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Further Examples of Non-geometric Sections of Arithmetic Fundamental Groups

by

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Abstract

We show the existence of group-theoretic sections of certain *geometrically pro-nilpotent by abelian* arithmetic fundamental groups of hyperbolic curves over p-adic local fields which are *non-geometric*, i.e., which do not arise from rational points. Among these quotients is the *geometrically metabelian* arithmetic fundamental group.

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§0. Introduction/statement of the main result

Grothendieck's anabelian section conjecture predicts that sections of arithmetic fundamental groups of hyperbolic curves over finitely generated fields over $\mathbb{Q}$ arise from rational points (cf. [Saïdi] for a more precise formulation of the conjecture). Accordingly, sections of arithmetic fundamental groups of hyperbolic curves over p-adic local fields, which are defined over number fields and which arise from global sections, should arise from rational points. In this context it is tempting to predict a *p-adic analog of Grothendieck's anabelian section conjecture*. In [Saïdi3] we investigated such an analog and exhibited two necessary and sufficient conditions for a section of the arithmetic fundamental group of a hyperbolic curve over a p-adic local field to be geometric, i.e., to arise from a rational point (cf. [Saïdi3, Thm. 4.5]).

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For the time being there are no examples of sections of (the full) arithmetic fundamental groups of hyperbolic curves over p-adic local fields which are non-geometric, and one can still hope for the validity of a p-adic analog of the section conjecture. On the other hand, recent examples were found of group-theoretic sections of certain (*geometrically characteristic*) *quotients* of arithmetic fundamental groups of curves over p-adic local fields which are *non-geometric*. Hoshi constructed examples of sections of the *geometrically pro-p* quotient of arithmetic fundamental groups of curves over p-adic local fields which are non-geometric (cf. [Hoshi]). (Actually, Hoshi's example arises from group-theoretic sections of geometrically pro-p fundamental groups of hyperbolic curves over number fields (cf. [Hoshi]).) In [Saïdi] we constructed examples of group-theoretic sections of *geometrically prime-to-p* fundamental groups of hyperbolic curves over p-adic local fields which are non-geometric (cf. [Saïdi, Sect. 3]). Further, in [Saïdi2] we provided examples of group-theoretic sections of the "étale by geometrically abelian" fundamental group of hyperbolic curves over p-adic local fields which are non-geometric. *The existence of these examples is crucial for our understanding of the p-adic section conjecture.* Indeed, if the p-adic version of the section conjecture holds true then it may possibly hold true even for smaller quotients of the arithmetic fundamental group, and one would like to know these quotients in this case. On the other hand, more elaborate examples of non-geometric sections as above may lead to a counterexample for the p-adic version of the section conjecture.

In this note we provide further examples of sections of certain quotients of arithmetic fundamental groups of curves over p-adic local fields which are non-geometric. These quotients include the *geometrically metabelian* and certain *geometrically pro-nilpotent by abelian* quotients.

Next we fix notation and state our main results:

- Let
$$1 \to H' \to H \xrightarrow{\text{pr}} G \to 1$$
be an exact sequence of profinite groups. We will refer to a continuous homomorphism $s\colon G \to H$ satisfying $\text{pr} \circ s = \text{id}_G$ as a (group-theoretic) *section*, or *splitting*, of the above sequence, or simply a section of the projection $\text{pr}\colon H \twoheadrightarrow G$. We denote by $\text{Sect}(H \twoheadrightarrow G)$ the set of sections of the projection $H \twoheadrightarrow G$.
- Given a profinite group H and a prime integer ℓ, we will denote by H^ℓ the maximal *pro-ℓ* quotient of H, by H^{ab} the maximal *abelian* quotient of H and by $H^{\text{ab},\ell}$ its maximal *abelian pro-ℓ* quotient. Thus $H^{\text{ab},\ell} = (H^\ell)^{\text{ab}}$.

  Let $p \geq 2$ be a *prime integer* and k a *p-adic local field*, meaning that $k/\mathbb{Q}_p$ is a finite extension, with ring of integers $\mathcal{O}_k$, and residue field F. Thus

F is a *finite field* of characteristic p. Let $X \to \operatorname{Spec} k$ be a proper, smooth, and geometrically connected *hyperbolic* (i.e., genus$(X) \geq 2$) *curve* over k. Let η be a geometric point of X above its generic point, which determines an algebraic closure $\bar{k}$ of k, and a geometric point $\bar{\eta}$ of $\overline{X} \overset{\text{def}}{=} X \times_k \bar{k}$. There exists a canonical exact sequence of profinite groups (cf. [Grothendieck, Exp. IX, Thm. 6.1])

$$1 \to \pi_1(\overline{X}, \bar{\eta}) \to \pi_1(X, \eta) \to G_k \to 1.$$

Here, $\pi_1(X, \eta)$ denotes the *arithmetic étale fundamental group* of X with base point η, $\pi_1(\overline{X}, \bar{\eta})$ the étale fundamental group of $\overline{X} \overset{\text{def}}{=} X \times_k \bar{k}$ with base point $\bar{\eta}$, and $G_k \overset{\text{def}}{=} \operatorname{Gal}(\bar{k}/k)$ the absolute Galois group of k.

