*Publ. RIMS Kyoto Univ.* **59** (2023), 339–391
DOI 10.4171/PRIMS/59-2-3

# Fractional Kolmogorov Operator and Desingularizing Weights

by

Damir Kinzebulatov and Yuliy A. Semënov

## Abstract

We establish sharp upper and lower bounds on the heat kernel of the fractional Laplace operator perturbed by Hardy-type drift by transferring it to an appropriate weighted space with singular weight.

*2020 Mathematics Subject Classification:* Primary 35K08, 47D07; Secondary 60J35.
*Keywords:* Non-local operators, heat kernel estimates, desingularization.

## Contents



Communicated by N. Ozawa. Received June 27, 2020. Revised May 12, 2021.

D. Kinzebulatov: Département de mathématiques et de statistique, Université Laval, 1045 av. de la Médecine, Québec, QC, G1V 0A6, Canada;
e-mail: `damir.kinzebulatov@mat.ulaval.ca`
Yu. A. Semënov: Department of Mathematics, University of Toronto, 40 St. George Str., Toronto, ON, M5S 2E4, Canada;
e-mail: `semenov.yu.a@gmail.com`

## §1. Introduction

The fractional Kolmogorov operator $(-\Delta)^{\frac{\alpha}{2}} + \mathsf{f} \cdot \nabla$, $1 < \alpha < 2$ with a (locally unbounded) vector field $\mathsf{f}\colon \mathbb{R}^d \to \mathbb{R}^d$ plays an important role in probability theory, where it arises as the generator of a symmetric $\alpha$-stable process with a drift (in contrast to diffusion processes, an $\alpha$-stable process has long-range interactions). It has been the subject of intensive study over the past two decades. There is now a well-developed theory of this operator, with $\mathsf{f}$ belonging to the corresponding Kato class. This class contains, in particular, the vector fields $\mathsf{f}$, with $|\mathsf{f}| \in L^p$, $p > \frac{d}{\alpha-1}$, and is responsible for the existence of the standard (local-in-time) two-sided bound on the heat kernel $e^{-t\Lambda}(x,y)$, $\Lambda = (-\Delta)^{\frac{\alpha}{2}} + \mathsf{f} \cdot \nabla$, in terms of $e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)$; see [BJ].

The authors in [KSS] studied in $\mathbb{R}^d$, $d \geq 3$ the fractional Kolmogorov operator

$$\Lambda = (-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla, \quad b(x) = \kappa |x|^{-\alpha} x, \quad 0 < \kappa < \kappa_0,$$

where $\kappa_0$ is the borderline constant for existence of $e^{-t\Lambda}(x,y) \geq 0$. The model vector field $b$ lies outside the scope of the Kato class, and exhibits critical behaviour both at $x = 0$ and at infinity, making the standard upper bound on $e^{-t\Lambda}(x,y)$ in terms of $e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)$ invalid. Instead, the two-sided bounds $e^{-t\Lambda}(x,y) \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\varphi_t(y)$ $(y \neq 0)$ hold for an appropriate weight $\varphi_t \geq \frac{1}{2}$ unbounded at $y = 0$ [KSS, Thm. 3].

The present paper continues [KSS]. Throughout this paper, $d \geq 3$ and $1 < \alpha < 2$. We study the heat kernel $e^{-t\Lambda}(x,y)$ of the fractional Kolmogorov operator with the drift of opposite sign ("repulsion case"),

$$\begin{aligned} \Lambda &= (-\Delta)^{\frac{\alpha}{2}} - b \cdot \nabla, \\ b(x) &= \kappa |x|^{-\alpha} x, \quad 0 < \kappa < \infty. \end{aligned} \tag{1.1}$$

Although the standard (global) upper bound in terms of $e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)$ holds true for $e^{-t\Lambda}(x,y)$ (Theorem 3.2 below), the singularity of $b$ at $x = 0$ makes it off the mark. Namely, in Theorems 3.3 and 3.4 below we establish sharp upper and lower bounds

$$e^{-t\Lambda}(x,y) \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\psi_t(y), \quad x, y \in \mathbb{R}^d,\ t > 0, \tag{ULB$_w$}$$

where the continuous weight $0 \leq \psi_t(y) \leq 2$ vanishes at $y = 0$ as $|y|^\beta$, $\beta > 0$ (Theorem 3.1). (Here, the notation $a(z) \approx b(z)$ means that $c^{-1}b(z) \leq a(z) \leq cb(z)$ for some constant $c > 1$ and all admissible $z$.) The order of vanishing $\beta$ $(< \alpha)$ depends explicitly on $\kappa > 0$ and tends to $\alpha$ as $\kappa \uparrow \infty$.

The key step in proving the upper and lower bound ($\text{ULB}_w$) is the weighted Nash initial estimate

$$(\text{NIE}_w) \qquad 0 \le e^{-t\Lambda}(x,y) \le Ct^{-\frac{d}{\alpha}}\psi_t(y), \quad x,y \in \mathbb{R}^d,\ t>0.$$

The proof of ($\text{NIE}_w$) uses the method of desingularizing weights [MS0, MS1, MS2] based on ideas set forth by Nash [N]: it depends on the "desingularizing" $(L^1, L^1)$ bound on the weighted semigroup $\psi_t e^{-t\Lambda}\psi_t^{-1}$.

Operator (1.1) in the local case $\alpha = 2$ has been studied in [MeSS, MeNS] by considering it in the space $L^2(\mathbb{R}^d, |x|^\gamma dx)$ for appropriate $\gamma$, where the operator becomes symmetric. This approach, however, does not work for $\alpha < 2$.

Recently, the authors in [CKSV, JW] considered the fractional Schrödinger operator

$$H_+ = (-\Delta)^{\frac{\alpha}{2}} + V, \quad V(x) = \kappa|x|^{-\alpha},\ 0<\alpha<2,\ \kappa>0,$$

and established, using different methods, sharp two-sided bounds

$$e^{-tH_+}(x,y) \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\psi_t(x)\psi_t(y)$$

for appropriate weights $\psi_t(x)$ vanishing at $x=0$. We apply some ideas from [JW] (the "method of self-improving estimates", in the proof of Theorem 3.3).

In contrast to the cited papers, this work deals with a purely non-local and non-symmetric situation. This leads to new difficulties, and requires new ideas. Even the proof of the standard upper bound $e^{-t\Lambda}(x,y) \le Ce^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)$ (Theorem 3.2), as well as the construction of semigroups $e^{-t\Lambda}$, $e^{-t\Lambda^*}$ (Sections 8 and 9), become non-trivial. The same applies to the Sobolev regularity of $e^{-t\Lambda}f$, $f \in C_c^\infty$ established in Section 8.2. We consider these results, along with Theorems 3.3 and 3.4, to be the main results of this article.

Below we apply the scheme of the proof of the upper and lower bounds in [KSS], although with comprehensive modifications in the method, both at the level of the abstract desingularization theorem (Theorem 2.1) and in the proofs of ($\text{NIE}_w$), ($\text{ULB}_w$) and of the standard upper bound.

We note that the heat kernel of the operator $(-\Delta)^{\frac{\alpha}{2}} + \mathsf{f}\cdot\nabla$ with $\operatorname{div}\mathsf{f} = 0$ was studied in [MM2, MM1]. Concerning the case $\operatorname{div}\mathsf{f}=0$ and $\alpha = 2$, see [Z, Z2].

For properties of the Feller process determined by (1.1), see [KM].

Let us mention that the vector field $b(x) = \kappa|x|^{-\alpha}x$ exhibits critical behaviour even if we remove the singularity of $b$ at the origin. Namely, if we consider $\Lambda$ with $b$ bounded in $B(0,1)$ but having slower decay at infinity, $b(x) = \kappa|x|^{-\alpha+\varepsilon}x$, $\varepsilon>0$ for $|x|\ge 1$, then the global-in-time upper bound $e^{-t\Lambda}(x,y) \le Ce^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)$ of Theorem 3.2 would no longer be valid.

### §1.1. Notation

- We denote by $\mathcal{B}(X,Y)$ the space of bounded linear operators between Banach spaces $X \to Y$, endowed with the operator norm $\|\cdot\|_{X\to Y}$. Set $\mathcal{B}(X) := \mathcal{B}(X,X)$.
- We write $T = s\text{-}X\text{-}\lim_n T_n$ for $T, T_n \in \mathcal{B}(X)$ if $Tf = \lim_n T_n f$ in $X$ for every $f \in X$. We also write $T_n \overset{s}{\to} T$ if $X = L^2$.
- Denote $\|\cdot\|_{p\to q} := \|\cdot\|_{L^p\to L^q}$.
- $L^p_+ := \{f \in L^p \mid f \geq 0 \text{ a.e.}\}$.
- $\mathcal{S}$ denotes the L. Schwartz space of test functions.
- $C_u := \{f \in C(\mathbb{R}^d) \mid f$ are uniformly continuous and bounded$\}$ (with the sup-norm).
- We write $c \neq c(\varepsilon)$ to emphasize that $c$ is independent of $\varepsilon$.
- $\operatorname{sprt} f$ denotes the support of function $f$.
- $]a,b[$ denotes an open interval.
- Given operators $A$, $B$, we write $B \supset A$ if $B$ is an extension of $A$.

## §2. Desingularization in abstract setting

We first prove a general desingularization theorem in the abstract setting. We will apply it in the next section to the fractional Kolmogorov operator.

Let $X$ be a locally compact topological space, and $\mu$ a $\sigma$-finite Borel measure on $X$. Set $L^p = L^p(X,\mu)$, $p \in [1,\infty]$, a (complex) Banach space. We use the notation

$$\langle u,v\rangle = \langle u\bar{v}\rangle := \int_X u\bar{v}\,d\mu, \quad \|\cdot\|_{p\to q} = \|\cdot\|_{L^p\to L^q}.$$

Let $-\Lambda$ be the generator of a contraction $C_0$ semigroup $e^{-t\Lambda}$, $t > 0$, in $L^2$.

Assume that, for some constants $M \geq 1$, $c_S > 0$, $j > 1$, $c > 0$,

$$(B_{11}) \qquad \|e^{-t\Lambda}f\|_1 \leq M\|f\|_1, \quad t \geq 0,\ f \in L^1 \cap L^2.$$

$$(B_{12}) \qquad \text{Sobolev embedding property:} \quad \operatorname{Re}\langle \Lambda u, u\rangle \geq c_S\|u\|^2_{2j},\ u \in D(\Lambda).$$

$$(B_{13}) \qquad \|e^{-t\Lambda}\|_{2\to\infty} \leq ct^{-\frac{j'}{2}}, \quad t > 0,\ j' = \frac{j}{j-1}.$$

Assume also that there exists a family of real-valued weights $\psi = \{\psi_s\}_{s>0}$ on $X$ such that, for all $s > 0$,

$$(B_{21}) \qquad 0 \leq \psi_s, \psi_s^{-1} \in L^1_{\text{loc}}(X - N, \mu), \quad \text{where } N \text{ is a closed null set,}$$

and there exist constants $\theta \in ]0,1[$, $\theta \neq \theta(s)$, $c_i \neq c_i(s)$ $(i=2,3)$ and a measurable set $\Omega^s \subset X$ such that

$$(B_{22}) \qquad \psi_s(x)^{-\theta} \leq c_2 \qquad \text{for all } x \in X - \Omega^s,$$

$$(B_{23}) \qquad \|\psi_s^{-\theta}\|_{L^{q'}(\Omega^s)} \leq c_3 s^{j'/q'}, \text{ where } q' = \frac{2}{1-\theta}.$$

**Theorem 2.1.** *In addition to* $(B_{11})$*–*$(B_{23})$*, assume that there exists a constant* $c_1 \neq c_1(s)$ *such that, for any* $s>0$ *and all* $\frac{s}{2} \leq t \leq s$*,*

$$(B_3) \qquad \|\psi_s e^{-t\Lambda}\psi_s^{-1} f\|_1 \leq c_1 \|f\|_1, \quad f \in L^1.$$

*Then there is a constant* $C$ *such that, for all* $t>0$ *and* $\mu$ *a.e.* $x,y \in X$*,*

$$|e^{-t\Lambda}(x,y)| \leq C t^{-j'} \psi_t(y).$$

**Remark 2.2.** In application of Theorem 2.1 to concrete operators, the main difficulty is in the verification of assumption $(B_3)$.

*Proof of Theorem 2.1.* Set $\psi \equiv \psi_s$ and put $L^2_\psi := L^2(X, \psi^2\, d\mu)$. In what follows, $\|\cdot\|_{2,\psi}$ and $\langle \cdot,\cdot\rangle_\psi$ denote the norm and the inner product in $L^2_\psi$, respectively, and $\|\cdot\|_{2,\psi\to 2,\psi}$ denotes the operator norm in $\mathcal{B}(L^2_\psi)$.

Define a unitary map $\Psi\colon L^2_\psi \to L^2$ by $\Psi f = \psi f$. Set $\Lambda_\psi = \Psi^{-1}\Lambda\Psi$ of domain $D(\Lambda_\psi) = \Psi^{-1}D(\Lambda)$. Then

$$e^{-t\Lambda_\psi} = \Psi^{-1} e^{-t\Lambda}\Psi, \quad \|e^{-t\Lambda_\psi}\|_{2,\psi\to 2,\psi} = \|e^{-t\Lambda}\|_{2\to 2}, \quad t \geq 0.$$

Here and below the subscript $\psi$ indicates that the corresponding quantities are related to the measure $\psi^2\, d\mu$.

Set $u_t = e^{-t\Lambda_\psi} f$, $f \in L^2_\psi \cap L^1_\psi$. Applying $(B_{12})$, and then the Hölder inequality, we have

$$\begin{aligned}
-\frac{1}{2}\frac{d}{dt}\langle u_t, u_t\rangle_\psi &= \operatorname{Re}\langle \Lambda_\psi u_t, u_t\rangle_\psi \quad \text{(see Remark 2.3 for the proof, if needed)}\\
&= \operatorname{Re}\langle \Lambda\psi u_t, \psi u_t\rangle\\
&\geq c_S \|\psi u_t\|_{2j}^2\\
&\geq c_S \frac{\langle u_t, u_t\rangle_\psi^r}{\|\psi u_t\|_q^{2(r-1)}},
\end{aligned}$$

where $q = \frac{2}{1+\theta}$ $(<2)$ and $r = \frac{(1+\theta)j-1}{j\theta}$.

Noticing that $(B_{11}) + (B_{12})$ implies the bound $\|e^{-t\Lambda}\|_{1\to 2} \leq \hat{c} t^{-\frac{j'}{2}}$ (for details, if needed, see Remark 2.4 below), we have by the interpolation inequality

$$\|e^{-t\Lambda}\|_{1\to q} \leq c_4 t^{-\frac{j'}{q'}}, \quad q' = \frac{q}{q-1}, \; c_4 = M^{\frac{2}{q}-1}\hat{c}^{\frac{2}{q'}}.$$

Also, by ($B_{11}$) and the contractivity of $e^{-t\Lambda}$ in $L^2$,

$$\|e^{-t\Lambda}\|_{q\to q} \le \|e^{-t\Lambda}\|_{1\to 1}^{1-a}\|e^{-t\Lambda}\|_{2\to 2}^{a} \le M^{\frac{2}{q}-1}$$

(for $a = 2 - \frac{2}{q}$). Therefore,

$$\begin{aligned}\|\psi u_t\|_q &= \|e^{-t\Lambda}\psi f\|_q = \|e^{-t\Lambda}|\psi|^{-\theta}|\psi|^{\frac{2}{q}} f\|_q \\ &\qquad \text{(we are applying } (B_{22}), (B_{23})) \\ &\le c_2\|e^{-t\Lambda}\|_{q\to q}\|f\|_{q,\psi} + \|e^{-t\Lambda}\|_{1\to q}\| \, |\psi|^{-\theta}\|_{L^{q'}(\Omega^s)}\|f\|_{q,\psi} \\ &\le \big(c_2 M^{\frac{2}{q}-1} + c_3c_4(s/t)^{\frac{j'}{q'}}\big)\|f\|_{q,\psi}.\end{aligned}$$

Thus, setting $w = \langle u_t, u_t\rangle_\psi$, we obtain

$$\frac{d}{dt}w^{1-r} \ge 2(r-1)c_S\big(c_2M^{\frac{2}{q}-1} + c_3c_4(s/t)^{\frac{j'}{q'}}\big)^{-2(r-1)}\|f\|_{q,\psi}^{-2(r-1)}.$$

Integrating this differential inequality yields

$$\|u_t\|_{2,\psi_s} \le C_1 t^{-j'(\frac{1}{q}-\frac{1}{2})}\|f\|_{q,\psi_s}, \quad s/2 \le t \le s.$$

The last inequality and ($B_3$) rewritten in the form $\|u_t\|_{1,\psi} \le c_1\|f\|_{1,\psi}$ yield, according to the Coulhon–Raynaud extrapolation theorem (Theorem B.1),

$$\|u_t\|_{2,\psi_s} \le C_2 t^{-\frac{j'}{2}}\|f\|_{1,\psi_s}, \quad s/2 \le t \le s,$$

or

$$\|e^{-t\Lambda}h\|_2 \le C_2 t^{-\frac{j'}{2}}\|h\|_{1,\sqrt{\psi_s}}, \quad h \in L^2 \cap L^1_{\sqrt{\psi_s}}, \; s/2 \le t \le s, \tag{2.1}$$

where $L^1_{\sqrt{\psi_s}} := L^1(X, \psi_s\, d\mu)$.

Since $\|e^{-2t\Lambda}h\|_\infty \le \|e^{-t\Lambda}\|_{2\to\infty}\|e^{-t\Lambda}h\|_2$, we have, employing ($B_{13}$),

$$\|e^{-2t\Lambda}h\|_\infty \le cC_2 t^{-j'}\|h\|_{1,\sqrt{\psi_s}},$$

and so the assertion of Theorem 2.1 follows. □

**Remark 2.3.** Above we evaluated

$$\frac{d}{dt}\langle u_t, u_t\rangle_\psi = \lim_{\tau\to 0}\frac{\langle u_{t+\tau}, u_{t+\tau}\rangle_\psi - \langle u_t, u_t\rangle_\psi}{\tau},$$

where

$$\begin{aligned}\frac{\langle u_{t+\tau}, u_{t+\tau}\rangle_\psi - \langle u_t, u_t\rangle_\psi}{\tau} &= \operatorname{Re}\Big\langle \frac{u_{t+\tau}-u_t}{\tau}, u_{t+\tau}\Big\rangle_\psi + \operatorname{Re}\Big\langle \frac{u_{t+\tau}-u_t}{\tau}, u_t\Big\rangle_\psi \\ &\to -2\operatorname{Re}\langle \Lambda_\psi u_t, u_t\rangle_\psi \quad (\tau \to 0)\end{aligned}$$

(using the strong continuity of $e^{-t\Lambda_\psi}$).

**Remark 2.4.** A standard argument yields ($B_{11}$) + ($B_{12}$) $\Rightarrow \|e^{-t\Lambda}\|_{1\to 2} \leq \hat{c}t^{-\frac{j'}{2}}$, $t > 0$. Indeed, setting $u_t := e^{-t\Lambda}f$, $f \in L^2 \cap L^1$, we have, applying ($B_{12}$), Hölder's inequality and ($B_{11}$),

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2 &= \mathrm{Re}\langle \Lambda u_t, u_t\rangle \\ &\geq c_S\|u_t\|_{2j}^2 \\ &\geq c_S\|u_t\|_2^{2+\frac{2}{j'}}\|u_t\|_1^{-\frac{2}{j'}} \\ &\geq c_S M^{-\frac{2}{j'}}\|u_t\|_2^{2+\frac{2}{j'}}\|f\|_1^{-\frac{2}{j'}}. \end{aligned}$$

Thus, $w := \|u_t\|_2^2$ satisfies

$$\frac{d}{dt}w^{-\frac{1}{j'}} \geq C\|f\|_1^{-\frac{2}{j'}}, \quad C = \frac{2c_S M^{-\frac{2}{j'}}}{j'},$$

so integrating this inequality we obtain $\|e^{-t\Lambda}\|_{1\to 2} \leq C^{-\frac{j'}{2}}t^{-\frac{j'}{2}}$.

It is now seen that $(B_1) \equiv$ ($B_{11}$)+($B_{12}$)+($B_{13}$) implies the bound $e^{-t\Lambda}(x,y) \leq \tilde{c}t^{-j'}$.

## §3. Heat kernel $e^{-t\Lambda}(x,y)$ for $\Lambda = (-\Delta)^{\frac{\alpha}{2}} - \kappa|x|^{-\alpha}x\cdot\nabla$, $1 < \alpha < 2$, $\kappa > 0$

We now state in detail our main result concerning the fractional Kolmogorov operator $(-\Delta)^{\frac{\alpha}{2}} - \kappa|x|^{-\alpha}x\cdot\nabla$, $1 < \alpha < 2$, $\kappa > 0$.

**(1)** Let us outline the construction of an appropriate operator realization $\Lambda_r$ of $(-\Delta)^{\frac{\alpha}{2}} - \kappa|x|^{-\alpha}x\cdot\nabla$ in $L^r$, $1 \leq r < \infty$. Set

$$b_\varepsilon(x) := \kappa|x|_\varepsilon^{-\alpha}x, \quad |x|_\varepsilon := \sqrt{|x|^2+\varepsilon},\ \varepsilon > 0,$$

define the approximating operators in $L^r$

$$\Lambda^\varepsilon \equiv \Lambda_r^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} - b_\varepsilon\cdot\nabla, \quad D(\Lambda_r^\varepsilon) = \mathcal{W}^{\alpha,r} := (1+(-\Delta)^{\frac{\alpha}{2}})^{-1}L^r,\ 1 \leq r < \infty,$$

and in $C_u$ (the space of uniformly continuous bounded functions with standard sup-norm),

$$\Lambda^\varepsilon \equiv \Lambda_{C_u}^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} - b_\varepsilon\cdot\nabla, \quad D(\Lambda_{C_u}^\varepsilon) = D((-\Delta)_{C_u}^{\frac{\alpha}{2}}).$$

The operator $-\Lambda^\varepsilon$ is the generator of a holomorphic semigroup in $L^r$ and in $C_u$. Moreover, the corresponding semigroups are positivity preserving:

$$e^{-t\Lambda^\varepsilon}L_+^r \subset L_+^r \quad \text{and} \quad e^{-t\Lambda^\varepsilon}C_u^+ \subset C_u^+$$

where $L^r_+ := \{f \in L^r \mid f \geq 0\}$, $C_u^+ := \{f \in C_u \mid f \geq 0\}$. Also,

$$\|e^{-t\Lambda^\varepsilon} f\|_\infty \leq \|f\|_\infty, \quad f \in L^r \cap L^\infty, \quad \text{or} \quad f \in C_u.$$

For details, if needed, see Section 8 below.

In Proposition 8.5 below we show that, for every $r \in [1,\infty[$, the limit

$$s\text{-}L^r\text{-}\lim_{\varepsilon \downarrow 0} e^{-t\Lambda_r^\varepsilon} \quad (\text{loc. uniformly in } t \geq 0)$$

exists and determines a positivity-preserving, contraction $C_0$ semigroup in $L^r$, say $e^{-t\Lambda_r}$; the (minus) generator $\Lambda_r$ is an appropriate operator realization of the fractional Kolmogorov operator $(-\Delta)^{\frac{\alpha}{2}} - \kappa|x|^{-\alpha}x\cdot\nabla$ in $L^r$; there exists a constant $c$ such that

$$\|e^{-t\Lambda_r}\|_{r\to q} \leq ct^{-\frac{d}{\alpha}(\frac{1}{r}-\frac{1}{q})}, \quad t > 0,$$

for all $1 \leq r < q \leq \infty$; by construction, the semigroups $e^{-t\Lambda_r}$ are consistent:

$$e^{-t\Lambda_r} \upharpoonright L^r \cap L^p = e^{-t\Lambda_p} \upharpoonright L^r \cap L^p.$$

Using Proposition 8.5 we obtain

$$\langle \Lambda_r u, h\rangle = \langle u, (-\Delta)^{\frac{\alpha}{2}} h\rangle + \langle u, b\cdot\nabla h\rangle + \langle u, (\operatorname{div} b)h\rangle, \quad u \in D(\Lambda_r),\ h \in C_c^\infty$$

(cf. [KSS, Prop. 9]).

