Publ. RIMS Kyoto Univ. **59** (2023), 423–486
DOI 10.4171/PRIMS/59-3-1

Construction of the Affine Super Yangian

by

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Abstract

In this paper, we define the affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ with a coproduct structure. We also obtain an evaluation homomorphism, that is, an algebra homomorphism from $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ to the completion of the universal enveloping algebra of $\widehat{\mathfrak{gl}}(m|n)$.

2020 Mathematics Subject Classification: 17B37.
Keywords: Quantum group, superalgebra, super Yangian, coproduct, evaluation map.

§1. Introduction

Drinfel'd ([5, 6]) defined the Yangian of a finite-dimensional simple Lie algebra $\mathfrak{g}$ in order to obtain a solution of the Yang–Baxter equation. The Yangian is a quantum group which is the deformation of the current algebra $\mathfrak{g}[z]$. He defined it by three different presentations. One of those presentations is called the Drinfel'd presentation, whose generators are $\{h_{i,r}, x^{\pm}_{i,r} \mid r \in \mathbb{Z}_{\geq 0}\}$, where $\{h_i, x^{\pm}_i\}$ are Chevalley generators of $\mathfrak{g}$. The definition of the Yangian as an associative algebra naturally extends to the case that $\mathfrak{g}$ is a symmetrizable Kac–Moody Lie algebra in the Drinfel'd presentation. Defining its quasi-Hopf algebra structure is more involved, but this problem has been settled for affine Kac–Moody Lie algebras in [12, 1, 25].

It is known that the Yangians are closely related to W-algebras. It was shown in [21] that there exist surjective homomorphisms from Yangians of type A to rectangular finite W-algebras of type A. More generally, Brundan and Kleshchev ([4]) constructed a surjective homomorphism from a shifted Yangian, a subalgebra of the Yangian of type A, to a finite W-algebra of type A. Using a geometric realization of the Yangian, Schiffmann and Vasserot ([23]) have constructed a

Communicated by T. Arakawa. Received October 17, 2020. Revised April 14, 2021; May 2, 2021.

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surjective homomorphism from the Yangian of $\widehat{\mathfrak{gl}}(1)$ to the universal enveloping algebra of the principal W-algebra of type A, and proved the celebrated AGT conjecture ([8, 2]).

In the case of the Lie superalgebra $\mathfrak{sl}(m|n)$, the corresponding Yangian in the Drinfel'd presentation was first introduced by Stukopin ([24], see also [9]). The relationship between Yangians and W-algebras was also studied in the case of finite Lie superalgebras by Briot and Ragoucy [3] for $\mathfrak{sl}(m|n)$ and by Peng [20] for $\mathfrak{gl}(1|n)$. In the recent paper [7], Gaberdiel, Li, Peng and H. Zhang defined the Yangian $\widehat{\mathfrak{gl}}(1|1)$ for the affine Lie superalgebra $\widehat{\mathfrak{gl}}(1|1)$ and obtained similar results to [23] in the super setting.

In this article we define the affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ as a quantum group (= an associative algebra equipped with a coproduct satisfying compatibility conditions) in the Drinfel'd presentation. We upgrade the definition of the Yangian associated with $\mathfrak{sl}(m|n)$ of Gow [9] to define the affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ as an associative algebra; see Definition 3.1. However, to define the coproduct for $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$, we need to obtain yet another presentation, that is, a *minimalistic presentation.*

Theorem 1.1. *The affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ is isomorphic to the associative superalgebra over $\mathbb{C}$ generated by $x^+_{i,r}$, $x^-_{i,r}$, $h_{i,r}$ ($0 \leq i \leq m+n-1$, $r = 0, 1$) subject to the defining relations* (3.17)–(3.25).

By Theorem 1.1, the following assertion gives a coproduct Δ for $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ that is compatible with the defining relations (3.17)–(3.25).

Theorem 1.2. *We can define an algebra homomorphism*

$$\Delta\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) \to Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) \mathbin{\widehat{\otimes}} Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$$

that satisfies the coassociativity. Here, $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) \mathbin{\widehat{\otimes}} Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ is the degreewise completion of $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) \otimes Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ in the sense of [18].

When $\mathfrak{g}$ is $\mathfrak{sl}(n)$, $Y_h(\mathfrak{sl}(n))$ has an evaluation map $\mathrm{ev}\colon Y_h(\mathfrak{sl}(n)) \twoheadrightarrow U(\mathfrak{sl}(n))$, which enables us to define actions of $Y_h(\mathfrak{sl}(n))$ on any highest weight representation of $\mathfrak{sl}(n)$. In [11], Guay showed that the affine Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))$ has the evaluation map $\mathrm{ev}\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n)) \to \widetilde{U}(\widehat{\mathfrak{gl}}(n))$, where $\widetilde{U}(\widehat{\mathfrak{gl}}(n))$ is a completion of the universal enveloping algebra of $\widehat{\mathfrak{gl}}(n)$. The surjectivity of Guay's evaluation map is not trivial and was recently shown in [15]. In the second half of this paper, we construct an evaluation map of the affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ (see Theorem 5.1).

Theorem 1.3. *Assume* $c\hbar = (-m+n)\varepsilon_1$. *Then there exists a nontrivial algebra homomorphism* $\mathrm{ev}\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) \to U(\widehat{\mathfrak{gl}}(m|n))_{\mathrm{comp},+}$ *determined by* (5.2)–(5.5), *where* $U(\widehat{\mathfrak{gl}}(m|n))_{\mathrm{comp},+}$ *is a completion of the universal enveloping algebra of* $\widehat{\mathfrak{gl}}(m|n)$.

This paper is organized as follows. In Section 2 we recall the definitions of the Lie superalgebras $\mathfrak{sl}(m|n)$ and $\widehat{\mathfrak{sl}}(m|n)$. In Section 3 we define the affine super Yangian of type A and give the minimalistic presentation. Note that the Yangian for the finite-dimensional Lie superalgebra is defined only for type A in the literature. In Section 4 we define its coproduct. Finally, we give the evaluation map for the affine super Yangian in Section 5.

§2. Preliminaries

In this section we recall the definition and presentation of the Lie superalgebra $\widehat{\mathfrak{sl}}(m|n)$ (see [13]). First, we recall the definitions of $\mathfrak{sl}(m|n)$ and $\mathfrak{gl}(m|n)$.

Definition 2.1. Let us set $M_{k,l}(\mathbb{C})$ as the set of $k\times l$ matrices over $\mathbb{C}$. We define the Lie superalgebras $\mathfrak{sl}(m|n)$ and $\mathfrak{gl}(m|n)$ as

$$\mathfrak{gl}(m|n) = \left\{ \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \,\middle|\, A\in M_{m,m}(\mathbb{C}),\ B\in M_{m,n}(\mathbb{C}),\ C\in M_{n,m}(\mathbb{C}),\ D\in M_{n,n}(\mathbb{C}) \right\},$$
$$\mathfrak{sl}(m|n) = \left\{ \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathfrak{gl}(m|n) \,\middle|\, \mathrm{tr}(A)-\mathrm{tr}(D)=0 \right\},$$

where we define $\left[\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right), \left(\begin{smallmatrix} E & F \\ G & H\end{smallmatrix}\right)\right]$ as

$$\left[\begin{pmatrix} A & B \\ C & D\end{pmatrix}, \begin{pmatrix} E & F \\ G & H\end{pmatrix}\right] = \begin{pmatrix} AE-EA+(BG+FC) & AF+BH-(EB+FD) \\ CE+DG-(GA+HC) & DH-HD+(CF+GB)\end{pmatrix}.$$

As with $\mathfrak{sl}(m)$, $\mathfrak{sl}(m|n)$ has a presentation whose generators are Chevalley generators (see [22, 10]).

Proposition 2.2. *We set* $p\colon \{1,\dots,m+n\}\to\{0,1\}$ *as*

$$p(i) = \begin{cases} 0 & (1\le i\le m),\\ 1 & (m+1\le i\le m+n).\end{cases}$$

Suppose that $m,n\ge 2$, $m\neq n$ *and* $A=(a_{i,j})_{1\le i,j\le m+n-1}$ *is an* $(m+n-1)\times(m+n-1)$ *matrix whose components are*

$$a_{i,j} = \begin{cases} (-1)^{p(i)}+(-1)^{p(i+1)} & \text{if } i=j,\\ -(-1)^{p(i+1)} & \text{if } j=i+1,\\ -(-1)^{p(i)} & \text{if } j=i-1,\\ 0 & \text{otherwise.}\end{cases}$$

Then $\mathfrak{sl}(m|n)$ *is isomorphic to the Lie superalgebra over* $\mathbb{C}$ *defined by the generators* $\{x_i^{\pm}, h_i \mid 1 \leq i \leq m+n-1\}$ *and by the relations*

$$[h_i, h_j] = 0, \quad [h_i, x_j^{\pm}] = \pm a_{i,j} x_j^{\pm}, \quad [x_i^+, x_j^-] = \delta_{i,j} h_i, \quad \mathrm{ad}(x_i^{\pm})^{1+|a_{i,j}|} x_j^{\pm} = 0,$$
$$[x_m^{\pm}, x_m^{\pm}] = 0, \quad [[x_{m-1}^{\pm}, x_m^{\pm}], [x_{m+1}^{\pm}, x_m^{\pm}]] = 0,$$

where the generators $x_m^{\pm}$ *are odd and all other generators are even.*

The isomorphism Ψ *is given by*

$$\Psi(h_i) = (-1)^{p(i)} E_{ii} - (-1)^{p(i+1)} E_{i+1,i+1},$$
$$\Psi(x_i^+) = E_{i,i+1},$$
$$\Psi(x_i^-) = (-1)^{p(i)} E_{i+1,i}.$$

Next we recall the definition of the affinization of $\mathfrak{sl}(m|n)$ and $\mathfrak{gl}(m|n)$ (see [19]). Lie superalgebra $\mathfrak{sl}(m|n)$ has a nondegenerate invariant bilinear form $\kappa\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$. The bilinear form is uniquely determined up to a scalar multiple, so we fix it.

Definition 2.3. Suppose that $\mathfrak{g}$ is $\mathfrak{sl}(m|n)$ or $\mathfrak{gl}(m|n)$. Then we set a Lie superalgebra $\tilde{\mathfrak{g}}$ as $\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ whose commutator relations are

$$[a \otimes t^s, b \otimes t^u] = [a, b] \otimes t^{s+u} + s\delta_{s+u,0}\kappa(a, b)c,$$
$$c \text{ is a central element of } \tilde{\mathfrak{g}},$$
$$[d, a \otimes t^s] = sa \otimes t^s.$$

We also set a subalgebra $\hat{\mathfrak{g}} \subset \tilde{\mathfrak{g}}$ as $\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}c$.

We have another presentation of $\widehat{\mathfrak{sl}}(m|n)$ (see [26]).

Proposition 2.4. *Suppose that* $m, n \geq 2$, $m \neq n$ *and* $A = (a_{i,j})_{0 \leq i,j \leq m+n-1}$ *is an* $(m+n) \times (m+n)$ *matrix whose components are*

$$a_{i,j} = \begin{cases} (-1)^{p(i)} + (-1)^{p(i+1)} & \text{if } i = j, \\ -(-1)^{p(i+1)} & \text{if } j = i+1, \\ -(-1)^{p(i)} & \text{if } j = i-1, \\ 1 & \text{if } (i,j) = (0, m+n-1), (m+n-1, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\widetilde{\mathfrak{sl}}(m|n)$ *is isomorphic to the Lie superalgebra over* $\mathbb{C}$ *defined by the generators* $\{x_i^{\pm}, h_i, d \mid 0 \leq i \leq m+n-1\}$ *and by the relations*

$$(2.5)\quad [d,h_i]=0,\quad [d,x_i^+]=\begin{cases}x_i^+ & (i=0),\\ 0 & (\textit{otherwise}),\end{cases}\quad [d,x_i^-]=\begin{cases}-x_i^- & (i=0),\\ 0 & (\textit{otherwise}),\end{cases}$$

$$(2.6)\quad [h_i,h_j]=0,\ \ [h_i,x_j^{\pm}]=\pm a_{i,j}x_j^{\pm},\quad [x_i^+,x_j^-]=\delta_{i,j}h_i,\ \ \mathrm{ad}(x_i^{\pm})^{1+|a_{i,j}|}x_j^{\pm}=0,$$

$$(2.7)\quad [x_0^{\pm},x_0^{\pm}]=0,\ \ [x_m^{\pm},x_m^{\pm}]=0,$$

$$(2.8)\quad [[x_{m-1}^{\pm},x_m^{\pm}],[x_{m+1}^{\pm},x_m^{\pm}]]=0,\quad [[x_{m+n-1}^{\pm},x_0^{\pm}],[x_1^{\pm},x_0^{\pm}]]=0,$$

where the generators $x_m^{\pm}$ *and* $x_0^{\pm}$ *are odd and all other generators are even.*

The isomorphism Ξ *is given by*

$$\Xi(h_i)=\begin{cases}-E_{1,1}-E_{m+n,m+n}+c & (i=0),\\ (-1)^{p(i)}E_{ii}-(-1)^{p(i+1)}E_{i+1,i+1} & (1\leq i\leq m+n-1),\end{cases}$$

$$\Xi(x_i^+)=\begin{cases}E_{m+n,1}\otimes t & (i=0),\\ E_{i,i+1} & (\textit{otherwise}),\end{cases}$$

$$\Xi(x_i^-)=\begin{cases}-E_{1,m+n}\otimes t^{-1} & (i=0),\\ (-1)^{p(i)}E_{i+1,i} & (\textit{otherwise}).\end{cases}$$

Moreover, $\widehat{\mathfrak{sl}}(m|n)$ *is isomorphic to the Lie superalgebra over* $\mathbb{C}$ *defined by the generators* $\{x_i^{\pm}, h_i \mid 0\leq i\leq m+n-1\}$ *and by the relations* (2.6)–(2.8).

Finally, we set some notation. Let us set $\{\alpha_i\}_{0\leq i\leq m+n-1}$ as a set of simple roots of $\widetilde{\mathfrak{sl}}(m|n)$ and δ as a positive root $\sum_{0\leq i\leq m+n-1}\alpha_i$. Moreover, we set Δ (resp. Δ_+) as a set of roots (resp. positive roots) of $\widetilde{\mathfrak{sl}}(m|n)$. We denote the parity of $E_{i,j}$ as $p(E_{i,j})$. Obviously, $p(E_{i,j})$ is equal to $p(i)+p(j)$. We also set Δ_+^{re} and Δ^{re} as $\Delta_+\setminus\mathbb{Z}_{>0}\delta$ and $\Delta\setminus\mathbb{Z}\delta$. We also take an inner product on $\bigoplus_{0\leq i\leq m+n-1}\mathbb{C}\alpha_i$ determined by $(\alpha_i,\alpha_j)=a_{i,j}$. Assume that $\mathfrak{g}=\widetilde{\mathfrak{sl}}(m|n)$ and let $\mathfrak{g}_\alpha$ be the root α space of $\mathfrak{g}$. We set $\{x_\alpha^{k_\alpha}\}_{1\leq k_\alpha\leq \dim\mathfrak{g}_\alpha}$ as a basis of $\mathfrak{g}_\alpha$ which satisfies $\kappa(x_\alpha^p,x_{-\alpha}^q)=\delta_{p,q}$ for all $\alpha\in\Delta_+$. We also denote the parity of $x_\alpha^{k_\alpha}$ by $p(\alpha)$. Moreover, we sometimes identify $\{0,\dots,m+n-1\}$ with $\mathbb{Z}/(m+n)\mathbb{Z}$ and denote it by I.

§3. The minimalistic presentation of the affine super Yangian

First, we define the affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$. This definition is a generalization of Stukopin's super Yangian ([24]). Let us set $\{x,y\}$ as $xy+yx$.

Definition 3.1. Suppose that $m,n\geq 2$ and $m\neq n$. The affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ is the associative superalgebra over $\mathbb{C}$ generated by $x_{i,r}^+$, $x_{i,r}^-$, $h_{i,r}$

($i \in \{0,1,\dots,m+n-1\}$, $r \in \mathbb{Z}_{\geq 0}$) with parameters $\varepsilon_1,\varepsilon_2 \in \mathbb{C}$ subject to the defining relations

$$[h_{i,r}, h_{j,s}] = 0, \tag{3.2}$$

$$[x^+_{i,r}, x^-_{j,s}] = \delta_{i,j} h_{i,r+s}, \tag{3.3}$$

$$[h_{i,0}, x^\pm_{j,r}] = \pm a_{i,j} x^\pm_{j,r}, \tag{3.4}$$

$$[h_{i,r+1}, x^\pm_{j,s}] - [h_{i,r}, x^\pm_{j,s+1}] = \pm a_{i,j}\frac{\varepsilon_1+\varepsilon_2}{2}\{h_{i,r}, x^\pm_{j,s}\} - b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}[h_{i,r}, x^\pm_{j,s}], \tag{3.5}$$

$$[x^\pm_{i,r+1}, x^\pm_{j,s}] - [x^\pm_{i,r}, x^\pm_{j,s+1}] = \pm a_{i,j}\frac{\varepsilon_1+\varepsilon_2}{2}\{x^\pm_{i,r}, x^\pm_{j,s}\} - b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}[x^\pm_{i,r}, x^\pm_{j,s}], \tag{3.6}$$

$$\sum_{w \in \mathfrak{S}_{1+|a_{i,j}|}} [x^\pm_{i,r_{w(1)}}, [x^\pm_{i,r_{w(2)}}, \dots, [x^\pm_{i,r_{w(1+|a_{i,j}|)}}, x^\pm_{j,s}], \dots]] = 0 \quad (i \neq j), \tag{3.7}$$

$$[x^\pm_{i,r}, x^\pm_{i,s}] = 0 \quad (i = 0, m), \tag{3.8}$$

$$[[x^\pm_{i-1,r}, x^\pm_{i,0}], [x^\pm_{i,0}, x^\pm_{i+1,s}]] = 0 \quad (i = 0, m), \tag{3.9}$$

where

$$a_{i,j} = \begin{cases} (-1)^{p(i)} + (-1)^{p(i+1)} & \text{if } i = j,\\ -(-1)^{p(i+1)} & \text{if } j = i+1,\\ -(-1)^{p(i)} & \text{if } j = i-1,\\ 1 & \text{if } (i,j) = (0, m+n-1), (m+n-1, 0),\\ 0 & \text{otherwise,} \end{cases}$$

$$b_{i,j} = \begin{cases} -(-1)^{p(i+1)} & \text{if } j = i+1,\\ (-1)^{p(i)} & \text{if } j = i-1,\\ -1 & \text{if } (i,j) = (0, m+n-1),\\ 1 & \text{if } (i,j) = (m+n-1, 0),\\ 0 & \text{otherwise,} \end{cases}$$

and the generators $x^\pm_{m,r}$ and $x^\pm_{0,r}$ are odd and all other generators are even.

Remark 3.10. In this paper, we set $[x,y]$ as $xy - (-1)^{p(x)p(y)}yx$ for all homogeneous elements x, y. Thus, (3.8) is nontrivial.

We also define the affine super Yangian associated with $\widetilde{\mathfrak{sl}}(m|n)$.

Definition 3.11. Suppose that $m, n \geq 2$ and $m \neq n$. We define $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ as the associative superalgebra over $\mathbb{C}$ generated by $\{x^\pm_{i,r}, h_{i,r}, d \mid i \in \{0,1,\dots, m+n-1\}, r \in \mathbb{Z}_{\geq 0}\}$ with parameters $\varepsilon_1,\varepsilon_2 \in \mathbb{C}$ subject to the defining relations

(3.2)–(3.9) and

$$[d,h_{i,r}]=0,\quad [d,x^+_{i,r}]=\begin{cases}1 & \text{if } i=0,\\ 0 & \text{if } i\neq 0,\end{cases}\quad [d,x^-_{i,r}]=\begin{cases}-1 & \text{if } i=0,\\ 0 & \text{if } i\neq 0,\end{cases} \tag{3.12}$$

where the generators $x^{\pm}_{m,r}$ and $x^{\pm}_{0,r}$ are odd and all other generators are even.

One of the difficulties of Definition 3.1 is that the number of generators is infinite. In the rest of this section, we construct a new presentation of the affine super Yangian such that the number of generators is finite.

Let us set $\tilde{h}_{i,1}=h_{i,1}-\frac{\varepsilon_1+\varepsilon_2}{2}h_{i,0}^2$. By the definition of $\tilde{h}_{i,1}$, we can rewrite (3.5) as

$$[\tilde{h}_{i,1},x^{\pm}_{j,r}]=\pm a_{i,j}\Big(x^{\pm}_{j,r+1}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,r}\Big). \tag{3.13}$$

By (3.13), we find that $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ is generated by $x^+_{i,r}$, $x^-_{i,r}$, $h_{i,r}$ ($i\in\{0,1,\dots,m+n-1\}$, $r=0,1$). In fact, by (3.13) and (3.3), we have the following relations:

$$\begin{aligned} x^{\pm}_{i,r+1}&=\pm\frac{1}{a_{i,i}}[\tilde{h}_{i,1},x^{\pm}_{i,r}],\\ h_{i,r+1}&=[x^+_{i,r+1},x^-_{i,0}]\quad \text{if } i\neq m,0,\end{aligned} \tag{3.14}$$

$$\begin{aligned} x^{\pm}_{i,r+1}&=\pm\frac{1}{a_{i+1,i}}[\tilde{h}_{i+1,1},x^{\pm}_{i,r}]+b_{i+1,i}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{i,r},\\ h_{i,r+1}&=[x^+_{i,r+1},x^-_{i,0}]\quad \text{if } i=m,0\end{aligned} \tag{3.15}$$

for all $r\geq 1$. In the following theorem, we construct the minimalistic presentation of the affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ whose generators are $x^+_{i,r}$, $x^-_{i,r}$, $h_{i,r}$ ($i\in\{0,1,\dots,m+n-1\}$, $r=0,1$). We remark that we have not checked that the presentation is minimalistic yet. However, we call this presentation a "minimalistic presentation" since, in the non-super case, the corresponding presentation is called a "minimalistic presentation".

Theorem 3.16. *Suppose that $m,n\geq 2$ and $m\neq n$. The affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ is isomorphic to the associative superalgebra generated by $x^+_{i,r}$, $x^-_{i,r}$, $h_{i,r}$ ($i\in\{0,1,\dots,m+n-1\}$, $r=0,1$) subject to the defining relations*

$$[h_{i,r},h_{j,s}]=0, \tag{3.17}$$

$$[x^+_{i,0},x^-_{j,0}]=\delta_{i,j}h_{i,0}, \tag{3.18}$$

$$[x^+_{i,1},x^-_{j,0}]=\delta_{i,j}h_{i,1}=[x^+_{i,0},x^-_{j,1}], \tag{3.19}$$

$$[h_{i,0},x^{\pm}_{j,r}]=\pm a_{i,j}x^{\pm}_{j,r}, \tag{3.20}$$

$$[\tilde{h}_{i,1}, x^{\pm}_{j,0}] = \pm a_{i,j}\Big(x^{\pm}_{j,1} - b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,0}\Big), \tag{3.21}$$

$$[x^{\pm}_{i,1}, x^{\pm}_{j,0}] - [x^{\pm}_{i,0}, x^{\pm}_{j,1}] = \pm\, a_{i,j}\frac{\varepsilon_1+\varepsilon_2}{2}\{x^{\pm}_{i,0}, x^{\pm}_{j,0}\} - b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}[x^{\pm}_{i,0}, x^{\pm}_{j,0}], \tag{3.22}$$

$$(\mathrm{ad}\, x^{\pm}_{i,0})^{1+|a_{i,j}|}(x^{\pm}_{j,0}) = 0 \quad (i \neq j), \tag{3.23}$$

$$[x^{\pm}_{i,0}, x^{\pm}_{i,0}] = 0 \quad (i = 0, m), \tag{3.24}$$

$$[[x^{\pm}_{i-1,0}, x^{\pm}_{i,0}], [x^{\pm}_{i,0}, x^{\pm}_{i+1,0}]] = 0 \quad (i = 0, m), \tag{3.25}$$

where the generators $x^{\pm}_{m,r}$ and $x^{\pm}_{0,r}$ are odd and all other generators are even.

The outline of the proof of Theorem 3.16 is similar to that of [12, Thm. 2.13]. To simplify the notation, we denote the associative superalgebra defined in Theorem 3.16 as $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$. We construct $x^{\pm}_{i,r}$ and $h_{i,r}$ as the elements of $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ inductively by (3.14) and (3.15). Since (3.17)–(3.25) are contained in the defining relations of the affine super Yangian, it is enough to check that the defining relations of the affine super Yangians (3.2)–(3.9) are deduced from (3.17)–(3.25) in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$. The proof of Theorem 3.16 is divided into eight lemmas, that is, Lemmas 3.26, 3.31, 3.35, 3.36, 3.37, 3.38, 3.57 and 3.58.

Most of the defining relations (3.2)-(3.9) are obtained in the same way as those of [17] or [12]. For example, we have the following lemma in a similar way to [12, Lem. 2.22].

Lemma 3.26.