- Let Π be a quotient of $\pi_1(X, \eta)$ such that the projection $\pi_1(X, \eta) \twoheadrightarrow G_k$ factors as $\pi_1(X, \eta) \twoheadrightarrow \Pi \twoheadrightarrow G_k$, and which is geometrically non-trivial, meaning that $\operatorname{Ker}(\Pi \twoheadrightarrow G_k)$ is non-trivial. Given a section $s \colon G_k \to \Pi$ of the projection $\Pi \twoheadrightarrow G_k$, we say that s is *geometric* if $s(G_k)$ is contained in (hence equal to) the decomposition group $D_x \subset \Pi$ associated to a rational point $x \in X(k)$. In this case we say s arises from the rational point x. We say that the section s is *non-geometric* if s is not geometric in the above sense, i.e., $s(G_k)$ is *not* contained in the decomposition group associated to a rational point $x \in X(k)$. (Note that in the above discussion the decomposition group D_x is only defined up to conjugation by elements of $\operatorname{Ker}(\Pi \twoheadrightarrow G_k)$.)
- We *assume* $X(k) \neq \emptyset$. We *fix* a k-rational point $x \in X(k)$ and $s \overset{\text{def}}{=} s_x \colon G_k \to \pi_1(X, \eta)$ a section of the projection $\pi_1(X, \eta) \twoheadrightarrow G_k$ associated to x. Thus s is defined only up to conjugation by $\pi_1(\overline{X}, \bar{\eta})$. Note that the section s induces a structure of G_k-group on any characteristic quotient of $\pi_1(\overline{X}, \bar{\eta})$.
- Let Δ be a quotient of $\pi_1(\overline{X}, \bar{\eta})$ which fits in an exact sequence

$$1 \to \widetilde{H} \to \pi_1(\overline{X}, \bar{\eta}) \to \Delta \to 1,$$

where $\widetilde{H} \overset{\text{def}}{=} \operatorname{Ker}(\pi_1(\overline{X}, \bar{\eta}) \twoheadrightarrow \Delta)$. We consider the following Condition $(\star)$ on Δ.

Condition $(\star)$**.**

(i) Δ *is* pro-nilpotent *and is a* characteristic *quotient of* $\pi_1(\overline{X}, \bar{\eta})$.

(ii) $H^0(U, \Delta) = 0$ *for every open subgroup* U *of* G_k.

(iii) *The quotient* $\pi_1(\overline{X}, \bar{\eta}) \twoheadrightarrow \pi_1(\overline{X}, \bar{\eta})^{\text{ab}}$ factors *as*

$$\pi_1(\overline{X}, \bar{\eta}) \twoheadrightarrow \Delta \twoheadrightarrow \pi_1(\overline{X}, \bar{\eta})^{\text{ab}}.$$

(iv) *Let* $H \overset{\text{def}}{=} \widetilde{H}^{\text{ab}}$ *and* $\Gamma \overset{\text{def}}{=} \pi_1(\overline{X}, \bar{\eta}) / \operatorname{Ker}(\widetilde{H} \twoheadrightarrow H)$. *We have a push-out diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widetilde{H} & \longrightarrow & \pi_1(\overline{X}, \bar{\eta}) & \longrightarrow & \Delta & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & H & \longrightarrow & \Gamma & \longrightarrow & \Delta & \longrightarrow & 1 \end{array}$$

where the middle and left vertical maps are surjective. There exists a prime integer $\ell \neq p$, *such that the natural surjective map*

$$\Gamma^\ell \twoheadrightarrow \Delta^\ell$$

is not an isomorphism.

Let

$$\Pi \overset{\text{def}}{=} \pi_1(X, \eta) / \operatorname{Ker}(\pi_1(\overline{X}, \bar{\eta}) \twoheadrightarrow \Gamma),$$
$$\widetilde{\Pi} \overset{\text{def}}{=} \pi_1(X, \eta) / \operatorname{Ker}(\pi_1(\overline{X}, \bar{\eta}) \twoheadrightarrow \Delta).$$

We have the following push-out diagrams:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\overline{X}, \bar{\eta}) & \longrightarrow & \pi_1(X, \eta) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Delta & \longrightarrow & \widetilde{\Pi} & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

where the vertical maps are surjective. Thus $\operatorname{Ker}(\Pi \twoheadrightarrow \widetilde{\Pi}) = \operatorname{Ker}(\Gamma \twoheadrightarrow \Delta) = H$.

- An example of a quotient Δ satisfying Condition $(\star)$ is $\Delta = \pi_1(\overline{X}, \bar{\eta})^{\text{ab}}$, the maximal abelian quotient of $\pi_1(\overline{X}, \bar{\eta})$ (cf. [Saïdi1, Lem. 1.3] for Condition $(\star)$(ii)). In this case $\widetilde{\Pi} \overset{\text{def}}{=} \pi_1(X, \eta)^{(\text{ab})}$ is the geometrically abelian quotient of $\pi_1(X, \eta)$, Γ is the maximal metabelian quotient of $\pi_1(\overline{X}, \bar{\eta})$, and Π is the *geometrically metabelian* quotient of $\pi_1(X, \eta)$. More generally, any pro-nilpotent characteristic quotient Δ of $\pi_1(\overline{X}, \bar{\eta})$ which satisfies Conditions $(\star)$(ii), (iii), and for which there exists a prime integer $\ell \neq p$ such that the natural projection $\pi_1(\overline{X}, \bar{\eta})^\ell \twoheadrightarrow \Delta^\ell$ is not an isomorphism, satisfies Condition $(\star)$.

Given a finite extension k'/k (all finite extensions of k we consider are contained in $\bar{k}$) and the corresponding open subgroup $G_{k'} \subseteq G_k$, we will denote by $\Pi_{k'}$ the pull-back of the group extension Π by $G_{k'} \hookrightarrow G_k$. Thus we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \Pi_{k'} & \longrightarrow & G_{k'} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

where the right square is cartesian. Likewise we write $\widetilde{\Pi}_{k'} \stackrel{\text{def}}{=} \widetilde{\Pi} \times_{G_k} G_{k'}$. Note that $\Pi_{k'}$ is a quotient of $\pi_1(X_{k'}, \eta)$, where $X_{k'} \stackrel{\text{def}}{=} X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ and η is naturally induced by the above geometric point η. Our first main result in this paper is the following.