**(2)** We now introduce the desingularizing weights for $e^{-t\Lambda}$. Define $\beta$ by

$$\beta\frac{d+\beta-2}{d+\beta-\alpha}\frac{\gamma(d+\beta-2)}{\gamma(d+\beta-\alpha)} = \kappa,$$

where

$$\gamma(\alpha) := \frac{2^\alpha \pi^{\frac{d}{2}}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d}{2}-\frac{\alpha}{2})}.$$

Direct calculations show that $\beta \in ]0,\alpha[$ exists (see Figure 1), and that $|x|^\beta$ is a Lyapunov function of the formal adjoint operator $\Lambda^* = (-\Delta)^{\frac{\alpha}{2}} + \nabla\cdot b$, i.e. $\Lambda^*|x|^\beta = 0$.

Set

$$\psi(x) \equiv \psi_s(x) := \eta(s^{-\frac{1}{\alpha}}|x|),$$

where $\eta$ is given by

$$\eta(t) = \begin{cases} t^\beta, & 0 < t < 1, \\ \beta t(2-\frac{t}{2}) + 1 - \frac{3}{2}\beta, & 1 \leq t \leq 2, \\ 1 + \frac{\beta}{2}, & t \geq 2. \end{cases}$$

In the proofs, we will be also using function

$$\tilde\psi(x) \equiv \tilde\psi_s(x) := s^{-\frac{\beta}{\alpha}}|x|^\beta.$$

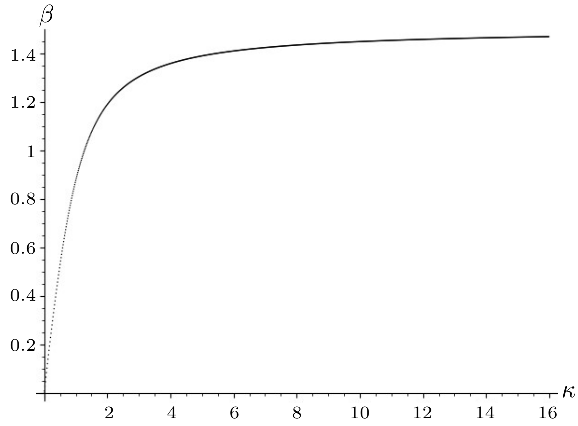


Figure 1. The function $\kappa \mapsto \beta$ for $d = 3$ and $\alpha = \frac{3}{2}$.

Applying Theorem 2.1 to the operator $\Lambda_r$ and the weights $\psi_s$, we obtain the following theorem:

**Theorem 3.1.** *We have that $e^{-t\Lambda_r}$ is an integral operator for each $t > 0$ with integral kernel $e^{-t\Lambda}(x,y) \geq 0$. There exists a constant $c_{N,w}$ such that, up to a change of $e^{-t\Lambda}(x,y)$ on a measure zero set, the weighted Nash initial estimate*

$$e^{-t\Lambda}(x,y) \leq c_{N,w} t^{-\frac{d}{\alpha}} \psi_t(y) \tag{NIE$_w$}$$

*is valid for all $x, y \in \mathbb{R}^d$ and $t > 0$.*

The next step is to deduce the following global-in-time "standard" upper bound on $e^{-t\Lambda}(x,y)$.

**Theorem 3.2.** (i) *There is a constant $C_1$ such that, up to a change of $e^{-t\Lambda}(x,y)$ on a measure zero set, for all $t > 0$, $x, y \in \mathbb{R}^d$,*

$$e^{-t\Lambda}(x,y) \leq C_1 e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y).$$

(ii) *Moreover, for a given $\delta \in ]0,1[$, there is a constant $D = D_\delta > 0$ such that*

$$e^{-t\Lambda}(x,y) \leq (1+\delta) e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y), \quad |x| > Dt^{\frac{1}{\alpha}}, \ y \in \mathbb{R}^d.$$

Theorems 3.1 and 3.2 are the key tools which allow us to establish the upper bound on $e^{-t\Lambda}(x,y)$:

**Theorem 3.3.** *There is a constant $C$ such that, up to a change of $e^{-t\Lambda}(x,y)$ on a measure zero set, for all $t>0$, $x,y\in\mathbb{R}^d$,*

$$e^{-t\Lambda}(x,y)\le Ce^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\psi_t(y). \tag{UB$_w$}$$

Using Theorem 3.3, we prove the lower bound on $e^{-t\Lambda}(x,y)$:

**Theorem 3.4.** *There is a constant $\widetilde{C}>0$ such that, up to a change of $e^{-t\Lambda}(x,y)$ on a measure zero set, for all $t>0$, $x,y\in\mathbb{R}^d$,*

$$e^{-t\Lambda}(x,y)\ge \widetilde{C}e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\psi_t(y). \tag{LB$_w$}$$

## §4. Proof of Theorem 3.1: The weighted Nash initial estimate

The proof follows by applying Theorem 2.1 to $e^{-t\Lambda_r}$.

Conditions ($B_{11}$) and ($B_{13}$) (with $j'=\frac{d}{\alpha}$) are satisfied by Proposition 8.5. Let us prove ($B_{12}$). By Proposition 8.1 ($\Lambda^\varepsilon\equiv\Lambda_2^\varepsilon$),

$$\operatorname{Re}\langle\Lambda^\varepsilon(1+\Lambda^\varepsilon)^{-1}g,(1+\Lambda^\varepsilon)^{-1}g\rangle\ge c_S\|(1+\Lambda^\varepsilon)^{-1}g\|_{2j}^2,\quad g\in L^2,$$

where $j=\frac{d}{d-\alpha}$, $c_S\neq c_S(\varepsilon)$, i.e.

$$\operatorname{Re}\langle g-(1+\Lambda^\varepsilon)^{-1}g,(1+\Lambda^\varepsilon)^{-1}g\rangle\ge c_S\|(1+\Lambda^\varepsilon)^{-1}g\|_{2j}^2.$$

Using the convergence $(1+\Lambda^\varepsilon)^{-1}\xrightarrow{s}(1+\Lambda)^{-1}$ in $L^2$ as $\varepsilon\downarrow 0$ (Proposition 8.5) in the LHS of the last inequality, and a weak compactness argument in $L^{2j}$ in its RHS, we obtain $\operatorname{Re}\langle\Lambda(1+\Lambda)^{-1}g,(1+\Lambda)^{-1}g\rangle\ge c_S\|(1+\Lambda)^{-1}g\|_{2j}^2$ for all $g\in L^2$, and so ($B_{12}$) is proved.

Condition ($B_{21}$) is evident from the definition of the weights $\psi_s$. It is easily seen that ($B_{22}$), ($B_{23}$) hold with $\Omega^s=B(0,s^{\frac{1}{\alpha}})$ and $\theta=\frac{(2-\alpha)d}{(2-\alpha)d+8\beta}$. It remains to prove the desingularizing $(L^1,L^1)$ bound ($B_3$), which presents the main difficulty.

The rest of this section is devoted to the verification of ($B_3$).

To verify ($B_3$), we modify the proof of the analogous $(L^1,L^1)$ bound in [KSS] (see also Remark 4.8 below). We will appeal to the Lumer–Phillips theorem applied to specially constructed $C_0$ semigroups in $L^1$, corresponding to operators with smooth coefficients and smooth weights, which approximate $\psi_s e^{-t\Lambda}\psi_s^{-1}$.

Recall that $b_\varepsilon(x):=\kappa|x|_\varepsilon^{-\alpha}x$, $|x|_\varepsilon:=\sqrt{|x|^2+\varepsilon}$, $\varepsilon>0$,

$$\Lambda^\varepsilon:=(-\Delta)^{\frac{\alpha}{2}}-b_\varepsilon\cdot\nabla,\quad D(\Lambda^\varepsilon)=\mathcal{W}^{\alpha,1}:=(1+(-\Delta)^{\frac{\alpha}{2}})^{-1}L^1,$$
$$(\Lambda^\varepsilon)^*=(-\Delta)^{\frac{\alpha}{2}}+\nabla\cdot b_\varepsilon,\quad D((\Lambda^\varepsilon)^*)=\mathcal{W}^{\alpha,1}.$$

By the Hille perturbation theorem, for each $\varepsilon > 0$, both $e^{-t\Lambda^\varepsilon}$ and $e^{-t(\Lambda^\varepsilon)^*}$ can be viewed as $C_0$ semigroups in $L^1$ and $C_u$ (see Sections 8 and 9).

Define approximating weights

$$\phi_{n,\varepsilon} := n^{-1} + e^{-\frac{(\Lambda^\varepsilon)^*}{n}}\psi, \quad \psi = \psi_s.$$

**Remark 4.1.** This choice of regularization of $\psi$ is dictated by the method: $e^{-\frac{(\Lambda^\varepsilon)^*}{n}}$ will be needed below to control the auxiliary potential $U_\varepsilon$. See also Remark 4.7 below.

In $L^1$ define operators

$$Q = \phi_{n,\varepsilon}\Lambda^\varepsilon\phi_{n,\varepsilon}^{-1}, \quad D(Q) = \phi_{n,\varepsilon}D(\Lambda^\varepsilon),$$

where $\phi_{n,\varepsilon}D(\Lambda^\varepsilon) := \{\phi_{n,\varepsilon}u \mid u \in D(\Lambda^\varepsilon)\}$,

$$F^t_{\varepsilon,n} = \phi_{n,\varepsilon}e^{-t\Lambda^\varepsilon}\phi_{n,\varepsilon}^{-1}.$$

Since $\phi_{n,\varepsilon}, \phi_{n,\varepsilon}^{-1} \in L^\infty$, these operators are well defined. In particular, $F^t_{\varepsilon,n}$ are bounded $C_0$ semigroups in $L^1$, say $F^t_{\varepsilon,n} = e^{-tG}$.

Set

$$\begin{aligned} M &:= \phi_{n,\varepsilon}(1 + (-\Delta)^{\frac{\alpha}{2}})^{-1}[L^1 \cap C_u] \\ &= \phi_{n,\varepsilon}(\lambda_\varepsilon + \Lambda^\varepsilon)^{-1}[L^1 \cap C_u], \quad 0 < \lambda_\varepsilon \in \rho(-\Lambda^\varepsilon). \end{aligned}$$

Clearly, $M$ is a dense subspace of $L^1$, $M \subset D(Q)$ and $M \subset D(G)$. Moreover, $Q \upharpoonright M \subset G$. Indeed, for $f = \phi_{n,\varepsilon}u \in M$,

$$Gf = s\text{-}L^1\text{-}\lim_{t\downarrow 0} t^{-1}(1 - e^{-tG})f = \phi_{n,\varepsilon}s\text{-}L^1\text{-}\lim_{t\downarrow 0} t^{-1}(1 - e^{-t\Lambda^\varepsilon})u = \phi_{n,\varepsilon}\Lambda^\varepsilon u = Qf.$$

Thus $Q \upharpoonright M$ is closable and $\widetilde{Q} := (Q \upharpoonright M)^{\mathrm{clos}} \subset G$.

**Proposition 4.2.** *The range $R(\lambda_\varepsilon + \widetilde{Q})$ is dense in $L^1$.*

*Proof.* If $\langle(\lambda_\varepsilon + \widetilde{Q})h, v\rangle = 0$ for all $h \in D(\widetilde{Q})$ and some $v \in L^\infty$, $\|v\|_\infty = 1$, then taking $h \in M$ we would have $\langle(\lambda_\varepsilon + Q)\phi_{n,\varepsilon}(\lambda_\varepsilon + \Lambda^\varepsilon)^{-1}g, v\rangle = 0$, $g \in L^1 \cap C_u$, or $\langle\phi_{n,\varepsilon}g, v\rangle = 0$. Choosing $g = e^{\frac{\Delta}{k}}(\chi_m v)$, where $\chi_m \in C_c^\infty$ with $\chi_m(x) = 1$ when $x \in B(0,m)$, we would have $\lim_{k\uparrow\infty}\langle\phi_{n,\varepsilon}g, v\rangle = \langle\phi_{n,\varepsilon}\chi_m, |v|^2\rangle = 0$, and so $v = 0$. Thus, $R(\lambda_\varepsilon + \widetilde{Q})$ is dense in $L^1$. □

**Proposition 4.3.** *There are constants $\hat{c} > 0$ and $\varepsilon_n > 0$ such that, for every $n$ and all $0 < \varepsilon \leq \varepsilon_n$,*

$$\lambda + \widetilde{Q} \textit{ is accretive whenever } \lambda \geq \hat{c}s^{-1} + n^{-1},$$

*where $s > 0$ is from the definition of the weight $\phi_{n,\varepsilon}$.*

*Proof.* Recall that both $e^{-t\Lambda^\varepsilon}$ and $e^{-t(\Lambda^\varepsilon)^*}$ are holomorphic in $L^1$ and $C_u$ due to Hille's perturbation theorem. We have

$$\psi = \psi_{(1)} + \psi_{(u)}, \quad 0 \leq \psi_{(1)} \in D((-\Delta)_1^{\frac{\alpha}{2}}), \quad 0 \leq \psi_{(u)} \in D((-\Delta)_{C_u}^{\frac{\alpha}{2}}).$$

For instance,

$$\psi_{(u)} := 1 + \frac{\beta}{2}, \quad \psi_{(1)} := \psi - 1 - \frac{\beta}{2} \quad (\text{so sprt}\, \psi_{(1)} \subset B(0, 2s^{\frac{1}{\alpha}})).$$

In $B(0, s^{\frac{1}{\alpha}})$, the weight $\psi$ coincides with $\tilde{\psi}(x) \equiv \tilde{\psi}_s(x) := s^{-\frac{\beta}{\alpha}}|x|^\beta$, so $\psi_{(1)} \in D((-\Delta)_1)$. Thus, $\psi_{(1)} \in D((-\Delta)_1^{\frac{\alpha}{2}})$ (see e.g. [Ka, Chap. V, Sect. 3.11]). Therefore,

$$(\Lambda^\varepsilon)^*\psi \quad (= (\Lambda^\varepsilon)^*_{L^1}\psi_{(1)} + (\Lambda^\varepsilon)^*_{C_u}\psi_{(u)})$$

is well defined and belongs to $L^1 + C_u = \{w + v \mid w \in L^1, v \in C_u\}$.

We verify that $\mathrm{Re}\langle(\lambda + \widetilde{Q})f, \frac{f}{|f|}\rangle \geq 0$ for all $f \in D(\widetilde{Q})$. For $f = \phi_{n,\varepsilon}u \in M$, we have

$$\Big\langle Qf, \frac{f}{|f|}\Big\rangle = \Big\langle \phi_{n,\varepsilon}\Lambda^\varepsilon u, \frac{f}{|f|}\Big\rangle = \lim_{t\downarrow 0} t^{-1}\Big\langle \phi_{n,\varepsilon}(1 - e^{-t\Lambda^\varepsilon})u, \frac{f}{|f|}\Big\rangle,$$

$$\begin{aligned}\mathrm{Re}\Big\langle Qf, \frac{f}{|f|}\Big\rangle &\geq \lim_{t\downarrow 0} t^{-1}\langle(1 - e^{-t\Lambda^\varepsilon})|u|, \phi_{n,\varepsilon}\rangle \\ &= \lim_{t\downarrow 0} t^{-1}\langle(1 - e^{-t\Lambda^\varepsilon})|u|, n^{-1}\rangle + \lim_{t\downarrow 0} t^{-1}\langle(1 - e^{-t\Lambda^\varepsilon})e^{-\frac{\Lambda^\varepsilon}{n}}|u|, \psi\rangle \\ &= \lim_{t\downarrow 0} t^{-1}\langle|u|, (1 - e^{-t(\Lambda^\varepsilon)^*})n^{-1}\rangle + \lim_{t\downarrow 0} t^{-1}\langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|, (1 - e^{-t(\Lambda^\varepsilon)^*})\psi\rangle \\ &= \langle|u|, (\Lambda^\varepsilon)^* n^{-1}\rangle + \langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|, (\Lambda^\varepsilon)^*\psi\rangle,\end{aligned}$$

where the first term is positive since

$$(\Lambda^\varepsilon)^* n^{-1} = n^{-1}\,\mathrm{div}\, b_\varepsilon = n^{-1}(d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2) \geq n^{-1}(d - \alpha)|x|_\varepsilon^{-\alpha} \geq 0.$$

Thus,

$$\mathrm{Re}\Big\langle Qf, \frac{f}{|f|}\Big\rangle \geq \langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|, (\Lambda^\varepsilon)^*\psi\rangle, \tag{4.1}$$

so it remains to bound $J := \langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|, (\Lambda^\varepsilon)^*\psi\rangle$ from below. For that, we estimate from below

$$(\Lambda^\varepsilon)^*\psi = (-\Delta)^{\frac{\alpha}{2}}\psi + \mathrm{div}(b_\varepsilon\psi).$$

**Claim 4.4.** *We have*

$$(-\Delta)^{\frac{\alpha}{2}}\psi \geq -\beta(d + \beta - 2)\frac{\gamma(d + \beta - 2)}{\gamma(d + \beta - \alpha)}|x|^{-\alpha}\tilde{\psi}.$$

*Proof.* All identities are in the sense of distributions:

$$(-\Delta)^{\frac{\alpha}{2}}\psi = -I_{2-\alpha}\Delta\psi$$
$$= -I_{2-\alpha}\Delta\tilde{\psi} - I_{2-\alpha}\Delta(\psi - \tilde{\psi}),$$

where $I_\nu = (-\Delta)^{-\frac{\nu}{2}}$ is the Riesz potential, and we evaluate the first term

$$-I_{2-\alpha}\Delta\tilde{\psi} = -s^{-\frac{\beta}{\alpha}}\beta(d+\beta-2)I_{2-\alpha}|x|^{\beta-2}$$
$$= -s^{-\frac{\beta}{\alpha}}\beta(d+\beta-2)\frac{\gamma(d+\beta-2)}{\gamma(d+\beta-\alpha)}|x|^{\beta-\alpha},$$

while the second term is positive and can be omitted:

$$-I_{2-\alpha}\Delta(\psi - \tilde{\psi}) \geq 0$$

(see Remark 4.6 below for detailed calculation). The proof of Claim 4.4 is completed. □

**Claim 4.5.** *We have*

$$\operatorname{div}(b_\varepsilon\psi) \geq \operatorname{div}(b\tilde{\psi}) - U_\varepsilon\tilde{\psi} - \hat{c}s^{-1}\psi$$

*for a constant* $\hat{c} \neq \hat{c}(\varepsilon, n)$, *where* $U_\varepsilon(x) := \kappa(d+\beta-\alpha)(|x|^{-\alpha} - |x|_\varepsilon^{-\alpha}) > 0$.

*Proof.* We represent

$$\operatorname{div}(b_\varepsilon\psi) = \operatorname{div}(b\tilde{\psi}) + \operatorname{div}(b_\varepsilon\psi) - \operatorname{div}(b\tilde{\psi})$$

and estimate the difference $\operatorname{div}(b_\varepsilon\psi) - \operatorname{div}(b\tilde{\psi})$:

$$\operatorname{div}(b_\varepsilon\psi) - \operatorname{div}(b\tilde{\psi}) = \operatorname{div}\big[b(\psi - \tilde{\psi})\big] + \operatorname{div}\big[(b_\varepsilon - b)\psi\big]$$
$$= h_1 + \operatorname{div}\big[(b_\varepsilon - b)\psi\big],$$

where $h_1 \in C_\infty$ (continuous functions vanishing at infinity), $h_1 = 0$ in $B(0, s^{\frac{1}{\alpha}})$. In turn,

$$\operatorname{div}\big[(b_\varepsilon - b)\psi\big] = (b_\varepsilon - b)\cdot\nabla\psi + (\operatorname{div} b_\varepsilon - \operatorname{div} b)\psi$$
$$= \kappa(|x|_\varepsilon^{-\alpha} - |x|^{-\alpha})x\cdot\nabla\tilde{\psi} + h_2 + \kappa\big[d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2 - (d-\alpha)|x|^{-\alpha}\big]\psi$$
$$\text{(where } h_2 := \kappa(|x|_\varepsilon^{-\alpha} - |x|^{-\alpha})x\cdot\nabla(\psi - \tilde{\psi}) \in C_\infty,\ h_2 = 0 \text{ in } B(0, s^{\frac{1}{\alpha}}))$$
$$= \kappa(|x|_\varepsilon^{-\alpha} - |x|^{-\alpha})\beta\tilde{\psi} + h_2 + \kappa\big[d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2 - (d-\alpha)|x|^{-\alpha}\big]\psi$$
$$\geq \kappa(|x|_\varepsilon^{-\alpha} - |x|^{-\alpha})\beta\tilde{\psi} + h_2 + \kappa(d-\alpha)(|x|_\varepsilon^{-\alpha} - |x|^{-\alpha})\psi.$$

Thus,

$$\operatorname{div}(b_\varepsilon\psi) \geq \operatorname{div}(b\tilde{\psi}) + \kappa(d+\beta-\alpha)(|x|_\varepsilon^{-\alpha} - |x|^{-\alpha})\tilde{\psi} + h_1 + h_2 + h_3,$$

where $h_3 := \kappa(d-\alpha)(|x|_\varepsilon^{-\alpha} - |x|^{-\alpha})(\psi - \tilde{\psi}) \in C_\infty$, $h_3 = 0$ in $B(0, s^{\frac{1}{\alpha}})$.

