{\mathrm{cl}}$ owns $C_\omega(f) = \int_{E'} C_0(f) \exp\{i\sqrt{2}\,\mathrm{Re}\, L(f)\}\, d\mu_\omega(L)$ then $C_{\nu_w \omega}(f) = \int_{E'} C_0(f) \exp\{i\sqrt{2}\,\mathrm{Re}\, L(f)\}\, d\mu_\omega^w(L)$, with $d\mu_\omega^w(L) := d\mu_\omega(L \circ w^{-1})$.

(e) Since $\mu_{\nu_w \omega} = \mu_\omega^w$ we have $\nu_w \mathcal{S}_{\mathrm{cl}} = \mathcal{S}_{\mathrm{cl}}$, as well as $\nu_w \partial_e \mathcal{S}_{\mathrm{cl}} = \partial_e \mathcal{S}_{\mathrm{cl}}$ for all $w \in G^\alpha$. Thus $\omega \in \mathcal{S}_{\mathrm{cl}}$ is ν_w-invariant if and only if $\mu_\omega^w = \mu_\omega$.

(f) Since $L \mapsto w' L$ respects (un-)boundedness, we conclude that each ν_w, $w \in G^\alpha$ leaves $\mathcal{S}_{\mathrm{Fcl}}$ and $\partial_e \mathcal{S}_{\mathrm{Fcl}}$, and also $\mathcal{S}_{\mathrm{scl}}$ and $\partial_e \mathcal{S}_{\mathrm{scl}}$, invariant.

§5.2. Effective symmetries and their Heisenberg generators

Effective features of symmetry transformations are revealed in a representation, for which we choose $(W_\mu(f), \mathcal{H}_\mu, \Omega_\mu)$ over a q-classical state ω_μ, $\mu \in M_p(E')$. Since a non-trivial representation of the simple Weyl algebra is faithful, the represented symmetry transformations in the observable picture are gained by setting

$$\gamma_s^\mu(\Pi_\mu(A)) := \Pi_\mu(\gamma_s(A)) \quad \forall\, A \in \mathcal{W}(E, \sigma),\; s \in \mathbb{R}. \tag{5.1}$$

If $\mu = \mu^{w_{\mathbb{R}}}$ *the* GNS unitary implementation is introduced as

$$U_s^\mu \Pi_\mu(A) \Omega_\mu := \Pi_\mu(\gamma_s(A)) \Omega_\mu, \quad A \in \mathcal{W}(E, \hbar\sigma),\; s \in \mathbb{R}. \tag{5.2}$$

By sandwiching the observables with the U_s^μ, the γ_s^μ are extended to $\mathcal{M}_\mu$, but keep their symbol. This mathematical standard procedure is physically of utmost significance in cases where $\mathcal{M}_\mu$ owns a non-trivial center. As we will demonstrate for the photons, the generators of U_s^μ may acquire thereby a uniquely determined classical part.

Notational Remark 5.5. If the *-algebra $\Delta(E, \sigma)$ of Definition 2.2 is restricted to the subspace $X \subset E$, we write $\Delta(X, \sigma)$, and if represented in $\mathcal{H}_\mu$, we write $\Delta_\mu(X, \sigma)$.

Proposition 5.6. *Consider $\mu \in M_p(E')$ invariant under $w_{\mathbb{R}} = \exp\{i\mathbb{R}K\} \subset G_c^\alpha$.*

(a) *Then $\mathbb{R} \ni s \mapsto U_s^\mu$ is strongly continuous and its generator K_μ annihilates Ω_μ.*

(b) *If ω_μ is a C^2-state (with twice continuously differentiable characteristic function $s \mapsto C_\mu(sf)$) then we get, by differentiation,*

$$K_\mu W_\mu(f)\Omega_\mu = \Phi_\mu(iKf)W_\mu(f)\Omega_\mu \quad \forall\, f \in E, \tag{5.3}$$

and $\Delta_\mu(E,\sigma)\Omega_\mu$ is a core for K_μ.

Proof. (a) It suffices to show that $C_0(w_s f)\int_{E'} \exp\{i\sqrt{2}\,\mathrm{Re}\, L(w_s f)\}\, d\mu(L)$ is continuous in $s \in \mathbb{R}$ (see [Rie2020, Lem. 2.4]), which results from the dominated convergence theorem.

(b) The right-hand side of equation (5.3) exists according to [HonRie15, I, Prop. 18.3-7] (concerning the domain of the field operator) and by K acting τ-continuously on E. The domain $\Delta_\mu(E,\sigma)\Omega_\mu$ is dense in $\mathcal{H}_\mu$ since $E \ni f \mapsto W_\mu(f)$ is τ-strong continuous, and it is $U_{\mathbb{R}}^\mu$-invariant, since E is $w_{\mathbb{R}}$-invariant. □

Observation 5.7. We remark that $E' \ni L \mapsto W^0(f)[L] = \exp\{i\sqrt{2}\,\mathrm{Re}\, L(f)\}$ is a more natural function of $F = \sqrt{2}\,\mathrm{Re}\, L$ taken from the *real-linear topological dual* $\mathcal{E}' := \mathrm{Re}(E')$. This leads to the following $\mathbb{R}$-linear w*-homeomorphism from E' onto $\mathcal{E}'$, the two dual spaces over (E,τ),

$$\begin{aligned} &E' \ni L \mapsto \sqrt{2}\,\mathrm{Re}\, L =: F_L \in \mathcal{E}', \\ &\mathcal{E}' \ni F \mapsto \frac{1}{\sqrt{2}}(F(\cdot) - i(F \circ i)(\cdot)) =: L_F \in E'. \end{aligned} \tag{5.4}$$

The function realizations of the classical field observables are then

$$\Phi^0(f)[F] = F(f), \quad W^0(f)[F] = \exp\{iF(f)\}, \quad \forall\, f \in E,\ \forall\, F \in \mathcal{E}', \tag{5.5}$$

$$a^{0*}(f)[F] := L_F(f) = \frac{1}{\sqrt{2}}(\Phi^0(f) - i\Phi^0(if))[F], \quad a^0(f)[F] := \overline{a^{0*}(f)[F]}. \tag{5.6}$$

We are going to develop a Poisson formalism on the whole of $\mathcal{E}'$ and may afterwards restrict the generators to functions on $\mathcal{E}'_{\mathrm{ess}}$ (or on E'_{ess}, if desired). Because of the duality procedure it is even sufficient to evaluate the symmetry generators on the embedded $\mathrm{Re}(E) \subset \mathcal{E}'$. It is important to keep the original complex test function space E, since in the present approach the classical fields are part of the total effective photon field, all of them smeared by the same test functions.

Notational Remark 5.8. If $\mu \in M_p(E')$ then we get the $\mathbb{R}$-transferred measure $\mu^{\mathbb{R}} \in M_p(\mathcal{E}')$. We *keep, however, the old symbols* of the functions and measures on

E' and identify the corresponding quantities on $\mathcal{E}'$ by means of their independent variables, or by context. In this manner it is not necessary to change the symbols for the dual dynamical symmetry transformations while going from E' to $\mathcal{E}'$.

Proposition 5.9. *Consider a $\mu \in M_p(\mathcal{E}'_{\text{ess}})$, invariant under $w_{\mathbb{R}} = \exp\{i\mathbb{R}K\} \subset G^\alpha$.*

(a) *We obtain for the unitary GNS-implementation $U^\mu_{\mathbb{R}}$ on $(\Pi_\mu, \mathcal{H}_\mu, \Omega_\mu)$,*

$$U^\mu_s W_\mu(f)\Omega_\mu := W_F(w_s f)\Omega_0 \otimes W^0_\mu(w_s f) 1_\mu, \quad f \in E,\ s \in \mathbb{R}. \tag{5.7}$$

(b) *It holds that ω_μ is a C^2-state if and only if $\int_{\mathcal{E}'_{\text{ess}}} F(f)^2\, d\mu[F] < \infty$ for all $f \in E$.*

(c) *If even $\exp\{i\mathbb{R}K\} \subset G^\alpha_c$ and ω_μ is C^2 then we obtain on a dense domain in $\mathcal{H}_\mu$, comprising $\Delta_\mu(E,\sigma)\Omega_\mu$,*

$$\begin{aligned}\frac{d}{ids} W_\mu(\exp\{isK\}f)|_{s=0} = [K_\mu, W_\mu(f)]|_- &= \Phi_\mu(iKf)W_\mu(f)\\ &= \Phi_F(iKf)W_F(f) \otimes W^0_\mu(f)\\ &\quad + W_F(f) \otimes \Phi^0_\mu(iKf)W^0_\mu(f) \quad \forall f \in E.\end{aligned} \tag{5.8}$$

Proof. (a) It follows from equation (4.1).

(b) The ω_μ is C^2 if and only if Ω_μ is in $\operatorname{dom}\Phi_\mu(f)$ for all $f \in E$, where the latter means for the classical part the existence of the indicated second moment for μ.

(c) $\Delta_\mu(E,\sigma)\Omega_\mu \subset \operatorname{dom}\Phi_\mu(f)$ for all $f \in E$. □

To evaluate $\Phi^0_\mu(iKf)W^0_\mu(f)$ we need a Poisson formalism.

§5.3. Poisson bracket formalism for classical generators

We introduce some notions for the Hamilton flows on the phase space $\mathcal{E}'$. The second dual $\mathcal{E}'' =: \mathcal{E}$, consisting of the $\sigma(\mathcal{E}', E)$-continuous, $\mathbb{R}$-linear functionals on $\mathcal{E}'$, may be realized via E and then identified with E. The tangent and cotangent bundles for the flat manifold $\mathcal{E}'$ are $T_F\mathcal{E}' = \mathcal{E}'$ and $T^*_F\mathcal{E}' = \mathcal{E}'' = E$ for each $F \in \mathcal{E}'$.

Definition 5.10. Consider $\mathcal{E}'$ as the phase space for classical fields.

(a) For each $f \in E$ we define an $\mathbb{R}$-linear, τ–w*-continuous, and w*-dense embedding $\sigma_\sharp\colon E \to \mathcal{E}'$ by $\sigma_\sharp(f)(g) := \sigma(f,g) = \operatorname{Im}(f|g)$ for all $g \in E$, as well as $\rho_\sharp(f)(g) := \operatorname{Re}(f|g) = -\sigma_\sharp(if)(g)$.

(b) The total differential $d_F A$ of a $\mathbb{C}$-valued function $A\colon \mathcal{E}' \to \mathbb{C}$ at $F \in \mathcal{E}'$ is given by $d_F A[G] := \frac{dA[F+sG]}{ds}\big|_{s=0}$ for all $G \in T_F\mathcal{E}'$, where the linear form

$d_F A\colon \mathcal{E}' \to \mathbb{C}$, $G \mapsto d_F A[G]$ is supposed w*-continuous, and thus $d_F A \in T_F^* \mathcal{E}' = E$ for all $F \in \mathcal{E}'$.

If $d_F A$ satisfies this condition for all $F \in \mathcal{E}'$, then A is called differentiable.

(c) We define the Poisson bracket $F \mapsto \{A, B\}[F]$ for two differentiable, $\mathbb{C}$-valued functions $A = A_1 + iA_2$, $B = B_1 + iB_2$ on $\mathcal{E}'$ in terms of a constant, $\mathbb{C}$-bilinear bivector field $\Sigma_F = \Sigma$ for all $F \in \mathcal{E}'$, called a *Poisson tensor*, as follows:

$$\begin{aligned} \{A, B\}[F] &:= \Sigma(d_F A, d_F B) \\ &= -\sigma(d_F A_1, d_F B_1) - i\sigma(d_F A_1, d_F B_2) \\ &\quad - i\sigma(d_F A_2, d_F B_1) + \sigma(d_F A_2, d_F B_2). \end{aligned} \tag{5.9}$$

For example, the total differentials of $\Phi^0(f)$ and $W^0(f)$ are

$$\begin{aligned} d_F \Phi^0(f) &= f \in T_F^* \mathcal{E}' = E \quad \forall\, f \in E,\ \forall\, F \in \mathcal{E}', \\ d_F W^0(f) &= if \exp\{iF(f)\} = (d_F \Phi^0(f)) i W^0(f)[F] \in E. \end{aligned} \tag{5.10}$$

Therefrom we obtain for all $f, g \in E$ (with 1 the unit function on $\mathcal{E}'$),

$$\{\Phi^0(f), \Phi^0(g)\} = -\sigma(f, g)1, \tag{5.11}$$

$$\{a^0(f), a^{0*}(g)\} = i(f|g)1, \quad \{a^0(f), \Phi^0(g)\} = \frac{i}{\sqrt{2}}(f|g)1, \tag{5.12}$$

$$\{\Phi^0(f), W^0(g)\} = \{\Phi^0(f), i\Phi^0(g)\} W^0(g), \tag{5.13}$$

$$\{W^0(f), W^0(g)\} = \sigma(f, g) W^0(f + g). \tag{5.14}$$

The last relation demonstrates that our present Poisson bracket, acting on complex functions on $\mathcal{E}'$, or on complex functions on any real-linear subspace $Y \subset \mathcal{E}'$, makes $\mathrm{LH}\{W^0(f) \mid f \in E\}$ into a *Poisson algebra* $\Delta^0(E, Y)$: a complex *-algebra of functions on Y, in which the anti-symmetric product $\{\cdot, \cdot\}$ satisfies the Leibniz rule and Jacobi identity and is *-real (i.e. $\{A, B\}^* = \{A^*, B^*\}$). The algebraic structure does in fact not depend on Y. If we restrict the $W^0(f)$ to test functions on a $\mathbb{C}$-linear subspace $X \subset E$ we write $\Delta^0(X, Y)$.

