Publ. RIMS Kyoto Univ. **59** (2023), 587–631
DOI 10.4171/PRIMS/59-3-6

Full Justification for the Extended Green–Naghdi System for an Uneven Bottom with/without Surface Tension

by

Bashar Khorbatly and Samer Israwi

Abstract

This paper is a continuation of a previous work on the extended Green–Naghdi system. We prolong the system, in arbitrary dimension, with/without surface tension, and for a general bottom topography. Confining the work to the one-dimensional case, well-posedness and consistency with respect to initial data and parameters are proved, taking into account the effect of surface tension. The results are local, but long term in the sense of dependence upon initial data. As a conclusion, our solution remains close to the exact solution of the full Euler system with a better (smaller) precision and therefore the full justification of the models.

2020 Mathematics Subject Classification: Primary 35Q35, 35L45, 35L60, 76B45; Secondary 76B55.
Keywords: Water waves, asymptotic models, extended Green–Naghdi system, bottom topography, surface tension.

§1. Introduction

§1.1. The water-wave problem

The two-dimensional full water-wave problem in arbitrary dimension with general bottom topography is considered. We assume that the fluid is of constant density ρ and denote by $\Omega_t = \{(X,z) \in \mathbb{R}^d \times \mathbb{R},\ -h_0 + b(X) < z < \zeta(t,X)\}$ the domain of the fluid for each time variable t. The surface of the fluid is a graph parametrized

Communicated by K. Nakanishi. Received October 21, 2020. Revised June 16, 2021; June 22, 2021; June 25, 2021; July 6, 2021; July 30, 2021.

B. Khorbatly: Lebanese American University (LAU), Graduate Studies and Research (GSR), School of Arts and Sciences, Department of Computer Science and Mathematics, Beirut, Lebanon;
e-mail: bashar-elkhorbatly@hotmail.com
S. Israwi: Laboratory of Mathematics-EDST, Department of Mathematics, Faculty of Sciences 1, Lebanese University, Beirut, Lebanon;
e-mail: s_israwi83@hotmail.com

© 2023 Research Institute for Mathematical Sciences, Kyoto University.
This work is licensed under a CC BY 4.0 license.

by ζ and the bottom topography is parametrized by $-h_0 + b(X)$ independent of time with h_0 the depth. Take $d = 1, 2$ as the spatial dimension of the surface of the fluid, where $X \in \mathbb{R}^d$, the spatial variable, is written as $X = (x, y)$ when $d = 2$ and $X = x$ when $d = 1$, while the vertical variable is denoted by z. The motion of an ideal moving fluid is described by the free-surface Euler equations for steady flow along a streamline (their well-posedness was recognized after the work of Nalimov [30], Yosihara [35], Craig [5], Wu [33, 34], and Lannes [21]) which is a connection between the velocity V, the pressure P, and the density ρ of the fluid that is based on Newton's second law of motion and can be written in the form

$$\partial_t V + (V \cdot \nabla_{X,z})V = -g\vec{e}_z - \frac{1}{\rho}\nabla_{X,z}P \quad \text{in } (X, z) \in \Omega_t,\ t \geq 0, \tag{1.1}$$

denoting by $-g\vec{e}_z$ the gravitational field which is acting vertically downward, with g greater than zero, and $\vec{e}_z$ is a unit vector in the vertical direction. This equation is combined with several physical assumptions. The incompressibility of the fluid is expressed by

$$\nabla_{X,z} \cdot V = 0 \quad \text{in } (X, z) \in \Omega_t,\ t \geq 0. \tag{1.2}$$

Another assumption is that the flow is irrotational:

$$\nabla_{X,z} \times V = 0 \quad \text{in } (X, z) \in \Omega_t,\ t \geq 0. \tag{1.3}$$

Denote $\nabla = \nabla_X$. These equations are complemented with boundary conditions. At the free surface, a dynamic condition is given by

$$P - P_{\text{atm}} = \sigma\kappa(\zeta) \quad \text{at } z = \zeta(t, X),\ t \geq 0,$$

where P_{atm} is the (constant) atmospheric pressure, $\sigma > 0$ is the surface tension coefficient, and $\kappa(\zeta) = -\nabla\cdot(\nabla\zeta/\sqrt{1 + |\nabla\zeta|^2})$ is the mean curvature of the surface. Another boundary condition at the top surface is the kinematic condition:

$$\partial_t\zeta - \sqrt{1 + |\nabla\zeta|^2}V \cdot n_+ = 0 \quad \text{at } z = \zeta(t, X),\ t \geq 0, \tag{1.4}$$

where the outward unit normal to the upper boundary is

$$n_+ = \frac{1}{\sqrt{1 + |\nabla\zeta|^2}}(\nabla\zeta^\top, 1)^\top.$$

This condition states that fluid particles cannot cross the surface. A similar condition on the velocity at the bottom is given by

$$V \cdot n_- = 0 \quad \text{at } z = -h_0 + b(X),\ t \geq 0, \tag{1.5}$$

where the outward unit normal to the lower boundary

$$n_- = \frac{1}{\sqrt{1+|\nabla b|^2}}(\nabla b^\top, -1)^\top.$$

Another assumption states that the fluid is at rest at infinity and is given by

$$\lim_{|(t,X,z)|\to+\infty} |\zeta(t,X,z)| + |V(t,X,z)| = 0 \quad \text{in } (X,z)\in\Omega_t,\ t\geq 0. \tag{1.6}$$

Equations (1.1)–(1.6) are called the free-surface Euler equations.

§1.2. Non-dimensionalized equations: Bernoulli and Zakharov/Craig–Sulem

The complexity of the Euler equations is embodied in the many unknowns on a moving-with-time domain Ω_t. Thus we will follow the Eulerian approach that focuses on specific locations in the space through which the fluid flows as time passes. More precisely, we derive simpler asymptotic models in some geophysical regimes that require identification of small parameters. Neglecting rotational effects is the starting point of the derivation process. In other words, assumption (1.3) ensures the existence of the potential velocity of the fluid $\varphi(t,X,z)$. Consequently, the Euler system can be rewritten under Bernoulli's formulation:

$$\begin{cases} \Delta_{X,z}\varphi = 0 & \text{at } -h_0 + b(X) < z < \zeta(t,X),\\ \partial_z\varphi - \nabla b\cdot\nabla\varphi = 0 & \text{at } z = -h_0 + b(X),\\ \partial_t\zeta + \nabla\varphi\cdot\nabla\zeta - \partial_z\varphi = 0 & \text{at } z = \zeta(t,X),\\ \partial_t\varphi + \dfrac{1}{2}|\nabla_{X,z}\varphi|^2 + g\zeta = -\dfrac{\sigma}{\rho}\kappa(\zeta) & \text{at } z = \zeta(t,X). \end{cases}$$

The Laplacian equation is obtained by taking the divergence of the potential velocity combined with (1.2). The second and third equations are written using the boundary conditions (1.5)–(1.4) respectively, while the last equation is established by commuting $V = \nabla_{X,z}\varphi$ in (1.1).

Although the present system has fewer unknowns, in order to solve the Laplacian equation we still need information from the boundary that moves with time, and its location is determined by two coupled non-linear partial differential equations, which is a basic difficulty. This leads us to identify some small parameters from which it is often possible to gain insight on the behavior of the flow. More precisely, let us introduce some dimensionless parameters that encode the various regimes of interest:

- non-linearity or the amplitude parameter: $\varepsilon = \frac{a}{h_0} \in (0,1)$,
- dispersion or the shallowness parameter: $\mu = (\frac{h_0}{\lambda})^2 \in (0,1)$,

- bottom topography parameter: $\beta = \frac{b_0}{h_0} \in (0,1)$,
- classical Bond number which measures the ratio of gravity forces over capillary forces: $\mathrm{Bo} = \frac{\rho g \lambda^2}{\sigma} > 1$,

where a is the amplitude of the wave, λ the wavelength of the wave, b_0 the size of the bottom topography variations, h_0 the reference depth, ρ the density of the fluid, and σ the surface tension coefficient. We now execute the classical shallow-water ($\mu \ll 1$) non-dimensionalization using the following variables:

$$X = \lambda X', \quad z = h_0 z', \quad \zeta = a\zeta', \quad \varphi = \frac{a}{h_0}\lambda\sqrt{gh_0}\varphi',$$
$$b = b_0 b', \qquad t = \frac{\lambda}{\sqrt{gh_0}}t'.$$

Therefore, the equations of motion (1.1)–(1.6) are rewritten in the dimensionless Bernoulli formulation (we eliminate the primes for the sake of clarity) as

$$(1.7)\qquad \begin{cases} \mu\partial_x^2\varphi + \mu\partial_y^2\varphi + \partial_z^2\varphi = 0 & \text{at } -1+\beta b(X) < z < \varepsilon\zeta(t,X), \\ \partial_z\varphi - \mu\beta\nabla b\cdot\nabla\varphi = 0 & \text{at } z = -1+\beta b(X), \\ \partial_t\zeta - \dfrac{1}{\mu}(-\mu\varepsilon\nabla\zeta\cdot\nabla\varphi + \partial_z\varphi) = 0 & \text{at } z = \varepsilon\zeta(t,X), \\ \partial_t\varphi + \dfrac{1}{2}\Big(\varepsilon|\nabla\varphi|^2 + \dfrac{\varepsilon}{\mu}(\partial_z\varphi)^2\Big) + \zeta & \\ \quad = -\dfrac{1}{\mathrm{Bo}}\dfrac{\kappa(\varepsilon\sqrt{\mu}\zeta)}{\varepsilon\sqrt{\mu}} & \text{at } z = \varepsilon\zeta(t,X). \end{cases}$$

Now system (1.7) is reduced to a system where all functions are evaluated at the free surface (i.e. in $\mathbb{R}_+ \times \mathbb{R}^d$). The new system is known as the dimensionless Zakharov/Craig–Sulem [36] formulation of the water-wave equations. We begin by introducing $\psi\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ the trace of the velocity potential at the free surface,

$$\psi(t,X) = \varphi(t,X,\varepsilon\zeta(t,X)) = \varphi_{|_{z=\varepsilon\zeta}}$$

and the Dirichlet–Neumann operator $\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\cdot$ defined by

$$\begin{aligned}\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi &= -\mu(\varepsilon\nabla\zeta)\cdot(\nabla\varphi)_{|_{z=\varepsilon\zeta}} + (\partial_z\varphi)_{|_{z=\varepsilon\zeta}} \\ &= \sqrt{1+\mu\varepsilon^2|\nabla\zeta|^2}(\partial_n\varphi)_{|_{z=\varepsilon\zeta}},\end{aligned}$$

with φ solving (see [22] for an accurate analysis) the Laplace equation with Neumann (at the bottom) and Dirichlet (at the surface) boundary conditions

$$(1.8)\qquad \begin{cases} \mu\partial_x^2\varphi + \mu\partial_y^2\varphi + \partial_z^2\varphi = 0 \quad \text{in } -1+\beta b(X) < z < \varepsilon\zeta(t,X), \\ \partial_n\varphi_{|_{z=-1+\beta b}} = 0, \quad \varphi_{|_{z=\varepsilon\zeta}} = \psi(t,X), \end{cases}$$

where $\partial_n \varphi = n_- \cdot \nabla_{X,z}\varphi$ refers to the upward normal derivative at the bottom. A set of two equations on the free-surface parametrization ζ and the trace of the velocity potential at the surface $\psi = \varphi_{|_{z=\varepsilon\zeta}}$ involving the Dirichlet–Neumann operator are given by

$$
\begin{cases}
\partial_t \zeta - \dfrac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = 0, \\
\partial_t \psi + \zeta + \dfrac{\varepsilon}{2}|\nabla\psi|^2 - \varepsilon\mu \dfrac{\left(\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi + \nabla(\varepsilon\zeta)\cdot\nabla\psi\right)^2}{2(1+\varepsilon^2\mu|\nabla\zeta|^2)} \\
\quad = -\dfrac{1}{\mathrm{Bo}}\dfrac{\kappa(\varepsilon\sqrt{\mu}\zeta)}{\varepsilon\sqrt{\mu}}.
\end{cases}
\tag{1.9}
$$

§1.3. Shallow-water, large-amplitude regime ($\mu \ll 1$, $\varepsilon \sim 1$, $\beta \sim 1$)

The solutions of system (1.9) are very hard to explain and appear more complex than necessary to model. At this stage, a traditional technique is to pick an asymptotic regime, in which we search for an approximate model and hence an approximate solution. In the sequel, we seek solutions up to third-order error in the dispersion parameter. Formally, under this condition one approach is based on a perturbation method with respect to a small parameter $\mu \ll 1$, where no assumptions are made on $\varepsilon \sim 1$ and $\beta \sim 1$. The Saint-Venant equations, first derived in 1871 [6, 7], are $O(\mu)$ corrections of the shallow-water equations. These equations couple the evolution of the free surface ζ to the evolution of the vertically averaged horizontal component of the velocity v, and are expressed as

$$
\begin{cases}
\partial_t \zeta + \nabla\cdot(hv) = 0, \\
\partial_t v + \nabla\zeta + \varepsilon(v\cdot\nabla)v = O(\mu),
\end{cases}
\tag{1.10}
$$

where $h(t,X) = 1 + \varepsilon\zeta(t,X) - \beta b(X)$ and

$$
v(t,X) = \frac{1}{h(t,X)} \int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)} \nabla\varphi(t,X,z)\,dz,
$$

the non-dimensional height of the liquid. The first justification of such a model goes back to Ovsjannikov [31], and Kano and Nishida [17] who prove that the solution of the shallow-water equations (1.10) converges to the solution of the water-wave equations as $\mu \to 0$ in the one-dimensional case, and under some restrictive assumptions. However, Li [25] removes these assumptions and gives rigorous justification. Note that, for such a symmetrizable hyperbolic system (multiply the second equation by h), classical local existence can be established; see [1, 32].

Up to second-order error in μ, the classical Green–Naghdi system takes into account dispersive effects ignored by (1.10). This system is derived in [28] through

Hamilton's principle and in [23] through a general method of derivation (see Section 2) and is exactly the same as the one found in [11], that is,

$$\begin{cases}\partial_t\zeta + \nabla\cdot(hv) = 0,\\ (1+\mu\mathcal{T}[h,\beta b])\partial_t v + \nabla\zeta + \varepsilon(v\cdot\nabla)v + \mu\varepsilon(Q[h]v + \beta Q_b[h]v) = O(\mu^2),\end{cases}$$

where we recall that $h = 1+\varepsilon\zeta-\beta b$, and $\mathcal{T}v = \mathcal{T}[h,\beta b]v$ and the quadratic form $Q[h]v$ are defined as

$$\begin{aligned}\mathcal{T}v &= \frac{-1}{3h}\nabla(h^3\nabla\cdot v) + \frac{1}{2h}\big(\nabla(h^2(\beta\nabla b)\cdot v) - h^2(\beta\nabla b)\nabla\cdot v\big) + \beta^2(\nabla b\cdot v)\nabla b,\\ Q[h]v &= \frac{-1}{3h}\nabla\big(h^3\big((v\cdot\nabla)(\nabla\cdot v) - (\nabla\cdot v)^2\big)\big),\\ Q_b[h]v &= \frac{1}{2h}\big[\nabla(h^2(v\cdot\nabla)^2 b) - h^2\big((v\cdot\nabla)(\nabla\cdot v) - (\nabla\cdot v)^2\big)\nabla b\big] + \beta((v\cdot\nabla)^2 b)\nabla b.\end{aligned}$$

For this approximation we refer to several works that give a rigorous justification of the well-posedness of this system using various arguments, such as Li [25], Alvarez-Samaniego and Lannes [2], Iguchi [12], Israwi [16, 15], Khorbatly [18], and Fujiwara and Iguchi [10]. We also refer to Lannes and Marche [24] for recent progress made with this system in the derivation of an alternative new class of equations having better mathematical structure, which makes them much more suitable for numerical simulation.

§1.4. Main results: Comments and organization

The aim of this paper is to derive and fully justify (i.e. local existence, consistency, and convergence) the extended Green–Naghdi (modified) system to one order higher in μ in the presence of a non-trivial bottom topography, thereby introducing quite a few new and non-trivial higher-order terms. In other words, this system is a much higher-order approximation of the full Euler system, with less error up to $O(\mu^3)$, i.e. it combines much better dispersive properties, and thus a wider range of application in oceanography. In the spirit of the work done in [26, 27], Matsuno derived the extended equations for a flat bottom by a slightly different method from the one used here [23, 20]. He showed that they permit a Hamiltonian structure and stated that the linear dispersion relation of his model does not have good structure, i.e. we cannot expect the well-posedness of the initial value problem, so that obtaining an error estimate of the solutions of $O(\mu^3)$ is hopeless. In fact, this is not the case in the presence of surface tension, as the linear part of the μ^3 modified model has better structure and the solution might approximate the solution to the full water-wave equations up to $O(\mu^3)$ (see Remark 2.2). Here it is worth noticing that in recent papers, Iguchi [13, 14] obtained and mathematically justified a model in a similar asymptotic regime to this work. Also it was shown that the so-called Isobe–Kakinuma model permits a Hamiltonian structure [8]. Note that one of its strong advantages is that it does not contain any higher-order derivatives and is thus less troublesome in terms of numerical computation. However, the only drawback of such a hierarchy is that Green–Naghdi/Boussinesq systems are not one of the models. Regarding this, a number of challenging/interesting problems associated with the extended system are worthy of additional study. We list some of them below:

- Identification of physically relevant models among various extended models, i.e. having asymptotic models of $O(\mu^3, \varepsilon^2\mu^2, \varepsilon\beta\mu^2, \beta^2)$ (medium amplitude and bathymetry models), and one can also derive higher-order versions of various shallow-water models, such as the well-known Boussinesq model in multiple surface and topography variations etc.
- Numerical computations of the initial value problems, as well as solitary and periodic wave solutions, for less complicated models mentioned above.
- The effect of higher-order dispersion on the wave characteristics and the effect of the presence, or lack thereof, of surface tension on the justification of the asymptotic models by means of rigorous mathematical analysis.