A straightforward calculation shows that $h_i \geq -c_i\psi s^{-1}$ with $c_i \neq c_i(\varepsilon, n)$, $i = 1, 2, 3$ (we have used that $h_i = 0$ in $B(0, s^{\frac{1}{\alpha}})$). The assertion of Claim 4.5 follows. □

Now we combine Claims 4.4 and 4.5: in view of the choice of $\beta$,

$$-\beta(d+\beta-2)\frac{\gamma(d+\beta-2)}{\gamma(d+\beta-\alpha)}|x|^{-\alpha}\tilde{\psi} + \operatorname{div}(b\tilde{\psi}) = 0$$

(that is, formally, $\Lambda^*\tilde{\psi} = 0$), and so

$$(\Lambda^\varepsilon)^*\psi \geq -U_\varepsilon\tilde{\psi} - \hat{c}s^{-1}\psi.$$

It follows that

$$\begin{aligned}
J \equiv \langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|, (\Lambda^\varepsilon)^*\psi\rangle &\geq -\hat{c}s^{-1}\langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|, \psi\rangle - \langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|, U_\varepsilon\tilde{\psi}\rangle \\
&= -\hat{c}s^{-1}\langle |u|, e^{-\frac{(\Lambda^\varepsilon)^*}{n}}\psi\rangle - \langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|, U_\varepsilon\tilde{\psi}\rangle \\
&\geq -\hat{c}s^{-1}\langle |u|, n^{-1} + e^{-\frac{(\Lambda^\varepsilon)^*}{n}}\psi\rangle - \langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|, U_\varepsilon\tilde{\psi}\rangle \\
&\qquad \text{(recall that } |u| = \phi_{n,\varepsilon}^{-1}|f| \text{ and } \phi_{n,\varepsilon} = n^{-1} + e^{-\frac{(\Lambda^\varepsilon)^*}{n}}\psi) \\
&= -\hat{c}s^{-1}\|f\|_1 - \langle |u|, e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(U_\varepsilon\tilde{\psi})\rangle.
\end{aligned}$$

Now, for every $n \geq 1$, we have

$$\begin{aligned}
\|e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(U_\varepsilon\tilde{\psi})\|_\infty &\leq \|e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(\mathbf{1}_{B^c(0,R)}U_\varepsilon\tilde{\psi})\|_\infty + \|e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(\mathbf{1}_{B(0,R)}U_\varepsilon\tilde{\psi})\|_\infty \\
&\qquad \text{(we are using that } e^{-t(\Lambda^\varepsilon)^*} \text{ is a } L^\infty \text{ contraction} \\
&\qquad \text{and ultracontraction; see Proposition 9.1)} \\
&\leq \|\mathbf{1}_{B^c(0,R)}U_\varepsilon\tilde{\psi}\|_\infty + c_N n^{\frac{d}{\alpha}}\|\mathbf{1}_{B(0,R)}U_\varepsilon\tilde{\psi}\|_1 \\
&\qquad \text{(we fix } R = R_n \text{ such that } \|\mathbf{1}_{B^c(0,R)}U_\varepsilon\tilde{\psi}\|_\infty \leq 2^{-1}n^{-2} \\
&\qquad \text{and choose } \varepsilon_n > 0 \text{ such that, for all } \varepsilon \leq \varepsilon_n, \\
&\qquad \|\mathbf{1}_{B(0,R)}U_\varepsilon\tilde{\psi}\|_1 \leq 2^{-1}n^{-2}(c_N n^{\frac{d}{\alpha}})^{-1}) \\
&\leq n^{-2}.
\end{aligned}$$

Therefore, since $\phi_{n,\varepsilon} \geq n^{-1}$, we have for every $n$ and all $\varepsilon \leq \varepsilon_n$,

$$\|\phi_{n,\varepsilon}^{-1}e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(U_\varepsilon\tilde{\psi})\|_\infty \leq n^{-1}$$

and so $\langle |u|, e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(U_\varepsilon\tilde{\psi})\rangle \leq n^{-1}\|f\|_1$. Thus,

$$J \geq -(\hat{c}s^{-1} + n^{-1})\|f\|_1.$$

Returning to (4.1), one can easily see that the latter yields the assertion of Proposition 4.3. □

**Remark 4.6.** Let us show that $-\Delta(\psi-\tilde{\psi}) \geq 0$. Without loss of generality, $s=1$. The inequality is evidently true on $\{0<|x|\leq 1\}\cup\{|x|\geq 2\}$. Now, let $1<|x|<2$. Then

$$\begin{aligned}\Delta(\tilde{\psi}-\psi) &= \beta(d+\beta-2)|x|^{\beta-2} - \eta''(|x|) - \eta'(|x|)(d-1)|x|^{-1} \\ &= \beta(d+\beta-2)|x|^{\beta-2} + \beta - \beta(2-|x|)(d-1)|x|^{-1} \\ &= \beta|x|^{-2}\big((d+\beta-2)|x|^{\beta} + |x|^2 - (d-1)(2-|x|)|x|\big) \\ &\geq \beta|x|^{-2}\big((d+\beta-2)+1-(d-1)\big) \geq 0.\end{aligned}$$

The fact that $\widetilde{Q}$ is closed, together with Propositions 4.2 and 4.3, implies that the range $R(\lambda_\varepsilon+\widetilde{Q}) = L^1$ (Appendix C). Then, by the Lumer–Phillips theorem, $\lambda+\widetilde{Q}$ is the (minus) generator of a contraction semigroup, and $\widetilde{Q}=G$ due to $\widetilde{Q}\subset G$. Thus, it follows that, for all $n$ and all $\varepsilon\leq\varepsilon_n$,

$$(\star) \qquad \|e^{-tG}\|_{1\to1} \equiv \|\phi_{n,\varepsilon}e^{-t\Lambda^\varepsilon}\phi_{n,\varepsilon}^{-1}\|_{1\to1} \leq e^{\omega t}, \quad \omega = \hat{c}s^{-1}+n^{-1}.$$

To obtain ($B_3$), it remains to pass to the limit in ($\star$): first in $\varepsilon\downarrow 0$ and then in $n\to\infty$. It suffices to prove ($B_3$) on positive functions. By ($\star$),

$$\|\phi_{n,\varepsilon}e^{-t\Lambda^\varepsilon}\phi_{n,\varepsilon}^{-1}f\|_1 \leq e^{\omega t}\|f\|_1, \quad 0\leq f\in L^1,$$

or taking $f=\phi_{n,\varepsilon}h$, $0\leq h\in L^1$,

$$\|\phi_{n,\varepsilon}e^{-t\Lambda^\varepsilon}h\|_1 \leq e^{\omega t}\|\phi_{n,\varepsilon}h\|_1.$$

Using Proposition 8.5 we have

$$\begin{aligned}\|\phi_{n,\varepsilon}e^{-t\Lambda^\varepsilon}h\|_1 &= \langle n^{-1}e^{-t\Lambda^\varepsilon}h\rangle + \langle\psi, e^{-(t+\frac{1}{n})\Lambda^\varepsilon}h\rangle \\ &\to \langle n^{-1}e^{-t\Lambda}h\rangle + \langle\psi, e^{-(t+\frac{1}{n})\Lambda}h\rangle \quad \text{as } \varepsilon\downarrow 0\end{aligned}$$

and

$$\|\phi_{n,\varepsilon}h\|_1 = n^{-1}\langle h\rangle + \langle\psi, e^{-\frac{\Lambda^\varepsilon}{n}}h\rangle \to n^{-1}\langle h\rangle + \langle\psi, e^{-\frac{\Lambda}{n}}h\rangle \quad \text{as } \varepsilon\downarrow 0.$$

Thus,

$$\langle n^{-1}e^{-t\Lambda}h\rangle + \langle\psi, e^{-(t+\frac{1}{n})\Lambda}h\rangle \leq e^{\omega t}(n^{-1}\langle h\rangle + \langle\psi, e^{-\frac{\Lambda}{n}}h\rangle).$$

Taking $n\to\infty$, we obtain $\langle\psi e^{-t\Lambda}h\rangle \leq e^{\hat{c}s^{-1}t}\langle\psi h\rangle$. Condition ($B_3$) now follows.

The proof of Theorem 3.1 is completed. □

**Remark 4.7** (On the choice of the regularization $\phi_{n,\varepsilon}$ of the weight $\psi$). In [KSS] we construct the regularization of the weight in the same way as above, although there the factor $e^{-\frac{1}{n}(\Lambda^\varepsilon)^*}$ serves a different purpose (in [KSS] the drift term $b\cdot\nabla$ has the opposite sign, and so the corresponding weight is unbounded). (As a by-product, this allows us to consider $(-\Delta)^{\frac{\alpha}{2}}$ perturbed by two drift terms, as in the present paper and as in [KSS], possibly having singularities at different points.)

**Remark 4.8.** In the proof of the analogous $(L^1,L^1)$ bound in [KSS, proof of Thm. 2], where we consider the vector field $b$ of the opposite sign, we first pass to the limit in $n\to\infty$, and then in $\varepsilon\downarrow 0$. In the proof of Theorem 3.1 above this order is naturally reversed.

As a consequence of the $(L^1,L^1)$ bound ($B_3$), we obtain, up to change of $e^{-t\Lambda}(x,y)$ on a measure zero set, the following corollary:

**Corollary 4.9.** *We have*

$$\langle e^{-t\Lambda}(\cdot,x)\psi_t(\cdot)\rangle\leq c_1\psi_t(x)$$

*for all* $x\in\mathbb{R}^d$, $x\neq 0$, $t>0$.

*Proof.* By ($B_3$),

$$\langle\psi_t e^{-t\Lambda}h\rangle\leq c_1\langle\psi_t h\rangle,\quad 0\leq h\in C_c^\infty,$$

i.e.

$$\langle\!\langle h(z)e^{-t\Lambda}(\cdot,z)\psi_t(\cdot)\rangle\!\rangle_z\leq c_1\langle\psi_t h\rangle.$$

The required result now follows upon selecting $h\to\delta_x$, $x\neq 0$ and applying the Lebesgue differentiation theorem in the LHS. □

As a consequence of Corollary 4.9 and ($\mathrm{NIE}_w$), we obtain the following corollary:

**Corollary 4.10.** *We have*

$$\langle e^{-t\Lambda}(\cdot,x)\rangle=\langle e^{-t\Lambda^*}(x,\cdot)\rangle\leq C_2\psi_t(x)$$

*for all* $x\in\mathbb{R}^d$, $x\neq 0$, $t>0$.

*Proof.* We have

$$\begin{aligned}\langle e^{-t\Lambda^*}(x,\cdot)\rangle &\leq \big\langle \mathbf{1}_{B(0,t^{\frac{1}{\alpha}})}(\cdot)e^{-t\Lambda^*}(x,\cdot)\big\rangle+\big\langle \mathbf{1}_{B^c(0,t^{\frac{1}{\alpha}})}(\cdot)e^{-t\Lambda^*}(x,\cdot)\psi_t(\cdot)\big\rangle\\ &=: I_1+I_2.\end{aligned}$$

By ($\mathrm{NIE}_w$), $I_1\leq c'\psi_t(x)$, and by Corollary 4.9, $I_2\leq c''\psi_t(x)$, for appropriate constants $c'$, $c''<\infty$. Set $C_2:=c'+c''$. □

## §5. Proof of Theorem 3.2: The standard upper bounds

**Proof of assertion (i).** For brevity, put $A := (-\Delta)^{\frac{\alpha}{2}}$. Recall that

$$k_0^{-1} t(|x-y|^{-d-\alpha} \wedge t^{-\frac{d+\alpha}{\alpha}}) \leq e^{-tA}(x,y) \leq k_0 t(|x-y|^{-d-\alpha} \wedge t^{-\frac{d+\alpha}{\alpha}})$$

for all $x, y \in \mathbb{R}^d$, $x \neq y$, $t > 0$, for a constant $k_0 = k_0(d,\alpha) > 1$.

In view of Proposition 8.5, it suffices to prove the a priori bound

$$e^{-t\Lambda^\varepsilon}(x,y) \leq C_1 e^{-tA}(x,y), \quad x, y \in \mathbb{R}^d,\ t > 0,\ C_1 \neq C_1(\varepsilon).$$

By duality, it suffices to prove

$$e^{-t(\Lambda^\varepsilon)^*}(x,y) \leq C_1 e^{-tA}(x,y), \quad x, y \in \mathbb{R}^d,\ t > 0,\ C_1 \neq C_1(\varepsilon).$$

***Step 1.*** For every $D > 1$ and all $t > 0$, $|x| \leq Dt^{\frac{1}{\alpha}}$, $|y| \leq Dt^{\frac{1}{\alpha}}$ the bound

$$e^{-t(\Lambda^\varepsilon)^*}(x,y) \leq k_0 c_N (2D)^{d+\alpha} e^{-tA}(x,y)$$

is valid.

In fact, we will prove the following lemma:

**Lemma 5.1.** *Let $t > 0$ and $D > 1$. Then*

(i) $e^{-t(\Lambda^\varepsilon)^*}(x,y) \leq k_0 c_N (2D)^{d+\alpha} e^{-tA}(x,y), \quad |x| \leq Dt^{\frac{1}{\alpha}},\ |y| \leq Dt^{\frac{1}{\alpha}}$.

(ii) $e^{-t\Lambda^*}(x,y) \leq k_0 c_{N,w} (1+D)^{d+\alpha} e^{-tA}(x,y)\psi_t(x), \quad |x| \leq t^{\frac{1}{\alpha}},\ |y| \leq Dt^{\frac{1}{\alpha}}$.

*Proof.* (i) Note that $(|x| \leq Dt^{\frac{1}{\alpha}}, |y| \leq Dt^{\frac{1}{\alpha}}) \Rightarrow t^{-\frac{d}{\alpha}} \leq (2D)^{d+\alpha} t|x-y|^{-d-\alpha}$. The latter means that $t^{-\frac{d}{\alpha}} \leq k_0(2D)^{d+\alpha} e^{-tA}(x,y)$. In Proposition 9.2, the Nash initial estimate

$$\text{(NIE)} \qquad e^{-t(\Lambda^\varepsilon)^*}(x,y) \leq c_N t^{-\frac{d}{\alpha}}, \quad x, y \in \mathbb{R}^d,\ t > 0$$

is proved. Therefore,

$$e^{-t(\Lambda^\varepsilon)^*}(x,y) \leq c_N t^{-\frac{d}{\alpha}} \leq k_0 c_N (2D)^{d+\alpha} e^{-tA}(x,y).$$

(ii) Clearly, $(|x| \leq t^{\frac{1}{\alpha}}, |y| \leq Dt^{\frac{1}{\alpha}}) \Rightarrow t^{-\frac{d}{\alpha}} \leq (1+D)^{d+\alpha} t|x-y|^{-d-\alpha}$, and so the inequality $t^{-\frac{d}{\alpha}} \leq k_0(1+D)^{d+\alpha} e^{-tA}(x,y)$ is valid. By ($\text{NIE}_w$) (Theorem 3.1), $e^{-t\Lambda^*}(x,y) \leq c_{N,w} t^{-\frac{d}{\alpha}} \psi_t(x)$ for all $t > 0$, $x, y \in \mathbb{R}^d$. Therefore,

$$e^{-t\Lambda^*}(x,y) \leq k_0 c_{N,w} (1+D)^{d+\alpha} e^{-tA}(x,y)\psi_t(x). \qquad \square$$

In what follows, we will need the following estimates: Set

$$E^t(x,y) = t(|x-y|^{-d-\alpha-1} \wedge t^{-\frac{d+\alpha+1}{\alpha}})$$

and

$$E^t f(x) := \langle E^t(x,\cdot) f(\cdot) \rangle, \quad t > 0.$$

**Lemma 5.2.** *There exist constants $k_i$ $(i=1,2,3)$ such that for all $0<t<\infty$, $x$, $y \in \mathbb{R}^d$,*

(i) $|\nabla_x e^{-tA}(x,y)| \le k_1 E^t(x,y)$;

(ii) $\int_0^t \langle e^{-(t-\tau)A}(x,\cdot)E^\tau(\cdot,y)\rangle \, d\tau \le k_2 t^{\frac{\alpha-1}{\alpha}} e^{-tA}(x,y)$;

(iii) $\int_0^t \langle E^{t-\tau}(x,\cdot)E^\tau(\cdot,y)\rangle \, d\tau \le k_3 t^{\frac{\alpha-1}{\alpha}} E^t(x,y)$.

*Proof.* For the proofs of (i), (ii), see e.g. [BJ]. Essentially the same argument yields (iii); see e.g. [KSS, Sect. 5] for details. □

***Step 2.*** Fix $\delta \in \,]0,2^{-1}[$. Set $C_g := \kappa k_1(2k_2+k_3)$, $R := (C_g\delta^{-1})^{\frac{1}{\alpha-1}}$ and $m = 1+2k_0k_1$.

*If $D \ge Rm$, then the bound*

$$e^{-t(\Lambda^\varepsilon)^*}(x,y) \le (1+\delta)e^{-tA}(x,y), \quad x \in \mathbb{R}^d,\ |y| > Dt^{\frac{1}{\alpha}},\ t>0 \tag{5.1}$$

*is valid.*

We use the Duhamel formula

$$\begin{aligned} e^{-t(\Lambda^\varepsilon)^*} &= e^{-tA} + \int_0^t e^{-\tau(\Lambda^\varepsilon)^*}(B^t_{\varepsilon,R}+B^{t,c}_{\varepsilon,R})e^{-(t-\tau)A}\,d\tau \\ &=: e^{-tA} + K^t_R + K^{t,c}_R, \quad R := (C_g\delta^{-1})^{\frac{1}{\alpha-1}}, \end{aligned} \tag{5.2}$$

where

$$B^t_{\varepsilon,R} := \mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}B_\varepsilon, \quad B^{t,c}_{\varepsilon,R} := \mathbf{1}_{B^c(0,Rt^{\frac{1}{\alpha}})}B_\varepsilon, \quad B_\varepsilon := -b_\varepsilon\cdot\nabla - W_\varepsilon,$$

and

$$W_\varepsilon(x) := \kappa(d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2) \ (=\operatorname{div} b_\varepsilon), \quad |b_\varepsilon(x)| = \kappa|x|_\varepsilon^{-\alpha}|x|.$$

Set

$$M^t_R(x,y) := (d-\alpha)\kappa\int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|\cdot|_\varepsilon^{-\alpha}e^{-(t-\tau)A}(\cdot,y)\rangle\,d\tau.$$

**Claim 5.3.** *For every $D \ge Rm$ and all $|y| > Dt^{\frac{1}{\alpha}}$, $x \in \mathbb{R}^d$, we have*

$$K^t_R(x,y) \le -\frac{1}{2}M^t_R(x,y).$$

*Proof.* Using Lemma 5.2(i), we obtain

$$\begin{aligned}K_R^t(x,y) &\equiv \int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)B_{\varepsilon,R}^t(\cdot)e^{-(t-\tau)A}(\cdot,y)\rangle\, d\tau \\ &\leq k_1 \int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|b_\varepsilon(\cdot)|E^{t-\tau}(\cdot,y)\rangle\, d\tau \\ &\quad - \int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)W_\varepsilon(\cdot)e^{-(t-\tau)A}(\cdot,y)\rangle\, d\tau =: I_1 + I_2.\end{aligned}$$

Using $E^{t-\tau}(z,y) \leq k_0 e^{-(t-\tau)A}(z,y)|z-y|^{-1}$, we obtain

$$\begin{aligned}I_1 &\leq k_0k_1 \int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|b_\varepsilon(\cdot)|e^{-(t-\tau)A}(\cdot,y)|\cdot-y|^{-1}\rangle\, d\tau \\ &\quad \text{(we are using } \mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|b_\varepsilon(\cdot)||\cdot-y|^{-1} \leq \mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)R(D-R)^{-1}\kappa|\cdot|_\varepsilon^{-\alpha}) \\ &\leq k_0k_1R(D-R)^{-1}\kappa \int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|\cdot|_\varepsilon^{-\alpha}e^{-(t-\tau)A}(\cdot,y)\rangle\, d\tau \\ &= k_0k_1R(D-R)^{-1}(d-\alpha)^{-1}M_R^t(x,y).\end{aligned}$$

We now compare the RHS of the last estimate with $I_2$. Since $W_\varepsilon(\cdot) \geq \kappa(d-\alpha)|\cdot|_\varepsilon^{-\alpha}$, we have

$$K_R^t(x,y) \leq \big(k_0k_1R(D-R)^{-1}(d-\alpha)^{-1}-1\big)M_R^t(x,y).$$

Since $k_0k_1R(D-R)^{-1} \leq \frac{k_0k_1}{m-1} \leq \frac{1}{2}$ and $d-\alpha > 1$ by our assumptions, we end the proof of Claim 5.3. ☐

**Claim 5.4.** *For every $D \geq Rm$ and all $|y| > Dt^{\frac{1}{\alpha}}$, $x \in \mathbb{R}^d$, we have*

$$K_R^{t,c}(x,y) \leq \delta(M_R^t(x,y) + e^{-tA}(x,y)).$$

*Proof.* Recall that

$$K_R^{t,c}(x,y) \equiv \int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)B_{\varepsilon,R}^{t,c}(\cdot)e^{-(t-\tau)A}(\cdot,y)\rangle\, d\tau,$$

where $B_{\varepsilon,R}^{t,c} = \mathbf{1}_{B^c(0,Rt^{\frac{1}{\alpha}})}(-b_\varepsilon\cdot\nabla - W_\varepsilon)$. Thus, discarding in $K_R^{t,c}$ the term containing $-W_\varepsilon$ and using Lemma 5.2(i), we obtain

$$(*) \qquad K_R^{t,c}(x,y) \leq k_1\kappa R^{1-\alpha}t^{-\frac{\alpha-1}{\alpha}} \int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)E^{t-\tau}(\cdot,y)\rangle\, d\tau.$$

We will have to estimate the integral in the RHS of (*).

Put

$$(e^{-\tau(\Lambda^\varepsilon)^*}E^{t-\tau})(x,y)=\langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)E^{t-\tau}(\cdot,y)\rangle,$$
$$(e^{-\tau(\Lambda^\varepsilon)^*}B_\varepsilon E^{t-\tau})(x,y)=\langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)B_\varepsilon(\cdot)E^{t-\tau}(\cdot,y)\rangle,$$

and analogously for $B_\varepsilon$ replaced by similar operators.

By the Duhamel formula,

$$\begin{aligned}&\int_0^t (e^{-\tau(\Lambda^\varepsilon)^*}E^{t-\tau})(x,y)\,d\tau\\ &\quad=\int_0^t (e^{-\tau A}E^{t-\tau})(x,y)\,d\tau\\ &\qquad+\int_0^t\int_0^\tau \big(e^{-\tau'(\Lambda^\varepsilon)^*}(B^t_{\varepsilon,R}+B^{t,c}_{\varepsilon,R})e^{-(\tau-\tau')A}\,d\tau' E^{t-\tau}\big)(x,y)\,d\tau\\ &\quad\equiv\int_0^t (e^{-\tau A}E^{t-\tau})(x,y)\,d\tau+J_R(x,y)+J_R^c(x,y),\end{aligned}$$

where, by Lemma 5.2(ii),

$$\int_0^t (\langle e^{-\tau A}(x,\cdot)E^{t-\tau}(\cdot,y)\rangle)(x,y)\,d\tau\le k_2 t^{\frac{\alpha-1}{\alpha}}e^{-tA}(x,y).$$

Let us estimate $J_R(x,y)$ and $J_R^c(x,y)$.

In $J_R(x,y)$, discarding the term containing $-W_\varepsilon$ and applying Lemma 5.2(i), we obtain

$$\begin{aligned}J_R(x,y)&\le k_1\int_0^t\int_0^\tau \big(e^{-\tau'(\Lambda^\varepsilon)^*}\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b_\varepsilon|E^{\tau-\tau'}\,d\tau' E^{t-\tau}\big)(x,y)\,d\tau\\ &\qquad\text{(we are changing the order of integration and applying Lemma 5.2(iii))}\\ &\le k_1k_3\int_0^t \big(e^{-\tau'(\Lambda^\varepsilon)^*}\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b_\varepsilon|(t-\tau')^{\frac{\alpha-1}{\alpha}}E^{t-\tau'}\big)(x,y)\,d\tau'\\ &\le k_1k_3t^{\frac{\alpha-1}{\alpha}}\int_0^t \big(e^{-\tau'(\Lambda^\varepsilon)^*}\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b_\varepsilon|E^{t-\tau'}\big)(x,y)\,d\tau'.\end{aligned}$$

Now, repeating the corresponding argument in the proof of Claim 5.3, we obtain

$$J_R(x,y)\le C_2t^{\frac{\alpha-1}{\alpha}}M_R^t(x,y),\quad C_2=k_0k_1k_3R(D-R)^{-1}(d-\alpha)^{-1}\le\frac{k_3}{2}.$$

($C_2\le\frac{k_0k_1k_3}{m-1}(d-\alpha)^{-1}\le\frac{k_3}{2}(d-\alpha)^{-1}\le\frac{k_3}{2}$.)

In turn, $J_R^c=\int_0^t(J_R^c)^\tau E^{t-\tau}\,d\tau$, where

$$(J_R^c)^\tau:=\int_0^\tau e^{-\tau'(\Lambda^\varepsilon)^*}B^c_{\varepsilon,R}e^{-(\tau-\tau')A}\,d\tau'.$$

Again, discarding the $-W_\varepsilon$ term in $B^c_{\varepsilon,R}$ and applying Lemma 5.2(i), we obtain

$$|(J_R^c)^\tau(x,y)| \le \kappa k_1 R^{1-\alpha}\tau^{-\frac{\alpha-1}{\alpha}} \int_0^\tau (e^{-\tau'(\Lambda^\varepsilon)^*}E^{\tau-\tau'})(x,y)\,d\tau'.$$

Due to Lemma 5.2(iii),

$$\begin{aligned}|J_R^c(x,y)| &\le \kappa k_1 k_3 R^{1-\alpha} t^{-\frac{\alpha-1}{\alpha}} \int_0^t \langle e^{-\tau'(\Lambda^\varepsilon)^*}(x,\cdot)(t-\tau')^{\frac{\alpha-1}{\alpha}} E^{t-\tau'}(\cdot,y)\rangle\,d\tau' \\ &\le \kappa k_1 k_3 R^{1-\alpha}\int_0^t \langle e^{-\tau'(\Lambda^\varepsilon)^*}(x,\cdot)E^{t-\tau'}(\cdot,y)\rangle\,d\tau'.\end{aligned}$$

Thus, due to $\kappa k_1 k_3 R^{1-\alpha} \le \delta < \frac{1}{2}$,

$$\begin{aligned}&\int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)E^{t-\tau}(\cdot,y)\rangle\,d\tau \\ &\quad\le k_2 t^{\frac{\alpha-1}{\alpha}} e^{-tA}(x,y) + \frac{k_3}{2} t^{\frac{\alpha-1}{\alpha}} M_R^t(x,y) + \frac{1}{2}\int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)E^{t-\tau}(\cdot,y)\rangle\,d\tau.\end{aligned}$$

Thus, we obtain

$$\int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)E^{t-\tau}(\cdot,y)\rangle\,d\tau \le 2k_2 t^{\frac{\alpha-1}{\alpha}} e^{-tA}(x,y) + k_3 t^{\frac{\alpha-1}{\alpha}} M_R^t(x,y).$$

Substituting the latter in $(*)$, we obtain Claim 5.4. □

Now, applying Claims 5.3 and 5.4 in (5.2), we have

$$\begin{aligned}e^{-t(\Lambda^\varepsilon)^*}(x,y) &\le e^{-tA}(x,y) - \frac{1}{2}M_R^t(x,y) + \delta(M_R^t(x,y) + e^{-tA}(x,y)) \\ &\le (1+\delta)e^{-tA}(x,y),\end{aligned}$$

thus ending the proof of Step 2.