A "Hamilton function" $H^0\colon \mathcal{E}' \to \mathbb{R}$ for a differentiable flow $w'_{\mathbb{R}}$ on $\mathcal{E}'$ is characterized by

$$\frac{d}{ds} A \circ w'_s = \{H^0, A\} \circ w'_s \quad \forall\, s \in \mathbb{R}, \tag{5.15}$$

valid for certain differentiable phase space functions A. We have, sometimes however, to restrict H^0 to a subspace $Y \subset \mathcal{E}'$ and the A to a Poisson algebra $\Delta^0(X, Y)$.

Let us introduce for a given flow $w_{\mathbb{R}} = \exp\{i\mathbb{R}K\} \subset G_c^\alpha$ (where K is τ-continuous) and a given ONB (orthonormal basis) $b \equiv \{e_n\} \subset E$ of $\mathcal{H}$ the $\mathbb{C}$-linear *subspace* in $\mathcal{H}$, $E_b := \mathrm{LH}\{e_n \mid n \in \mathbb{N}\}$, which is norm dense in $\mathcal{H}$.

Theorem 5.11. *Consider a flow $w_{\mathbb{R}} = \exp\{i\mathbb{R}K\} \subset G_c^\alpha$ with its dual action $w'_{\mathbb{R}} = \exp\{i\mathbb{R}K'\}$ on $\mathcal{E}'$. Fix an ONB $b = \{e_n\} \subset E$ of $\mathcal{H}$ and form $E_b \subset E$.*

(a) *The phase space function*

$$F \mapsto d\Gamma^0(K)[F] := \Big(\sum_{n=1}^{\infty} a^{0*}(Ke_n)a^0(e_n)\Big)[F]$$
$$= \sum_{n=1}^{\infty} L_F(Ke_n)\overline{L_F(e_n)} \tag{5.16}$$

is defined on a domain which includes $\sigma_\sharp(E) \ni F$. On the latter we have

$$d\Gamma^0(K)[\sigma_\sharp(f)] = \frac{1}{2}\,\mathrm{Im}(f|iKf)$$
$$= \frac{1}{2}\,\mathrm{Re}(f|Kf) =: H_K^0[\sigma_\sharp(f)] \quad \forall f \in E \tag{5.17}$$

(setting $H_K^0 := d\Gamma^0(K)|_{\sigma_\sharp(E)}$).

(b) *Denote $d\Gamma_N^0(K) := (\sum_{n=1}^N a^{0*}(Ke_n)a^0(e_n))$ for $N \in \mathbb{N}$. Then we get for all $h \in E_b$,*

$$\lim_{N\to\infty}\{d\Gamma_N^0(K), \Phi^0(h)\}[F] = \Phi^0(iKh)[F] = F(iKh) \quad \forall F \in \mathcal{E}', \tag{5.18}$$

$$\lim_{N\to\infty}\{d\Gamma_N^0(K), W^0(h)\}[F] = (i\Phi^0(iKh)W^0(h))[F] \quad \forall F \in \mathcal{E}'. \tag{5.19}$$

(c) *For all $A \in \Delta^0(E_b, \mathcal{E}')$ it holds that*

$$\frac{d}{ds}A[w'_sF]\big|_{s=0} = \lim_{N\to\infty}\{d\Gamma_N^0(K), A\}[F] \quad \forall F \in \mathcal{E}'. \tag{5.20}$$

For all $A \in \Delta^0(E, \sigma_\sharp(E))$ we obtain

$$\frac{d}{ds}A[w'_sF]\big|_{s=0} = \{H_K^0, A\}[F] \quad \forall F \in \sigma_\sharp(E). \tag{5.21}$$

Proof. (a) Insert $F = \sigma_\sharp(f)$ into the last expression of equation (5.16) and find via Observation 5.7 that $\sqrt{2}L_F(e_n) = \mathrm{Im}(f|e_n) - i\,\mathrm{Re}(f|e_n) = -i(f|e_n)$. Then we obtain the series $\sum_n^\infty (f|Ke_n)(e_n|f)/2$.

(b) To show equation (5.18) use the last equation of (5.12) and find

$$\sum_n^N \Big[\frac{-i}{\sqrt{2}}(h|Ke_n)a^0(e_n) + \frac{i}{\sqrt{2}}a^{0*}(Ke_n)(e_n|h)\Big][F]$$

for the left-hand side. One can draw the sums into the field arguments and perform $N \to \infty$, which in fact involves only finitely many terms, and ends up with

$$\Big[\frac{-i}{\sqrt{2}} a^0(Kh) + \frac{i}{\sqrt{2}} a^{0*}(Kh)\Big][F] = \Phi^0(iKh)[F].$$

To show equation (5.19) use equation (5.13) and employ the preceding calculation.

(c) Make linear combinations of equation (5.19) to get equation (5.20).

To obtain equation (5.21) pull the N-limit into the Poisson bracket of equation (5.20) and employ equation (5.17). $\square$

We can now supplement Proposition 5.9 by the following assertion.

Theorem 5.12. *Choose a $\mu \in M_p(\mathcal{E}'_{\text{ess}})$, invariant under $w_{\mathbb{R}} = \exp\{i\mathbb{R}K\} \subset G_c^\alpha$, and such that $\omega_\mu \in \mathcal{S}_{\text{ess}}$ is a C^2-state. Fix an ONB $b = \{e_n\} \subset E$ of $\mathcal{H}$ and form $E_b \subset E$.*

We then obtain for the self-adjoint generator of the unitary GNS-implementation $U_{\mathbb{R}}^\mu = \exp\{i\mathbb{R}K_\mu\}$ on $\mathcal{H}_\mu = F_+(\mathcal{H}) \otimes L^2(\mathcal{E}'_{\text{ess}}, \mu)$ that K_μ is uniquely determined on the dense set $\Delta_\mu(E_b, \mathcal{E}'_{\text{ess}})\Omega_\mu \subset \mathcal{H}_\mu$ by the following relations, valid for any $f \in E_b$:

$$\begin{aligned} K_\mu(W_F(f) \otimes W_\mu^0(f))\Omega_\mu &= [d\Gamma(K), W_F(f)]\Omega_0 \otimes W_\mu^0(f)1_\mu \\ &\quad + W_F(f)\Omega_0 \otimes \Big(\lim_{N\to\infty}\{d\Gamma_N^0(K), W^0(f)\}\Big)_\mu 1_\mu, \end{aligned} \tag{5.22}$$

where $d\Gamma(K) = \sum_{n=1}^\infty a_F^*(Ke_n)a_F(e_n)$ and $d\Gamma_N^0(K) = \sum_{n=1}^N a^{*0}(Ke_n)a^0(e_n)$. Here A_μ is the multiplication operator on $L^2(\mathcal{E}'_{\text{ess}}, \mu)$ by the function $A\colon \mathcal{E}'_{\text{ess}} \to \mathbb{C}$.

In a certain formal sense, we may say that the ($\mathbb{R}$-linear, positive) mapping

$$K = K^* \mapsto \sum_{n=1}^\infty a_F^*(Ke_n)a_F(e_n) + \sum_{n=1}^\infty a^{*0}(Ke_n)a^0(e_n) \tag{5.23}$$

is a "classically extended second quantization", generating the symmetry transformations and dynamics of collectively dressed photon fields.

Proof. The first part of equation (5.22) is Fock space formalism. The classical part follows from the fact that equation (5.19), together with equation (5.8), gives the s-derivative of $W^0(\exp\{isK\}f)$ at $s = 0$, acting as multiplication operator on $L^2(\mathcal{E}'_{\text{ess}}, \mu)$.

Since E_b is norm dense in $\mathcal{H}$, the Fock part $\Delta_F(E_b, \mathcal{E}'_{\text{ess}})\Omega_0$ is dense in $F_+(\mathcal{H})$. In the classical part, $\Delta^0(E_b, \mathcal{E}'_{\text{ess}})$ is a *-algebra linearly generated by the characters $\chi_f[F] = \exp\{iF(f)\}$, $f \in E_b$ on $\mathcal{E}' \ni F$, which separates the points of $\mathcal{E}'$. This is so, because $\rho_\sharp(E_b)$ is w*-dense in $\mathcal{E}'$; see the proof of Theorem 3.3(c). Thus we can apply the Stone–Weierstrass theorem as in the proof of Theorem 4.1(c) and obtain the denseness of $\Delta^0_\mu(E_b,\mathcal{E}'_{\text{ess}})1_\mu$ in $L^2(\mathcal{E}'_{\text{ess}},\mu)$. □

Irrespectively of some technical restrictions, the effective photon dynamics and the symmetry transformations, together with their generators, appear in a uniform manner for the quantized particle and classical wave part. Due to the Poisson bracket structure for the classical generators, the K_μ are not affiliated with the representation von Neumann algebra $\mathcal{M}_\mu$, which means mathematically that $U^\mu_s \notin \mathcal{M}_\mu$, for $s \neq 0$. In axiomatic quantum field theory, such K_μ are sometimes discredited as not being proper observables, but their physical importance is confirmed in the following subsection.

§6. Maxwellian shape of the central generators and classical Maxwell fields

We recall now that there is a natural decomposition of a field theory into canonical components. We decompose $E \ni f = f_R + if_I \in E_R + iE_I$, where $E_R = E_I = \mathsf{S}^\top_0(\mathbb{R}^3,\mathbb{R}^3)$ and obtain from the Fock space relations equation (2.5) and from equation (5.11),

$$[\Phi_\mu(f_R),\Phi_\mu(f_I)] \subset i\hbar(f_R|g_I)\mathbf{1}_\mu, \quad \{\Phi^0_\mu(f_R),\Phi^0_\mu(f_I)\} = -(f_R|g_I)1_\mu. \tag{6.1}$$

These CCR relations fit to f_R indexing the "position field" and f_I the "momentum field", but this identification can be symplectically transformed.

In the real dual space $\mathcal{E}' \ni F$, underlying the classical central theory as phase space, the decomposition $F(f) = F_R(f_R) + F_I(f_I)$ is given by the unique restrictions of F to E_R and of $F \circ i$ to E_I. We write then $F(f) = (F_R,F_I)(f) = F_R(f_R) + F_I(f_I)$, and denote $\rho_\sharp(f) = (f_R,f_I)$ and $\sigma_\sharp(f) = (-f_I,f_R)$, both being elements (F_R,F_I) in $\mathcal{E}'$.

In E it holds by transversality $-\Delta = \text{curl}^2$, and the involvement of the curl seems to be at the heart of the spin-one phenomenon. By the *$\mathbb{C}$-linearity of B*, its dual B' acts in the same manner on F_R and F_I (and so do the related transformations on $\Phi^0_\mu(f_R)$ and $\Phi^0_\mu(f_I)$). But two real fields, with equal phase spaces and the same energy and other fundamental symmetry generators, are a hardly conceivable phenomenon. The transition to real field theory should, therefore, be accompanied by a different scaling of the two components. So $F = (F_R,F_I) \in \mathcal{E}'$ should be replaced by

$$\breve{F} = T'^{-1}F := (\mathbf{s}'F_R, \mathbf{s}^{-1'}F_I) \in T'^{-1}\mathcal{E}' \equiv \breve{\mathcal{E}}', \tag{6.2}$$

where $\mathbf{s}'$ denotes a "scaling" operator, and where we discriminate conceptually $T'^{-1}\mathcal{E}'$ from the mathematically equal $\mathcal{E}'$. The pre-dual operators are then $T^{-1}f = (\mathbf{s}f_R + i\mathbf{s}^{-1}f_I)$ and lead to a symplectic transformation $T^{-1}\colon E \to E$, where we denote $T^{-1}E \equiv \breve{E}$ if we want to indicate the different physical meaning of the transformed test functions. The operator $\mathbf{s}\colon E_R \to E_R$ should involve the curl and should, together with its inverse, distribute the energy differently onto the two field components. We make the most simple ansatz $\mathbf{s} = (\eta B)^{1/2}$, with $B = c\sqrt{-\Delta}$ and $\eta > 0$ a numerical factor, and observe that $(\eta B)^{\pm 1/2}$ is self-adjoint and τ-continuous. This gives

$$T^{-1}(f) = T^{-1}(f_R + if_I) = (\eta B)^{1/2} f_R + i(\eta B)^{-1/2} f_I =: \breve{f}_R + i\breve{f}_I = \breve{f}, \tag{6.3}$$

$$T\breve{f} = T(\breve{f}_R + i\breve{f}_I) = (\eta B)^{-1/2}\breve{f}_R + i(\eta B)^{1/2}\breve{f}_I = f. \tag{6.4}$$

In Maxwell theory the conserved quantities energy, momentum, and angular momentum ("Hamilton functions" in mathematical mechanics [AbraMars]) are expressed as integrals over assumedly square-integrable electrodynamic fields. We face the dilemma that we gained here autonomous classical fields only, if they just are not square integrable. But it is general experience that a microscopically derived macrodynamics requires a new scaling of position, and perhaps time. For example, the electrodynamic field of a ring laser is macroscopically square integrable, but singular if viewed from microscopic photon theory. We employ therefore fields from $\sigma_\sharp(E)$, tacitly assuming the macroscopic length scale.