Theorem A. *We use notation as above. Let X be a proper, smooth, and geometrically connected hyperbolic curve over the p-adic local field k. Assume that $X(k) \neq \emptyset$ and that X has* potentially good reduction*. Let Δ be a quotient of $\pi_1(\overline{X}, \bar{\eta})$* satisfying Condition *($\star$), and Π the corresponding quotient of $\pi_1(X, \eta)$ as in the above discussion which fits in the exact sequence $1 \to \Gamma \to \Pi \to G_k \to 1$. Then there exists a finite extension $\tilde{k}/k$ such that the following holds. For every finite extension $k'/\tilde{k}$, there exists a section $s\colon G_{k'} \to \Pi_{k'}$ of the projection $\Pi_{k'} \twoheadrightarrow G_{k'}$ which is* non-geometric.

As a corollary of Theorem A we obtain the following (cf. examples discussed after introducing Condition ($\star$)).

Corollary B. *There exist* non-geometric *sections of geometrically metabelian arithmetic fundamental groups of hyperbolic curves over p-adic local fields.*

Let $m \geq 1$ be an integer. With the notation above, let $\Delta_m \stackrel{\text{def}}{=} \Delta_{m,X}$ be the maximal *m-step solvable pro-p* quotient of $\pi_1(\overline{X}, \bar{\eta})$, and $\Pi_m \stackrel{\text{def}}{=} \Pi_{m,X}$ the *geometrically m-step solvable pro-p* quotient of $\pi_1(X, \eta)$ which sits in the exact sequence

$$1 \to \Delta_m \to \Pi_m \to G_k \to 1$$

(cf. [Saïdi1, Sect. 1]). Note that Δ_m does not satisfy Condition ($\star$)(iii). It is plausible, in light of Hoshi's example in [Hoshi] (cf. above discussion), that there exist non-geometric sections of Π_m for a suitable X/k as above (this is easily seen if $m = 1$, using the Kummer exact sequence associated to Pic^0_X). In this context we prove the following.

Theorem C. *We use notation as above. There exists an integer $N \geq 2$, such that the following holds. For every prime integer $p \geq N$ there exists a proper,* *smooth, and geometrically connected hyperbolic curve X over a p-adic local field k, an integer $m \geq 2$, and a section $s\colon G_k \to \Pi_{m,X}$ of the projection $\Pi_{m,X} \twoheadrightarrow G_k$ which is* non-geometric.

§1. Proof of Theorem A

We use the notation introduced in Section 0, as well as the notation and assumptions in Theorem A. Thus X is a proper, smooth, and geometrically connected hyperbolic curve over the p-adic local field k, $X(k) \neq \emptyset$, and we assume (without loss of generality) that X has good reduction over $\mathcal{O}_k$. Further, Δ is a characteristic quotient of $\pi_1(\overline{X}, \bar{\eta})$ satisfying Condition $(\star)$, and Π is the corresponding quotient of $\pi_1(X, \eta)$ as above which fits in the exact sequence $1 \to \Gamma \to \Pi \to G_k \to 1$ (cf. Section 0). We have the following commutative diagram of exact sequences:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H & = & H & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & | \\
1 & \longrightarrow & \Delta & \longrightarrow & \widetilde{\Pi} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{1.1}
$$

Recall that $x \in X(k)$ is a k-rational point and $s \overset{\text{def}}{=} s_x\colon G_k \to \pi_1(X, \eta)$ is a section of the projection $\pi_1(X, \eta) \twoheadrightarrow G_k$ associated to x. Further, s induces sections $s_1 \overset{\text{def}}{=} s_{1,x}\colon G_k \to \widetilde{\Pi}$ of the projection $\widetilde{\Pi} \twoheadrightarrow G_k$ and $s_2 \overset{\text{def}}{=} s_{2,x}\colon G_k \to \Pi$ of the projection $\Pi \twoheadrightarrow G_k$, which fit in a commutative diagram

$$
\begin{array}{ccc}
G_k & \xrightarrow{s_2} & \Pi \\
\| & & \downarrow \\
G_k & \xrightarrow{s_1} & \widetilde{\Pi}
\end{array}
$$

where the right vertical map is the one in diagram (1.1).

The profinite group Δ is *finitely generated*, as follows from the well-known finite generation of the profinite group $\pi_1(\overline{X}, \bar{\eta})$ which projects onto Δ. Let $\{\Delta^i\}_{i\geq 1}$ be a countable system of *characteristic open* subgroups of Δ such that

$$\Delta^{i+1} \subseteq \Delta^i, \quad \Delta^1 = \Delta, \quad \text{and} \quad \bigcap_{i\geq 1} \Delta^i = \{1\}.$$

Write $\Delta_i \stackrel{\text{def}}{=} \Delta/\Delta^i$. Thus Δ_i is a *finite characteristic* quotient of Δ, and we have a push-out diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & \widetilde{\Pi} & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Delta_i & \longrightarrow & \Pi_i & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

which defines a (*geometrically finite*) quotient Π_i of $\widetilde{\Pi}$. The section s_1 induces a section

$$\rho_i \colon G_k \to \Pi_i$$

of the projection $\Pi_i \twoheadrightarrow G_k$ for all $i \geq 1$. Write

$$\widetilde{\Pi}^i \stackrel{\text{def}}{=} \widetilde{\Pi}^i[s_1] \stackrel{\text{def}}{=} \Delta^i.s_1(G_k).$$

Note that $\widetilde{\Pi}^i \subseteq \widetilde{\Pi}$ is an open subgroup which contains the image $s_1(G_k)$ of s_1. Write Π^i for the inverse image of $\widetilde{\Pi}^i$ in $\pi_1(X, \eta)$. Thus $\Pi^i \subseteq \pi_1(X, \eta)$ is an open subgroup corresponding to an étale cover

$$X_i \to X_1 \stackrel{\text{def}}{=} X$$

defined over k (since Π^i maps onto G_k via the natural projection $\pi_1(X, \eta) \twoheadrightarrow G_k$, by the very definition of Π^i).