(1) *The defining relation (3.4) holds for all $i, j \in I$ in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.*

(2) *For all $i, j \in I$, we obtain*

$$[\tilde{h}_{i,1}, x^{\pm}_{j,r}] = \pm a_{i,j}\Big(x^{\pm}_{j,r+1} - b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,r}\Big) \tag{3.27}$$

in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

Proof. We only show the case that $j = 0, m$. The other case is proven in the same way as [12, Lem. 2.22]. We prove (1), (2) by induction on r. When $r = 0$, they are nothing but (3.20) and (3.21). Suppose that (3.4) and (3.27) hold when $r = k$. First, let us show that (3.4) holds when $r = k+1$. By (3.15), we obtain

$$[h_{i,0}, x^{\pm}_{j,k+1}] = \pm\frac{1}{a_{j,j+1}}[h_{i,0}, [\tilde{h}_{j+1,1}, x^{\pm}_{j,k}]] + b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}[h_{i,0}, x^{\pm}_{j,k}]. \tag{3.28}$$

By $[h_{i,0},h_{j,1}]=0$, we find that the first term on the right-hand side of (3.28) is equal to

$$\pm\frac{1}{a_{j,j+1}}[h_{i,0},[\tilde{h}_{j+1,1},x^{\pm}_{j,k}]]=\pm\frac{1}{a_{j,j+1}}[\tilde{h}_{j+1,1},[h_{i,0},x^{\pm}_{j,k}]].$$

By the induction hypothesis on r, we can rewrite the right-hand side of (3.28) as

$$\begin{aligned}&\pm\frac{1}{a_{j,j+1}}[\tilde{h}_{j+1,1},[h_{i,0},x^{\pm}_{j,k}]]+b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}[h_{i,0},x^{\pm}_{j,k}]\\&\quad=\frac{a_{i,j}}{a_{j,j+1}}[\tilde{h}_{j+1,1},x^{\pm}_{j,k}]\pm a_{i,j}b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k}\\&\quad=\frac{a_{i,j}}{a_{j,j+1}}\Big(\pm a_{j,j+1}\Big(x^{\pm}_{j,k+1}-b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k}\Big)\Big)\pm a_{i,j}b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k}\\&\quad=\pm a_{i,j}x^{\pm}_{j,k+1}.\end{aligned}$$

Thus, we have shown that $[h_{i,0},x^{\pm}_{j,k+1}]=\pm a_{i,j}x^{\pm}_{j,k+1}$.

Next we show that (3.4) holds when $r=k+1$. Since we have already proved that (3.4) holds when $r=k+1$, it is enough to check the relation

$$[\tilde{h}_{i,1},x^{\pm}_{j,k+1}]=\pm a_{i,j}\Big(x^{\pm}_{j,k+2}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k+1}\Big).$$

By (3.15), we obtain

$$[\tilde{h}_{i,1},x^{\pm}_{j,k+1}]=\pm\frac{1}{a_{j,j+1}}[\tilde{h}_{i,1},[\tilde{h}_{j+1,1},x^{\pm}_{j,k}]]+b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}[\tilde{h}_{i,1},x^{\pm}_{j,k}]. \tag{3.29}$$

By $[h_{i,1},h_{j,1}]=0$, we find that the right-hand side of (3.29) is equal to

$$\pm\frac{1}{a_{j,j+1}}[\tilde{h}_{j+1,1},[\tilde{h}_{i,1},x^{\pm}_{j,k}]]+b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}[\tilde{h}_{i,1},x^{\pm}_{j,k}].$$

By the induction hypothesis on r, we can rewrite the right-hand side of (3.29) as

$$\begin{aligned}&\pm\frac{1}{a_{j,j+1}}[\tilde{h}_{j+1,1},[\tilde{h}_{i,1},x^{\pm}_{j,k}]]+b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}[\tilde{h}_{i,1},x^{\pm}_{j,k}]\\&\quad=\frac{a_{i,j}}{a_{j,j+1}}\Big([\tilde{h}_{j+1,1},x^{\pm}_{j,k+1}]-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}[\tilde{h}_{j+1,1},x^{\pm}_{j,k}]\Big)\\&\qquad\pm a_{i,j}b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}\Big(x^{\pm}_{j,k+1}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k}\Big).\end{aligned} \tag{3.30}$$

Since $x^{\pm}_{j,k+2}$ is defined by (3.15), we find that the right-hand side of (3.30) is equal to

$$\begin{aligned}&\pm a_{i,j}\Big(x^{\pm}_{j,k+2}-b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k+1}\Big)\\&\qquad\mp a_{i,j}b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}\Big(x^{\pm}_{j,k+1}-b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k}\Big)\\&\qquad\pm a_{i,j}b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}\Big(x^{\pm}_{j,k+1}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k}\Big).\end{aligned}$$

By direct computation, it is equal to

$$\pm a_{i,j}\Big(x^{\pm}_{j,k+2} - b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k+1}\Big).$$

This completes the proof. □

We also obtain the following lemma in a similar way to [12].

Lemma 3.31.

(1) *The relation (3.3) holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i=j$ *and* $r+s\leq 2$.

(2) *Suppose that* $i,j\in I$ *and* $i\neq j$. *Then relations (3.3) and (3.6) hold for any* r *and* s *in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

(3) *The relation (3.6) holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i=j$, $(r,s)=(1,0)$.

(4) *The relation (3.5) holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i=j$, $(r,s)=(1,0)$.

(5) *For all* $i,j\in I$, *the relation (3.5) holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $(r,s)=(1,0)$.

(6) *Set* $\tilde{h}_{i,2}=h_{i,2}-h_{i,0}h_{i,1}+\frac{1}{3}h^3_{i,0}$. *Then the following equation holds for all* $i,j\in I$ *in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$:

$$\begin{aligned}[\tilde{h}_{i,2},x^{\pm}_{j,0}] &= \pm a_{i,j}x^{\pm}_{j,2} \pm \frac{1}{12}a^3_{i,j}x^{\pm}_{j,0}\\ &\quad \mp a_{i,j}b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}\Big(x^{\pm}_{j,1}-\frac{1}{2}x^{\pm}_j h_i - b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_j\Big).\end{aligned}$$

(7) *For all* $i,j\in I$, *the relation (3.7) holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when*

  (a) $r_1=\cdots=r_b=0$, $s\in\mathbb{Z}_{\geq 0}$;

  (b) $r_1=1$, $r_2=\cdots=r_b=0$, $s\in\mathbb{Z}_{\geq 0}$;

  (c) $r_1=2$, $r_2=\cdots=r_b=0$, $s\in\mathbb{Z}_{\geq 0}$;

  (d) *($b\geq 2$ and)* $r_1=r_2=1$, $r_3=\cdots=r_b=0$, $s\in\mathbb{Z}_{\geq 0}$.

(8) *In* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$, *we have*

$$[h_{j,1},x^{\pm}_{i,1}] = \frac{a_{i,j}}{a_{i,i}}[h_{i,1},x^{\pm}_{i,1}] \pm \frac{a_{i,j}}{2}(\{h_{j,0},x^{\pm}_{i,1}\}-\{h_{i,0},x^{\pm}_{i,1}\}) \mp a_{j,i}m_{j,i}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{i,1}$$

for all $i,j\in I$ *such that* $a_{i,i}\neq 0$.

(9) *For all* $i,j\in I$, *we have*

$$[h_{i,2},h_{j,0}]=0$$

in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

(10) *Suppose that* $i,j\in I$ *such that* $a_{i,i}=2$ *and* $a_{i,j}=-1$. *Then*

$$[h_{i,2},h_{i,1}]=0$$

holds in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

Proof. We only prove (1)–(5) since the proof of (6) (resp. (7), (8), (9), (10)) is the same as that of [12, Lem. 2.31 (resp. Lem. 2.33, Lem. 2.34, Lem. 2.35, Prop. 2.36)].

The proofs of (1) and (2) are the same as those of [12, Lems. 2.22 and 2.26]. In the case where $i,j\neq 0,m$, the proof of (3) (resp. (4) and (5)) is also the same as that of [12, Lem. 2.23 (resp. Lems. 2.24 and 2.28)]. We omit it. We only show that (3) holds since (4) and (5) are derived from (3) in a similar way to [12].

Suppose that $i=j=0,m$. First, we show that $[x^+_{i,1},x^+_{i,0}]=[x^+_{i,0},x^+_{i,1}]=0$ holds. Applying $\mathrm{ad}(\tilde{h}_{i+1,1})$ to (3.24), we have $\pm a_{i,i+1}[x^\pm_{i,1},x^\pm_{i,0}]\pm a_{i,i+1}[x^\pm_{i,0},x^\pm_{i,1}]$. Since $[x^\pm_{i,1},x^\pm_{i,0}]$ is equal to $[x^\pm_{i,0},x^\pm_{i,1}]$, we obtain $[x^\pm_{i,1},x^\pm_{i,0}]=[x^\pm_{i,0},x^\pm_{i,1}]=0$. Next we show that $[x^\pm_{i,2},x^\pm_{i,0}]=[x^\pm_{i,1},x^\pm_{i,1}]=[x^\pm_{i,0},x^\pm_{i,2}]$ holds. Applying $\mathrm{ad}(\tilde{h}_{i+1,1})$ to $[x^\pm_{i,1},x^\pm_{i,0}]=[x^\pm_{i,0},x^\pm_{i,1}]=0$, we obtain

$$\pm a_{i,i+1}([x^\pm_{i,2},x^\pm_{i,0}]+[x^\pm_{i,1},x^\pm_{i,1}])=0, \tag{3.32}$$

$$\pm a_{i,i+1}([x^\pm_{i,1},x^\pm_{i,1}]+[x^\pm_{i,0},x^\pm_{i,2}])=0. \tag{3.33}$$

In the case where $j=0,m$ and $i=j+1$, we can prove (5) in a similar way to [12, Lem. 2.28]. Then, in a discussion similar to that of [17, Lem. 1.4], there exists $\hat{h}_{i+1,2}$ such that

$$[\hat{h}_{i+1,2},x^\pm_{i,0}]=\pm a_{i,i+1}x^\pm_{i,2}.$$

Applying $\mathrm{ad}(\hat{h}_{i+1,2})$ to (3.24), we obtain

$$\pm a_{i,i+1}([x^\pm_{i,2},x^\pm_{i,0}]+[x^\pm_{i,0},x^\pm_{i,2}])=0. \tag{3.34}$$

Since (3.32), (3.33) and (3.34) are linearly independent, we obtain $[x^\pm_{i,2},x^\pm_{i,0}]=[x^\pm_{i,1},x^\pm_{i,1}]=[x^\pm_{i,0},x^\pm_{i,2}]$. We have proved (3). □

In the case where $a_{i,i}=-2$ and $a_{i,j}=1$, we obtain $[h_{i,2},h_{i,1}]=0$ by changing the proof of [12, Prop. 2.36] a little.

Lemma 3.35. *Suppose that $i,j\in I$ such that $a_{i,i}=-2$ and $a_{i,j}=1$. Then we obtain*

$$[h_{i,2},h_{i,1}]=0$$

in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

Proof. We change $h_{i,r}$, $x^+_{i,r}$, and $x^-_{i,r}$, which are written in the proof of [12, Prop. 2.36], into $-h_{i,r}$, $-x^+_{i,r}$, and $x^-_{i,r}$. Then we obtain $[-h_{i,2},-h_{i,1}]=0$. □

By Lemma 3.31(10) and Lemma 3.35, we obtain the following lemma in the same way as [12, Prop. 2.39] since we only need the condition that $a_{i,i}\neq 0$ and $a_{i,j}\neq 0$. We omit the proof.

Lemma 3.36. *Suppose that $i, j \in I$ such that $a_{i,i} \neq 0$ and $a_{i,j} \neq 0$. Then we have*

$$[h_{j,2}, h_{j,1}] = 0$$

in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

Therefore, we know that $[h_{i,2}, h_{i,1}] = 0$ holds for all $i \in I$. By using the relation $[h_{i,2}, h_{i,1}] = 0$, we obtain the following lemma in a similar way to [17, Thm. 1.2] since the proof of these statements needs only the condition that $a_{i,i} \neq 0$.

Lemma 3.37.

(1) *The relation (3.2) holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i = j \neq 0, m$.
(2) *The relation (3.3) holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i = j \neq 0, m$.
(3) *The relation (3.6) holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i = j \neq 0, m$.
(4) *The relation (3.5) holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i = j \neq 0, m$.

Next we prove the same statement as that of Lemma 3.37 in the case that $i = j = 0, m$.

Lemma 3.38.

(1) *The relation (3.6) holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i = j = 0, m$. *In particular, (3.8) holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.
(2) *The relation (3.3) holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i = j = 0, m$.
(3) *We obtain* $[h_{i,r}, x^{\pm}_{i,0}] = 0$ *when* $i = 0, m$ *in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.
(4) *The relation (3.5) holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i = j = 0, m$.
(5) *The relation (3.2) holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i = j = 0, m$.

Proof. (1) It is enough to check the equality $[x^{\pm}_{i,r}, x^{\pm}_{i,s}] = 0$. We only show that $[x^{+}_{i,r}, x^{+}_{i,s}] = 0$ holds. We can obtain $[x^{-}_{i,r}, x^{-}_{i,s}] = 0$ in a similar way. We prove (3.6) holds by induction on $k = r + s$. When $k = 0$, it is nothing but (3.24). Applying $\mathrm{ad}(\tilde{h}_{i+1,1})$ to $[x^{+}_{i,0}, x^{+}_{i,0}] = 0$, we obtain

$$a_{i,i+1}([x^{+}_{i,1}, x^{+}_{i,0}] + [x^{+}_{i,0}, x^{+}_{i,1}]) = 0.$$

Since $[x^{+}_{i,1}, x^{+}_{i,0}] = [x^{+}_{i,0}, x^{+}_{i,1}]$, we have $[x^{+}_{i,1}, x^{+}_{i,0}] = [x^{+}_{i,0}, x^{+}_{i,1}] = 0$.

Suppose that $[x^{+}_{i,r}, x^{+}_{i,s}] = 0$ holds for all r, s such that $r + s = k, k + 1$. Applying $\mathrm{ad}(\tilde{h}_{i+1,1})$ to $[x^{+}_{i,u}, x^{+}_{i,k+1-u}] = 0$, we have

$$[\tilde{h}_{i+1,1}, [x^{+}_{i,u}, x^{+}_{i,k+1-u}]] = 0. \tag{3.39}$$

By Lemma 3.31(4) and the induction hypothesis, we have

$$[\tilde{h}_{i+1,1}, [x^{+}_{i,u}, x^{+}_{i,k+1-u}]] = a_{i,i+1}([x^{+}_{i,u+1}, x^{+}_{i,k+1-u}] + [x^{+}_{i,u}, x^{+}_{i,k+2-u}]). \tag{3.40}$$

Since $a_{i,i+1}\neq 0$, we find the relation

$$[x^+_{i,u+1},x^+_{i,k+1-u}]=-[x^+_{i,u},x^+_{i,k+2-u}] \tag{3.41}$$

by (3.39) and (3.40). In particular, we obtain

$$[x^+_{i,u+2},x^+_{i,k-u}]=[x^+_{i,u},x^+_{i,k+2-u}]. \tag{3.42}$$

Applying $\mathrm{ad}(\tilde{h}_{i+1,2})$ to $[x^+_{i,u},x^+_{i,k-u}]=0$, we have

$$[\tilde{h}_{i+1,2},[x^+_{i,u},x^+_{i,k-u}]]=0 \tag{3.43}$$

by the induction hypothesis. By Lemma 3.31(7), Lemma 3.36 and the induction hypothesis, we have

$$[\tilde{h}_{i+1,2},[x^+_{i,u},x^+_{i,k-u}]]=a_{i,i+1}([x^+_{i,u+2},x^+_{i,k-u}]+[x^+_{i,u},x^+_{i,k+2-u}]). \tag{3.44}$$

Since $a_{i,i+1}\neq 0$, we obtain the relation

$$[x^+_{i,u+2},x^+_{i,k-u}]=-[x^+_{i,u},x^+_{i,k+2-u}] \tag{3.45}$$

by (3.43) and (3.44). Since (3.45) and (3.42) are linearly independent, we have shown that $[x^+_{i,u},x^+_{i,k+2-u}]=0$ holds.

(2) We prove the statement by induction on $r+s=k$. When $k=0$, it is nothing but (3.24). Suppose that $[x^+_{i,r},x^-_{i,s}]=h_{i,r+s}$ for all r, s such that $r+s\leq k$. Then we have the following claim.

Claim 3.46.

(a) For all r, s, we obtain

$$\begin{aligned}[h_{i,r+1},x^+_{i+1,s}]-[h_{i,r},x^+_{i+1,s+1}]&=a_{i,i+1}\frac{\varepsilon_1+\varepsilon_2}{2}\{h_{i,r},x^+_{i+1,s}\}\\ &\quad-b_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}[h_{i,r},x^+_{i+1,s}].\end{aligned} \tag{3.47}$$

(b) For all $r+s=k-1$, we obtain

$$\begin{aligned}[h_{i,r+1},x^-_{i+1,s}]-[h_{i,r},x^-_{i+1,s+1}]&=-a_{i,i+1}\frac{\varepsilon_1+\varepsilon_2}{2}\{h_{i,r},x^-_{i+1,s}\}\\ &\quad-b_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}[h_{i,r},x^-_{i+1,s}].\end{aligned} \tag{3.48}$$

Proof. (a) By the definition of $h_{i,r}$, we have

$$[h_{i,r+1},x^+_{i+1,s}]-[h_{i,r},x^+_{i+1,s+1}]=[[x^+_{i,r+1},x^-_{i,0}],x^+_{i+1,s}]-[[x^+_{i,r},x^-_{i,0}],x^+_{i+1,s+1}].$$

By the Jacobi identity and Lemma 3.31(4), we obtain

$$[h_{i,r+1}, x^+_{i+1,s}] - [h_{i,r}, x^+_{i+1,s+1}] = [\{[x^+_{i,r+1}, x^+_{i+1,s}] - [x^+_{i,r}, x^+_{i+1,s+1}]\}, x^-_{i,0}].$$

By Lemma 3.31(4), we have

$$\begin{aligned}&[h_{i,r+1}, x^+_{i+1,s}] - [h_{i,r}, x^+_{i+1,s+1}]\\ &\quad = \Big[\pm a_{i,i+1}\frac{\varepsilon_1+\varepsilon_2}{2}\{x^+_{i,r}, x^+_{i+1,s}\} - b_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}[x^+_{i,r}, x^+_{i+1,s}], x^-_{i,0}\Big].\end{aligned}$$

By Lemma 3.31(4), we obtain

$$\begin{aligned}&[h_{i,r+1}, x^+_{i+1,s}] - [h_{m,r}, x^+_{m+1,s+1}]\\ &\quad = \pm a_{i,i+1}\frac{\varepsilon_1+\varepsilon_2}{2}\{h_{i,r}, x^+_{i+1,s}\} - b_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}[h_{i,r}, x^+_{i+1,s}].\end{aligned}$$

(b) By the assumption that $[x^+_{i,p}, x^-_{i,q}] = h_{i,p+q}$ holds for all $p+q \leq k$, we have

$$[h_{i,r+1}, x^-_{i+1,s}] - [h_{i,r}, x^-_{i+1,s+1}] = [[x^+_{i,r}, x^-_{i,1}], x^-_{i+1,s}] - [[x^+_{i,r}, x^-_{i,0}], x^-_{i+1,s+1}]$$

since $r+1 \leq k$. By a similar discussion to (a), we have

$$[h_{i,r+1}, x^-_{i+1,s}] - [h_{i,r}, x^-_{i+1,s+1}] = [x^+_{i,r}, \{[x^-_{i,1}, x^-_{i+1,s}] - [x^-_{i,0}, x^-_{i+1,s+1}]\}].$$

By Lemma 3.31(4), we obtain

$$\begin{aligned}&[h_{i,r+1}, x^-_{i+1,s}] - [h_{i,r}, x^-_{i+1,s+1}]\\ &\quad = \Big[x^+_{i,r}, -a_{i,i+1}\frac{\varepsilon_1+\varepsilon_2}{2}\{x^-_{i,0}, x^-_{i+1,s}\} - b_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}[x^-_{i,0}, x^-_{i+1,s}]\Big].\end{aligned}$$

Then, by Lemma 3.31(4), we have

$$\begin{aligned}&[h_{i,r+1}, x^-_{i+1,s}] - [h_{i,r}, x^-_{i+1,s+1}]\\ &\quad = -a_{i,i+1}\frac{\varepsilon_1+\varepsilon_2}{2}\{h_{i,r}, x^-_{i+1,s}\} - b_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}[h_{i,r}, x^-_{i+1,s}].\end{aligned}$$ □

By a similar discussion to [17, Lem. 1.4], there exists $\tilde{h}_{i,k}$ such that

$$\begin{aligned}&\tilde{h}_{i,k} = h_{i,k} + \text{polynomial of } \{h_{i,t} \mid 0 \leq t \leq k-1\},\\ &[\tilde{h}_{i,k}, x^+_{i+1,1}] = a_{i,i+1}x^+_{i+1,k+1},\\ &[\tilde{h}_{i,k}, x^-_{i+1,0}] = -a_{i,i+1}x^-_{i+1,k}.\end{aligned}$$

Claim 3.49. The following equation holds:

$$[\tilde{h}_{i+1,1}, h_{i,k}] = 0. \tag{3.50}$$

Proof. By the assumption that $[x^+_{i,p}, x^-_{i,q}] = h_{i,k}$ holds for all $p+q \leq k$ we have

$$[\tilde{h}_{i+1,1}, h_{i,s}] = [[\tilde{h}_{i+1,1}, x^+_{i,s}], x^-_{i,0}] + [x^+_{i,s}, [\tilde{h}_{i+1,1}, x^-_{i,0}]] = 0$$

for all $s < k$. Thus, it is enough to show that $[\tilde{h}_{i,k}, h_{i+1,1}] = 0$ holds. By the definition of $h_{i+1,1}$, we obtain

$$\begin{aligned}[\tilde{h}_{i,k}, h_{i+1,1}] &= [\tilde{h}_{i,k}, [x^+_{i+1,1}, x^-_{i+1,0}]] \\ &= a_{i,i+1}[x^+_{i+1,k+1}, x^-_{i+1,0}] - a_{i,i+1}[x^+_{i+1,1}, x^-_{i+1,k}].\end{aligned} \tag{3.51}$$

By Lemma 3.37, it is equal to zero. □

Applying $\mathrm{ad}(\tilde{h}_{i+1,1})$ to $[x^+_{i,r}, x^-_{i,k-r}] = h_{i,k}$, we obtain

$$[\tilde{h}_{i+1,1}, [x^+_{i,r}, x^-_{i,k-r}]] = [\tilde{h}_{i+1,1}, h_{i,k}] \tag{3.52}$$

by the induction hypothesis. By Lemma 3.31(4), we can rewrite (3.52) as

$$a_{i,i+1}([x^+_{i,r+1}, x^-_{i,k-r}] - [x^+_{i,r}, x^-_{i,k-r+1}]) = [\tilde{h}_{i+1,1}, h_{i,k}] = 0. \tag{3.53}$$

It is nothing but the statement.

(3) We only show the statement for $+$. The other case is proven in a similar way. By (2), $[h_{i,r}, x^+_{i,0}]$ is equal to $[[x^+_{i,r}, x^-_{i,0}], x^+_{i,0}]$. By (1) and the Jacobi identity, we have

$$[[x^+_{i,r}, x^-_{i,0}], x^+_{i,0}] = [x^+_{i,r}, [x^-_{i,0}, x^+_{i,0}]]. \tag{3.54}$$

The right-hand side of (3.54) is equal to $[x^+_{i,r}, h_{i,0}]$. By Lemma 3.26(1), the right-hand side is equal to zero since $a_{i,i} = 0$.

(4) It is enough to check the equality $[h_{i,r}, x^\pm_{i,s}] = 0$. We only show the statement for $+$. The other case is proven in a similar way. We use proof by induction on s. When $s = 0$, it is nothing but (3). Suppose that $[h_{i,r}, x^+_{i,s}] = 0$ holds. Applying $\mathrm{ad}(\tilde{h}_{i+1,1})$ to $[h_{i,r}, x^+_{i,s}] = 0$, we find the equality

$$[\tilde{h}_{i+1,1}, [h_{i,r}, x^+_{i,s}]] = 0 \tag{3.55}$$

by the induction hypothesis. By the proof of (2), we obtain $[\tilde{h}_{i+1,1}, h_{i,n}] = 0$. Thus, the right-hand side of (3.55) is equal to $[h_{i,r}, [\tilde{h}_{i+1,1}, x^+_{i,s}]]$. By Lemma 3.31(4), we obtain

$$[h_{i,r}, [\tilde{h}_{i+1,1}, x^+_{i,s}]] = a_{i,i+1}\Big[h_{i,r}, \Big(x^+_{i,s+1} - \frac{\varepsilon_1 - \varepsilon_2}{2} b_{i+1,i} x^+_{i,s}\Big)\Big]. \tag{3.56}$$

By the induction hypothesis, we find that the right-hand side of (3.56) is equal to $a_{i,i+1}[h_{i,r}, x^+_{i,s+1}]$. Since $a_{i,i+1} \neq 0$, we obtain $[h_{i,r}, x^+_{i,s+1}] = 0$.

(5) By (2), $[h_{i,r}, h_{i,s}]$ is equal to $[h_{i,r}, [x^+_{i,s}, x^-_{i,0}]]$. By the Jacobi identity, we have

$$[h_{i,r}, [x^+_{i,s}, x^-_{i,0}]] = [[h_{i,r}, x^+_{i,s}], x^-_{i,0}] + [x^+_{i,s}, [h_{i,r}, x^-_{i,0}]].$$

By (4), the right-hand side is equal to zero. We have shown the relation

$$[h_{i,r}, h_{i,s}] = 0. \qquad \square$$

We obtain the relation (3.6) by Lemmas 3.31(2), 3.37(3) and 3.38(1). We also find that the relation (3.3) holds by Lemmas 3.31(2), 3.37(2) and 3.38(2).