Following the general method for the derivation of asymptotic non-linear models in shallow and deep water first introduced in [23], we derive the new system generalizing (in the presence of an arbitrary topography) the investigations of [20] done by the authors. What is different for this system is the pure intricacy and number of terms that have not yet been derived or analyzed, especially in the case when the bottom is not flat. For the sake of constructing the solution by an energy estimate method, one must go through an impressive set of estimates and calculations to obtain the correct energy estimate. Due to this, the work is to be confined to one space dimension. The strategy of the proof is to write the system in a quasilinear form, i.e. to be treated as a hyperbolic system, and then use a symmetrizer to derive a good energy estimate of the solution. The existence is then obtained through a fixed point argument [1, 32]. On behalf of the inadequate linear dispersion structure of the original extended system (2.9), the study is to be done on a modified asymptotic variant:

$$(1.11)\quad \begin{cases} \partial_t\zeta + \partial_x(hv) = 0, \\ \Im(\partial_t v + \varepsilon v v_x) + h\partial_x\zeta - \dfrac{\mu}{\mathrm{bo}} h\zeta_{xxx} + \dfrac{2}{45}\mu^2 h\partial_x^2(h^4\zeta_{xxx}) + \mu^2\mathcal{I}[h,\beta b]\zeta_x \\ \qquad - \varepsilon^2\mu^2\dfrac{1}{\mathrm{bo}}T[U]\zeta_x + \varepsilon\mu\mathcal{Q}_1[U]v_x + \varepsilon\mu\beta\mathcal{B}_1[U]v_x + \varepsilon\mu^2\mathcal{Q}_2[U]v_x \\ \qquad + \varepsilon\mu^2\mathcal{Q}_3[U]v_x + \varepsilon\mu^2\beta\mathcal{B}_2[U]v_x + \varepsilon\mu^2\beta^2\mathcal{B}_3[U]v_x + \mathcal{R}[\mu,\varepsilon h,\beta b](U) \\ \quad = O(\mu^3), \end{cases}$$

where $U=(\zeta,v)^\top$, $h=1+\varepsilon\zeta-\beta b$, and $\Im=h+\mu\mathcal{T}[h,\beta b]-\mu^2\mathfrak{T}[h,\beta b]$. Denoting

$$\begin{aligned}
\mathcal{T}[h,\beta b]v &= -\frac{1}{3}\partial_x(h^3\partial_x v)+\frac{\beta}{2}\big[\partial_x(h^2 b_x v)-h^2 b_x v_x Vr\big]+\beta^2 h b_x^2 v,\\
T[U]\zeta_x &= -\frac{1}{2}h\partial_x^2(\zeta_x^2\zeta_x),\\
\mathfrak{T}[h,\beta b]v &= -\frac{1}{45}\partial_x^2(h^5\partial_x^2 v)\\
&\quad+\frac{\beta}{24}\big[\partial_x(\partial_x(h^4 b_x)\partial_x v)+\partial_x^2(h^4\partial_x(b_x v))-b_x\partial_x(h^4\partial_x^2 v)\big]\\
&\quad+\frac{\beta^2}{12}\big[2\partial_x(h^3 b_x^2\partial_x v)+\partial_x(h^3 b_x b_{xx} v)+2b_x\partial_x(h^3 b_x\partial_x v)+b_x\partial_x(h^3 b_{xx} v)\big],
\end{aligned}$$

where the non-topographical terms $\mathcal{Q}_1[U]$, $\mathcal{Q}_2[U]$, $\mathcal{Q}_3[U]$ are represented by

$$\begin{aligned}
\mathcal{Q}_1[U]f &= \frac{2}{3}\partial_x(h^3 v_x f),\quad \mathcal{Q}_2[U]f=\frac{8}{45}\partial_x^2(h^5 v_{xx} f),\quad \mathcal{Q}_3[U]f=\frac{1}{15}\partial_x(h^5 v_{xx}\partial_x f),\\
\mathcal{I}[h,\beta b]f &= \frac{4}{45}h_x\partial_x(h^4\partial_x^2 f)+\frac{2}{45}h^4 h_{xx}\partial_x^2 f\\
&\quad-\frac{\beta}{12}\big[\partial_x(\partial_x(h^4 b_x)\partial_x f)+\partial_x^2(h^4 b_{xx} f)+\partial_x^2(h^4 b_x\partial_x f)-b_x\partial_x(h^4\partial_x^2 f)\big]\\
&\quad-\frac{\beta^2}{6}\big[2\partial_x(h^3 b_x^2\partial_x f)+\partial_x(h^3 b_x b_{xx} f)+2b_x\partial_x(h^3 b_x\partial_x f)+b_x\partial_x(h^3 b_{xx} f)\big],
\end{aligned}$$

while the purely topographical terms are represented by $\mathcal{B}_1[U]f=h^2 b_x v_x f$ and

$$\begin{aligned}
\mathcal{B}_2[U]f &= -\frac{1}{24}\partial_x^2\big\{h^4(2b_{xx}v+9b_x v_x)f\big\}-\frac{1}{4}h^4 b_{xx} v_{xx} f-\frac{1}{24}\partial_x((h^4)_x b_x v_x f)\\
&\quad-\frac{1}{24}\partial_x(h^4 b_{xx} v_x f)-\frac{1}{4}\partial_x(h^4 b_{xx} v\partial_x f)+\frac{1}{12}b_x\partial_x(h^4 v_{xx} f),\\
\mathcal{B}_3[U]f &= \frac{3}{4}\partial_x(h^3 b_x b_{xx} v f)+\frac{1}{12}h^3(b_{xx}v+2b_x v_x)b_{xx} f\\
&\quad-\frac{1}{12}b_x\partial_x\big\{h^3(b_{xx} v f+8b_x v_x f)\big\},\\
\mathcal{R}[\mu,\varepsilon h,\beta b](U) &= \frac{1}{2}\varepsilon\mu\beta\partial_x(h^2 b_{xx} v^2)+\varepsilon\mu\beta^2 h b_x b_{xx} v^2+\frac{1}{24}\varepsilon\mu^2\beta\partial_x^2(h^4 b_{xxx} v^2)\\
&\quad+\frac{1}{12}\varepsilon\mu^2\beta^2\partial_x(h^3 b_x b_{xxx} v^2)+\frac{1}{4}\varepsilon\mu^2\beta^2\partial_x(h^3 b_{xx}^2 v^2)\\
&\quad+\frac{1}{12}\varepsilon\mu^2\beta^2 b_x\partial_x(h^3 b_{xxx} v^2).
\end{aligned}$$

This new variant shares the same order of precision as the original, but has a mathematical structure that is more suitable for well-posedness, in particular, its linear dispersion relation (see for instance Remark 2.2). Consequently, a problematic term appears in the form of fifth-order derivatives on the surface elevation, which require special treatment. Several different techniques are used throughout the proofs to generalize the flat bottom case [20], such as invertibility of the $\Im = h + \mu\mathcal{T}[h,\beta b] - \mu^2\mathfrak{T}[h,\beta b]$ operator, and re-expression of terms. The main point regarding the justification of such a higher-order asymptotic model is the following well-posedness result:

Long-time existence with surface tension. Theorem 5.1 states that (1.11) admits a unique solution in $U \in X^s = H^{s+2}(\mathbb{R}) \times H^{s+2}(\mathbb{R})$ on existence time-scales $T_{\max}$ up to order $\frac{1}{\max(\varepsilon,\beta)}$ such that $1/T_{\max}$ depends on initial data and the rescaled bond number (2.7) bo^{-1} as the energy estimate constant (4.4) depends on the same $\mathrm{bo}^{-1} \in [0,1)$.

In the sequel, Section 2 is devoted to the derivation of the extended two-dimensional Green–Naghdi system for non-flat bottom topography and the modification process held for the one-dimensional system to be studied. In Section 3, some properties on the fourth-order elliptic operator $\Im$ and its inverse are given. In Section 4, a suitable energy norm and symmetrizer are sought, and the linearized system with surface tension is studied for an appropriate energy preservation. The main results (Theorem 5.1: existence, Theorem 5.2: stability property, Proposition 5.3: consistency, and Theorem 5.4: convergence) are deduced and proved in Section 5.

§1.5. Notation

We denote by $C(\lambda_1, \lambda_2, \dots)$ a constant depending on the parameters $\lambda_1, \lambda_2, \dots$, and *whose dependence on the λ_j is always assumed to be non-decreasing*. The notation $a \lesssim b$ means that $a \le Cb$, for some non-negative constant C whose exact expression is of no importance (*in particular, it is independent of the small parameters involved*). Also, the notation $a \vee b$ stands for the maximum between a and b.

We denote the norm $|\cdot|_{L^2}$ simply by $|\cdot|_2$. The inner product of any functions f_1 and f_2 in the Hilbert space $L^2(\mathbb{R}^d)$ is denoted by $(f_1, f_2) = \int_{\mathbb{R}^d} f_1(X) f_2(X)\,dX$. The space $L^\infty = L^\infty(\mathbb{R}^d)$ consists of all essentially bounded, Lebesgue-measurable functions f with the norm $|f|_{L^\infty} = \operatorname{ess\,sup}|f(X)| < \infty$. We denote $W^{2,\infty} = W^{2,\infty}(\mathbb{R}) = \{f \in L^\infty,\ f_x \text{ and } f_{xx} \in L^\infty\}$, endowed with its canonical norm.

For any real constant s, $H^s = H^s(\mathbb{R}^d)$ denotes the Sobolev space of all tempered distributions f with the norm $|f|_{H^s} = |\Lambda^s f|_2 < \infty$, where Λ is the pseudo-differential operator $\Lambda^s = (1 - \partial_x^2)^{s/2}$.

For any functions $u = u(t, X)$ and $v(t, X)$ defined on $[0,T) \times \mathbb{R}^d$ with $T > 0$, we denote the inner product, the L^p-norm, and in particular the L^2-norm, as well as the Sobolev norm, with respect to the spatial variable, by $(u,v) = (u(\cdot,t), v(\cdot,t))$, $|u|_{L^p} = |u(\cdot,t)|_{L^p}$, $|u|_{L^2} = |u(\cdot,t)|_{L^2}$, and $|u|_{H^s} = |u(\cdot,t)|_{H^s}$, respectively.

Let $C^k(\mathbb{R}^d)$ denote the space of k-times continuously differentiable functions and $C_0^\infty(\mathbb{R}^d)$ denote the space of infinitely differentiable functions, with compact support in $\mathbb{R}^d$. We denote by $C_b^\infty(\mathbb{R})$ the space of infinitely differentiable functions that are bounded together with all their derivatives. For any closed operator T defined on a Banach space Y of functions, the commutator $[T, f]$ is defined by $[T, f]g = T(fg) - fT(g)$ with f, g, and fg belonging to the domain of T.

§2. Derivation of the extended Green–Naghdi model with general bottom topography

§2.1. Derivation of the system in two space dimensions

We start by averaging the horizontal velocity over depth (reducing the dimension by 1), i.e. we introduce a depth-averaged horizontal velocity such as

$$v(t,X) = \frac{1}{h(t,X)} \int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)} \nabla\varphi(t,X,z)\,dz, \tag{2.1}$$

where $h(t, X) = 1 + \varepsilon\zeta(t, X) - \beta b(X)$ is the non-dimensionalized height of the liquid. The exact expression for $-\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = \nabla \cdot (hv)$ stems from a clear outcome of Green's identity or by a straightforward calculation and rearranging terms using (1.7). The first evolution equation on ζ in terms of (ζ, v) reads

$$\partial_t\zeta + \nabla \cdot (hv) = 0.$$

Note that this equation exactly coincides with the first equation of (1.9). Now, proceeding as in [23], for the second evolution equation on v, $\nabla\psi$ does not have an exact expression in terms of (ζ, v). Therefore, since $\mu \ll 1$, we look for an asymptotic expansion with respect to μ on $\nabla\psi$ in terms of (ζ, v), and this is obtained through an asymptotic description of φ in the fluid by constructing

$$\varphi_{\text{app}}(t,X,z) = \varphi_0 + \mu\varphi_1 + \mu^2\varphi_2 + \cdots + \mu^N\varphi_N = \sum_{j=0}^{N} \mu^j\varphi_j. \tag{2.2}$$

Plugging expression (2.2) into the boundary value problem (1.8), and dropping all terms of $O(\mu^{N+1})$, one gets

$$\forall\, j = 0,1,\ldots,N, \quad \partial_z^2\varphi_j = -\partial_x^2\varphi_{j-1} - \partial_y^2\varphi_{j-1}, \tag{2.3}$$

with the convention $\varphi_{-1} = 0$ by definition and the boundary condition

$$\forall\, j = 0,1,\ldots,N, \quad \begin{cases} -\beta\nabla b\nabla\varphi_{j-1} + \partial_z\varphi_j = 0 & \text{at } z = -1+\beta b, \\ (\varphi_j)_{|z=\varepsilon\zeta} = \delta_{0,j}\psi, \end{cases} \tag{2.4}$$

where $\delta_{0,j} = 1$ if $j = 0$ and 0 otherwise. Solving the ODE (2.3)–(2.4) yields three solutions that are polynomials of order 0, 2, 4 in z such that

$$
\begin{aligned}
\varphi_0(t,X,z) &= \psi(t,X),\\
\varphi_1(t,X,z) &= (z-\varepsilon\zeta)\Big(-\frac{1}{2}(z+\varepsilon\zeta)-1+\beta b\Big)\nabla\cdot(\nabla\psi)+\beta(z-\varepsilon\zeta)\nabla b\cdot\nabla\psi,\\
\varphi_2(t,X,z) &= (z-\varepsilon\zeta)\beta\nabla b\cdot(\nabla\varphi_1)_{|_{z=-1+\beta b}}\\
&\quad+\frac{1}{2}((z+1-\beta b)^2-h^2)(\varepsilon\nabla\zeta)(-\varepsilon\nabla\zeta+2(\beta\nabla b))\Delta\psi\\
&\quad-2\Big[\frac{1}{2}((z+1-\beta b)^2-h^2)h(\varepsilon\nabla\zeta)\\
&\qquad+\frac{1}{2}\Big(\frac{1}{3}(z-\varepsilon\zeta)^3-(z-\varepsilon\zeta)h^2\Big)(\beta\nabla b)\Big]\nabla(\Delta\psi)\\
&\quad-\Big[\frac{1}{2}((z+1-\beta b)^2-h^2)h\nabla\cdot(\varepsilon\nabla\zeta)\\
&\qquad+\frac{1}{2}\Big(\frac{1}{3}(z-\varepsilon\zeta)^3-(z-\varepsilon\zeta)h^2\Big)\nabla\cdot(\beta\nabla b)\Big]\Delta\psi\\
&\quad+\Big[\frac{1}{24}(z^4-(\varepsilon\zeta)^4)-\frac{1}{6}(-1+\beta b)^3(z-\varepsilon\zeta)\\
&\qquad-\frac{(\varepsilon\zeta)^2}{4}((z+1-\beta b)^2-h^2)\\
&\qquad-\frac{1}{2}\Big(\frac{1}{3}(z-\varepsilon\zeta)^3-h^2(z-\varepsilon\zeta)\Big)(-1+\beta b)\Big]\nabla\cdot(\nabla(\Delta\psi))\\
&\quad+((z+1-\beta b)^2-h^2)(\varepsilon\nabla\zeta)\nabla(\beta\nabla b\cdot\nabla\psi)\\
&\quad+\frac{1}{2}((z+1-\beta b)^2-h^2)\nabla\cdot(\varepsilon\nabla\zeta)\beta\nabla b\cdot\nabla\psi\\
&\quad-\frac{1}{2}\Big(\frac{1}{3}(z-\varepsilon\zeta)^3-(z-\varepsilon\zeta)h^2\Big)\nabla\cdot(\nabla(\beta\nabla b\cdot\nabla\psi)).
\end{aligned}
$$

The horizontal component of the velocity in the fluid domain now reads

$$
\begin{aligned}
V(t,X,z) &= \nabla\varphi_{\mathrm{app}}\\
&= \nabla\varphi_0(t,X,z)+\mu\nabla\varphi_1(t,X,z)+\mu^2\nabla\varphi_2(t,X,z)+O(\mu^3).
\end{aligned}
$$

The averaged velocity is thus

$$
v(t,X) = \nabla\psi+\frac{\mu}{h}\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)}\nabla\varphi_1\,dz+\frac{\mu^2}{h}\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)}\nabla\varphi_2\,dz+O(\mu^3).
$$

As in [23], the second-order approximation $O(\mu^2)$ of $\nabla\psi$ in terms of ζ and v reads

$$
\begin{aligned}
\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)}\nabla\varphi_1\,dz &= -\frac{1}{3}\nabla(h^3\Delta\psi)+\frac{\beta}{2}[\nabla(h^2\nabla b\cdot\nabla\psi)-h^2\nabla b\Delta\psi]+\beta^2 h\nabla b\nabla b\cdot\nabla\psi\\
&= \mathcal{T}[h,\beta b]\nabla\psi.
\end{aligned}
$$

The new ingredient is writing a third-order approximation in μ of $\nabla\psi$ in terms of (ζ, v). Thus we need to compute the integral $J_2[h,\beta b]\nabla\psi = \int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)} \nabla\varphi_2\, dz$. As the terms are many, we refer to Appendix A.1 for detailed calculations and a clear presentation of J_2.

Remark 2.1. Here and throughout the rest of this paper we will introduce the two new notions "non-topographical" and "purely topographical" to differentiate many terms. Apart from the expression for the height of the fluid $h = 1+\varepsilon\zeta-\beta b = O(\varepsilon,\beta)$, the non-topographical expressions are the terms that do not include a bottom parameter $\beta^{k\in\mathbb{N}}$ in front of the term. Otherwise, the terms are purely topographical.

The averaged velocity expression becomes

$$v = \nabla\psi - \frac{\mu}{h}\mathcal{T}[h,\beta b]\nabla\psi + \frac{\mu^2}{h}J_2[h,\beta b]\nabla\psi + O(\mu^3),$$

but

$$\nabla\psi = v + \frac{\mu}{h}\mathcal{T}[h,\beta b]v + \frac{\mu^2}{h}\Big[\mathcal{T}[h,\beta b]\Big(\frac{1}{h}\mathcal{T}[h,\beta b]v\Big) - J_2[h,\beta b]v\Big].$$

Hence (see the details of $\mathcal{T}(h^{-1}\mathcal{T})$ in Appendix A.2 and of $\mathfrak{T} = \mathcal{T}(h^{-1}\mathcal{T}) - J_2$ in Appendix A.3), we obtain

$$\nabla\psi = v + \frac{\mu}{h}\mathcal{T}[h,\beta b]v + \frac{\mu^2}{h}\mathfrak{T}[h,\beta b]v + O(\mu^3), \tag{2.5}$$

where

$$\mathcal{T}[h,\beta b]w = -\frac{1}{3}\nabla(h^3\nabla\cdot w) + \frac{\beta}{2}\big[\nabla(h^2\nabla b\cdot w) - h^2\nabla\cdot w\nabla b\big] + \beta^2 h(\nabla b\cdot w)\nabla b$$

and

$$\begin{aligned}\mathfrak{T}[h,\beta b]w = {}& -\frac{1}{45}\nabla\big(\nabla\cdot(h^5\nabla(\nabla\cdot w))\big) + \frac{1}{24}\beta\nabla\big(\nabla\cdot(h^4\nabla(\nabla b\cdot w))\big)\\ &+ \frac{1}{24}\beta\nabla(\nabla\cdot w\nabla\cdot(h^4\nabla b)) - \frac{1}{24}\beta\nabla\cdot(h^4\nabla(\nabla\cdot w))\nabla b\\ &+ \frac{1}{12}\beta^2\nabla(h^3\nabla\cdot w(\nabla b\nabla b)) + \frac{1}{12}\beta^2\nabla(h^3\nabla b\nabla(\nabla b\cdot w))\\ &+ \frac{1}{12}\beta^2\nabla\cdot(h^3\nabla\cdot w\nabla b)\nabla b + \frac{1}{12}\beta^2\nabla\cdot(h^3\nabla(\nabla b\cdot w))\nabla b.\end{aligned}$$

It follows the lines of derivation using (multiple) scales in general, and the derivation of the Green–Naghdi system itself in particular. As such, it is formal and essentially algebraic (there is no functional analytic framework), but it is also lengthy and a feat in thoroughness and endurance, as the terms of the extended topography order are many, involved, and require clever tricks to gather in a suitable form. The main steps are

(1) take the gradient of the second equation of (1.9) then multiply it by h;
(2) replace $\nabla\psi$ by its expression (2.5) in the second equation of (1.9);
(3) replace $\mathcal{G}[\varepsilon\zeta,\beta b]\psi$ by $-\mu\nabla\cdot(hv)$ in the resulting equations;
(4) drop the $O(\mu^3)$ terms;
(5) expand then reduce terms of the same size;
(6) take advantage of the following vector triple products and vector identities:

$$u\times(\nu\times\omega)=(u\cdot\omega)\nu-(u\cdot\nu)\omega,$$
$$\nabla\times(\nabla G)=0\quad\text{and}\quad\nabla\times(GF)=G\nabla\times F+\nabla G\times F,$$

where G is a differentiable scalar function and u, ν, ω, and F are differentiable vector fields.