Note that the étale cover $\overline{X}_i \stackrel{\text{def}}{=} X_i \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k} \to \overline{X}$ is Galois with Galois group Δ_i, and we have a commutative diagram of étale covers

$$\begin{array}{ccc} \overline{X}_i & \longrightarrow & \overline{X} \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X \end{array}$$

where $\overline{X}_i \to X$ is Galois with Galois group Π_i, and $\overline{X}_i \to X_i$ is Galois with Galois group $\rho_i(G_k)$. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} & & 1 & & 1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & \widetilde{\Delta}^i = \pi_1(\overline{X}_i, \bar{\eta}) & \longrightarrow & \Pi^i = \pi_1(X_i, \eta) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \pi_1(\overline{X}, \bar{\eta}) & \longrightarrow & \pi_1(X, \eta) & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

where $\widetilde{\Delta}^i$ is the inverse image of Δ^i in $\pi_1(\overline{X}, \bar{\eta})$, and the equalities $\widetilde{\Delta}^i = \pi_1(\overline{X}_i, \bar{\eta})$, $\Pi^i = \pi_1(X_i, \eta)$, are natural identifications; the base points η (resp. $\bar{\eta}$) of X_i (resp. $\overline{X}_i$) are those induced by the base points η (resp. $\bar{\eta}$) of X (resp. $\overline{X}$). Note that $\Pi^{i+1} \subseteq \Pi^i$ and $\widetilde{\Delta}^{i+1} \subseteq \widetilde{\Delta}^i$, as follows from the various definitions.

Lemma 1.1. *With the above notation and that in Section 0, the following holds:*

$$\widetilde{H} = \bigcap_{i \geq 1} \widetilde{\Delta}^i.$$

Proof. Follows from the various definitions. □

We take this opportunity to correct a mistake that occurred in [Saïdi2, Lem. 1.1]. The claim there that $\mathcal{I}_X = \bigcap_{i \geq 1} \Pi_i$ is false; however, this does not affect the validity of the results or other assertions made in [Saïdi2].

For each integer $i \geq 1$, consider the push-out diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widetilde{\Delta}^i = \pi_1(\overline{X}_i, \bar{\eta}) & \longrightarrow & \Pi^i = \pi_1(X_i, \eta) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \widetilde{\Delta}^{i,\mathrm{ab}} & \longrightarrow & \Pi^{(i,\mathrm{ab})} & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

where $\widetilde{\Delta}^{i,\mathrm{ab}}$ is the maximal abelian quotient of $\widetilde{\Delta}^i$ and $\Pi^{(i,\mathrm{ab})}$ is the *geometrically abelian* fundamental group of X_i. Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & \mathcal{H} \stackrel{\mathrm{def}}{=} \mathcal{H}[s_1] & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle s_1} & & \\ 1 & \longrightarrow & H & \longrightarrow & \Pi & \longrightarrow & \widetilde{\Pi} & \longrightarrow & 1 \end{array} \tag{1.2}$$

where the right square is cartesian. Thus (the group extension) $\mathcal{H}$ is the pull-back of (the group extension) Π via the section $s_1 \colon G_k \to \widetilde{\Pi}$.

Lemma 1.2. *We have natural identifications*

$$H \xrightarrow{\sim} \varprojlim_{i\geq 1} \widetilde{\Delta}^{i,\mathrm{ab}} \quad \textit{and} \quad \mathcal{H} \xrightarrow{\sim} \varprojlim_{i\geq 1} \Pi^{(i,\mathrm{ab})}.$$

Proof. Similar to the proof of [Saïdi2, Lem. 1.3]. □

The section $s_2\colon G_k \to \Pi$, which lifts the section s_1, induces a section $s_2\colon G_k \to \mathcal{H}$ of the projection $\mathcal{H} \twoheadrightarrow G_k$ (since $s_2(G_k) \subset \mathcal{H}$). We fix the section $s_2\colon G_k \to \mathcal{H}$ as a base point of the torsor of splittings of the upper sequence in diagram (1.2). Thus the set of splittings of the group extension $\mathcal{H}$, modulo conjugation by elements of H, is a torsor under $H^1(G_k, H)$, the G_k-module structure of H being deduced from diagram (1.2). The splitting $s_2\colon G_k \to \mathcal{H}$ thus corresponds to $0 \in H^1(G_k, H)$. Note that the set of splittings $\tilde{s}\colon G_k \to \mathcal{H}$ of the group extension $\mathcal{H}$ is in one-to-one correspondence with the set of sections $\tilde{s}\colon G_k \to \Pi$ of the projection $\Pi \twoheadrightarrow G_k$ which lift the section s_1.

Let $\tilde{s}\colon G_k \to \mathcal{H}$ be a section of the group extension $\mathcal{H}$, which induces a section $\tilde{s}\colon G_k \to \Pi$ of the projection $\Pi \twoheadrightarrow G_k$ which lift the section s_1. Let $[\tilde{s}]$ be the class of $\tilde{s}$ in $H^1(G_k, H)$ (cf. above discussion).

Fact 1.3. The section $\tilde{s}\colon G_k \to \Pi$ is geometric if and only if $[\tilde{s}] = 0$. In this case the section $\tilde{s}$ is associated to the rational point $x \in X(k)$.