In the same way as [17, Thm. 1.2], we obtain the defining relations (3.5), (3.2) and (3.7). Thus, we omit the proof.

Lemma 3.57.

(1) *The relations* (3.5) *and* (3.2) *hold in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i \neq j$.
(2) *The relation* (3.7) *holds for all* $i, j \in I$ *in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

We remark that the relation (3.2) holds by Lemmas 3.37(1), 3.38(5) and 3.57(1). We also find that the relation (3.5) holds by Lemmas 3.37(4), 3.38(4) and 3.57(1).

Now, it is enough to show that (3.8) and (3.9) are deduced from (3.17)–(3.25). However, we have already obtained (3.8), since (3.8) is equivalent to (3.6) when $i = j = 0, m$. Thus, to accomplish the proof, we only need to show that (3.9) holds.

Lemma 3.58. *The relation* (3.9) *holds for* $i = 0, m$ *in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

Proof. We prove it by induction on $k = r+s$. When $k = 0$, it is nothing but (3.25). Suppose that (3.25) holds for all r, s such that $r + s = k$. Applying $\mathrm{ad}(\tilde{h}_{i+2,1})$ to $[[x^\pm_{i-1,r}, x^\pm_{i,0}], [x^\pm_{i,0}, x^\pm_{i+1,s}]] = 0$, we obtain

$$a_{i-2,i-1}[[x^\pm_{i-1,r+1}, x^\pm_{i,0}], [x^\pm_{i,0}, x^\pm_{i+1,s}]] = 0.$$

Similarly, applying $\mathrm{ad}(\tilde{h}_{i+2,1})$ to $[[x^\pm_{i-1,r}, x^\pm_{i,0}], [x^\pm_{i,0}, x^\pm_{i+1,s}]] = 0$, we have

$$a_{i+2,i+1}[[x^\pm_{i-1,r}, x^\pm_{i,0}], [x^\pm_{i,0}, x^\pm_{i+1,s+1}]] = 0.$$

Thus, we have shown that (3.9) holds for all r, s such that $r + s = k + 1$. $\square$

This completes the proof of Theorem 3.16.

By Theorem 3.16, we also obtain the minimalistic presentation of $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$.

Theorem 3.59. *Suppose that $m,n\geq 2$ and $m\neq n$. Then $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ is isomorphic to the superalgebra generated by $x^+_{i,r}$, $x^-_{i,r}$, $h_{i,r}$ ($i\in\{0,1,\dots,m+n-1\}$, $r=0,1$) subject to the defining relations (3.17)–(3.25) and*

$$[d,h_{i,r}]=0,\quad [d,x^+_{i,r}]=\begin{cases}x^+_{i,r} & \text{if } i=0,\\ 0 & \text{if } i\neq 0,\end{cases}\quad [d,x^-_{i,r}]=\begin{cases}-x^-_{i,r} & \text{if } i=0,\\ 0 & \text{if } i\neq 0,\end{cases}\tag{3.60}$$

where the generators $x^\pm_{m,r}$ and $x^\pm_{0,r}$ are odd and all other generators are even.

The relation (3.12) is derived from (3.60) in a similar way to that of Lemma 3.26. We omit the proof.

§4. Coproduct for the affine super Yangian

In this section we define the coproduct for the affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$. We recall the definition of standard degreewise completion (see [18]).

Definition 4.1. Let $A=\bigoplus_{i\in\mathbb{Z}}A(i)$ be a graded algebra. For all $i\in\mathbb{Z}$, we set a topology on $A(i)$ such that for $a\in A(i)$ the set

$$\left\{a+\textstyle\sum_{r>N}A(i-r)\cdot A(r)\;\middle|\;N\in\mathbb{Z}_{\geq 0}\right\}$$

forms a fundamental system of open neighborhoods of a. The standard degreewise completion of A is $\bigoplus_{i\in\mathbb{Z}}\hat{A}(i)$, where $\hat{A}(i)$ is the completion of the space $A(i)$. By the definition of $\hat{A}(i)$, we find that

$$\hat{A}=\bigoplus_{i\in\mathbb{Z}}\varprojlim_{N}A(i)\Big/\sum_{r>N}A(i-r)\cdot A(r).$$

Let us set the degree on $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ determined by

$$\deg(h_{i,r})=0,\quad \deg(x^+_{i,r})=\begin{cases}1 & \text{if } i=0,\\ 0 & \text{if } i\neq 0,\end{cases}\quad \deg(x^-_{i,r})=\begin{cases}-1 & \text{if } i=0,\\ 0 & \text{if } i\neq 0.\end{cases}\tag{4.2}$$

Then $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ and $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))^{\otimes 2}$ become the graded algebra. We define $\widehat{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ (resp. $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))\mathbin{\widehat{\otimes}}Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$) as the standard degreewise completion of $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ (resp. $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))^{\otimes 2}$) in the sense of Definition 4.1.

We prepare some notation. There exists a homomorphism from $\widetilde{\mathfrak{sl}}(m|n)$ to $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ determined by $\Phi(h_i)=h_{i,0}$, $\Phi(x^\pm_i)=x^\pm_{i,0}$ and $\Phi(d)=d$. We sometimes denote $\Phi(x)$ by x in order to simplify the notation. In particular, we denote $\Phi(x^p_\alpha)$ by x^p_α for all $\alpha\in\Delta$. By Theorem 5.1, we note that $\dim(\Phi(\mathfrak{g}_\alpha))=1$ for all $\alpha\in\Delta_{\mathrm{re}}$.

Theorem 4.3. *The linear map*

$$\Delta\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) \to Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) \mathbin{\widehat{\otimes}} Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)),$$

uniquely determined by

$$\begin{aligned} &\Delta(h_{i,0}) = h_{i,0} \otimes 1 + 1 \otimes h_{i,0},\\ &\Delta(x^{\pm}_{i,0}) = x^{\pm}_{i,0} \otimes 1 + 1 \otimes x^{\pm}_{i,0},\\ &\Delta(h_{i,1}) = h_{i,1} \otimes 1 + 1 \otimes h_{i,1} + (\varepsilon_1+\varepsilon_2) h_{i,0} \otimes h_{i,0}\\ &\qquad - (\varepsilon_1+\varepsilon_2) \sum_{\alpha\in\Delta_+} \sum_{1\leq k_\alpha \leq \dim \mathfrak{g}_\alpha} (\alpha,\alpha_i) x^{k_\alpha}_{-\alpha} \otimes x^{k_\alpha}_{\alpha} \end{aligned} \tag{4.4}$$

is an algebra homomorphism. Moreover, Δ satisfies the coassociativity.

The rest of this section is devoted to the proof of Theorem 4.3. The outline of the proof is similar to that of [12, Thm. 4.9]. In [12], the analogy of the Drinfel'd J presentation is considered in order to prove the existence of the coproduct for the affine Yangian. We construct elements similar to those constructed in [12, equation (3.7)].

Definition 4.5. We set

$$J(h_i) = h_{i,1} + v_i, \quad J(x^{\pm}_i) = x^{\pm}_{i,1} + w^{\pm}_i,$$

where

$$\begin{aligned} v_i &= \frac{\varepsilon_1+\varepsilon_2}{2} \sum_{\alpha\in\Delta_+} \sum_{1\leq k_\alpha\leq \dim \mathfrak{g}_\alpha} (\alpha,\alpha_i) x^{k_\alpha}_{-\alpha} x^{k_\alpha}_{\alpha} - \frac{\varepsilon_1+\varepsilon_2}{2} h_i^2,\\ w^+_i &= -\frac{\varepsilon_1+\varepsilon_2}{2} \sum_{\alpha\in\Delta_+} \sum_{1\leq k_\alpha\leq \dim \mathfrak{g}_\alpha} [x^+_i, x^{k_\alpha}_{-\alpha}] x^{k_\alpha}_{\alpha},\\ w^-_i &= \frac{\varepsilon_1+\varepsilon_2}{2} \sum_{\alpha\in\Delta_+} \sum_{1\leq k_\alpha\leq \dim \mathfrak{g}_\alpha} x^{k_\alpha}_{-\alpha} [x^{k_\alpha}_{\alpha}, x^-_i]. \end{aligned}$$

Then $J(h_i)$ and $J(x^{\pm}_i)$ are elements of $\widehat{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

Next we prove similar results to [12, Lem. 3.9 and Prop. 3.21]. In fact, they are (4.8)–(4.11) and (4.27). We prepare one lemma in order to obtain (4.8)–(4.11) and (4.27). It is an analogy of [14, Prop. 2.4].

Lemma 4.6 ([19, Lem. 18.4.1]). *For all $\alpha,\beta\in\Delta_+$, we obtain*

$$\sum_{1\leq k_\beta\leq \dim \mathfrak{g}_\beta} [x^{k_\beta}_{\beta}, z] \otimes x^{k_\beta}_{-\beta} = \sum_{1\leq k_\alpha\leq \dim \mathfrak{g}_\alpha} x^{k_\alpha}_{\alpha} \otimes [z, x^{k_\alpha}_{-\alpha}]$$

if $z \in \mathfrak{g}_{\beta-\alpha}$.

Lemma 4.7. *The following relations hold:*

$$[J(h_i),h_j]=0, \tag{4.8}$$

$$[J(h_i),x_j^{\pm}]=\pm(\alpha_i,\alpha_j)J(x_j^{\pm})\mp a_{i,j}b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x_{j,0}^{\pm}, \tag{4.9}$$

$$[J(x_i^{\pm}),x_j^{\pm}]=[x_i^{\pm},J(x_j^{\pm})]-\frac{\varepsilon_1-\varepsilon_2}{2}b_{i,j}[x_{i,0}^{\pm},x_{j,0}^{\pm}], \tag{4.10}$$

$$[J(x_i^{\pm}),x_j^{\mp}]=[x_i^{\pm},J(x_j^{\mp})]=\delta_{i,j}J(h_i). \tag{4.11}$$

Proof. Since $h_{i,1}$ commutes with h_j by (3.2) and v_i commutes with h_j by the definition of v_i, we obtain (4.8). We only show the other relations hold for $+$. In a similar way, we obtain them for $-$. First, we prove (4.9) holds for $+$. By (3.21), the left-hand side of (4.9) is equal to

$$\begin{aligned}\Big[\tilde{h}_{i,1}+v_i+\frac{\varepsilon_1+\varepsilon_2}{2}h_{i,0}^2,x_j^{+}\Big]&=a_{i,j}\Big(x_{j,1}^{+}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x_{j,0}^{+}\Big)\\&\quad+\Big[\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha,\beta\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(\alpha,\alpha_i)x_{-\alpha}^{k_\alpha}x_{\alpha}^{k_\alpha},x_j^{+}\Big].\end{aligned} \tag{4.12}$$

By direct computation, the second term on the right-hand side of (4.12) is equal to

$$\begin{aligned}&\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(\alpha,\alpha_i)x_{-\alpha}^{k_\alpha}[x_{\alpha}^{k_\alpha},x_j^{+}]\\&\quad+\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)p(\alpha_j)}(\alpha,\alpha_i)[x_{-\alpha}^{k_\alpha},x_j^{+}]x_{\alpha}^{k_\alpha}.\end{aligned} \tag{4.13}$$

By Lemma 4.6, (4.13) is equal to

$$\begin{aligned}&\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(\alpha-\alpha_j,\alpha_i)[x_j^{+},x_{-\alpha}^{k_\alpha}]x_{\alpha}^{k_\alpha}\\&\quad+\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)p(\alpha_j)}(\alpha,\alpha_i)[x_{-\alpha}^{k_\alpha},x_j^{+}]x_{\alpha}^{k_\alpha}.\end{aligned} \tag{4.14}$$

Since $(-1)^{p(\alpha)p(\alpha_j)}[x_{-\alpha}^{k_\alpha},x_j^{+}]+[x_j^{+},x_{-\alpha}^{k_\alpha}]=0$ holds, the sum of the first and second terms of (4.14) is equal to $-\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(\alpha_j,\alpha_i)[x_j^{+},x_{-\alpha}^{k_\alpha}]x_{\alpha}^{k_\alpha}$. Thus, we obtain

$$\begin{aligned}[J(h_i),x_j^{+}]&=a_{i,j}\Big(x_{j,1}^{+}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x_{j,0}^{+}\Big)\\&\quad-\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(\alpha_j,\alpha_i)[x_j^{+},x_{-\alpha}^{k_\alpha}]x_{\alpha}^{k_\alpha}.\end{aligned}$$

Thus, we have obtained (4.9) for $+$.

Next we show that (4.10) holds for $+$. By the definition of $J(x_i^+)$, $[J(x_i^+), x_j^+] - [x_i^+, J(x_j^+)]$ is equal to

$$[x_{i,1}^+, x_{j,0}^+] - [x_{i,0}^+, x_{j,1}^+] + [w_i^+, x_j^+] - [x_i^+, w_j^+].$$

By (3.22), $[x_{i,1}^+, x_{j,0}^+] - [x_{i,0}^+, x_{j,1}^+]$ is equal to $\frac{\varepsilon_1+\varepsilon_2}{2} a_{i,j}\{x_{i,0}^+, x_{j,0}^+\} - \frac{\varepsilon_1-\varepsilon_2}{2} b_{i,j}[x_{i,0}^+, x_{j,0}^+]$. By the definition of w_i^+, we obtain

$$\begin{aligned}
&[w_i^+, x_j^+] - [x_i^+, w_j^+] \\
&\quad = -\frac{\varepsilon_1+\varepsilon_2}{2} \sum_{\alpha\in\Delta_+} \sum_{1\le k_\alpha \le \dim \mathfrak{g}_\alpha} [x_i^+, x_{-\alpha}^{k_\alpha}][x_\alpha^{k_\alpha}, x_j^+] \\
&\qquad - \frac{\varepsilon_1+\varepsilon_2}{2} \sum_{\alpha\in\Delta_+} \sum_{1\le k_\alpha \le \dim \mathfrak{g}_\alpha} (-1)^{p(\alpha)p(\alpha_j)} [[x_i^+, x_{-\alpha}^{k_\alpha}], x_j^+] x_\alpha^{k_\alpha} \\
&\qquad + \frac{\varepsilon_1+\varepsilon_2}{2} \sum_{\alpha\in\Delta_+} \sum_{1\le k_\alpha \le \dim \mathfrak{g}_\alpha} [x_i^+, [x_j^+, x_{-\alpha}^{k_\alpha}]] x_\alpha^{k_\alpha} \\
&\qquad + \frac{\varepsilon_1+\varepsilon_2}{2} \sum_{\alpha\in\Delta_+} \sum_{1\le k_\alpha \le \dim \mathfrak{g}_\alpha} (-1)^{p(\alpha)p(\alpha_i)+p(\alpha_j)p(\alpha_i)} [x_j^+, x_{-\alpha}^{k_\alpha}][x_i^+, x_\alpha^{k_\alpha}].
\end{aligned} \tag{4.15}$$

By Lemma 4.6, we find the equality

$$\begin{aligned}
&\text{the first term on the right-hand side of (4.15)} \\
&\quad = -\frac{\varepsilon_1+\varepsilon_2}{2} \sum_{\alpha\in\Delta_+} \sum_{1\le k_\alpha \le \dim \mathfrak{g}_\alpha} [x_i^+, [x_j^+, x_{-\alpha}^{k_\alpha}]] x_\alpha^{k_\alpha} + \frac{\varepsilon_1+\varepsilon_2}{2} [x_i^+, h_j] x_j^+.
\end{aligned} \tag{4.16}$$

We also find the relation

$$\begin{aligned}
&\text{the fourth term on the right-hand side of (4.15)} \\
&\quad = \frac{\varepsilon_1+\varepsilon_2}{2} \sum_{\alpha\in\Delta_+} \sum_{1\le k_\alpha \le \dim \mathfrak{g}_\alpha} (-1)^{p(\alpha)p(\alpha_i)+p(\alpha_j)p(\alpha_i)} [x_j^+, [x_{-\alpha}^{k_\alpha}, x_i^+]] x_\alpha^{k_\alpha} \\
&\qquad + \frac{\varepsilon_1+\varepsilon_2}{2} (-1)^{p(\alpha_i)p(\alpha_j)} [x_j^+, h_i] x_i^+
\end{aligned} \tag{4.17}$$

by Lemma 4.6. Applying (4.16) and (4.17) to (4.15), we obtain

$$[w_i^+, x_j^+] - [x_i^+, w_j^+] = \frac{\varepsilon_1+\varepsilon_2}{2} [x_i^+, h_j] x_j^+ + \frac{\varepsilon_1+\varepsilon_2}{2} (-1)^{p(\alpha_i)p(\alpha_j)} [x_j^+, h_i] x_i^+.$$

Since $m, n \ge 2$, there exist no i, j such that $a_{i,j} \neq 0$ and $p(\alpha_i)p(\alpha_j) = 1$. Thus, we obtain

$$\frac{\varepsilon_1+\varepsilon_2}{2} [x_i^+, h_j] x_j^+ + \frac{\varepsilon_1+\varepsilon_2}{2} (-1)^{p(\alpha_i)p(\alpha_j)} [x_j^+, h_i] x_i^+ = -\frac{\varepsilon_1+\varepsilon_2}{2} a_{i,j} \{x_i^+, x_j^+\}.$$

Hence, we have obtained

$$[J(x_i^+), x_j^+] - [x_i^+, J(x_j^+)] = -\frac{\varepsilon_1-\varepsilon_2}{2} b_{i,j} [x_{i,0}^+, x_{j,0}^+].$$

Finally, we show that $[J(x_i^+), x_j^-] = \delta_{i,j}J(h_i)$ holds. By the definition of $J(x_i^+)$, $[J(x_i^+), x_j^-]$ is equal to $[x_{i,1}^+, x_{j,0}^-] + [w_i^+, x_{j,0}^-]$. By (3.3), $[x_{i,1}^+, x_{j,0}^-]$ is $\delta_{i,j}h_{i,1}$. By direct computation, we have

$$\begin{aligned}[w_i^+, x_{j,0}^-] &= -\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\le k_\alpha\le \dim\mathfrak{g}_\alpha}[x_i^+, x_{-\alpha}^{k_\alpha}][x_\alpha^{k_\alpha}, x_j^-]\\ &\quad -\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\le k_\alpha\le \dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)p(\alpha_j)}[[x_i^+, x_{-\alpha}^{k_\alpha}], x_j^-]x_\alpha^{k_\alpha}.\end{aligned}\tag{4.18}$$

By Lemma 4.6, we have

$$\begin{aligned}&\text{the first term on the right-hand side of (4.18)}\\ &= -\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\le k_\alpha\le \dim\mathfrak{g}_\alpha}[x_i^+, [x_j^-, x_{-\alpha}^{k_\alpha}]]x_\alpha^{k_\alpha} - \frac{\varepsilon_1+\varepsilon_2}{2}\delta_{i,j}h_i^2.\end{aligned}\tag{4.19}$$

By the Jacobi identity, we find the equality

$$\begin{aligned}[x_i^+, [x_j^-, x_{-\alpha}^{k_\alpha}]] &= -(-1)^{p(\alpha)p(\alpha_j)}[x_i^+, [x_{-\alpha}^{k_\alpha}, x_j^-]]\\ &= -(-1)^{p(\alpha)p(\alpha_j)}[[x_i^+, x_{-\alpha}^{k_\alpha}], x_j^-]x_\alpha^{k_\alpha}\\ &\quad -(-1)^{p(\alpha)p(\alpha_j)}(-1)^{p(\alpha)p(\alpha_i)}[x_{-\alpha}^{k_\alpha}, [x_i^+, x_j^-]]x_\alpha^{k_\alpha}.\end{aligned}\tag{4.20}$$

Thus, we obtain

$$\begin{aligned}[w_i^+, x_{j,0}^-] &= \frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\le k_\alpha\le \dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)p(\alpha_j)}[[x_i^+, x_{-\alpha}^{k_\alpha}], x_j^-]x_\alpha^{k_\alpha}\\ &\quad +\frac{\varepsilon_1+\varepsilon_2}{2}\delta_{i,j}\sum_{\alpha\in\Delta_+}\sum_{1\le k_\alpha\le \dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)p(\alpha_j)}(-1)^{p(\alpha)p(\alpha_j)}\\ &\qquad\qquad \times[x_{-\alpha}^{k_\alpha}, [x_i^+, x_j^-]]x_\alpha^{k_\alpha} - \frac{\varepsilon_1+\varepsilon_2}{2}\delta_{i,j}h_i^2\\ &\quad -\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\le k_\alpha\le \dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)p(\alpha_j)}[[x_i^+, x_{-\alpha}^{k_\alpha}], x_j^-]x_\alpha^{k_\alpha}\\ &= \frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\le k_\alpha\le \dim\mathfrak{g}_\alpha}\delta_{i,j}(\alpha_i,\alpha)x_{-\alpha}^{k_\alpha}x_\alpha^{k_\alpha} - \frac{\varepsilon_1+\varepsilon_2}{2}\delta_{i,j}h_i^2,\end{aligned}$$

where the first equality is due to (4.19) and the second equality is due to (4.20). Then we have shown that $[J(x_i^+), x_i^-] = \delta_{i,j}J(h_i)$. Similarly, we can obtain $[x_i^+, J(x_j^-)] = \delta_{i,j}J(h_i)$. This completes the proof. □

By (4.8)–(4.11), we obtain the following convenient relation.

Corollary 4.21.

(1) *When* $i \neq j, j \pm 1$, $[J(x_i^{\pm}), x_j^{\pm}] = 0$ *holds.*

(2) *Suppose that* $j < i-1$. *We have the following relation:*

$$\begin{aligned}
&\mathrm{ad}(J(x_i^{\pm})) \prod_{i+1 \le k \le m+n-1} \mathrm{ad}(x_k^{\pm}) \prod_{0 \le k \le j-1} \mathrm{ad}(x_k^{\pm})(x_j^{\pm}) \\
&\quad = \mathrm{ad}(x_i^{\pm})\,\mathrm{ad}(J(x_{i+1}^{\pm})) \prod_{i+2 \le k \le m+n-1} \mathrm{ad}(x_k^{\pm}) \prod_{0 \le k \le j-1} \mathrm{ad}(x_k^{\pm})(x_j^{\pm}) \\
&\qquad - b_{i,i+1} \frac{\varepsilon_1 - \varepsilon_2}{2} \prod_{i \le k \le m+n-1} \mathrm{ad}(x_k^{\pm}) \prod_{0 \le k \le j-1} \mathrm{ad}(x_k^{\pm})(x_j^{\pm}).
\end{aligned}$$

(3) *For all* $\alpha \in \sum_{1 \le l \le m+n-1} \mathbb{Z}_{\ge 0} \alpha_i$ *and* $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$, *there exists a number* $d_{i,j}^{\alpha}$ *such that*

$$(\alpha_j, \alpha)[J(h_i), x_{\pm\alpha}] - (\alpha_i, \alpha)[J(h_j), x_{\pm\alpha}] = \pm d_{i,j}^{\alpha} x_{\pm\alpha}.$$

(4) *Suppose that* $j < i-1$. *We have*

$$\begin{aligned}
&\Big[J(h_s), \prod_{i \le k \le m+n-1} \mathrm{ad}(x_k^{\pm}) \prod_{0 \le k \le j-1} \mathrm{ad}(x_k^{\pm})(x_j^{\pm})\Big] \\
&\quad = \pm(\alpha_s, \alpha) \prod_{i \le k \le m+n-1} \mathrm{ad}(x_k^{\pm}) \prod_{0 \le k \le j-1} \mathrm{ad}(x_k^{\pm}) J(x_j^{\pm}) \\
&\qquad \pm c_2 \prod_{i \le k \le m+n-1} \mathrm{ad}(x_k^{\pm}) \prod_{0 \le k \le j-1} \mathrm{ad}(x_k^{\pm})(x_j^{\pm}),
\end{aligned}$$

where $\alpha = \sum_{i \le k \le m+n-1} \alpha_k + \sum_{0 \le k \le j} \alpha_k$ *and* c_2 *is a complex number.*

Proof. We only show the relations for $+$. The other case is proven in a similar way.

(1) By the definition of the commutator relations of $\widehat{\mathfrak{sl}}(m|n)$, $[x_i^+, x_j^+] = 0$ holds when $i \neq j, j \pm 1$. There exists an index p such that $a_{i,p} \neq 0$ and $a_{j,p} = 0$. Applying $\mathrm{ad}(J(h_p))$ to $[x_i^+, x_j^+] = 0$, we obtain

$$a_{i,p}([J(x_i^+), x_j^+] - b_{i,p} \frac{\varepsilon_1 - \varepsilon_2}{2} [x_i^+, x_j^+]) = 0$$

by (4.9). Since $a_{i,p} \neq 0$, we have shown that $[J(x_i^+), x_j^+] = 0$ holds.