***Step 3.*** *Set $R = 1 \vee (2\kappa k_3)^{\frac{1}{\alpha-1}}$ and let $D \ge 2R$. Then there is a constant $C = C(d,\alpha,\kappa,R)$ such that the bound*

$$e^{-t(\Lambda^\varepsilon)^*}(x,y) \le Ce^{-tA}(x,y), \quad |x| > 2Dt^{\frac{1}{\alpha}},\ |y| \le Dt^{\frac{1}{\alpha}},\ t>0$$

*is valid.*

(See the proof below for the explicit formula for $C(d,\alpha,\kappa,R)$.)

Using the Duhamel formula and applying Lemma 5.2(i), we have

$$\begin{aligned}e^{-t(\Lambda^\varepsilon)^*}(x,y) &\le e^{-tA}(x,y) + k_1\int_0^t (E^\tau |b_\varepsilon| e^{-(t-\tau)(\Lambda^\varepsilon)^*})(x,y)\,d\tau \\ &\le e^{-tA}(x,y) + k_1 L^t_{\varepsilon,R}(x,y) + k_1 L^{t,c}_{\varepsilon,R}(x,y),\end{aligned} \tag{5.3}$$

where

$$L^t_{\varepsilon,R}(x,y) := \int_0^t (E^\tau \mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b_\varepsilon| e^{-(t-\tau)(\Lambda^\varepsilon)^*})(x,y)\,d\tau,$$

$$L^{t,c}_{\varepsilon,R}(x,y) := \int_0^t (E^\tau \mathbf{1}_{B^c(0,Rt^{\frac{1}{\alpha}})}|b_\varepsilon| e^{-(t-\tau)(\Lambda^\varepsilon)^*})(x,y)\,d\tau.$$

Let us estimate $L^t_{\varepsilon,R}(x,y)$. Recalling that $E^t(x,z) = t(|x-z|^{-d-\alpha-1} \wedge t^{-\frac{d+\alpha+1}{\alpha}})$ and taking into account that $|x| \geq 2Dt^{\frac{1}{\alpha}}$, $|z| \leq Rt^{\frac{1}{\alpha}}$, we obtain

$$E^\tau(x,z) \leq t|x-z|^{-d-\alpha-1} \leq t|x-z|^{-d-\alpha}(3R)^{-1}t^{-\frac{1}{\alpha}}.$$

Therefore,

$$\begin{aligned}
L^t_{\varepsilon,R}(x,y) &\leq (3R)^{-1}t^{-\frac{1}{\alpha}} \int_0^t \langle t|x-\cdot|^{-\alpha-d}\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|b_\varepsilon(\cdot)|e^{-(t-\tau)(\Lambda^\varepsilon)^*}(\cdot,y)\rangle\,d\tau \\
&\quad \text{(we are using that } |x| > 2Dt^{\frac{1}{\alpha}},\ |\cdot| \leq Rt^{\frac{1}{\alpha}}) \\
&\leq (3R)^{-1}(4/3)^{d+\alpha}t^{-\frac{1}{\alpha}}t|x|^{-\alpha-d}\int_0^t \langle \mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|b_\varepsilon(\cdot)|e^{-(t-\tau)(\Lambda^\varepsilon)^*}(\cdot,y)\rangle\,d\tau \\
&\quad \text{(we are using that } |y| \leq Dt^{\frac{1}{\alpha}},\ D \geq 2R \text{ and setting } c = 3^{-1}(16/9)^{d+\alpha}) \\
&\leq cR^{-1}t^{-\frac{1}{\alpha}}t|x-y|^{-\alpha-d}\int_0^t \langle \mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|b_\varepsilon(\cdot)|e^{-(t-\tau)(\Lambda^\varepsilon)^*}(\cdot,y)\rangle\,d\tau \\
&\quad \text{(we are using } t|x-y|^{-\alpha-d} = t(|x-y|^{-\alpha-d} \wedge t^{-\frac{d+\alpha}{\alpha}}) \\
&\quad \text{since } |x-y|^{-\alpha-d} \leq (2R)^{-d-\alpha}t^{-\frac{d+\alpha}{\alpha}} < t^{-\frac{d+\alpha}{\alpha}}, \\
&\quad \text{and are re-denoting } t-\tau \text{ by } \tau) \\
&\leq k_0 cR^{-1}t^{-\frac{1}{\alpha}}e^{-tA}(x,y)\int_0^t \|e^{-\tau\Lambda^\varepsilon}\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b|\|_\infty\,d\tau \\
&\quad \text{(we are applying Proposition 8.1)} \\
&\leq k_0 cR^{-1}t^{-\frac{1}{\alpha}}e^{-tA}(x,y)c_N\int_0^t \tau^{-\frac{d}{\alpha p}}\,d\tau\, \|\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b|\|_p \quad \Big(p = \frac{d}{\alpha-\frac{1}{2}}\Big).
\end{aligned}$$

Since $\int_0^t \tau^{-\frac{d}{\alpha p}}\,d\tau = 2\alpha t^{\frac{1}{2\alpha}}$ and $\|\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b|\,\|_p = \kappa R^{\frac{1}{2}}t^{\frac{1}{2\alpha}}\tilde{c}$, $\tilde{c} = \tilde{c}(d) < \infty$, we have

$$L^t_{\varepsilon,R}(x,y) \leq C'R^{-\frac{1}{2}}e^{-tA}(x,y), \quad C' = 2\kappa\alpha k_0 c c_N \tilde{c}$$

or, for convenience,

$$L^t_{\varepsilon,R}(x,y) \leq C'e^{-tA}(x,y). \tag{5.4}$$

In turn, clearly,

$$L^{t,c}_{\varepsilon,R}(x,y) \leq \kappa R^{1-\alpha}t^{-\frac{\alpha-1}{\alpha}}\int_0^t E^\tau e^{-(t-\tau)(\Lambda^\varepsilon)*}\,d\tau.$$

Let us estimate the integral in the RHS. Using the Duhamel formula, we obtain

$$\int_0^t (E^\tau e^{-(t-\tau)(\Lambda^\varepsilon)^*})(x,y)\,d\tau$$
$$\leq \int_0^t (E^\tau e^{-(t-\tau)A})(x,y)\,d\tau + \int_0^t \Big(E^\tau \int_0^{t-\tau} E^{t-\tau-s}|b_\varepsilon| e^{-s(\Lambda^\varepsilon)^*}\,ds\Big)(x,y)\,d\tau$$

(we are applying Lemma 5.2(ii) and changing the order of integration)

$$\leq k_2 t^{\frac{\alpha-1}{\alpha}} e^{-tA}(x,y) + \int_0^t \int_0^{t-s} (E^\tau E^{t-s-\tau}|b_\varepsilon| e^{-s(\Lambda^\varepsilon)^*})(x,y)\,d\tau\,ds$$

(we are applying Lemma 5.2(iii))

$$\leq k_2 t^{\frac{\alpha-1}{\alpha}} e^{-tA}(x,y) + k_3 \int_0^t (t-s)^{\frac{\alpha-1}{\alpha}} (E^{t-s}|b_\varepsilon| e^{-s(\Lambda^\varepsilon)^*})(x,y)\,ds$$
$$\leq k_2 t^{\frac{\alpha-1}{\alpha}} e^{-tA}(x,y) + k_3 t^{\frac{\alpha-1}{\alpha}} \int_0^t (E^{t-s}\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b_\varepsilon| e^{-s(\Lambda^\varepsilon)^*})(x,y)\,ds$$
$$+ k_3 t^{\frac{\alpha-1}{\alpha}} \int_0^t (E^{t-s}\mathbf{1}_{B^c(0,Rt^{\frac{1}{\alpha}})}|b| e^{-s(\Lambda^\varepsilon)^*})(x,y)\,ds$$
$$\leq k_2 t^{\frac{\alpha-1}{\alpha}} e^{-tA}(x,y) + k_3 t^{\frac{\alpha-1}{\alpha}} L^t_{\varepsilon,R}(x,y) + k_3\kappa R^{1-\alpha} \int_0^t (E^{t-s} e^{-s(\Lambda^\varepsilon)^*})(x,y)\,ds$$

(we are applying (5.4) to the second term, and noting that $k_3\kappa R^{1-\alpha} \leq \frac{1}{2}$)

$$\leq (k_2 + k_3 C') t^{\frac{\alpha-1}{\alpha}} e^{-tA}(x,y) + \frac{1}{2}\int_0^t (E^{t-s} e^{-s(\Lambda^\varepsilon)^*})(x,y)\,ds.$$

Therefore,

$$\int_0^t E^\tau (e^{-(t-\tau)(\Lambda^\varepsilon)*})(x,y)\,d\tau \leq 2(k_2 + k_3C')t^{\frac{\alpha-1}{\alpha}} e^{-tA}(x,y),$$

and so

$$L^{c,t}_{\varepsilon,R}(x,y) \leq 2\kappa(k_2 + k_3C')R^{1-\alpha} e^{-tA}(x,y). \tag{5.5}$$

Applying (5.4) and (5.5) in (5.3), we obtain the desired bound

$$e^{-t(\Lambda^\varepsilon)^*}(x,y) \leq Ce^{-tA}(x,y), \quad |x| > 2Dt^{\frac{1}{\alpha}},\ |y| \leq Dt^{\frac{1}{\alpha}},$$

for all $R > 1$ such that $k_3\kappa R^{1-\alpha} \leq \frac{1}{2}$, $D \geq 2R$, where $C := 1 + k_1C' + k_1 2\kappa(k_2 + k_3C')R^{1-\alpha}$. The assertion of Step 3 follows.

We are in position to complete the proof of Theorem 3.2(i), i.e. to prove the bound

$$e^{-t(\Lambda^\varepsilon)^*}(x,y) \leq C_1 e^{-tA}(x,y), \quad x,y \in \mathbb{R}^d,\ t > 0, \tag{5.6}$$

for an appropriate constant $C_1 = C_1(d,\alpha,\kappa)$.

To prove (5.6), we combine Steps 1–3 as follows. *Fix* $D$ large enough so that the assertions of both Steps 2 and 3 hold.

Without loss of generality, the assertion of Step 3 holds for all $|x| > Dt^{\frac{1}{\alpha}}$, $|y| \leq Dt^{\frac{1}{\alpha}}$ (indeed, by Step 1, (5.6) is true for all $|x| \leq 2Dt^{\frac{1}{\alpha}}$, $|y| \leq 2Dt^{\frac{1}{\alpha}}$ (with $C_1 = C_0'(4D)^{d+\alpha}$) and so, in particular, for all $Dt^{\frac{1}{\alpha}} < |x| \leq 2Dt^{\frac{1}{\alpha}}$, $|y| \leq Dt^{\frac{1}{\alpha}}$; the rest follows from the assertion of Step 3 as stated). Thus, the desired bound (5.6) is true for all $|x| > Dt^{\frac{1}{\alpha}}$, $|y| \leq Dt^{\frac{1}{\alpha}}$ and, by Step 2, for all $x \in \mathbb{R}^d$, $|y| > Dt^{\frac{1}{\alpha}}$.

It remains to prove (5.6) in the case $|x| \leq Dt^{\frac{1}{\alpha}}$, $|y| \leq Dt^{\frac{1}{\alpha}}$. But this is the assertion of Step 1.

Thus, (5.6) is true, with constant $C_1$ equal to the maximum of the constants in Step 1 (with $2D$ in place of $D$) and in Steps 2, 3.

**Proof of assertion (ii).** The result follows immediately from Step 2 in the proof of (i) upon taking $\varepsilon \downarrow 0$ (cf. Proposition 9.2).

The proof of Theorem 3.2 is completed. □

## §6. Proof of Theorem 3.3: The weighted upper bound

Recall $A \equiv (-\Delta)^{\frac{\alpha}{2}}$. The estimates below are after a modification of $e^{-t\Lambda}(x,y)$, $e^{-t\Lambda^*}(x,y)$ on a measure zero set, if necessary. We are going to prove that there is a constant $C < \infty$ such that

$$e^{-t\Lambda}(x,y) \leq Ce^{-tA}(x,y)\psi_t(y), \quad t > 0,\ x,y \in \mathbb{R}^d. \tag{6.1}$$

Clearly, Theorems 3.1 and 3.2(i) combined yield

$$e^{-t\Lambda}(x,y) \leq C_1 c_{N,w}\big(e^{-tA}(x,y) \wedge \big(t^{-\frac{d}{\alpha}}\psi_t(y)\big)\big), \quad t > 0,\ x,y \in \mathbb{R}^d. \tag{6.2}$$

(1) If $|y| \geq t^{\frac{1}{\alpha}}$, then $\psi_t(y) \geq 1$. Then, by (6.2),

$$e^{-t\Lambda}(x,y) \leq C_1 c_{N,w} e^{-tA}(x,y) \leq C_1 c_{N,w} e^{-tA}(x,y)\psi_t(y),$$

i.e. (6.1) holds.

(2) If $|x| \leq Dt^{\frac{1}{\alpha}}$, $|y| < t^{\frac{1}{\alpha}}$ for some constant $D > 1$, then by (6.2) (cf. Lemma 5.1(i))

$$e^{-t\Lambda}(x,y) \leq C_1 c_{N,w} t^{-\frac{d}{\alpha}}\psi_t(y) \leq C_1 c_{N,w} k_0^{-1}(D+1)^{d+\alpha} e^{-tA}(x,y)\psi_t(y),$$

i.e. (6.1) holds.

(3) It remains therefore to consider the case $|x| > Dt^{\frac{1}{\alpha}}$, $|y| < t^{\frac{1}{\alpha}}$.

By duality (cf. Proposition 9.2), it suffices to prove the estimate

$$e^{-t\Lambda^*}(x,y) \le Ce^{-tA}(x,y)\psi_t(x) \tag{6.3}$$

for all $|x| < t^{\frac{1}{\alpha}}$, $|y| > Dt^{\frac{1}{\alpha}}$, $t > 0$, for some $D > 1$.

We will use Corollary 4.10,

$$\langle e^{-t\Lambda^*}(x,\cdot)\rangle \le C_2\psi_t(x) \quad \text{for all } x \in \mathbb{R}^d,\ t > 0,$$

the "standard" upper bound (Theorem 3.2(i))

$$e^{-t\Lambda^*}(x,y) \le C_1 e^{-tA}(x,y) \quad \text{for all } x, y \in \mathbb{R}^d,\ t > 0,$$

and its partial improvement (Theorem 3.2(ii)): for every $\delta > 0$ there exists a sufficiently large $D$ such that for all $|x| < t^{\frac{1}{\alpha}}$, $|y| > Dt^{\frac{1}{\alpha}}$, and all $z \in B(y, \frac{|y-x|}{2})$,

$$e^{-t\Lambda^*}(x,z) \le C_\delta e^{-tA}(x,z), \quad e^{-t\Lambda^*}(z,y) \le C_\delta e^{-tA}(z,y), \quad C_\delta := 1+\delta. \tag{6.4}$$

We will need the following inequality (this is [JW, Lem. 3.3]):

$$2\langle \mathbf{1}_{B(y,\frac{|x-y|}{2})}(\cdot)e^{-\frac{t}{2}A}(x,\cdot)e^{-\frac{t}{2}A}(\cdot,y)\rangle \le e^{-tA}(x,y). \tag{6.5}$$

Indeed, by symmetry, the LHS of (6.5) coincides with

$$\begin{aligned}\langle \mathbf{1}_{B(y,\frac{|x-y|}{2})}(\cdot)e^{-\frac{t}{2}A}(x,\cdot)e^{-\frac{t}{2}A}(\cdot,y)\rangle + \langle \mathbf{1}_{B(x,\frac{|x-y|}{2})}(\cdot)e^{-\frac{t}{2}A}(x,\cdot)e^{-\frac{t}{2}A}(\cdot,y)\rangle \\ \le \langle e^{-\frac{t}{2}A}(x,\cdot)e^{-\frac{t}{2}A}(\cdot,y)\rangle = e^{-tA}(x,y),\end{aligned}$$

i.e. (6.5) follows.

**Proposition 6.1.** (i) *There exists a constant $c_5$ such that*

$$e^{-t\Lambda^*}(x,y) \le \langle \mathbf{1}_{B(y,\frac{|x-y|}{2})}(\cdot)e^{-\frac{t}{2}\Lambda^*}(x,\cdot)e^{-\frac{t}{2}\Lambda^*}(\cdot,y)\rangle + c_5 e^{-tA}(x,y)\psi_t(x).$$

(ii) *If $|x| < t^{\frac{1}{\alpha}}$, $|y| > Dt^{\frac{1}{\alpha}}$ with $D > 1$ sufficiently large, then*

$$e^{-t\Lambda^*}(x,y) \le \Big(\frac{C_\delta^2}{2} + c_5\psi_t(x)\Big)e^{-tA}(x,y).$$

*Proof.* We have

$$\begin{aligned} e^{-t\Lambda^*}(x,y) = &\ \langle \mathbf{1}_{B(y,\frac{|x-y|}{2})}(\cdot)e^{-\frac{t}{2}\Lambda^*}(x,\cdot)e^{-\frac{t}{2}\Lambda^*}(\cdot,y)\rangle \\ &+ \langle \mathbf{1}_{B^c(y,\frac{|x-y|}{2})}e^{-\frac{t}{2}\Lambda^*}(x,\cdot)e^{-\frac{t}{2}\Lambda^*}(\cdot,y)\rangle \\ =: &\ J_1 + J_2.\end{aligned}$$

(i) For $z \in B^c(y, \frac{|x-y|}{2})$,

$$e^{-\frac{t}{2}\Lambda^*}(z,y) \le C_1 e^{-\frac{t}{2}A}(z,y) \le k e^{-tA}(x,y).$$

Thus,

$$J_2 \le k e^{-tA}(x,y)\langle \mathbf{1}_{B^c(y,\frac{|x-y|}{2})}(\cdot) e^{-\frac{t}{2}\Lambda^*}(x,\cdot)\rangle$$

(we are applying Corollary 4.10)

$$\le kC_2 e^{-tA}(x,y)\psi_{\frac{t}{2}}(x) \le c_5 e^{-tA}(x,y)\psi_t(x),$$

and so (i) follows.

(ii) Using (i), it remains to estimate $J_1$. Applying (6.4), we have

$$J_1 \le C_\delta^2 \big\langle \mathbf{1}_{B(y,\frac{|x-y|}{2})}(\cdot) e^{-\frac{t}{2}A}(x,\cdot) e^{-\frac{t}{2}A}(\cdot,y)\big\rangle$$

Finally, we use (6.5). □

Let us complete the proof of Theorem 3.3.

By Proposition 6.1(ii),

$$e^{-t\Lambda^*}(x,y) \le \Big(\frac{C_\delta^2}{2} + c_5\psi_t(x)\Big) e^{-tA}(x,y).$$

Set $\nu := \frac{C_\delta}{2} 2^{\frac{\beta}{\alpha}}$. Fix $\delta \in ]0, (\sqrt{2}-1) \wedge (2^{1-\frac{\alpha}{\beta}} - 1)[$. Then $\frac{C_\delta^2}{2} < 1$ and $\nu < 1$. Now, suppose that, for $n = 2, 3, \dots$,

$$e^{-t\Lambda^*}(x,y) \le \Big(\frac{C_\delta^{n+1}}{2^n} + c_5(1+\nu+\dots+\nu^{n-1})\psi_t(x)\Big) e^{-tA}(x,y). \tag{6.6}$$

Then, using Proposition 6.1(i), we have

$$\begin{aligned} e^{-t\Lambda^*}(x,y) &\le \langle \mathbf{1}_{B(y,\frac{|x-y|}{2})}(\cdot) e^{-\frac{t}{2}\Lambda^*}(x,\cdot) C_\delta e^{-\frac{t}{2}A}(\cdot,y)\rangle + c_5 e^{-tA}(x,y)\psi_t(x) \\ &\le \Big\langle \mathbf{1}_{B(y,\frac{|x-y|}{2})}(\cdot) C_\delta \Big(\frac{C_\delta^{n+1}}{2^n} + c_5(1+\nu+\dots+\nu^{n-1})\psi_{\frac{t}{2}}(x)\Big) \\ &\qquad \times e^{-\frac{t}{2}A}(x,\cdot) e^{-\frac{t}{2}A}(\cdot,y)\Big\rangle + c_5 e^{-tA}(x,y)\psi_t(x) \\ &\qquad \text{(we are applying (6.5) and inequality } \psi_{\frac{t}{2}}(x) \le 2^{\frac{\beta}{\alpha}}\psi_t(x)) \\ &\le \Big(\frac{C_\delta^{n+2}}{2^{n+1}} + c_5(\nu+\nu^2+\dots+\nu^n)\psi_t(x)\Big) e^{-tA}(x,y) + c_5 e^{-tA}(x,y)\psi_t(x) \\ &= \Big(\frac{C_\delta^{n+2}}{2^{n+1}} + c_5(1+\nu+\nu^2+\dots+\nu^n)\psi_t(x)\Big) e^{-tA}(x,y). \end{aligned}$$

Thus by induction, (6.6) holds for $n+1$. Sending $n \to \infty$ there, we obtain

$$e^{-t\Lambda^*}(x,y) \le c_5(1-\nu)^{-1} e^{-tA}(x,y)\psi_t(x),$$

as needed. The proof of (6.3) is completed. The proof of Theorem 3.3 is completed.

**Remark 6.2.** Let us prove that $\psi_{\frac{t}{2}}(x) \leq 2^{\frac{\beta}{\alpha}}\psi_t(x)$. This is equivalent to $2^{\frac{\beta}{\alpha}}\eta(t) \geq \eta(2^{\frac{1}{\alpha}}t)$ ($\eta$ was defined in Section 3). For $0 < t \leq 2^{-\frac{1}{\alpha}}$, the latter is an equality and is obvious. For $t \geq 1$ and all $0 < \beta < \alpha$, the required inequality is almost evident: clearly, for $r = 1 - \varepsilon$,

$$2^{\frac{\beta}{\alpha}} > (1+r)^{\frac{\beta}{\alpha}} > 1 + \frac{\beta}{\alpha}r; \quad \frac{\beta}{\alpha}(1-\varepsilon) \geq \frac{\beta}{2}$$

if $\alpha \leq 2(1-\varepsilon)$.

Thus, it remains to consider the case $2^{-\frac{1}{\alpha}} < t < 1$ and $0 < \beta < \alpha$. We have to prove, for $\tau = 2^{\frac{1}{\alpha}}t$, that

$$\tau^\beta \geq \beta\Big(2 - \frac{\tau}{2}\Big)\tau + 1 - \frac{3}{2}\beta \quad (1 < \tau < 2^{\frac{1}{\alpha}}).$$

If $\tau = 1$, this inequality is satisfied. Now, set $f(\tau) = \tau^\beta + \beta(\frac{\tau}{2} - 2)\tau + \frac{3}{2}\beta - 1$. Then $f'(\tau) = \beta(\tau^{\beta-1} + \tau - 2) > 0$ due to $\beta > 0$, $\tau > 1$, and $\tau^{\beta-1} + \tau \geq 2\tau^{\frac{\beta}{2}} > 2$. Thus $f(\tau) > 0$ whenever $1 < \tau < 2^{\frac{1}{\alpha}}$. □

## §7. Proof of Theorem 3.4: The weighted lower bound

Recall that

$$k_0^{-1}t(|x-y|^{-d-\alpha} \wedge t^{-\frac{d+\alpha}{\alpha}}) \leq e^{-tA}(x,y) \leq k_0 t(|x-y|^{-d-\alpha} \wedge t^{-\frac{d+\alpha}{\alpha}}) \tag{7.1}$$

for all $x, y \in \mathbb{R}^d$, $x \neq y$, $t > 0$, for a constant $k_0 = k_0(d,\alpha) > 1$.