First we want to calculate the energetic Hamilton function in the scaled theory. Since the scale transformation $T^{-1}\colon E \to \breve{E}$ is symplectic and τ-differentiable, the Poisson formalism is transferred isomorphically to the real dual $\breve{\mathcal{E}}'$ of the still complex $\breve{E}$. Then $H^0_B[\sigma_\sharp(f)]$ transforms to $\breve{H}^0_{\breve{B}}[\sigma_\sharp(\breve{f})]$ with $\breve{B} = T^{-1}iBT$ the anti-self-adjoint generator of the dynamics on $\breve{E}$. Referring to the first version of equation (5.17) we obtain $\breve{H}^0_{\breve{B}}[\sigma_\sharp(\breve{f})] = \frac{1}{2}\,\mathrm{Im}(\breve{f}|T^{-1}iBT\breve{f})$. We calculate

$$\begin{aligned}\breve{B}\breve{f} = T^{-1}iBT(\breve{f}_R + i\breve{f}_I) &= T^{-1}iB((\eta B)^{-1/2}\breve{f}_R + i(\eta B)^{1/2}\breve{f}_I)\\ &= -\eta B^2\breve{f}_I + \frac{i}{\eta}\breve{f}_R.\end{aligned} \tag{6.5}$$

With $(\breve{F}_R, \breve{F}_I) = (-\breve{f}_I, \breve{f}_R)$ we get

$$\begin{aligned}\breve{H}^0_{\breve{B}}[\sigma_\sharp(\breve{f})] &= \frac{1}{2}(\breve{f}_I|\eta B^2\breve{f}_I) + \frac{1}{2}\Big(\breve{f}_R\Big|\frac{1}{\eta}\breve{f}_R\Big)\\ &= \frac{1}{2}(\breve{F}_R|\eta B^2\breve{F}_R) + \frac{1}{2}\Big(\breve{F}_I\Big|\frac{1}{\eta}\breve{F}_I\Big).\end{aligned}$$

We now *denote* $\breve{\mathcal{E}}' \ni (\breve{F}_R, \breve{F}_I) \equiv (\mathbf{A}, \mathbf{Y})$.

By the self-adjointness of curl we arrive at the usual classical Hamiltonian for the transverse canonical and force fields of sourceless electrodynamics,

$$\breve{H}^0_{\breve{B}}[(\mathbf{A},\mathbf{Y})] = \frac{1}{2\eta}(\mathbf{Y}|\mathbf{Y}) + \frac{\eta c^2}{2}(\operatorname{curl}\mathbf{A}|\operatorname{curl}\mathbf{A}) \quad (\forall\,(\mathbf{A},\mathbf{Y}) \in \sigma_\sharp(E) \subset \mathcal{E}')$$

$$(6.6) \qquad = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} (\mathbf{E}(x)^2 + c^2\mathbf{B}(x)^2)\, d^3x, \quad \epsilon_0 := \eta,\ \mathbf{B} := \operatorname{curl}\mathbf{A},\ \mathbf{E} := -\mathbf{Y}/\epsilon_0.$$

Since $\hbar$ dropped out from the generators, the two physical constants $\hbar$ and c of the photon theory are in the classical subtheory replaced by ϵ_0 and c, where ϵ_0 is the value of the scaling parameter η determined classically by methods of field measurements.

For calculating the momentum "Hamilton function" we remark that $\breve{p}_n = T^{-1}ip_nT = n{\cdot}\nabla$ is a real operator and the subsequent calculation can be performed without accent,

$$\begin{aligned}\breve{H}^0_{\breve{p}_n}[\sigma_\sharp(\breve{f})] &= \frac{1}{2}\operatorname{Re}(\breve{f}|(-i)(n\cdot\nabla)\breve{f}) = \frac{1}{2}\operatorname{Im}(f|(n\cdot\nabla)f) \\ &= \frac{1}{2}\big\{(f_R|(n\cdot\nabla)f_I) - (f_I|(n\cdot\nabla)f_R)\big\} = -(\mathbf{Y}|(n\cdot\nabla)\mathbf{A}),\end{aligned}$$

using that $\partial/\partial x_j$ is anti-self-adjoint. The relations

$$n\cdot[\mathbf{Y}\times(\nabla\times\mathbf{A})] = \mathbf{Y}\cdot[(n\cdot\nabla)\mathbf{A}] - n\cdot[(\mathbf{Y}\cdot\nabla)\mathbf{A}],$$

$$\int_{\mathbb{R}^3} n\cdot[(\mathbf{Y}\cdot\nabla)\mathbf{A}]\,d^3x = -\int_{\mathbb{R}^3}[n\cdot\mathbf{A}][\underbrace{\nabla\cdot\mathbf{Y}}_{=0}]\,d^3x = 0$$

(observing the transversality of $\mathbf{Y}$) imply that

$$(6.7) \qquad \breve{H}^0_{\breve{p}_n}[(\mathbf{A},\mathbf{Y})] = -\int_{\mathbb{R}^3}\mathbf{Y}\cdot[(n\cdot\nabla)\mathbf{A}]\,d^3x = -\int_{\mathbb{R}^3} n\cdot[\mathbf{Y}\times(\nabla\times\mathbf{A})]\,d^3x$$

$$(6.8) \qquad = \epsilon_0\mu_0\int_{\mathbb{R}^3} n\cdot\Big[\mathbf{E}(x)\times\frac{1}{\mu_0}\mathbf{B}(x)\Big]\,d^3x = \frac{1}{c^2}\int_{\mathbb{R}^3} n\cdot\mathbf{S}(x)\,d^3x,$$

where $\mathbf{S}(x) := \mathbf{E}(x)\times\frac{1}{\mu_0}\mathbf{B}(x)$, with $\mu_0 = (\epsilon_0c^2)^{-1}$, denotes the Poynting vector of the transverse fields.

For calculating the "Hamilton function" for $\mathsf{L_n} = -in\cdot(\mathbf{I} - (x\times\mathbf{p}))$ (with again a real $i\mathsf{L_n}$), we need a more complicated vector analysis (similar to that given in [HonRie15, I] in a different context, and to certain pieces in [CohTann, Sect. I, C.5]). According to Appendix B we obtain in fact

$$\begin{aligned}\breve{H}^0_{\breve{\mathsf{L}}_n}[(\mathbf{A},\mathbf{Y})] &= \frac{1}{2}\operatorname{Im}(f|i\mathsf{L_n}f) = \epsilon_0\int_{\mathbb{R}^3} n\cdot\{x\times[\mathbf{E}(x)\times\mathbf{B}(x)]\}\,d^3x \\ (6.9) \qquad &= \frac{1}{c^2}\int_{\mathbb{R}^3} n\cdot\{x\times\mathbf{S}(x)\}\,d^3x.\end{aligned}$$

Altogether we have gained through a general formula the field "Hamilton functions" from the photonic symmetry generators.

If we want to evaluate the Poisson bracket transformations for, e.g., the dynamics, we have to use the canonical variables. In the original Poisson formalism this is given in equation (5.18). For $F \in \mathcal{E}'$ in $\sigma_\sharp(E)$ we have that $\lim_{N\to\infty} d\Gamma_N^0(B)$ converges to H_B^0 and find that

$$\begin{aligned}\{H_B^0, \Phi^0(f)\}[F] &= \{H_B^0[F], F(f)\} = F(iBf)\\ &= \frac{d}{dt}F(\exp\{itB\}f)_{t=0} \quad \forall\, f \in E.\end{aligned} \tag{6.10}$$

The second Poisson bracket is written in the usual way, identifying the observable $\Phi^0(f)[F]$ with the smeared phase space element. Application of the scaling transformation produces

$$\{\breve{H}_{\breve{B}}^0[\breve{F}], \breve{F}(\breve{f})\} = \breve{F}(T^{-1}iBf) = \breve{F}(\breve{B}\breve{f}) = \frac{d}{dt}\breve{F}(\exp\{t\breve{B}\}\breve{f})_{t=0} \quad \forall\, \breve{f} \in \breve{E}. \tag{6.11}$$

Using equation (6.5) for $\breve{B}\breve{f}$ we find in matrix notation

$$\breve{B}\begin{pmatrix}\breve{f}_R\\ \breve{f}_I\end{pmatrix} = \begin{pmatrix}0 & -\eta B^2\\ \frac{1}{\eta}\mathbf{1} & 0\end{pmatrix}\begin{pmatrix}\breve{f}_R\\ \breve{f}_I\end{pmatrix} = \begin{pmatrix}0 & \eta c^2\Delta\\ \frac{1}{\eta}\mathbf{1} & 0\end{pmatrix}\begin{pmatrix}\breve{f}_R\\ \breve{f}_I\end{pmatrix}. \tag{6.12}$$

The generator on the dual variables $(\mathbf{A}, \mathbf{Y}) \in \breve{\mathcal{E}}'$ is gained by the transposed matrix

$$\begin{pmatrix}\dot{\mathbf{A}}\\ \dot{\mathbf{Y}}\end{pmatrix} = \begin{pmatrix}\dot{\mathbf{A}}\\ -\epsilon_0\dot{\mathbf{E}}\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{\epsilon_0}\mathbf{1}\\ -\frac{1}{\mu_0}\operatorname{curl}^2 & 0\end{pmatrix}\begin{pmatrix}\mathbf{A}\\ \mathbf{Y}\end{pmatrix} = \begin{pmatrix}-\mathbf{E}\\ -\frac{1}{\mu_0}\operatorname{curl}\mathbf{B}\end{pmatrix}. \tag{6.13}$$

But the first line $\dot{\mathbf{A}} = -\mathbf{E}$ is equivalent to $\dot{\mathbf{B}} = -\operatorname{curl}\mathbf{E}$ since curl is bijective on the transverse fields. Together with $\nabla\cdot\mathbf{B} = 0$ we read off the three sourceless transverse Maxwell equations. To add on the left-hand side in the second line a transverse current density $-\mathbf{j}^\top$ would require in the present quantum–particle foundation of the classical theory a minimal coupling ansatz to charged fermions (see Appendix D).

The decisive point is that the calculations have been performed in the test function space and then were transposed to the dual space. But this can be executed with the smeared *quantized fields* too.

§7. Quantum Maxwell fields as transformed photon fields and discussion

First we sketch the transition to the quantum Maxwell fields algebraically.

Definition 7.1. As before, we call $\mathcal{W}(E,\sigma)$ the "photon Weyl algebra" and assume for convenience $\hbar = 1$.

(a) We distinguish conceptually the "quantized Maxwell Weyl algebra" $\mathcal{W}(\breve{E},\sigma)$ from $\mathcal{W}(E,\sigma)$ and denote $\mathcal{W}(\breve{E},\sigma) \ni W^M(\breve{f}) := W(T\breve{f}) =: (\alpha_M W)(\breve{f})$.

Clearly, $\alpha_M \colon \mathcal{W}(E,\sigma) \to \mathcal{W}(\breve{E},\sigma)$ is a *-isomorphism "onto" (in fact, a differently written Bogoliubov automorphism on the photon algebra $\mathcal{W}(E,\sigma)$).

(b) The photon dynamics α_t on $\mathcal{W}(E,\sigma)$ induces the "quantized Maxwell dynamics" $W_t^M(\breve{f}) := (\alpha_t W)(T\breve{f}) = W(\exp\{itB\}T\breve{f}) = W^M(\exp\{t\breve{B}\}\breve{f})$.