(2) By (1), the left-hand side is equal to

$$\mathrm{ad}([J(x_i^+), x_{i+1}^+]) \prod_{i+2 \le k \le m+n-1} \mathrm{ad}(x_k^+) \prod_{0 \le k \le j-1} \mathrm{ad}(x_k^+)(x_j^+).$$

By (4.11), it is equal to

$$\begin{aligned}
&\mathrm{ad}([x_i^+, J(x_{i+1}^+)]) \prod_{i+2\le k\le m+n-1} \mathrm{ad}(x_k^+) \prod_{0\le k\le j-1} \mathrm{ad}(x_k^+)(x_j^+)\\
&\quad - b_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}\,\mathrm{ad}([x_i^+, x_{i+1}^+]) \prod_{i+2\le k\le m+n-1} \mathrm{ad}(x_k^+) \prod_{0\le k\le j-1} \mathrm{ad}(x_k^+)(x_j^+).
\end{aligned} \tag{4.22}$$

By the Jacobi identity, the first term of (4.22) is equal to

$$\mathrm{ad}(x_i^+)\,\mathrm{ad}(J(x_{i+1}^+)) \prod_{i+2\le k\le m+n-1} \mathrm{ad}(x_k^+) \prod_{0\le k\le j-1} \mathrm{ad}(x_k^+)(x_j^+).$$

This completes the proof.

(3) It is enough to assume that $x_{\pm\alpha} = \prod_{s\le k\le t-1}\mathrm{ad}(x_k^\pm)x_t^\pm$. By (4.9), we have

$$\begin{aligned}
[J(h_i), x_{\pm\alpha}] &= \pm\delta(s\ge i+1\ge t)a_{i,i+1} \prod_{s\le k\le i} \mathrm{ad}(x_k^\pm)J(x_{i+1}^\pm) \prod_{i+2\le k\le t-1} \mathrm{ad}(x_k^\pm)x_t^\pm\\
&\quad \pm \delta(s\ge i\ge t)a_{i,i} \prod_{s\le k\le i-1} \mathrm{ad}(x_k^\pm)J(x_i^\pm) \prod_{i+1\le k\le t-1} \mathrm{ad}(x_k^\pm)x_t^\pm\\
&\quad \pm \delta(s\ge i-1\ge t)a_{i,i-1} \prod_{s\le k\le i-2} \mathrm{ad}(x_k^\pm)J(x_{i-1}^\pm) \prod_{i+1\le k\le t-1} \mathrm{ad}(x_k^\pm)x_t^\pm\\
&\quad \pm d_i^1(\alpha_i,\alpha) \prod_{s\le k\le t-1} \mathrm{ad}(x_k^\pm)x_t^\pm,
\end{aligned}$$

where d_i^1 is a complex number. By a discussion similar to the one in the proof of (2), we find that there exists a complex number d_i^2 such that the sum of the first three terms is equal to

$$\pm(\alpha_i,\alpha) \prod_{s\le k\le t-1} \mathrm{ad}(x_k^\pm)J(x_t^\pm) \pm d_i^2(\alpha_i,\alpha) \prod_{s\le k\le t-1} \mathrm{ad}(x_k^\pm)x_t^\pm.$$

Then we obtain

$$\begin{aligned}
&(\alpha_j,\alpha)[J(h_i), x_{\pm\alpha}] - (\alpha_k,\alpha)[J(h_j), x_{\pm\alpha}]\\
&\qquad = \pm\{(\alpha_j,\alpha)(d_i^1+d_i^2) - (\alpha_i,\alpha)(d_j^1+d_j^2)\}x_{\pm\alpha}.
\end{aligned}$$

We complete the proof.

(4) It is proven in a similar way to (3). □

Next, in order to obtain (4.27), we prepare $\{\tau_i\}_{i\neq 0,m}$, which are automorphisms of the affine super Yangian. Let us set $\{s_i\}_{i\neq 0,m}$ as an automorphism of

Δ such that $s_i(\alpha) = \alpha - \frac{2(\alpha_i,\alpha)}{(\alpha_i,\alpha_i)}\alpha_i$. By the definition of $\widehat{\mathfrak{sl}}(m|n)$, we can rewrite s_i explicitly as

$$s_i(\alpha_j) = \begin{cases} -\alpha_j & \text{if } i = j, \\ \alpha_i + \alpha_j & \text{if } j = i \pm 1, \\ \alpha_j & \text{otherwise.} \end{cases}$$

It is called a simple reflection. We also define $\{\tau_i\}_{i\neq 0,m}$ as an operator on the affine super Yangian determined by

$$\tau_i(x) = \exp(\mathrm{ad}(x_i^+))\exp(-\mathrm{ad}(x_i^-))\exp(\mathrm{ad}(x_i^+))x. \tag{4.23}$$

By the defining relation (3.7), τ_i is well defined as an operator on the affine super Yangian. The following lemma is well known (see [14]).

Lemma 4.24.

(1) *The action of τ_i preserves the inner product κ.*
(2) *For all $\alpha \in \Delta$, $\tau_i(\mathfrak{g}_\alpha) = \mathfrak{g}_{s_i(\alpha)}$.*

Then, in a similar way to of [12, Lems. 3.17 and 3.19], we can compute the action of τ_i on $J(h_j)$ and write it explicitly.

Lemma 4.25. *When $i \neq 0, m$, we obtain*

$$\tau_i(J(h_j)) = J(h_j) - \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}J(h_i) + a_{i,j}b_{j,i}(\varepsilon_1 - \varepsilon_2)h_i.$$

Since $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta^{\mathrm{re}}$, we sometimes denote $x_{-\alpha}^{k_\alpha}$ and $x_\alpha^{k_\alpha}$ as $x_{-\alpha}$ and x_α for all $\alpha \in \Delta_+^{\mathrm{re}}$.

Proposition 4.26. *For $i, j \in I$ and a positive real root α, the following equation holds:*

$$(\alpha_j,\alpha)[J(h_i), x_\alpha] - (\alpha_i,\alpha)[J(h_j), x_\alpha] = c_{i,j}^\alpha x_\alpha, \tag{4.27}$$

where $c_{i,j}^\alpha$ is a complex number such that $c_{i,j}^\alpha = -c_{i,j}^{-\alpha}$.

Proof. We divide the proof into two cases: one is that α is even, the other is that α is odd.

Case 1, α is even. Suppose that α is even. Then there exists $s \in \mathbb{Z}$ such that α is an element of $\sum_{1\leq l\leq m-1}\mathbb{Z}\alpha_i + s\delta$ or $\sum_{m+1\leq l\leq m+n-1}\mathbb{Z}\alpha_i + s\delta$. We only prove the case where $\alpha \in \sum_{1\leq l\leq m-1}\mathbb{Z}_{\geq 0}\alpha_i + \mathbb{Z}_{\geq 0}\delta$. The other cases are proven in a similar way.

First, we prove the case where $\alpha = \alpha_k + s\delta$, where $k \neq 0, m$.

Claim 4.28. Suppose that $\alpha=\alpha_k+s\delta$ such that $k\neq 0,m$. Then we have

$$[J(h_i),x_\alpha]=\frac{(\alpha_i,\alpha_k)}{(\alpha_k,\alpha_k)}[J(h_k),x_\alpha]+d_\alpha x_\alpha,$$
$$[J(h_i),x_{-\alpha}]=\frac{(\alpha_i,\alpha_k)}{(\alpha_k,\alpha_k)}[J(h_k),x_{-\alpha}]-d_\alpha x_{-\alpha},$$

where d_α is a complex number.

Proof. Let us set

$$x^1_{\pm\delta}=\Big[x^\pm_k,\prod_{k+1\leq p\leq m+n-1}\mathrm{ad}(x^\pm_p)\prod_{0\leq p\leq k-2}\mathrm{ad}(x^\pm_p)(x^\pm_{k-1})\Big].$$

It is enough to suppose that $x_{\pm\alpha}=\mathrm{ad}(x^1_{\pm\delta})^s x^\pm_k$ since $\mathrm{ad}(x^1_{\pm\delta})^s x^\pm_k$ is nonzero. By the Jacobi identity, we obtain

$$\begin{aligned}&[(\alpha_k,\alpha)J(h_i)-(\alpha_i,\alpha)J(h_k),\mathrm{ad}(x^1_{\pm\delta})^s x^\pm_k]\\&=\sum_{0\leq t\leq s-1}\mathrm{ad}(x^1_{\pm\delta})^t\,\mathrm{ad}([(\alpha_k,\alpha)J(h_i)-(\alpha_i,\alpha)J(h_k),x^1_{\pm\delta}])\,\mathrm{ad}(x^1_{\pm\delta})^{s-1-t}x^\pm_k\\&\quad+\mathrm{ad}(x^1_{\pm\delta})^s[(\alpha_k,\alpha)J(h_i)-(\alpha_i,\alpha)J(h_k),x^\pm_k].\end{aligned}\tag{4.29}$$

By (4.9), $[(\alpha_k,\alpha)J(h_i)-(\alpha_i,\alpha)J(h_k),x^\pm_k]$ can be written as $\pm f_k x^\pm_k$, where f_k is a complex number. Then we have

$$\begin{aligned}[J(h_i),x^1_{\pm\delta}]=\Big[x^\pm_k,\Big[&(\alpha_k,\alpha)J(h_i)-(\alpha_i,\alpha)J(h_k),\\&\prod_{k+1\leq p\leq m+n-1}\mathrm{ad}(x^\pm_p)\prod_{0\leq p\leq k-2}\mathrm{ad}(x^\pm_p)(x^\pm_{k-1})\Big]\Big]\pm f_k x^1_{\pm\delta}.\end{aligned}$$

By Corollary 4.21(4), we can rewrite the first term as

$$\begin{aligned}&\pm(\alpha_k,\alpha)(\alpha_i,\alpha)\Big[x^\pm_k,\prod_{k+1\leq p\leq m+n-1}\mathrm{ad}(x^\pm_p)\prod_{0\leq p\leq k-2}\mathrm{ad}(x^\pm_p)J(x^\pm_{k-1})\Big]\\&\qquad\mp(\alpha_k,\alpha)(\alpha_i,\alpha)\Big[x^\pm_k,\prod_{k+1\leq p\leq m+n-1}\mathrm{ad}(x^\pm_p)\prod_{0\leq p\leq k-2}\mathrm{ad}(x^\pm_p)J(x^\pm_{k-1})\Big]\pm g_k x^1_{\pm\delta}\\&=\pm g_k x^1_{\pm\delta},\end{aligned}$$

where g_k is a complex number. We have obtained the statement. □

Now let us consider the case where α is a general even root. Any even root $\alpha=\sum_{0\leq k\leq l}\alpha_{p+k}$ can be written as $\prod_{0\leq k\leq l-1}s_{p+k}(\alpha_{p+l})$ by the explicit presentation of s_i. Let us prove that the statement of Proposition 4.26 holds by induction on l.

When $l=1$, it is nothing but Claim 4.28. Assume that (4.27) holds when $l=q$. We set α and β as $\prod_{0\leq k\leq q} s_{p+k}(\alpha_{p+q+1})$ and $\prod_{1\leq k\leq q} s_{p+k}(\alpha_{p+q+1})$. Suppose that x_β is a nonzero element of $\mathfrak{g}_\beta$. By Lemma 4.24, $\mathfrak{g}_\alpha$ contains a nonzero element $\tau_{s_p}(x_\beta)$. Thus, we obtain

$$\begin{aligned}
&(\alpha_j,\alpha)[J(h_i),\tau_{s_p}(x_{\pm\beta})]-(\alpha_i,\alpha)[J(h_j),\tau_{s_p}(x_{\pm\beta})]\\
&=\tau_{s_p}\Big\{(\alpha_j,\alpha)\Big[J(h_i)-\frac{2(\alpha_i,\alpha_p)}{(\alpha_p,\alpha_p)}J(h_p),x_{\pm\beta}\Big]\\
&\qquad -(\alpha_i,\alpha)\Big[J(h_j)-\frac{2(\alpha_i,\alpha_p)}{(\alpha_p,\alpha_p)}J(h_p),(x_{\pm\beta})\Big]\Big\}\\
&\mp\{(\alpha_j,\alpha)a_{p,i}b_{i,p}-(\alpha_i,\alpha)a_{p,j}b_{j,p}\}(\varepsilon_1-\varepsilon_2)x_{\pm\alpha}
\end{aligned}\tag{4.30}$$

by Lemma 4.25. Let us suppose that $(\alpha_t,\beta)\neq 0$. Then, by the induction hypothesis, we find the relation

$$[J(h_u),x_{\pm\beta}]=\pm\frac{(\alpha_u,\beta)}{(\alpha_t,\beta)}[J(h_t),x_{\pm\beta}]\pm c^{\beta}_{u,t}x_{\pm\beta}.\tag{4.31}$$

Applying (4.31) to (4.30), we obtain

$$\begin{aligned}
&\Big[J(h_i)-\frac{2(\alpha_i,\alpha_p)}{(\alpha_p,\alpha_p)}J(h_p),x_{\pm\beta}\Big]\\
&=\pm\Big\{\frac{(\alpha_i,\beta)}{(\alpha_t,\beta)}-\frac{2(\alpha_i,\alpha_p)}{(\alpha_p,\alpha_p)}\cdot\frac{(\alpha_p,\beta)}{(\alpha_t,\beta)}\Big\}([J(h_t),x_{\pm\beta}]+c^{\beta}_{i,t}x_{\pm\beta})\\
&=\pm\frac{(\alpha_i,\beta-\frac{2(\alpha_i,\alpha_p)}{(\alpha_p,\alpha_p)}\alpha_p)}{(\alpha_t,\beta)}([J(h_t),x_{\pm\beta}]+c^{\beta}_{i,t}x_{\pm\beta}).
\end{aligned}$$

By the definition of s_p, α is equal to $\beta-\frac{2(\alpha_i,\alpha_p)}{(\alpha_p,\alpha_p)}\alpha_p$. Then we have

$$\Big[J(h_i)-\frac{2(\alpha_i,\alpha_p)}{(\alpha_p,\alpha_p)}J(h_p),x_{\pm\beta}\Big]=\pm\frac{(\alpha,\alpha_i)}{(\alpha_t,\beta)}([J(h_t),x_{\pm\beta}]+c^{\beta}_{i,t}x_{\pm\beta}).\tag{4.32}$$

Similarly, we find the relation

$$\Big[J(h_j)-\frac{2(\alpha_j,\alpha_p)}{(\alpha_p,\alpha_p)}J(h_p),x_{\pm\beta}\Big]=\pm\frac{(\alpha,\alpha_j)}{(\alpha_t,\beta)}([J(h_t),x_{\pm\beta}]+c^{\beta}_{j,t}x_{\pm\beta}).\tag{4.33}$$

Applying (4.32) and (4.33) to the right-hand side of (4.30),

$$\begin{aligned}
&(\alpha_j,\alpha)[J(h_i),\tau_{s_p}(x_{\pm\beta})]-(\alpha_i,\alpha)[J(h_j),\tau_{s_p}(x_{\pm\beta})]\\
&=\pm\tau_{s_p}\Big\{(\alpha_j,\alpha)\frac{(\alpha,\alpha_i)}{(\alpha_t,\beta)}c^{\beta}_{i,t}x_{\pm\beta}-(\alpha_i,\alpha)\frac{(\alpha,\alpha_j)}{(\alpha_t,\beta)}c^{\beta}_{j,t}x_{\pm\beta}\Big\}\\
&\mp\{(\alpha_j,\alpha)a_{p,i}b_{i,p}-(\alpha_i,\alpha)a_{p,j}b_{j,p}\}(\varepsilon_1-\varepsilon_2)x_{\pm\alpha}.
\end{aligned}$$

This completes the proof of the case where α is even.

Case 2, α is odd. Hereafter, we suppose that m is greater than 3. The other case is proven in a similar way. First, we consider the case where $\alpha = \sum_{1\leq l\leq m-1}\alpha_i+\alpha_m+s\delta$.

Claim 4.34.

(1) When $i\neq 0,1,m,m+1$, $[J(h_i),x_{\pm\alpha}]=\pm c^i_\alpha x_{\pm\alpha}$, where c_α is a complex number.

(2) We obtain the following equations:

$$[J(h_0),x_{\pm\alpha}]=\frac{(\alpha_0,\alpha)}{(\alpha_1,\alpha)}[J(h_1),x_{\pm\alpha}]\pm d_{0,1}x_{\pm\alpha}, \tag{4.35}$$

$$[J(h_m),x_{\pm\alpha}]=\frac{(\alpha_m,\alpha)}{(\alpha_1,\alpha)}[J(h_1),x_{\pm\alpha}]\pm d_{m,1}x_{\pm\alpha}, \tag{4.36}$$

$$[J(h_{m+1}),x_{\pm\alpha}]=\frac{(\alpha_{m+1},\alpha)}{(\alpha_m,\alpha)}[J(h_m),x_{\pm\alpha}]\pm d_{m,m+1}x_{\pm\alpha}, \tag{4.37}$$

where $d_{0,1}$, $d_{m,1}$ and $d_{m,m+1}$ are complex numbers.

Proof. (1) When $i\neq 0,1,2,m,m+1$, we set $x^2_{\pm\delta}=[x^\pm_1,\prod_{2\leq p\leq m+n-1}\mathrm{ad}(x^\pm_p)(x^\pm_0)]$. It is sufficient to assume that

$$x_{\pm\alpha}=\mathrm{ad}(x^2_{\pm\delta})^s\sum_{1\leq l\leq m-1}\mathrm{ad}(x^\pm_i)(x^\pm_m)$$

since the right-hand side is nonzero. In a similar way to Claim 4.28, we also have

$$[J(h_i),x^2_{\pm\delta}]=\pm h_\delta x^2_{\pm\delta}, \tag{4.38}$$

$$\Bigg[J(h_i),\sum_{1\leq l\leq m-1}\mathrm{ad}(x^\pm_i)(x^\pm_m)\Bigg]=\pm i_\alpha\sum_{1\leq l\leq m-1}\mathrm{ad}(x^\pm_i)(x^\pm_m), \tag{4.39}$$

where h_δ and i_α are complex numbers. Thus, we find the equality

$$[J(h_i),x^{k_\alpha}_\alpha]=\pm(sh_\delta+i_\alpha)\,\mathrm{ad}(x^2_{\pm\delta})^s\sum_{1\leq l\leq m-1}\mathrm{ad}(x^\pm_i)(x^\pm_m)$$

by the Jacobi identity, (4.38) and (4.39). We have proved the statement when $i\neq 0,1,2,m,m+1$. When $i=2$, we set $x^3_{\pm\delta}$ as

$$\Bigg[x^\pm_{m+1},\prod_{m+2\leq p\leq m+n-1}\mathrm{ad}(x^\pm_p)\prod_{0\leq p\leq m-1}\mathrm{ad}(x^\pm_p)\,\mathrm{ad}(x^\pm_m)\Bigg].$$

It is enough to assume that

$$x_{\pm\alpha}=\mathrm{ad}(x^3_{\pm\delta})^s\sum_{1\leq l\leq m-1}\mathrm{ad}(x^\pm_i)(x^\pm_m)$$

since the right-hand side is nonzero. In a similar way to Claim 4.28, we also have

$$[J(h_i), x^3_{\pm\delta}] = \pm j_\delta x^3_{\pm\delta}, \tag{4.40}$$

$$\left[J(h_i), \sum_{1\leq l\leq m-1} \mathrm{ad}(x_i^{\pm})(x_m^{\pm})\right] = \pm k_\alpha \sum_{1\leq l\leq m-1} \mathrm{ad}(x_i^{\pm})(x_m^{\pm}), \tag{4.41}$$

where j_δ and k_α are complex numbers. Thus, we find the relation

$$[J(h_i), x_\alpha] = \pm(sj_\delta + k_\alpha)\,\mathrm{ad}(x^2_{\pm\delta})^s \sum_{1\leq l\leq m-1} \mathrm{ad}(x_i^{\pm})(x_m^{\pm})$$

by the Jacobi identity, (4.40) and (4.41). We have proved the statement when $i = 2$.

(2) First, we prove that (4.35) holds. By the definition of α, $x_{\pm\alpha}$ can be written as $[x_{\pm\beta}, x_m^{\pm}]$, where $x_{\pm\beta}$ is a nonzero element of $\mathfrak{g}_{\alpha-\alpha_m}$. Since $[J(h_0), x_m^{\pm}]$ and $[J(h_1), x_m^{\pm}]$ are equal to zero by (4.9), we obtain

$$[J(h_0), x_{\pm\alpha}] = [[J(h_0), x_{\pm\beta}], x_m^{\pm}], \tag{4.42}$$

$$[J(h_1), x_{\pm\alpha}] = [[J(h_1), x_{\pm\beta}], x_m^{\pm}]. \tag{4.43}$$

Then, because β is even, we have

$$[[J(h_0), x_{\pm\beta}], x_m^{\pm}] = \frac{(\alpha_0, \beta)}{(\alpha_1, \beta)}[[J(h_1), x_{\pm\beta}], x_m^{\pm}] + \frac{(\alpha_0, \beta)}{(\alpha_1, \beta)}[x_{\pm\beta}, x_m^{\pm}] \tag{4.44}$$

by Case 1. By (4.42), (4.43) and (4.44), we find the equality

$$[J(h_0), x_{\pm\alpha}] = \frac{(\alpha_0, \beta)}{(\alpha_1, \beta)}[J(h_1), x_{\pm\alpha}] + \frac{(\alpha_0, \beta)}{(\alpha_1, \beta)}[x_{\pm\beta}, x_m^{\pm}].$$

Thus we have shown that (4.35) holds. Similarly, we obtain (4.36) since also $[J(h_m), x_m^{+}] = 0$ holds.

Finally, we prove that (4.37) holds. We set

$$x^4_{\pm\delta} = \left[x_1^{\pm}, \prod_{2\leq p\leq m+n-1} \mathrm{ad}(x_p^{\pm})(x_0^{\pm})\right].$$

It is enough to check the relation under the assumption that

$$x_{\pm\alpha} = \mathrm{ad}(x^4_{\pm\delta})^s \prod_{1\leq p\leq m-1} \mathrm{ad}(x_p^{\pm})(x_m^{\pm})$$

since the right-hand side is nonzero. Then we obtain

$$
\begin{aligned}
&[J(h_m), x_{\pm\alpha}]\\
&\quad = \sum_{1\leq t\leq s} \mathrm{ad}(x^4_{\pm\delta})^{t-1}\,\mathrm{ad}([J(h_m), x^4_{\pm\delta}])\,\mathrm{ad}(x^4_{\pm\delta})^{s-t} \prod_{1\leq p\leq m-1} \mathrm{ad}(x^{\pm}_p)(x^{\pm}_m)\\
&\qquad + \Big[J(h_m), \mathrm{ad}(x^4_{\pm\delta})^s \prod_{1\leq p\leq m-1} \mathrm{ad}(x^{\pm}_p)(x^{\pm}_m)\Big],
\end{aligned}
\tag{4.45}
$$

$$
\begin{aligned}
&[J(h_{m+1}), x_{\pm\alpha}]\\
&\quad = \sum_{1\leq t\leq s} \mathrm{ad}(x^4_{\pm\delta})^{t-1}\,\mathrm{ad}([J(h_{m+1}), x^4_{\pm\delta}])\,\mathrm{ad}(x^4_{\pm\delta})^{s-t} \prod_{1\leq p\leq m-1} \mathrm{ad}(x^{\pm}_p)(x^{\pm}_m)\\
&\qquad + \Big[J(h_{m+1}), \mathrm{ad}(x^4_{\pm\delta})^s \prod_{1\leq p\leq m-1} \mathrm{ad}(x^{\pm}_p)(x^{\pm}_m)\Big]
\end{aligned}
\tag{4.46}
$$

by the Jacobi identity. First, we rewrite the first term on the right-hand sides of (4.45) and (4.46). By the assumption m is greater than 3, $[J(h_m), x^{\pm}_1] = 0$ holds by (4.9). Then, in a similar way to Claim 4.28, we find the equalities

$$
[J(h_m), x^4_{\pm\delta}] = \pm t_\delta x^4_{\pm\delta}, \tag{4.47}
$$

$$
[J(h_{m+1}), x^4_{\pm\delta}] = \pm u_\delta x^4_{\pm\delta}, \tag{4.48}
$$

where t_δ and u_δ are complex numbers. Then we obtain

$$
\begin{aligned}
&\text{the first term on the right-hand side of (4.45)}\\
&\quad = \pm t_\delta\, \mathrm{ad}(x^4_{\pm\delta})^s \prod_{1\leq p\leq m-1} \mathrm{ad}(x^{\pm}_p)(x^{\pm}_m),
\end{aligned}
\tag{4.49}
$$

$$
\begin{aligned}
&\text{the first term on the right-hand side of (4.46)}\\
&\quad = \pm u_\delta\, \mathrm{ad}(x^4_{\pm\delta})^s \prod_{1\leq p\leq m-1} \mathrm{ad}(x^{\pm}_p)(x^{\pm}_m)
\end{aligned}
\tag{4.50}
$$

by (4.47) and (4.48). Next we rewrite the second term on the right-hand sides of (4.45) and (4.46). By (4.9), we obtain

$$
\begin{aligned}
&\text{the second term on the right-hand side of (4.45)}\\
&\quad = \mathrm{ad}(x^4_{\pm\delta})^s \prod_{1\leq p\leq m-2} \mathrm{ad}(x^{\pm}_p)[J(h_m), [x^{\pm}_{m-1}, x^{\pm}_m]],
\end{aligned}
\tag{4.51}
$$

$$
\begin{aligned}
&\text{the second term on the right-hand side of (4.46)}\\
&\quad = \mathrm{ad}(x^4_{\pm\delta})^s \prod_{1\leq p\leq m-2} \mathrm{ad}(x^{\pm}_p)[J(h_{m+1}), [x^{\pm}_{m-1}, x^{\pm}_m]].
\end{aligned}
\tag{4.52}
$$

By (4.9) and (4.10), we find that

$$
\begin{aligned}
&[J(h_m),[x^\pm_{m-1},x^\pm_m]]\\
&\quad= \pm a_{m,m-1}[J(x^\pm_{m-1}),x^\pm_m] \mp a_{m,m-1}b_{m,m-1}\frac{\varepsilon_1-\varepsilon_2}{2}[x^\pm_{m-1},x^\pm_m]\\
&\quad= \pm a_{m,m-1}[x^\pm_{m-1},J(x^\pm_m)]\\
&\qquad \mp a_{m,m-1}(b_{m-1,m}+b_{m,m-1})\frac{\varepsilon_1-\varepsilon_2}{2}[x^\pm_{m-1},x^\pm_m], \qquad (4.53)\\
&[J(h_{m+1}),[x^\pm_{m-1},x^\pm_m]]\\
&\quad= \pm a_{m+1,m}[x^\pm_{m-1},J(x^\pm_m)] \mp a_{m+1,m}b_{m,m+1}\frac{\varepsilon_1-\varepsilon_2}{2}[x^\pm_{m-1},x^\pm_m]. \qquad (4.54)
\end{aligned}
$$

Since $a_{m,m-1}=(\alpha,\alpha_m)$ and $a_{m+1,m}=(\alpha,\alpha_{m+1})$, by (4.53) and (4.54), we obtain

$$
\begin{aligned}
&(\alpha,\alpha_{m+1})[J(h_m),[x^\pm_{m-1},x^\pm_m]]-(\alpha,\alpha_m)[J(h_{m+1}),[x^\pm_{m-1},x^\pm_m]]\\
&\quad= \pm u_\alpha[x^\pm_{m-1},x^\pm_m], \qquad (4.55)
\end{aligned}
$$

where u_α is a complex number. Thus, we know that

$$
\begin{aligned}
&(\alpha,\alpha_{m+1})(\text{the second term on the right-hand side of (4.45)})\\
&\qquad -(\alpha,\alpha_m)(\text{the second term on the right-hand side of (4.46)})\\
&\quad= u_\alpha\,\mathrm{ad}(x^4_{\pm\delta})^s\prod_{1\leq p\leq m-1}\mathrm{ad}(x^\pm_p)(x^\pm_m) \qquad (4.56)
\end{aligned}
$$

holds. By (4.49), (4.50) and (4.56), we have

$$
\begin{aligned}
&(\alpha,\alpha_{m+1})[J(h_m),x_{\pm\alpha}]-(\alpha,\alpha_m)[J(h_{m+1}),x_{\pm\alpha}]\\
&\quad= \pm(s(\alpha,\alpha_{m+1})t_\delta - s(\alpha,\alpha_m)u_\delta + u_\alpha)\,\mathrm{ad}(x^4_{\pm\delta})^s\prod_{1\leq p\leq m-1}\mathrm{ad}(x^\pm_p)(x^\pm_m).
\end{aligned}
$$

Then we have obtained (4.37). □

Next, let us consider the case where α is a general odd root. We only show the case where $\alpha\in\alpha_m+\sum_{1\leq t\leq m+n-1,t\neq m}\mathbb{Z}_{\geq 0}\alpha_t+s\delta$. The other case is proven in a similar way.