**(1)** First, we prove the "standard" lower bound away from the origin. The estimates below are after a modification of $e^{-t\Lambda}(x,y)$, $e^{-t\Lambda^*}(x,y)$ on a measure zero set, if necessary.

**Lemma 7.1.** *There exists a generic constant $0 < \gamma < \frac{1}{2}$ such that, for all $r \geq \gamma^{-2}$ and $t > 0$,*

$$e^{-t\Lambda^*}(x,y) \geq \frac{1}{2}e^{-tA}(x,y)$$

*whenever $|x| \geq rt^{\frac{1}{\alpha}}$, $|y| \geq rt^{\frac{1}{\alpha}}$.*

*Proof.* In view of Proposition 8.5 it suffices to prove the inequality $e^{-t(\Lambda^\varepsilon)^*}(x,y) \geq \frac{1}{2}e^{-tA}(x,y)$.

By the Duhamel formula,

$$e^{-t(\Lambda^\varepsilon)^*}(x,y) \geq e^{-tA}(x,y) - |M_t(x,y)|,$$

where

$$M_t(x,y) := \int_0^t (e^{-(t-\tau)A}\nabla \cdot b_\varepsilon e^{-\tau(\Lambda^\varepsilon)^*})(x,y)\, d\tau.$$

Using Lemma 5.2(i), we have

$$|M_t(x,y)| \leq k_1\kappa \int_0^t \langle E^{t-\tau}(x,\cdot)|\cdot|^{-\alpha+1}e^{-\tau(\Lambda^\varepsilon)^*}(\cdot,y)\rangle\, d\tau$$

(we are using Theorem 3.2(i) – the standard upper bound)

$$\leq k_1\kappa C_1 \int_0^t \langle E^{t-\tau}(x,\cdot)|\cdot|^{-\alpha+1}e^{-\tau A}(\cdot,y)\rangle\, d\tau.$$

Set

$$J(\mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}|\cdot|^{1-\alpha}) := \int_0^t \langle \mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}(\cdot)E^{t-\tau}(x,\cdot)|\cdot|^{-\alpha+1}e^{-\tau A}(\cdot,y)\rangle\, d\tau,$$
$$J(\mathbf{1}_{B^c(0,\gamma r t^{\frac{1}{\alpha}})}|\cdot|^{1-\alpha}) := \int_0^t \langle \mathbf{1}_{B^c(0,\gamma r t^{\frac{1}{\alpha}})}(\cdot)E^{t-\tau}(x,\cdot)|\cdot|^{-\alpha+1}e^{-\tau A}(\cdot,y)\rangle\, d\tau,$$

where $0<\gamma<2^{-1}$.

Note that if $|x| \geq r t^{\frac{1}{\alpha}}$, then for every $z \in B(0,\gamma r t^{\frac{1}{\alpha}})$,

$$E^{t-\tau}(x,z) \leq C_5 e^{-(t-\tau)A}(x,z)|x-z|^{-1} \leq C_5 2 r^{-1} t^{-\frac{1}{\alpha}} e^{-(t-\tau)A}(x,z).$$

Thus, using the inequality

$$e^{-tA}(x,z)e^{-sA}(z,y) \leq K e^{-(t+s)A}(x,y)(e^{-tA}(x,z)+e^{-sA}(z,y)), \tag{7.2}$$

which holds for a constant $K=K(d,\alpha)$, all $x,z,y\in\mathbb{R}^d$, and $t,s>0$ (see e.g. [BJ]), we have

$$J(\mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}|\cdot|^{1-\alpha})$$
$$\leq C_5 2 r^{-1} t^{-\frac{1}{\alpha}} K e^{-tA}(x,y) \int_0^t \langle \mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}(\cdot)|\cdot|^{1-\alpha}(e^{-(t-\tau)A}(x,\cdot) + e^{-\tau A}(\cdot,y))\rangle\, d\tau.$$

Next, for all $0<\tau<t$, $|x|\geq r t^{\frac{1}{\alpha}}$, $|y| \geq r t^{\frac{1}{\alpha}}$,

$$\mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}(\cdot)e^{-\tau A}(\cdot,y) \leq C_6 t^{-\frac{d}{\alpha}} r^{-d-\alpha} \quad \text{if } (1-\gamma)r>1,$$
$$\mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}(\cdot)e^{-(t-\tau)A}(x,\cdot) \leq C_7 t^{-\frac{d}{\alpha}} r^{-d-\alpha} \quad \text{if } (1-\gamma)r>1,$$

and so

$$J(\mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}|\cdot|^{1-\alpha}) \leq C_8 t^{-\frac{d+1}{\alpha}} r^{-d-\alpha-1} e^{-tA}(x,y) \int_0^t \langle \mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}(\cdot)|\cdot|^{1-\alpha}\rangle\, d\tau$$
$$\leq C_9 r^{-2\alpha}\gamma^{d-\alpha+1}e^{-tA}(x,y)$$
$$\leq C_9 2^{2\alpha}\gamma^{d-\alpha+1}e^{-tA}(x,y) \quad \text{if } r>(1-\gamma)^{-1}.$$

Therefore,

$$(*)\quad J(\mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}|\cdot|^{1-\alpha}) \leq C_{10}\gamma^{d-\alpha+1}e^{-tA}(x,y) \quad \text{if } r > (1-\gamma)^{-1},\ 0<\gamma<2^{-1}.$$

In turn,

$$\begin{aligned} J(\mathbf{1}_{B^c(0,\gamma r t^{\frac{1}{\alpha}})}|\cdot|^{1-\alpha}) &\leq \frac{c_1 C}{2} C_0(\gamma r t^{\frac{1}{\alpha}})^{1-\alpha} t^{1-\frac{1}{\alpha}} e^{-tA}(x,y) \\ &= C_{11}(\gamma r)^{1-\alpha} e^{-tA}(x,y) \end{aligned}$$

as follows immediately from Lemma 5.2(ii):

$$\int_0^t \langle e^{-(t-\tau)A}(x,\cdot)E^\tau(\cdot,y)\rangle\, d\tau \leq C_0 t^{1-\frac{1}{\alpha}} e^{-tA}(x,y).$$

Thus, if $r \geq \gamma^{-2}$, then

$$(**)\qquad J(\mathbf{1}_{B^c(0,\gamma r t^{\frac{1}{\alpha}})}|\cdot|^{1-\alpha}) \leq C_{11}\gamma^{\alpha-1}e^{-tA}(x,y).$$

Finally, selecting $\gamma > 0$ sufficiently small, $k_1\kappa C(C_{10} \vee C_{11})\gamma^{\alpha-1} \leq \frac{1}{4}$, and using $(*)$, $(**)$, we have

$$|M_t(x,y)| \leq \frac{1}{2}e^{-tA}(x,y),$$

which ends the proof. □

**Corollary 7.2.** *For every $r > 0$, there is a constant $c(r) > 0$ such that*

$$e^{-t\Lambda^*}(x,y) \geq c(r)e^{-tA}(x,y)$$

*whenever $|x| \geq rt^{\frac{1}{\alpha}}$, $|y| \geq rt^{\frac{1}{\alpha}}$, $t > 0$.*

*Proof.* In Lemma 7.1, fix some $r \geq \gamma^{-2}$, so that

$$\text{(7.3)}\qquad e^{-t\Lambda^*}(x,y) \geq 2^{-1}e^{-tA}(x,y), \quad |x| \geq rt^{\frac{1}{\alpha}},\ |y| \geq rt^{\frac{1}{\alpha}},$$

$$\text{(7.4)}\qquad e^{-t\frac{1}{2}\Lambda^*}(x,y) \geq 2^{-1}e^{-\frac{t}{2}A}(x,y), \quad |x| \geq r\Big(\frac{t}{2}\Big)^{\frac{1}{\alpha}},\ |y| \geq r\Big(\frac{t}{2}\Big)^{\frac{1}{\alpha}}.$$

We now extend (7.3), by proving existence of a constant $0 < c_1 < 2^{-1}$ such that

$$\text{(7.3}'\text{)}\qquad e^{-t\Lambda^*}(x,y) \geq c_1 e^{-tA}(x,y), \quad |x| \geq r\Big(\frac{t}{2}\Big)^{\frac{1}{\alpha}},\ |y| \geq r\Big(\frac{t}{2}\Big)^{\frac{1}{\alpha}}.$$

Clearly, we need to consider only the case $rt^{\frac{1}{\alpha}} \geq |x| \geq r(\frac{t}{2})^{\frac{1}{\alpha}}$, $r \geq |y| \geq r(\frac{t}{2})^{\frac{1}{\alpha}}$. By the reproduction property,

$$\begin{aligned}
e^{-t\Lambda^*}(x,y) &\geq \langle e^{-\frac{1}{2}t\Lambda^*}(x,\cdot)\mathbf{1}_{B^c(0,r(\frac{t}{2})^{\frac{1}{\alpha}})}(\cdot)e^{-\frac{1}{2}t\Lambda^*}(\cdot,y)\rangle \\
&\quad \text{(we are applying (7.4))} \\
&\geq 2^{-2}\langle e^{-\frac{1}{2}tA}(x,\cdot)\mathbf{1}_{B^c(0,r(\frac{t}{2})^{\frac{1}{\alpha}})}(\cdot)e^{-\frac{1}{2}tA}(\cdot,y)\rangle \\
&> 2^{-2}\langle e^{-\frac{1}{2}tA}(x,\cdot)\mathbf{1}_{B(0,(r+1)(\frac{t}{2})^{\frac{1}{\alpha}})-B(0,r(\frac{t}{2})^{\frac{1}{\alpha}})}(\cdot)e^{-\frac{1}{2}tA}(\cdot,y)\rangle \\
&\quad \text{(we are using the lower bound in (7.1))} \\
&\geq 2^{-2}\tilde{c}t^{-\frac{d}{\alpha}} \quad (\tilde{c}=\tilde{c}(r)>0) \\
&\quad \text{(we are using the upper bound in (7.1))} \\
&\geq c_1 e^{-tA}(x,y) \quad \text{for appropriate } 0<c_1=c_1(r)<2^{-1},
\end{aligned}$$

i.e. we have proved (7.3′).

The same argument yields

$$e^{-\frac{1}{2}t\Lambda^*}(x,y) \geq c_1 e^{-\frac{1}{2}tA}(x,y), \quad |x| \geq r\Big(\frac{t}{2^2}\Big)^{\frac{1}{\alpha}}, \; |y| \geq r\Big(\frac{t}{2^2}\Big)^{\frac{1}{\alpha}}. \tag{7.4′}$$

Thus, we can repeat the above procedure $m-1$ times obtaining

$$e^{-t\Lambda^*}(x,y) \geq c_m e^{-tA}(x,y), \quad |x| \geq r\Big(\frac{t}{2^m}\Big)^{\frac{1}{\alpha}}, \; |y| \geq r\Big(\frac{t}{2^m}\Big)^{\frac{1}{\alpha}}$$

for appropriate $c_m > 0$, from which the assertion of Corollary 7.2 follows. ☐

**(2)** Next, in Proposition 7.4 we will prove an "integral lower bound". We need the following lemma:

**Lemma 7.3.** *For every $0 \leq h \in L^1$, $t > 0$,*

$$t^{-1}\int_0^t \|\psi_\tau h\|_1 \, d\tau \leq \widehat{C}\|\psi_t h\|_1$$

*for a constant $\widehat{C} = \widehat{C}(\alpha,\beta)$.*

*Proof.* Define $\psi_{0,t}(y) = \eta_0(t^{-\frac{1}{\alpha}}|y|)$, where

$$\eta_0(u) = \begin{cases} u^\beta, & 0<u<1, \\ 1, & u \geq 1. \end{cases}$$

Since $c^{-1}\psi_t \leq \psi_{0,t} \leq c\psi_t$, $c>1$, it suffices to prove Lemma 7.3 for weight $\psi_{0,t}$.

For brevity, write $\psi_t := \psi_{0,t}$. We have

$$\|\psi_\tau h\|_1 = \langle \mathbf{1}_{B(0,\tau^{\frac{1}{\alpha}})}(\tau^{-\frac{1}{\alpha}}|\cdot|)^\beta h\rangle + \langle \mathbf{1}_{B^c(0,\tau^{\frac{1}{\alpha}})} h\rangle,$$

and so

$$\int_0^t \|\psi_\tau h\|_1 \, d\tau = \left\langle \left( \int_0^t \mathbf{1}_{B(0,\tau^{\frac{1}{\alpha}})} \tau^{-\frac{\beta}{\alpha}} \, d\tau \right) |\cdot|^\beta h \right\rangle + \left\langle \left( \int_0^t \mathbf{1}_{B^c(0,\tau^{\frac{1}{\alpha}})} \, d\tau \right) h \right\rangle.$$

If $|x| \leq t^{\frac{1}{\alpha}}$, then

$$\int_0^t \mathbf{1}_{B(0,\tau^{\frac{1}{\alpha}})}(x) \tau^{-\frac{\beta}{\alpha}} \, d\tau = \int_{|x|^\alpha}^t \tau^{-\frac{\beta}{\alpha}} \, d\tau = \frac{1}{1-\frac{\beta}{\alpha}} (t^{-\frac{\beta}{\alpha}+1} - |x|^{-\beta+\alpha})$$

and

$$\int_0^t \mathbf{1}_{B^c(0,\tau^{\frac{1}{\alpha}})}(x) \, d\tau = \int_0^{|x|^\alpha} d\tau = |x|^\alpha.$$

If $|x| > t^{\frac{1}{\alpha}}$, then

$$\int_0^t \mathbf{1}_{B(0,\tau^{\frac{1}{\alpha}})}(x) \tau^{-\frac{\beta}{\alpha}} \, d\tau = 0, \quad \int_0^t \mathbf{1}_{B^c(0,\tau^{\frac{1}{\alpha}})}(x) \, d\tau = t.$$

Thus,

$$\begin{aligned}\int_0^t \|\psi_\tau h\|_1 \, d\tau &= \left\langle \mathbf{1}_{B(0,t^{\frac{1}{\alpha}})} \frac{\alpha}{\alpha-\beta} (t^{-\frac{\beta}{\alpha}+1} - |\cdot|^{-\beta+\alpha}) |\cdot|^\beta h \right\rangle \\ &\quad + \langle \mathbf{1}_{B(0,t^{\frac{1}{\alpha}})} |\cdot|^\alpha h \rangle + t \langle \mathbf{1}_{B^c(0,t^{\frac{1}{\alpha}})} h \rangle \\ &= t \frac{\alpha}{\alpha-\beta} \langle \mathbf{1}_{B(0,t^{\frac{1}{\alpha}})} \psi_t h \rangle - \frac{\beta}{\alpha-\beta} \langle \mathbf{1}_{B(0,t^{\frac{1}{\alpha}})} |\cdot|^\alpha h \rangle + t \langle \mathbf{1}_{B^c(0,t^{\frac{1}{\alpha}})} \psi_t h \rangle \\ &\leq t \frac{2\alpha-\beta}{\alpha-\beta} \langle \psi_t h \rangle.\end{aligned}$$

□

**Proposition 7.4.** *Define $g_t = \psi_t h$, $0 \leq h \in \mathcal{S}$ – the L. Schwartz space of test functions. Then there exists a generic constant $\nu > 0$ such that, for all $t > 0$,*

$$\langle \psi_t e^{-t\Lambda} \psi_t^{-1} g_t \rangle \geq \nu \langle g_t \rangle.$$

*Proof.* Recall that both $e^{-t\Lambda^\varepsilon}$ and $e^{-t(\Lambda^\varepsilon)^*}$ are holomorphic in $L^1$ and $C_u$ due to Hille's perturbation theorem. We have $\psi = \psi_{(1)} + \psi_{(u)}$, where

$$\begin{aligned}\psi_{(1)} &\in D((-\Delta)_1^{\frac{\alpha}{2}}) \quad \big(= D((\Lambda^\varepsilon)_1^*) = D(\Lambda_1^\varepsilon)\big), \\ \psi_{(u)} &\in D((-\Delta)_{C_u}^{\frac{\alpha}{2}}) \quad \big(= D((\Lambda^\varepsilon)_{C_u}^*) = D(\Lambda_{C_u}^\varepsilon)\big)\end{aligned}$$

(see the proof of Proposition 4.3 for details), so $(\Lambda^\varepsilon)^* \psi = (\Lambda^\varepsilon)_{L^1}^* \psi_{(1)} + (\Lambda^\varepsilon)_{C_u}^* \psi_{(u)}$ belongs to $L^1 + C_u$.

Now, set $g_{s,n} = \phi_{s,n}h$, $\phi_{s,n}(x) = (e^{-\frac{(\Lambda^\varepsilon)^*}{n}}\psi_s)(x)$. We have, for $s > t > 0$,

$$\begin{aligned}\langle g_{s,n}\rangle - \langle \phi_{s,n}e^{-t\Lambda^\varepsilon}h\rangle &= \int_0^t \langle \psi_s, \Lambda^\varepsilon e^{-\tau\Lambda^\varepsilon}e^{-\frac{\Lambda^\varepsilon}{n}}h\rangle\, d\tau \\ &= \lim_{r\downarrow 0} r^{-1}\int_0^t \langle \psi_s, (1-e^{-r\Lambda^\varepsilon})e^{-\tau\Lambda^\varepsilon}e^{-\frac{\Lambda^\varepsilon}{n}}h\rangle\, d\tau \\ &= \lim_{r\downarrow 0} r^{-1}\int_0^t \langle (1-e^{-r(\Lambda^\varepsilon)^*})\psi_s, e^{-\tau\Lambda^\varepsilon}e^{-\frac{\Lambda^\varepsilon}{n}}h\rangle\, d\tau \\ &= \int_0^t \langle (\Lambda^\varepsilon)^*\psi_s, e^{-\tau\Lambda^\varepsilon}e^{-\frac{\Lambda^\varepsilon}{n}}h\rangle\, d\tau.\end{aligned}$$

Arguing as in the proof of Proposition 4.3, we represent

$$(\Lambda^\varepsilon)^*\psi_s = \mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}W_\varepsilon\psi_s + v_\varepsilon,$$

where

$$W_\varepsilon(x) = \kappa(|x|_\varepsilon^{-\alpha} - |x|^{-\alpha})\beta + \kappa\big[d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2 - (d-\alpha)|x|^{-\alpha}\big]$$

and $0 \le v_\varepsilon \in L^\infty$, $\|v_\varepsilon\|_\infty \le \frac{c'}{s}$, $c' \ne c'(\varepsilon)$ (see Remark 7.5 below for a detailed calculation).

Then

$$\begin{aligned}&\langle g_{s,n}\rangle - \langle \phi_{s,n}e^{-t\Lambda^\varepsilon}h\rangle \\ &\quad\le \int_0^t \langle \mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}W_\varepsilon\psi_s, e^{-(\tau+\frac{1}{n})\Lambda^\varepsilon}h\rangle\, d\tau + \int_0^t \langle v_\varepsilon, e^{-\tau\Lambda^\varepsilon}e^{-\frac{\Lambda^\varepsilon}{n}}h\rangle\, d\tau\end{aligned}$$

or, sending $n \to \infty$,

$$\begin{aligned}\langle g_s\rangle - \langle \psi_s e^{-t\Lambda^\varepsilon}h\rangle &\le \int_0^t \langle \mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}W_\varepsilon\psi_s, e^{-\tau\Lambda^\varepsilon}h\rangle\, d\tau + \int_0^t \langle v_\varepsilon, e^{-\tau\Lambda^\varepsilon}h\rangle\, d\tau \\ &\le \int_0^t \langle \mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}W_\varepsilon\psi_s, e^{-\tau\Lambda^\varepsilon}h\rangle\, d\tau + c's^{-1}\int_0^t \|e^{-\tau\Lambda^\varepsilon}h\|_1\, d\tau.\end{aligned}$$

Next, we pass to the limit $\varepsilon \downarrow 0$:

$$(\star) \qquad \langle g_s\rangle - \langle \psi_s e^{-t\Lambda}h\rangle \le c's^{-1}\int_0^t \|e^{-\tau\Lambda}h\|_1\, d\tau.$$

We estimate the RHS of ($\star$) using the upper bound (Theorem 3.3):

$$\begin{aligned}c's^{-1}\int_0^t \|e^{-\tau\Lambda}h\|_1\, d\tau \le c's^{-1}C\int_0^t \|e^{-\tau A}\psi_\tau h\|_1\, d\tau &\le c's^{-1}C\int_0^t \|\psi_\tau h\|_1\, d\tau \\ &\text{(we are applying Lemma 7.3)} \\ &\le c'C\widehat{C}\frac{t}{s}\|\psi_t h\|_1.\end{aligned}$$

Therefore, using $\psi_s \geq (\frac{t}{s})^{\frac{\beta}{\alpha}}\psi_t$, we obtain

$$c's^{-1}\int_0^t \|e^{-\tau\Lambda}h\|_1\,d\tau \leq c'C\widehat{C}\frac{t}{s}\Big(\frac{t}{s}\Big)^{-\frac{\beta}{\alpha}}\|g_s\|_1.$$

Thus, by ($\star$), $(1-c'C\widehat{C}(\frac{t}{s})^{\frac{\alpha-\beta}{\alpha}})\langle g_s\rangle \leq \langle \psi_s e^{-t\Lambda}h\rangle$. Since $\beta<\alpha$, we can select $s>t$ such that $c'C\widehat{C}(\frac{t}{s})^{\frac{\alpha-\beta}{\alpha}}=\frac{1}{2}$, which yields the bound

$$\langle \psi_s e^{-t\Lambda}\psi_s^{-1}g_s\rangle \geq \frac{1}{2}\langle g_s\rangle.$$

Finally, using $\psi_t \geq \psi_s \geq (\frac{t}{s})^{\frac{\beta}{\alpha}}\psi_t$ and setting $2\nu := (\frac{t}{s})^{\frac{\beta}{\alpha}} = (2c'C\widehat{C})^{-\frac{\beta}{\alpha-\beta}}$, we have

$$\langle \psi_t e^{-t\Lambda}\psi_t^{-1}g_t\rangle = \langle \psi_t e^{-t\Lambda}\psi_s^{-1}g_s\rangle \geq \langle \psi_s e^{-t\Lambda}\psi_s^{-1}g_s\rangle \geq \frac{1}{2}\langle g_s\rangle \geq \frac{1}{2}\Big(\frac{t}{s}\Big)^{\frac{\beta}{\alpha}}\langle g_t\rangle = \nu\langle g_t\rangle.$$

□

**Remark 7.5.** In the proof of Proposition 7.4, we calculate $(\Lambda^\varepsilon)^*\psi_s$ arguing as in the proof of Proposition 4.3:

$$(\Lambda^\varepsilon)^*\psi = (-\Delta)^{\frac{\alpha}{2}}\psi + \operatorname{div}(b_\varepsilon\psi), \quad \psi=\psi_s,$$

where

$$(-\Delta)^{\frac{\alpha}{2}}\psi = -s^{-\frac{\beta}{\alpha}}\beta(d+\beta-2)\frac{\gamma(d+\beta-2)}{\gamma(d+\beta-\alpha)}|x|^{\beta-\alpha}+h_0$$

for $h_0 := -I_{2-\alpha}\Delta(\psi-\tilde{\psi}) \in L^\infty$, $\|h_0\|_\infty \leq c_0 s^{-1}$. In turn,

$$\operatorname{div}(b_\varepsilon\psi) = \operatorname{div}(b\tilde{\psi}) + W_\varepsilon + h_1 + h_2 + h_3,$$

where $\|h_i\|_\infty \leq c_i s^{-1}$, $i=1,2,3$. Since, by the choice of $\beta$,

$$-\beta(d+\beta-2)\frac{\gamma(d+\beta-2)}{\gamma(d+\beta-\alpha)}|x|^{-\alpha}\tilde{\psi} + \operatorname{div}(b\tilde{\psi}) = 0,$$

we have

$$(\Lambda^\varepsilon)^*\psi = \mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}W_\varepsilon + v_\varepsilon, \quad v_\varepsilon := \mathbf{1}_{B^c(0,s^{\frac{1}{\alpha}})}W_\varepsilon + h_0 + h_1 + h_2 + h_3,$$

where, it is easily seen, $\|v_\varepsilon\|_\infty \leq c's^{-1}$, as claimed.