Proof. First, assume that the section $\tilde{s}\colon G_k \to \Pi$ is geometric and arises from a rational point $\tilde{x} \in X(k)$. Both sections, $\tilde{s}\colon G_k \to \Pi$ and $s_2\colon G_k \to \Pi$, induce splittings $\tilde{s}^{\mathrm{ab}}\colon G_k \to \pi_1(X,\eta)^{(\mathrm{ab})}$ and $s_2^{\mathrm{ab}}\colon G_k \to \pi_1(X,\eta)^{(\mathrm{ab})}$ of the group extension $1 \to \pi_1(\overline{X},\bar{\eta})^{\mathrm{ab}} \to \pi_1(X,\eta)^{(\mathrm{ab})} \to G_k \to 1$, where $\pi_1(X,\eta)^{(\mathrm{ab})}$ is the geometrically abelian quotient of $\pi_1(X,\eta)$. Further, one has $\tilde{s}^{\mathrm{ab}} = s_2^{\mathrm{ab}}$ (cf. Condition $(\star)$(iii) and the fact that both $\tilde{s}$ and s_2 lift the section s_1). A standard argument, resorting to the Kummer exact sequence associated to the jacobian Pic^0_X of X, shows that $\tilde{x} = x$ (cf. [Tamagawa, Prop. 2.8]).

Next we claim $[\tilde{s}] = 0$. Indeed, the classes of $\tilde{s}$ and s_2 in $H^1(G_k, \Gamma)$ coincide as both sections are geometric and associated to the same rational point x, hence $\tilde{s}$ and s_2 are conjugate by an element of Γ. Here we view the set of splittings of the group extension Π (of Γ by G_k) as a torsor under $H^1(G_k, \Gamma)$, with base point the class of the section s_2. Further, the natural map $H^1(G_k, H) \to H^1(G_k, \Gamma)$ of pointed cohomology sets is injective as follows from Condition $(\star)$(ii) (cf. [Serre, I.§5, Prop. 38,] and diagram (1.1)). (Here, the G_k-module structure on H (resp. G_k-group structure on Γ) is induced by the section s_1 (resp. s_2) (cf. diagram (1.1)).) Thus $[\tilde{s}] = 0$.

Conversely, if $[\tilde{s}] = 0$, then $\tilde{s}$ is conjugate to s_2 by an element of H, hence is geometric and associated to the rational point x. □

As a consequence we obtain the following.

Lemma 1.4. *Let k'/k be a finite extension. There exists a section $\tilde{s}\colon G_{k'} \to \Pi_{k'}$ of the projection $\Pi_{k'} \twoheadrightarrow G_{k'}$, which lifts the section $s_{1,k'}\colon G_{k'} \to \widetilde{\Pi}_{k'}$ of the projection $\widetilde{\Pi}_{k'} \twoheadrightarrow G_k$ induced by s_1, and which is* non-geometric, *if and only if $H^1(G_{k'}, H) \neq 0$.*

Thus proving Theorem A reduces to proving the following.

Proposition 1.5. *There exists a finite extension $\tilde{k}/k$ such that $H^1(G_{k'}, H) \neq 0$ for every finite extension $k'/\tilde{k}$.*

The rest of this section is devoted to proving Proposition 1.5. Let $\ell \neq p$ be a prime integer such that the map $\Gamma^\ell \twoheadrightarrow \Delta^\ell$ is not an isomorphism (cf. Condition $(\star)$(iv)). Write $\pi_1(\overline{X}, \bar{\eta})^\ell$ for the maximal *pro-ℓ* quotient of $\pi_1(\overline{X}, \bar{\eta})$, and

$$\pi_1(X, \eta)^{(\ell)} \overset{\text{def}}{=} \pi_1(X, \eta)/\operatorname{Ker}(\pi_1(\overline{X}, \bar{\eta}) \twoheadrightarrow \pi_1(\overline{X}, \bar{\eta})^\ell)$$

for the *geometrically pro-ℓ* quotient of $\pi_1(X, \eta)$, which fits in the exact sequence

$$1 \to \pi_1(\overline{X}, \bar{\eta})^\ell \to \pi_1(X, \eta)^{(\ell)} \to G_k \to 1.$$

Let $s^\ell = s_x^\ell\colon G_k \to \pi_1(X, \eta)^{(\ell)}$ be the section of the projection $\pi_1(X, \eta)^{(\ell)} \twoheadrightarrow G_k$ induced by the section $s = s_x$. This section induces a representation

$$\rho^\ell\colon G_k \to \operatorname{Aut}(\pi_1(\overline{X}, \bar{\eta})^\ell)$$

which factors as $G_k \twoheadrightarrow G_F \to \operatorname{Aut}(\pi_1(\overline{X}, \bar{\eta})^\ell)$, where G_F is the quotient of G_k by its inertia subgroup, since X has good reduction over $\mathcal{O}_k$. Further, the image of the representation ρ^ℓ is almost pro-ℓ, i.e., $\rho^\ell(G_k)$ possesses an open subgroup which is pro-ℓ. In particular, there exists a finite extension $\tilde{k}/k$ such that the restriction $\rho^\ell_{\tilde{k}}\colon G_{\tilde{k}} \to \operatorname{Aut}(\pi_1(\overline{X}, \bar{\eta})^\ell)$ of ρ^ℓ to $G_{\tilde{k}}$ has a pro-ℓ image. In order to prove Proposition 1.5 we will, without loss of generality, *assume that the image of ρ^ℓ is pro-ℓ* and will show $H^1(G_k, H) \neq 0$.