(c) To $\omega \in \mathcal{S}(W(E,\sigma)) \equiv \mathcal{S}$ we associate a $\nu_M\omega \in \breve{\mathcal{S}}$ by setting $\langle \nu_M\omega; W^M(\breve{f})\rangle := \langle \omega; (\alpha_M^{-1}W^M)(T\breve{f})\rangle$, with $\breve{\mathcal{S}} := \mathcal{S}(W(\breve{E},\sigma))$.

(d) If $(\Pi, \mathcal{H}_\Pi)$ is a representation of $\mathcal{W}(E,\sigma)$, induced by the $\{W_\Pi(f) \mid f \in E\}$, then the Weyl operators $\{W^M_{\breve{\Pi}}(\breve{f}) := W_\Pi(T\breve{f}) \mid \breve{f} \in \breve{E}\}$ generate a represented Maxwell Weyl algebra $\mathcal{W}_{\breve{\Pi}}(\breve{E},\sigma)$, which acts also on $\mathcal{H}_\Pi$ and for which we write $(\breve{\Pi}, \mathcal{H}_\Pi) = (\Pi \circ \alpha_M^{-1}, \mathcal{H}_\Pi)$, and inversely $(\breve{\Pi} \circ \alpha_M, \mathcal{H}_\Pi) = (\Pi, \mathcal{H}_\Pi)$.

Especially for the photonic Fock representation $(\Pi_F, F_+(\mathcal{H}))$ we obtain the quantum Maxwell representation $(\breve{\Pi}_F, F_+(\mathcal{H}))$ inheriting the Weyl operators $W_F^M(\breve{f}) = W_F(T\breve{f})$.

(e) For a given regular photonic representation $\mathcal{W}_\Pi(E,\sigma)$, where automatically also $\mathbb{R} \ni s \mapsto W^M_{\breve{\Pi}}(s\breve{f})$ is weak operator-continuous for all $\breve{f} \in \breve{E}$, we define the self-adjoint *quantized Maxwell field* $\Phi^M_{\breve{\Pi}}(\breve{f})$ as the Stone generator (*which has no abstract counterpart*!).

We introduce the splitting $\Phi^M_{\breve{\Pi}}(\breve{f}) = \Phi_{\breve{\Pi},\mathbf{A}}(\breve{f}_R) + \Phi_{\breve{\Pi},\mathbf{Y}}(\breve{f}_I)$ and obtain thereby the canonical field operator components. We set further on $\Phi_{\breve{\Pi},\mathbf{B}}(\breve{f}_R) := \operatorname{curl}\Phi_{\breve{\Pi},\mathbf{A}}(\breve{f}_R) = \Phi_{\breve{\Pi},\mathbf{A}}(\operatorname{curl}\breve{f}_R)$ and $\Phi_{\breve{\Pi},\mathbf{E}}(\breve{f}_I) := -\Phi_{\breve{\Pi},\mathbf{Y}}(\breve{f}_I)/\epsilon_0$.

Theorem 7.2. *In terms of the foregoing definitions we derive the following relations.*

(a) *ν_M is an affine isometry which maps the orthomodular lattice $F(\mathcal{S})$ of norm-closed state faces isomorphically onto $F(\breve{\mathcal{S}})$ (see Appendix A.1). In particular, the folium $\mathcal{F}_\tau \subset \mathcal{S}$ (of τ-continuous states on $\mathcal{W}(E,\sigma)$) is mapped onto the folium $\breve{\mathcal{F}}_\tau \subset \breve{\mathcal{S}}$ (of τ-continuous states on $\mathcal{W}(\breve{E},\sigma)$).*

(b) *The Fock representation Π_F and the Maxwell Fock representation $\breve{\Pi}_F$, which both act irreducibly on $F_+(\mathcal{H})$, cannot be unitarily transformed into each other, meaning disjointness.*

(c) *The Maxwellian field operators in a GNS-representation $\breve{\Pi}$ of a time-invariant C^2-state $\breve{\omega}$ satisfy the Maxwell equations for transverse fields on a dense time-*

invariant domain in the following way if we write the Maxwell field in tuple notation:

$$\dot{\Phi}^M_{\breve{\Pi}}(\breve{f}) = \begin{pmatrix} \dot{\Phi}_{\breve{\Pi},\mathbf{A}}(\breve{f}_R) \\ \dot{\Phi}_{\breve{\Pi},\mathbf{Y}}(\breve{f}_I) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\epsilon_0}\mathbf{1} \\ -c^2 \operatorname{curl}^2 & 0 \end{pmatrix} \begin{pmatrix} \Phi_{\breve{\Pi},\mathbf{A}}(\breve{f}_R) \\ \Phi_{\breve{\Pi},\mathbf{Y}}(\breve{f}_I) \end{pmatrix}. \tag{7.1}$$

The remaining two Maxwell equations

$$\nabla \cdot \Phi_{\breve{\Pi},\mathbf{B}} = 0, \quad \nabla \cdot \Phi_{\breve{\Pi};\mathbf{E}} = 0, \tag{7.2}$$

hold in some sense since we have no electric or magnetic material sources, which would microscopically be situated in another field space (see Appendix D).

(d) *If $\mathcal{S}_{\mathrm{ess}}$ is associated with $E'_{\mathrm{ess}} \subset E'_{\mathrm{s}}$ according to Definition 3.13(d) then $\nu_M \mathcal{S}_{\mathrm{ess}}$ is associated with $(T^{-1})'\mathcal{E}'_{\mathrm{ess}}$, where $\mathcal{E}'_{\mathrm{ess}} = \{\sqrt{2}\,\mathrm{Re}\, L \mid L \in E'_{\mathrm{ess}}\}$.*

Thus a photonic fusion representation $(\Pi_\mu, \mathcal{H}_\mu)$ of $\mathcal{W}(E,\sigma)$ gives rise to a fusion representation of the quantized Maxwell field algebra $\mathcal{W}(\breve{E},\sigma)$, and vice versa.

Proof. (a) According to [Kadison65], ν_M is an affine isometry and, therefore, maps the lattice of state faces ortho-isomorphically onto itself.

Observe that for the characteristic function of $\nu_M\omega$ it holds that $C_{\nu_M\omega}(\breve{f}) = C_\omega(T\breve{f})$ and that T is τ-continuous.

(b) Since [Shale62] one knows that a Bogoliubov transformation α_T is unitary implementable on Fock space, if and only if the anti-linear part $1/2(T + iTi)$ is Hilbert–Schmidt, which is not the case for the present α^M.

(c) The linear hull $\mathrm{LH}\{\Phi^M_{\breve{\Pi}}(\breve{f})\Omega_{\breve{\omega}} \mid \breve{f} \in \breve{E}\}$ is dense in the representation space and time invariant since $\exp\{t\breve{B}\}$ leaves $\breve{E}$ invariant. On this domain, $t \mapsto \Phi^M_{\breve{\Pi}}(\exp\{t\breve{B}\}\breve{g})$ is defined and differentiable for all $\breve{g} \in \breve{E}$ since also $\breve{B}$ leaves $\breve{E}$ invariant due to the infrared regularization.

Explicitly, we find for the time derivative (at $t = 0$) $\dot{\Phi}^M_{\breve{\Pi}}(f) = \Phi^M_{\breve{\Pi}}(\breve{B}\breve{f}) = \Phi^M_{\breve{\Pi}}(-\eta B^2 \breve{f}_I + (1/\eta)\breve{f}_R)$ (see equation (6.5)). This, rewritten in the canonical components (like in equation (6.12)) and transposed, leads to equation (7.1).

(d) This is only a matter of notation. □

For gaining the contact to the usual introduction of the quantized Maxwell fields in non-relativistic QED we generalize equation (2.4) to any regular representation $\breve{\Pi}$ expressing $\Phi^M_{\breve{\Pi}}$ by the creation and annihilation operators of Π,

$$\begin{aligned}\Phi^M_{\breve{\Pi}}(\exp\{t\breve{B}\}\breve{f}) &= \Phi_\Pi(\exp\{itB\}T\breve{f}) \\ &= \sqrt{\hbar/2}[a_\Pi(\exp\{itB\}T\breve{f}) + a^*_\Pi(\exp\{itB\}T\breve{f})]\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\hbar/2}\sum_{n=1}^{\infty}[\exp\{-it\omega_n\}a_\Pi(e_n)(T\breve{f}|e_n) \\
&\qquad + \exp\{it\omega_n\}a_\Pi^*(e_n)(e_n|T\breve{f})],
\end{aligned}
\tag{7.3}
$$

where we formally have assumed an orthonormal eigensystem $\{e_n\} \subset E$ of B with eigenvalues ω_n. This usually is only written down for $\Pi = \Pi_F$, in which case we have the creation and annihilation operators of Section 2. The quantized $\mathbf{A}$ potential is obtained by restricting the $\breve{f}$ in (7.3) to $\breve{f}_R \in E_R$ and the quantized $\mathbf{E}$ by restricting $\breve{f}$ in (7.3) to $\breve{f}_I \in E_I$, and by first using equation (6.4) and then $\Phi_{\mathbf{Y},t} = -\epsilon_0 \Phi_{\mathbf{E},t}$. One finds

$$
\begin{aligned}
\Phi_{\breve{\Pi},\mathbf{A},t}(\breve{f}) &= \sum_{n=1}^{\infty} \frac{\sqrt{\hbar}}{\sqrt{2\epsilon_0\omega_n}}[\exp\{-it\omega_n\}a_\Pi(e_n)(\breve{f}_R|e_n) \\
&\qquad + \exp\{it\omega_n\}a_\Pi^*(e_n)(e_n|\breve{f}_R)], \\
\Phi_{\breve{\Pi},\mathbf{E},t}(\breve{f}) &= \sum_{n=1}^{\infty} \frac{\sqrt{\hbar\epsilon_0\omega_n}}{\sqrt{2}}[\exp\{-it\omega_n\}a_\Pi(e_n)(\breve{f}_R|e_n) \\
&\qquad + \exp\{it\omega_n\}a_\Pi^*(e_n)(e_n|\breve{f}_R)].
\end{aligned}
$$

In this form the Maxwellian field operators arise usually through "quantizing" the corresponding classical fields in their Fourier decompositions by replacing the numerical coefficients through the Fock annihilation and creation operators (cf. e.g. [Vogel, Sect. 2.2.2]). This is a well-approved, but in first line a structural-theoretical concept. It works trouble-free for the transverse fields only. The relativistic extension, including the longitudinal and scalar fields, leads in the Gupta–Bleuler theory to "ghosts" ([Schweb62, Chap. 9b]).

In contrast to the quantization program which presupposes the classical field theory, the present treatment deduces the quantized and classical transverse Maxwell fields from the experimentally well-confirmed photon concept. This strategy also has certain mathematical advantages, since, e.g., the Maxwellian dynamics is (also classically) easier to handle in the complexified form resulting from the photon ansatz. In particular, the *algebraic* photon theory also gives us, for the quantized Maxwell fields, interesting structural, as well as interpretational, insights into various Hilbert space representations.

It is not the place here to give more details. Let us only mention that our "fusion representations" $(\Pi_\mu, \mathcal{H}_\mu)$, $\mu \in M_p(E'_{\mathrm{ess}})$ lead in the Maxwellian form to a fusion of quantized and classical fields, where the quantized part is again subdivided into a strictly quantum and a transient component. The main characteristic of the quantum components is the incompatibility of the magnetic and electric fields, where only one of the two fields may be actualized. By comparison with the particle formulation one may assume that the strictly quantum part is sharply localized in absorption processes, and its magnetic and electric field components are experimentally not discernible. The observation of the non-classical features of light, with, e.g., a squeezed quadrature component (e.g. [Vogel]), should be attributed to the transient part. If we transform equation (4.10) into the form of Maxwell fields it shows a continuous increase of the compatibility of their magnetic and electric components in approaching a classical wave by exhibiting a decreasing commutator. If, in the Planck law, one varies the temperature T from 0 to ∞, for constant frequency $\omega = c|k|$, one comes from the particle-like Wien asymptotics to the wave-like Rayleigh–Jeans asymptotics. The same region is bridged by the transient wave functions $\hat{f}_{k,\lambda}/\|\hat{f}_{k,\lambda}\| \in E_e \subset \overline{E_p}$, $0 \leq \lambda \leq \infty$ (see the example preceding Section 3.3), tending from 0 – and leaving only photon particles there – to the plane wave $\hat{e}_k$. The interior of this region is also the domain of Einstein's "fields of force". Large T was for Einstein equivalent to large radiation density (see [Haar69, p. 30]) and thus equivalent to especially many overlapping "fields of force". If one uses n exemplars of $\hat{f}_{k,\lambda}/\|\hat{f}_{k,\lambda}\|$ for a macroscopic filling of the k-mode (cf. [HonRie15, I, p. 752]), then the increase rate of $\mathbb{N} \ni n \mapsto \lambda(n)$ can be related to the photon density. The "transient photon" concept leads, therefore, to a more detailed microscopic reconsideration of the Planck law.