Finally, after capturing the information above we obtain the extended two-dimensional Green–Naghdi system for an uneven bottom topography ($\beta\neq 0$) without surface tension ($\sigma=0$) with an error of order μ^3:

$$\begin{cases}\partial_t\zeta+\nabla\cdot(hv)=0,\\ (h+\mu\mathcal{T}[h,\beta b]+\mu^2\mathfrak{T}[h,\beta b])\partial_t v+h\nabla\zeta+\varepsilon h(v\cdot\nabla)v+\varepsilon\mu Q_1[U]v\\ \quad+\varepsilon\mu\beta B_1[U]v+\varepsilon\mu\beta^2B_2[U]v+\varepsilon\mu^2Q_2[U]v+\varepsilon\mu^2\beta B_3[U]v\\ \quad+\varepsilon\mu^2\beta^2B_4[U]v=O(\mu^3),\end{cases}\tag{2.6}$$

where $v=(v_1,v_2)^\top$, $U=(\zeta,v)^\top$, and $h(t,X)=1+\varepsilon\zeta(t,X)-\beta b(X)$, and we denote

$$\begin{aligned}\mathcal{T}[h,\beta b]w&=-\frac{1}{3}\nabla(h^3\nabla\cdot w)+\frac{\beta}{2}\big[\nabla(h^2\nabla b\cdot w)-h^2\nabla\cdot w\nabla b\big]+\beta^2h(\nabla b\cdot w)\nabla b,\\ \mathfrak{T}[h,\beta b]w&=-\frac{1}{45}\nabla\big(\nabla\cdot(h^5\nabla(\nabla\cdot w))\big)+\frac{1}{24}\beta\nabla\big(\nabla\cdot(h^4\nabla(\nabla b\cdot w))\big)\\ &\quad+\frac{1}{24}\beta\nabla(\nabla\cdot w\nabla\cdot(h^4\nabla b))-\frac{1}{24}\beta\nabla\cdot(h^4\nabla(\nabla\cdot w))\nabla b\\ &\quad+\frac{1}{12}\beta^2\nabla(h^3\nabla\cdot w(\nabla b\nabla b))+\frac{1}{12}\beta^2\nabla(h^3\nabla b\nabla(\nabla b\cdot w))\\ &\quad+\frac{1}{12}\beta^2\nabla\cdot(h^3\nabla\cdot w\nabla b)\nabla b+\frac{1}{12}\beta^2\nabla\cdot(h^3\nabla(\nabla b\cdot w))\nabla b.\end{aligned}$$

Here, the non-topographical terms $Q_1[U]$, $Q_2[U]$ are represented by

$$\begin{aligned}Q_1[U]v&=-\frac{1}{3}\nabla\big(h^3\big((v\cdot\nabla)(\nabla\cdot v)-(\nabla\cdot v)^2\big)\big),\\ Q_2[U]v&=-\frac{1}{45}\nabla\big[\nabla\cdot\big\{h^5(\nabla^2(\nabla\cdot v))v-5h^5(\nabla\cdot v)\nabla(\nabla\cdot v)\\ &\qquad+\nabla h^5\times(v\times\nabla(\nabla\cdot v))\big\}\big]\end{aligned}$$

$$+ \frac{2}{45}\nabla\big(h^5(\nabla(\nabla\cdot v))^2\big) + \frac{1}{45}\nabla\cdot\big(h^5\nabla(\nabla\cdot v)\big)\nabla(\nabla\cdot v)$$
$$+ \frac{1}{90}h^5\nabla\big\{(\nabla(\nabla\cdot v))^2\big\},$$

while the purely topographical terms $B_1[U]$, $B_2[U]$, $B_3[U]$, $B_4[U]$ are represented by

$$\begin{aligned}
B_1[U]v &= \frac{1}{2}\nabla(h^2(v\cdot\nabla)^2 b) - \frac{1}{2}h^2\big((v\cdot\nabla)(\nabla\cdot v) - (\nabla\cdot v)^2\big)\nabla b,\\
B_2[U]v &= h((v\cdot\nabla)^2 b)\nabla b,\\
B_3[U]v &= +\frac{1}{24}\nabla\big\{\nabla\cdot\big(h^4\nabla^2(\nabla b\cdot v)v + h^4(\nabla\cdot v\nabla\cdot\nabla b)v\\
&\qquad - 4h^4(\nabla\cdot v)^2\nabla b - 4h^4\nabla\cdot v\nabla(\nabla b\cdot v)\\
&\qquad + \nabla\cdot v\nabla h^4\times(v\times\nabla b) + \nabla h^4\times(v\times\nabla(\nabla b\cdot v))\big)\big\}\\
&\quad + \frac{1}{48}\nabla b\times(\nabla h^4\times\nabla(\nabla\cdot v)^2) - \frac{1}{48}h^4\nabla\cdot\nabla b\nabla(\nabla\cdot v)^2\\
&\quad - \frac{1}{24}\nabla\cdot(h^4\nabla(\nabla b\cdot v))\nabla(\nabla\cdot v) - \frac{1}{24}h^4\nabla^2(\nabla\cdot v)\nabla(\nabla b\cdot v)\\
&\quad + \frac{1}{24}\nabla(\nabla\cdot v)\times(\nabla(\nabla b\cdot v)\times\nabla h^4) - \frac{1}{6}\nabla(h^4\nabla(\nabla\cdot v)\nabla(\nabla b\cdot v))\\
&\quad + \frac{1}{24}\nabla\big((\nabla h^4\cdot v)(\nabla b\nabla(\nabla\cdot v))\big)\\
&\quad - \frac{1}{24}\nabla\cdot\big\{h^4(\nabla^2(\nabla\cdot v)v - 2\nabla(\nabla\cdot v)^2) + \nabla h^4\times(v\times\nabla(\nabla\cdot v))\big\}\nabla b,\\
B_4[U]v &= \frac{1}{12}\nabla\big\{h^3\nabla\cdot\big(\nabla b\times(v\times\nabla(\nabla b\cdot v))\big) + h^3\nabla\cdot(\nabla\cdot v\nabla b\times(v\times\nabla b))\\
&\qquad + h^3(\nabla b\cdot v)\nabla^2(\nabla b\cdot v) + h^3(\nabla b\cdot v)\nabla\cdot(\nabla\cdot v\nabla b)\\
&\qquad + 2h^3(\nabla(\nabla b\cdot v))^2 - 2h^3(\nabla\cdot v)^2\nabla b\nabla b\big\}\\
&\quad + \frac{1}{12}\nabla\cdot(h^3\nabla\cdot v\nabla b)\nabla(\nabla b\cdot v) + \frac{1}{12}\nabla\cdot(h^3\nabla(\nabla b\cdot v))\nabla(\nabla b\cdot v)\\
&\quad + \frac{1}{12}h^3\nabla\cdot v\nabla(\nabla b\nabla(\nabla b\cdot v)) + \frac{1}{24}h^3(\nabla\cdot v)^2\nabla(\nabla b\nabla b)\\
&\quad + \frac{1}{24}h^3\nabla\big\{(\nabla(\nabla b\cdot v))^2\big\}\\
&\quad + \frac{1}{12}\nabla\cdot\big\{-3h^3(\nabla\cdot v)^2\nabla b + \nabla\cdot v\nabla h^3\times(v\times\nabla b)\\
&\qquad + h^3 v\nabla\cdot(\nabla\cdot v\nabla b)\big\}\nabla b\\
&\quad + \frac{1}{12}\nabla\cdot\big\{h^3\nabla^2(\nabla b\cdot v)v - 3h^3\nabla\cdot v\nabla(\nabla b\cdot v)\\
&\qquad + \nabla h^3\times(v\times\nabla(\nabla b\cdot v))\big\}\nabla b,
\end{aligned}$$

where the expression for Q_2 introduces the Laplacian operator $\nabla^2 = \nabla\cdot\nabla = \Delta$.

§2.2. The capillary components

Different strategies exist to deal with the presence of surface tension ($\sigma \neq 0$) in the water-wave problem, such as [36, 29, 3, 4]. The main contrast in our work is that the gradient of the capillary term $-\frac{1}{\mathrm{Bo}}\frac{\kappa(\varepsilon\sqrt{\mu}\zeta)}{\varepsilon\sqrt{\mu}}$ multiplied by h must be added to the right-hand side of the second equation of (2.6). From a physical point of view, the effect of surface tension on the water surface is negligible, so the only condition we need is that there is a small amount of surface tension. More precisely, since the Bond number Bo is generally large, we assume that the capillary parameter Bo^{-1} is of the same order as the shallowness parameter $\mu \ll 1$. And therefore we define the *rescaled Bond number* bo instead of the *classical Bond number* Bo, as follows:

$$0 < \mathrm{bo} = \mu\mathrm{Bo} = \frac{\rho g h_0^2}{\sigma} \ll \mathrm{Bo}, \tag{2.7}$$

where h_0 is the reference depth, ρ the positive constant density of the fluid, g the acceleration of gravity, and σ the surface tension coefficient. Regarding this, bo is not assumed too small so that $\mathrm{Bo}^{-1} = \mu\mathrm{bo}^{-1} = O(\mu)$, and the two-dimensional capillary terms that should be added stand for

$$\begin{aligned} -\frac{1}{\mathrm{Bo}}h\nabla\Big\{\frac{\kappa(\varepsilon\sqrt{\mu}\zeta)}{\varepsilon\sqrt{\mu}}\Big\} &= \frac{1}{\mathrm{bo}}\mu h\nabla(\nabla\cdot(\nabla\zeta)) \\ &\quad - \frac{1}{2\mathrm{bo}}\varepsilon^2\mu^2 h\nabla(\nabla\cdot(|\nabla\zeta|^2\nabla\zeta)) + O(\varepsilon^4\mu^3). \end{aligned} \tag{2.8}$$

§2.3. The one-dimensional case

For the mathematical analysis of the model we will confine the work to one space dimension. The extended Green–Naghdi system (2.6) with surface tension is rearranged after a few calculations, taking into account the capillary terms (2.8), as

$$\begin{cases} \partial_t\zeta + \partial_x(hv) = 0, \\ \big(h + \mu\mathcal{T}[h,\beta b] + \mu^2\mathfrak{T}[h,\beta b]\big)\partial_t v + h\partial_x\zeta + \varepsilon h v v_x + \varepsilon\mu Q_1[U]v \\ \quad + \varepsilon\mu\beta B_1[U]v + \varepsilon\mu\beta^2 B_2[U]v + \varepsilon\mu^2 Q_2[U]v + \varepsilon\mu^2\beta B_3[U]v \\ \quad + \varepsilon\mu^2\beta^2 B_4[U]v = \dfrac{1}{\mathrm{bo}}\mu h\zeta_{xxx} + \varepsilon^2\mu^2\dfrac{1}{\mathrm{bo}}T[U]\zeta_x + O(\mu^3), \end{cases} \tag{2.9}$$

where $U = (\zeta, v)^\top$ and denoting by $h = h(t,x) = 1 + \varepsilon\zeta(t,x) - \beta b(x)$ the total non-dimensional height of the liquid, with

$$\mathcal{T}[h,\beta b]v = -\frac{1}{3}\partial_x(h^3\partial_x v) + \frac{\beta}{2}\big[\partial_x(h^2 b_x v) - h^2 b_x v_x\big] + \beta^2 h b_x^2 v,$$

$$
\begin{aligned}
T[U]\zeta_x &= -\frac{1}{2}h\partial_x^2(\zeta_x^2\zeta_x),\\
\mathfrak{T}[h,\beta b]v &= -\frac{1}{45}\partial_x^2(h^5\partial_x^2 v)\\
&\quad+\frac{\beta}{24}\big[\partial_x(\partial_x(h^4 b_x)\partial_x v)+\partial_x^2(h^4\partial_x(b_x v))-b_x\partial_x(h^4\partial_x^2 v)\big]\\
&\quad+\frac{\beta^2}{12}\big[2\partial_x(h^3 b_x^2\partial_x v)+\partial_x(h^3 b_x b_{xx} v)+2b_x\partial_x(h^3 b_x\partial_x v)+b_x\partial_x(h^3 b_{xx} v)\big].
\end{aligned}
$$

The non-topographical terms $Q_1[U]$, $Q_2[U]$ are represented by

$$
\begin{aligned}
Q_1[U]v &= -\frac{1}{3}\partial_x(h^3(vv_{xx}-v_x^2)),\\
Q_2[U]v &= -\frac{1}{45}\partial_x\big\{\partial_x(h^5(vv_{xxx}-5v_x v_{xx}))-3h^5(v_{xx})^2\big\},
\end{aligned}
$$

while the purely topographical terms $B_1[U]$, $B_2[U]$, $B_3[U]$, $B_4[U]$ are represented by

$$
\begin{aligned}
B_1[U]v &= \frac{1}{2}\big[\partial_x(h^2 b_{xx} v^2)+\partial_x(h^2 b_x v v_x)-h^2(vv_{xx}-v_x^2)b_x\big],\\
B_2[U]v &= h\big\{b_{xx}v^2+b_x v v_x\big\}b_x,\\
B_3[U]v &= \frac{1}{24}\partial_x^2\big\{h^4(b_{xxx}v^2-b_{xx}vv_x+b_x vv_{xx}-8b_x v_x^2)\big\}-\frac{1}{4}h^4 b_{xx}v_x v_{xx}\\
&\quad+\frac{1}{24}\partial_x(h_x^4 b_x vv_{xx})-\frac{5}{24}\partial_x(h^4 b_{xx}vv_{xx})-\frac{1}{24}b_x\partial_x\big\{h^4(vv_{xxx}+v_x v_{xx})\big\},\\
B_4[U]v &= \frac{1}{12}\partial_x\big\{h^3(b_{xxx}b_x v^2+2b_x^2 vv_{xx}+10b_x b_{xx}vv_x+2b_x^2 v_x^2+3b_{xx}^2 v^2)\big\}\\
&\quad+\frac{1}{12}h^3(b_{xx}v+2b_x v_x)b_{xx}v_x+\frac{1}{12}b_x\partial_x\big\{h^3(b_{xxx}v^2+2b_x vv_{xx}-6b_x v_x^2)\big\}.
\end{aligned}
$$

§2.4. New invariant of (2.9)

The main interest of this reformulation is to gather all terms of fifth-order derivatives in the leftmost term, that is, $(h+\mu\mathcal{T}[h]+\mu^2\mathfrak{T}[h])(\partial_t v+\varepsilon vv_x)$. This is reachable by setting $\pm\varepsilon\mu\mathcal{T}[h,\beta b](vv_x)$ and $\pm\varepsilon\mu^2\mathfrak{T}[h,\beta b](vv_x)$ in the second equation of (2.9). The new formulation then reads

$$
\begin{cases}
\partial_t\zeta+\partial_x(hv)=0,\\
\big(h+\mu\mathcal{T}[h,\beta b]+\mu^2\mathfrak{T}[h,\beta b]\big)(\partial_t v+\varepsilon vv_x)+h\partial_x\zeta+\varepsilon\mu\mathbf{Q}_1[U]v\\
\quad+\varepsilon\mu\beta\mathbf{B}_1[U]v+\varepsilon\mu\beta^2\mathbf{B}_2[U]v+\varepsilon\mu^2\mathbf{Q}_2[U]v+\varepsilon\mu^2\beta\mathbf{B}_3[U]v\\
\quad+\varepsilon\mu^2\beta^2\mathbf{B}_4[U]v=\dfrac{1}{\mathrm{bo}}\mu h\zeta_{xxx}+\varepsilon^2\mu^2\dfrac{1}{\mathrm{bo}}T[U]\zeta_x+O(\mu^3),
\end{cases}
\tag{2.10}
$$

where $U = (\zeta, v)^\top$ and $h(t,x) = 1 + \varepsilon\zeta(t,x) - \beta b(x)$, and one may write the above expressions as

$$\begin{aligned}
\mathcal{T}[h,\beta b]v &= -\frac{1}{3}\partial_x(h^3\partial_x v) + \frac{\beta}{2}\big[\partial_x(h^2 b_x v) - h^2 b_x v_x\big] + \beta^2 h b_x^2 v,\\
T[U]\zeta_x &= -\frac{1}{2}h\partial_x^2(\zeta_x^2\zeta_x),\\
\mathfrak{T}[h,\beta b]v &= -\frac{1}{45}\partial_x^2(h^5\partial_x^2 v)\\
&\quad + \frac{\beta}{24}\big[\partial_x(\partial_x(h^4 b_x)\partial_x v) + \partial_x^2(h^4\partial_x(b_x v)) - b_x\partial_x(h^4\partial_x^2 v)\big]\\
&\quad + \frac{\beta^2}{12}\big[2\partial_x(h^3 b_x^2\partial_x v) + \partial_x(h^3 b_x b_{xx} v) + 2b_x\partial_x(h^3 b_x\partial_x v) + b_x\partial_x(h^3 b_{xx} v)\big],
\end{aligned}$$

where the reformulated non-topographical terms $\mathbf{Q}_1[U]$, $\mathbf{Q}_2[U]$ are represented by

$$\mathbf{Q}_1[U]v = \frac{2}{3}\partial_x(h^3 v_x^2), \qquad \mathbf{Q}_2[U]v = \frac{1}{45}\partial_x\big\{8\partial_x(h^5 v_x v_{xx}) + 3h^5 v_{xx}^2\big\},$$

while the reformulated purely topographical terms are represented by $\mathbf{B}_1[U]v = \frac{1}{2}\partial_x(h^2 b_{xx} v^2) + h^2 b_x v_x^2$, and $\mathbf{B}_2[U]v = h b_{xx} b_x v^2$ with

$$\begin{aligned}
\mathbf{B}_3[U]v &= \frac{1}{24}\partial_x^2\big\{h^4(b_{xxx}v^2 - 2b_{xx}vv_x - 9b_x v_x^2)\big\} - \frac{1}{4}h^4 b_{xx} v_x v_{xx}\\
&\quad - \frac{1}{24}\partial_x((h^4)_x b_x v_x^2) - \frac{1}{24}\partial_x(h^4 b_{xx} v_x^2)\\
&\quad - \frac{1}{4}\partial_x(h^4 b_{xx} v v_{xx}) + \frac{1}{12}b_x\partial_x(h^4 v_x v_{xx}),\\
\mathbf{B}_4[U]v &= \frac{1}{12}\partial_x\big\{h^3(b_{xxx}b_x v^2 + 9b_x b_{xx} v v_x + 3b_{xx}^2 v^2)\big\} + \frac{1}{12}h^3(b_{xx}v + 2b_x v_x)b_{xx}v_x\\
&\quad + \frac{1}{12}b_x\partial_x\big\{h^3(b_{xxx}v^2 - b_{xx}vv_x - 8b_x v_x^2)\big\}.
\end{aligned}$$

§2.5. The modified system to be studied (2.11)

In this subsection we will state in Remark 2.2 the reason behind the use of the BBM trick and consequently the corresponding modified model (2.11).

Remark 2.2. The linear dispersion relation of the original/reformulated extended model (2.9)–(2.10) reads

$$\Big(1 + \frac{1}{3}\mu|\xi|^2 - \frac{1}{45}\mu^2|\xi|^4\Big)\omega^2 - |\xi|^2 = 0.$$

In fact, in the high-frequency regime $|\xi| \gg 1$, an instability appears. As a result, the Cauchy problem for the μ^3 model is ill posed. Therefore, a modification is required to the structure of the model, so that the dispersion relation does not give rise to any singularities.

In other words, the positive sign of $\mathfrak{T}[h,\beta b]$ in the μ^2 fifth-order factorized term is problematic and prevents the invertibility proof of the operator $h+\mu\mathcal{T}[h,\beta b]+\mu^2\mathfrak{T}[h,\beta b]$ by a Lax–Milgram theorem. This property is of highest interest for the well-posedness demonstration. To overcome this difficulty we replace the positive sign by a negative sign. A remainder term in the expression $2\mu^2\mathfrak{T}[h,\beta b](\partial_t v+\varepsilon v v_x)$ appears. At this stage, in order to trade the time-dependent derivative on v for a spatial one, a BBM trick is used and is represented by the approximate equation $\partial_t v+\varepsilon v v_x=-\zeta_x+O(\mu)$.

It is noteworthy that one may replace the relation $h+\mu\mathcal{T}[h,\beta b]+\mu^2\mathfrak{T}[h,\beta b]=\Im+2\mu^2\mathfrak{T}[h,\beta b]$, which defines the new operator $\Im$, by an alternative one $\Im+(\alpha+1)\mu^2\mathfrak{T}[h,\beta b]=h+\mu\mathcal{T}[h,\beta b]+\mu^2\mathfrak{T}[h,\beta b]$, with an arbitrary real parameter $\alpha>0$. The special case $\alpha=1$ recovers the present definition. In this general setting, the linear dispersion relation would also not give rise to any singularities.