Let Δ^ℓ be the maximal pro-ℓ quotient of Δ, and $\widetilde{\Pi}^{(\ell)} \overset{\text{def}}{=} \widetilde{\Pi}/\operatorname{Ker}(\Delta \twoheadrightarrow \Delta^\ell)$ the geometrically pro-ℓ quotient of $\widetilde{\Pi}$. We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta & \longrightarrow & \widetilde{\Pi} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta^\ell & \longrightarrow & \widetilde{\Pi}^{(\ell)} & \longrightarrow & G_k & \longrightarrow & 1
\end{array}$$

where the left and middle vertical maps are surjective. For $i \geq 1$, let N^i be the image of Δ^i in Δ^ℓ, and $\widetilde{N}^i$, $\widehat{N}^i$ the pre-images of N^i in $\pi_1(\overline{X}, \bar{\eta})$ and $\pi_1(\overline{X}, \bar{\eta})^\ell$, respectively. Note that $\widetilde{N}^i$ is a characteristic subgroup of $\pi_1(\overline{X}, \bar{\eta})$, and $\widehat{N}^i$ is stable by the action of $s^\ell(G_k)$.

Let $\widetilde{U}^i \stackrel{\text{def}}{=} \widetilde{N}^i.s(G_k)$ and $\widehat{U}^i \stackrel{\text{def}}{=} \widehat{N}^i.s^\ell(G_k)$, for $i \geq 1$. Thus $\widetilde{U}^i$ is an open subgroup of $\pi_1(X, \eta)$ corresponding to an étale cover $Y_i \to X$, and the étale cover $X_i \to X$ factorises as

$$X_i \to Y_i \to X$$

[$\pi_1(X_i, \eta) \subset \pi_1(Y_i, \eta)$ as follows from the various definitions], where $X_i \to Y_i$ is an étale cover of degree prime-to-ℓ, since Δ (hence also Δ_i, for $i \geq 1$) is pro-nilpotent (see Condition $(\star)$(i)). Further, $\widetilde{U}^i$ and $\widehat{U}^i$, are naturally identified with $\pi_1(Y_i, \eta)$ and $\pi_1(Y_i, \eta)^{(\ell)}$, respectively, for all $i \geq 1$. Here, $\pi_1(Y_i, \eta)^{(\ell)}$ is the geometrically pro-ℓ quotient of $\pi_1(Y_i, \eta)$, and sits in an exact sequence

$$1 \to \pi_1(\overline{Y}_i, \bar{\eta})^\ell \to \pi_1(Y_i, \eta)^{(\ell)} \to G_k \to 1,$$

where $\overline{Y}_i \stackrel{\text{def}}{=} Y \times_k \bar{k}$.

The natural action of G_k on $\pi_1(\overline{X}, \bar{\eta})^\ell$, and which factorises through G_F^ℓ by our assumption on the representation ρ^ℓ, is compatible with its action on the open subgroup $\pi_1(\overline{Y}_i, \bar{\eta})^\ell$, hence this latter action also factorises through G_F^ℓ.

There is a surjective homomorphism (recall Lemma 1.2)

$$H^1(G_k, H) = \varprojlim_{i \geq 1} H^1(G_k, \widetilde{\Delta}^{i,\text{ab}}) \twoheadrightarrow \varprojlim_{i \geq 1} H^1(G_k, \widetilde{\Delta}^{i,\text{ab},\ell}).$$

(Indeed, $\varprojlim_{i \geq 1} H^1(G_k, \widetilde{\Delta}^{i,\text{ab}}) \xrightarrow{\sim} \prod_{l \in \mathfrak{Primes}} \varprojlim_{i \geq 1} H^1(G_k, \widetilde{\Delta}^{i,\text{ab},l})$, where the product is over all prime integers l and the above homomorphism is the projection onto the ℓth factor.) Further, the étale covers $\{X_i \to Y_i\}_{i \geq 1}$ induce a homomorphism (recall $\widetilde{\Delta}^i = \pi_1(\overline{X}_i, \bar{\eta})$)

$$\varprojlim_{i \geq 1} H^1(G_k, \widetilde{\Delta}^{i,\text{ab},\ell}) \to \varprojlim_{i \geq 1} H^1(G_k, \pi_1(\overline{Y}_i, \bar{\eta})^{\text{ab},\ell}),$$

which is surjective. More precisely, the map

$$H^1(G_k, \widetilde{\Delta}^{i,\text{ab},\ell}) \to H^1(G_k, \pi_1(\overline{Y}_i, \bar{\eta})^{\text{ab},\ell})$$

is surjective for all $i \geq 1$, as follows easily from a restriction–corestriction argument using the fact that the degree of the cover $X_i \to Y_i$ is prime-to-ℓ (observe the maps on cohomology induced by the natural maps $\pi_1(\overline{Y}_i, \bar{\eta})^{\text{ab},\ell} \xrightarrow{\text{res}} \widetilde{\Delta}^{i,\text{ab},\ell} \xrightarrow{\text{cor}} \pi_1(\overline{Y}_i, \bar{\eta})^{\text{ab},\ell}$ arising from the morphisms $\text{Pic}^0(Y_i) \to \text{Pic}^0(X_i) \to \text{Pic}^0(Y_i)$, where the first one is the pull-back map of line bundles and the second is the norm map). Thus in order to prove Proposition 1.5 it suffices to show the following:

Proposition 1.6. *With the above notation, it holds that*

$$\varprojlim_{i\geq 1} H^1(G_k, \pi_1(\overline{Y}_i, \bar{\eta})^{\mathrm{ab},\ell}) \neq 0.$$