In many investigations of quantum optics, one and the same phenomenon, as, e.g., the light-electric effect or laser radiation emission, is described partly quantum mechanically and partly classically (see "semiclassical approximation" as, e.g., in [ScullyZub]). It is therefore useful to have a consistent theoretical frame which covers both areas simultaneously.

The *general aspects* of a fusion of photons and fields are most smoothly expressed in the state space $\mathcal{S}$ of the photonic field algebra $\mathcal{W}(E, \hbar\sigma)$. That notion is immediately given by the abstract C*-algebra and does not require representation theory. A dynamical symmetry group $\nu^K_{\mathbb{R}}$ in the algebraic Schrödinger picture (see Definition 5.2) leaves the stable simplicial face $\mathcal{S}_{\text{eff}}$ of the pertinent singular q-classical states invariant. Since it is the dual of a "global" automorphism group it can shift extremal $\omega \in \partial_e\mathcal{S}_{\text{eff}}$ into disjoint extremal states. In fact, Maxwell solutions appear now as trajectories in $\partial_e\mathcal{S}_{\text{eff}}$.

But also, more practical results are profitably treated in the algebraic state picture. In the original parametrization $\partial_e\mathcal{S}_{\text{eff}} = \{\omega_L \mid L \in E'_{\text{eff}}\}$ we may denote by $\varphi_\mu = \int^{\oplus}_{E'_{\text{eff}}} \varphi_L \, d\mu(L)$, with φ_L normal states on $\mathcal{M}_L$, the central decomposition of any normal state on $\mathcal{M}_\mu$, with $\mu \in M_p(E'_{\text{eff}})$ (see [Takes79] for the direct decomposition of normal states). By virtue of our singular, possibly infinite-dimensional field spaces E'_{eff}, we obtain a direct, but substantial, generalization of the one-dimensional integration domain in [HonRie15, III, formula (48.4.48)] for the square root of the C*-algebraic transition probability (see Appendix A):

$$T^{\frac{1}{2}}_{\mathcal{W}(E,\hbar\sigma)}(\varphi_\mu,\varphi'_{\mu'}) = \int_{E'_{\mathrm{eff}}} T^{\frac{1}{2}}_{\mathcal{W}(E,\hbar\sigma)}(\varphi_L,\varphi'_L)\Big[\frac{d\mu}{d\bar{\mu}}\frac{d\mu'}{d\bar{\mu}}\Big]^{\frac{1}{2}}(L)\,d\bar{\mu}(L), \tag{7.4}$$

where $\bar{\mu}$ signifies any (positive) measure to which μ and μ' are absolutely continuous.

Since the transition probabilities describe the uncontrollable state fluctuations, formula (7.4) gives us a neat specification of how the total noise in an information-carrying electromagnetic field is decomposed into the classical field noise and the photonic quantum noise. So we find that in $[\cdot]^{\frac{1}{2}}$ the classical noise is expressed by the classic-statistical overlap of the electrodynamic field distributions, mostly known in mathematical statistics (cf. [Hida80, Sect. I.3]). In each sector L, $T_{\mathcal{W}(E,\hbar\sigma)}(\varphi_L,\varphi'_L)$ gives the quantum mechanical transition probability of the (not necessarily pure) photon states φ_L, $\varphi'_L \in \mathcal{S}_n(\mathcal{M}_L)$. This is the part which would be affected by our discrimination between particle and transient photons and could refine quantum noise theory.

We have shown here that the division of the total photonic state space $\mathcal{S}$ into sectorial folia, separated by vanishing transition probabilities between photonic states with macroscopic differences of intensities (and not by the influence of a hypothetical noisy environment, e.g. [Schlossh]) has led us to a foliation of $\mathcal{S}$ which represents (the influence of) classical observables. Stable simplicial folia, like the present $\mathcal{S}_{\mathrm{ess}} \subset \mathcal{S}$, indicate the emergence of autonomous subtheories. If they display extremal boundaries which consist of pairwise disjoint pure states, then any continuous trajectory in this boundary describes a classical flow. Such pure boundary states constitute a sharpening of the concept of “ontic quantum states” discussed in [Primas]. Since also the decomposition measures and transition probabilities are formulated in the state picture, we may conclude that the convex state space approach ([HonRie15, III, Chap. 47]) offers a most elegant form, not only for the photonic fusion theory, but also quite generally for a unification of the quantum mechanical and classical world view.

Appendix A. Basics for the state space

Appendix A.1. Disjoint states, projections, global automorphisms

We sketch some basic concepts for the states $\omega \in \mathcal{S}(\mathcal{A}) \equiv \mathcal{S}$ on a unital C*-algebra $\mathcal{A}$, that are linear, positive, and normalized functionals on $\mathcal{A}$, and introduce our notation.

If $\mathcal{A}$ is the C*-Weyl algebra $\mathcal{W}(E,\hbar\sigma)$, $E \neq 0$, as introduced in Definition 2.2, then there is an affine bijection $\mathcal{S} \ni \omega \mapsto C_\omega \in \mathcal{C}(E,\hbar\sigma)$ which is w*-Δ continuous, where the convex set of characteristic functions $\mathcal{C}(E,\hbar\sigma)$ contains the normalized twisted-positive definite functions (cf. equation (2.11)) on E, and Δ denotes the topology of pointwise convergence (see [HonRie05]).

For each $\omega \in \mathcal{S}$ there exists a triple $(\Pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ consisting of a representation $(\Pi_\omega, \mathcal{H}_\omega)$, with a normalized cyclic vector $\Omega_\omega \in \mathcal{H}_\omega$, reproducing the expectations $\langle \omega; A\rangle = (\Omega_\omega | \Pi_\omega(A)\Omega_\omega)$ for all $A \in \mathcal{A}$. This *GNS-triple* is unique up to unitary equivalence. We obtain by weak closure, or double commutant, the von Neumann algebra $\mathcal{M}_\omega := \Pi_\omega(\mathcal{A})''$ with center $\mathcal{Z}_\omega$. A state ω is *normal* to a representation $(\Pi, \mathcal{H})$ if it is representable by a density operator, a normalized, positive trace class operator $\varrho \in \mathcal{T}_1^+(\mathcal{H})$. We denote by $\mathcal{S}_n(\Pi) \subset \mathcal{S}$ the convex set of all normal states on $(\Pi, \mathcal{H})$.

A *face* F of $\mathcal{S}$ is a convex set of states, the components of their (finite) convex decompositions stay in F. The face is *stable* if the same holds also for decompositions via arbitrary orthogonal probability measures. The set of all norm-closed faces is a complete orthomodular lattice $F(\mathcal{S})$. If a face F, together with its orthocomplement $F^\perp$, provides a unique convex decomposition $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ with $\omega_1 \in F$ and $\omega_2 \in F^\perp$ for all $\omega \in \mathcal{S}$, then F, respectively $F^\perp$, is called a *split face*. We term a norm-closed split face a *folium* $\mathcal{F}$. The set of all folia $\mathcal{F}(\mathcal{S})$ is a complete Boolean sublattice of $F(\mathcal{S})$. Each $\mathcal{S}_n(\Pi)$ is a folium, and vice versa. The smallest face (split face) containing ω is denoted F_ω ($\mathcal{F}_\omega$), and $\mathcal{F}_\omega$ equals $\mathcal{S}_n(\Pi_\omega)$.

The weak closure of the universal representation $(\Pi_u, \mathcal{H}_u) := \bigoplus_{\omega\in\mathcal{S}}(\Pi_\omega, \mathcal{H}_\omega)$ is the universal von Neumann algebra $\mathcal{M}_u := \Pi_u(\mathcal{A})''$ which has a rich center $\mathcal{Z}_u$ for an anti-liminary C*-algebra like the Weyl algebra $\mathcal{A} = \mathcal{W}(E,\hbar\sigma)$, $E \neq 0$. Each $\omega \in \mathcal{S}$ is representable by $\Omega_\omega \in \mathcal{H}_u$ and is thus normal on $(\Pi_u, \mathcal{H}_u)$.

We denote the complete orthomodular lattice of projections in $\mathcal{M}_u$ by $\mathcal{P}_u$ and the central projections by $\mathcal{P}_u^c$. If we define $\zeta\colon \mathcal{P}_u \xrightarrow{\text{onto}} F(\mathcal{S})$ by $\zeta(P) := \{\omega \in \mathcal{S} \mid \langle\omega; A\rangle = \langle\omega; PAP\rangle\ \forall\, A \in \mathcal{A}\} =: F_P$ we obtain an ortho-lattice isomorphism with the restriction $\zeta\colon \mathcal{P}_u^c \xrightarrow{\text{onto}} \mathcal{F}(\mathcal{S})$, where $\zeta(P_1 \wedge P_2) = \zeta(P_1) \cap \zeta(P_2)$, $\zeta(\mathbf{1}) = \mathcal{S}$, $\zeta(0) = \emptyset$, and $F^\perp = \zeta(P^\perp)$.

We call $\zeta^{-1}(F_\omega) =: S_\omega$ the *support* of ω, and $\zeta^{-1}(\mathcal{F}_\omega) =: Z_\omega$ the *central support* of ω. A state ω is by definition pure, if it is extremal, written $\omega \in \partial_e\mathcal{S}$, and that is equivalent to $\mathcal{S}_\omega$ being atomic. A state ω with Z_ω being an atom in $\mathcal{P}_u^c$ is a *factor state*. Two states ω and φ are called *orthogonal* if $F_\omega \wedge F_\varphi = \emptyset$, and *disjoint* if $\mathcal{F}_\omega \wedge \mathcal{F}_\varphi = \emptyset$. The algebra $\mathcal{M}_\omega$ has trivial center if and only if ω is a factor state.

Two pure states ω and φ have unitary equivalent GNS-representations if and only if $S_\omega = uS_\varphi u^{-1}$ for some unitary $u \in \mathcal{A}$, and then are *unitary equivalent*. An automorphism $\alpha \in {}^*\text{-aut}\,\mathcal{A}$ is *inner* if it is implemented by a unitary $u \in \mathcal{A}$. Then $\alpha^*\omega$ is unitary equivalent to ω for all $\omega \in \partial_e\mathcal{S}$ and $\alpha^*\omega \in \mathcal{F}_\omega$. The next "stronger" automorphisms α are those which are represented in any faithful representation $(\Pi, \mathcal{H})$ as $\alpha^\Pi = \Pi\alpha\Pi^{-1} = u\Pi(\cdot)u^{-1}$ by a $u \in \Pi(\mathcal{A})''$. Only if that is not possible can α^Π move elements of the center, especially of $\mathcal{Z}_u$, and dually map a folium into a different one. In Proposition 3.6 we named these "global automorphisms" and gave a necessary condition. They may be unitarily represented in faithful representations by $u \notin \Pi(\mathcal{A})''$.

Appendix A.2. Orthogonal decompositions

The diagonalization of a density operator by a "complete set of commuting observables" is on $\mathcal{S} \ni \omega$ generalized by a convex decomposition $\omega = \int_{\mathcal{S}} \varphi \, d\mu(\varphi)$ by means of an orthogonal probability measure $\mu \in \mathcal{O}(\mathcal{S})$ on the w*-Borel sets $\mathsf{B}(\mathcal{S})$. Each $\mu \in \mathcal{O}(\mathcal{S})$ is bi-univocally associated with an abelian von Neumann algebra $\mathcal{N}_\mu \subset \Pi_\omega(\mathcal{A})'$. If $\mathcal{N}_\mu \subset \mathcal{Z}_\omega$ then μ is called a *subcentral decomposition* of ω and if $\mathcal{N}_\mu = \mathcal{Z}_\omega$ then $\mu = \mu_\omega$ is called the *central decomposition of* ω. If $\mathcal{A}$ is norm separable then $(\mathcal{S}, \mathsf{B}(\mathcal{S}))$ is a standard Borel space and the support $\operatorname{supp}\mu$ is defined as usual. In addition, $\mu \in \mathcal{O}(\mathcal{S})$ is called extremal if it is supported by pure states, which then are pairwise orthogonal. A subcentral decomposition contains, roughly speaking, only pairwise disjoint states, which are factor states in the case of the central decomposition. If $\mathcal{A}$ is not norm separable, like $\mathcal{W}(E, \hbar\sigma)$, the foregoing assertions hold only in the sense of quasi-supportedness.