**Proposition 7.6.** *For every $R_0>0$ there exist constants $0<r<R_0<R$ such that for all $t>0$,*

$$\frac{\nu}{2}\psi_t(x) \leq e^{-t\Lambda^*}\psi_t\mathbf{1}_{R_t,r_t}(x) \quad \text{for all } x\in B(0,R_{0,t}),\ x\neq 0,$$

*where $r_t := rt^{\frac{1}{\alpha}}$, $R_{0,t} := R_0 t^{\frac{1}{\alpha}}$, $R_t := Rt^{\frac{1}{\alpha}}$, $\mathbf{1}_{R_t,r_t} := \mathbf{1}_{B(0,R_t)} - \mathbf{1}_{B(0,r_t)}$.*

*Proof.* It suffices to prove that, for all $g := \psi_t h$, $0 \leq h \in \mathcal{S}$ with $\operatorname{sprt} h \subset B(0, R_{0,t})$,

$$\frac{\nu}{2}\langle g\rangle \leq \langle \mathbf{1}_{R_t,r_t}\psi_t e^{-t\Lambda}\psi_t^{-1} g\rangle.$$

By the upper bound (Theorem 3.3),

$$\begin{aligned}
\langle \mathbf{1}_{B(0,r_t)}\psi_t e^{-t\Lambda}\psi_t^{-1} g\rangle &\leq C\langle \mathbf{1}_{B(0,r_t)}\psi_t, e^{-tA} g\rangle \\
&\leq CC_1 t^{-\frac{d}{\alpha}}\|\mathbf{1}_{B(0,r_t)}\psi_t\|_1\|g\|_1 \\
&= CC_1\|\mathbf{1}_{B(0,r)}\psi_1\|_1\|g\|_1, \quad \|\mathbf{1}_{B(0,r)}\psi_1\|_1 \to 0 \text{ as } r \downarrow 0. \\
\langle \mathbf{1}_{B^c(0,R_t)}\psi_t e^{-t\Lambda}\psi_t^{-1} g\rangle &\leq C\langle \mathbf{1}_{B^c(0,R_t)}\psi_t, e^{-tA} g\rangle \\
&\leq C\langle e^{-tA}\mathbf{1}_{B^c(0,R_t)}, g\mathbf{1}_{B(0,R_{0,t})}\rangle \\
&\leq 2C \sup_{x\in B(0,R_{0,t})} e^{-tA}\mathbf{1}_{B^c(0,R_t)}(x)\|g\|_1 \\
&\leq C(R_0,R)\|g\|_1, \quad C(R_0,R) \to 0 \text{ as } R - R_0 \uparrow \infty,
\end{aligned}$$

where at the last step we have used, for $x \in B(0, R_{0,t})$, $y \in B^c(0, R_t)$ and $\tilde{x} = R_0^{-1}t^{-\frac{1}{\alpha}}x \in B(0,1)$, $\tilde{y} = R^{-1}t^{-\frac{1}{\alpha}}y \in B^c(0,1)$,

$$\begin{aligned}
e^{-tA}(x,y) &\leq k_0 t|x-y|^{-d-\alpha} \\
&\leq k_0 t|R_0 t^{\frac{1}{\alpha}}\tilde{x} - Rt^{\frac{1}{\alpha}}\tilde{y}|^{-d-\alpha} \\
&< 2k_0 t^{-\frac{d}{\alpha}}(R-R_0)^{-d-\alpha}|\tilde{y}|^{-d-\alpha}.
\end{aligned}$$

It remains to apply Proposition 7.4 to obtain $\frac{\nu}{2}\langle g\rangle \leq \langle \mathbf{1}_{R_t,r_t}\psi_t e^{-t\Lambda}\psi_t^{-1} g\rangle$. □

**Proposition 7.7.** *We have* $\langle h\rangle = \langle e^{-t\Lambda^*}h\rangle$ *for every* $h \in L^1$, $t > 0$.

*Proof.* Proposition 7.7 follows from $\langle h\rangle = \langle e^{-t(\Lambda^\varepsilon)^*}h\rangle$ and Proposition 8.5. □

**Proposition 7.8.** *For every* $R_0 > 0$ *there exist constants* $0 < r < R_0 < R$ *such that for all* $t > 0$,

$$\frac{1}{2} \leq e^{-t\Lambda}\mathbf{1}_{R_t,r_t}(x) \quad \textit{for all } x \in B(0, R_{0,t}),$$

*where* $r_t := rt^{\frac{1}{\alpha}}$, $R_{0,t} := R_0 t^{\frac{1}{\alpha}}$, $R_t := Rt^{\frac{1}{\alpha}}$, $\mathbf{1}_{R_t,r_t} := \mathbf{1}_{B(0,R_t)} - \mathbf{1}_{B(0,r_t)}$.

*Proof.* We essentially repeat the proof of Proposition 7.6. It suffices to prove that, for all $0 \leq h \in \mathcal{S}$ with $\operatorname{sprt} h \subset B(0, R_{0,t})$,

$$\frac{1}{2}\langle h\rangle \leq \langle \mathbf{1}_{R_t,r_t}e^{-t\Lambda^*}h\rangle.$$

By the upper bound (Theorem 3.3),

$$\begin{aligned}
\langle \mathbf{1}_{B(0,r_t)} e^{-t\Lambda^*} h\rangle &\le C\langle \mathbf{1}_{B(0,r_t)}\psi_t, e^{-tA}h\rangle \\
&\le CC_1 t^{-\frac{d}{\alpha}}\|\mathbf{1}_{B(0,r_t)}\psi_t\|_1\|h\|_1 \\
&= o(r)\|h\|_1, \quad o(r)\to 0 \text{ as } r\downarrow 0; \\
\langle \mathbf{1}_{B^c(0,R_t)} e^{-t\Lambda^*} h\rangle &\le C\langle \mathbf{1}_{B^c(0,R_t)}\psi_t, e^{-tA}h\rangle \\
&\le C\langle e^{-tA}\mathbf{1}_{B^c(0,R_t)}, h\mathbf{1}_{B(0,R_{0,t})}\rangle \\
&\le C \sup_{x\in B(0,R_{0,t})} e^{-tA}\mathbf{1}_{B^c(0,R_t)}(x)\|h\|_1 \\
&= C(R_0,R)\|h\|_1, \quad C(R_0,R)\to 0 \text{ as } R-R_0\uparrow\infty.
\end{aligned}$$

The last two estimates and Proposition 7.7 yield $\frac{1}{2}\langle h\rangle \le \langle \mathbf{1}_{R_t,r_t} e^{-t\Lambda^*} h\rangle$. □

**(3)** We are in position to complete the proof of the lower bound using the so-called $3q$ argument.

Set $q_t(x,y) := \psi_t^{-1}(x)e^{-t\Lambda^*}(x,y)$, $x\neq 0$.

(a) Let $x,y\in B^c(0,t^{\frac{1}{\alpha}})$, $x\neq y$. Then, using that $\psi_{3t}^{-1}\ge \frac{1}{1+\beta/2}$, we have by Corollary 7.2,

$$q_{3t}(x,y)\ge \frac{1}{1+\beta/2}e^{-3t\Lambda^*}(x,y)\ge ce^{-3tA}(x,y).$$

Let $r_t = rt^{\frac{1}{\alpha}}$, $R_t = Rt^{\frac{1}{\alpha}}$ be as in Propositions 7.6 and 7.8, where we fix $R_0 = 1$ (hence $r<1$).

(b) Let $x\in B(0,t^{\frac{1}{\alpha}})$, $|y|\ge rt^{\frac{1}{\alpha}}$, $x\neq y$. By the reproduction property,

$$\begin{aligned}
q_{2t}(x,y) &\ge \psi_{2t}^{-1}(x)\langle e^{-t\Lambda^*}(x,\cdot)\psi_t^{-1}(\cdot)\psi_t(\cdot)e^{-t\Lambda^*}(\cdot,y)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\
&\ge \psi_{2t}^{-1}(x)\psi_t^{-1}(R_t)\langle e^{-t\Lambda^*}(x,\cdot)\psi_t(\cdot)e^{-t\Lambda^*}(\cdot,y)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\
&\ge \psi_{2t}^{-1}(x)\psi_t^{-1}(R_t)(e^{-t\Lambda^*}\psi_t\mathbf{1}_{R_t,r_t})(x)\inf_{r_t\le|z|\le R_t} e^{-t\Lambda^*}(z,y) \\
&\qquad \text{(we are applying Corollary 7.2, Proposition 7.6,} \\
&\qquad \text{and using } \psi_t^{-1}(R_t)=\tfrac{1}{1+\beta/2}) \\
&\ge \frac{\nu}{2}\frac{1}{1+\beta/2}\psi_{2t}^{-1}(x)\psi_t(x)c(r)\inf_{r_t\le|z|\le R_t} e^{-tA}(z,y) \\
&\qquad \text{(we are using } \psi_t\ge\psi_{2t}) \\
&\ge C_1 e^{-2tA}(x,y).
\end{aligned}$$

(b′) Let $x\in B(0,t^{\frac{1}{\alpha}})$, $|y|\ge t^{\frac{1}{\alpha}}$, $x\neq y$. Arguing as in (b), we obtain

$$q_{3t}(x,y)\ge C_2 e^{-3tA}(x,y).$$

(c) Let $|x| \geq rt^{\frac{1}{\alpha}}$, $y \in B(0, t^{\frac{1}{\alpha}})$, $x \neq y$. We have

$$\begin{aligned} q_{2t}(x,y) &\geq \psi_{2t}^{-1}(x)\langle e^{-t\Lambda^*}(x,\cdot)e^{-t\Lambda^*}(\cdot,y)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\ &= \psi_{2t}^{-1}(x)\langle e^{-t\Lambda^*}(x,\cdot)e^{-t\Lambda}(y,\cdot)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\ &\quad \text{(we are using } \psi_{2t}^{-1} \geq \tfrac{1}{1+\beta/2} \text{ and applying Corollary 7.2)} \\ &\geq c(r)\frac{1}{1+\beta/2}\langle e^{-tA}(x,\cdot)e^{-t\Lambda}(y,\cdot)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\ &\quad \text{(we are applying (7.1))} \\ &\geq C_3(r)t(Rt^{\frac{1}{\alpha}} + |x|)^{-d-\alpha}\langle e^{-t\Lambda}(y,\cdot)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\ &\quad \text{(we are applying Proposition 7.8)} \\ &\geq C_3(r)2^{-1}t(Rt^{\frac{1}{\alpha}} + |x|)^{-d-\alpha} \geq C_4(r)e^{-2tA}(x,y). \end{aligned}$$

(c′) Let $|x| \geq t^{\frac{1}{\alpha}}$, $y \in B(0, t^{\frac{1}{\alpha}})$, $x \neq y$. Arguing as in (c), we obtain

$$q_{3t}(x,y) \geq C_5(r)e^{-3tA}(x,y).$$

(d) Let $x, y \in B(0, t^{\frac{1}{\alpha}})$, $x \neq y$. By the reproduction property,

$$\begin{aligned} q_{3t}(x,y) &\geq \psi_{3t}^{-1}(x)\langle e^{-t\Lambda^*}(x,\cdot)e^{-2t\Lambda^*}(\cdot,y)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\ &\quad \text{(we are using (c))} \\ &\geq C_4(r)\psi_{3t}^{-1}(x)\langle e^{-t\Lambda^*}(x,\cdot)\psi_{2t}(\cdot)e^{-2tA}(\cdot,y)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\ &\quad \text{(we are using } \psi_{2t} \geq 2^{\frac{\beta}{\alpha}}\psi_t \text{ and } e^{-2tA}(z,y) \geq c(r,R)t^{-\frac{d}{\alpha}} > 0 \\ &\quad\ \text{for } r_t \leq |z| \leq R_t,\ |y| \leq t^{\frac{1}{\alpha}}) \\ &\geq c(r,R)C_4 2^{\frac{\beta}{\alpha}}\psi_{3t}^{-1}(x)t^{-\frac{d}{\alpha}}\langle e^{-t\Lambda^*}(x,\cdot)\mathbf{1}_{R_t,r_t}(\cdot)\psi_t(\cdot)\rangle \\ &\quad \text{(we are applying Proposition 7.6 and using } \psi_t \geq \psi_{3t}) \\ &\geq c(r,R)C_4 2^{\frac{\beta}{\alpha}}\frac{\nu}{2}t^{-\frac{d}{\alpha}} \\ &\quad \text{(we are applying (7.1))} \\ &\geq C_5(r,R)e^{-3tA}(x,y). \end{aligned}$$

By (a), (b′), (c′), (d), $q_{3t}(x,y) \geq Ce^{-3tA}(x,y)$ for all $x, y \in \mathbb{R}^d$, $x \neq y$, $x \neq 0$, and so

$$e^{-3t\Lambda^*}(x,y) \geq Ce^{-3tA}(x,y)\psi_{3t}(x), \quad t > 0.$$

The lower bound is proved.

## §8. Construction of the semigroup $e^{-t\Lambda_r}$, $\Lambda_r = (-\Delta)^{\frac{\alpha}{2}} - b \cdot \nabla$ in $L^r$, $1 \leq r < \infty$

Set $b_\varepsilon(x) := \kappa |x|_\varepsilon^{-\alpha} x$, $\kappa > 0$, $|x|_\varepsilon := \sqrt{|x|^2 + \varepsilon}$, $\varepsilon > 0$,

$$\Lambda_r^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} - b_\varepsilon \cdot \nabla, \quad D(\Lambda_r^\varepsilon) = \mathcal{W}^{\alpha,r} := (1 + (-\Delta)^{\frac{\alpha}{2}})^{-1} L^r.$$

To prove that $-\Lambda^\varepsilon \equiv -\Lambda_r^\varepsilon$ is the generator of a holomorphic semigroup in $L^r$, $1 \leq r < \infty$, we appeal to the Hille perturbation theorem [Ka, Chap. IX, Sect. 2.2]. To verify its assumptions, we use a well-known estimate ($A \equiv (-\Delta)^{\frac{\alpha}{2}}$)

$$|\nabla(\zeta + A)^{-1}(x, y)| \leq C(\operatorname{Re}\zeta + A)^{-\frac{\alpha-1}{\alpha}}(x, y), \quad \operatorname{Re}\zeta > 0, \ C = C(d, \alpha)$$

(for the proof see e.g. [KM, App. A]). Then for $Y = L^r$,

$$\|b_\varepsilon \cdot \nabla(\zeta + A)^{-1}\|_{Y \to Y} \leq C\|b_\varepsilon\|_\infty \|(\operatorname{Re}\zeta + A)^{-\frac{\alpha-1}{\alpha}}\|_{Y \to Y} \leq C\|b_\varepsilon\|_\infty (\operatorname{Re}\zeta)^{-\frac{\alpha-1}{\alpha}},$$

and so $\|b_\varepsilon \cdot \nabla(\zeta + A)^{-1}\|_{Y \to Y}$, $\operatorname{Re}\zeta \geq c_\varepsilon$, can be made arbitrarily small by selecting $c_\varepsilon$ sufficiently large. It follows that the Neumann series for

$$(\zeta + \Lambda^\varepsilon)^{-1} = (\zeta + A)^{-1}(1 + T)^{-1}, \quad T := -b_\varepsilon \cdot \nabla(\zeta + A)^{-1}$$

converges in $L^p$ and $C_u$ and satisfies $\|(\zeta + \Lambda^\varepsilon)^{-1}\|_{Y \to Y} \leq C_\varepsilon |\zeta|^{-1}$, $\operatorname{Re}\zeta \geq c_\varepsilon$, i.e. $-\Lambda^\varepsilon$ is the generator of a holomorphic semigroup.

The same argument (with $Y = C_u$) shows that $\Lambda^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} - b_\varepsilon \cdot \nabla$, with $D(\Lambda^\varepsilon) := D((-\Delta)^{\frac{\alpha}{2}}_{C_u})$, generates a holomorphic semigroup in $C_u$.

To prove that these semigroups are positivity preserving, it suffices to consider $Y = L^2$. It is not difficult to verify that $e^{-t\Lambda^\varepsilon}$ are contractions in $L^2$ (see Proposition 8.1 below). Thus, the required result will follow from the Phillips criterion: $e^{-t\Lambda^\varepsilon} \operatorname{Re} L^2 \subset \operatorname{Re} L^2$ (clear) and

$$\langle \Lambda^\varepsilon u, u_+ \rangle \geq 0, \quad u \in D(\Lambda^\varepsilon) = \mathcal{W}^{\alpha,2},$$

where $u_+ := u \vee 0$. Indeed, since $e^{-tA}$ is positivity preserving, $\langle Au, u_+ \rangle \geq 0$. On the other hand, taking into account that $\alpha > 1$ and integrating by parts, we find

$$\langle -b_\varepsilon \cdot \nabla u, u_+ \rangle = \langle -b_\varepsilon \cdot \nabla u_+, u_+ \rangle = \frac{1}{2}\langle \operatorname{div} b_\varepsilon, u_+^2 \rangle \geq 0$$

(recall $\operatorname{div} b_\varepsilon = \kappa(d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2) \geq 0$), so $\langle \Lambda^\varepsilon u, u_+ \rangle \geq 0$, as needed.

**Proposition 8.1.** *For every $r \in [1, \infty[$ and $\varepsilon > 0$, $e^{-t\Lambda_r^\varepsilon}$ is a contraction $C_0$ semigroup in $L^r$. There exists a constant $c_N \neq c_N(\varepsilon)$ such that*

$$\|e^{-t\Lambda_r^\varepsilon}\|_{r \to q} \leq c_N t^{-\frac{d}{\alpha}(\frac{1}{r} - \frac{1}{q})}, \quad t > 0,$$

*for all $1 \leq r < q \leq \infty$.*

*In particular, there is a constant $c_S > 0$, $c_S \neq c_S(\varepsilon)$ such that ($\Lambda^\varepsilon \equiv \Lambda^\varepsilon_2$)*

$$\operatorname{Re}\langle \Lambda^\varepsilon u, u\rangle \geq c_S \|u\|_{2j}^2, \quad u \in D(\Lambda^\varepsilon),\ j = \frac{d}{d-\alpha}.$$

*Proof.* First, let $1 < r < \infty$. Set $u \equiv u(t) := e^{-t\Lambda^\varepsilon_r} f$, $f \in L^1 \cap L^\infty$, and write $A := (-\Delta)^{\frac{\alpha}{2}}$. Multiplying the equation $\partial_t u + \Lambda^\varepsilon_r u = 0$ by $\bar{u}|u|^{r-2}$ and integrating in the spatial variables, we obtain (taking into account that $D(\Lambda^\varepsilon_r) = D(A_r) \subset W^{1,r}$)

$$\frac{1}{r}\partial_t \|u\|_r^r + \operatorname{Re}\langle Au, u|u|^{r-2}\rangle - \operatorname{Re}\langle b_\varepsilon \cdot \nabla u, u|u|^{r-2}\rangle = 0.$$

Note that, since $-A$ is a Markov generator,

$$\operatorname{Re}\langle Au, u|u|^{r-2}\rangle \geq \frac{4}{rr'}\|A^{\frac{1}{2}}|u|^{\frac{r}{2}}\|_2^2$$

(indeed, by [LS, Thm. 2.1] or by Theorem A.1, $\operatorname{Re}\langle Au, u|u|^{r-2}\rangle \geq \frac{4}{rr'}\|A^{\frac{1}{2}}u^{\frac{r}{2}}\|_2^2$, $u^{\frac{r}{2}} := u|u|^{\frac{r}{2}-1}$, and by the Beurling–Deny theory $\|A^{\frac{1}{2}}u^{\frac{r}{2}}\|_2^2 \geq \|A^{\frac{1}{2}}|u|^{\frac{r}{2}}\|_2^2$). Integration by parts yields

$$-\operatorname{Re}\langle b_\varepsilon \cdot \nabla u, u|u|^{r-2}\rangle = \frac{\kappa}{r}\langle (d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2)|u|^r\rangle \geq \kappa\frac{d-\alpha}{r}\langle |x|_\varepsilon^{-\alpha}|u|^r\rangle \geq 0.$$

Thus,

$$-\partial_t \|u\|_r^r \geq \frac{4}{r'}\|A^{\frac{1}{2}}|u|^{\frac{r}{2}}\|_2^2. \tag{8.1}$$

From (8.1) we obtain $\|u(t)\|_r \leq \|f\|_r$, $t \geq 0$ and since $L^1 \cap L^\infty$ is dense in $L^r$, $\|e^{-t\Lambda^\varepsilon_r}\|_{r\to r} \leq 1$ as needed.

Since $e^{-t\Lambda^\varepsilon_1} \upharpoonright L^1 \cap L^r = e^{-t\Lambda^\varepsilon_r} \upharpoonright L^1 \cap L^r$, the latter clearly yields

$$\|e^{-t\Lambda^\varepsilon_1} f\|_r \leq \|f\|_r, \quad f \in L^1 \cap L^\infty.$$

Sending $r \uparrow \infty$, we have $\|e^{-t\Lambda^\varepsilon_r} f\|_\infty \leq \|f\|_\infty$, and sending $r \downarrow 1$, we have $\|e^{-t\Lambda^\varepsilon_1}\|_{1\to 1} \leq 1$.

Let us prove the ultracontractivity of $e^{-t\Lambda^\varepsilon_r}$. By (8.1),

$$-\partial_t \|u\|_{2r}^{2r} \geq \frac{4}{(2r)'}\|A^{\frac{1}{2}}|u|^r\|_2^2, \quad 1 \leq r < \infty.$$

Using the Nash inequality $\|A^{\frac{1}{2}}h\|_2^2 \geq C_N\|h\|_2^{2+\frac{2\alpha}{d}}\|h\|_1^{-\frac{2\alpha}{d}}$ and $\|u(t)\|_r \leq \|f\|_r$, we have, setting $v := \|u\|_{2r}^{2r}$,

$$\partial_t(v^{-\frac{\alpha}{d}}) \geq c_1 \|f\|_r^{-\frac{2r\alpha}{d}},$$

where $c_1 = C_N \frac{\alpha}{d}\frac{4}{(2r)'}$. Integrating this inequality yields

$$\|e^{-t\Lambda^\varepsilon_r}\|_{r\to 2r} \leq c_1^{-\frac{d}{2\alpha r}} t^{-\frac{d}{\alpha}(\frac{1}{r}-\frac{1}{2r})}, \quad t > 0, \tag{$*$}$$

and so, by the semigroup property,

$$\|e^{-t\Lambda_r^\varepsilon}\|_{1\to 2^m} \leq c_N t^{-\frac{d}{\alpha}(1-\frac{1}{2^m})}, \quad t>0,\ m\geq 1,$$

where the constant $c_N \neq c_N(m)$. Thus, sending $m$ to infinity we arrive at

$$\|e^{-t\Lambda_r^\varepsilon}\|_{1\to\infty} \leq c_N t^{-\frac{d}{\alpha}}, \quad t>0.$$

The latter and the contractivity of $e^{-t\Lambda_r^\varepsilon}$ in all $L^q$, $1\leq q\leq\infty$ yield via interpolation the desired bound $\|e^{-t\Lambda_p^\varepsilon}\|_{p\to q} \leq c_N t^{-\frac{d}{\alpha}(\frac{1}{p}-\frac{1}{q})}$, $t>0$, for all $1\leq p<q\leq\infty$.

Finally, since $D(\Lambda^\varepsilon)=D(A)$, we have, for $u\in D(A)$, $\mathrm{Re}\langle\Lambda^\varepsilon u,u\rangle \geq \|A^{\frac{1}{2}}u\|_2^2 \geq c_S\|u\|_{2j}^2$ (the last inequality is proved e.g. in [Zi, Sect. 2.8]). □

## §8.1. Case $d\geq 4$

We will first provide an elementary argument that allows us to treat all $d=4,5,\dots$, except the particularly important case $d=3$.

**Proposition 8.2.** *For every $r\in[1,\infty[$ the limit*

$$s\text{-}L^r\text{-}\lim_{\varepsilon\downarrow 0} e^{-t\Lambda_r^\varepsilon} \quad (\textit{loc. uniformly in } t\geq 0)$$

*exists and determines a contraction $C_0$ semigroup on $L^r$, say $e^{-t\Lambda_r}$.*

*For all $1\leq r<q\leq\infty$,*

$$\|e^{-t\Lambda_r}\|_{r\to q} \leq c_N t^{-\frac{d}{\alpha}(\frac{1}{r}-\frac{1}{q})}, \quad t>0,$$

*with $c_N$ from Proposition 8.1.*

*Proof.* First, let $r=2$. Set $u^\varepsilon(t):=e^{-t\Lambda^\varepsilon}f$, $f\in C_c^\infty$.