In this case, the modified linear dispersion relation of the new system (2.11) (with $\mathrm{bo}^{-1}=0$) exhibits no singularity and reads

$$\Big(1+\frac{1}{3}\mu|\xi|^2+\frac{1}{45}\mu^2|\xi|^4\Big)\omega^2-|\xi|^2-\frac{2}{45}\mu^2|\xi|^6=0.$$

On the other hand, in the presence of surface tension (with $\mathrm{bo}^{-1}\neq 0$), the modified linear dispersion relation of the new system (2.11) reads

$$\Big(1+\frac{1}{3}\mu|\xi|^2+\frac{1}{45}\mu^2|\xi|^4\Big)\omega^2-|\xi|^2-\frac{1}{\mathrm{bo}}\mu|\xi|^4-\frac{2}{45}\mu^2|\xi|^6=0.$$

In view of the above remark, we introduce a new operator $\Im=h+\mu\mathcal{T}[h,\beta b]-\mu^2\mathfrak{T}[h,\beta b]$. This has to be followed by some necessary rearrangements due to a suitable specification of an appropriate symmetrizer (4.2). The modified one-dimensional extended Green–Naghdi system with surface tension then reads

$$\begin{cases}\partial_t\zeta+\partial_x(hv)=0,\\ \Im(\partial_t v+\varepsilon v v_x)+h\partial_x\zeta-\dfrac{\mu}{\mathrm{bo}}h\zeta_{xxx}+\dfrac{2}{45}\mu^2 h\partial_x^2(h^4\zeta_{xxx})+\mu^2\mathcal{I}[h,\beta b]\zeta_x\\ \qquad -\varepsilon^2\mu^2\dfrac{1}{\mathrm{bo}}T[U]\zeta_x+\varepsilon\mu\mathcal{Q}_1[U]v_x+\varepsilon\mu\beta\mathcal{B}_1[U]v_x+\varepsilon\mu^2\mathcal{Q}_2[U]v_x\\ \qquad +\varepsilon\mu^2\mathcal{Q}_3[U]v_x+\varepsilon\mu^2\beta\mathcal{B}_2[U]v_x+\varepsilon\mu^2\beta^2\mathcal{B}_3[U]v_x+\mathcal{R}[\mu,\varepsilon h,\beta b](U)\\ \quad =O(\mu^3),\end{cases}\tag{2.11}$$

where $U=(\zeta,v)^\top$ and $h(t,x)=1+\varepsilon\zeta(t,x)-\beta b(x)$. We denote

$$\mathcal{T}[h,\beta b]v=-\frac{1}{3}\partial_x(h^3\partial_x v)+\frac{\beta}{2}\big[\partial_x(h^2 b_x v)-h^2 b_x v_x\big]+\beta^2 h b_x^2 v,\tag{2.12}$$

$$T[U]\zeta_x=-\frac{1}{2}h\partial_x^2(\zeta_x^2\zeta_x),\tag{2.13}$$

$$\begin{aligned}\mathfrak{T}[h,\beta b]v = &-\frac{1}{45}\partial_x^2(h^5\partial_x^2 v)\\ &+\frac{\beta}{24}\big[\partial_x(\partial_x(h^4 b_x)\partial_x v)+\partial_x^2(h^4\partial_x(b_x v))-b_x\partial_x(h^4\partial_x^2 v)\big]\\ &+\frac{\beta^2}{12}\big[2\partial_x(h^3 b_x^2\partial_x v)+\partial_x(h^3 b_x b_{xx} v)\\ &\qquad +2b_x\partial_x(h^3 b_x\partial_x v)+b_x\partial_x(h^3 b_{xx} v)\big],\end{aligned}\tag{2.14}$$

where the non-topographical terms $\mathcal{Q}_1[U]$, $\mathcal{Q}_2[U]$, $\mathcal{Q}_3[U]$ are represented by

$$\mathcal{Q}_1[U]f=\frac{2}{3}\partial_x(h^3 v_x f),\quad \mathcal{Q}_2[U]f=\frac{8}{45}\partial_x^2(h^5 v_{xx} f),\quad \mathcal{Q}_3[U]f=\frac{1}{15}\partial_x(h^5 v_{xx}\partial_x f),$$

$$\begin{aligned}\mathcal{I}[h,\beta b]f=&\frac{4}{45}h_x\partial_x(h^4\partial_x^2 f)+\frac{2}{45}h^4 h_{xx}\partial_x^2 f\\ &-\frac{\beta}{12}\big[\partial_x(\partial_x(h^4 b_x)\partial_x f)+\partial_x^2(h^4 b_{xx} f)+\partial_x^2(h^4 b_x\partial_x f)-b_x\partial_x(h^4\partial_x^2 f)\big]\\ &-\frac{\beta^2}{6}\big[2\partial_x(h^3 b_x^2\partial_x f)+\partial_x(h^3 b_x b_{xx} f)\\ &\qquad +2b_x\partial_x(h^3 b_x\partial_x f)+b_x\partial_x(h^3 b_{xx} f)\big],\end{aligned}$$

while the purely topographical terms are represented by $\mathcal{B}_1[U]f=h^2 b_x v_x f$ and

$$\begin{aligned}\mathcal{B}_2[U]f=&-\frac{1}{24}\partial_x^2\big\{h^4(2b_{xx}v+9b_x v_x)f\big\}-\frac{1}{4}h^4 b_{xx} v_{xx} f\\ &-\frac{1}{24}\partial_x((h^4)_x b_x v_x f)-\frac{1}{24}\partial_x(h^4 b_{xx} v_x f)\\ &-\frac{1}{4}\partial_x(h^4 b_{xx} v\partial_x f)+\frac{1}{12}b_x\partial_x(h^4 v_{xx} f),\\ \mathcal{B}_3[U]f=&\frac{3}{4}\partial_x(h^3 b_x b_{xx} v f)+\frac{1}{12}h^3(b_{xx}v+2b_x v_x)b_{xx} f\\ &-\frac{1}{12}b_x\partial_x\big\{h^3(b_{xx} v f+8b_x v_x f)\big\},\\ \mathcal{R}[\mu,\varepsilon h,\beta b](U)=&\frac{1}{2}\varepsilon\mu\beta\partial_x(h^2 b_{xx} v^2)+\varepsilon\mu\beta^2 h b_x b_{xx} v^2+\frac{1}{24}\varepsilon\mu^2\beta\partial_x^2(h^4 b_{xxx} v^2)\\ &+\frac{1}{12}\varepsilon\mu^2\beta^2\partial_x(h^3 b_x b_{xxx} v^2)+\frac{1}{4}\varepsilon\mu^2\beta^2\partial_x(h^3 b_{xx}^2 v^2)\\ &+\frac{1}{12}\varepsilon\mu^2\beta^2 b_x\partial_x(h^3 b_{xxx} v^2).\end{aligned}$$

§3. Preliminary results

From a physical point of view we will assume that the fluid depth is constantly limited. This assumption is essential for the mathematical analysis. Therefore, the analysis is to be done under a non-zero depth condition:

$$\text{there exist } h_{\min} > 0, \quad \inf_{x\in\mathbb{R}} h \geq h_{\min} \quad \text{where } h(t,x) = 1 + \varepsilon\zeta(t,x) - \beta b(x). \tag{3.1}$$

We will intensively use two formulations of the leftmost operator $\Im = h + \mu\mathcal{T}[h,\beta b] - \mu^2\mathfrak{T}[h,\beta b]$ (we recall (2.12)–(2.14) the definitions of $\mathcal{T}$ and $\mathfrak{T}$) which at some points provide more convenience in the energy estimate derivation and the analysis of the operator itself. These two expressions correspond to two formulations of the operator $\mathfrak{T}[h,\beta b]$, one defined by (2.14) and the other by

$$\begin{aligned}\mathfrak{T}[h,\beta b] = &-\frac{1}{45}\partial_x^2(h^5\partial_x^2\cdot)\\ &+\frac{\beta}{24}\big[2\partial_x^2(h^4 b_x\partial_x\cdot) + \partial_x^2(h^4 b_{xx}(\cdot)) - 2\partial_x(h^4 b_x\partial_x^2\cdot) + h^4 b_{xx}\partial_x^2\cdot\big]\\ &+\frac{\beta^2}{12}\big[4\partial_x(h^3 b_x^2\partial_x\cdot) + 2\partial_x(h^3 b_x b_{xx}(\cdot)) - 2h^3 b_x^2\partial_x(\cdot) - h^3 b_x b_{xx}(\cdot)\big].\end{aligned} \tag{3.2}$$

The following two lemmas provide important invertibility results on $\Im$ and specify some properties on its inverse $\Im^{-1}$.

Lemma 3.1. *Let $b \in C_b^\infty(\mathbb{R})$ and assume that $\zeta(t,\cdot) \in W^{2,\infty}(\mathbb{R})$ is a differentiable scalar function under condition (3.1). Then the operator*

$$\Im\colon H^4(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

is well defined, one-to-one, and onto.

Proof. The proof of the invertibility of $\Im$ is a direct application of the Lax–Milgram theorem. We define by $H^2_\mu(\mathbb{R})$ the space $H^2(\mathbb{R})$ endowed with the norm $|\cdot|_\mu$ as

$$H^2_\mu(\mathbb{R}) = \big\{v \in H^2(\mathbb{R});\ |v|_\mu^2 = |v|_2^2 + \mu|v_x|_2^2 + \mu^2|v_{xx}|_2^2 < \infty\big\},$$

where $|\cdot|_\mu$ is equivalent to $|\cdot|_{H^2}$ but not uniformly with respect to $\mu \in (0,1)$. Let $f \in L^2(\mathbb{R})$. Consider the weak problem

$$\begin{cases}\text{find } v \in H^2_\mu(\mathbb{R}) \text{ such that}\\ a(v,u) = L(u)\ \forall\, u \in H^2_\mu(\mathbb{R}),\end{cases}$$

with $L(u) = (f,u)$ and the bilinear form $a(v,u) = (\Im v, u)$ which can be written as

$$\begin{aligned}a(v,u) = (hv,u) + \mu\Big(h\Big(\frac{\sqrt{3}}{3}hv_x - \frac{\sqrt{3}}{2}\beta b_x v\Big), \frac{\sqrt{3}}{3}hu_x - \frac{\sqrt{3}}{2}\beta b_x u\Big)\\ + \frac{\mu\beta^2}{4}(hb_x v, b_x u)\end{aligned}$$

$$
\begin{aligned}
&+\mu^2\Big(h\Big(\frac{\sqrt{5}}{15}h^2v_{xx}-\frac{\sqrt{5}}{4}\beta hb_xv_x-\frac{\sqrt{5}}{8}\beta hb_{xx}v\Big),\\
&\qquad\frac{\sqrt{5}}{15}h^2u_{xx}-\frac{\sqrt{5}}{4}\beta hb_xu_x-\frac{\sqrt{5}}{8}\beta hb_{xx}u\Big)\\
&+\mu^2\beta^2\Big(h\Big(\frac{\sqrt{3}}{12}hb_xv_x+\frac{\sqrt{3}}{24}hb_{xx}v\Big),\frac{\sqrt{3}}{12}hb_xu_x+\frac{\sqrt{3}}{24}hb_{xx}u\Big).
\end{aligned}
$$

It is easy to see that a and L are continuous on $H^2_\mu(\mathbb{R})\times H^2_\mu(\mathbb{R})$ and $H^2_\mu(\mathbb{R})$ respectively. In addition, using (3.1) we have

$$
\begin{aligned}
a(v,v)\geq{}& h_{\min}|v|_2^2+\mu h_{\min}\Big|\frac{\sqrt{3}}{3}hv_x-\frac{\sqrt{3}}{2}\beta b_xv\Big|_2^2+\frac{\mu\beta^2}{4}|b_xv|_2^2\\
&+\mu^2h_{\min}\Big|\frac{\sqrt{5}}{15}h^2v_{xx}-\frac{\sqrt{5}}{4}\beta hb_xv_x-\frac{\sqrt{5}}{8}\beta hb_{xx}v\Big|_2^2\\
&+\mu^2\beta^2h_{\min}\Big|\frac{\sqrt{3}}{12}hb_xv_x+\frac{\sqrt{3}}{24}hb_{xx}v\Big|_2^2,
\end{aligned}
$$

but it holds that

$$
\begin{aligned}
|v|_\mu^2&\leq|v|_2^2+\frac{3\mu}{h_{\min}^2}\Big|\frac{\sqrt{3}}{3}hv_x\Big|_2^2+\frac{45\mu^2}{h_{\min}^4}\Big|\frac{\sqrt{5}}{15}h^2v_{xx}\Big|_2^2\\
&\leq|v|_2^2+\frac{18}{h_{\min}^2}\mu\Big(\Big|\frac{\sqrt{3}}{3}hv_x-\frac{\sqrt{3}}{2}\beta b_xv\Big|_2^2+\frac{\beta^2}{4}|b_xv|_2^2\Big)\\
&\quad+\frac{90}{h_{\min}^4}\mu^2\Big|\frac{\sqrt{5}}{15}h^2v_{xx}-\frac{\sqrt{5}}{4}\beta hb_xv_x-\frac{\sqrt{5}}{8}\beta hb_{xx}v\Big|_2^2\\
&\quad+\frac{1350}{h_{\min}^4}\mu^2\beta^2\Big|\frac{\sqrt{3}}{12}hb_xv_x+\frac{\sqrt{3}}{24}hb_{xx}v\Big|_2^2.
\end{aligned}
$$

Therefore, after denoting $\mathfrak{M}_{h_{\min}}=\max\{1,\frac{18}{h_{\min}^2},\frac{90}{h_{\min}^4},\frac{1350}{h_{\min}^4}\}$, we deduce the coercivity condition on $H^2_\mu(\mathbb{R})\times H^2_\mu(\mathbb{R})$ represented by the inequality

$$
a(v,v)\geq\frac{h_{\min}}{\mathfrak{M}_{h_{\min}}}|v|_\mu^2. \tag{3.3}
$$

Therefore by the Lax–Milgram theorem, for every $f\in L^2(\mathbb{R})$, there exists a unique $v\in H^2_\mu(\mathbb{R})$ such that for all $u\in H^2_\mu(\mathbb{R})$ we have $a(v,u)=(\Im v,u)=L(u)=(f,u)$. Equivalently, there is a unique variational solution to the equation

$$
\Im v=f. \tag{3.4}
$$

It remains to prove that $v\in H^4(\mathbb{R})$. Indeed, fix $\mu\in(0,1)$ and introduce the well-defined invertible operator $J\colon H^2(\mathbb{R})\to L^2(\mathbb{R})$ defined by $J\phi=\phi-\frac{\mu^2}{45}\partial_x(h^5\partial_x\phi)$ such that (see Appendix A.4 for further discussion), for all $q,g\in L^2(\mathbb{R})$, it holds that

$$\begin{cases} \mu|J^{-1}\partial_x g|_{H^1_\mu} \lesssim |g|_{L^2}, \\ |J^{-1}q|_{H^1_\mu} \lesssim |q|_{L^2}. \end{cases} \tag{3.5}$$

From the definition of operators $\Im$ and J combined with (3.4), it holds that $v_{xx} = J^{-1}\psi$ such that

$$\begin{aligned}\psi &= hv + v_{xx} + \mu\mathcal{T}[h,\beta b]v + \frac{1}{45}\mu^2\partial_x(h_x^5 v_{xx}) \\ &\quad - \frac{\mu^2\beta}{24}\big[\partial_x(\partial_x(h^4 b_x)\partial_x v) + \partial_x^2(h^4\partial_x(b_x v)) - b_x\partial_x(h^4\partial_x^2 v)\big] \\ &\quad - \frac{\mu^2\beta^2}{12}\big[2\partial_x(h^3 b_x^2\partial_x v) + \partial_x(h^3 b_x b_{xx} v) + 2b_x\partial_x(h^3 b_x\partial_x v) + b_x\partial_x(h^3 b_{xx} v)\big] - f.\end{aligned}$$

Since $v \in H^2(\mathbb{R})$, $f \in L^2(\mathbb{R})$ and using (3.5), one may deduce that $J^{-1}\psi = v_{xx} \in H^1(\mathbb{R})$. Thus by (3.4) and (3.1) we have

$$\begin{aligned} v_{xxxx} = &-\frac{45}{h^4\mu^2}v - \frac{45}{h^5\mu}\mathcal{T}[h,\beta b]v - \frac{h^5_{xx}}{h^5}v_{xx} - \frac{2h_x^5}{h^5}v_{xxx} \\ &+ \frac{15\beta}{8h^5}\big[\partial_x(\partial_x(h^4 b_x)\partial_x v) + \partial_x^2(h^4\partial_x(b_x v)) - b_x\partial_x(h^4\partial_x^2 v)\big] \\ &+ \frac{15\beta^2}{4h^5}\big[2\partial_x(h^3 b_x^2\partial_x v) + \partial_x(h^3 b_x b_{xx} v) + 2b_x\partial_x(h^3 b_x\partial_x v) + b_x\partial_x(h^3 b_{xx} v)\big] \\ &+ f \in L^2(\mathbb{R}).\end{aligned}$$

Hence the proof is complete. □

The following lemma gives functional properties to the operator $\Im^{-1}$.

Lemma 3.2. *Let $\zeta \in H^{t_0+1}(\mathbb{R})$ and $b \in H^{t_0+3}(\mathbb{R})$ be such that (3.1) is satisfied. Then we have the following:*

(i) *For all $0 \le s \le t_0 + 1$,*

$$\begin{aligned}|\Im^{-1}f|_{H^s} + \sqrt{\mu}|\partial_x\Im^{-1}f|_{H^s} + \mu|\partial_x^2\Im^{-1}f|_{H^s} \\ \le C(h_{\min}^{-1}, |h-1|_{H^{t_0+1}}, |b|_{H^{t_0+3}})|f|_{H^s}.\end{aligned}$$

(ii) *For all $0 \le s \le t_0 + 1$,*

$$\begin{aligned}\sqrt{\mu}|\Im^{-1}\partial_x f|_{H^s} + \mu|\partial_x\Im^{-1}\partial_x f|_{H^s} + \mu\sqrt{\mu}|\partial_x^2\Im^{-1}\partial_x f|_{H^s} \\ \le C(h_{\min}^{-1}, |h-1|_{H^{t_0+1}}, |b|_{H^{t_0+3}})|f|_{H^s}, \\ \mu|\Im^{-1}\partial_x^2 f|_{H^s} + \mu\sqrt{\mu}|\partial_x\Im^{-1}\partial_x^2 f|_{H^s} + \mu^2|\partial_x^2\Im^{-1}\partial_x^2 f|_{H^s} \\ \le C(h_{\min}^{-1}, |h-1|_{H^{t_0+1}}, |b|_{H^{t_0+3}})|f|_{H^s}.\end{aligned}$$

(iii) *For all* $s \geq t_0 + 1$,

$$\|\Im^{-1}\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} + \sqrt{\mu}\|\Im^{-1}\partial_x\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} + \mu\|\Im^{-1}\partial_x^2\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} \leq C_s,$$
$$\mu\|\partial_x\Im^{-1}\partial_x\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} + \mu\sqrt{\mu}\|\partial_x^2\Im^{-1}\partial_x\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} \leq C_s,$$
$$\mu\sqrt{\mu}\|\partial_x\Im^{-1}\partial_x^2\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} + \mu^2\|\partial_x^2\Im^{-1}\partial_x^2 f\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} \leq C_s,$$

where C_s *is a constant depending on* $h_{\min}^{-1}$, $|h-1|_{H^s}$, $|b|_{H^{s+3}}$ *and independent of* $(\varepsilon,\mu) \in (0,1)^2$.

Proof. The proof is a generalization of the proof when $\beta = 0$ of operator $\Im$ in [20]. Note that here it is more convenient to use the second formulation (3.2) of the operator $\Im$.