According to the Effros theorem, a GNS–von Neumann algebra $\mathcal{M}_\omega$ is spatially decomposable if $\int_{\mathcal{S}}^{\oplus} \Omega_\varphi \, d\mu_\omega(\varphi)$ is cyclic for the direct integral representation $\int_{\mathcal{S}}^{\oplus} (\Pi_\varphi, \mathcal{H}_\varphi) \, d\mu_\omega(\varphi)$ (with μ_ω the central measure), which makes the latter unitary equivalent to the GNS-representation. For non-separable $\mathcal{A}$, μ_ω is only quasi-supported on the factor states and we need for a spatial decomposition Theorem 4.1(d). So it is remarkable that our photonic central decompositions are *supported* on pure (pairwise disjoint) states, which are moved into each other by the symmetry operations.

Appendix A.3. Algebraic transition probabilities

Algebraic transition probabilities for states on a C*-algebra $\mathcal{A}$ (as e.g. elaborated on in detail in [HonRie15, III]) start from a general definition of [Cantoni75] which is based on probability distributions for observable values, instead of state vectors. For any self-adjoint $A \in \mathcal{A} \equiv \Pi_u(\mathcal{A}) \subset \mathcal{M}_u$, there exists the spectral representation $A = \int_{\mathbb{R}} t \, dP_A(t)$ with the projection-valued measure $P_A(B) \in \mathcal{P}_u$ for all real Borel sets $B \in \mathsf{B}(\mathbb{R})$. For a state $\omega \in \mathcal{S}(\mathcal{A})$ we define the numerical measure $\omega^A(B) := \langle \omega; P_A(B) \rangle$.

The transition probability between such states ω and φ is then defined as

$$T_{\mathcal{A}}(\omega,\varphi) \equiv T_{\mathcal{M}_u}(\omega,\varphi) := \inf_{A\in\mathcal{M}_{u\,\mathrm{sa}}} \Big\{ \Big(\int_{\mathbb{R}} \sqrt{\frac{d\omega^A}{d\sigma}} \sqrt{\frac{d\varphi^A}{d\sigma}}\, d\sigma \Big)^2 \Big\}, \tag{A.1}$$

where σ is any finite positive measure, with respect to which ω^A and φ^A are absolutely continuous. The original meaning of a transition probability is thus the infimum of the overlaps of the spectral distributions of the states, taken over all self-adjoint observables.

One knows that $(\omega,\varphi) \mapsto T_{\mathcal{A}}(\omega,\varphi)$ is separately convex, jointly continuous in the norm topology, and jointly lower semicontinuous in the w*-topology.

Proposition A.1. *Let ω, φ be states on the C^*-algebra $\mathcal{A}$. It holds the following result from [Alberti83],*

$$T_{\mathcal{A}}(\omega,\varphi) = \sup_{U\in U(\Pi(\mathcal{A})')} \big\{ |(\Omega|U\Phi)_\Pi|^2 \big\}, \tag{A.2}$$

with $U(\Pi(\mathcal{A})')$ the unitary elements in the commutant $\Pi(\mathcal{A})'$ of a representation Π, in which both states are given by vectors.

If ω, φ are pure then $\Pi(\mathcal{A})' = \mathbb{C}\mathbf{1}_\Pi$ and we have the traditional expression $T_{\mathcal{A}}(\omega,\varphi) = |(\Omega|\Phi)_\Pi|^2$, because the scalar unitaries are eaten by the modulus operation.

Quite generally, we find the following result holds:

Proposition A.2. *Consider two states ω, φ on the C^*-algebra $\mathcal{A}$.*

(a) *Then $T_{\mathcal{A}}(\omega,\varphi) = 0$ if and only if ω and φ are orthogonal.*
(b) *There hold the following equivalent conditions for the disjointness of the two states:*

$$\mathcal{F}_\omega \cap \mathcal{F}_\varphi = \emptyset \;\Leftrightarrow\; Z_\omega Z_\varphi = 0 \;\Leftrightarrow\; T_{\mathcal{A}}(\omega',\varphi') = 0 \;\; \forall\,(\omega',\varphi') \in \mathcal{F}_\omega \times \mathcal{F}_\varphi. \tag{A.3}$$

More generally, all pairs $(\omega',\varphi') \in \mathcal{F}_1 \times \mathcal{F}_2$ are orthogonal if and only if the folia $\mathcal{F}_1$ and $\mathcal{F}_2$ are intersection-free.

In [HonRie15, III, Sect. 48.4] the following indispensable result is proved:

Theorem A.3. *Let $\mathcal{M} = \int_X^\oplus \mathcal{M}(x)\, d\mu_0(x)$ be a direct integral of von Neumann algebras in terms of a σ-finite measure μ_0 on a standard Borel space $(X, \mathsf{B}(\mathsf{X}))$ (cf. [Takes79, Chap. IV]). Then for $\omega = \int_X^\oplus \omega(x)\, d\mu(x)$, $\varphi = \int_X^\oplus \varphi(x)\, d\mu'(x) \in \mathcal{S}_n(\mathcal{M})$, the measures μ and μ' are absolutely continuous to μ_0 and*

$$T_{\mathcal{M}}^{\frac{1}{2}}(\omega,\varphi) = \int_X T_{\mathcal{M}(x)}^{\frac{1}{2}}(\omega(x),\varphi(x)) \Big[\frac{d\mu}{d\mu_0} \frac{d\mu'}{d\mu_0} \Big]^{\frac{1}{2}}(x)\, d\mu_0(x). \tag{A.4}$$

Since the diagonal algebra is a subalgebra of the center of $\mathcal{M}$, $d\mu_0(x)$ is a subcentral measure in a certain parametrization.

Appendix B. Photonic derivation of angular field momentum

For calculating the "Hamilton function" for the photonic *angular momentum* $\mathsf{L_n}$ about the axis $n \in \mathbb{R}^3$ we recall that $\mathsf{L_n} = -i\hbar n \cdot (\mathbf{I} - (x \times \nabla))$ (taking henceforth $\hbar = 1$ discussing the generator). Explicitly, it holds that $n \cdot \mathbf{I} := n_1\ell_1 + n_2\ell_2 + n_3\ell_3$. The 3×3-matrices ℓ_j may be chosen as

$$\ell_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \ell_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \ell_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{B.1}$$

and constitute a basis for the real Lie algebra $\mathcal{SO}(3)$ (with matrix commutator as Lie product).

They satisfy the commutation relations $[\ell_i, \ell_j] = \varepsilon_{ijk}\,\ell_k, i, j, k \in \{1, 2, 3\}$. It holds, with $\mathbf{A} = (A_1, A_2, A_3)$, that $(\ell_i \mathbf{A})_j = \sum_k \varepsilon_{ijk} A_k$ and $[(n \cdot \mathbf{I})\mathbf{A}]_j = \sum_{i,k} n_i \varepsilon_{ijk} A_k$.

The finite rotations $R \in \mathrm{SO}(3)$ are in some sense composed of rotations around the coordinate axis, where the real orthogonal matrix $\exp\{s\, n \cdot \mathbf{I}\} \equiv R_n(s)$, $s \in \mathbb{R}$ rotates about n. Acting as $R^{-1}f(Rx)$ on one-photon wave functions $f \in E$ (and equally as $R^{-1}\hat{f}(Rk)$ in the k-representation) these transformations constitute a unitary representation of SO(3). Because $(Rk) \cdot \hat{f}(Rk) = 0 = k \cdot R^{-1}f(Rk)$, they in fact conserve transversality. We discuss in the following Appendix C topological properties of the generator $\mathsf{L_n} = -in \cdot (\mathbf{I} - (k \times \nabla_k))$, $k \in \mathbb{R}^3$ in the k-representation.

Working in position space, we remark that $i\mathsf{L_n} f = \{n \cdot [\mathbf{I} - (x \times \nabla)]\} f$ is real valued if $f \in E$ is so and use $\mathsf{L_n} = T\mathsf{L}T^{-1}$, where $T^{-1}\colon E \to \breve{E}$ is the scaling transformation. We get

$$\begin{aligned} \breve{H}^0_{\mathsf{L}_n}[\sigma_\sharp(\breve{f})] &= \frac{1}{2}\,\mathrm{Re}(\breve{f}|\mathsf{L_n}\breve{f}) = -\frac{1}{2}\,\mathrm{Im}(f|i\mathsf{L_n}f) = -(f_R|iL_n f_I) \\ &= (\mathbf{Y}|i\mathsf{L_n}\mathbf{A}) = (\mathbf{Y}|(n \cdot \mathbf{I})\mathbf{A}) - (\mathbf{Y}|[n \cdot (x \times \nabla)]\mathbf{A}). \end{aligned}$$

The second term, referring to the "orbital angular momentum", already has some similarity to the electrodynamic formulas. We have to evaluate further the first term pertaining to the "spin part" (both parts being physically *not clearly defined* since photons lack a rest system). It holds that

$$\mathbf{Y} \cdot [(n \cdot \mathbf{I})\mathbf{A}] = \sum_{i,j,k} n_i Y_j \varepsilon_{ijk} A_k = n \cdot (\mathbf{Y} \times \mathbf{A}) = \sum_{ijkl} n_i \varepsilon_{ijk} \delta_{jl} Y_l A_k.$$

We use the transversality of $\mathbf{Y}$ as well as $x_j \frac{\partial}{\partial x_l} f = \frac{\partial}{\partial x_l} x_j f - \delta_{jl} f$ to manipulate the following integrals, where the first is an artificially supplemented vanishing expression:

$$
\begin{aligned}
(\mathbf{Y}|(n\cdot\mathbf{I})\mathbf{A}) &= -\int_{\mathbb{R}^3}[n\cdot(x\times\mathbf{A})][\underbrace{\nabla\cdot\mathbf{Y}}_{=0}]\,d^3x + \int_{\mathbb{R}^3} n\cdot(\mathbf{Y}\times\mathbf{A})\,d^3x \\
&= -\sum_{i,j,k,l} n_i\varepsilon_{ijk}\Big\{\Big(\frac{\partial}{\partial x_l}\mathbf{Y}_l\Big| x_j\mathbf{A}_k\Big) + \delta_{jl}(\mathbf{Y}_l|\mathbf{A}_k)\Big\} \\
&= -\sum_{i,j,k,l} n_i\varepsilon_{ijk}\Big(\mathbf{Y}\Big| x_j\frac{\partial}{\partial x_l}\mathbf{A}\Big) \\
&= -\int_{\mathbb{R}^3} n\cdot\{x\times[(\mathbf{Y}\cdot\nabla)\mathbf{A}]\}\,d^3x.
\end{aligned}
$$

We combine this with the "orbital angular-momentum part" to finally reach

$$
\begin{aligned}
\breve{H}^0_{\mathsf{L}_n}[(\mathbf{A},\mathbf{Y})] &= -\int_{\mathbb{R}^3} n\cdot\{x\times[(\mathbf{Y}\cdot\nabla)\mathbf{A}]\}\,d^3x - \int_{\mathbb{R}^3}\mathbf{Y}\cdot\{[n\cdot(x\times\nabla)]\mathbf{A}\}\,d^3x \\
&= -\int_{\mathbb{R}^3} n\cdot\{x\times[\mathbf{Y}\times(\nabla\times\mathbf{A})]\}\,d^3x \\
&= \epsilon_0\int_{\mathbb{R}^3} n\cdot\{x\times[\mathbf{E}(x)\times\mathbf{B}(x)]\}\,d^3x \\
&= \frac{1}{c^2}\int_{\mathbb{R}^3} n\cdot\{x\times\mathbf{S}(x)\}\,d^3x. \qquad \text{(B.2)}
\end{aligned}
$$

So we have gained a formal, but not a physical, explanation for how the two angular-momentum parts of the photon submerge into the uniform expression for the field angular momentum. There seems to exist no published comment by Einstein on the angular momentum of the photon (see [Pais82, p. 426]). This stands in sharp contrast to Einstein's well-known extensive studies on the photon momentum statistics, which he immediately connected with the pressure of light.

Appendix C. Proof of Theorem 3.4

(a) Taking $\hat{f} = (\hat{\varphi}_\nu) \in \hat{E}$ we see for the mentioned K that $(\exp\{isK\}\hat{f})(k) =: \hat{f}_s(k)$ equals respectively the expressions $\exp\{isc|k|\}\hat{f}(k)$, $\exp\{isn\cdot k\}\hat{f}(k)$, and $R_n^{-1}(s)\hat{f}(R_n(s)k)$, with $R_n(s)$ the rotation matrices about the n-axis. Thus $\hat{f}_s$ is in $\hat{E}$ and is differentiable by s for each fixed k. Then for each k it holds that $(\exp\{isK\}-\mathbf{1})\hat{f}(k) = s\hat{g}(s,k) \in \hat{E}$, with s in a compact neighborhood $0 \in J \subset \mathbb{R}$. For each k, $\lim_{s\to 0}\hat{g}(s,k) = \frac{d}{ds}\hat{f}_s(k)|_{s=0} = (K\hat{f})(k)$ which is in $\hat{E}$ by inspection. Thus we have for each component that $\sup_k|k^\alpha D^\beta \pi_\nu \hat{g}(s,k)| \le \Gamma > 0$ for $s \in J$

and all $\alpha, \beta \in \mathsf{A}$. So $\|(\exp\{isK\} - \mathbf{1})\hat{\varphi}_\nu\|_{\alpha,\beta}$ tends with s to 0 for each component $\pi_\nu \hat{f} = \hat{\varphi}_\nu \in \hat{\mathsf{S}}^1_0$ and we find that $s \mapsto \exp\{isK\}\hat{f}$ is τ-continuous.