Since $\alpha\in\alpha_m+\sum_{1\leq t\leq m+n-1,t\neq m}\mathbb{Z}_{\geq 0}\alpha_t+s\delta$, the root α can be written as $\prod_{1\leq t\leq p}s_{i_t}(\sum_{1\leq i\leq m}\alpha_i+\alpha_m)$. Then we prove the statement by induction on p. When $p=0$, it is nothing but Claim 4.34. Other cases are proven in a similar way to Case 1. □

We easily obtain the following corollary.

Corollary 4.57. *The following equations hold:*

$$[J(h_i),\widetilde{v_j}]+[\widetilde{v_i},J(h_j)]=0, \tag{4.58}$$

$$[J(h_i),J(h_j)]+[\widetilde{v_i},\widetilde{v_j}]=0, \tag{4.59}$$

where $\widetilde{v_i}=v_i+\frac{\varepsilon_1+\varepsilon_2}{2}{h_i}^2$.

Proof. First, we show that (4.58) holds. Since $\widetilde{v_i}=\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha_j,\alpha)x_{-\alpha}^{k_\alpha}x_\alpha^{k_\alpha}$ holds, we obtain

$$\begin{aligned}[J(h_i),\widetilde{v_j}]+[\widetilde{v_i},J(h_j)]&=\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha_j,\alpha)[J(h_i),x_{-\alpha}]x_\alpha\\&\quad+\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha_j,\alpha)x_{-\alpha}[J(h_i),x_\alpha]\\&\quad+\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha_i,\alpha)[x_{-\alpha},J(h_j)]x_\alpha\\&\quad+\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha_i,\alpha)x_{-\alpha}[x_\alpha,J(h_j)].\end{aligned} \tag{4.60}$$

By Proposition 4.26, there exists $c_{i,j}^\alpha\in\mathbb{C}$ such that

$$\begin{aligned}&\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha_j,\alpha)[J(h_i),x_{-\alpha}]x_\alpha+\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha_i,\alpha)[x_{-\alpha},J(h_j)]x_\alpha\\&\qquad=-\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+^{\mathrm{re}}}c_{i,j}^\alpha x_{-\alpha}x_\alpha,\end{aligned} \tag{4.61}$$

$$\begin{aligned}&\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha_j,\alpha)x_{-\alpha}[J(h_i),x_\alpha]+\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha_i,\alpha)x_{-\alpha}[x_\alpha,J(h_j)]\\&\qquad=\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+^{\mathrm{re}}}c_{i,j}^\alpha x_{-\alpha}x_\alpha.\end{aligned} \tag{4.62}$$

Therefore, applying (4.61) and (4.62) to (4.60), we have obtained the relation (4.58). By the defining relation (3.3), we find the equality

$$[J(h_i)-\widetilde{v_i},J(h_j)-\widetilde{v_j}]=[h_{i,1},h_{j,1}]=0. \tag{4.63}$$

On the other hand, we find the relation

$$[J(h_i)-\widetilde{v_i},J(h_j)-\widetilde{v_j}]=[J(h_i),J(h_j)]-[\widetilde{v_i},J(h_j)]-[J(h_i),\widetilde{v_j}]+[\widetilde{v_i},\widetilde{v_j}]. \tag{4.64}$$

By (4.58), the right-hand side of (4.64) is equal to the left-hand side of (4.59). Thus, by (4.63), we have found that (4.59) holds. □

Now we are in position to obtain the proof of Theorem 4.3. To simplify the notation, we set $\square(x)$ to $x\otimes 1+1\otimes x$ for all $x\in Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

Proof of Theorem 4.3. It is enough to check that Δ is compatible with (3.17)–(3.25), which are the defining relations of the minimalistic presentation of the affine super Yangian. Since the restriction of Δ to $\widehat{\mathfrak{sl}}(m|n)$ is nothing but the usual coproduct of $\widehat{\mathfrak{sl}}(m|n)$, Δ is compatible with (3.18), (3.23), (3.24) and (3.25). We also know that Δ is compatible with (3.20) since $\Delta(x^{\pm}_{i,1})$ is defined as

$$\begin{cases}\pm\dfrac{1}{a_{i,i}}[\Delta(\tilde{h}_{i,1}),\Delta(x^{\pm}_{i,0})] & \text{if } i\neq m,0,\\ \pm\dfrac{1}{a_{i+1,i}}[\Delta(\tilde{h}_{i+1,1}),\Delta(x^{\pm}_{i,0})]+b_{i+1,i}\dfrac{\varepsilon_1-\varepsilon_2}{2}\Delta(x^{\pm}_{i,0}) & \text{if } i=m,0,\end{cases}$$

and $\Delta(\tilde{h}_{i+1,1})$ and $\Delta(\tilde{h}_{i,1})$ commute with $\Delta(h_{j,0})$ by the definition. We find that the defining relation (3.19) (resp. (3.21), (3.22)) is equivalent to (4.11) (resp. (4.9), (4.10)) by the proof of Lemma 4.7. It is easy to show that Δ is compatible with (4.11), (4.9) and (4.10) in the same way as [12, Thm. 4.9]. Thus, it is enough to show that Δ is compatible with (3.17). By the definition of $J(h_i)$, we obtain

$$\begin{aligned}[\Delta(h_{i1}),\Delta(h_{j1})]&=[\Delta(J(h_i))-\Delta(\widetilde{v_i}),\Delta(J(h_j))-\Delta(\widetilde{v_j})]\\&=[\Delta(J(h_i)),\Delta(J(h_j))]+[\Delta(\widetilde{v_i}),\Delta(\widetilde{v_j})]\\&\quad-[\Delta(J(h_i)),\Delta(\widetilde{v_j})]-[\Delta(\widetilde{v_i}),\Delta(J(h_j))],\end{aligned}\tag{4.65}$$

where $\widetilde{v_i}=v_i+\frac{\varepsilon_1+\varepsilon_2}{2}h_i^2$. It is enough to show that

$$[\Delta(J(h_i)),\Delta(J(h_j))]+[\Delta(\widetilde{v_i}),\Delta(\widetilde{v_j})]=0\tag{4.66}$$

and

$$[\Delta(J(h_i)),\Delta(\widetilde{v_j})]+[\Delta(\widetilde{v_i}),\Delta(J(h_j))]=0\tag{4.67}$$

hold. We only show that (4.66) holds. The outline of the proof of (4.67) is the same as that of [12, Thm. 4.9]. In order to simplify the computation, we define

$$\begin{aligned}\Omega_+&=\sum_{1\leq k\leq \dim\mathfrak{h}}u^k\otimes u_k+\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)}x^{k_\alpha}_{\alpha}\otimes x^{k_\alpha}_{-\alpha},\\ \Omega_-&=\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}x^{k_\alpha}_{-\alpha}\otimes x^{k_\alpha}_{\alpha},\\ \Omega&=\sum_{1\leq k\leq \dim\mathfrak{h}}u^k\otimes u_k+\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}((-1)^{p(\alpha)}x^{k_\alpha}_{\alpha}\otimes x^{k_\alpha}_{-\alpha}+x^{k_\alpha}_{-\alpha}\otimes x^{k_\alpha}_{\alpha}),\end{aligned}$$

where $\{u^k\}$ and $\{u_k\}$ are basis of $\mathfrak{h}$ such that $\kappa(u_k, u^l) = \delta_{k,l}$. By the definition of $J(h_i)$, it is easy to obtain

$$\Delta(J(h_i)) = \square(J(h_i)) + \frac{\varepsilon_1+\varepsilon_2}{2}[h_{i,0}\otimes 1, \Omega] \tag{4.68}$$

since we have

$$\begin{aligned}\Delta(xy) &= (x\otimes 1 + 1\otimes x)(y\otimes 1 + 1\otimes y)\\ &= (-1)^{p(1)p(y)}xy\otimes 1 + (-1)^{p(x)p(1)}1\otimes xy + (-1)^{p(1)p(1)}x\otimes y\\ &\quad + (-1)^{p(x)p(y)}y\otimes x\end{aligned}$$

by the relation $(x\otimes y)(z\otimes w) = (-1)^{p(y)p(z)}xz\otimes yw$ for all homogeneous elements x, y, z, w. Thus, by (4.68), we obtain

$$\begin{aligned}&[\Delta(J(h_i)), \Delta(J(h_j))]\\ &\quad = \square([J(h_i), J(h_j)]) + \frac{\varepsilon_1+\varepsilon_2}{2}[\square(J(h_i)), [h_{j,0}\otimes 1, \Omega]]\\ &\qquad - \frac{\varepsilon_1+\varepsilon_2}{2}[\square(J(h_j)), [h_{i,0}\otimes 1, \Omega]] + \frac{(\varepsilon_1+\varepsilon_2)^2}{4}[[h_{i,0}\otimes 1, \Omega], [h_{j,0}\otimes 1, \Omega]].\end{aligned}$$

First, we prove that

$$\frac{\varepsilon_1+\varepsilon_2}{2}[\square(J(h_i)), [h_{j,0}\otimes 1, \Omega]] - \frac{\varepsilon_1+\varepsilon_2}{2}[\square(J(h_j)), [h_{i,0}\otimes 1, \Omega]] = 0 \tag{4.69}$$

holds. Since $[h_{j,0}\otimes 1, \Omega] = \sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha,\alpha_i)(x_{-\alpha}\otimes x_\alpha - x_\alpha\otimes x_{-\alpha})$ holds, we have

$$\begin{aligned}&[\square(J(h_i)), [h_{j,0}\otimes 1, \Omega]] - [\square(J(h_j)), [h_{i,0}\otimes 1, \Omega]]\\ &\quad = \sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha,\alpha_j)\big((-1)^{p(\alpha)}[J(h_i), x_\alpha]\otimes x_{-\alpha} - (-1)^{p(\alpha)}x_\alpha\otimes[J(h_i), x_{-\alpha}]\\ &\qquad\qquad + [J(h_i), x_{-\alpha}]\otimes x_\alpha - x_{-\alpha}\otimes[J(h_i), x_\alpha]\big)\\ &\qquad - \sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha,\alpha_i)\big((-1)^{p(\alpha)}[J(h_j), x_\alpha]\otimes x_{-\alpha} - (-1)^{p(\alpha)}x_\alpha\otimes[J(h_j), x_{-\alpha}]\\ &\qquad\qquad + [J(h_j), x_{-\alpha}]\otimes x_\alpha - x_{-\alpha}\otimes[J(h_j), x_\alpha]\big)\\ &\quad = \sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha,\alpha_j)(\alpha,\alpha_i)c^\alpha_{i,j}\\ &\qquad\qquad \times\big((-1)^{p(\alpha)}x_\alpha\otimes x_{-\alpha} - (-1)^{p(\alpha)}x_\alpha\otimes x_{-\alpha} + x_{-\alpha}\otimes x_\alpha - x_{-\alpha}\otimes x_\alpha\big)\\ &\qquad - \sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha,\alpha_i)(\alpha,\alpha_j)c^\alpha_{j,i}\\ &\qquad\qquad \times\big((-1)^{p(\alpha)}x_\alpha\otimes x_{-\alpha} - (-1)^{p(\alpha)}x_\alpha\otimes x_{-\alpha} + x_{-\alpha}\otimes x_\alpha - x_{-\alpha}\otimes x_\alpha\big)\\ &\quad = 0,\end{aligned}$$

where the third equality is due to Proposition 4.26. Therefore (4.69) holds. Since $\Delta(\widetilde{v}_i)=\square(\widetilde{v}_i)-\frac{\varepsilon_1+\varepsilon_2}{2}[h_{i,0}\otimes 1,\Omega_+-\Omega_-]$ holds, we obtain

$$\begin{aligned}[\Delta(\widetilde{v}_i),\Delta(\widetilde{v}_j)]=\square([\widetilde{v}_i,\widetilde{v}_j])+\frac{\varepsilon_1+\varepsilon_2}{2}\big(&-[\square(\widetilde{v}_i),[h_{j,0}\otimes 1,\Omega_+-\Omega_-]]\\&+[\square(\widetilde{v}_j),[h_{i,0}\otimes 1,\Omega_+-\Omega_-]]\big)\\+\frac{(\varepsilon_1+\varepsilon_2)^2}{4}&[[h_{i,0}\otimes 1,\Omega_+-\Omega_-],[h_{j,0}\otimes 1,\Omega_+-\Omega_-]].\end{aligned}$$

Using this along with $\Omega=\Omega_++\Omega_-$ and (4.69), we find the equality

$$\begin{aligned}&[\Delta(J(h_i)),\Delta(J(h_j))]+[\Delta(\widetilde{v}_i),\Delta(\widetilde{v}_j)]\\&\quad=\square([J(h_i),J(h_j)]+[\widetilde{v}_i,\widetilde{v}_j])\\&\qquad+\frac{\varepsilon_1+\varepsilon_2}{2}\big(-[\square(\widetilde{v}_i),[h_{j,0}\otimes 1,\Omega_+-\Omega_-]]\\&\qquad\qquad+[\square(\widetilde{v}_j),[h_{i,0}\otimes 1,\Omega_+-\Omega_-]]\big)\\&\qquad+\frac{(\varepsilon_1+\varepsilon_2)^2}{2}\big([[h_{i,0}\otimes 1,\Omega_+],[h_{j,0}\otimes 1,\Omega_+]]\\&\qquad\qquad+[[h_{i,0}\otimes 1,\Omega_-],[h_{j,0}\otimes 1,\Omega_-]]\big).\end{aligned}\tag{4.70}$$

In the same way as [12, Thm. 4.9], we can check that the sum of the last four terms on the right-hand side of (4.70) vanishes. By Corollary 4.57, $\square([J(h_i),J(h_j)]+[\widetilde{v}_i,\widetilde{v}_j])=0$ holds. The coassociativity is proven in a similar way to [12]. We complete the proof. □

By setting the degree on $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ determined by (4.2) and $\deg(d)=0$, we can define the $\widehat{Y}_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ (resp. $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))\mathbin{\widehat{\otimes}}Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$) as the degreewise completion of $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ (resp. $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))^{\otimes 2}$) in the sense of [18]. We regard a representation of $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ as that of $\widetilde{\mathfrak{sl}}(m|n)$ via Φ. By Theorem 4.3, we easily obtain the following corollary.

Corollary 4.71. *There exists a linear map* $\Delta\colon Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))\to Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))\mathbin{\widehat{\otimes}} Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$*, uniquely determined by*

$$\Delta(h_{i,0})=h_{i,0}\otimes 1+1\otimes h_{i,0},\ \ \Delta(x^{\pm}_{i,0})=x^{\pm}_{i,0}\otimes 1+1\otimes x^{\pm}_{i,0},\ \Delta(d)=d\otimes 1+1\otimes d,$$

$$\begin{aligned}\Delta(h_{i,1})&=h_{i,1}\otimes 1+1\otimes h_{i,1}+(\varepsilon_1+\varepsilon_2)h_{i,0}\otimes h_{i,0}\\&\quad-(\varepsilon_1+\varepsilon_2)\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(\alpha,\alpha_i)x^{k_\alpha}_{-\alpha}\otimes x^{k_\alpha}_{\alpha},\end{aligned}$$

which is an algebra homomorphism. Moreover, Δ satisfies the coassociativity.

In particular, Δ defines an action on $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ on $V\otimes W$ for any $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$-modules V, W which are in the category $\mathcal{O}$ as $\widetilde{\mathfrak{sl}}(m|n)$-modules.

§5. Evaluation map for the affine super Yangian

Since the definition of the affine super Yangian is very complicated, it is not clear whether the affine super Yangian is trivial or not. In this section, we construct the nontrivial homomorphism from the affine super Yangian to the completion of $U(\widehat{\mathfrak{gl}}(m|n))$. In this section, we set $\widehat{\mathfrak{gl}}(m|n) = \mathfrak{gl}(m|n)\otimes\mathbb{C}[t^{\pm 1}]\oplus\mathbb{C}c\oplus\mathbb{C}z$ as a Lie superalgebra whose defining relations are

$$c,\ z \text{ are central elements,}$$

$$[x\otimes t^s, y\otimes t^u] = \begin{cases} [x,y]\otimes t^{s+u} + s\delta_{s+u,0}\operatorname{str}(xy)c & \text{if } x,y\in\mathfrak{sl}(m|n),\\ [e_{a,b},e_{i,i}]\otimes t^{s+u} + s\delta_{s+u,0}\operatorname{str}(e_{a,b}e_{i,i})c & \\ \quad +s\delta_{a,b}(-1)^{p(a)+p(i)}z & \text{if } x=e_{a,b},\ y=e_{i,i}.\end{cases}$$

For all $s\in\mathbb{Z}$, we denote $E_{i,j}\otimes t^s$ by $E_{i,j}(s)$. We also set the grading of $U(\widehat{\mathfrak{gl}}(m|n))/U(\widehat{\mathfrak{gl}}(m|n))(z-1)$ as $\deg(X(s)) = s$ and $\deg(c) = 0$. We define $U(\widehat{\mathfrak{gl}}(m|n))_{\mathrm{comp},+}$ as the standard degreewise completion of $U(\widehat{\mathfrak{gl}}(m|n))/U(\widehat{\mathfrak{gl}}(m|n))(z-1)$ in the sense of Definition 4.1.

Let us state the main result of this section. In order to simplify the notation, we denote $\varepsilon_1+\varepsilon_2$ as $\hbar$.

Theorem 5.1. *Assume $c\hbar = (-m+n)\varepsilon_1$ and $z = 1$. Let α be a complex number. Then there exists an algebra homomorphism*

$$\mathrm{ev}\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) \to U(\widehat{\mathfrak{gl}}(m|n))_{\mathrm{comp},+}$$

uniquely determined by

$$\mathrm{ev}(x^+_{i,0}) = x^+_i,\quad \mathrm{ev}(x^-_{i,0}) = x^-_i,\quad \mathrm{ev}(h_{i,0}) = h_i, \tag{5.2}$$

$$\mathrm{ev}(x^+_{i,1}) = \begin{cases} (\alpha-(m-n)\varepsilon_1)x^+_0 & \\ \quad +\hbar\displaystyle\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,1}(s+1) & \text{if } i=0,\\ \big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x^+_i & \\ \quad +\hbar\displaystyle\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s) & \\ \quad +\hbar\displaystyle\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i+1}(s+1) & \text{if } i\neq 0,\end{cases} \tag{5.3}$$

$$
(5.4)\quad \mathrm{ev}(x_{i,1}^-) = \begin{cases} (\alpha-(m-n)\varepsilon_1)x_0^- \\ \quad -\hbar\displaystyle\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{1,k}(-s-1)E_{k,m+n}(s) & \text{if } i=0,\\ \big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x_i^- \\ \quad +(-1)^{p(i)}\hbar\displaystyle\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)E_{k,i}(s) \\ \quad +(-1)^{p(i)}\hbar\displaystyle\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i}(s+1) & \text{if } i\neq 0,\end{cases}
$$

$$
(5.5)\quad \mathrm{ev}(h_{i,1}) = \begin{cases} (\alpha-(m-n)\varepsilon_1)h_0+\hbar E_{m+n,m+n}(E_{1,1}-c) \\ \quad -\hbar\displaystyle\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,m+n}(s) \\ \quad -\hbar\displaystyle\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{1,k}(-s-1)E_{k,1}(s+1) & \text{if } i=0,\\ (\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)h_i \\ \quad -(-1)^{p(E_{i,i+1})}\hbar E_{i,i}E_{i+1,i+1} \\ \quad +\hbar(-1)^{p(i)}\displaystyle\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s) \\ \quad +\hbar(-1)^{p(i)}\displaystyle\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i}(s+1) \\ \quad -\hbar(-1)^{p(i+1)}\displaystyle\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)E_{k,i+1}(s) \\ \quad -\hbar(-1)^{p(i+1)}\displaystyle\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i+1}(s+1) & \text{if } i\neq 0.\end{cases}
$$

The outline of the proof is the same as that of [16]. It is enough to check that ev is compatible with (3.17)–(3.25), which are the defining relations of the minimalistic presentation of the affine super Yangian. When we restrict ev to $\widehat{\mathfrak{sl}}(m|n)$, ev is an identity map on $\widehat{\mathfrak{sl}}(m|n)$. Thus, ev is compatible with (3.18), (3.20), (3.23)–(3.25).

We set an anti-automorphism $\omega\colon U(\widehat{\mathfrak{gl}}(m|n))\to U(\widehat{\mathfrak{gl}}(m|n))$ as

$$
\omega(X\otimes t^r)=(-1)^rX^T\otimes t^r,\quad \omega(c)=c,
$$

where X^T is the transpose of a matrix X. Then the compatibility of ev with (3.21) and (3.22) for $-$ is deduced from that for $+$ by applying the anti-automorphism ω

since we have $\omega(\mathrm{ev}(h_{i,1}))=\mathrm{ev}(h_{i,1})$ and $\omega(\mathrm{ev}(x^+_{i,1}))=(-1)^{p(i)}\,\mathrm{ev}(x^-_{i,1})$. Therefore, it is enough to check the following lemma.

Lemma 5.6. *The following equations hold:*

$$[\mathrm{ev}(x^+_{i,1}),\mathrm{ev}(x^-_{j,0})]=\delta_{i,j}\,\mathrm{ev}(h_{i,1}),\tag{5.7}$$

$$[\mathrm{ev}(\tilde{h}_{i,1}),x^+_j]=a_{i,j}\Big(\mathrm{ev}(x^+_{j,1})-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^+_j\Big),\tag{5.8}$$

$$[\mathrm{ev}(x^+_{i,1}),x^+_j]-[x^+_i,\mathrm{ev}(x^+_{j,1})]=a_{i,j}\frac{\varepsilon_1+\varepsilon_2}{2}\{x^+_i,x^+_j\}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}[x^+_i,x^+_j],\tag{5.9}$$

$$[\mathrm{ev}(h_{i,1}),\mathrm{ev}(h_{j,1})]=0.\tag{5.10}$$

The rest of the paper is devoted to the proof of Lemma 5.6.

§5.1. The proof of (5.7)

We prepare one claim before starting the proof.