Assume that $f \in H^s(\mathbb{R})$ and $u = \Im^{-1}f$; then $\Im u = f$. Apply Λ^s to both sides, then multiply by $\Lambda^s u$, which yields the following equality (note that $\Im\Lambda^s u = \Lambda^s f - [\Lambda^s, \Im]u$):

$$a(\Lambda^s u, \Lambda^s u) = (\tilde{f}, \Lambda^s u) + \sqrt{\mu}(\partial_x\tilde{g}, \Lambda^s u) + \mu(\partial_x^2\tilde{p}, \Lambda^s u),$$

such that $\tilde{f}$, $\tilde{g}$, and $\tilde{p}$ read

$$\begin{aligned}
\tilde{f} &= \Lambda^s f - [\Lambda^s, h]u + \frac{\mu\beta}{2}[\Lambda^s, h^2 b_x]u_x - \mu\beta^2[\Lambda^s, hb_x^2]u + \frac{\mu^2\beta}{24}[\Lambda^s, h^4 b_{xx}]u_{xx} \\
&\quad - \frac{\mu^2\beta^2}{6}[\Lambda^s, h^3 b_x^2]u_x - \frac{\mu^2\beta^2}{12}[\Lambda^s, h^3 b_x b_{xx}]u, \\
\tilde{g} &= \frac{\sqrt{\mu}}{3}[\Lambda^s, h^3]u_x - \frac{\sqrt{\mu}\beta}{2}[\Lambda^s, h^2 b_x]u - \frac{\mu\sqrt{\mu}\beta}{12}[\Lambda^s, h^4 b_x]u_{xx} \\
&\quad + \frac{\mu\sqrt{\mu}\beta^2}{3}[\Lambda^s, h^3 b_x^2]u_x + \frac{\mu\sqrt{\mu}\beta^2}{6}[\Lambda^s, h^3 b_x b_{xx}]u, \\
\tilde{p} &= -\frac{\mu}{45}[\Lambda^s, h^5]u_{xx} + \frac{\mu\beta}{12}[\Lambda^s h^4 b_x]u_x + \frac{\mu\beta}{24}[\Lambda^s, h^4 b_{xx}]u.
\end{aligned}$$

Integrating by parts and using (3.3), we get

$$\frac{h_{\min}}{\mathfrak{M}_{h_{\min}}}|\Lambda^s u|_\mu \leq |\tilde{f}|_2 + |\tilde{g}|_2 + |\tilde{p}|_2.$$

Now, using the necessary Kato–Ponce commutator estimates below (see [2, Lem. 4.6]),

$$|[\Lambda^s, f]u|_2 \lesssim |\nabla f|_{H^{t_0}}|u|_{H^{s-1}} \quad \text{if } 0 \leq s \leq t_0 + 1, \tag{3.6}$$
$$|[\Lambda^s, f]u|_2 \lesssim |\nabla f|_{H^{s-1}}|u|_{H^{s-1}} \quad \text{if } s \geq t_0 + 1, \tag{3.7}$$

it holds that

$$|\tilde{f}|_2 + |\tilde{g}|_2 + |\tilde{p}|_2 \leq |f|_{H^s} + C(|h-1|_{H^{t_0+1}})|\Lambda^{s-1}u|_\mu.$$

Hence, the inequality (i) holds after a continuous induction on s. For the proof of (ii), one has to replace $u = \sqrt{\mu}\Im^{-1}\partial_x f$ and $u = \mu\Im^{-1}\partial_x^2 f$ for a second time. The general strategy is the same as in (i), noticing that Λ^s commutes with ∂_x, ∂_x^2. The only difference is in the expression for $\tilde{f}$, $\tilde{g}$, $\tilde{p}$ when setting $u = \sqrt{\mu}\Im^{-1}\partial_x f$ and similarly when $u = \mu\Im^{-1}\partial_x^2 f$. The rest of the proof is as in [20]. □

§4. The linearized system

In order to rewrite the extended Green–Naghdi system for an uneven bottom with surface tension in a condensed form, we introduce a new symmetric operator $\mathfrak{J}_{\mathrm{bo}}$:

$$\mathfrak{J}_{\mathrm{bo}} = 1 - \frac{\mu}{\mathrm{bo}}\partial_x^2(\cdot) + \frac{2}{45}\mu^2\partial_x^2(h^4\partial_x^2\cdot), \quad \text{with } h(t,x) = 1 + \varepsilon\zeta(t,x) - \beta b(x).$$

The first equation in (2.11) can be written as

$$\partial_t\zeta + \varepsilon v\partial_x\zeta + h\partial_x v - \beta v\partial_x b = 0.$$

For the second equation in (2.11), applying $\Im^{-1}$ to both sides we get

$$\begin{aligned}
&\partial_t v + \varepsilon v v_x + \Im^{-1}(h\mathfrak{J}_{\mathrm{bo}}\zeta_x) + \mu^2\Im^{-1}(\mathcal{I}[h,\beta b]\zeta_x) - \varepsilon^2\mu^2\frac{1}{\mathrm{bo}}\Im^{-1}(T[U]\zeta_x)\\
&\quad + \varepsilon\mu\Im^{-1}(\mathcal{Q}_1[U]v_x) + \varepsilon\mu\beta\Im^{-1}(\mathcal{B}_1[U]v_x) + \varepsilon\mu^2\Im^{-1}(\mathcal{Q}_2[U]v_x)\\
&\quad + \varepsilon\mu^2\Im^{-1}(\mathcal{Q}_3[U]v_x) + \varepsilon\mu^2\beta\Im^{-1}(\mathcal{B}_2[U]v_x)\\
&\quad + \varepsilon\mu^2\beta^2\Im^{-1}(\mathcal{B}_3[U]v_x) + \Im^{-1}(\mathcal{R}[\mu,\varepsilon h,\beta b](U)) = O(\mu^3).
\end{aligned}$$

Hence, the extended Green–Naghdi system ($\beta \neq 0$) with surface tension can be written in the form

$$\partial_t U + A[U]\partial_x U + B(U) = 0,$$

where $U = (\zeta, v)^\top$ and

$$A[U] = \begin{pmatrix} \varepsilon v & h \\ \begin{array}{c}\Im^{-1}(h\mathfrak{J}_{\mathrm{bo}}\cdot) + \mu^2\Im^{-1}(\mathcal{I}[h,\beta b]\cdot)\\ -\varepsilon^2\mu^2\frac{1}{\mathrm{bo}}\Im^{-1}(T[U]\cdot)\end{array} & \varepsilon v + \varepsilon\mu Q_b[U]\cdot + \varepsilon\mu^2 Q_{bb}[U]\cdot \end{pmatrix},$$

with

$$\begin{aligned}
Q_b[U]\cdot &= \Im^{-1}(\mathcal{Q}_1[U]\cdot) + \Im^{-1}(\beta\mathcal{B}_1[U]\cdot),\\
Q_{bb}[U]\cdot &= \Im^{-1}(\mathcal{Q}_2[U]\cdot + \mathcal{Q}_3[U]\cdot + \beta\mathcal{B}_2[U]\cdot + \beta^2\mathcal{B}_3[U]\cdot),
\end{aligned}$$

and

$$B(U) = \begin{pmatrix} -\beta b_x v \\ \Im^{-1}(\mathcal{R}[\mu,\varepsilon h,\beta b](U)) \end{pmatrix}.$$

We consider now the linearized system around some reference state $\underline{U} = (\underline{\zeta}, \underline{v})^{\top}$:

$$\begin{cases} \partial_t U + A[\underline{U}]\partial_x U + B(\underline{U}) = 0, \\ U_{|t=0} = U_0. \end{cases} \tag{4.1}$$

The proof of the energy estimate which permits the convergence of an iterative scheme to construct a solution to the extended system (2.11) for the initial value problem (4.1) requires us to define the X^s spaces, which are the energy spaces for this problem.

Definition 4.1. For all $s \geq 0$ and $T > 0$, we denote by X^s the vector space $H^{s+2}(\mathbb{R}) \times H^{s+2}(\mathbb{R})$ endowed with the norm

$$|U|_{X^s}^2 := |\zeta|_{H^s}^2 + \frac{\mu}{\mathrm{bo}}|\zeta_x|_{H^s}^2 + \mu^2|\zeta_{xx}|_{H^s}^2 + |v|_{H^s}^2 + \mu|v_x|_{H^s}^2 + \mu^2|v_{xx}|_{H^s}^2,$$

while $\frac{\mu}{\mathrm{bo}} = O(\mu)$ and X_T^s stands for $C([0, \frac{T}{\varepsilon\vee\beta}]; X^s)$ endowed with its canonical norm.

First, recall that a pseudo-symmetrizer for $A[\underline{U}]$ is given by

$$\begin{aligned} S &= \begin{pmatrix} \underline{\mathfrak{I}}_{\mathrm{bo}} & 0 \\ 0 & \underline{\mathfrak{I}} \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{\mu}{\mathrm{bo}}\partial_x^2(\cdot) + \frac{2}{45}\mu^2\partial_x^2(\underline{h}^4\partial_x^2\cdot) & 0 \\ 0 & \underline{h} + \mu\mathcal{T}[\underline{h}, \beta b] - \mu^2\mathfrak{T}[\underline{h}, \beta b] \end{pmatrix}, \end{aligned} \tag{4.2}$$

where $\underline{h} = 1 + \varepsilon\underline{\zeta} - \beta b$. A natural energy for the initial value problem (4.1) is suggested to be

$$E^s(U)^2 = (\Lambda^s U, S\Lambda^s U).$$

The connection between $E^s(U)$ and the X^s-norm is examined using the lemma below.

Lemma 4.2. *Let $s \geq 0$, $b \in C_b^\infty(\mathbb{R})$, and $\underline{\zeta} \in L^\infty(\mathbb{R})$. Under the non-zero-depth condition*

$$\text{there exists } h_{\min} > 0, \ \inf_{x\in\mathbb{R}} h \geq h_{\min}, \quad h(t,x) = 1 + \varepsilon\zeta(t,x) - \beta b(x), \tag{4.3}$$

$E^s(U)$ is uniformly equivalent to the $|\cdot|_{X^s}$-norm with respect to $(\mu, \varepsilon, \mathrm{bo}^{-1}) \in (0,1)^3$:

$$E^s(U) \leq C(|\underline{\zeta}|_\infty)|U|_{X^s} \quad \textit{and} \quad |U|_{X^s} \leq C(h_{\min}^{-1})E^s(U).$$

Proof. First note that $E^s(U)^2 = (\Lambda^s U, S\Lambda^s U)$ with $S\Lambda^s U = (\underline{\mathfrak{J}}_{\mathrm{bo}}\Lambda^s\zeta, \underline{\mathfrak{S}}\Lambda^s v)$. Then we get

$$E^s(U)^2 = (\Lambda^s\zeta, \underline{\mathfrak{J}}_{\mathrm{bo}}\Lambda^s\zeta) + (\Lambda^s v, \underline{\mathfrak{S}}\Lambda^s v).$$

Using the expressions for $\underline{\mathfrak{S}}$, $\underline{\mathfrak{J}}_{\mathrm{bo}}$, and the proof of Lemma 3.1, by integrating by parts it holds that

$$E^s(U)^2 = (\Lambda^s\zeta, \Lambda^s\zeta) + \frac{\mu}{\mathrm{bo}}(\Lambda^s\zeta_x, \Lambda^s\zeta_x) + \frac{2}{45}\mu^2(\underline{h}^4\Lambda^s\zeta_{xx}, \Lambda^s\zeta_{xx}) + \underline{a}(\Lambda^s v, \Lambda^s v),$$

where

$$\begin{aligned}\underline{a}(v,u) = (hv,u) &+ \mu\Big(\underline{h}\Big(\frac{\sqrt{3}}{3}\underline{h}v_x - \frac{\sqrt{3}}{2}\beta b_x v\Big), \frac{\sqrt{3}}{3}\underline{h}u_x - \frac{\sqrt{3}}{2}\beta b_x u\Big)\\ &+ \frac{\mu\beta^2}{4}(\underline{h}b_x v, b_x u)\\ &+ \mu^2\Big(\underline{h}\Big(\frac{\sqrt{5}}{15}\underline{h}^2 v_{xx} - \frac{\sqrt{5}}{4}\beta\underline{h}b_x v_x - \frac{\sqrt{5}}{8}\beta\underline{h}b_{xx}v\Big),\\ &\qquad \frac{\sqrt{5}}{15}\underline{h}^2 u_{xx} - \frac{\sqrt{5}}{4}\beta\underline{h}b_x u_x - \frac{\sqrt{5}}{8}\beta\underline{h}b_{xx}u\Big)\\ &+ \mu^2\beta^2\Big(\underline{h}\Big(\frac{\sqrt{3}}{12}\underline{h}b_x v_x + \frac{\sqrt{3}}{24}\underline{h}b_{xx}v\Big), \frac{\sqrt{3}}{12}\underline{h}b_x u_x + \frac{\sqrt{3}}{24}\underline{h}b_{xx}u\Big).\end{aligned}$$

From the assumption of the lemma we know that $b \in C_b^\infty(\mathbb{R})$, $\underline{\zeta} \in L^\infty(\mathbb{R})$, and that the water depth is constantly limited (4.3). Then by the Cauchy–Schwarz inequality and with the help of the proof of Lemma 3.1, we get the two inequalities of the desired lemma. □

A derivation of the prior energy estimate is given in the proposition below.

Proposition 4.3. *Let $s > 3/2$, $\mathrm{bo}^{-1} \in [0,1)$, and $b(x) \in H^{s+3}(\mathbb{R})$. Also let $\underline{U} = (\underline{\zeta}, \underline{v})^\top \in X_T^s$ be such that $\partial_t\underline{U} \in X_T^{s-1}$ and satisfy condition (3.1) on $[0, \frac{T}{\varepsilon\vee\beta}]$. Then, for all $U_0 \in X^s$, there exists a unique solution $U = (\zeta, v)^\top \in X_T^s$ to (4.1) and for all $0 \le t \le \frac{T}{\varepsilon\vee\beta}$ satisfying*

$$\begin{aligned}E^s(U(t)) \le{}& (e^{(\varepsilon\vee\beta)\lambda_T t})^{1/2}E^s(U_0)\\ &+ (\varepsilon\vee\beta)\int_0^t (e^{(\varepsilon\vee\beta)\lambda_T(t-t')})^{1/2}C(E^s(\underline{U})(t'), \mathrm{bo}^{-1})\,dt',\end{aligned}\tag{4.4}$$

for some $\lambda_T = \lambda_T(\sup_{0\le t\le T/\varepsilon\vee\beta} E^s(\underline{U}(t)), \sup_{0\le t\le T/\varepsilon\vee\beta} |\partial_t\underline{h}(t)|_{L^\infty}, \mathrm{bo}^{-1})$, where $\varepsilon\vee\beta = \max\{\varepsilon,\beta\}$.

Remark 4.4. In the following proof and for the sake of simplicity, we will not attempt to show the dependence on the bottom parametrization $b \in H^{s+3}(\mathbb{R})$ in all of the verifications. For an explanation of why $s+3$ regularization is required, consider the control of term $D_3 + D_4$.

Proof of Proposition 4.3. The existence and uniqueness of the solution is a direct adaptation of the proof in [16, App. A] (one may see also [32, 1] for general details). Our attention targets mainly the demonstration of the energy estimate. Consider any $\lambda \in \mathbb{R}$; the key point is to bound from above, in terms of $E^s(U)$, the component below:

$$e^{(\varepsilon\vee\beta)\lambda t}\partial_t(e^{-(\varepsilon\vee\beta)\lambda t}E^s(U)^2) = -(\varepsilon\vee\beta)\lambda E^s(U)^2 + \partial_t(E^s(U)^2).$$

Using the fact that $\underline{\mathfrak{I}}$ and $\underline{\mathfrak{J}}_{\mathrm{bo}}$ are symmetric, in addition to the identities

$$\partial_t(\underline{\mathfrak{I}}\Lambda^s v) = [\partial_t, \underline{\mathfrak{I}}]\Lambda^s v + \underline{\mathfrak{I}}\Lambda^s v_t, \quad \partial_t(\underline{\mathfrak{J}}_{\mathrm{bo}}\Lambda^s\zeta) = [\partial_t, \underline{\mathfrak{J}}_{\mathrm{bo}}]\Lambda^s\zeta + \underline{\mathfrak{J}}_{\mathrm{bo}}\Lambda^s\zeta_t,$$

one gets after using (4.1) that

$$\begin{aligned}\partial_t(E^s(U)^2) = &-2(S\Lambda^s U, [\Lambda^s, A[\underline{U}]]\partial_x U) - 2(S\Lambda^s U, A[\underline{U}]\Lambda^s\partial_x U)\\ &- 2(\Lambda^s B(\underline{U}), S\Lambda^s U) + (\Lambda^s\zeta, [\partial_t, \underline{\mathfrak{J}}_{\mathrm{bo}}]\Lambda^s\zeta) + (\Lambda^s v, [\partial_t, \underline{\mathfrak{I}}]\Lambda^s v).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\frac{1}{2}e^{(\varepsilon\vee\beta)\lambda t}\partial_t(e^{-(\varepsilon\vee\beta)\lambda t}E^s(U)^2) = &-\frac{(\varepsilon\vee\beta)\lambda}{2}E^s(U)^2 - (SA[\underline{U}]\Lambda^s\partial_x U, \Lambda^s U)\\ &- ([\Lambda^s, A[\underline{U}]]\partial_x U, S\Lambda^s U) - (\Lambda^s B(\underline{U}), S\Lambda^s U)\\ &+ \frac{1}{2}(\Lambda^s\zeta, [\partial_t, \underline{\mathfrak{J}}_{\mathrm{bo}}]\Lambda^s\zeta) + \frac{1}{2}(\Lambda^s v, [\partial_t, \underline{\mathfrak{I}}]\Lambda^s v).\end{aligned} \tag{4.5}$$

We will focus now on bounding from above the purely topographical components of the right-hand side of (4.5), knowing that the non-topographical expressions have been controlled in [20]. Note that by Parseval's identity, the Cauchy–Schwarz inequality, and then Young's inequality, we will use the inequality

$$\mu|\zeta_x|^2_{H^s} \le \frac{1}{2}|\zeta|^2_{H^s} + \frac{1}{2}\mu^2|\zeta_{xx}|^2_{H^s}. \tag{4.6}$$

Estimation of $(SA[\underline{U}]\Lambda^s\partial_x U, \Lambda^s U)$. Put

$$R[\underline{U}]\cdot = \varepsilon\mu\mathcal{Q}_1[\underline{U}]\cdot + \varepsilon\mu^2\mathcal{Q}_2[\underline{U}]\cdot + \varepsilon\mu^2\mathcal{Q}_3[\underline{U}]\cdot.$$

By definition we have

$$SA[\underline{U}] = \begin{pmatrix} \varepsilon\underline{\mathfrak{J}}_{\mathrm{bo}}(\underline{v}\cdot) & \underline{\mathfrak{J}}_{\mathrm{bo}}(\underline{h}\cdot) \\ \begin{array}{c}\underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}\cdot + \mu^2\mathcal{I}[\underline{h}, \beta b]\cdot \\ -\varepsilon^2\mu^2\frac{1}{\mathrm{bo}}T[\underline{U}]\cdot\end{array} & \begin{array}{c}\varepsilon\underline{\mathfrak{I}}(\underline{v}\cdot) + R[\underline{U}]\cdot + \varepsilon\mu\beta\mathcal{B}_1[\underline{U}]\cdot \\ +\varepsilon\mu^2\beta\mathcal{B}_2[\underline{U}]\cdot + \varepsilon\mu^2\beta^2\mathcal{B}_3[\underline{U}]\cdot\end{array}\end{pmatrix}.$$

Then it holds that

$$\begin{aligned}(SA[\underline{U}]\Lambda^s\partial_x U,\Lambda^s U) &= \varepsilon(\mathfrak{J}_{\mathrm{bo}}(\underline{v}\Lambda^s\zeta_x),\Lambda^s\zeta) + (\mathfrak{J}_{\mathrm{bo}}(\underline{h}\Lambda^s v_x),\Lambda^s\zeta)\\ &\quad + (\underline{h}\mathfrak{J}_{\mathrm{bo}}\Lambda^s\zeta_x,\Lambda^s v) + \mu^2(\mathcal{I}[\underline{h},\beta b]\Lambda^s\zeta_x,\Lambda^s v)\\ &\quad - \varepsilon^2\mu^2\frac{1}{\mathrm{bo}}(T[\underline{U}]\Lambda^s\zeta_x,\Lambda^s v) + \varepsilon(\mathfrak{S}(\underline{v}\Lambda^s v_x),\Lambda^s v)\\ &\quad + (R[\underline{U}]\Lambda^s v_x,\Lambda^s v) + \varepsilon\mu\beta(\mathcal{B}_1[\underline{U}]\Lambda^s v_x,\Lambda^s v)\\ &\quad + \varepsilon\mu^2\beta(\mathcal{B}_2[\underline{U}]\Lambda^s v_x,\Lambda^s v) + \varepsilon\mu^2\beta^2(\mathcal{B}_3[\underline{U}]\Lambda^s v_x,\Lambda^s v)\\ &= A_1 + A_2 + \cdots + A_{10}.\end{aligned}$$

From [20] and with inequality (4.6) in hand, one may deduce that the non-topographical terms A_1, A_2, A_3, A_5, A_7 are controlled as

$$|A_1 + A_2 + A_3 + A_5 + A_7| \lesssim (\varepsilon\vee\beta)C(E^s(\underline{U}),\mathrm{bo}^{-1})E^s(U)^2.$$

To control A_4, an integration by parts yields

$$\begin{aligned}A_4 &= \frac{4}{45}\mu^2(\underline{h}_x\partial_x(\underline{h}^4\Lambda^s\zeta_{xxx}),\Lambda^s v) + \frac{2}{45}\mu^2(\underline{h}^4\underline{h}_{xx}\Lambda^s\zeta_{xxx},\Lambda^s v)\\ &\quad + \frac{1}{12}\mu^2\beta(\partial_x(\underline{h}^4 b_x)\Lambda^s\zeta_{xx},\Lambda^s v_x) - \frac{1}{12}\mu^2\beta(\underline{h}^4 b_{xx}\Lambda^s\zeta_x,\Lambda^s v_{xx})\\ &\quad - \frac{1}{12}\mu^2\beta(\underline{h}^4 b_x\Lambda^s\zeta_{xx},\Lambda^s v_{xx}) - \frac{1}{12}\mu^2\beta(\underline{h}^4\Lambda^s\zeta_{xxx},\partial_x(b_x\Lambda^s v))\\ &\quad + \frac{1}{3}\mu^2\beta^2(\underline{h}^3 b_x^2\Lambda^s\zeta_{xx},\Lambda^s v_x) + \frac{1}{6}\mu^2\beta^2(\underline{h}^3 b_x b_{xx}\Lambda^s\zeta_x,\Lambda^s v_x)\\ &\quad + \frac{1}{3}\mu^2\beta^2(\underline{h}^3 b_x\Lambda^s\zeta_{xx},\partial_x(b_x\Lambda^s v_x)) + \frac{1}{6}\mu^2\beta^2(\underline{h}^3 b_{xx}\Lambda^s\zeta_x,\partial_x(b_x\Lambda^s v_x))\\ &= A_{41} + A_{42} + \cdots + A_{4(10)}.\end{aligned}$$