To prove the τ-continuity of $\exp\{isK\}\colon E \to E$ for fixed $s \in \mathbb{R}$, we must investigate the $D^\beta \exp\{isK\}\hat{\varphi}_\nu$, $\beta \in \mathsf{A}$. For $K \in \{B, p_n\}$, terms of the form of equation (C.1) result, multiplied by $\exp\{isK(|k|, k)\}$, where the exponential is linear in $|k|$ or k and does not influence the estimation.

For $K = L_n$ we have to estimate $D^\beta[R^{-1}\hat{f}(Rk)]_\nu$, where the rotation $R \equiv R_n(s)$ mixes the components, and we introduce the invertible function $k \mapsto l(k) := Rk$. Inspection of the case $\beta = (1,0,0)$, that is, $\sum_{\kappa=1}^3 \sum_{\lambda=1}^3 R^{-1}_{\nu\kappa} \frac{\partial l_\lambda(k)}{\partial k_1} \frac{\partial}{\partial l_\lambda} \hat{\varphi}_\kappa(l(k))$ gives by iteration the finite sum $D^\beta[R^{-1}\hat{f}(Rk)]_\nu = \sum_{\kappa=1}^3 \sum_n N_{n\kappa\nu} D^{\beta_{n\kappa\nu}} \hat{\varphi}_\kappa(l(k))$ with $N_{n\kappa\nu} \in \mathbb{R}$. Observing that $k^\alpha = \sum_m M_m l^{\alpha_m}$, $M_m \in \mathbb{R}$, we find that $k^\alpha D^\beta[R^{-1}\hat{f}(Rk)]_\nu = \sum_{\kappa=1}^3 \sum_m \sum_n M_m N_{n\kappa\nu} l(k)^{\alpha_m} D^{\beta_{n\alpha\nu}} \hat{\varphi}_\kappa(l(k))$. Now $\sup_k$ of this expression in absolute is also reached by $\sup_l$ so

$$\|[R^{-1}\hat{f} \circ R]_\nu\|_{\alpha\beta} \le \sum_{\kappa=1}^3 \sum_m \sum_n |M_m N_{n\kappa\nu}| \, \|\hat{\varphi}_\kappa\|_{\alpha_m \beta_{n\kappa\nu}},$$

expressing τ-continuity.

(b) First we have for $\hat{\varphi} \in \hat{\mathsf{S}}^1_0$ to evaluate $D^\beta(\hat{\varphi}(k)/|k|^m)$, with $m \in \mathbb{N}$ odd. By iterated application of the Leibniz rule, we obtain for all $\beta \in \mathsf{A}$ qualitative expressions in the form of finite sums, involving some $r_n \in \mathbb{R}$, $\alpha_n, \beta_n \in \mathsf{A}$, $m_n \in \mathbb{N}$,

$$D^\beta(\hat{\varphi}(k)/|k|^m) = \sum_n r_n \, k^{\alpha_n} |k|^{-m_n} D^{\beta_n} \hat{\varphi}(k). \tag{C.1}$$

(Differentiation by k_ν gives the next higher similar expression, where the m_n remain odd.)

Discussing B first, we remark that $D^\beta |k|\hat{\varphi}$ gives for $\beta > 0$ an expression like equation (C.1). Amplifying the fraction by $|k|$ makes $m_n + 1$ even. In the numerator we replace $|k|$ by the larger $(1 + k^2)$. Altogether we find finite families $(c_p \in \mathbb{R})$, $(\gamma_p \in \mathsf{A})$, $p \in \mathbb{N}$ so that we have only positive-integer k-powers, leading to

$$\begin{aligned}
\| |k|\hat{\varphi}\|_{\alpha,\beta} &= \sup_k |k^\alpha D^\beta |k| \hat{\varphi}(k)| \\
&\le \sum_n |r_n| \sup_k |k^{\alpha+\alpha_n}(1+k^2)(k^2)^{-\frac{m_n+1}{2}} D^{\beta_n}\hat{\varphi}(k)| \\
&\le \sum_p |c_p| \, \|\hat{\varphi}\|_{\gamma_p, \beta_p}. \qquad \text{(C.2)}
\end{aligned}$$

Thus B is τ-continuous, and the continuity of $P^\top$ follows similarly. For $p_n = -n \cdot k$, continuity is even easier. Concerning L_n we have to study the components $D^\beta[(n \cdot \mathbf{I} - k \times \nabla)\hat{f}(k)]_\nu$. Since the derivatives are on the right, no new complication arises.

Appendix D. Remarks on gauged non-relativistic QED

To quantize Maxwell theory in topologically non-trivial cavities $\Lambda \subset \mathbb{R}^3$, one is automatically led to concepts of *gauge theory* (e.g. [Frampton] and references therein). From the present approach it seems natural to first "gauge" the electromagnetic field and to couple it to a "gauged" spinor field in a second step.

If the first Betti number b_1 (giving the number of cut surfaces to make Λ simply connected) or the second Betti number b_2 (giving the number of connected parts of the surface $\partial\Lambda$) is greater than zero, the open Λ allows a coordination only on local charts $U_\alpha \subset \Lambda$, $\alpha \in J$ with J some index set. The vector potential $\mathbf{A}$ over Λ can, therefore, be given only by a family of local functions $U_\alpha \ni x \mapsto \mathbf{A}_\alpha(x)$, $\alpha \in J$. In overlapping chart regions $U_\alpha \cap U_\beta \ni x$, the differences $\mathbf{A}_\alpha(x) - \mathbf{A}_\beta(x)$ are determined via upgrading Λ to a principal $G = U(1)$-bundle $\mathcal{P}(\pi, \Lambda, G)$ with projection $\pi\colon \mathcal{P} \to \Lambda$, typical fiber G, and global bundle group G. The U_α should then be arranged to provide local factorizations $\tau_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times U(1)$ and to fit to the geometric structure (e.g. by covering just one of the mutually disjoint cut surfaces). The coordinated bundle [Steenrod51] is characterized by the transition coefficients $z_{\alpha,\beta}(x) = \exp\{i\lambda_{\alpha,\beta}(x)\} \in U(1)$ for $x \in U_\alpha \cap U_\beta$ which have to satisfy the chain rule.

The family of local potentials $\mathbf{A}_\alpha(x)$, $\alpha \in J$ is made into a geometric object by interpreting them as the local coordinates of a horizontal ($U(1)$-invariant) connection one-form $\omega \in \Omega^1(\Lambda, \mathfrak{g})$ with values in the Lie algebra $\mathfrak{g} = i\mathbb{R}$. This implies $\mathbf{A}_\beta(x) - \mathbf{A}_\alpha(x) = d\lambda_{\beta,\alpha}(x)$ and looks like a gauge transformation if one takes into account that in coordinates the cotangent vector $d\lambda_{\beta,\alpha}(x)$ is $\nabla\lambda_{\beta,\alpha}(x)$. In this manner the geometric object ω corresponds to the physical object "gauge class of a vector potential". The bundle is topologically trivial if $z_{\beta,\alpha}(x) = z_\beta(x) z_\alpha^{-1}(x)$ giving $\mathbf{A}_\beta(x) - d\lambda_\beta(x) = \mathbf{A}_\alpha(x) - d\lambda_\alpha(x) = \mathbf{A}_\alpha(x) + i z_\alpha^{-1}(x)\, dz_\alpha(x)$.

If $\mathbf{A}_\alpha(x)$ is initially transverse, the $\mathbf{A}_\alpha(x) + \nabla\lambda_{\beta,\alpha}(x)$ acquire a *longitudinal part*. For a flat but topologically non-trivial bundle with $b_1 > 0$, $b_2 = 0$ there exist phase gradients $\nabla\lambda$ with vanishing divergence in the interior of the cut Λ. They lead to so-called *cohomological vector potentials* which supplement the Helmholtz–Hodge decomposition of $\mathbf{A}$ into transverse and longitudinal parts [HonRie15].

The group G acts on $b \in \mathcal{P}(\pi, \Lambda, G) \equiv \mathcal{P}$ by right multiplication bg^{-1}. The action of G on a section $s\colon \Lambda \to \mathcal{P}$ is performed by the *local gauge group* $\mathcal{G} = \mathrm{Map}(\Lambda, G) \ni \chi$.

For a *fermion* of mass m and charge q one has likewise only local realizations of its spinor wave function $\mathcal{U}_\alpha \ni x \mapsto \psi_\alpha(x) \in \mathbb{C}^n$. In the presence of a vector potential, its kinetic energy $\frac{\hbar^2\nabla^2}{-2m}$ goes locally over to $\frac{\hbar^2}{-2m}(\nabla - \frac{iq}{\hbar}\mathbf{A}_\alpha)^2$ by means of the *minimal coupling principle*. This expression stays invariant under the combined gauge transformation $\mathbf{A}_\alpha(x) \to \mathbf{A}_\alpha(x) + \nabla\lambda_\alpha(x)$ and $\psi_\alpha(x) \to \exp\{iq\lambda_\alpha(x)/\hbar\}\psi_\alpha(x)$. Since the gauge behavior of the vector potential is geometrically associated with $\mathcal{P} \equiv \mathcal{P}(\pi, \Lambda, G)$ one has to arrange the same with the spinors.

Since the spinors take values in $\mathbb{C}^n \equiv \mathcal{W}$ we have a smooth action $l\colon G \times \mathcal{W} \to \mathcal{W}$ by the local phase multiplication. A point of great mathematical concern is the fact that the phase value is irrelevant for the physical meaning of ψ. Then the associated bundle is not simply defined by $\mathcal{P} \times \mathcal{W}$ but by $(\mathcal{P} \times \mathcal{W})/G \equiv \mathcal{P} \times_G \mathcal{W}$, where $G \ni g$ acts as $T_g(b, \psi) = (bg^{-1}, l_g\psi)$. Working with fermion sections one averages over $\mathcal{G}$.

This setup so far has important applications, not only for single fermions in an external vector potential, but also in quantum electronics where the c-number current is often built on averaged spinor wave functions or on the macroscopic wave functions of the superconducting Cooper pair condensate. Particularly for the latter *ordered electron collectives*, the horizontal lift of a path in Λ to a section in $\mathcal{P}(\pi, \Lambda, U(1))$ is important. It is induced through a given connection ω which defines a horizontal tangent bundle. This associates a unique phase at each point along the lifted path. Such phase observables lead to interesting physical consequences like flux quantization, Aharanov–Bohm effects, etc., but often are treated heuristically (and are named “non-integrable phases” in [Tonomura]).

The decisive mathematical problem is the quantization of the hitherto real-valued $\mathbf{A}$ and the second quantization of the fermions. The extensive literature on this topic refers predominantly to the relativistic covariant theory. To the fewer non-relativistic mathematical elaborations belongs the advanced paper [Hannabuss], which appeals to C*-algebraic notions in connection with gauge theory. (In its extensive references one also finds titles on relativistic algebraic QED.) We try to give a very rough impression of its first part.

As is usual in particle physics, one starts there with the material fermion algebra and afterwards associates the boson algebra, but the principal gauge bundle $\mathcal{P}(\pi, \Lambda, G) \equiv \mathcal{P}$ is present from the beginning. One specializes Λ to $\mathbb{R}^3$ (giving factorized transition coefficients), whereas G is mostly an arbitrary connected compact Lie group (to include non-abelian gauge theories).

Since the connections $\omega \in \Omega^1(\Lambda, \mathfrak{g})$ of $\mathcal{P}$ are real valued, one treats the fermions by a Clifford algebra, but translates the results into the usual fermion algebra of smeared fields $\Psi(\xi)$ which satisfy the CAR (canonical anti-commutation relations) at equal time $[\Psi^*(\xi), \Psi(\eta)]_+ = (\xi|\eta) = \int_\Lambda (\bar{\xi}(x)|\eta(x))_{\mathbb{C}^n}\, d^3x$. We denote *here* the spinor algebra in the complex or Clifford form simply by $\mathcal{F}$. The gauge group G acts on the Fock-represented $\mathcal{F}$ via automorphisms.