Proof. As discussed above the natural action of G_k on $\pi_1(\overline{Y}_i, \bar{\eta})^{\mathrm{ab},\ell}$ factors through G_F^ℓ (which is isomorphic to $\mathbb{Z}_\ell$). There is an injective inflation map

$$\inf\colon \varprojlim_{i\geq 1} H^1(G_F^\ell, \pi_1(\overline{Y}_i, \bar{\eta})^{\mathrm{ab},\ell}) \hookrightarrow \varprojlim_{i\geq 1} H^1(G_k, \pi_1(\overline{Y}_i, \bar{\eta})^{\mathrm{ab},\ell}).$$

Further,

$$(\varprojlim_{i\geq 1} \pi_1(\overline{Y}_i, \bar{\eta})^{\mathrm{ab},\ell})_{G_F^\ell} = H^1(G_F^\ell, \varprojlim_{i\geq 1} \pi_1(\overline{Y}_i, \bar{\eta})^{\mathrm{ab},\ell}) = \varprojlim_{i\geq 1} H^1(G_F^\ell, \pi_1(\overline{Y}_i, \bar{\eta})^{\mathrm{ab},\ell}),$$

where the notation $(\ )_{G_F^\ell}$ stands for the co-invariant module, the first equality follows from the fact that G_F^ℓ is procyclic, and the second equality follows from [Neukirch-Schmidt-Winberg, Cor. (2.3.5)].

There is a natural isomorphism

$$(\mathrm{Ker}(\Gamma^\ell \to \Delta^\ell)) \xrightarrow{\sim} \varprojlim_{i\geq 1} \pi_1(\overline{Y}_i, \bar{\eta})^{\mathrm{ab},\ell}.$$

(The proof is similar to the proof of Lemma 1.2.) Thus $\varprojlim_{i\geq 1} \pi_1(\overline{Y}_i, \eta)^{\mathrm{ab},\ell} \neq 0$ (cf. Condition $(\star)$(iii) and our choice of ℓ). The proof of Proposition 1.6 is from the following.

Lemma 1.7. *Let T be an abelian pro-ℓ group and P an infinite pro-ℓ cyclic group. Assume T is a continuous P-module. Then, the co-invariant module $(T)_P = \{0\}$ is trivial if and only if $T = \{0\}$ itself is trivial.*

Proof. Let $T^\wedge$ be the Pontryagin dual of T which is an ℓ-primary torsion group. The dual of $(T)_P$ is the invariant group $(T^\wedge)^P$. It suffices to show that $(T^\wedge)^P$ is trivial if and only if $(T^\wedge)$ is trivial. The action of P on $T^\wedge$ is discrete, in particular $T^\wedge$ is the union of finite ℓ-groups which are stable P-submodules. We can thus reduce to the case where $T^\wedge$ and P are finite. Suppose $(T^\wedge)$ is finite and non-trivial; then $(T^\wedge)^P$ is non-trivial since its order is divisible by ℓ, and $(T^\wedge)^P$ contains 0. □

This finishes the proof of Proposition 1.6, hence the proof of Proposition 1.5, and the proof of Theorem A. □

§2. Proof of Theorem C

The rest of this paper is devoted to proving Theorem C. We use the notation introduced in Section 0 and the statement of Theorem C.

Let K be a number field (finite extension of $\mathbb{Q}$) and $\overline{K}$ an algebraic closure of K. Let X be a proper, smooth, and geometrically connected hyperbolic curve over K. Write $J \stackrel{\text{def}}{=} \operatorname{Pic}^0_X$ for the jacobian of X. Assume $X(K) \neq \emptyset$. Fix a rational point $x \in X(K)$, and consider the embedding $\iota\colon X \hookrightarrow J$ defined by $\iota(x) = 0_J$. For any field extension $\overline{K} \subset L$, with L algebraically closed, let $J^{\text{tor}} \stackrel{\text{def}}{=} J(L)^{\text{tor}} = J(\overline{K})^{\text{tor}}$ be the torsion subgroup of J. The intersection $X \cap J^{\text{tor}}$ is finite by [Raynaud]. Let M be the cardinality of the subgroup of J^{tor} generated by $X \cap J^{\text{tor}}$. We *assume* $M \geq 2$ (this M will be the integer N required in Theorem C).

Let $p > M$ be a prime integer, k a p-adic completion of K, $\bar{k}$ an algebraic closure of k, $X_k \stackrel{\text{def}}{=} X \times_K k$, and $X_{\bar{k}} \stackrel{\text{def}}{=} X \times_K \bar{k}$. Recall the exact sequence of fundamental groups (cf. Section 0)

$$1 \to \pi_1(X_{\bar{k}}, \bar{\eta}) \to \pi_1(X_k, \eta) \to G_k \to 1.$$

Let Δ be the *maximal pro-p quotient* of $\pi_1(X_{\bar{k}}, \bar{\eta})$, and

$$\Pi \stackrel{\text{def}}{=} \pi_1(X_k, \eta)/\operatorname{Ker}(\pi_1(X_{\bar{k}}, \bar{\eta}) \twoheadrightarrow \Delta)$$

the *geometrically pro-p* arithmetic fundamental group of X_k.