To control A_{41} and A_{42}, by integration by parts and (4.6) it holds that

$$|A_{41} + A_{42}| \le (\varepsilon\vee\beta)C(|\underline{\zeta}|_{W^{1,\infty}},|\underline{\zeta}|_{H^s},\mu|\underline{\zeta}_{xx}|_{H^s})E^s(U)^2.$$

Again, using integration by parts one can write

$$\begin{aligned}|A_{46}| &= \frac{1}{12}\mu^2\beta\left|(\Lambda^s\zeta_{xx},\partial_x(\underline{h}^4 b_{xx}\Lambda^s v)) + \frac{1}{12}\mu^2\beta(\Lambda^s\zeta_{xx},\partial_x(\underline{h}^4 b_x\Lambda^s v_x))\right|\\ &\le \beta C(|\underline{\zeta}|_{W^{1,\infty}})E^s(U)^2.\end{aligned}$$

The rest of the components are controlled similarly, so it holds that

$$|A_4| \le (\varepsilon\vee\beta)C(|\underline{\zeta}|_{W^{1,\infty}},|\underline{\zeta}|_{H^s},\mu|\underline{\zeta}_{xx}|_{H^s})E^s(U)^2.$$

To control A_6 one should notice that the non-topographical terms are bounded from above in [20] by

$$(\varepsilon \vee \beta) C(|\underline{\zeta}|_{W^{1,\infty}}, |\underline{v}_x|_\infty, \sqrt{\mu}|\underline{v}_{xx}|_\infty) E^s(U)^2,$$

while the purely topographical terms can be written as

$$\begin{aligned}
&-\frac{1}{2}\varepsilon\mu\beta(\underline{h}^2 b_x \underline{v}\Lambda^s v_x, \Lambda^s v_x) - \frac{1}{2}\varepsilon\mu\beta(\underline{h}^2 b_x \partial_x(\underline{v}\Lambda^s v_x), \Lambda^s v) \\
&\qquad + \varepsilon\mu\beta^2(\underline{h} b_x^2 \underline{v}\Lambda^s v_x, \Lambda^s v) + \frac{1}{24}\varepsilon\mu^2\beta((\underline{h}^4 b_x)_x \partial_x(\underline{v}\Lambda^s v_x), \Lambda^s v_x) \\
&\qquad - \frac{1}{24}\varepsilon\mu^2\beta(\underline{h}^4 \partial_x(b_x \underline{v}\Lambda^s v_x), \Lambda^s v_{xx}) - \frac{1}{24}\varepsilon\mu^2\beta(\underline{h}^4 \partial_x^2(\underline{v}\Lambda^s v_x), \partial_x(b_x \Lambda^s v)) \\
&\qquad + \frac{1}{6}\varepsilon\mu^2\beta^2(\underline{h}^3 b_x^2 \partial_x(\underline{v}\Lambda^s v_x), \Lambda^s v_x) + \frac{1}{12}\varepsilon\mu^2\beta^2(\underline{h}^3 b_x b_{xx} \underline{v}\Lambda^s v_x, \Lambda^s v_x) \\
&\qquad + \frac{1}{6}\varepsilon\mu^2\beta^2(\underline{h}^3 b_x \partial_x(\underline{v}\Lambda^s v_x), \partial_x(b_x\Lambda^s v)) + \frac{1}{6}\varepsilon\mu^2\beta^2(\underline{h}^3 b_{xx} \underline{v}\Lambda^s v_x, \partial_x(b_x \Lambda^s v)) \\
&\quad = A_{61} + A_{62} + \cdots + A_{6(10)}.
\end{aligned}$$

Again, by integration by parts one has

$$\begin{aligned}
|A_{66}| &= \frac{1}{24}\varepsilon\mu^2\beta\big|(\partial_x(\underline{v}\Lambda^s v_x), \partial_x(\underline{h}^4 b_{xx}\Lambda^s v)) + (\partial_x(\underline{v}\Lambda^s v_x), \partial_x(\underline{h}^4 b_x \Lambda^s v_x))\big| \\
&\le \varepsilon\beta C(|\underline{\zeta}|_{W^{1,\infty}}, |\underline{v}|_{W^{1,\infty}}) E^s(U)^2.
\end{aligned}$$

Hence, we get

$$|A_6| \le (\varepsilon \vee \beta) C(|\underline{\zeta}_{xx}|_{W^{1,\infty}}, |\underline{v}|_{W^{1,\infty}}, \sqrt{\mu}|\underline{v}_{xx}|_\infty) E^s(U)^2.$$

Similarly, using the expressions for $\mathcal{B}_1[U]$, $\mathcal{B}_2[U]$, $\mathcal{B}_3[U]$ and integrations by parts one has

$$\begin{aligned}
|A_8| &\le \varepsilon\beta C(|\underline{\zeta}|_\infty, |\underline{v}_x|_\infty) E^s(U)^2, \\
|A_9| &\le \varepsilon\beta C(|\underline{\zeta}|_{W^{1,\infty}}, |\underline{v}|_{W^{1,\infty}}, \sqrt{\mu}|\underline{v}_{xx}|_\infty) E^s(U)^2, \\
|A_{10}| &\le \varepsilon\beta C(|\underline{\zeta}|_\infty, |\underline{v}|_{W^{1,\infty}}) E^s(U)^2.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&|(SA[\underline{U}]\Lambda^s \partial_x U, \Lambda^s U)| \\
&\qquad \le (\varepsilon \vee \beta) C\big(|\zeta|_{W^{1,\infty}}, |\underline{\zeta}|_{H^s}, \mu|\underline{\zeta}_{xx}|_{H^s}, |\underline{v}|_{W^{1,\infty}}, \sqrt{\mu}|\underline{v}_{xx}|_\infty, \mathrm{bo}^{-1}\big) E^s(U)^2.
\end{aligned}$$

Estimation of $([\Lambda^s, A[\underline{U}]]\partial_x U, S\Lambda^s U)$. First of all, we have

$$\begin{aligned}
&([\Lambda^s, A[\underline{U}]]\partial_x U, S\Lambda^s U) \\
&\qquad = \varepsilon([\Lambda^s, \underline{v}]\zeta_x, \mathfrak{J}_{\mathrm{bo}}\Lambda^s\zeta) + \varepsilon([\Lambda^s, \underline{h}]v_x, \mathfrak{J}_{\mathrm{bo}}\Lambda^s\zeta) \\
&\qquad\quad + ([\Lambda^s, \underline{\mathfrak{F}}^{-1}(\underline{h}\mathfrak{J}_{\mathrm{bo}}\cdot)]\zeta_x, \underline{\mathfrak{F}}\Lambda^s v) + \mu^2([\Lambda^s, \underline{\mathfrak{F}}^{-1}(\mathcal{I}[\underline{h}, \beta b]\cdot)]\zeta_x, \underline{\mathfrak{F}}\Lambda^s v)
\end{aligned}$$

$$
\begin{aligned}
&- \varepsilon^2\mu^2 \frac{1}{\mathrm{bo}}([\Lambda^s, \underline{\mathfrak{T}}^{-1}(T[\underline{U}]\cdot)]\zeta_x, \underline{\mathfrak{T}}\Lambda^s v) + \varepsilon([\Lambda^s, \underline{v}]v_x, \underline{\mathfrak{T}}\Lambda^s v)\\
&+ \varepsilon\mu([\Lambda^s, \underline{\mathfrak{T}}^{-1}(Q_b[\underline{U}]\cdot)]v_x, \underline{\mathfrak{T}}\Lambda^s v) + \varepsilon\mu^2([\Lambda^s, \underline{\mathfrak{T}}^{-1}(Q_{bb}[\underline{U}]\cdot)]v_x, \underline{\mathfrak{T}}\Lambda^s v)\\
= {} & D_1 + D_2 + \cdots + D_8.
\end{aligned}
$$

To control D_1, D_2 we use the expression for $\underline{\mathfrak{J}}_{\mathrm{bo}}$, the commutator estimate (3.7), and the fact that

$$
\partial_x^2[\Lambda^s, M]N = [\Lambda^s, M_{xx}]N + 2[\Lambda^s, M_x]N_x + [\Lambda^s, M]N_{xx}. \tag{4.7}
$$

Then, with (4.6) in hand, it holds that

$$
|D_1+D_2| \le (\varepsilon\vee\beta)C\big(|\underline{\zeta}|_\infty, |\underline{h}-1|_{H^s}, |\underline{\zeta}|_{H^s}, \mu|\underline{\zeta}_{xx}|_{H^s}, |\underline{v}|_{H^s}, \mu|\underline{v}_{xx}|_{H^s}, \mathrm{bo}^{-1}\big)E^s(U)^2.
$$

To control $D_3 + D_4$, remark that $\underline{\mathfrak{T}}$ is symmetric to

$$
\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}]\underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}\zeta_x = \underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}(\underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}\cdot)]\zeta_x - [\Lambda^s, \underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}]\zeta_x.
$$

Moreover, since $[\Lambda^s, \underline{\mathfrak{T}}^{-1}] = -\underline{\mathfrak{T}}^{-1}[\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}$, one gets

$$
\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}(\underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}\cdot)]\zeta_x = -[\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}\underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}\zeta_x + [\Lambda^s, \underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}]\zeta_x.
$$

By using the explicit expression for $\underline{\mathfrak{T}}$, integration by parts, and the facts

$$
[\Lambda^s, \partial_x(M\partial_x\cdot)]N = \partial_x[\Lambda^s, M]N_x, \quad [\Lambda^s, \partial_x^2(M\partial_x^2\cdot)]N = \partial_x^2[\Lambda^s, M]N_{xx}, \tag{4.8}
$$

one can write

$$
\begin{aligned}
D_3 + D_4 = {} & ([\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}(\underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}\zeta_x), \Lambda^s v) + ([\Lambda^s, \underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}]\zeta_x, \Lambda^s v)\\
& + \mu^2([\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}(\mathcal{I}[\underline{h}, \beta b]\zeta_x), \Lambda^s v) + \mu^2([\Lambda^s, \mathcal{I}[\underline{h}, \beta b]]\zeta_x, \Lambda^s v).
\end{aligned}
$$

Now, using the expression for $\underline{\mathfrak{J}}_{\mathrm{bo}}$ one has

$$
\begin{aligned}
\frac{2}{45}\mu^2\underline{h}\partial_x^2(\underline{h}^4\zeta_{xxx}) = {} & 2\underline{\mathfrak{T}}\zeta_x - 2\underline{h}\zeta_x + \frac{\mu}{3}\partial_x(\underline{h}^3\zeta_{xx}) - \mu\beta[\partial_x(\underline{h}^2 b_x\zeta_x) - \underline{h}^2 b_x\zeta_{xx}]\\
& - 2\mu\beta^2\underline{h}b_x^2\zeta_x - \mu^2\mathcal{I}[\underline{h}, \beta b]\zeta_x.
\end{aligned}
$$

Then it holds that

$$
\begin{aligned}
&\underline{\mathfrak{T}}^{-1}(\underline{h}\underline{\mathfrak{J}}_{\mathrm{bo}}\zeta_x) + \mu^2\underline{\mathfrak{T}}^{-1}(\mathcal{I}[\underline{h}, \beta b]\zeta_x)\\
&\quad = 2\zeta_x - \underline{\mathfrak{T}}^{-1}(\underline{h}\zeta_x) - \frac{\mu}{\mathrm{bo}}\underline{\mathfrak{T}}^{-1}(\underline{h}\zeta_{xxx}) + \frac{2\mu}{3}\underline{\mathfrak{T}}^{-1}\partial_x(\underline{h}^3\zeta_{xx})\\
&\quad\quad - \mu\beta\underline{\mathfrak{T}}^{-1}\big(\partial_x(\underline{h}^2 b_x\zeta_x) - \underline{h}^2 b_x\zeta_{xx} + 2\beta\underline{h}b_x^2\zeta_x\big).
\end{aligned}
$$

The above equality then implies

$$\begin{aligned}
D_3 + D_4 &= 2([\Lambda^s, \underline{\mathfrak{T}}]\zeta_x, \Lambda^s v) - ([\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}(\underline{h}\zeta_x), \Lambda^s v) \\
&\quad + \frac{2\mu}{3}([\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}\partial_x(\underline{h}^3\zeta_{xx}), \Lambda^s v) - \frac{\mu}{\mathrm{bo}}([\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}(\underline{h}\zeta_{xxx}), \Lambda^s v) \\
&\quad + ([\Lambda^s, \underline{h}\mathfrak{J}_{\mathrm{bo}}]\zeta_x, \Lambda^s v) \\
&\quad - \mu\beta\big([\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}\big(\partial_x(\underline{h}^2 b_x\zeta_x) - \underline{h}^2 b_x\zeta_{xx} + 2\beta\underline{h}b_x^2\zeta_x\big), \Lambda^s v\big) \\
&\quad + \mu^2([\Lambda^s, \mathcal{I}[\underline{h}, \beta b]]\zeta_x, \Lambda^s v) \\
&= D_{341} + D_{342} + \cdots + D_{347}.
\end{aligned}$$

In order to facilitate our way of controlling $D_3 + D_4$, we will use the re-expression (3.2) of $\underline{\mathfrak{T}}$, which reads

$$\begin{aligned}
\underline{\mathfrak{T}} &= \underline{h} - \frac{1}{3}\mu\partial_x(\underline{h}^3\partial_x\cdot) + \frac{\mu\beta}{2}\partial_x(\underline{h}^2 b_x(\cdot)) - \frac{\mu\beta}{2}\underline{h}^2 b_x\partial_x\cdot \\
&\quad + \mu\beta^2\underline{h}b_x^2(\cdot) + \frac{1}{45}\mu^2\partial_x^2(\underline{h}^5\partial_x^2\cdot) \\
&\quad - \frac{\mu^2\beta}{24}\big[2\partial_x^2(\underline{h}^4 b_x\partial_x^2\cdot) + \partial_x^2(\underline{h}^4 b_{xx}(\cdot)) - 2\partial_x(\underline{h}^4 b_x\partial_x^2\cdot) + \underline{h}^4 b_{xx}\partial_x^2\cdot\big] \\
&\quad - \frac{\mu^2\beta^2}{12}\big[4\partial_x(\underline{h}^3 b_x^2\partial_x\cdot) + \partial_x(\underline{h}^3 b_x b_{xx}(\cdot)) - 2\underline{h}^3 b_x b_{xx}\partial_x\cdot - \underline{h}^3 b_{xx}^2(\cdot)\big].
\end{aligned}$$

Using this, the commutator estimate (3.7), identities (4.8), and Lemma 3.2, it holds that

$$\begin{aligned}
|D_{341}| &\le (\varepsilon \vee \beta)C(|\underline{h} - 1|_{H^s})E^s(U)^2, \\
|D_{342} + D_{343} + D_{344}| &\le (\varepsilon \vee \beta)C(|\underline{h} - 1|_{H^s}, C_s)E^s(U)^2
\end{aligned}$$

To control D_{345}, one should use the explicit expression for $\mathfrak{J}_{\mathrm{bo}}$ and the fact that

$$\begin{aligned}
[\Lambda^s, M\partial_x^{(\imath)}(N\partial_x^2\cdot)]P &= [\Lambda^s, M\partial_x^{(\imath)}]N\partial_x^2 P \\
&\quad + M\partial_x^{(\imath)}[\Lambda^s, N]\partial_x^2 P \quad \text{with } \imath = \{1, 2\},
\end{aligned} \tag{4.9}$$

to write D_{345} as

$$\begin{aligned}
D_{345} &= ([\Lambda^s, \underline{h}]\zeta_x, \Lambda^s v) - \frac{\mu}{\mathrm{bo}}([\Lambda^s, \underline{h}]\zeta_{xxx}, \Lambda^s v) \\
&\quad + \frac{2}{45}\mu^2([\Lambda^s, \underline{h}^5]\zeta_{xxx}, \partial_x^2(\underline{h}\Lambda^s v)) + \frac{2}{45}\mu^2([\Lambda^s, \underline{h}\partial_x^2]\underline{h}^5\zeta_{xxx}, \Lambda^s v) \\
&= D_{3451} + D_{3452} + D_{3453} + D_{3454},
\end{aligned}$$

such that

$$|D_{3451} + D_{3452} + D_{3453}| \le (\varepsilon \vee \beta)C(|\underline{h} - 1|_{H^s}, |\underline{\zeta}|_\infty, \mathrm{bo}^{-1})E^s(U)^2.$$

To control D_{3454}, one should note the commutator identity

$$[\Lambda^s, M\partial_x^2]N = \partial_x^2[\Lambda^s, M]N - 2\partial_x[\Lambda^s, \partial_x M]N + [\Lambda^s, \partial_x^2 M]N \\ = [\Lambda^s, M]\partial_x^2 N. \tag{4.10}$$

Then, with (4.6) in hand, it holds that

$$|D_{3454}| \leq (\varepsilon \vee \beta)C(|\underline{h} - 1|_{H^s}, |\underline{\zeta}|_{H^s}, \mu|\underline{\zeta}_{xxx}|_{H^{s-1}})E^s(U)^2.$$

Therefore, we obtain

$$|D_{345}| \leq (\varepsilon \vee \beta)C(|\underline{h} - 1|_{H^s}, |\underline{\zeta}|_\infty, |\underline{\zeta}|_{H^s}, \mu|\underline{\zeta}_{xxx}|_{H^{s-1}}, C_s, \mathrm{bo}^{-1})E^s(U)^2.$$

To control D_{346}, by integration by parts and using the explicit expression for $\underline{\Im}$, the commutator estimate (3.7), identities (4.8), and Lemma 3.2, we get

$$|D_{346}| \leq \beta C(|\underline{h} - 1|_{H^s}, C_s)E^s(U)^2.$$

To control D_{347}, we use integration by parts, the commutator (3.7) with the help of (4.9), (4.10), (4.6), and the commutator identity below,

$$[\Lambda^s, M\partial_x]N = \partial_x[\Lambda^s, M]N - [\Lambda^s, \partial_x M]N = [\Lambda^s, M]\partial_x N.$$

The non-topographical terms are bounded from above by

$$(\varepsilon \vee \beta)C(|\underline{h} - 1|_{H^s}, |\underline{\zeta}|_{H^s}, \mu|\underline{\zeta}_{xx}|_{H^s})E^s(U)^2.$$

On the other side, for the purely topographical terms in D_{347} we use

$$[\Lambda^s, \partial_x(M\cdot)]N = \partial_x[\Lambda^s, M]N \quad \text{and} \quad [\Lambda^s, \partial_x^2(M\cdot)]N = \partial_x^2[\Lambda^s, M]N, \tag{4.11}$$

and by integration by parts and (3.7) we write

$$\mu^2\beta - \frac{1}{6}([\Lambda^s, \underline{h}^4 b_x]\zeta_{xxx}, \Lambda^s v_{xx}) - \frac{1}{12}([\Lambda^s, \underline{h}^4 b_{xx}]\zeta_x, \Lambda^s v_{xx}) \\ - \frac{1}{6}([\Lambda^s, \underline{h}^4 b_x]\zeta_{xxx}, \Lambda^s v_x) - \frac{1}{12}([\Lambda^s, \underline{h}^4 b_{xx}]\zeta_{xxx}, \Lambda^s v) \\ - \frac{2}{3}\beta([\Lambda^s, \underline{h}^3 b_x^2]\zeta_{xx}, \Lambda^s v_x) + \frac{1}{6}\beta([\Lambda^s, \underline{h}^3 b_x b_{xx}]\zeta_x, \Lambda^s v_x) \\ + \frac{1}{3}\beta([\Lambda^s, \underline{h}^3 b_x b_{xx}]\zeta_{xx}, \Lambda^s v) + \frac{1}{6}\beta([\Lambda^s, \underline{h}^3 b_{xx}^2]\zeta_x, \Lambda^s v) \\ \leq (\varepsilon \vee \beta)C(|\underline{h} - 1|_{H^s})E^s(U)^2.$$

Thus, after collecting the information above, we obtain

$$|D_3 + D_4| \leq (\varepsilon \vee \beta)C(|\underline{h} - 1|_{H^s}, |\underline{\zeta}|_\infty, |\underline{\zeta}|_{H^s}, \mu|\underline{\zeta}_{xxx}|_{H^{s-1}}, C_s, \mathrm{bo}^{-1})E^s(U)^2.$$