This G-action is through the gauge principle connected with the transformation of the connections $\omega \in \Omega^1(\Lambda, \mathfrak{g}) \equiv \Omega$ and influences the fermion dynamics. The fermions are therefore represented, not by sections over Λ, but rather by sections $\omega \mapsto \Xi(\omega) \in W$ which satisfy $\Xi(\omega + i\chi^{-1}\, d\chi) = \alpha_\chi[\Xi](\omega)$ for all $\chi \in \mathcal{G}$. This is the condition to define an induced fermion algebra $\mathcal{F}_{\mathrm{ind}}$. Any function F on $\Omega/\mathcal{G}$ lifts to a function $\widetilde{F}$ on Ω which acts by pointwise multiplication on the spinor sections in $\mathcal{F}_{\mathrm{ind}}$. The quantized $\mathbf{A}$ is defined by giving the multiplication action $\widetilde{F}(q\mathbf{A}/\hbar)$ on the sections in $\mathcal{F}_{\mathrm{ind}}$. For these sections one can similarly use linear combinations of products of spinors with the $\phi_a(\omega) = \exp\{ia(\omega)\}$, where a is taken from a dual group of the not locally compact additive connection group Ω.

There is also the action $\tau_u[\Xi](\omega) = \Xi(\omega + u)$ which allows one to form the crossed product algebra $\mathcal{A} = \mathcal{F}_{\mathrm{ind}} \times_\tau \Omega$. The elements of $\mathcal{A}$ can be viewed as generated by products of algebraic elements with point measures δ_u concentrated at $u \in \Omega$. They are denoted by $\phi_{a,u}$, with $u \in \Omega$ and a in the dual of Ω, and satisfy via an introduced $\star$-product the relations $\phi_{a,u} \star \phi_{b,v} = \mathrm{e}^{i[b(u)-a(v)]}\phi_{b,v} \star \phi_{a,u}$, similarly to the Weyl relations for canonical boson fields. The $\mathcal{G}$-fixed algebra is denoted by $\mathcal{B}$.

The $\mathbf{E}$-fields are introduced via the τ operations. This comes for $G = U(1)$ by comparison with the usual commutation relations

$$[\mathbf{E}(u), \mathbf{A}(a)] = -i\frac{\hbar}{\epsilon_0}\int_\Lambda u(x)\cdot a^\top(x)\, d^3x \tag{D.1}$$

(cf. the first part in equation (6.1) and insert there – after symplectic scaling – $\Phi_\mu(f_R) = \mathbf{A}_\mu(f_R)$ and $\Phi_\mu(f_I) = -\epsilon_0\mathbf{E}_\mu(f_I)$), where the $\mathbf{A}$-fields are transversal, but not necessarily the $\mathbf{E}$-fields. This leads to

$$\exp\{i\mathbf{E}(u)\}\Big(\frac{q}{\hbar}\mathbf{A}(a)\Big)\exp\{-i\mathbf{E}(u)\} = \frac{\hbar}{\epsilon_0}\mathbf{A}(a) + \frac{q}{\epsilon_0}\int_\Lambda u(x)\cdot a^\top(x)\, d^3x, \tag{D.2}$$

so that τ_{qu/ϵ_0} is implemented by $\exp\{\frac{i\epsilon_0}{q}\mathbf{E}(u)\}$.

The most spectacular consequence is the validity of the Poisson–Gauss law in operator form (with $(\nabla\cdot\mathbf{E})(f)$ in a different algebra and differently smeared than $\mathbf{E}(u)$)

$$(\nabla\cdot\mathbf{E})(f) = \varrho(f)/\epsilon_0, \quad \varrho(f) = \int_\Lambda f(x)\Psi(x)^*\cdot\Psi(x)\, d^3x. \tag{D.3}$$

As an operation on the spinor fields, $\mathbf{E}$ does not commute with them. In unsmeared form one gets from equation (D.3) the commutation relation

$$\mathbf{E}(x)\Psi(y) = \Psi(y)\Big[\mathbf{E}(x) + \nabla\frac{q}{4\pi\epsilon_0|x-y|}\Big]. \tag{D.4}$$

This is interpreted as "creating a fermion, also creates its Coulomb field". So, in the end, we have arrived at the scalar potential, duly attached to the fermions.

Acknowledgements

The author thanks Gordon McCabe for sending him his work "What is light?" prior to publication and for the exchange of ideas on the present subject.

Thanks are also due to the referee for the suggestion to include the quantized Maxwell field, together with the connection to gauging, to locate the conceptual place of the present photonic approach.

References

[AbraMars] R. Abraham and J. E. Marsden, *Foundations of mechanics*, Benjamin/Cummings, Advanced Book Program, Reading, MA, 1978. Zbl 0393.70001 MR 515141

[Alberti83] P. M. Alberti, A note on the transition probability over C^*-algebras, Lett. Math. Phys. **7** (1983), 25–32. Zbl 0538.46051 MR 691969

[Araki] H. Araki, *Mathematical theory of quantum fields*, International Series of Monographs on Physics 101, Oxford University Press, Oxford, 2009. Zbl 1170.81001 MR 2542202

[BogShir] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the theory of quantized fields*, Interscience Monographs in Physics and Astronomy 3, Interscience Publishers, New York; Interscience Publishers, London, 1959 Zbl 0088.21701 MR 0110471

[BratRob1] O. Bratteli and D. W. Robinson, *Operator algebras and quantum-statistical mechanics I*, Texts and Monographs in Physics, Springer, New York, 1987. Zbl 0905.46046 MR 887100

[Cantoni75] V. Cantoni, Generalized "transition probability", Comm. Math. Phys. **44** (1975), 125–128. MR 389088

[CohTann] C. Cohen-Tannoudji, J. Dupont-Roc and G. Grynberg, *Photons and atoms, introduction to QED*, John Wiley & Sons, New York, Toronto, Singapore, 1989.

[Conway85] J. B. Conway, *A course in functional analysis*, Graduate Texts in Mathematics 96, Springer, New York, 1985. Zbl 0558.46001 MR 768926

[DaviesOp] E. B. Davies, *Quantum theory of open systems*, Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1976. Zbl 0388.46044 MR 0489429

[Duncan] A. Duncan, *The conceptual framework of quantum field theory*, Oxford University Press, Oxford, 2012. Zbl 1252.81001 MR 3088481

[Earman] J. Earman, Superselection rules for philosophers, Erkenntnis **69** (2008), 377–414. Zbl 1163.81301 MR 2453623

[Einst05P] A. Einstein, Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt, Ann. der Phys. **4** (1905), 132–148. Zbl 36.0883.01

[Einst09a] A. Einstein, Zum gegenwärtigen Stand des Strahlungsproblems, Phys. Zs. **10** (1909), 185–193. Zbl 40.0985.01

[Frampton] P. H. Frampton, *Gauge field theories*, Wiley-Interscience [John Wiley & Sons], New York, 2000. Zbl 1059.81122 MR 1884969

[Haar69] D. Ter Haar, *Quantentheorie, Einführung und Originaltexte*, Vieweg, Braunschweig, 1969.

[Hannabuss] K. C. Hannabuss, Some C^*-algebras associated to quantum gauge theories, J. Aust. Math. Soc. **90** (2011), 9–24. Zbl 1217.81126 MR 2810941

[HeppReg] K. Hepp, Proof of the Bogoliobov–Parasiuk theorem on renormalization, Commun. Math. Phys. **2** (1966), 301-326. Zbl 1222.81219

[Hida80] T. Hida, *Brownian motion*, Applications of Mathematics 11, Springer, New York-Berlin, 1980. Zbl 0432.60002 MR 562914

[HonRie05] R. Honegger and A. Rieckers, Some continuous field quantizations, equivalent to the C^*-Weyl quantization, Publ. Res. Inst. Math. Sci. **41** (2005), 113–138. Zbl 1158.46311 MR 2115969

[HonRie15] R. Honegger and A. Rieckers, *Photons in Fock space and beyond I, II, III*, World Scientific, Hackensack, NJ, 2015. Zbl 1381.81005 MR 3616065

[Kadison65] R. V. Kadison, Transformations of states in operator theory and dynamics, Topology **3** (1965), 177–198. Zbl 0129.08705 MR 169073

[KadR67] R. V. Kadison and J. R. Ringrose, Derivations and automorphisms of operator algebras, Comm. Math. Phys. **4** (1967), 32–63. Zbl 0149.34403 MR 206735

[KadR83] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras I, II*, Pure and Applied Mathematics 100, 100-2, Academic Press, Orlando, FL, 1983, 1986. Zbl 0518.46046 Zbl 0601.46054 MR 719020 MR 859186

[KleinF] M. V. Klein and T. E. Furtak, *Optics*, John Wiley & Sons, New York, Toronto, Singapore, 1986.

[McCabe] G. McCabe, What is light?, Phil. Sci. Archive **9.9** (2020), 0–75.

[MehR82] J. Mehra and H. Rechenberg, *The historical development of quantum theory I, II, III*, Springer, New York, 1982. MR 0676476 MR 0676477 MR 0687388 MR 0678922

[Pais82] A. Pais, *"Subtle is the Lord...": The science and the life of Albert Einstein*, The Clarendon Press, Oxford University Press, New York, 1982. Zbl 0525.01017 MR 690419

[Peders79] G. K. Pedersen, *C^*-algebras and their automorphism groups*, London Mathematical Society Monographs 14, Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1979. Zbl 0416.46043 MR 548006

[Primas] H. Primas, *Chemistry, quantum mechanics and reductionism*, Springer, Berlin, 1983. MR 905875

[ReedSim1] M. Reed and B. Simon, *Methods of modern mathematical physics I: Functional analysis*, Academic Press, New York, 1972. Zbl 0242.46001 MR 493419

[ReedSim2] M. Reed and B. Simon, *Methods of modern mathematical physics II: Fourier analysis, self-adjointness*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Zbl 0308.47002 MR 0493420

[Rie2020] A. Rieckers, The regular ground states of the linear Boson field in terms of soft modes, Publ. Res. Inst. Math. Sci. **56** (2020), 137–171. Zbl 1433.81114 MR 4055976

[Schaef66] H. H. Schaefer, *Topological vector spaces*, Macmillan Series in Advanced Mathematics and Theoretical Physics, Macmillan, New York; Collier Macmillan, London, 1966. Zbl 0141.30503 MR 0193469

[Schlossh] M. Schlosshauer, *Decoherence and the quantum-to-classical transition*, Springer, Berlin, Heidelberg, New York, 2008.

[Schweb62] S. S. Schweber, *An introduction to relativistic quantum field theory*, Harper and Row, New York, 1962. Zbl 0111.43102 MR 0127796

[ScullyZub] M. O. Scully and M. S. Zubairy, *Quantum optics*, Cambridge University Press, Cambridge, 1997.

[Segal62] I. E. Segal, Mathematical characterization of the physical vacuum for a linear Bose–Einstein field. (Foundations of the dynamics of infinite systems III), Illinois J. Math. **6** (1962), 500–523. Zbl 0106.42804 MR 143519

[Sewell02] G. L. Sewell, *Quantum mechanics and its emergent macrophysics*, Princeton University Press, Princeton, NJ, 2002. Zbl 1007.82001 MR 1919619

[Shale62] D. Shale, Linear symmetries of free boson fields, Trans. Amer. Math. Soc. **103** (1962), 149–167. Zbl 0171.46901 MR 137504

[Steenrod51] N. Steenrod, *The topology of fiber bundles*, Princeton Mathematical Series 14, Princeton University Press, Princeton, NJ, 1951. Zbl 0054.07103 MR 0039258

[StrWigh] R. F. Streater and A. S. Wightman, *PCT, spin and statistics, and all that*, W. A. Benjamin, New York, 1964. Zbl 0135.44305 MR 0161603

[Takes79] M. Takesaki, *Theory of operator algebras I*, Springer, New York-Heidelberg, 1979. Zbl 0436.46043 MR 548728

[Tonomura] A. Tonomura, *Electron holography*, Springer Series in Optical Sciences 70, Springer, Berlin, Heidelberg, New York, 1993. Zbl 0997.78502

[Trifonov] A. Trifonov, D. Subacius, A. Berzanskis and A. Zavriyev, Single photon counting at telecom wavelength and quantum key distribution, J. Mod. Optics **51** (2004), 1399–1415.

[Vogel] W. Vogel, D.-G. Welsch and S. Wallentowitz, *Quantum optics, an introduction*, Wiley-VCH, Berlin, New York, Singapore, 2001. Zbl 1034.81057

[vNeum] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Grundlehren der mathematischen Wissenschaften 38, Springer, Berlin, 1932. Zbl 0005.09104 MR 0223138

[Weinl69] M. Weinless, Existence and uniqueness of the vacuum for linear quantized fields, J. Functional Analysis **4** (1969), 350–379. Zbl 0205.57504 MR 0253687