Claim 5.11. The following relations hold:

$$\begin{aligned}
&\Big[\sum_{s\geq p}\sum_{k=1}^{a}(-1)^{p(k)}E_{i,k}(-s)E_{k,j}(s),E_{x,y}\Big]\\
&\quad=\delta_{j,x}\sum_{s\geq p}\sum_{k=1}^{a}(-1)^{p(k)}E_{i,k}(-s)E_{k,y}(s)\\
&\qquad-(-1)^{p(E_{i,j})p(E_{x,y})}\sum_{s\geq p}\sum_{k=1}^{a}(-1)^{p(k)}E_{x,k}(-s)E_{k,j}(s)\\
&\qquad+\{\delta(x\leq a<y)-\delta(x>a\geq y)\}\\
&\qquad\quad\times\sum_{s\geq p}(-1)^{p(x)+p(E_{x,j})p(E_{x,y})}E_{i,y}(-s)E_{x,j}(s),
\end{aligned}\tag{5.12}$$

$$\begin{aligned}
&\Big[\sum_{s\geq p}\sum_{k=a}^{m+n}(-1)^{p(k)}E_{i,k}(-s)E_{k,j}(s),E_{x,y}\Big]\\
&\quad=\delta_{j,x}\sum_{s\geq p}\sum_{k=a}^{m+n}(-1)^{p(k)}E_{i,k}(-s)E_{k,y}(s)\\
&\qquad-(-1)^{p(E_{i,j})p(E_{x,y})}\sum_{s\geq p}\sum_{k=a}^{m+n}(-1)^{p(k)}E_{x,k}(-s)E_{k,j}(s)\\
&\qquad+\{\delta(x\geq a>y)-\delta(x<a\leq y)\}\\
&\qquad\quad\times\sum_{s\geq p}(-1)^{p(x)+p(E_{x,j})p(E_{x,y})}E_{i,y}(-s)E_{x,j}(s).
\end{aligned}\tag{5.13}$$

Proof. We prove only (5.12) since (5.13) is proven in a similar way. By direct computation, the first term of (5.12) is equal to

$$\begin{aligned}
&\delta_{j,x}\sum_{s\geq p}\sum_{k=1}^{a}(-1)^{p(k)}E_{i,k}(-s)E_{k,y}(s)\\
&\quad-\delta(y\leq a)\sum_{s\geq p}(-1)^{p(y)+p(E_{y,j})p(E_{x,y})}E_{i,y}(-s)E_{x,j}(s)\\
&\quad+\delta(x\leq a)\sum_{s\geq p}(-1)^{p(x)+p(E_{x,j})p(E_{x,y})}E_{i,y}(-s)E_{x,j}(s)\\
&\quad-(-1)^{p(E_{i,j})p(E_{x,y})}\sum_{s\geq p}\sum_{k=1}^{a}(-1)^{p(k)}E_{x,k}(-s)E_{k,j}(s). \qquad (5.14)
\end{aligned}$$

Since $p(y)+p(E_{y,j})p(E_{x,y})=p(x)+p(E_{x,j})p(E_{x,y})$, the sum of the second and third terms of (5.14) is equal to

$$\{\delta(x\leq a<y)-\delta(x>a\geq y)\}\sum_{s\geq p}(-1)^{p(x)+p(E_{x,j})p(E_{x,y})}E_{i,y}(-s)E_{x,j}(s).$$

Then we obtain (5.13). □

Suppose that $i,j\neq 0$. Other cases are proven in a similar way. By the definition of $\mathrm{ev}(x^+_{i,1})$, we obtain

$$\begin{aligned}
&[\mathrm{ev}(x^+_{i,1}),\mathrm{ev}(x^-_{j,0})]\\
&\quad=\big[\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x^+_i,(-1)^{p(j)}E_{j+1,j}\big]\\
&\qquad+\left[\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s),(-1)^{p(j)}E_{j+1,j}\right]\\
&\qquad+\left[\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i+1}(s+1),(-1)^{p(j)}E_{j+1,j}\right]. \qquad (5.15)
\end{aligned}$$

By (5.12), $[\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s),(-1)^{p(j)}E_{j+1,j}]$, the second term on the right-hand side of (5.15) is equal to

$$\begin{aligned}
&\left[\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s),(-1)^{p(j)}E_{j+1,j}\right]\\
&\quad=\delta_{i,j}\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(i)+p(k)}E_{i,k}(-s)E_{k,i}(s)
\end{aligned}$$

$$- \delta_{i,j}\hbar \sum_{s\geq 0}\sum_{k=1}^{i} (-1)^{p(k)+p(i)+p(E_{i,i+1})} E_{i+1,k}(-s)E_{k,i+1}(s)$$

$$- \delta_{i,j}\hbar \sum_{s\geq 0} (-1)^{p(E_{i,i+1})p(E_{i,i+1})} E_{i,i}(-s)E_{i+1,i+1}(s). \tag{5.16}$$

Similarly, by (5.13), $[\hbar \sum_{s\geq 0}\sum_{k=i+1}^{m+n} (-1)^{p(k)} E_{i,k}(-s-1)E_{k,i+1}(s+1), (-1)^{p(j)} \times E_{j+1,j}]$, the third term on the right-hand side of (5.15) is equal to

$$\left[\hbar \sum_{s\geq 0}\sum_{k=i+1}^{m+n} (-1)^{p(k)} E_{i,k}(-s-1)E_{k,i+1}(s+1), (-1)^{p(j)} E_{j+1,j}\right]$$

$$= \hbar \sum_{s\geq 0}\sum_{k=i+1}^{m+n} \delta_{i,j}(-1)^{p(k)+p(i)} E_{i,k}(-s-1)E_{k,i}(s+1)$$

$$- \hbar \sum_{s\geq 0}\sum_{k=i+1}^{m+n} \delta_{i,j}(-1)^{p(k)+p(i)+p(E_{i,i+1})} E_{i+1,k}(-s-1)E_{k,i+1}(s+1)$$

$$+ \hbar \sum_{s\geq 0} \delta_{i,j}(-1)^{p(i+1)+p(i)} E_{i,i}(-s-1)E_{i+1,i+1}(s+1). \tag{5.17}$$

We can rewrite the sum of the last term of (5.16) and the last term of (5.17). Since $p(E_{i,i+1}) = p(i) + p(i+1)$ holds, we obtain

$$- \hbar \sum_{s\geq 0} (-1)^{p(E_{i,i+1})p(E_{i,i+1})} E_{i,i}(-s)E_{i+1,i+1}(s)$$

$$+ \hbar \sum_{s\geq 0} (-1)^{p(i+1)+p(i)+p(E_{i+1,i+1})p(E_{i,i+1})} E_{i,i}(-s-1)E_{i+1,i+1}(s+1)$$

$$= -\hbar \sum_{s\geq 0} (-1)^{p(E_{i,i+1})} E_{i,i}(-s)E_{i+1,i+1}(s)$$

$$+ \hbar \sum_{s\geq 0} (-1)^{p(E_{i,i+1})} E_{i,i}(-s-1)E_{i+1,i+1}(s+1)$$

$$= -\hbar(-1)^{p(E_{i,i+1})} E_{i,i}E_{i+1,i+1}. \tag{5.18}$$

Thus, we have shown that $[\mathrm{ev}(x^+_{i,1}), \mathrm{ev}(x^-_{j,0})] = \delta_{i,j}\,\mathrm{ev}(h_{i,1})$ holds by (5.16), (5.17) and (5.18).

§5.2. The proof of (5.8)

We only show the case where $i, j \neq 0$ and when $i = 0$ and $j \neq 0$. The other case is proven in a similar way.

Case 1, $i,j\neq 0$. First, we show the case where $i,j\neq 0$. By the definition of $\mathrm{ev}(h_{i,1})$, we obtain

$$
\begin{aligned}
&[\mathrm{ev}(\tilde{h}_{i,1}),\mathrm{ev}(x^+_{j,0})]\\
&=\Big[(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)h_i\\
&\qquad -\frac{1}{2}\hbar((E_{i,i})^2+(E_{i+1,i+1})^2),E_{j,j+1}\Big]\\
&\quad+\Big[\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s),E_{j,j+1}\Big]\\
&\quad+\Big[\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i}(s+1),E_{j,j+1}\Big]\\
&\quad-\Big[\hbar(-1)^{p(i+1)}\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)E_{k,i+1}(s),E_{j,j+1}\Big]\\
&\quad-\Big[\hbar(-1)^{p(i+1)}\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i+1}(s+1),E_{j,j+1}\Big].
\end{aligned}
\tag{5.19}
$$

Let us compute these terms respectively. By direct computation, the first term on the right-hand side of (5.19) is equal to

$$
\begin{aligned}
&\Big[\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)h_i\\
&\qquad -\frac{1}{2}\hbar((E_{i,i})^2+(E_{i+1,i+1})^2),E_{j,j+1}\Big]\\
&\quad=\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)a_{i,j}x^+_j\\
&\qquad-\frac{\hbar}{2}\big(\delta_{i,j}(\{E_{i,i+1},E_{i,i}\}-\{E_{i,i+1},E_{i+1,i+1}\})\\
&\qquad\qquad-\delta_{i,j+1}\{E_{i-1,i},E_{i,i}\}+\delta_{i+1,j}\{E_{i+1,i+2},E_{i+1,i+1}\}\big).
\end{aligned}
\tag{5.20}
$$

By (5.12) and (5.13), we also find that the sum of the second and third terms on the right-hand side of (5.19) is equal to

$$
\begin{aligned}
&\Big[\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s),E_{j,j+1}\Big]\\
&\qquad+\Big[\hbar(-1)^{p(i)}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i}(s+1),E_{j,j+1}\Big]
\end{aligned}
$$

$$\begin{aligned}
&= \hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i,j}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s)\\
&\quad - \hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i,j+1}(-1)^{p(k)}E_{j,k}(-s)E_{k,i}(s)\\
&\quad + \hbar(-1)^{p(i)}\sum_{s\geq 0}\delta_{i,j}(-1)^{p(i)}E_{i,i+1}(-s)E_{i,i}(s)\\
&\quad + \hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i,j}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,j+1}(s+1)\\
&\quad - \hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i,j+1}(-1)^{p(k)}E_{j,k}(-s-1)E_{k,i}(s+1)\\
&\quad - \hbar(-1)^{p(i)}\sum_{s\geq 0}\delta_{i,j}(-1)^{p(i+1)+p(E_{i,i+1})p(E_{i+1,i})}\\
&\qquad\qquad \times E_{i,i+1}(-s-1)E_{i,i}(s+1).
\end{aligned} \tag{5.21}$$

By a direct computation, we obtain

$$\text{the sum of the third and sixth terms of } (5.21) = \hbar\delta_{i,j}E_{i,i+1}E_{i,i}. \tag{5.22}$$

Next, let us rewrite the sum of the first and fourth terms of (5.21). By the definition of $\mathrm{ev}(x_{i,1}^+)$, we obtain

$$\begin{aligned}
&\text{the first term of } (5.21) + \text{the fourth term of } (5.21)\\
&\quad = \delta_{i,j}\big(\mathrm{ev}(x_{i,1}^+) - \big(\alpha - (i - 2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x_i^+\big).
\end{aligned} \tag{5.23}$$

By the definition of $\mathrm{ev}(x_{i,1}^+)$, we also obtain

$$\begin{aligned}
&\text{the second term of } (5.21) + \text{the fifth term of } (5.21)\\
&\quad = -\delta_{i,j+1}\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=1}^{j}(-1)^{p(k)}E_{j,k}(-s)E_{k,i}(s)\\
&\qquad - \delta_{i,j+1}\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=j+1}^{m+n}(-1)^{p(k)}E_{j,k}(-s-1)E_{k,i}(s+1)\\
&\qquad - \hbar\delta_{i,j+1}E_{j,i}E_{i,i}\\
&\quad = -\delta_{i,j+1}(-1)^{p(i)}\big(\mathrm{ev}(x_{j,1}^+) - \big(\alpha - (j - 2\delta(i\geq m+1)(j-m))\varepsilon_1\big)x_j^+\big)\\
&\qquad - \hbar\delta_{i,j+1}E_{j,i}E_{i,i}.
\end{aligned} \tag{5.24}$$

Therefore, by (5.22), (5.23) and (5.24), the sum of first, second and third terms on the right-hand side of (5.19) is equal to

$$\begin{aligned}
&\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)a_{i,j}x_j^+\\
&\quad-\frac{\hbar}{2}\big(\delta_{i,j}(\{E_{i,i+1},E_{i,i}\}-\{E_{i,i+1},E_{i+1,i+1}\})-\delta_{i,j+1}\{E_{i-1,i},E_{i,i}\}\\
&\qquad\quad+\delta_{i+1,j}\{E_{i+1,i+2},E_{i+1,i+1}\}\big)\\
&\quad+\hbar\delta_{i,j}E_{i,i+1}E_{i,i}\\
&\quad+(-1)^{p(i)}\delta_{i,j}\big(\mathrm{ev}(x_{i,1}^+)-\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x_i^+\big)\\
&\quad-(-1)^{p(i)}\delta_{i,j+1}\big(\mathrm{ev}(x_{j,1}^+)-\big(\alpha-(j-2\delta(i\geq m+1)(j-m))\varepsilon_1\big)x_j^+\big).
\end{aligned}\tag{5.25}$$

Similarly to (5.25), we find that the sum of the fourth and fifth terms on the right-hand side of (5.19) is equal to

$$\begin{aligned}
&-\hbar\delta_{j,i}E_{i+1,i+1}E_{i,i+1}\\
&\quad-(-1)^{p(i+1)}\delta_{i+1,j}\big(\mathrm{ev}(x_{j,1}^+)-\big(\alpha-(i-2\delta(j\geq m+1)(j-m))\varepsilon_1\big)x_j^+\big)\\
&\quad+\delta_{i+1,j}\hbar E_{i+1,i+1}E_{i+1,j+1}\\
&\quad+\delta_{i,j}(-1)^{p(i+1)}\big(\mathrm{ev}(x_{i,1}^+)-\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x_i^+\big).
\end{aligned}\tag{5.26}$$

Then $[\mathrm{ev}(\tilde{h}_{i,1}),\mathrm{ev}(x_{j,0}^+)]$ is equal to the sum of (5.20), (5.25) and (5.26):

$$\begin{aligned}
&\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)a_{i,j}x_j^+\\
&\quad-\frac{\hbar}{2}\big(\delta_{i,j}(\{E_{i,i+1},E_{i,i}\}-\{E_{i,i+1},E_{i+1,i+1}\})\\
&\qquad\quad-\delta_{i,j+1}\{E_{i-1,i},E_{i,i}\}+\delta_{i+1,j}\{E_{i+1,i+2},E_{i+1,i+1}\}\big)\\
&\quad+\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)a_{i,j}x_j^+\\
&\quad-\frac{\hbar}{2}\big(\delta_{i,j}(\{E_{i,i+1},E_{i,i}\}-\{E_{i,i+1},E_{i+1,i+1}\})-\delta_{i,j+1}\{E_{i-1,i},E_{i,i}\}\\
&\qquad\quad+\delta_{i+1,j}\{E_{i+1,i+2},E_{i+1,i+1}\}\big)\\
&\quad+\hbar\delta_{i,j}E_{i,i+1}E_{i,i}\\
&\quad+(-1)^{p(i)}\delta_{i,j}\big(\mathrm{ev}(x_{i,1}^+)-\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x_i^+\big)\\
&\quad-(-1)^{p(i)}\delta_{i,j+1}\big(\mathrm{ev}(x_{j,1}^+)-\big(\alpha-(j-2\delta(i\geq m+1)(j-m))\varepsilon_1\big)x_j^+\big)\\
&\quad-\hbar\delta_{j,i}E_{i+1,i+1}E_{i,i+1}\\
&\quad-(-1)^{p(i+1)}\delta_{i+1,j}\big(\mathrm{ev}(x_{j,1}^+)-\big(\alpha-(i-2\delta(j\geq m+1)(j-m))\varepsilon_1\big)x_j^+\big)\\
&\quad+\delta_{i+1,j}\hbar E_{i+1,i+1}E_{i+1,j+1}\\
&\quad+\delta_{i,j}(-1)^{p(i+1)}\big(\mathrm{ev}(x_{i,1}^+)-\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x_i^+\big).
\end{aligned}$$

By (5.20), (5.25) and (5.26), when $i \neq j, j\pm 1$, $[\mathrm{ev}(\tilde{h}_{i,1}), \mathrm{ev}(x^+_{j,0})]$ is zero. Provided that $i = j$, $[\mathrm{ev}(\tilde{h}_{i,1}), \mathrm{ev}(x^+_{i,0})]$ is equal to

$$\begin{aligned}
&\big(\alpha - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1\big)a_{i,i}x^+_i \\
&\quad - \frac{\hbar}{2}(\{E_{i,i+1}, E_{i,i}\} - \{E_{i,i+1}, E_{i+1,i+1}\}) \\
&\quad + \hbar E_{i,i+1}E_{i,i} \\
&\quad + (-1)^{p(i)}\big(\mathrm{ev}(x^+_{i,1}) - \big(\alpha - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1\big)x^+_i\big) \\
&\quad - \hbar E_{i+1,i+1}E_{i,i+1} \\
&\quad + (-1)^{p(i+1)}\big(\mathrm{ev}(x^+_{i,1}) - \big(\alpha - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1\big)x^+_i\big).
\end{aligned} \tag{5.27}$$

Since $a_{i,i} = (-1)^{p(i)} + (-1)^{p(i+1)}$ holds, we have

$$\begin{aligned}
&\big(\alpha - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1\big)a_{i,i}x^+_i \\
&\quad - (-1)^{p(i)}\big(\alpha - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1\big)x^+_i \\
&\quad - (-1)^{p(i+1)}\big(\alpha - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1\big)x^+_i = 0
\end{aligned}$$

and

$$\begin{aligned}
&- \frac{\hbar}{2}(\{E_{i,i+1}, E_{i,i}\} - \{E_{i,i+1}, E_{i+1,i+1}\}) + \hbar E_{i,i+1}E_{i,i} - \hbar E_{i+1,i+1}E_{i,i+1} \\
&\quad = -\frac{\hbar}{2}(E_{i,i}E_{i,i+1} - E_{i,i+1}E_{i,i} + E_{i+1,i+1}E_{i,i+1} - E_{i,i+1}E_{i+1,i+1}) \\
&\quad = -\frac{\hbar}{2}(E_{i,i+1} - E_{i,i+1}) = 0.
\end{aligned}$$

Then we find that $[\mathrm{ev}(\tilde{h}_{i,1}), \mathrm{ev}(x^+_{i,0})]$ is equal to $a_{i,i}\,\mathrm{ev}(x^+_{i,1})$.

When $i = j+1$, $[\mathrm{ev}(\tilde{h}_{i,1}), \mathrm{ev}(x^+_{j,0})]$ is equal to

$$\begin{aligned}
&\big(\alpha - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1\big)a_{i,j}x^+_j + \frac{\hbar}{2}\{E_{i-1,i}, E_{i,i}\} \\
&\quad - (-1)^{p(i)}\big(\mathrm{ev}(x^+_{j,1}) - \big(\alpha - (j - 2\delta(i \geq m+1)(j-m))\varepsilon_1\big)x^+_j\big) - \hbar E_{j,i}E_{i,i}.
\end{aligned}$$

Since $a_{i,j} = -(-1)^{p(i)}$ holds, we have

$$\frac{\hbar}{2}\{E_{i-1,i}, E_{i,i}\} - \hbar E_{j,i}E_{i,i} = \frac{\hbar}{2}[E_{i,i}, E_{i-1,i}] = -\frac{\hbar}{2}E_{i-1,i}$$

and

$$\begin{aligned}
&\big(\alpha - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1\big)a_{i,j}x^+_j \\
&\quad + (-1)^{p(i)}\big(\alpha - (j - 2\delta(i \geq m+1)(j-m)\varepsilon_1)x^+_j\big) = \varepsilon_1 x^+_j.
\end{aligned}$$

Then we find that $[\mathrm{ev}(\tilde{h}_{i,1}), \mathrm{ev}(x^+_{j,0})]$ is equal to $a_{i,i-1}(\mathrm{ev}(x^+_{i-1}) + a_{i,i-1}\frac{\varepsilon_1 - \varepsilon_2}{2}E_{i-1,i})$.

When $i=j-1$, $[\mathrm{ev}(\tilde{h}_{i,1}),\mathrm{ev}(x^+_{j,0})]$ is equal to

$$\begin{aligned}&\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)a_{i,j}x_j^+-\frac{\hbar}{2}(\{E_{i+1,i+2},E_{i+1,i+1}\})\\&\quad-(-1)^{p(i+1)}\big(\mathrm{ev}(x^+_{j,1})-\big(\alpha-(i-2\delta(j\geq m+1)(j-m))\varepsilon_1\big)x_j^+\big)\\&\quad+\hbar E_{i+1,i+1}E_{i+1,j+1}.\end{aligned}$$

Since $a_{i,j}=-(-1)^{p(j)}$ holds, we have

$$-\frac{\hbar}{2}\{E_{i+1,i+2},E_{i+1,i+1}\}+\hbar E_{i+1,i+1}E_{i+1,j+1}=\frac{\hbar}{2}[E_{i+1,i+1},E_{i+1,i+2}]=\frac{\hbar}{2}E_{i+1,i+2}$$

and

$$\begin{aligned}&\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)a_{i,j}x_j^+\\&\quad+(-1)^{p(i+1)}\big(\alpha-(i-2\delta(j\geq m+1)(j-m))\varepsilon_1\big)x_j^+=-\varepsilon_1E_{i+1,i+2}.\end{aligned}$$

Then $[\mathrm{ev}(\tilde{h}_{i,1}),\mathrm{ev}(x^+_{i+1,0})]$ is equal to $a_{i,i+1}(\mathrm{ev}(x^+_{i+1,1})-a_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}E_{i+1,i+2})$.

Case 2, $i=0$ and $j\neq 0$. By the definition of ev, we obtain

$$\begin{aligned}&[\mathrm{ev}(\tilde{h}_{0,1}),\mathrm{ev}(x^+_{j,0})]\\&\quad=\Big[(\alpha-(m-n)\varepsilon_1)h_0-\frac{1}{2}\hbar((E_{m+n,m+n})^2+(E_{1,1}-c)^2),E_{j,j+1}\Big]\\&\qquad-\Big[\hbar\sum_{s\geq0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,m+n}(s),E_{j,j+1}\Big]\\&\qquad-\Big[\hbar\sum_{s\geq0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{1,k}(-s-1)E_{k,1}(s+1),E_{j,j+1}\Big].\end{aligned}\tag{5.28}$$

By direct computation, the first term of (5.28) is equal to

$$\begin{aligned}&(\alpha-(m-n)\varepsilon_1)a_{0,j}x_j^+-\frac{\hbar}{2}\big(-\delta_{m+n-1,j}(\{E_{m+n,m+n},E_{m+n-1,m+n}\}\\&\qquad+\delta_{1,j}\{E_{1,2},(E_{1,1}-c)\})\big).\end{aligned}\tag{5.29}$$

We also find that the second term of (5.28) is equal to

$$\begin{aligned}&\hbar\sum_{s\geq0}(-1)^{p(j+1)+p(E_{j,j+1})p(E_{j+1,m+n})}E_{m+n,j+1}(-s)E_{j,m+n}(s)\\&\quad-\hbar\sum_{s\geq0}(-1)^{p(j)+p(E_{j,j+1})p(E_{j,m+n})}E_{m+n,j+1}(-s)E_{j,m+n}(s)\\&\quad+\hbar\sum_{s\geq0}\sum_{k=1}^{m+n}\delta_{m+n,j+1}(-1)^{p(k)}E_{m+n-1,k}(-s)E_{k,m+n}(s).\end{aligned}\tag{5.30}$$

By direct computation, we also know that the third term of (5.28) is equal to

$$
\begin{aligned}
&-\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}\delta_{1,j}(-1)^{p(k)}E_{1,k}(-s-1)E_{k,2}(s+1)\\
&\quad+\hbar\sum_{s\geq 0}(-1)^{p(j+1)+p(E_{j,j+1})p(E_{j+1,1})}E_{1,j+1}(-s-1)E_{j,1}(s+1)\\
&\quad-\hbar\sum_{s\geq 0}(-1)^{p(j)+p(E_{j,j+1})p(E_{j,1})}E_{1,j+1}(-s-1)E_{j,1}(s+1).
\end{aligned}\tag{5.31}
$$

First, we show that the sum of the first and second terms of (5.30) is equal to zero. By direct computation, we have

$$
\begin{aligned}
&\text{the first term of (5.30)}+\text{the second term of (5.30)}\\
&\quad=\hbar\sum_{s\geq 0}(-1)^{p(j+1)+p(E_{j,j+1})p(E_{j+1,m+n})}E_{m+n,j+1}(-s)E_{j,m+n}(s)\\
&\quad\quad-\hbar\sum_{s\geq 0}(-1)^{p(j)+p(E_{j,j+1})p(E_{j,m+n})}E_{m+n,j+1}(-s)E_{j,m+n}(s)=0.
\end{aligned}\tag{5.32}
$$

Similarly, by direct computation, we also obtain

$$
\begin{aligned}
&\text{the second term of (5.31)}+\text{the third term of (5.31)}\\
&\quad=\hbar\sum_{s\geq 0}(-1)^{p(j+1)+p(E_{j,j+1})p(E_{j+1,1})}E_{1,j+1}(-s-1)E_{j,1}(s+1)\\
&\quad\quad-\hbar\sum_{s\geq 0}(-1)^{p(j)+p(E_{j,j+1})p(E_{j,1})}E_{1,j+1}(-s-1)E_{j,1}(s+1)=0.
\end{aligned}\tag{5.33}
$$