**Claim 8.3.** $\|\nabla u^\varepsilon(t)\|_2 \leq \|\nabla f\|_2$, $t\geq 0$.

*Proof.* Denote $u\equiv u^\varepsilon$, $w:=\nabla u$, $w_i:=\nabla_i u$. Due to $f\in C_c^\infty$ and $\nabla_i^n b_\varepsilon^i\in C^\infty\cap L^\infty$, $i=1,\dots,d$, $n\geq 1$ we can and will differentiate the equation $\partial_t u+\Lambda^\varepsilon u=0$ in $x_i$, obtaining

$$\partial_t w_i + (-\Delta)^{\frac{\alpha}{2}} w_i - b_\varepsilon\cdot\nabla w_i - (\nabla_i b_\varepsilon)\cdot w = 0.$$

Multiplying the latter by $\bar{w}_i$, integrating by parts and summing up in $i=1,\dots,d$ we have

$$\frac{1}{2}\partial_t\|w\|_2^2 + \sum_{i=1}^d \|(-\Delta)^{\frac{\alpha}{4}}w_i\|_2^2 - \mathrm{Re}\sum_{i=1}^d\langle b_\varepsilon\cdot\nabla w_i,w_i\rangle - \mathrm{Re}\sum_{i=1}^d\langle(\nabla_i b_\varepsilon)\cdot w,w_i\rangle = 0,$$

$$-\mathrm{Re}\langle b_\varepsilon\cdot\nabla w_i,w_i\rangle = \frac{\kappa}{2}\langle(d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2)w_i,w_i\rangle,$$

$$-\langle(\nabla_i b_\varepsilon)\cdot w,w_i\rangle = -\kappa\langle|x|_\varepsilon^{-\alpha}w_i,w_i\rangle + \kappa\alpha\langle|x|_\varepsilon^{-\alpha-2}x_i\bar{w}_i(x\cdot w)\rangle.$$

Thus,

$$\frac{1}{2}\partial_t\|w\|_2^2 + \sum_{i=1}^d \|(-\Delta)^{\frac{\alpha}{4}} w_i\|_2^2 + \kappa\frac{d-\alpha}{2}\langle |x|_\varepsilon^{-\alpha}|w|^2\rangle + \frac{\kappa\alpha\varepsilon}{2}\langle |x|_\varepsilon^{-\alpha-2}|w|^2\rangle$$
$$- \kappa\langle |x|_\varepsilon^{-\alpha}|w|^2\rangle + \kappa\alpha\langle |x|_\varepsilon^{-\alpha-2}|x\cdot w|^2\rangle = 0,$$

and so, since $\kappa>0$,

$$\frac{1}{2}\partial_t\|w\|_2^2 + \kappa\frac{d-\alpha-2}{2}\langle |x|_\varepsilon^{-\alpha}|w|^2\rangle \leq 0.$$

Since $d\geq 4$, $\alpha<2$, we have $d-\alpha-2>0$. Thus, integrating in $t$, we obtain $\|w(t)\|_2^2 \leq \|\nabla f\|_2^2$, $t\geq 0$, as needed. □

Next, set $u_n := u^{\varepsilon_n}$, $b_n := b_{\varepsilon_n}$, where $\varepsilon_n \downarrow 0$, and put

$$g(t) := u_n(t) - u_m(t), \quad t\geq 0.$$

**Claim 8.4.** *We have $\|g(t)\|_2 \to 0$ uniformly in $t\in[0,1]$ as $n,m\to\infty$.*

*Proof.* We subtract the equations for $u_n$ and $u_m$ and obtain

$$\partial_t g + (-\Delta)^{\frac{\alpha}{2}} g - b_n\cdot\nabla g - (b_n-b_m)\cdot\nabla u_m = 0,$$
$$\frac{1}{2}\partial_t\|g\|_2^2 + \|(-\Delta)^{\frac{\alpha}{4}} g\|_2^2 - \mathrm{Re}\langle b_n\cdot\nabla g, g\rangle - \mathrm{Re}\langle (b_n-b_m)\cdot\nabla u_m, g\rangle = 0. \tag{8.2}$$

Concerning the last two terms, we have

$$-\mathrm{Re}\langle b_n\cdot\nabla g, g\rangle = \frac{\kappa}{2}\langle (d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2)g, g\rangle \geq \kappa\frac{d-\alpha}{2}\langle |x|_\varepsilon^{-\alpha}, |g|^2\rangle,$$

$$\begin{aligned}
|\langle (b_n-b_m)\cdot\nabla u_m, g\rangle| &\leq |\langle \mathbf{1}_{B(0,1)}(b_n-b_m)\cdot\nabla u_m, g\rangle| \\
&\quad + |\langle \mathbf{1}^c_{B(0,1)}(b_n-b_m)\cdot\nabla u_m, g\rangle| \\
&\quad (\text{we are using } \|g\|_\infty \leq 2\|f\|_\infty,\ \|g\|_2\leq 2\|f\|_2) \\
&\leq \|\mathbf{1}_{B(0,1)}(b_n-b_m)\|_2\|\nabla u_m\|_2 2\|f\|_\infty \\
&\quad + \|\mathbf{1}^c_{B(0,1)}(b_n-b_m)\|_\infty\|\nabla u_m\|_2 2\|f\|_2 \\
&\quad (\text{we are using Claim 8.3}) \\
&\leq \|\mathbf{1}_{B(0,1)}(b_n-b_m)\|_2\|\nabla f\|_2 2\|f\|_\infty \\
&\quad + \|\mathbf{1}^c_{B(0,1)}(b_n-b_m)\|_\infty\|\nabla f\|_2 2\|f\|_2 \\
&\to 0 \text{ as } n,m\to\infty.
\end{aligned}$$

Thus, integrating (8.2) in $t$ and using the last two observations, we end the proof of Claim 8.4. □

By Claim 8.4, $\{e^{-t\Lambda^{\varepsilon_n}} f\}_{n=1}^{\infty}$, $f \in C_c^\infty$ is a Cauchy sequence in $L^\infty([0,1], L^2)$. Set

$$T_2^t f := s\text{-}L^2\text{-}\lim_n e^{-t\Lambda^{\varepsilon_n}} f \;\text{ uniformly in } 0 \le t \le 1. \tag{8.3}$$

(Clearly, the limit does not depend on the choice of $\{\varepsilon_n\} \downarrow 0$.) Since $e^{-t\Lambda^{\varepsilon_n}}$ are contractions in $L^2$, we have $\|T_2^t f\|_2 \le \|f\|_2$, $t \in [0,1]$. Extending $T_2^t$ by continuity to $L^2$, we obtain that $T_2^t$ is strongly continuous. Furthermore,

$$T_2^t f = \lim_n e^{-t\Lambda^{\varepsilon_n}} f \quad \text{in } L^2 \text{ for all } f \in L^2,\, 0 \le t \le 1.$$

Finally, extending $T_2^t$ to all $t \ge 0$ using the reproduction property, we obtain a contraction $C_0$ semigroup $T_2^t =: e^{-t\Lambda}$, $t \ge 0$.

Now, let $1 \le r < \infty$. Since $e^{-t\Lambda^\varepsilon}$ is a contraction in $L^r$, we obtain, by construction (8.3) of $e^{-t\Lambda} f$, $f \in C_c^\infty$, appealing e.g. to Fatou's lemma, that

$$\|e^{-t\Lambda} f\|_r \le \|f\|_r, \quad t \ge 0.$$

Thus, extending $e^{-t\Lambda}$ by continuity to $L^r$, we can define contraction semigroups $T_r^t := [e^{-t\Lambda}]^{\text{clos}}_{L^r \to L^r}$, $t \ge 0$. The strong continuity of $T_r^t$ in $L^r$ is a consequence of strong continuity of $e^{-t\Lambda}$, contractivity of $T_r^t$, and Fatou's lemma. Write $T_r^t =: e^{-t\Lambda_r}$. Clearly,

$$e^{-t\Lambda_r} = s\text{-}L^r\text{-}\lim_n e^{-t\Lambda_r^{\varepsilon_n}}, \quad t \ge 0.$$

The latter and Proposition 8.1 complete the proof of Proposition 8.2. □

## §8.2. Case $d = 3$

The proof of the next proposition works in all dimensions $d \ge 3$.

**Proposition 8.5.** *For every $r \in [1, \infty[$ the limit*

$$s\text{-}L^r\text{-}\lim_{\varepsilon \downarrow 0} e^{-t\Lambda_r^\varepsilon} \quad (\textit{loc. uniformly in } t \ge 0)$$

*exists and determines a contraction $C_0$ semigroup on $L^r$, say, $e^{-t\Lambda_r}$. For all $1 \le r \le q \le \infty$,*

$$\|e^{-t\Lambda_r}\|_{r \to q} \le c_N t^{-\frac{d}{\alpha}(\frac{1}{r} - \frac{1}{q})}, \quad t > 0,$$

*with $c_N$ from Proposition 8.1.*

*Proof.* Denote $u^\varepsilon(t) := e^{-t\Lambda_r^\varepsilon} f$, $f \in C_c^\infty$. For brevity, write $u \equiv u^\varepsilon$ and $w := \nabla u$.

**Claim 8.6.** *For every* $r \in ]1,\infty[$,

$$\frac{1}{r}\|w(t_1)\|_r^r + \frac{4}{rr'}\int_0^{t_1}\sum_{i=1}^d \|(-\Delta)^{\frac{\alpha}{4}}(w_i|w|^{\frac{r-2}{2}})\|_2^2\,dt$$
$$+\kappa\frac{d-\alpha-r}{r}\int_0^{t_1}\langle |x|_\varepsilon^{-\alpha}|w|^r\rangle\,dt$$
$$+\alpha\kappa\int_0^{t_1}\langle |x|_\varepsilon^{\alpha-2}|x\cdot w|^2|w|^{r-2}\rangle\,dt \le \frac{1}{r}\|\nabla f\|_r^r, \quad t_1>0.$$

*In particular, for* $1<r<d-\alpha$,

$$\|w(t_1)\|_r^r + \frac{4}{r'}c_S d^{-\frac{\alpha}{d}}\int_0^{t_1}\|w\|_{rj}^r\,dt \le \|\nabla f\|_r^r, \quad t_1>0,\ j:=\frac{d}{d-\alpha}.$$

*Proof.* Set $w_i := \nabla_i u$. We differentiate $\partial_t u + \Lambda_r^\varepsilon u = 0$ in $x_i$, obtaining the identity

$$\partial_t w_i + (-\Delta)^{\frac{\alpha}{2}}w_i - b_\varepsilon\cdot\nabla w_i - (\nabla_i b_\varepsilon)\cdot w = 0,$$

which we multiply by $\bar{w}_i|w|^{r-2}$, integrate in the spatial variables, and then sum in $1\le i\le d$ to obtain

$$\frac{1}{r}\partial_t\|w\|_r^r + \mathrm{Re}\langle(-\Delta)^{\frac{\alpha}{2}}w, w|w|^{r-2}\rangle$$
$$-\mathrm{Re}\sum_{i=1}^d\langle b_\varepsilon\cdot\nabla w_i, w_i|w|^{r-2}\rangle - \mathrm{Re}\sum_{i=1}^d\langle(\nabla_i b_\varepsilon)\cdot w, w_i|w|^{r-2}\rangle = 0.$$

By Theorem A.1,

$$\mathrm{Re}\langle(-\Delta)^{\frac{\alpha}{2}}w, w|w|^{r-2}\rangle \ge \frac{4}{rr'}\sum_{i=1}^d\|(-\Delta)^{\frac{\alpha}{4}}(w_i|w|^{\frac{r-2}{2}})\|_2^2.$$

Next, integrating by parts, we obtain

$$-\mathrm{Re}\sum_{i=1}^d\langle b_\varepsilon\cdot\nabla w_i, w_i|w|^{r-2}\rangle = \frac{\kappa}{r}\langle(d|x|_\varepsilon^{-\alpha}-\alpha|x|_\varepsilon^{-\alpha-2}|x|^2)|w|^r\rangle$$
$$\ge \kappa\frac{d-\alpha}{r}\langle|x|_\varepsilon^{-\alpha}|w|^r\rangle,$$

and

$$\mathrm{Re}\sum_{i=1}^d\langle(\nabla_i b_\varepsilon)\cdot w, w_i|w|^{r-2}\rangle = \kappa\langle|x|_\varepsilon^{-\alpha}|w|^r\rangle - \alpha\kappa\langle|x|_\varepsilon^{-\alpha-2}(x\cdot w)^2|w|^{r-2}\rangle.$$

The first required inequality follows.

Now, let $1 < r < d - \alpha$. Note that

$$\sum_{i=1}^d \|(-\Delta)^{\frac{\alpha}{4}}(w_i|w|^{\frac{r-2}{2}})\|_2^2 \geq c_S \sum_{i=1}^d \|w_i|w|^{\frac{r-2}{2}}\|_{2j}^2 = c_S \sum_{i=1}^d \langle |w_i|^{2j}|w|^{(r-2)j}\rangle^{\frac{1}{j}}$$

$$\geq c_S \Big( \langle |w|^{(r-2)j} \sum_{i=1}^d |w_i|^{2j}\rangle\Big)^{\frac{1}{j}}$$

$$\text{(we use } (\textstyle\sum_{i=1}^d |w|^{2j})^{1/j} \geq (\sum_{i=1}^d |w_i|^2) d^{-1/j'} = |w|^2 d^{-1/j'})$$

$$\geq c_S d^{-1/j'} \langle |w|^{rj}\rangle^{\frac{1}{j}} = c_S d^{-\frac{\alpha}{d}} \|w\|_{rj}^r.$$

The second required inequality follows. □

Set $u_n := u^{\varepsilon_n}$, $b_n := b_{\varepsilon_n}$, where $\varepsilon_n \downarrow 0$. Let $g(t) := u_n(t) - u_m(t)$, $t \geq 0$.

**Claim 8.7.** *We have $\|g(t)\|_2 \to 0$ uniformly in $t \in [0,1]$ as $n, m \to \infty$.*

*Proof.* We subtract the equations for $u_n$ and $u_m$:

$$\partial_t g + (-\Delta)^{\frac{\alpha}{2}} g - b_n \cdot \nabla g - (b_n - b_m) \cdot \nabla u_m = 0.$$

Multiplying the latter by $\bar{g}$ and integrating, we obtain

$$\|g(t_1)\|_2^2 + \int_0^{t_1} \|(-\Delta)^{\frac{\alpha}{4}} g\|_2^2 \, dt - \operatorname{Re} \int_0^{t_1} \langle b_n \cdot \nabla g, g\rangle \, dt - \operatorname{Re} \int_0^{t_1} \langle (b_n - b_m) \cdot \nabla u_m, g\rangle \, dt = 0$$

for every $t_1 > 0$. Since

$$-\operatorname{Re}\langle b_n \cdot \nabla g, g\rangle = \frac{\kappa}{2}\langle (d|x|_\varepsilon^{-\alpha} - \alpha |x|_\varepsilon^{-\alpha-2}|x|^2) g, g\rangle \geq \kappa \frac{d-\alpha}{2}\langle |x|_\varepsilon^{-\alpha}, |g|^2\rangle,$$

we have

$$\|g(t_1)\|_2^2 + \int_0^{t_1} \|(-\Delta)^{\frac{\alpha}{4}} g\|_2^2 \, dt + \kappa \frac{d-\alpha}{2} \int_0^{t_1} \langle |x|^{-\alpha}, |g|^2\rangle \, dt \leq \int_0^{t_1} |\langle (b_n - b_m) \cdot \nabla u_m, g\rangle| \, dt. \tag{8.4}$$

Let us estimate the RHS of (8.4). Fix $1 < r < d - \alpha$ (as in the second assertion of Claim 8.6). Then

$$\begin{aligned}
|\langle (b_n - b_m) \cdot \nabla u_m, g\rangle| &\leq |\langle \mathbf{1}_{B(0,1)}(b_n - b_m) \cdot \nabla u_m, g\rangle| \\
&\quad + |\langle \mathbf{1}_{B^c(0,1)}(b_n - b_m) \cdot \nabla u_m, g\rangle| \\
&\quad (\text{we apply estimates } \|g\|_\infty \leq 2\|f\|_\infty,\ \|g\|_{(rj)'} \leq 2\|f\|_{(rj)'}) \\
&\leq \|\mathbf{1}_{B(0,1)}(b_n - b_m)\|_{(rj)'} \|\nabla u_m\|_{rj} 2\|f\|_\infty \\
&\quad + \|\mathbf{1}_{B^c(0,1)}(b_n - b_m)\|_\infty \|\nabla u_m\|_{rj} 2\|f\|_{(rj)'}.
\end{aligned}$$

Clearly $\|\mathbf{1}_{B^c(0,1)}(b_n - b_m)\|_\infty \to 0$ as $n, m \to \infty$. The same is true for $\|\mathbf{1}_{B(0,1)}(b_n - b_m)\|_{(rj)'}$ since $(rj)' = \frac{rd}{rd-d+\alpha} < \frac{d}{\alpha-1}$. Thus, in view of Claim 8.6,

$$\begin{aligned}
&\int_0^{t_1} |\langle (b_n - b_m) \cdot \nabla u_m, g\rangle| \, dt \\
&\quad \leq (\|\mathbf{1}_{B(0,1)}(b_n - b_m)\|_{(rj)'} \|f\|_\infty \\
&\qquad + \|\mathbf{1}_{B^c(0,1)}(b_n - b_m)\|_\infty \|f\|_{(rj)'}) 2 \int_0^{t_1} \|\nabla u_m\|_{rj} \, dt \to 0
\end{aligned}$$

as $n, m \to \infty$. □

Now, we argue as in the proof of Proposition 8.2 to obtain that for every $r \in [1, \infty[$ the limit $s\text{-}L^r\text{-}\lim_n e^{-t\Lambda_r^{\varepsilon_n}}$, $t \geq 0$ exists and determines a contraction $C_0$ semigroup on $L^r$. It is easily seen that the limit does not depend on the choice of $\varepsilon_n$.

The last assertion follows now from Proposition 8.1.

The proof of Proposition 8.5 is completed. □

## §9. Construction of the semigroup $e^{-t\Lambda_r^*}$, $\Lambda_r^* = (-\Delta)^{\frac{\alpha}{2}} + \nabla \cdot b$ in $L^r$, $1 \leq r < \infty$

Set $(\Lambda^\varepsilon)_r^* := (-\Delta)^{\frac{\alpha}{2}} + \nabla \cdot b_\varepsilon$, $D((\Lambda^\varepsilon)_r^*) = \mathcal{W}^{\alpha,r}$. By the Hille perturbation theorem, $-(\Lambda^\varepsilon)_r^*$ is the generator of a holomorphic $C_0$ semigroup in $L^r$ (arguing as in Section 8; the argument there also shows that $(\Lambda^\varepsilon)^* := (-\Delta)^{\frac{\alpha}{2}} + \nabla \cdot b_\varepsilon$, $D((\Lambda^\varepsilon)^*) = D((-\Delta)^{\frac{\alpha}{2}}_{C_u})$ is the generator of a holomorphic semigroup in $C_u$).

**Proposition 9.1.** *For every $r \in [1, \infty[$ and $\varepsilon > 0$, $e^{-t(\Lambda^\varepsilon)_r^*}$ is a contraction $C_0$ semigroup. For all $1 \leq r \leq q \leq \infty$,*

$$\|e^{-t(\Lambda^\varepsilon)_r^*}\|_{r \to q} \leq c_N t^{-\frac{d}{\alpha}(\frac{1}{r} - \frac{1}{q})}, \quad t > 0,$$

*with $c_N$ from Proposition 8.1.*

*Proof.* The semigroup $e^{-t(\Lambda^\varepsilon)^*_r}$ is constructed in $L^r$ repeating the argument in Section 8. The ultracontractivity estimate for $1 < r \leq q < \infty$ follows from Proposition 8.1 by duality, and for all $1 \leq r \leq q \leq \infty$ upon taking limits $r \downarrow 1$, $q \uparrow \infty$. □

**Proposition 9.2.** *For every $r \in [1, \infty[$ the limit*

$$s\text{-}L^r\text{-}\lim_{\varepsilon \downarrow 0} e^{-t(\Lambda^\varepsilon)^*_r} \quad (\textit{loc. uniformly in } t \geq 0)$$

*exists and determines a contraction $C_0$ semigroup in $L^r$, say, $e^{-t\Lambda^*_r}$. For all $1 \leq r \leq q \leq \infty$,*

$$\|e^{-t\Lambda^*_r}\|_{r \to q} \leq c_N t^{-\frac{d}{\alpha}(\frac{1}{r} - \frac{1}{q})}, \quad t > 0,$$

*with $c_N$ from Proposition 8.1.*

*We have for $1 < r < \infty$,*

$$\langle e^{-t\Lambda_{r'}(b)} f, g\rangle = \langle f, e^{-t\Lambda^*_r(b)} g\rangle, \quad t > 0, \ f \in L^{r'}, \ r' = \frac{r}{r-1}, \ g \in L^r.$$

*Proof.* First, let $r = 2$. In view of Proposition 9.1, we can argue as in the proof of [KSS, Prop. 10], appealing to the Rellich–Kondrashov theorem, to obtain the following: for every sequence $\varepsilon_n \downarrow 0$ there exists a subsequence $\varepsilon_{n_m}$ such that the limit

$$s\text{-}L^2\text{-}\lim_m e^{-t(\Lambda^{\varepsilon_{n_m}})^*} \quad (\text{loc. uniformly in } t \geq 0) \tag{9.1}$$

exists and determines a $C_0$ semigroup in $L^2$.

On the other hand, since

$$\langle e^{-t\Lambda^\varepsilon} f, g\rangle = \langle f, e^{-t(\Lambda^\varepsilon)^*} g\rangle, \quad t > 0, \ f, g \in L^2,$$

it follows from Proposition 8.5 that, for every $g \in L^2$, $e^{-t(\Lambda^\varepsilon)^*} g$ converges weakly in $L^2$ as $\varepsilon \downarrow 0$. Thus, the limit in (9.1) does not depend on the choice of $\varepsilon_{n_m}$ and $\varepsilon_n$.

For $1 \leq r < \infty$, we repeat the argument at the end of the proof of Proposition 8.2, appealing to Proposition 9.1.

The ultracontractivity estimate now follows from Proposition 9.1.

The last assertion follows from the analogous property of $e^{-t\Lambda^\varepsilon_{r'}}$, $e^{-t(\Lambda^\varepsilon)^*_r}$, $\varepsilon > 0$ and Propositions 8.5, 9.1. □

## Appendix A. $L^r$ (vector) inequalities for symmetric Markov generators

Let $X$ be a set and $\mu$ a $\sigma$-finite measure on $X$. Let $T^t = e^{-tA}$, $t \geq 0$ be a symmetric Markov semigroup in $L^2(X, \mu)$. Let

$$T_r^t := \left[T^t \upharpoonright L^2 \cap L^r\right]_{L^r \to L^r}, \quad t \geq 0,$$

a contraction $C_0$ semigroup on $L^r$, $r \in [1, \infty[$. Put $T_r^t =: e^{-tA_r}$.

**Theorem A.1.** *Let $f_i \in D(A_r)$ $(1 \leq i \leq m)$, $r \in ]1, \infty[$. Set $f := (f_i)_{i=1}^m$, $f_{(r)} := f|f|^{\frac{r-2}{2}}$. Then $f_i|f|^{\frac{r-2}{2}} \in D(A^{\frac{1}{2}})$ $(1 \leq i \leq m)$ and, applying the operators coordinate-wise, we have*

(i) $$\frac{4}{rr'}\langle A^{\frac{1}{2}} f_{(r)}, A^{\frac{1}{2}} f_{(r)}\rangle \leq \operatorname{Re}\langle A_r f, f|f|^{r-2}\rangle \leq \varkappa(r)\langle A^{\frac{1}{2}} f_{(r)}, A^{\frac{1}{2}} f_{(r)}\rangle,$$

(ii) $$\left|\operatorname{Im}\langle A_r f, f|f|^{r-2}\rangle\right| \leq \frac{|r-2|}{2\sqrt{r-1}} \operatorname{Re}\langle A_r f, f|f|^{r-2}\rangle,$$

*where*

$$\varkappa(r) := \sup_{s \in ]0,1[}\left[(1+s^{\frac{1}{r}})(1+s^{\frac{1}{r'}})(1+s^{\frac{1}{2}})^{-2}\right], \quad r' = \frac{r}{r-1},$$

$$\langle A^{\frac{1}{2}} f_{(r)}, A^{\frac{1}{2}} f_{(r)}\rangle = \sum_{i=1}^m \|A^{\frac{1}{2}}(f_i|f|^{\frac{r-2}{2}})\|_2^2,$$

$$\langle A_r f, f|f|^{r-2}\rangle = \sum_{i=1}^m \langle A_r f_i, f_i|f|^{r-2}\rangle.$$

Theorem A.1 is a prompt but useful modification of [LS, Thm. 2.1] (corresponding to the case $m = 1$): it allows us to control higher-order derivatives of $u(t) = e^{-t\Lambda} f$, $\Lambda \supset (-\Delta)^{\frac{\alpha}{2}} - b \cdot \nabla$, $f \in C_c^\infty$ in the proof of Proposition 8.5 (see Claim 8.6 there).