To control D_5, let us note the commutator identities (4.11) and

$$\begin{aligned}[\Lambda^s, M\partial_x^{(\imath)}(N\cdot)]P &= [\Lambda^s, M\partial_x^{(\imath)}]NP + M\partial_x^{(\imath)}[\Lambda^s, N]P\\ &= [\Lambda^s, M]\partial_x^{(\imath)}(NP) + M\partial_x^{(\imath)}[\Lambda^s, N]P,\end{aligned}$$

with $\imath = \{1,2\}$. Now, as in D_3, using (4.8), by integration by parts, Lemma 3.2, and the Kato–Ponce commutator estimate (3.7), it holds that

$$|D_5| \le (\varepsilon \vee \beta)C(|\underline{h}-1|_{H^{s-1}}, |\underline{\zeta}_x|_{H^{s-1}}, |\underline{\zeta}|_{H^s}, \mu|\underline{\zeta}_{xx}|_{H^s}, C_s, \mathrm{bo}^{-1})E^s(U)^2.$$

To control D_6, one can write, after checking the expression for $\Im$ and using (4.7) with integration by parts and the fact that $\partial_x[\Lambda^s, M]N = [\Lambda^s, M_x]N + [\Lambda^s, M]N_x$,

$$\begin{aligned}|D_6| = \varepsilon\Big|&([\Lambda^s, \underline{v}]v_x, \underline{h}\Lambda^s v) + \frac{1}{3}\mu([\Lambda^s, \underline{v}_x]v_x, \underline{h}^3\Lambda^s v_x)\\
&+ \frac{1}{3}\mu([\Lambda^s, \underline{v}]v_{xx}, \underline{h}^3\Lambda^s v_x) - \frac{1}{2}\mu\beta([\Lambda^s, \underline{v}_x]v_x, \underline{h}^2 b_x\Lambda^s v)\\
&- \frac{1}{2}\mu\beta([\Lambda^s, \underline{v}]v_{xx}, \underline{h}^2 b_x\Lambda^s v) + \mu\beta^2([\Lambda^s, \underline{v}]v_x, \underline{h}b_x^2\Lambda^s v)\\
&+ \frac{1}{45}\mu^2([\Lambda^s, \underline{v}_{xx}]v_x, \underline{h}^5\Lambda^s v_{xx}) + \frac{2}{45}\mu^2([\Lambda^s, \underline{v}_x]v_{xx}, \underline{h}^5\Lambda^s v_{xx})\\
&+ \frac{1}{45}\mu^2([\Lambda^s, \underline{v}]v_{xxx}, \underline{h}^5\Lambda^s v_{xx}) - \frac{1}{12}\mu^2\beta([\Lambda^s, \underline{v}_{xx}]v_x, \underline{h}^4 b_x\Lambda^s v_{xx})\\
&- \frac{1}{12}\mu^2\beta([\Lambda^s, \underline{v}]v_{xxx}, \underline{h}^4 b_x\Lambda^s v_{xx}) - \frac{1}{6}\mu^2\beta([\Lambda^s, \underline{v}_x]v_{xx}, \underline{h}^4 b_x\Lambda^s v_{xx})\\
&- \frac{1}{24}\mu^2\beta([\Lambda^s, \underline{v}_{xx}]v_x, \underline{h}^4 b_{xx}\Lambda^s v) - \frac{1}{24}\mu^2\beta([\Lambda^s, \underline{v}]v_{xxx}, \underline{h}^4 b_{xx}\Lambda^s v)\\
&- \frac{1}{12}\mu^2\beta([\Lambda^s, \underline{v}_x]v_{xx}, \underline{h}^4 b_{xx}\Lambda^s v) - \frac{1}{12}\mu^2\beta([\Lambda^s, \underline{v}_x]v_x, \underline{h}^4 b_x\Lambda^s v_{xx})\\
&- \frac{1}{12}\mu^2\beta([\Lambda^s, \underline{v}]v_{xx}, \underline{h}^4 b_x\Lambda^s v_{xx}) - \frac{1}{24}\mu^2\beta([\Lambda^s, \underline{v}]v_x, \underline{h}^4 b_{xx}\Lambda^s v_{xx})\\
&+ \frac{2}{3}\mu^2\beta^2([\Lambda^s, \underline{v}_x]v_x, \underline{h}^3 b_x^2\Lambda^s v_x) + \frac{2}{3}\mu^2\beta^2([\Lambda^s, \underline{v}]v_{xx}, \underline{h}^3 b_x^2\Lambda^s v_x)\\
&+ \frac{1}{12}\mu^2\beta^2([\Lambda^s, \underline{v}_x]v_x, \underline{h}^3 b_x b_{xx}\Lambda^s v) + \frac{1}{12}\mu^2\beta^2([\Lambda^s, \underline{v}]v_{xx}, \underline{h}^3 b_x b_{xx}\Lambda^s v)\\
&+ \frac{1}{6}\mu^2\beta^2([\Lambda^s, \underline{v}]v_x, \underline{h}^3 b_x b_{xx}\Lambda^s v) + \frac{1}{12}\mu^2\beta^2([\Lambda^s, \underline{v}]v_x, \underline{h}^3 b_{xx}^2\Lambda^s v)\Big|\\
\le (\varepsilon \vee \beta)&C(|\underline{\zeta}|_\infty, |\underline{v}|_{H^s}, \sqrt{\mu}|\underline{v}_{xx}|_{H^{s-1}}, \mu|\underline{v}_{xxx}|_{H^{s-1}})E^s(U)^2.\end{aligned}$$

To control D_7, one can realize D_3 and integrate by parts with the help of (4.8) to write

$$\begin{aligned}D_7 = &-\varepsilon\mu([\Lambda^s, \Im]\Im^{-1}\mathcal{Q}_1[\underline{U}]v_x, \Lambda^s v) + \varepsilon\mu([\Lambda^s, \mathcal{Q}_1[\underline{U}]]v_x, \Lambda^s v)\\ &- \varepsilon\mu\beta([\Lambda^s, \Im]\Im^{-1}\mathcal{B}_1[\underline{U}]v_x, \Lambda^s v) + \varepsilon\mu\beta([\Lambda^s, \mathcal{B}_1[\underline{U}]]v_x, \Lambda^s v).\end{aligned}$$

Furthermore, as above, using the expressions for $\underline{\mathfrak{T}}$, $\mathcal{Q}_1[\underline{U}]$, $\mathcal{B}_1[\underline{U}]$, with the help of Lemma 3.2, the commutator (3.7), and (4.8), (4.11), one gets

$$|D_7| \leq (\varepsilon \vee \beta) C(|\underline{h} - 1|_{H^s}, |\underline{v}_x|_{H^{s-1}}, \sqrt{\mu}|\underline{v}_{xx}|_{H^{s-1}}, C_s) E^s(U)^2.$$

To control D_8, one can write, by integration by parts,

$$\begin{aligned} D_8 = &-\varepsilon\mu^2([\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}\mathcal{Q}_2[\underline{U}]v_x, \Lambda^s v) + \varepsilon\mu^2([\Lambda^s, \mathcal{Q}_2[\underline{U}]]v_x, \Lambda^s v) \\ &- \varepsilon\mu^2([\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}\mathcal{Q}_3[\underline{U}]v_x, \Lambda^s v) + \varepsilon\mu^2([\Lambda^s, \mathcal{Q}_3[\underline{U}]]v_x, \Lambda^s v) \\ &- \varepsilon\mu^2\beta([\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}\mathcal{B}_2[\underline{U}]v_x, \Lambda^s v) + \varepsilon\mu^2\beta([\Lambda^s, \mathcal{B}_2[\underline{U}]]v_x, \Lambda^s v) \\ &- \varepsilon\mu^2\beta^2([\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}\mathcal{B}_3[\underline{U}]v_x, \Lambda^s v) + \varepsilon\mu^2\beta^2([\Lambda^s, \mathcal{B}_3[\underline{U}]]v_x, \Lambda^s v). \end{aligned}$$

Now, as above, using the expressions for $\underline{\mathfrak{T}}$, $\mathcal{Q}_2[\underline{U}]$, $\mathcal{Q}_3[\underline{U}]$, $\mathcal{B}_2[\underline{U}]$, $\mathcal{B}_3[\underline{U}]$, with the help of Lemma 3.2, the commutator (3.7), and (4.8)–(4.11), one gets

$$|D_8| \leq (\varepsilon \vee \beta) C(|\underline{h} - 1|_{H^s}, |\underline{v}_x|_{H^{s-1}}, \sqrt{\mu}|\underline{v}_{xx}|_{H^{s-1}}, \mu|\underline{v}_{xx}|, C_s) E^s(U)^2.$$

Eventually, as a conclusion, it holds that

$$\begin{aligned} |([\Lambda^s, A[\underline{U}]]\partial_x U, S\Lambda^s U)| \leq \varepsilon C(&|\underline{h} - 1|_{H^s}, |\zeta|_{H^s}, |\underline{\zeta}|_{H^s}, \mu|\underline{\zeta}_{xx}|_{H^s}, |\underline{v}|_{H^s}, \\ &\sqrt{\mu}|\underline{v}_x|_{H^s}, \mu|\underline{v}_{xx}|_{H^s}, C_s, \mathrm{bo}^{-1}) E^s(U)^2. \end{aligned}$$

Estimation of $(\Lambda^s B(\underline{U}), S\Lambda^s U)$. We recall that

$$B(\underline{U}) = \begin{pmatrix} -\beta b_x \underline{v} \\ \underline{\mathfrak{T}}^{-1}(\mathcal{R}[\mu, \varepsilon\underline{h}, \beta b](\underline{U})) \end{pmatrix},$$

knowing that the expressions for the operators $\underline{\mathfrak{T}}$ and $\mathcal{R}$ read

$$\begin{aligned} \underline{\mathfrak{T}} = \underline{h} &- \frac{1}{3}\mu\partial_x(\underline{h}^3\partial_x \cdot) + \frac{\mu\beta}{2}\partial_x(\underline{h}^2 b_x(\cdot)) - \frac{\mu\beta}{2}\underline{h}^2 b_x \partial_x \cdot \\ &+ \mu\beta^2 \underline{h} b_x^2(\cdot) + \frac{1}{45}\mu^2\partial_x^2(\underline{h}^5\partial_x^2 \cdot) \\ &- \frac{\mu^2\beta}{24}\left[2\partial_x^2(\underline{h}^4 b_x\partial_x \cdot) + \partial_x^2(\underline{h}^4 b_{xx}(\cdot)) - 2\partial_x(\underline{h}^4 b_x \partial_x^2 \cdot) + \underline{h}^4 b_{xx}\partial_x^2 \cdot\right] \\ &- \frac{\mu^2\beta^2}{12}\left[4\partial_x(\underline{h}^3 b_x^2 \partial_x \cdot) + \partial_x(\underline{h}^3 b_x b_{xx}(\cdot)) - 2\underline{h}^3 b_x b_{xx}\partial_x \cdot - \underline{h}^3 b_{xx}^2(\cdot)\right], \end{aligned}$$

$$\begin{aligned} \mathcal{R}[\mu, \varepsilon\underline{h}, \beta b](\underline{U}) = &\frac{1}{2}\varepsilon\mu\beta\partial_x(\underline{h}^2 b_{xx} v^2) + \varepsilon\mu\beta^2 \underline{h} b_x b_{xx}\underline{v}^2 + \frac{1}{24}\varepsilon\mu^2\beta\partial_x^2(\underline{h}^4 b_{xxx}\underline{v}^2) \\ &+ \frac{1}{12}\varepsilon\mu^2\beta^2\partial_x(\underline{h}^3 b_x b_{xxx}\underline{v}^2) + \frac{1}{4}\varepsilon\mu^2\beta^2\partial_x(\underline{h}^3 b_{xx}^2 \underline{v}^2). \end{aligned}$$

Now, as in D_3, one may write

$$\begin{aligned} (\Lambda^s B(\underline{U}), S\Lambda^s U) = -\beta(\Lambda^s(b_x\underline{v}), \underline{\mathfrak{J}}_{\mathrm{bo}}\Lambda^s\zeta) &+ (\Lambda^s\mathcal{R}[\mu, \varepsilon\underline{h}, \beta b](\underline{U}), \Lambda^s v) \\ &- ([\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}\mathcal{R}[\mu, \varepsilon\underline{h}, \beta b](\underline{U}), \Lambda^s v). \end{aligned}$$

Using the expressions for $\mathfrak{S}$, $\mathcal{R}[\mu, \varepsilon \underline{h}, \beta b](\underline{U})$, $\underline{\mathfrak{J}}_{\mathrm{bo}}$, with Lemma 3.2 and the commutator estimate (3.7) in hand, in addition to (4.8)–(4.11), it holds that

$$|(\Lambda^s B(\underline{U}), S\Lambda^s U)| \leq (\varepsilon \vee \beta) C(E^s(\underline{U}), \mathrm{bo}^{-1}) E^s(U).$$

Estimation of $(\boldsymbol{\Lambda^s \zeta}, [\boldsymbol{\partial_t}, \underline{\boldsymbol{\mathfrak{J}}}_{\mathbf{bo}}]\boldsymbol{\Lambda^s \zeta})$. One can write, after checking the expression for $\underline{\mathfrak{J}}_{\mathrm{bo}}$ and performing an integration by parts, that

$$|(\Lambda^s \zeta, [\partial_t, \underline{\mathfrak{J}}_{\mathrm{bo}}]\Lambda^s \zeta)| = \left|\frac{2}{45}\mu^2 (\partial_t \underline{h}^4 \Lambda^s \zeta_{xx}, \Lambda^s \zeta_{xx})\right| \leq (\varepsilon \vee \beta) C(|\partial_t \underline{h}|_\infty) E^s(U)^2.$$

Estimation of $(\boldsymbol{\Lambda^s v}, [\boldsymbol{\partial_t}, \underline{\boldsymbol{\mathfrak{S}}}]\boldsymbol{\Lambda^s v})$. Note that we have

$$[\partial_t, \underline{h}]\Lambda^s v = \partial_t \underline{h} \Lambda^s v, \quad [\partial_t, \partial_x(\underline{h}^3 \partial_x \cdot)]\Lambda^s v = \partial_x(\partial_t \underline{h}^3 \Lambda^s v_x),$$

and

$$[\partial_t, \partial_x^2(\underline{h}^5 \partial_x^2 \cdot)]\Lambda^s v = \partial_x^2(\partial_t \underline{h}^5 \Lambda^s v_{xx}).$$

Thus, by integration by parts, it holds that

$$\begin{aligned} &|(\Lambda^s v, [\partial_t, \underline{\mathfrak{S}}]\Lambda^s v)| \\ &\quad = \left|(\partial_t \underline{h} \Lambda^s v, \Lambda^s v) + \frac{\mu}{3}(\partial_t \underline{h}^3 \Lambda^s v_x, \Lambda^s v_x) + \frac{\mu^2}{45}(\partial_t \underline{h}^5 \Lambda^s v_{xx}, \Lambda^s v_{xx})\right| \\ &\quad \leq (\varepsilon \vee \beta) C(|\partial_t \underline{h}|_\infty, E^s(\underline{U})) E^s(U)^2. \end{aligned}$$

Gathering the information provided by the above estimates and using the fact that $H^s(\mathbb{R})$ is continuously embedded in $W^{1,\infty}(\mathbb{R})$, it holds that

$$\begin{aligned} \frac{1}{2} e^{(\varepsilon \vee \beta)\lambda t} \partial_t ((\varepsilon \vee \beta)^{-\varepsilon \lambda t} E^s(U)^2) &\leq (\varepsilon \vee \beta)(C(E^s(\underline{U}), |\partial_t \underline{h}|_{L^\infty}, \mathrm{bo}^{-1}) - \lambda) E^s(U)^2 \\ &\quad + (\varepsilon \vee \beta) C(E^s(\underline{U}), \mathrm{bo}^{-1}) E^s(U). \end{aligned}$$

Taking $\lambda = \lambda_T$ large enough (how large depends on $\sup_{t \in [0, \frac{T}{\varepsilon \vee \beta}]} C(E^s(\underline{U}), |\partial_t \underline{h}|_{L^\infty}, \mathrm{bo}^{-1})$) to have the first term of the right-hand side negative for all $t \in [0, \frac{T}{\varepsilon \vee \beta}]$, one deduces that

$$\forall t \in \left[0, \frac{T}{\varepsilon \vee \beta}\right], \quad \frac{1}{2} e^{(\varepsilon \vee \beta)\lambda t} \partial_t (e^{-(\varepsilon \vee \beta)\lambda t} E^s(U)^2) \leq (\varepsilon \vee \beta) C(E^s(\underline{U}), \mathrm{bo}^{-1}) E^s(U).$$

Integrating this differential inequality with the help of Grönwall's inequality yields therefore

$$\begin{aligned} \forall t \in \left[0, \frac{T}{\varepsilon \vee \beta}\right], \quad E^s(U(t)) &\leq e^{\frac{(\varepsilon \vee \beta)}{2}\lambda_T t} E^s(U_0) \\ &\quad + (\varepsilon \vee \beta) \int_0^t e^{\frac{(\varepsilon \vee \beta)}{2}\lambda_T (t-t')} C(E^s(\underline{U})(t'), \mathrm{bo}^{-1})\, dt', \end{aligned}$$

which is the desired estimate. □

§5. Full justification of the asymptotic model (2.11)

The main result of this paper, i.e. the long-time existence of a solution to the extended Green–Naghdi system (2.11) in $X^s = H^{s+2}(\mathbb{R}) \times H^{s+2}(\mathbb{R})$ with $s > 3/2$ and time of order $t = O(\frac{1}{\varepsilon \vee \beta})$ is stated below. Note that if some smallness assumption is made on $\varepsilon \vee \beta$, at that point the presence time ends up bigger.

It is worth noticing that the case when $\mathrm{bo}^{-1} = 0$ holds for a new energy space $Y^{s>3/2}$ endowed with the norm $|U|^2_{Y^s} := |\zeta|^2_{H^s} + \mu^2|\zeta_{xx}|^2_{H^s} + |v|^2_{H^s} + \mu|v_x|^2_{H^s} + \mu^2|v_{xx}|^2_{H^s}$. In view of inequality (4.6), it is not hard to check that a similar energy estimate holds with $\mathrm{bo}^{-1} = 0$.

Theorem 5.1 (Long-term local existence). *Fix any $s > 3/2$ and $b \in H^{s+3}(\mathbb{R})$. Let $U_0 = (\zeta_0, v_0)^\top \in X^s$ be such that the depth condition (3.1) is satisfied.*

Then there exists a maximal $T_{\max} = T(|U_0|_{X^s}) > 0$, uniformly bounded from below with respect to $\varepsilon, \mu, \beta \in (0,1)^3$ and $\mathrm{bo}^{-1} \in [0,1)$, such that the extended one-dimensional Green–Naghdi equations (2.11) with surface tension admit a unique solution $U = (\zeta, v)^\top \in X^s_{T_{\max}}$ with the initial condition $(\zeta_0, v_0)^\top$ and preserving the non-vanishing depth condition (3.1) for any $t \in [0, \frac{T_{\max}}{\varepsilon \vee \beta})$. In particular, if $T_{\max} < \infty$ one has

$$|U(t,\cdot)|_{X^s} \longrightarrow \infty \quad \textit{as } t \longrightarrow \frac{T_{\max}}{\varepsilon \vee \beta},$$

or

$$\inf_{\mathbb{R}} h(t,\cdot) = \inf_{\mathbb{R}} [1 + \varepsilon\zeta(t,\cdot) - \beta b(\cdot)] \longrightarrow 0 \quad \textit{as } t \longrightarrow \frac{T_{\max}}{\varepsilon \vee \beta}.$$

Proof. The proof of the well-posedness is a straightforward readjustment of the proof of [9, Thm. 7.3] or of [20, Thm. 1] using the energy estimate from the linear analysis proved in Proposition 4.3 (see also [19] for similar proofs of unidirectional equations). The techniques here are those used for hyperbolic systems (see [1, 32] for general details) with additional standard arguments, where no smallness assumption on the parameters ε, μ, β is required in the theorem. Note that, when a sequence of non-linear problems is devised, the difference with respect to the flat bottom case (see [20, Thm. 1] for details of the proof) occurs in the convergence of solutions that is established using the energy estimate. So, one has to deal with topography terms such as $(B(U_n) - B(U_{n-1}), S(U_{n+1} - U_n))$ to be controlled by $(\varepsilon \vee \beta) C_0 |U_{n+1} - U_n|_{X^0} |U_n - U_{n-1}|_{X^0}$, where C_0 is a constant depending on initial data U_0. It is worth noticing that the constant λ_T appearing in Proposition 4.3 is independent of $\partial_t \zeta^n$ and depends only on $|U_0|_{X^s}$. Indeed, by induction on n and using the mass conserved equation, it holds that $|\partial_t h^{n+1}|_{L^\infty} \lesssim \varepsilon |U^{n+1}|_{X^s}$. □

Theorem 5.1 is complemented by the following result that shows the stability of the solution with respect to perturbations, which is very useful for the justification of asymptotic approximations of the exact solution. (The solution $U=(\zeta,v)^\top$ and time $T_{\max}$ that appear in the statement below are those furnished by Theorem 5.1).