Next we rewrite the third term of (5.30). By direct computation, we have

$$
\begin{aligned}
&\text{the third term of (5.30)}\\
&\quad=\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n-1}\delta_{m+n,j+1}(-1)^{p(k)}E_{m+n-1,k}(-s)E_{k,m+n}(s)\\
&\quad\quad+\hbar\sum_{s\geq 0}\delta_{m+n,j+1}(-1)^{p(m+n)}E_{m+n-1,m+n}(-s-1)E_{m+n,m+n}(s+1)\\
&\quad\quad+\hbar\delta_{m+n,j+1}(-1)^{p(m+n)}E_{m+n-1,m+n}E_{m+n,m+n}\\
&\quad=\delta_{m+n,j+1}\big(\mathrm{ev}(x^+_{m+n-1,1})-(\alpha-(m-n+1)\varepsilon_1)x^+_{m+n-1}\big)\\
&\quad\quad+\hbar\delta_{m+n,j+1}(-1)^{p(m+n)}E_{m+n-1,m+n}E_{m+n,m+n}.
\end{aligned}\tag{5.34}
$$

Similarly, we rewrite the first term of (5.31) as

$$\begin{aligned}\text{the first term of (5.31)} &= -\hbar\sum_{s\geq 0}\delta_{1,j}E_{1,1}(-s)E_{1,2}(s)\\ &\quad -\hbar\sum_{s\geq 0}\sum_{k=2}^{m+n}\delta_{1,j}(-1)^{p(k)}E_{1,k}(-s-1)E_{k,2}(s+1)\\ &\quad +\hbar\sum_{s\geq 0}\delta_{1,j}E_{1,1}E_{1,2}\\ &= -\delta_{j,1}(\mathrm{ev}(x^+_{1,1})-(\alpha-\varepsilon_1)x^+_1)+\hbar\delta_{j,1}E_{1,1}E_{1,2}. \end{aligned}\tag{5.35}$$

Then, by (5.28), (5.32), (5.33), (5.34) and (5.35), we can rewrite $[\mathrm{ev}(\tilde{h}_{0,1}),x^+_{j,0}]$ as

$$\begin{aligned}&(\alpha-(m-n)\varepsilon_1)a_{0,j}x^+_j\\ &\quad -\frac{\hbar}{2}\big(-\delta_{m+n,j+1}\{E_{m+n,m+n},E_{j,m+n}\}+\delta_{1,j}\{E_{1,j+1},(E_{1,1}-c)\}\big)\\ &\quad +\delta_{m+n,j+1}\big(\mathrm{ev}(x^+_{m+n-1,1})-(\alpha-(m-n-1)\varepsilon_1)x^+_{m+n-1}\big)\\ &\quad +\hbar\delta_{m+n,j+1}(-1)^{p(m+n)}E_{m+n-1,m+n}E_{m+n,m+n}\\ &\quad -\delta_{j,1}\big(\mathrm{ev}(x^+_{1,1}-(\alpha-\varepsilon_1)x^+_1)\big)+\hbar\delta_{j,1}E_{1,1}E_{1,2}. \end{aligned}\tag{5.36}$$

By (5.36), when $j\neq 0,1,m+n-1$, $[\mathrm{ev}(\tilde{h}_{0,1}),\mathrm{ev}(x^+_{j,0})]$ is equal to zero. When $j=m+n-1$, $[\mathrm{ev}(\tilde{h}_{0,1}),\mathrm{ev}(x^+_{j,0})]$ is equal to

$$\begin{aligned}&(\alpha-(m-n)\varepsilon_1)x^+_{m+n-1}+\frac{\hbar}{2}\{E_{m+n,m+n},E_{j,m+n}\}\\ &\quad +\mathrm{ev}(x^+_{m+n-1,1})-(\alpha-(m-n+1)\varepsilon_1)x^+_{m+n-1}\\ &\quad +\hbar(-1)^{p(m+n)}E_{m+n-1,m+n}E_{m+n,m+n}. \end{aligned}$$

Since

$$\begin{aligned}&\frac{\hbar}{2}\{E_{m+n,m+n},E_{m+n-1,m+n}\}+\hbar(-1)^{p(m+n)}E_{m+n-1,m+n}E_{m+n,m+n}\\ &\quad =\frac{\hbar}{2}[E_{m+n,m+n},E_{m+n-1,m+n}]-\frac{\hbar}{2}E_{m+n-1,m+n} \end{aligned}$$

holds, $[\mathrm{ev}(\tilde{h}_{0,1}),\mathrm{ev}(x^+_{m+n-1,0})]$ is equal to

$$a_{m+n-1,0}\Big(\mathrm{ev}(x^+_{m+n-1,1})+a_{m+n-1,0}\frac{\varepsilon_1-\varepsilon_2}{2}E_{m+n-1,m+n}\Big).$$

By (5.36), when $j=1$, $[\mathrm{ev}(\tilde{h}_{0,1}),\mathrm{ev}(x^+_{1,0})]$ can be written as

$$-(\alpha-(m-n)\varepsilon_1)x^+_1-\frac{\hbar}{2}\{E_{1,j+1},(E_{1,1}-c)\}-\mathrm{ev}(x^+_{1,1})-(\alpha-\varepsilon_1)x^+_1+\hbar E_{1,1}E_{1,2}.$$

Since

$$\hbar E_{1,1}E_{1,2}-\frac{\hbar}{2}\{E_{1,2},(E_{1,1}-c)\}=\frac{\hbar}{2}[E_{1,1},E_{1,2}]+\hbar cE_{1,2}=\Big(\frac{\hbar}{2}+\hbar c\Big)E_{1,2}$$

holds, $[\mathrm{ev}(\tilde{h}_{0,1}),x^+_{1,0}]=a_{0,1}(\mathrm{ev}(x^+_{1,1})-a_{0,1}\frac{\varepsilon_1-\varepsilon_2}{2}\mathrm{ev}(x^+_{1,0}))$ is equivalent to the relation $c\hbar=(m-n)\varepsilon_1$. It is nothing but assumption. This completes the proof of the case $j\neq 0$ and $i=0$.

Other cases are proven in the same way. Thus, we show that

$$[\mathrm{ev}(\tilde{h}_{i,1}),\mathrm{ev}(x^+_{j,0})]=a_{i,j}\Big(\mathrm{ev}(x^+_{j,1})-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}\mathrm{ev}(x^+_{j,0})\Big)$$

holds.

§5.3. The proof of (5.9)

We only show the cases where $i,j\neq 0$ and $i=0,j\neq 0$. The other case is proven in a similar way.

Case 1, $i,j\neq 0$. Suppose that $i,j\neq 0$. First, we compute $[\mathrm{ev}(x^+_{i,1}),\mathrm{ev}(x^+_{j,0})]$. By the definition of $\mathrm{ev}(x^+_{i,1})$, we have

$$\begin{aligned}[\mathrm{ev}(x^+_{i,1}),\mathrm{ev}(x^+_{j,0})]&=\big[\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x^+_i,E_{j,j+1}\big]\\&\quad+\Big[\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s),E_{j,j+1}\Big]\\&\quad+\Big[\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i+1}(s+1),E_{j,j+1}\Big].\end{aligned}\tag{5.37}$$

By direct computation, the second term of (5.37) is equal to

$$\begin{aligned}&\Big[\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s),E_{j,j+1}\Big]\\&=\hbar\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s)E_{k,j+1}(s)\\&\quad-\hbar\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i,j+1}(-1)^{p(k)+p(E_{i+1,i})p(E_{j,j+1})}E_{j,k}(-s)E_{k,i+1}(s)\\&\quad+\hbar\sum_{s\geq 0}\delta_{j,i}(-1)^{p(i)+p(E_{i,i+1})p(E_{i,i+1})}E_{i,i+1}(-s)E_{i,i+1}(s).\end{aligned}\tag{5.38}$$

We also find that the third term of (5.37) is equal to

$$\Big[\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i+1}(s+1),E_{j,j+1}\Big]$$
$$=\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,j+1}(s+1)$$
$$-\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i,j+1}(-1)^{p(k)+p(E_{i+1,i})p(E_{j,j+1})}E_{j,k}(-s-1)E_{k,i+1}(s+1)$$
$$-\hbar\sum_{s\geq 0}\delta_{i,j}(-1)^{p(i+1)+p(E_{i+1,i})p(E_{i,i+1})}E_{i,i+1}(-s-1)E_{i,i+1}(s+1). \tag{5.39}$$

Thus, we can rewrite $[\mathrm{ev}(x_{i,1}^+),\mathrm{ev}(x_{j,0}^+)]$ as

$$\big[\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x_i^+,E_{j,j+1}Vr\big]$$
$$+\hbar\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s)E_{k,j+1}(s)$$
$$-\hbar\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i,j+1}(-1)^{p(k)+(p(E_{i+1,k})+p(E_{k,i}))p(E_{j,j+1})}E_{j,k}(-s)E_{k,i+1}(s)$$
$$+\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,j+1}(s+1)$$
$$-\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i,j+1}(-1)^{p(k)+(p(E_{i+1,k})+p(E_{k,i}))p(E_{j,j+1})}E_{j,k}(-s-1)\times E_{k,i+1}(s+1)$$
$$+\hbar\sum_{s\geq 0}\delta_{j,i}(-1)^{p(i)+p(E_{i,i+1})p(E_{j,i+1})}E_{i,i+1}(-s)E_{i,i+1}(s)$$
$$-\hbar\sum_{s\geq 0}\delta_{i,j}(-1)^{p(i+1)+p(E_{i+1,i})p(E_{i,i+1})}E_{i,i+1}(-s-1)E_{i,i+1}(s+1). \tag{5.40}$$

Next, let us compute $[\mathrm{ev}(x_{i,0}^+),\mathrm{ev}(x_{j,1}^+)]$. Since it is equal to

$$-(-1)^{p(E_{i,i+1})p(E_{j,j+1})}[\mathrm{ev}(x_{j,1}^+),\mathrm{ev}(x_{i,0}^+)],$$

we can rewrite $[\mathrm{ev}(x_{i,0}^+),\mathrm{ev}(x_{j,1}^+)]$ as

$$\big[E_{i,i+1},\big(\alpha-(j-2\delta(j\geq m+1)(j-m))\varepsilon_1\big)x_j^+\big]$$
$$-\hbar\sum_{s\geq 0}\sum_{k=1}^{j}\delta_{i,j+1}(-1)^{p(k)+p(E_{i,i+1})(p(E_{j,k})+p(E_{k,j+1}))}E_{j,k}(-s)E_{k,i+1}(s)$$

$$
\begin{aligned}
&+\hbar\sum_{s\geq 0}\sum_{k=1}^{j}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s)E_{k,j+1}(s)\\
&-\hbar\sum_{s\geq 0}\sum_{k=j+1}^{m+n}\delta_{i,j+1}(-1)^{p(k)+p(E_{i,i+1})(p(E_{j,k})+p(E_{k,j+1}))}E_{j,k}(-s-1)\\
&\qquad\qquad\times E_{k,i+1}(s+1)\\
&+\hbar\sum_{s\geq 0}\sum_{k=j+1}^{m+n}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,j+1}(s+1)\\
&-\hbar\sum_{s\geq 0}\delta_{i,j}(-1)^{p(i)}E_{i,i+1}(-s)E_{i,i+1}(s)\\
&+\hbar\sum_{s\geq 0}\delta_{i,j}(-1)^{p(i+1)}E_{i,i+1}(-s-1)E_{i,i+1}(s+1). \qquad (5.41)
\end{aligned}
$$

By (5.40) and (5.41), when $i\neq j, j\pm 1$, $[\mathrm{ev}(x^+_{i,1}),\mathrm{ev}(x^+_{j,0})]-[\mathrm{ev}(x^+_{i,0}),\mathrm{ev}(x^+_{j,1})]$ is equal to zero.

When $i=j$, $[\mathrm{ev}(x^+_{i,1}),\mathrm{ev}(x^+_{j,0})]-[\mathrm{ev}(x^+_{i,0}),\mathrm{ev}(x^+_{j,1})]$ is equal to

$$
\begin{aligned}
&\big[\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x^+_i, E_{j,j+1}\big]\\
&\quad-\big[E_{i,i+1},\big(\alpha-(j-2\delta(j\geq m+1)(j-m))\varepsilon_1\big)x^+_j\big]\\
&\quad+\hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s)E_{i,i+1}(s)\\
&\quad-\hbar\sum_{s\geq 0}(-1)^{p(i)}E_{i,i+1}(-s-1)E_{i,i+1}(s+1)\\
&\quad+\hbar\sum_{s\geq 0}(-1)^{p(i)}E_{i,i+1}(-s)E_{i,i+1}(s)\\
&\quad-\hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s-1)E_{i,i+1}(s+1). \qquad (5.42)
\end{aligned}
$$

Since $[x^+_i,x^+_i]=0$ holds, the first and second terms are zero. We also obtain

$$\text{the third term of (5.42)}+\text{the fourth term of (5.42)}=\hbar(-1)^{p(i)}E_{i,i+1}E_{i,i+1}$$

and

$$\text{the fifth term of (5.42)}+\text{the sixth term of (5.42)}=\hbar(-1)^{p(i+1)}E_{i,i+1}E_{i,i+1}$$

by direct computation. Thus, $[\mathrm{ev}(x^+_{i,1}),\mathrm{ev}(x^+_{j,0})]-[\mathrm{ev}(x^+_{i,0}),\mathrm{ev}(x^+_{j,1})]$ is equal to $\hbar a_{i,i}E_{i,i+1}E_{i,i+1}$ since $a_{i,i}=(-1)^{p(i)}+(-1)^{p(i+1)}$ holds.

When $i=j-1$, $[\mathrm{ev}(x^+_{i,1}),\mathrm{ev}(x^+_{j,0})]-[\mathrm{ev}(x^+_{i,0}),\mathrm{ev}(x^+_{j,1})]$ is equal to

$$\begin{aligned}
&\bigl[(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x_i^+,E_{j,j+1}\bigr]\\
&\quad-\bigl[E_{i,i+1},(\alpha-(j-2\delta(j\geq m+1)(j-m))\varepsilon_1)x_j^+\bigr]\\
&\quad+\hbar\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s)E_{k,j+1}(s)\\
&\quad+\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,j+1}(s+1)\\
&\quad-\hbar\sum_{s\geq 0}\sum_{k=1}^{j}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s)E_{k,j+1}(s)\\
&\quad-\hbar\sum_{s\geq 0}\sum_{k=j+1}^{m+n}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,j+1}(s+1).
\end{aligned}\tag{5.43}$$

By direct computation, we obtain

$$\begin{aligned}
&\text{the third term of (5.43)}+\text{the fifth term of (5.43)}\\
&\quad=-\hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s)E_{i+1,i+2}(s)
\end{aligned}$$

and

$$\begin{aligned}
&\text{the fourth term of (5.42)}+\text{the sixth term of (5.42)}\\
&\quad=\hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s-1)E_{i+1,i+2}(s+1).
\end{aligned}$$

Then $[\mathrm{ev}(x^+_{i,1}),\mathrm{ev}(x^+_{j,0})]-[\mathrm{ev}(x^+_{i,0}),\mathrm{ev}(x^+_{j,1})]$ is equal to

$$\begin{aligned}
&\bigl[(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x_i^+,E_{j,j+1}\bigr]\\
&\quad-\bigl[E_{i,i+1},(\alpha-(j-2\delta(j\geq m+1)(j-m))\varepsilon_1)x_j^+\bigr]\\
&\quad-\hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s)E_{i+1,i+2}(s)\\
&\quad+\hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s-1)E_{i+1,i+2}(s+1).
\end{aligned}$$

Since $a_{i,i+1}=-(-1)^{p(i+1)}$ holds, we have

$$\begin{aligned}
&-\hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s)E_{i+1,i+2}(s)\\
&\qquad+\hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s-1)E_{i+1,i+2}(s+1)
\end{aligned}$$

$$
\begin{aligned}
&= -(-1)^{p(i+1)}\hbar E_{i,i+1}E_{i+1,i+2}\\
&= a_{i,i+1}\frac{\hbar}{2}\{E_{i,i+1},E_{i+1,i+2}\} + a_{i,i+1}\frac{\hbar}{2}[E_{i,i+1},E_{i+1,i+2}]
\end{aligned}
$$

and

$$
\begin{aligned}
&\big[\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x_i^+, E_{j,j+1}\big]\\
&\qquad -\big[E_{i,i+1},\big(\alpha-(j-2\delta(j\geq m+1)(j-m))\varepsilon_1\big)x_j^+\big]\\
&\quad= (-1)^{p(i+1)}\varepsilon_1[E_{i,i+1},E_{i+1,i+2}]\\
&\quad= -a_{i,i+1}\varepsilon_1[E_{i,i+1},E_{i+1,i+2}].
\end{aligned}
$$

Then $[\mathrm{ev}(x_{i,1}^+),\mathrm{ev}(x_{j,0}^+)] - [\mathrm{ev}(x_{i,0}^+),\mathrm{ev}(x_{j,1}^+)]$ is equal to

$$
a_{i,i+1}\frac{\hbar}{2}\{E_{i,i+1},E_{i+1,i+2}\} - a_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}[E_{i,i+1},E_{i+1,i+2}].
$$

When $i=j+1$, $[\mathrm{ev}(x_{i,1}^+),\mathrm{ev}(x_{j,0}^+)] - [\mathrm{ev}(x_{i,0}^+),\mathrm{ev}(x_{j,1}^+)]$ is equal to

$$
\begin{aligned}
&\big[\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x_i^+, E_{j,j+1}\big]\\
&\quad -\big[E_{i,i+1},\big(\alpha-(j-2\delta(j\geq m+1)(j-m))\varepsilon_1\big)x_j^+\big]\\
&\quad -\hbar\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i,j+1}(-1)^{p(k)+p(E_{i+1,i})p(E_{j,j+1})}E_{j,k}(-s)E_{k,i+1}(s)\\
&\quad -\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i,j+1}(-1)^{p(k)+p(E_{i+1,i})p(E_{j,j+1})}E_{j,k}(-s-1)E_{k,i+1}(s+1)\\
&\quad +\hbar\sum_{s\geq 0}\sum_{k=1}^{j}\delta_{i,j+1}(-1)^{p(k)+p(E_{i+1,i})p(E_{j,j+1})}E_{j,k}(-s)E_{k,i+1}(s)\\
&\quad +\hbar\sum_{s\geq 0}\sum_{k=j+1}^{m+n}\delta_{i,j+1}(-1)^{p(k)+p(E_{i+1,i})p(E_{j,j+1})}E_{j,k}(-s-1)E_{k,i+1}(s+1).
\end{aligned}
\tag{5.44}
$$

By direct computation, we find that

$$
\begin{aligned}
&\text{the third term of (5.44)} + \text{the fifth term of (5.44)}\\
&\quad= -\hbar\sum_{s\geq 0}(-1)^{p(i)}E_{i-1,i}(-s)E_{i,i+1}(s)
\end{aligned}
$$

and

$$
\begin{aligned}
&\text{the fourth term of (5.44)} + \text{the sixth term of (5.44)}\\
&\quad= \hbar\sum_{s\geq 0}(-1)^{p(i)}E_{i-1,i}(-s-1)E_{i,i+1}(s+1)
\end{aligned}
$$

hold. Since $a_{i,i-1}=-(-1)^{p(i)}$ holds, we have

$$\begin{aligned}
&-\hbar\sum_{s\geq 0}(-1)^{p(i)}E_{i-1,i}(-s)E_{i,i+1}(s)+\hbar\sum_{s\geq 0}(-1)^{p(i)}E_{i-1,i}(-s-1)E_{i,i+1}(s+1)\\
&\quad=-\hbar(-1)^{p(i)}E_{i-1,i}E_{i,i+1}\\
&\quad=\frac{\hbar}{2}a_{i-1,i}\{E_{i,i+1},E_{i-1,i}\}-\frac{\hbar}{2}a_{i-1,i}[E_{i,i+1},E_{i-1,i}]
\end{aligned}$$

and

$$\begin{aligned}
&\big[\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x_i^+,E_{j,j+1}\big]\\
&\qquad-\big[E_{i,i+1},\big(\alpha-(j-2\delta(j\geq m+1)(j-m))\varepsilon_1\big)x_j^+\big]\\
&\quad=-(-1)^{p(i)}\varepsilon_1[E_{i,i+1},E_{i-1,i}]\\
&\quad=a_{i,i-1}\varepsilon_1[E_{i,i+1},E_{i-1,i}]
\end{aligned}$$

holds, $[\mathrm{ev}(x_{i,1}^+),\mathrm{ev}(x_{j,0}^+)]-[\mathrm{ev}(x_{i,0}^+),\mathrm{ev}(x_{j,1}^+)]$ is equal to

$$\begin{aligned}
&\big[\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x_i^+,E_{j,j+1}\big]\\
&\qquad-\big[E_{i,i+1},\big(\alpha-(j-2\delta(j\geq m+1)(j-m))\varepsilon_1\big)x_j^+\big]\\
&\qquad-\hbar(-1)^{p(i)}\{E_{i-1,i},E_{i,i+1}\}+(-1)^{p(i)}\frac{\hbar}{2}E_{i-1,i+1}.
\end{aligned}$$

Therefore, it is equal to $-a_{i,i-1}\frac{\hbar}{2}\{E_{i-1,i},E_{i,i+1}\}+a_{i,i-1}\frac{\varepsilon_1-\varepsilon_2}{2}E_{i-1,i+1}$.

Case 2, $i\neq 0$ and $j=0$. Suppose that $i\neq 0$. First, we compute $[\mathrm{ev}(x_{i,1}^+),\mathrm{ev}(x_{0,0}^+)]$. By the definition of ev, we obtain

$$\begin{aligned}
[\mathrm{ev}(x_{i,1}^+),\mathrm{ev}(x_{0,0}^+)]&=\big[\big(\alpha-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1\big)x_i^+,E_{m+n,1}(1)\big]\\
&\quad+\Big[\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s),E_{m+n,1}(1)\Big]\\
&\quad+\Big[\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i+1}(s+1),E_{m+n,1}(1)\Big].
\end{aligned}\tag{5.45}$$

By direct computation, the second term of (5.45) is equal to

$$\begin{aligned}
&\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n-1}\delta_{m+n,i+1}(-1)^{p(k)}E_{m+n-1,k}(-s)E_{k,1}(s+1)\\
&\qquad-\hbar\sum_{s\geq 0}(-1)^{p(E_{m+n,1})p(E_{1,i})+p(1)}E_{i,1}(-s)E_{m+n,i}(s+1)\\
&\qquad-\hbar\sum_{s\geq 0}\delta_{1,i}E_{m+n,1}(1-s)E_{1,2}(s)
\end{aligned}\tag{5.46}$$

and the third term of (5.45) is equal to

$$\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(m+n)}E_{m+n-1,m+n}(-s-1)E_{m+n,1}(s+2)$$
$$+\hbar\sum_{s\geq 0}(-1)^{p(E_{m+n,1})p(E_{m+n,i})+p(m+n)}E_{i,1}(-s)E_{m+n,i}(s+1)$$
$$-\hbar\sum_{s\geq 0}\sum_{k=2}^{m+n}\delta_{1,i}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,1}(s+1)+\delta_{i,1}cE_{m+n,2}(1). \tag{5.47}$$

Next we rewrite the sum of the second term of (5.46) and the second term of (5.47):

the second term of (5.46) + the second term of (5.47) = 0.