For an integer $m \geq 1$, let Δ_m be the maximal *m-step solvable pro-p* quotient of $\pi_1(X_{\bar{k}}, \bar{\eta})$, and

$$\Pi_m \stackrel{\text{def}}{=} \pi_1(X_k, \eta)/\operatorname{Ker}(\pi_1(X_{\bar{k}}, \bar{\eta}) \twoheadrightarrow \Delta_m)$$

the *geometrically m-step solvable pro-p* arithmetic fundamental group of X_k. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Delta[m] & = & \Delta[m] & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Delta_{m+1} & \longrightarrow & \Pi_{m+1} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta_m & \longrightarrow & \Pi_m & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where $\Delta[m] \overset{\text{def}}{=} \text{Ker}(\Delta_{m+1} \twoheadrightarrow \Delta_m) = \text{Ker}(\Pi_{m+1} \twoheadrightarrow \Pi_m)$ (cf. [Saïdi1, Sect. 1] for more details). Further, we have natural identifications

$$\Delta = \varprojlim_{m \geq 1} \Delta_m \quad \text{and} \quad \Pi = \varprojlim_{m \geq 1} \Pi_m.$$

Let $s_x : G_k \to \Pi$ be a section of the projection $\Pi \twoheadrightarrow G_k$ associated to the k-rational point (image in X_k of) x, which induces sections $s_{x,m} : G_k \to \Pi_m$ of the projections $\Pi_m \twoheadrightarrow G_k$ for all $m \geq 1$. (Thus s_x is defined up to conjugation by Δ.) We fix the section $s_{x,1}$ as a base point of the torsor of splittings of the exact sequence $1 \to \Delta_1 \to \Pi_1 \to G_k \to 1$ (Π_1 is the geometrically abelian pro-p arithmetic fundamental group of X_k), which is a torsor under $H^1(G_k, \Delta_1)$.

Let $y \in (X \cap J^{\text{tor}})(K) \setminus \{0_J\}$ (the existence of y follows from our assumption $M \geq 2$), $s_y : G_k \to \Pi$ be a section of the projection $\Pi \twoheadrightarrow G_k$ associated to the k-rational point (image in X_k of) y, which induces sections $s_{y,m} : G_k \to \Pi_m$ of the projections $\Pi_m \twoheadrightarrow G_k$ for all $m \geq 1$. The classes $[s_{x,1}] = 0$ and $[s_{y,1}]$, of the sections $s_{x,1}$ and $s_{y,1}$, respectively, in $H^1(G_k, \Delta_1)$ coincide. Indeed, this follows easily from the (pro-p) Kummer exact sequence associated to J and the fact that $\iota(y)$ is a torsion point of order prime-to-p (recall $p > M$).

More generally, for $m \geq 1$, consider the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta[m+1] & \longrightarrow & E[m+1] & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle s_{x,m}} & & \\ 1 & \longrightarrow & \Delta[m+1] & \longrightarrow & \Pi_{m+1} & \longrightarrow & \Pi_m & \longrightarrow & 1, \end{array} \tag{2.1}$$

where the right square is cartesian. Thus the group extension $E[m+1]$ is the pull-back of the group extension Π_{m+1} via the section $s_{x,m}$.

The upper exact sequence in diagram (2.1) splits. Indeed, this follows from the existence of the section $s_{x,m+1} : G_k \to \Pi_{m+1}$ which lifts the section $s_{x,m}$ and induces a splitting $s_{x,m+1} : G_k \to E[m+1]$ of the group extension $E[m+1]$. We fix the section $s_{x,m+1}$ as a base point for the torsor of splittings of the group extension $E[m+1]$, which is a torsor under $H^1(G_k, \Delta[m+1])$; the G_k-module structure of $\Delta[m+1]$ is deduced from diagram (2.1). If $z \in X(k)$ and $s_{z,m} = s_{x,m} : G_k \to \Pi_m$, then the section $s_{z,m+1}$ gives rise to a splitting $s_{z,m+1} : G_k \to E[m+1]$ of the upper exact sequence in diagram (2.1), hence to a class $[s_{z,m+1}] \in H^1(G_k, \Delta[m+1])$.

Define $\mathcal{S}_m$ to be the set of rational points $z \in X(k)$ such that $s_{x,m}(G_k)$ coincide with a decomposition group of Π_m associated to z. We have the following inclusions:

$$\cdots \subseteq \mathcal{S}_{m+1} \subseteq \mathcal{S}_m \subseteq \cdots \subseteq \mathcal{S}_2 \subseteq \mathcal{S}_1 = X(k) \cap J^{\text{tor},p'} \subseteq X \cap J^{\text{tor}}.$$

The equality $\mathcal{S}_1 = X(k) \cap J^{\mathrm{tor},p'}$ follows from the (pro-p) Kummer exact sequence associated to J and from the well-known structure of $J(k)$.

Lemma 2.1. *The equality* $\bigcap_{m\geq 1} \mathcal{S}_m = \{x\}$ *holds.*

Proof. Follows from [Mochizuki, Thm. C] and a limit argument using the fact that $\Pi = \varprojlim_{m\geq 1} \Pi_m$. □

It follows from Lemma 2.1, and the above discussion, that there exists $m \geq 1$ such that

$$\{x\} \subsetneq \mathcal{S}_m \quad \text{and} \quad \{x\} = \mathcal{S}_{m+1}.$$

Let

$$A \stackrel{\text{def}}{=} \big\{[s_{z,m+1}] \colon z \in \mathcal{S}_m\big\} \subset H^1(G_k, \Delta[m+1]).$$

Note that $\{0\} \subsetneq A$, which follows from the facts that $\{x\} \subsetneq \mathcal{S}_m$ and $\{x\} = \mathcal{S}_{m+1}$. Further, $\mathrm{Card}(A) \leq \mathrm{Card}(\mathcal{S}_m) \leq M < p$. In particular,

$$\exists\, \alpha \in H^1(G_k, \Delta[m+1]) \setminus A,$$

since $H^1(G_k, \Delta[m+1])$ is p-primary. Thus α corresponds to a section $\alpha\colon G_k \to \Pi_{m+1}$ of the projection $\Pi_{m+1} \twoheadrightarrow G_k$, which lifts the section $s_{x,m}$.

Lemma 2.2. *The section* $\alpha\colon G_k \to \Pi_{m+1}$ *is* non-geometric*.*

Proof. Follows from the various definitions and the fact that $\alpha \notin A$. □

This finishes the proof of Theorem C. □

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