For the sake of completeness, we included the detailed proof below.

**(1)** We will need the following claim:

**Claim A.2.** *There exists a finitely additive measure $\mu_t$ on $X \times X$, symmetric in the sense that $\mu_t(A \times B) = \mu_t(B \times A)$ on any $\mu$-measurable sets of finite measure $A$ and $B$, and satisfying*

$$\langle T^t f, g\rangle = \int_{X \times X} f(x)\overline{g(x)}\, d\mu_t(x, y) \quad (f, g \in L^1 \cap L^\infty).$$

In order to justify the claim, let us introduce the space $\mathcal{L}^\infty = \mathcal{L}^\infty(X, \mathcal{M}_\mu)$, i.e. the Banach space of all bounded $\mu$-measurable ($\equiv \mathcal{M}_\mu$-measurable) functions, endowed with the norm $|||f||| \coloneqq \sup\{|f(x)| \mid x \in X\}$. Here $\mathcal{M}_\mu$ is the $\sigma$-algebra of $\mu$-measurable sets.

Let $N^\infty \equiv \mathcal{N}^\infty(X, \mathcal{M}_\mu)$ be the set of all $\mu$-negligible functions, so that $L^\infty = \mathcal{L}^\infty/\mathcal{N}^\infty$. Denoting by $\pi\colon f \to \tilde{f}$ the canonical mapping of $\mathcal{L}^\infty$ onto $L^\infty$, we can identify $L^\infty$ with $\pi(\mathcal{L}^\infty)$. Since $\mu$ is $\sigma$-finite, there exists a lifting $\rho\colon L^\infty \to \mathcal{L}^\infty$, a linear multiplicative positivity-preserving map such that

$$\rho(\mathbf{1}_G) = \mathbf{1}_G \quad \text{for all } G \in \mathcal{M}_\mu \text{ with } \mu(G) < \infty.$$

Given $t > 0$ define $T^t_\rho\colon \mathcal{L}^\infty \to \mathcal{L}^\infty$ by

$$T^t_\rho f \coloneqq \rho(T^t_\infty f),$$

and so $T^t_\rho$ is a positivity-preserving semigroup, and

$$\langle T^t_\rho f, g\rangle = \langle T^t \tilde{f}, \tilde{g}\rangle \quad (\tilde{f}, \tilde{g} \in L^\infty \cap L^1).$$

The following set function is associated with the semigroup $T^t_\infty$:

$$P(t, x, G) \coloneqq (T^t_\rho \mathbf{1}_G)(x) \quad (t > 0,\ x \in X,\ G \in \mathcal{M}_\mu).$$

This function satisfies the following evident properties:

(1) $P(t, x, G)$ $(G \in \mathcal{M}_\mu)$ is finitely additive.
(2) $P(t, x, X) \le 1$.
(3) $\int f(y) P(t, \cdot, dy)$ exists and equals $T^t_\rho f(\cdot)$ $(f \in \mathcal{L}^\infty)$.

Set by definition

$$\mu_t(A \times B) = \int_A P(t, x, B)\, d\mu(x) \quad (A, B \in \mathcal{M}_\mu).$$

The claimed symmetry of $\mu_t$ is a direct consequence of the self-adjointness of $T^t$ and the fact that we can identify $T^t_\infty \mathbf{1}_G$ and $T^t \mathbf{1}_G$ for every $G \in \mathcal{M}_\mu$ of finite measure.

**(2)** We are in position to complete the proof of Theorem A.1.

*Proof of Theorem A.1.* We are going to establish the following inequalities: for all $f \in L^r$,

$$\text{(A.1)} \quad \frac{4}{rr'}\langle (1 - T^t_2) f_{(r)}, f_{(r)}\rangle \le \operatorname{Re}\langle (1 - T^t_r) f, f|f|^{r-2}\rangle \le \varkappa(r)\langle (1 - T^t_2) f_{(r)}, f_{(r)}\rangle,$$

$$\text{(A.2)} \quad \left|\operatorname{Im}\langle (1 - T^t_r) f, f|f|^{r-2}\rangle\right| \le \frac{|r-2|}{2\sqrt{r-1}} \operatorname{Re}\langle (1 - T^t_r) f, f|f|^{r-2}\rangle.$$

Then the required estimates will follow from the definitions of $A_r$ and $A^{\frac{1}{2}}$. Indeed, for $f \in D(A_r)$,

$$s\text{-}L^r\text{-}\lim_{t\downarrow 0} \frac{1}{t}(1-T_r^t)f \text{ exists and equals } A_r f.$$

On the other hand, e.g. by the spectral theorem for self-adjoint operators,

$$s\text{-}L^2\text{-}\lim_{t\downarrow 0} \frac{1}{t}\langle(1-T_2^t)g,g\rangle = \mathfrak{t}_A[g], \quad g \in D(\mathfrak{t}_A),$$

where $\mathfrak{t}_A[g] := \langle A^{\frac{1}{2}}g, A^{\frac{1}{2}}g\rangle$, $D(\mathfrak{t}_A) = D(A^{\frac{1}{2}})$, and

$$D(\mathfrak{t}_A) = \big\{g \in L^2 \mid \sup_{t>0} \tfrac{1}{t}\langle(1-T_2^t)g,g\rangle < \infty\big\}.$$

It is now clear that (A.1), (A.2) yield (i) and (ii).

First, let $f \in L^1 \cap L^\infty$ with $\operatorname{sprt} f \subset G$, $G \in \mathcal{M}_\mu$, $\mu(G) < \infty$. Using Claim A.2, we have

$$\begin{aligned}
\langle T^t f, f|f|^{r-2}\rangle &= \frac{1}{2}\langle T^t f, f|f|^{r-2}\rangle + \frac{1}{2}\langle f, T^t(f|f|^{r-2})\rangle \\
&= \frac{1}{2}\int \big[f(x)\cdot\bar f(y)|f(y)|^{r-2} + f(y)\cdot \bar f(x)|f(x)|^{r-2}\big]\,d\mu_t(x,y), \\
\langle T^t f_{(r)}, f_{(r)}\rangle &= \frac{1}{2}\int f_{(r)}(x)\cdot \bar f_{(r)}(y)\,d\mu_t(x,y) + \frac{1}{2}\int \bar f_{(r)}(x)\cdot f_{(r)}(y)\,d\mu_t(x,y), \\
\langle T^t \mathbf{1}_G, |f|^r\rangle &= \langle \mathbf{1}_G, T^t|f|^r\rangle \\
&= \frac{1}{2}\langle P(t,\cdot,G)|f(\cdot)|^r\rangle + \frac{1}{2}\Big\langle \mathbf{1}_G(\cdot)\int |f(y)|^r P(t,\cdot,dy)\Big\rangle \\
&= \frac{1}{2}\int \big[|f(x)|^r + |f(y)|^r\big]\,d\mu_t(x,y), \\
\|f\|_r^r &= \langle T^t\mathbf{1}_G, |f|^r\rangle + \langle(1-T^t\mathbf{1}_G), |f|^r\rangle.
\end{aligned}$$

Setting $s := |f(x)|$, $l := |f(y)|$, $\beta := \frac{f(x)\cdot\bar f(y)}{|f(x)|\,|f(y)|}$, $b := \operatorname{Re}\beta$, $a := \operatorname{Im}\beta$, we obtain

$$\begin{aligned}
\langle(1-T^t)f, f|f|^{r-2}\rangle &= \langle(1-T^t\mathbf{1}_G), |f|^r\rangle + \frac{1}{2}\int [s^r + l^r - \beta s l^{r-1} - \bar\beta l s^{r-1}]\,d\mu_t, \\
\operatorname{Re}\langle(1-T^t)f, f|f|^{r-2}\rangle &= \langle(1-T^t\mathbf{1}_G), |f|^r\rangle + \frac{1}{2}\int [s^r + l^r - b(sl^{r-1} + ls^{r-1})]\,d\mu_t, \\
\langle(1-T^t)f_{(r)}, f_{(r)}\rangle &= \langle(1-T^t\mathbf{1}_G), |f|^r\rangle + \frac{1}{2}\int [s^r + l^r - 2b(st)^{\frac{r}{2}}]\,d\mu_t, \\
\operatorname{Im}\langle(1-T^t)f, f|f|^{r-2}\rangle &= \frac{1}{2}\int a(sl^{r-1} - ls^{r-1})\,d\mu_t.
\end{aligned}$$

Now we employ the following elementary inequalities: for all $s,t \in [0,\infty[$, $r \in [1,\infty[$, $b \in [-1,1]$,

$$\frac{4}{rr'}(s^r + t^r - 2b(st)^{\frac{r}{2}}) \le s^r + t^r - b(st^{r-1} + ts^{r-1}) \le \varkappa(r)(s^r + t^r - 2b(st)^{\frac{r}{2}})$$

(Lemma A.4$(l_3)$,$(l_5)$ below),

$$|a|\,|st^{r-1} - ts^{r-1}| \le \frac{|r-2|}{2\sqrt{r-1}}\big[s^r + t^r - \sqrt{1-a^2}(st^{r-1} + ts^{r-1})\big], \quad a \in [-1,1]$$

(Lemma A.4$(l_4)$ below). We obtain (A.1), (A.2) but for $f \in L^1 \cap L^\infty$ with $\operatorname{sprt} f \in G$, $\mu(G) < \infty$.

To end the proof, we note that $\mu$ is a $\sigma$-finite measure, and so we can first get rid of the condition "$\operatorname{sprt} f \in G$, $\mu(G) < \infty$", and then, using the truncated functions

$$g_n = \begin{cases} g & \text{if } |g| \le n, \\ 0 & \text{if } |g| > n, \end{cases} \qquad n = 1, 2, \dots$$

and the dominated convergence theorem, get rid of "$f \in L^1 \cap L^\infty$". □

For the sake of completeness, we also include the following result concerning the scalar case.

**Theorem A.3.** *If $0 \le f \in D(A_r)$, then*

(iii) $$\frac{4}{rr'}\|A^{\frac{1}{2}} f^{\frac{r}{2}}\|_2^2 \le \langle A_r f, f^{r-1}\rangle \le \|A^{\frac{1}{2}} f^{\frac{r}{2}}\|_2^2.$$

*If $f \in D(A) \cap L^\infty$, then $f_{(r)} := |f|^{\frac{r}{2}} \operatorname{sgn} f \in D(A^{\frac{1}{2}})$, $r \in [2,\infty[$ and*

(i′) $$\frac{4}{rr'}\|A^{\frac{1}{2}} f_{(r)}\|_2^2 \le \operatorname{Re}\langle Af, |f|^{r-1}\operatorname{sgn} f\rangle \le \varkappa(r)\|A^{\frac{1}{2}} f_{(r)}\|_2^2, \quad \operatorname{sgn} f := \frac{f}{|f|}.$$

*If $0 \le f \in D(A) \cap L^\infty$, then $f^{\frac{r}{2}} \in D(A^{\frac{1}{2}})$, $r \in [2,\infty[$ and*

(iii′) $$\frac{4}{rr'}\|A^{\frac{1}{2}} f^{\frac{r}{2}}\|_2^2 \le \langle Af, f^{r-1}\rangle \le \|A^{\frac{1}{2}} f^{\frac{r}{2}}\|_2^2.$$

*Proof.* Following closely the proof of Theorem A.1, we obtain

$$\frac{4}{rr'}\langle (1-T^t) f^{\frac{r}{2}}, f^{\frac{r}{2}}\rangle \le \langle (1-T^t)f, f^{r-1}\rangle \le \langle (1-T^t) f^{\frac{r}{2}}, f^{\frac{r}{2}}\rangle \quad (f \in L^r_+),$$

which yields the required. □

In the proofs of Theorems A.1 and A.3 we use the following lemma:

**Lemma A.4.** *Let $s, t \in [0, \infty[$, $r \in [1, \infty[$ and $b \in [-1, 1]$. Then*

$$(l_1) \qquad \frac{4}{rr'}(s^{\frac{r}{2}} - t^{\frac{r}{2}})^2 \leq (s-t)(s^{r-1} - t^{r-1}) \leq (s^{\frac{r}{2}} - t^{\frac{r}{2}})^2,$$

$$(l_2) \qquad (s^{\frac{r}{2}} + t^{\frac{r}{2}})^2 \leq (s+t)(s^{r-1} + t^{r-1}) \leq \varkappa(r)(s^{\frac{r}{2}} + t^{\frac{r}{2}})^2,$$

$$(l_3) \qquad \frac{4}{rr'}(s^r + t^r + 2b(st)^{\frac{r}{2}}) \leq s^r + t^r + b(st^{r-1} + ts^{r-1}),$$

$$(l_4) \qquad |b|\,|st^{r-1} - ts^{r-1}| \leq \frac{|r-2|}{2\sqrt{r-1}}\big[s^r + t^r - \sqrt{1-b^2}(st^{r-1} + ts^{r-1})\big],$$

$$(l_5) \qquad s^r + t^r + b(st^{r-1} + ts^{r-1}) \leq \varkappa(r)(s^r + t^r + 2b(st)^{\frac{r}{2}}).$$

*Proof.*

- The RHS of $(l_1)$ and the LHS of $(l_2)$ are consequences of the inequality $2|\alpha|\,|\beta| \leq \alpha^2 + \beta^2$.
- The RHS of $(l_2)$ follows from the definition of $\varkappa(r)$.
- The LHS of $(l_1)$ follows from

$$\frac{4}{r^2}(s^{\frac{r}{2}} - t^{\frac{r}{2}})^2 = \left(\int_t^s z^{\frac{r}{2}-1}\,dz\right)^2 \leq \int_t^s dz \cdot \int_t^s z^{r-2}\,dz.$$

- $(l_3)$ is a consequence of the LHS of $(l_1)$.
- To derive $(l_4)$ set

$$A = st^{r-1} - ts^{r-1}, \quad B = \frac{|r-2|}{2\sqrt{r-1}}(st^{r-1} + ts^{r-1}), \quad C = \frac{|r-2|}{2\sqrt{r-1}}(s^r + t^r),$$

and note that $A^2 + B^2 \leq C^2 \Rightarrow |Ab| + |B\sqrt{1-b^2}| \leq C$.

The inequality $A^2 + B^2 \leq C^2$ follows from

$$(\star) \qquad (st^{r-1} - ts^{r-1})^2 \leq \left(\frac{r-2}{r}\right)^2 (s^r - t^r)^2$$

and the LHS's of $(l_1)$ and $(l_2)$.

Setting $v = s/t$, $(\star)$ takes the form

$$|v^{r-1} - v| \leq \frac{|r-2|}{r}|v^r - 1|.$$

All possible cases are reduced to the case where $v > 1$ and $r > 2$.

If $\frac{r-2}{r}v \geq 1$, then the inequality $v^{r-1} - v \leq \frac{r-2}{r}v^r - \frac{r-2}{r}$ is self-evident. If $1 < v < \frac{r}{r-2}$, we set $\psi(v) = \frac{r-2}{r}v^r - v^{r-1} + v - \frac{r-2}{r}$ and note that $\frac{d}{dv}\psi(v) \geq 0$ by Young's inequality.

- Finally, ($l_5$) follows from the RHS of ($l_2$) and the following elementary inequality:

$$\frac{A+bB}{A+bC} \leq \frac{A+B}{A+C} \quad (b \in [-1,1]), \text{ provided that } A > C \text{ and } B \geq C > 0,$$

where we take $A = s^r + t^r$, $B = st^{r-1} + ts^{r-1}$, $C = 2(st)^{\frac{r}{2}}$. □

## Appendix B. Extrapolation theorem

**Theorem B.1** (T. Coulhon–Y. Raynaud [VSC, Props. II.2.1, II.2.2]). *Let $U^{t,s}$: $L^1 \cap L^\infty \to L^1 + L^\infty$ be a two-parameter evolution family of operators:*

$$U^{t,s} = U^{t,\tau} U^{\tau,s}, \quad 0 \leq s < \tau < t \leq \infty.$$

*Suppose that, for some $1 \leq p < q < r \leq \infty$, $\nu > 0$, $M_1$, and $M_2$, the inequalities*

$$\|U^{t,s} f\|_p \leq M_1 \|f\|_p \quad \text{and} \quad \|U^{t,s} f\|_r \leq M_2 (t-s)^{-\nu} \|f\|_q$$

*are valid for all $(t,s)$ and $f \in L^1 \cap L^\infty$. Then*

$$\|U^{t,s} f\|_r \leq M (t-s)^{-\nu/(1-\beta)} \|f\|_p,$$

*where $\beta = \frac{r}{q}\frac{q-p}{r-p}$ and $M = 2^{\nu/(1-\beta)^2} M_1 M_2^{1/(1-\beta)}$.*

*Proof.* Set $2t_s = t + s$. The hypotheses and Hölder's inequality imply

$$\begin{aligned} \|U^{t,s} f\|_r &\leq M_2 (t-t_s)^{-\nu} \|U^{t_s,s} f\|_q \\ &\leq M_2 (t-t_s)^{-\nu} \|U^{t_s,s} f\|_r^\beta \|U^{t_s,s} f\|_p^{1-\beta} \\ &\leq M_2 M_1^{1-\beta} (t-t_s)^{-\nu} \|U^{t_s,s} f\|_r^\beta \|f\|_p^{1-\beta}, \end{aligned}$$

and hence

$$(t-s)^{\nu/(1-\beta)} \|U^{t,s} f\|_r / \|f\|_p \leq M_2 M_1^{1-\beta} 2^{\nu/(1-\beta)} \left[ (t_s - s)^{\nu/(1-\beta)} \|U^{t_s,s} f\|_r \,/ \|f\|_p \right]^\beta.$$

Setting $R_{2T} := \sup_{t-s \in ]0,T]} [(t-s)^{\nu/(1-\beta)} \|U^{t,s} f\|_r / \|f\|_p]$, we obtain from the last inequality that $R_{2T} \leq M^{1-\beta} (R_T)^\beta$. But $R_T \leq R_{2T}$, and so $R_{2T} \leq M$. □

## Appendix C. The range of an accretive operator

In the proof of Theorem 3.1 we use the following well-known result.

Let $P$ be a closed operator on $L^1$ such that $\mathrm{Re}\langle (\lambda + P) f, \frac{f}{|f|} \rangle \geq 0$ for all $f \in D(P)$, and $R(\mu + P)$ is dense in $L^1$ for some $\mu > \lambda$.

Then $R(\mu + P) = L^1$.

Indeed, let $y_n \in R(\mu + P)$, $n = 1, 2, \dots$, be a Cauchy sequence in $L^1$; $y_n = (\mu + P)x_n$, $x_n \in D(P)$. Write $[f, g] \coloneqq \langle f, \frac{g}{|g|}\rangle$. Then

$$\begin{aligned}(\mu - \lambda)\|x_n - x_m\|_1 &= (\mu - \lambda)\big[x_n - x_m, x_n - x_m\big] \\ &\leq (\mu - \lambda)\big[x_n - x_m, x_n - x_m\big] + \big[(\lambda + P)(x_n - x_m), x_n - x_m\big] \\ &= \big[(\mu + P)(x_n - x_m), x_n - x_m\big] \leq \|y_n - y_m\|_1.\end{aligned}$$

Thus, $\{x_n\}$ is itself a Cauchy sequence in $L^1$. Since $P$ is closed, the result follows.

## Acknowledgements

The research of D.K. was supported by NSERC (grant RGPIN-2017-05567) and FRQNT (grant 2019-NC-254946).

The authors express their gratitude to Karol Szczypkowski, Krzysztof Bogdan, Tomasz Jakubowski, Renming Song, and Jian Wang for useful discussions. We also thank the anonymous referee for very careful reading of the manuscript and for making a number of valuable comments.

## References

[BJ] K. Bogdan and T. Jakubowski, Estimates of heat kernel of fractional Laplacian perturbed by gradient operators, Comm. Math. Phys. **271** (2007), 179–198. Zbl 1129.47033 MR 2283957

[CKSV] S. Cho, P. Kim, R. Song and Z. Vondraček, Factorization and estimates of Dirichlet heat kernels for non-local operators with critical killings, J. Math. Pures Appl. **143** (2020), 208–256. Zbl 1460.60087 MR 4163128

[JW] T. Jakubowski and J. Wang, Heat kernel estimates for fractional Schrödinger operators with negative Hardy potential, Potential Anal. **53** (2020), 997–1024. Zbl 1459.60153 MR 4140086

[Ka] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer, Berlin, 1995. Zbl 0836.47009 MR 1335452

[KM] D. Kinzebulatov and K. R. Madou, On admissible singular drifts of symmetric $\alpha$-stable process, Math. Nachr. **295** (2022), 2036–2064. MR 4515442

[KSS] D. Kinzebulatov, Yu. A. Semënov and K. Szczypkowski, Heat kernel of fractional Laplacian with Hardy drift via desingularizing weights, J. London Math. Soc.(2) **104** (2021), 1861–1900. Zbl 1484.35252 MR 4339954

[LS] V. A. Liskevich and Yu. A. Semënov, *Some problems on Markov semigroups*, in *Schrödinger operators, Markov semigroups, wavelet analysis, operator algebras* (M. Demuth et al. eds), Mathematical Topics: Advances in Partial Differential Equations 11, Akademie, Berlin (1996), 163–217. Zbl 0854.47027 MR 1409835

[MM1] Y. Maekawa and H. Miura, On fundamental solutions for non-local parabolic equations with divergence free drift, Adv. Math. **247** (2013), 123–191. Zbl 1284.35208 MR 3096796

[MM2] Y. Maekawa and H. Miura, Upper bounds for fundamental solutions to non-local diffusion equations with divergence free drift, J. Funct. Anal. **264** (2013), 2245–2268. Zbl 1278.35262 MR 3035054

[MeNS] G. Metafune, L. Negro and C. Spina, Sharp kernel estimates for elliptic operators with second-order discontinuous coefficients, J. Evol. Equ. **18** (2018), 467–514. Zbl 06932111 MR 3820411

[MeSS] G. Metafune, M. Sobajima and C. Spina, Kernel estimates for elliptic operators with second order discontinuous coefficients, J. Evol. Equ. **17** (2017), 485–522. Zbl 06722115 MR 3630331

[MS0] P. D. Milman and Yu. A. Semënov, Desingularizing weights and heat kernel bounds, Preprint (1998).

[MS1] P. D. Milman and Yu. A. Semënov, Heat kernel bounds and desingularizing weights, J. Funct. Anal. **202** (2003), 1–24. Zbl 1036.35044 MR 1994762

[MS2] P. D. Milman and Yu. A. Semënov, Global heat kernel bounds via desingularizing weights, J. Funct. Anal. **212** (2004), 373–398. Zbl 1057.47043 MR 2064932

[N] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. Math. J. **80** (1958), 931–954. Zbl 0096.06902 MR 100158

[VSC] N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon, *Analysis and geometry on groups*, Cambridge Tracts in Mathematics 100, Cambridge University Press, 1992. Zbl 0813.22003 MR 1218884

[Z] Q. S. Zhang, A strong regularity result for parabolic equations, Comm. Math. Phys. **244** (2004), 245–260. Zbl 1061.35026 MR 2031029

[Z2] Q. S. Zhang, Local estimates on two linear parabolic equations with singular coefficients, Pacific J. Math. **223** (2006), 367–396. Zbl 1113.35041 MR 2221033

[Zi] W. P. Ziemer, *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*, Graduate Texts in Mathematics 120, Springer, New York, 1989. Zbl 0692.46022 MR 1014685