Therefore, $[\mathrm{ev}(x^+_{i,1}),\mathrm{ev}(x^+_{0,0})]$ is equal to

$$\big[(\alpha-i\varepsilon_1)x^+_i,E_{m+n,1}(1)\big]$$
$$+\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n-1}\delta_{m+n,i+1}(-1)^{p(k)}E_{m+n-1,k}(-s)E_{k,1}(s+1)$$
$$-\hbar\sum_{s\geq 0}\delta_{1,i}E_{m+n,1}(1-s)E_{1,2}(s)$$
$$+\hbar\sum_{s\geq 0}\delta_{m+n,i+1}(-1)^{p(m+n)}E_{m+n-1,m+n}(-s-1)E_{m+n,1}(s+2)$$
$$-\hbar\sum_{s\geq 0}\sum_{k=2}^{m+n}\delta_{1,i}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,1}(s+1)+\delta_{i,1}cE_{m+n,2}(1). \tag{5.48}$$

Next, let us compute $[\mathrm{ev}(x^+_{i,0}),\mathrm{ev}(x^+_{0,1})]$. By direct computation, we have

$$[\mathrm{ev}(x^+_{i,0}),\mathrm{ev}(x^+_{0,1})]$$
$$=[E_{i,i+1},(\alpha-(m-n)\varepsilon_1)x^+_0]$$
$$+\Big[E_{i,i+1},\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,1}(s+1)\Big]. \tag{5.49}$$

By direct computation, the second term of (5.49) is equal to

$$\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}\delta_{m+n,i+1}(-1)^{p(k)}E_{m+n-1,k}(-s)E_{k,1}(s+1)$$
$$-\hbar\sum_{s\geq 0}(-1)^{p(i)+p(E_{i,i+1})p(E_{m+n,i})}E_{m+n,i+1}(-s)E_{i,1}(s+1)$$

$$+\hbar\sum_{s\geq 0}(-1)^{p(i+1)+p(E_{i,i+1})p(E_{m+n,i+1})}E_{m+n,i+1}(-s)E_{i,1}(s+1)$$

$$-\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}\delta_{1,i}(-1)^{p(k)+p(E_{1,2})p(E_{m+n,1})}E_{m+n,k}(-s)E_{k,i+1}(s+1). \tag{5.50}$$

The sum of the second term of (5.50) and the third term of (5.50) is equal to zero. Thus, $[\mathrm{ev}(x^+_{i,0}),\mathrm{ev}(x^+_{0,1})]$ is equal to

$$[E_{i,i+1},(\alpha-(m-n)\varepsilon_1)x^+_0]$$

$$+\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}\delta_{m+n,i+1}(-1)^{p(k)}E_{m+n-1,k}(-s)E_{k,1}(s+1)$$

$$-\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}\delta_{1,i}(-1)^{p(k)+p(E_{1,2})p(E_{m+n,1})}E_{m+n,k}(-s)E_{k,2}(s+1). \tag{5.51}$$

Therefore, when $i\neq 0,1,m+n-1$, $[\mathrm{ev}(x^+_{i,1}),\mathrm{ev}(x^+_{0,0})]-[\mathrm{ev}(x^+_{i,0}),\mathrm{ev}(x^+_{0,1})]$ is zero. When $i=1$, $[\mathrm{ev}(x^+_{1,1}),\mathrm{ev}(x^+_{0,0})]-[\mathrm{ev}(x^+_{1,0}),\mathrm{ev}(x^+_{0,1})]$ is equal to

$$[(\alpha-\varepsilon_1)x^+_1,E_{m+n,1}(1)]-[E_{1,2},(\alpha-(m-n)\varepsilon_1)x^+_0]$$

$$-\hbar\sum_{s\geq 0}E_{m+n,1}(1-s)E_{1,2}(s)$$

$$-\hbar\sum_{s\geq 0}\sum_{k=2}^{m+n}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,2}(s+1)$$

$$+cE_{m+n,2}(1)+\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,2}(s+1). \tag{5.52}$$

By direct computation, we obtain

the third term of (5.52) + the fourth term of (5.52) + the sixth term of (5.52)

$$=-\hbar E_{m+n,1}(1)E_{1,2}(0)$$

$$=-\frac{\hbar}{2}\{E_{1,2}(0),E_{m+n,1}(1)\}+\frac{\hbar}{2}[E_{1,2}(0),E_{m+n,1}(1)].$$

Moreover, by direct computation, we obtain

$$[(\alpha-\varepsilon_1)x^+_1,E_{m+n,1}(1)]-[E_{1,2},(\alpha-(m-n)\varepsilon_1)x^+_0]$$

$$=(m-n-1)\varepsilon_1[x^+_1,E_{m+n,1}(1)].$$

Therefore, $[\mathrm{ev}(x^+_{1,1}),\mathrm{ev}(x^+_{0,0})]-[\mathrm{ev}(x^+_{1,0}),\mathrm{ev}(x^+_{0,1})]$ is equal to

$$-\frac{\hbar}{2}\{E_{1,2}(0),E_{m+n,1}(1)\}-\frac{\varepsilon_1-\varepsilon_2}{2}[x^+_1,E_{m+n,1}(1)]$$

by the assumption $\hbar c=(m-n)\varepsilon_1$.

When $i=m+n-1$, $[\mathrm{ev}(x^+_{m+n-1,1}),\mathrm{ev}(x^+_{0,0})]-[\mathrm{ev}(x^+_{m+n-1,0}),\mathrm{ev}(x^+_{0,1})]$ is equal to

$$\begin{aligned}
&[(\alpha-(m-n+1)\varepsilon_1)x^+_{m+n-1},E_{m+n,1}(1)]-[E_{m+n-1,m+n},(\alpha-(m-n)\varepsilon_1)x^+_0]\\
&\quad+\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n-1}(-1)^{p(k)}E_{m+n-1,k}(-s)E_{k,1}(s+1)\\
&\quad+\hbar\sum_{s\geq 0}\sum_{k=m+n}^{m+n}(-1)^{p(k)}E_{m+n-1,k}(-s-1)E_{k,1}(s+2)\\
&\quad-\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{m+n-1,k}(-s)E_{k,1}(s+1).
\end{aligned}$$

By direct computation, we obtain

$$\begin{aligned}
&\text{the third term of (5.52)}+\text{the fourth term of (5.52)}+\text{the fifth term of (5.52)}\\
&\quad=\hbar E_{m+n-1,m+n}(0)E_{m+n,1}(1)\\
&\quad=\frac{\hbar}{2}\{E_{m+n-1,m+n}(0),E_{m+n,1}(1)\}+\frac{\hbar}{2}[E_{m+n-1,m+n}(0),E_{m+n,1}(1)].
\end{aligned}$$

Moreover, by direct computation, we have

$$[(\alpha-(m-n+1)\varepsilon_1)x^+_i,E_{m+n,1}(1)]-[E_{i,i+1},(\alpha-(m-n)\varepsilon_1)x^+_0]=-\varepsilon_1[x^+_i,x^+_0].$$

Then $[\mathrm{ev}(x^+_{m+n-1,1}),\mathrm{ev}(x^+_{0,0})]-[\mathrm{ev}(x^+_{m+n-1,0}),\mathrm{ev}(x^+_{0,1})]$ is equal to

$$\frac{\hbar}{2}\{E_{m+n-1,m+n}(0),E_{m+n,1}(1)\}-\frac{\varepsilon_1-\varepsilon_2}{2}[x^+_{m+n-1},x^+_0].$$

This completes the proof of (5.9).

§5.4. The proof of (5.10)

Finally, we show $[\mathrm{ev}(h_{i,1}),\mathrm{ev}(h_{j,1})]=0$. Suppose that $i,j\neq 0$. It is enough to show the case where $i<j$. We set

$$\begin{aligned}
A_i&=\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s),\\
B_i&=\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i}(s+1),
\end{aligned}$$

$$C_i = \sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)E_{k,i+1}(s),$$
$$D_i = \sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i+1}(s+1).$$

Then, by the definition of $\mathrm{ev}(h_{i,1})$, we have

$$\begin{aligned}[\mathrm{ev}(h_{i,1}),\mathrm{ev}(h_{j,1})] &= (-1)^{p(i)+p(j)}\{[A_i,A_j]+[B_i,A_j]+[B_i,B_j]+[A_i,B_j]\}\\ &\quad+(-1)^{p(i)+p(j+1)}\{[A_i,C_j]+[B_i,C_j]+[B_i,D_j]+[A_i,D_j]\}\\ &\quad+(-1)^{p(i+1)+p(j)}\{[C_i,A_j]+[D_i,B_j]+[D_i,A_j]+[C_i,B_j]\}\\ &\quad+(-1)^{p(i+1)+p(j+1)}\{[C_i,C_j]+[D_i,C_j]+[D_i,D_j]+[C_i,D_j]\}.\end{aligned}$$

By the definitions of A_i, B_i, C_i and D_i, we obtain

$$[A_i,B_j]=[A_i,D_j]=0.$$

Thus, it is enough to show the following lemma.

Lemma 5.53. *The following relations hold:*

$$\begin{aligned}[A_i,A_j]+[B_i,A_j]+[B_i,B_j]&=0,\\ [A_i,C_j]+[B_i,C_j]+[B_i,D_j]+[A_i,D_j]&=0,\\ [C_i,A_j]+[D_i,B_j]+[D_i,B_j]+[C_i,B_j]&=0,\\ [C_i,C_j]+[D_i,C_j]+[D_i,D_j]&=0.\end{aligned}$$

Proof. We only show that $[A_i,A_j]+[B_i,A_j]+[B_i,B_j]=0$ holds. The other relations are obtained in the same way. By direct computation, we can rewrite $[A_i,A_j]$ as

$$\begin{aligned}[A_i,A_j] = &-\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(k)+p(i)+p(E_{i,k})p(E_{j,i})}E_{j,k}(-s-t)E_{i,j}(t)E_{k,i}(s)\\ &+\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-t)E_{i,j}(-s+t)E_{k,i}(s)\\ &-\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s)E_{j,i}(s-t)E_{k,j}(t)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(k)+p(i)+p(E_{k,i})p(E_{i,j})}E_{i,k}(-s)E_{j,i}(-t)E_{k,j}(s+t)\\
&+\sum_{s\geq 0}(sE_{i,i}(-s)E_{j,j}(s)-sE_{j,j}(-s)E_{i,i}(s)). \qquad (5.54)
\end{aligned}
$$

Since we find the two relations

the second term of (5.54)

$$
\begin{aligned}
&=\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s-t)E_{i,j}(t)E_{k,i}(s)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s)E_{i,j}(-t-1)E_{k,i}(s+t+1),
\end{aligned}
$$

the third term of (5.54)

$$
\begin{aligned}
&=\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{j,l})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{k,j}(t)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{k,j})}E_{i,k}(-s)E_{j,i}(-t)E_{k,j}(s+t),
\end{aligned}
$$

we have

$$
\begin{aligned}
[A_i,A_j]=&-\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(k)+p(i)+p(E_{i,k})p(E_{j,i})}E_{j,k}(-s-t)E_{i,j}(t)E_{k,i}(s)\\
&+\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s-t)E_{i,j}(t)E_{k,i}(s)\\
&+\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s)E_{i,j}(-t-1)E_{k,i}(s+t+1)\\
&-\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{j,l})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{k,j}(t)\\
&-\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{k,j})}E_{i,k}(-s)E_{j,i}(-t)E_{k,j}(s+t)\\
&+\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(k)+p(i)+p(E_{k,i})p(E_{i,j})}E_{i,k}(-s)E_{j,i}(-t)E_{k,j}(s+t)\\
&+\sum_{s\geq 0}(sE_{i,i}(-s)E_{j,j}(s)-sE_{j,j}(-s)E_{i,i}(s)). \qquad (5.55)
\end{aligned}
$$

We simplify the right-hand side of (5.55). By direct computation, we obtain

$$\begin{aligned}
&\text{the first term of (5.55)} + \text{the second term of (5.55)}\\
&= -\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(k)+p(i)+p(E_{i,k})p(E_{j,i})}E_{j,k}(-s-t)E_{i,j}(t)E_{k,i}(s)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s-t)E_{i,j}(t)E_{k,i}(s)=0
\end{aligned}\tag{5.56}$$

since $p(k)+p(i)+p(E_{i,k})p(E_{j,i})=p(E_{i,k})p(E_{j,k})$. Similarly, we have

$$\text{the fourth term of (5.55)} + \text{the sixth term of (5.55)} = 0.\tag{5.57}$$

By (5.56) and (5.57), we find the equality

$$\begin{aligned}
&[A_i, A_j]\\
&=\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(k)+p(l)+p(E_{i,k})p(E_{j,l})}E_{j,l}(-s)E_{i,j}(-t-1)E_{k,i}(s+t+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(k)+p(l)+p(E_{k,i})p(E_{j,l})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{l,j}(t)\\
&\quad+\sum_{s\geq 0}(sE_{i,i}(-s)E_{j,j}(s)-sE_{j,j}(-s)E_{i,i}(s)).
\end{aligned}$$

Computing the parity, we obtain

$$\begin{aligned}
[A_i, A_j] &= \sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s)E_{i,j}(-t-1)E_{k,i}(s+t+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{k,j}(t)\\
&\quad+\sum_{s\geq 0}(sE_{i,i}(-s)E_{j,j}(s)-sE_{j,j}(-s)E_{i,i}(s)).
\end{aligned}\tag{5.58}$$

Similarly, by direct computation, we have

$$\begin{aligned}
[B_i, B_j] &= \sum_{s,t\geq 0}\sum_{l=j+1}^{m+n}(-1)^{p(j)+p(l)+p(E_{j,l})p(E_{j,i})}E_{i,l}(-s-t-2)E_{j,i}(s+1)\\
&\qquad\times E_{l,j}(t+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=j+1}^{m+n}\delta_{k,l}(-1)^{p(E_{j,k})p(E_{k,i})}E_{i,k}(-s-t-2)E_{j,i}(s+1)\\
&\qquad\times E_{k,j}(t+1)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=j+1}^{m+n}\delta_{k,l}(-1)^{p(E_{j,k})p(E_{k,i})}E_{i,k}(-s-1)E_{j,i}(-t)\\
&\qquad\qquad\times E_{k,j}(s+t+1)\\
&+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=j+1}^{m+n}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{k,j})}E_{j,l}(-s-t-1)E_{i,j}(t)\\
&\qquad\qquad\times E_{k,i}(s+1)\\
&+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=j+1}^{m+n}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{k,j})}E_{j,k}(-t-1)E_{i,j}(-s-1)\\
&\qquad\qquad\times E_{k,i}(s+t+2)\\
&-\sum_{s,t\geq 0}\sum_{l=j+1}^{m+n}(-1)^{p(j)+p(l)+p(E_{j,i})p(E_{l,j})}E_{j,l}(-t-1)E_{i,j}(-s-1)\\
&\qquad\qquad\times E_{l,i}(s+t+2).
\end{aligned}\tag{5.59}
$$

We simplify the right-hand side of (5.59). By direct computation, we obtain

$$\text{the first term of } (5.59) + \text{the second term of } (5.59) = 0 \tag{5.60}$$

and

$$\text{the fifth term of } (5.59) + \text{the sixth term of } (5.59) = 0. \tag{5.61}$$

By (5.60) and (5.61), we find the equality

$$
\begin{aligned}
[B_i, B_j] &= -\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=j+1}^{m+n}\delta_{k,l}(-1)^{p(E_{j,k})p(E_{k,i})}E_{i,k}(-s-1)E_{j,i}(-t)\\
&\qquad\qquad\times E_{k,j}(s+t+1)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=j+1}^{m+n}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{k,j})}E_{j,k}(-s-t-1)E_{i,j}(t)\\
&\qquad\qquad\times E_{k,i}(s+1)\\
&= -\sum_{s,t\geq 0}\sum_{l=j+1}^{m+n}(-1)^{p(E_{j,l})p(E_{l,i})}E_{i,l}(-s-1)E_{j,i}(-t)E_{l,j}(s+t+1)\\
&\quad+\sum_{s,t\geq 0}\sum_{l=j+1}^{m+n}(-1)^{p(E_{l,i})p(E_{l,j})}E_{j,l}(-s-t-1)E_{i,j}(t)E_{l,i}(s+1).
\end{aligned}\tag{5.62}
$$

By direct computation, we also obtain

$$
\begin{aligned}
[B_i, A_j] &= \sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(j)+p(l)}E_{i,l}(-s-t-1)E_{l,j}(t)E_{j,i}(s+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)+p(i)+p(E_{i,k})p(E_{j,i})}E_{j,k}(-s-t-1)E_{i,j}(t)\\
&\qquad\qquad\times E_{k,i}(s+1)
\end{aligned}
$$

$$+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,l})}E_{j,l}(-s-t-1)E_{i,j}(t)\times E_{k,i}(s+1)$$

$$+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,l})}E_{j,l}(-s)E_{i,j}(-t-1)\times E_{k,i}(s+t+1)$$

$$-\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{j,l})}E_{i,k}(-s-1)E_{j,i}(-t)\times E_{l,j}(s+t+1)$$

$$-\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{j,l})}E_{i,k}(-s-t-1)E_{j,i}(s+1)\times E_{l,j}(t)$$

$$+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)+p(i)+p(E_{k,i})p(E_{i,j})}E_{i,k}(-s-1)E_{j,i}(-t)\times E_{k,j}(s+t+1)$$

$$-\sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(j)+p(l)}E_{i,j}(-s-1)E_{j,l}(-t)E_{l,i}(s+t+1). \tag{5.63}$$

Let us simplify the right-hand side of (5.63). We prepare the following four relations by direct computation:

the second term of (5.63) + the third term of (5.63)

$$=-\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)+p(i)+p(E_{i,k})p(E_{j,i})}E_{j,k}(-s-t-1)E_{i,j}(t)\times E_{k,i}(s+1)$$

$$+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s-t-1)E_{i,j}(t)\times E_{k,i}(s+1)$$

$$=-\sum_{s,t\geq 0}\sum_{k=j+1}^{m+n}(-1)^{p(E_{i,k})p(E_{j,i})}E_{j,k}(-s-t-1)E_{i,j}(t)E_{k,i}(s+1), \tag{5.64}$$

the first term of (5.63) + the sixth term of (5.63)

$$=\sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(j)+p(l)}E_{i,l}(-s-t-1)E_{l,j}(t)E_{j,i}(s+1)$$

$$-\sum_{s,t\geq 0}\sum_{l=1}^{j}\sum_{k=i+1}^{m+n}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-t-1)E_{j,i}(s+1)\times E_{k,j}(t)$$

$$
\begin{aligned}
&= \sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{k,j}(t)\\
&\quad+\sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(j)+p(l)}E_{i,l}(-s-t-1)[E_{l,j}(t),E_{j,i}(s+1)]\\
&= \sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{k,j}(t)\\
&\quad+\sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(j)+p(l)}E_{i,l}(-s-t-1)E_{l,i}(s+t+1)\\
&\quad-\sum_{s,t\geq 0}E_{i,i}(-s-t-1)E_{j,j}(s+t+1),
\end{aligned}
\tag{5.65}
$$

the fourth term of (5.63) + the eighth term of (5.63)

$$
\begin{aligned}
&= \sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s)E_{i,j}(-t-1)\\
&\qquad\qquad\qquad\times E_{k,i}(s+t+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(k)+p(l)}E_{i,j}(-s-1)E_{j,l}(-t)E_{l,i}(s+t+1)\\
&= -\sum_{s,t\geq 0}\sum_{l=1}^{i}(-1)^{p(E_{i,l})p(E_{j,l})}E_{j,l}(-s)E_{i,j}(-t-1)E_{l,i}(s+t+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(j)+p(l)}[E_{i,j}(-s-1),E_{j,l}(-t)]E_{l,i}(s+t+1)\\
&= -\sum_{s,t\geq 0}\sum_{l=1}^{i}(-1)^{p(E_{i,l})p(E_{j,l})}E_{j,l}(-s)E_{i,j}(-t-1)E_{l,i}(s+t+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(j)+p(l)}E_{i,l}(-s-t-1)E_{l,i}(s+t+1)\\
&\quad+\sum_{s,t\geq 0}E_{j,j}(-s-t-1)E_{i,i}(s+t+1),
\end{aligned}
\tag{5.66}
$$

the fifth term of (5.63) + the seventh term of (5.63)

$$
\begin{aligned}
&= -\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-1)E_{j,i}(-t)\\
&\qquad\qquad\qquad\times E_{k,j}(s+t+1)
\end{aligned}
$$

$$+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)+p(i)+p(E_{k,i})p(E_{i,j})}E_{i,k}(-s-1)E_{j,i}(-t)\times E_{k,j}(s+t+1)$$

$$=\sum_{s,t\geq 0}\sum_{k=j+1}^{m+n}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-1)E_{j,i}(-t)E_{k,j}(s+t+1). \tag{5.67}$$

Thus, by (5.64)–(5.67), we have

$$\begin{aligned}[B_i,A_j]=&-\sum_{s,t\geq 0}\sum_{k=j+1}^{m+n}(-1)^{p(E_{i,k})p(E_{j,i})}E_{j,k}(-s-t-1)E_{i,j}(t)E_{k,i}(s+1)\\&+\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{k,j}(t)\\&-\sum_{s,t\geq 0}\sum_{l=1}^{i}(-1)^{p(E_{i,l})p(E_{j,l})}E_{j,l}(-s)E_{i,j}(-t-1)E_{l,i}(s+t+1)\\&+\sum_{s,t\geq 0}\sum_{k=j+1}^{m+n}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-1)E_{j,i}(-t)E_{k,j}(s+t+1)\\&-\sum_{s\geq 0}\big(sE_{i,i}(-s)E_{j,j}(s)-sE_{j,j}(-s)E_{i,i}(s)\big).\end{aligned} \tag{5.68}$$

Adding (5.58), (5.62) and (5.68), we obtain $[A_i,A_j]+[B_i,A_j]+[B_i,B_j]=0$. □

This completes the proof of Lemma 5.6.

Acknowledgements

The author wishes to express his gratitude to his supervisor Tomoyuki Arakawa for providing lots of advice to improve this paper, and he would like to thank Jiaking Bao for pointing out a mistake in the definition of $b_{i,j}$. The author is also grateful for the support and encouragement of Junichi Matsuzawa. This work was supported by an Iwadare Scholarship and JSPS KAKENHI, Grant-in-Aid for JSPS Fellows, Grant Number JP20J12072.

References

[1] S. I. Boyarchenko and S. Z. Levendorskiĭ, On affine Yangians, Lett. Math. Phys. **32** (1994), 269–274. Zbl 0834.17012 MR 1310291

[2] A. Braverman, B. Feigin, M. Finkelberg and L. Rybnikov, A finite analog of the AGT relation I: Finite W-algebras and quasimaps' spaces, Comm. Math. Phys. **308** (2011), 457–478. Zbl 1247.81169 MR 2851149

[3] C. Briot and E. Ragoucy, $\mathcal{W}$-superalgebras as truncations of super-Yangians, J. Phys. A **36** (2003), 1057–1081. Zbl 1057.17019 MR 1965473

[4] J. Brundan and A. Kleshchev, Shifted Yangians and finite W-algebras, Adv. Math. **200** (2006), 136–195. Zbl 1083.17006 MR 2199632

[5] V. G. Drinfel'd, Hopf algebras and the quantum Yang–Baxter equation, Dokl. Akad. Nauk SSSR **283** (1985), 1060–1064. Zbl 0588.17015 MR 802128

[6] V. G. Drinfel'd, A new realization of Yangians and of quantum affine algebras, Dokl. Akad. Nauk SSSR **296** (1987), 13–17. MR 914215

[7] M. R. Gaberdiel, W. Li, C. Peng and H. Zhang, The supersymmetric affine Yangian, J. High Energy Phys. **2018** (2018), art. no. 200, 32 pp. Zbl 1391.81161 MR 3814987

[8] D. Gaiotto, Asymptotically free $N = 2$ theories and irregular conformal blocks, J. Phys. Conf. Ser. **462** (2013), art. no. 012014.

[9] L. Gow, Yangians of Lie superalgebras, PhD thesis, University of Sydney, 2007.

[10] P. Grozman and D. Leites, Defining relations for Lie superalgebras with Cartan matrix, Czechoslovak J. Phys. **51** (2001), 1–21. Zbl 0979.17002 MR 1833993

[11] N. Guay, Affine Yangians and deformed double current algebras in type A, Adv. Math. **211** (2007), 436–484. Zbl 1142.17008 MR 2323534

[12] N. Guay, H. Nakajima and C. Wendlandt, Coproduct for Yangians of affine Kac–Moody algebras, Adv. Math. **338** (2018), 865–911. Zbl 1447.17012 MR 3861718

[13] V. G. Kac, Lie superalgebras, Adv. Math. **26** (1977), 8–96. Zbl 0366.17012 MR 486011

[14] V. G. Kac, *Infinite-dimensional Lie algebras*, 3rd edn., Cambridge University Press, Cambridge, 1990. Zbl 0716.17022 MR 1104219

[15] R. Kodera, Braid group action on affine Yangian, SIGMA Symmetry Integrability Geom. Methods Appl. **15** (2019), art. no. 020, 28 pp. Zbl 1481.17012 MR 3923494

[16] R. Kodera, On Guay's evaluation map for affine Yangians, Algebr. Represent. Theory **24** (2021), 253–267. Zbl 1502.17010 MR 4207398

[17] S. Z. Levendorskiĭ, On generators and defining relations of Yangians, J. Geom. Phys. **12** (1993), 1–11. Zbl 0785.17019 MR 1226802

[18] A. Matsuo, K. Nagatomo and A. Tsuchiya, Quasi-finite algebras graded by Hamiltonian and vertex operator algebras, in *Moonshine: The first quarter century and beyond*, London Mathematical Society Lecture Note Series 372, Cambridge University Press, Cambridge, 2010, 282–329. Zbl 1227.17013 MR 2681785

[19] I. M. Musson, *Lie superalgebras and enveloping algebras*, Graduate Studies in Mathematics 131, American Mathematical Society, Providence, RI, 2012. Zbl 1255.17001 MR 2906817

[20] Y.-N. Peng, On shifted super Yangians and a class of finite W-superalgebras, J. Algebra **422** (2015), 520–562. Zbl 1302.17015 MR 3272090

[21] E. Ragoucy and P. Sorba, Yangian realisations from finite $\mathcal{W}$-algebras, Comm. Math. Phys. **203** (1999), 551–572. Zbl 0943.17009 MR 1700166

[22] M. Scheunert, Serre type presentation of special linear Lie superalgebras, in *Quantum symmetries (Clausthal, 1991)*, World Scientific, River Edge, NJ, 1993, 259–263. MR 1339535

[23] O. Schiffmann and E. Vasserot, Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on $\mathbf{A}^2$, Publ. Math. Inst. Hautes Études Sci. **118** (2013), 213–342. Zbl 1284.14008 MR 3150250

[24] V. Stukopin, Yangians of classical Lie superalgebras: Basic constructions, quantum double and universal R-matrix, in *Symmetry in nonlinear mathematical physics. Part 1, 2, 3*, Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos. 50, Part 1, 2 3, Natsīonal. Akad. Nauk Ukraïni, Īnst. Mat., Kiev, 2004, 1195–1201. Zbl 1122.17011 MR 2077938

[25] M. Ueda, Coproduct for the Yangian of type $A_2^{(2)}$, In preparation.

[26] H. Yamane, On defining relations of affine Lie superalgebras and affine quantized universal enveloping superalgebras, Publ. Res. Inst. Math. Sci. **35** (1999), 321–390. Zbl 0987.17007 MR 1710748