Theorem 5.2 (A stability property). *Let the assumption of Theorem 5.1 be satisfied and moreover assume that there exists* $\widetilde{U}=(\tilde{\zeta},\tilde{v})^\top\in C([0,\frac{T_{\max}}{\varepsilon\vee\beta}],X^{s+1}(\mathbb{R}))$ *such that*

$$\begin{cases}\partial_t\tilde{\zeta}+\partial_x(\tilde{h}\tilde{v})=f_1,\\ \widetilde{\Im}(\partial_t\tilde{v}+\varepsilon\tilde{v}\tilde{v}_x)+\tilde{h}\partial_x\tilde{\zeta}-\dfrac{1}{\mathrm{bo}}\mu\tilde{h}\tilde{\zeta}_{xxx}+\dfrac{2}{45}\mu^2\tilde{h}\partial_x^2(\tilde{h}^4\tilde{\zeta}_{xxx})+\mu^2\mathcal{I}[\tilde{h},\beta b]\tilde{\zeta}_x\\ \quad -\varepsilon^2\mu^2\dfrac{1}{\mathrm{bo}}T[\widetilde{U}]\tilde{\zeta}_x+\varepsilon\mu\mathcal{Q}_1[\widetilde{U}]v_x+\varepsilon\mu\beta\mathcal{B}_1[\widetilde{U}]\tilde{v}_x+\varepsilon\mu^2\mathcal{Q}_2[\widetilde{U}]\tilde{v}_x+\varepsilon\mu^2\mathcal{Q}_3[\widetilde{U}]\tilde{v}_x\\ \quad +\varepsilon\mu^2\beta\mathcal{B}_2[\widetilde{U}]\tilde{v}_x+\varepsilon\mu^2\beta^2\mathcal{B}_3[\widetilde{U}]\tilde{v}_x+\mathcal{R}[\mu,\varepsilon\tilde{h},\beta b](\widetilde{U})=\widetilde{\Im}f_2,\end{cases}$$

with $\widetilde{\Im}=\tilde{h}+\mu\mathcal{T}[\tilde{h},\beta b]-\mu^2\mathfrak{T}[\tilde{h},\beta b]$, $\tilde{h}=1+\varepsilon\tilde{\zeta}-\beta b$, *and* $\widetilde{F}=(f_1,f_2)^\top\in L^\infty([0,\frac{T_{\max}}{\varepsilon\vee\beta}],X^s(\mathbb{R})^2)$*. Then, for all* $t\in[0,\frac{T_{\max}}{\varepsilon\vee\beta}]$*, the error* $\mathfrak{E}=U-\widetilde{U}=(\zeta,v)^\top-(\tilde{\zeta},\tilde{v})^\top$ *with respect to* U *given by Theorem 5.1 satisfies for all* $0\le(\varepsilon\vee\beta)t\le T_{\max}$ *the inequality*

$$|\mathfrak{E}|_{L^\infty([0,t],X^s(\mathbb{R}))}\le(\varepsilon\vee\beta)\widetilde{C}\big(|\mathfrak{E}_{|t=0}|_{X^s(\mathbb{R})}+t|\widetilde{F}|_{L^\infty([0,t],X^s(\mathbb{R}))}\big),$$

where the constant $\widetilde{C}$ *depends on* $|U|_{L^\infty([0,T_{\max}/\varepsilon],X^s(\mathbb{R}))}$, $|\widetilde{U}|_{L^\infty([0,T_{\max}/\varepsilon\vee\beta],X^{s+1}(\mathbb{R}))}$.

Proof. The proof is a direct and classical consequence of the same energy estimate as evaluated in Theorem 5.1 which is itself similar to the energy estimate proved in Proposition 4.3. Subtracting the equations satisfied by $U=(\zeta,v)^\top$ and $\widetilde{U}=(\tilde{\zeta},\tilde{v})^\top$ we get the following system:

$$\begin{cases}\partial_t\mathfrak{E}+A[U]\partial_x\mathfrak{E}=-(A[U]-A[\widetilde{U}])\partial_x\widetilde{U}-[B(U)-B(\widetilde{U})]-\widetilde{F},\\ \mathfrak{E}_{|t=0}=U_0-\widetilde{U}_0.\end{cases}$$

Therefore, in view of the proof of Proposition 4.3, one may similarly deduce the estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\mathfrak{E}|^2_{X^s(\mathbb{R})}&=-(SA[U]\Lambda^s\partial_x\mathfrak{E},\Lambda^s\mathfrak{E})-([\Lambda^s,A[U]]\partial_x\mathfrak{E},S\Lambda^s\mathfrak{E})\\&\quad+([\Lambda^s,A[\widetilde{U}]]\partial_x\widetilde{U},S\Lambda^s\mathfrak{E})-([\Lambda^s,A[U]]\partial_x\widetilde{U},S\Lambda^s\mathfrak{E})\\&\quad+(SA[U]\partial_x\Lambda^s\widetilde{U},\Lambda^s\mathfrak{E})-(SA[\widetilde{U}]\partial_x\Lambda^s\widetilde{U},\Lambda^s\mathfrak{E})-(\Lambda^s\widetilde{F},S\Lambda^s\mathfrak{E})\\&\quad+\frac{1}{2}([\partial_t,S]\Lambda^s\mathfrak{E},\Lambda^s\mathfrak{E})-(\Lambda^sB(U),S\Lambda^s\mathfrak{E})+(\Lambda^sB(\widetilde{U}),S\Lambda^s\mathfrak{E})\\&\le(\varepsilon\vee\beta)\widetilde{C}[|\mathfrak{E}|^2_{X^s(\mathbb{R})}+|\mathfrak{E}|_{X^s(\mathbb{R})}|\widetilde{F}|_{X^s(\mathbb{R})}],\end{aligned}\tag{5.1}$$

where $\widetilde{C} = C(|U|_{L^\infty([0,T_{\max}/\varepsilon\vee\beta],X^s(\mathbb{R}))}, |\widetilde{U}|_{L^\infty([0,T_{\max}/\varepsilon\vee\beta],X^{s+1}(\mathbb{R}))})$ and we use the fact that $s-1>t_0$ with $|\partial_t h|_\infty = (\varepsilon\vee\beta)|vh_x + hv_x|_\infty \lesssim |U|^2_{X^s(\mathbb{R})}$. Integrating the differential inequality (5.1) by applying Grönwall's inequality therefore yields the desired result. □

We state here that the solutions to the water-wave equations (1.9) are consistent at $O(\mu^3)$ with the extended Green–Naghdi equations (2.9).

Proposition 5.3 (Consistency). *Let $U^{\text{euler}} = (\zeta,\psi)^\top$ be a family of solutions to the full Euler system (1.9) such that there exists $T>0$, $s>3/2$ for which $(\zeta,\psi')^\top$ is bounded in $L^\infty([0;T);H^{s+N})^2$ with N sufficiently large, uniformly with respect to $\varepsilon,\mu,\beta\in(0,1)^3$ and $\text{bo}^{-1}\in[0,1)$. Moreover, assume that $b\in H^{s+N}$ and that ζ satisfies (3.1). Define v as in (2.1). Then $(\zeta,v)^\top$ satisfies (2.11) up to a remainder R, bounded by*

$$\|R\|_{(L^\infty[0,T);H^s)} \leq \mu^3 C,$$

where $C = C(h_{\min}^{-1}, |b|_{H^{s+N}}, \|\zeta\|_{L^\infty([0,T[;H^{s+N})}, \|\psi'\|_{L^\infty([0,T[;H^{s+N})})$.

Proof. The proof follows the same lines as the proof of [22, Thm. 6.10] with N sufficiently large. In fact, it is sufficient to show that the second equation of (2.11) is satisfied up to a term $\mu^3 R$, with R as in the statement of the proposition. Indeed, taking the derivative of the second equation of (1.9) and replacing $\mathcal{G}[\varepsilon\zeta,\beta b]\psi$ by $-\mu\partial_x(hv)$ and in view of (2.5), replace ψ' by $v + \frac{\mu}{h}\mathcal{T}[h,\beta b]v + \frac{\mu^2}{h}\mathfrak{T}[h,\beta b]v + \mu^3 R_3$. At this stage, denote by $\mu^3 R$ all terms of order μ^3; then taking advantage of similar estimates to those of [22, Lem. 5.11], such that

$$|R_3|_{H^s} \leq C(h_{\min}^{-1}, |\zeta|_{H^{s+6}}, |b|_{H^{s+6}})|\psi'|_{H^{s+6}},$$

and

$$|\partial_t R_3|_{H^s} \leq C(h_{\min}^{-1}, |\zeta|_{H^{s+8}}, |b|_{H^{s+8}}, |\psi'|_{H^{s+8}}),$$

yields the desired control of R. □

Finally, the following convergence result states that the solutions of the full Euler system remain close to those of our system (2.11), with a little more precision as μ^3 is smaller.

Theorem 5.4 (Convergence). *Let $\varepsilon,\mu,\beta\in(0,1)^3$ and $\text{bo}^{-1}\in[0,1)$, $s>3/2$, $b\in H^{s+N}$, and $U_0=(\zeta_0,\psi_0)^\top\in H^{s+N}(\mathbb{R})^2$ satisfy condition (3.1) where N is sufficiently large. Moreover, assume $U^{\text{euler}}=(\zeta,\psi)^\top$ is a unique solution of the full Euler system (1.9) that satisfies the assumption of Proposition 5.3. Then there exist $C,T>0$, independent of ε, μ, β, bo^{-1}, such that*

- *there exists a unique solution* $U_{\text{ex}} = (\zeta_{\text{ex}}, v_{\text{ex}})^\top$ *to our new model* (2.11)*, defined on* $[0, \frac{T}{\varepsilon\vee\beta}]$ *and with initial data* $(\zeta^0, v^0)^\top$ *(provided by Theorem* 5.1*);*
- *the following error estimate holds, for all* $0 \leq t \leq T/(\varepsilon \vee \beta)$*:*

$$|(\zeta, v) - (\zeta_{\text{ex}}, v_{\text{ex}})|_{L^\infty([0,t];X^s)} \leq C\mu^3 t.$$

Proof. The existence of U_{ex} is given by our Theorem 5.1 (we choose T as the minimum of the existence time of both solutions; it is bounded from below, independently of $\varepsilon, \mu, \beta, \text{bo}^{-1} \in (0,1)^4$). Assuming that U^{euler} satisfies the assumptions of our consistency result, Proposition 5.3, therefore $(\zeta, v)^\top$ solves (2.11) up to a residual R bounded from above by μ^3. The result then follows from the stability property (Theorem 5.2). □

Appendix A. Derivation of the extended Green–Naghdi system

In this section we denote $w = \nabla\psi$ (note that w is independent of z).

Appendix A.1. Computation of the integral $J_2[h, \beta b]\nabla\psi$

Note that the following integrals are essential to our computation:

$$\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)} (z - \varepsilon\zeta)\, dz = -\frac{1}{2}h^2,$$

$$\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)} [(z + 1 - \beta b)\beta\nabla b + h\nabla h]\, dz = h^2\nabla h + \frac{1}{2}\beta h^2\nabla b,$$

$$\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)} \left[\frac{1}{3}(z - \varepsilon\zeta)^3 - (z - \varepsilon\zeta)h^2\right] dz = \frac{5}{12}h^4,$$

$$\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)} [(z + 1 - \beta b)^2 - h^2]\, dz = -\frac{2}{3}h^3,$$

$$\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)} [(z - \varepsilon\zeta)^2\varepsilon\nabla\zeta - \varepsilon h^2\nabla\zeta + 2(z - \varepsilon\zeta)h\nabla h]\, dz = -\frac{5}{3}h^3\nabla h - \frac{2}{3}\beta h^3\nabla b,$$

$$\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)} f_1(z)\, dz = \frac{2}{15}h^5,$$

$$\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)} f_2(z)\, dz = \frac{2}{15}\nabla h^5 + \frac{5}{24}h^4\beta\nabla b,$$

where the fourth-order polynomials in z, f_1 and f_2, are given by

$$f_1(z) = \frac{1}{24}(z^4 - (\varepsilon\zeta)^4) - \frac{1}{6}(-1 + \beta b)^3(z - \varepsilon\zeta) - \frac{(\varepsilon\zeta)^2}{4}((z + 1 - \beta b)^2 - h^2)$$
$$- \frac{1}{2}\Big(\frac{1}{3}(z - \varepsilon\zeta)^3 - h^2(z - \varepsilon\zeta)\Big)(-1 + \beta b),$$

$$f_2(z) = -\frac{1}{6}(\varepsilon\zeta)^3\varepsilon\nabla\zeta - \frac{1}{2}(-1+\beta b)^2(z-\varepsilon\zeta)\beta\nabla b + \frac{1}{6}(-1+\beta b)^3\varepsilon\nabla\zeta$$
$$- \frac{(\varepsilon\zeta)}{2}\varepsilon\nabla\zeta((z+1-\beta b)^2 - h^2) + \frac{(\varepsilon\zeta)^2}{2}((z+1-\beta b)\beta\nabla b + h\nabla h)$$
$$+ \frac{1}{2}((z-\varepsilon\zeta)^2 - h^2)(-1+\beta b)\varepsilon\nabla\zeta + (z-\varepsilon\zeta)h\nabla h(-1+\beta b)$$
$$- \frac{1}{2}\Big(\frac{1}{3}(z-\varepsilon\zeta)^3 - (z-\varepsilon\zeta)h^2\Big)\beta\nabla b.$$

The strategy is to expand then reduce all terms of the same size. First, find $\nabla\varphi_2$, and note that using the expression for $\nabla\varphi_1$ we have

$$\nabla[(z-\varepsilon\zeta)\beta\nabla b\cdot(\nabla\varphi_1)_{|_{z=1-\beta b}}]$$
$$= -\beta h\nabla\cdot w(\nabla b\cdot\nabla h)\nabla h - \beta^2 h\nabla\cdot w(\nabla h\cdot\nabla b)\nabla b$$
$$+ \beta(z-\varepsilon\zeta)\nabla(h\nabla\cdot w(\nabla h\cdot\nabla b)) - \frac{1}{2}\beta h^2\nabla h(\nabla b\cdot\nabla(\nabla\cdot w))$$
$$- \frac{1}{2}\beta^2 h^2(\nabla b\cdot\nabla(\nabla\cdot w))\nabla b + \frac{1}{2}(z-\varepsilon\zeta)\beta\nabla\big(h^2(\nabla b\cdot\nabla(\nabla\cdot w))\big)$$
$$+ \beta^2\nabla h((\nabla h\cdot\nabla b)(\nabla b\cdot w)) + \beta^3((\nabla h\cdot\nabla b)(\nabla b\cdot w))\nabla b$$
$$- (z-\varepsilon\zeta)\beta^2\nabla((\nabla h\cdot\nabla b)(\nabla b\cdot w)) + \beta^3\nabla h((\nabla b\cdot\nabla b)(\nabla b\cdot w))$$
$$+ \beta^4((\nabla b\cdot\nabla b)(\nabla b\cdot w))\nabla b - (z-\varepsilon\zeta)\beta^3\nabla((\nabla b\cdot\nabla b)(\nabla b\cdot w))$$
$$+ \beta^2 h\nabla h(\nabla b\cdot\nabla(\nabla b\cdot w)) + \beta^3 h(\nabla b\nabla(\nabla b\cdot w))\nabla b$$
$$- (z-\varepsilon\zeta)\beta^2\nabla\big(h\nabla(\nabla b\cdot\nabla(\nabla b\cdot w))\big).$$

Rearranging the many expressions for $\nabla\varphi_2$, then integrating over $]-1+\beta b,\varepsilon\zeta[$ and taking advantage of the integrals above one may simplify J_2 as follows.

The non-topographical expressions (see Remark 2.1) of

$$J_2[h,\beta b]w = \int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)} \nabla\varphi_2\,dz$$

are factorized into the two terms

$$\frac{2}{15}\nabla\big(\nabla\cdot(h^5\nabla(\nabla\cdot w))\big) + \frac{1}{3}\nabla(h^3\nabla\cdot(h\nabla h)\nabla\cdot w).$$

The purely topographical expressions (see Remark 2.1) of $J_2[h,\beta b]w$ are separated into four categories where they are multiplied by β, β^2, β^3, β^4 respectively.

The β-contributions are

$$-\frac{1}{2}\nabla(h^3\nabla\cdot w(\nabla h\nabla b)) + \frac{1}{8}\nabla(h^4\nabla\cdot w\nabla\cdot(\nabla b))$$
$$-\frac{2}{3}\nabla(h^3\nabla(\nabla b\cdot w)\nabla h) - \frac{1}{3}\nabla(h^3\nabla\cdot(\nabla h)(\nabla b\cdot w))$$

$$
\begin{aligned}
&- \frac{5}{24}\nabla\big(h^4\nabla\cdot(\nabla(\nabla b\cdot w))\big) + \frac{5}{24}h^4\nabla\cdot(\nabla(\nabla\cdot w))\nabla b\\
&+ \frac{1}{2}(\nabla\cdot w\nabla\cdot(\nabla h))\nabla b + \frac{1}{2}h^2\nabla\cdot w(\nabla h\nabla h)\nabla b + h^3(\nabla h\nabla(\nabla\cdot w))\nabla b\\
&= T_1 + T_2 + \cdots + T_9.
\end{aligned}
$$

The β^2-contributions are

$$
\begin{aligned}
&\frac{1}{2}\nabla(h^2(\nabla h\nabla b)(\nabla b\cdot w)) - \frac{1}{6}\nabla(h^3\nabla b\nabla(\nabla b\cdot w)) - \frac{1}{3}\nabla(h^3\nabla\cdot w(\nabla b\nabla b))\\
&\quad - \frac{1}{3}\nabla(h^3(\nabla b\cdot w)\nabla\cdot(\nabla b)) - h^2\nabla\cdot w(\nabla h\nabla b)\nabla b - \frac{1}{6}h^3(\nabla b\nabla(\nabla\cdot w))\nabla b\\
&\quad + \frac{1}{6}h^3(\nabla\cdot w\nabla\cdot(\nabla b))\nabla b - h^2(\nabla h\nabla(\nabla b\cdot w))\nabla b\\
&\quad - \frac{1}{2}h^2((\nabla b\cdot w)\nabla\cdot(\nabla h))\nabla b - \frac{1}{3}\nabla\cdot(\nabla(\nabla b\cdot w))\nabla b\\
&= P_1 + P_2 + \cdots + P_{10}.
\end{aligned}
$$

The β^3-contributions are

$$
\begin{aligned}
&h((\nabla h\nabla b)(\nabla b\cdot w))\nabla b + h((\nabla b\nabla b)(\nabla b\cdot w))\nabla h - \frac{1}{2}h^2\nabla\cdot w(\nabla b\nabla b)\nabla b\\
&\quad + \frac{1}{2}h^2\nabla((\nabla b\nabla b)(\nabla b\cdot w)) - \frac{1}{2}h^2((\nabla b\cdot w)\nabla\cdot(\nabla b))\nabla b.
\end{aligned}
$$

Lastly, the only term of size β^4 is $h(\nabla b\cdot w)(\nabla b\nabla b)\nabla b$.

Appendix A.2. Computation of the operator $\mathcal{T}[h,\beta b](h^{-1}\mathcal{T}[h,\beta b]w)$

Expanding then reducing terms of the same size, the expressions for $\mathcal{T}[h,\beta b]\times(\frac{1}{h}\mathcal{T}[h,\beta b]w)$ will be simplified as follows. Recall that

$$
\mathcal{T}[h,\beta b]w = -\frac{1}{3}\nabla(h^3\nabla\cdot w) + \frac{\beta}{2}[\nabla(h^2\nabla b\cdot w) - h^2\nabla b\nabla\cdot w] + \beta^2 h\nabla b\nabla b\cdot w.
$$

The non-topographical terms (see Remark 2.1) are

$$
\frac{1}{9}\nabla\Big(h^3\nabla\cdot\Big(\frac{1}{h}\nabla(h^3\nabla\cdot w)\Big)\Big) = \frac{1}{9}\nabla\big(\nabla\cdot(h^5\nabla(\nabla\cdot w))\big) + \frac{1}{3}\nabla(h^3\nabla\cdot(h\nabla h)\nabla\cdot w).
$$

The purely topographical expressions (see Remark 2.1) of $\mathcal{T}[h,\beta b](\frac{1}{h}\mathcal{T}[h,\beta b]w)$ will be separated into four categories where they are of size β, β^2, β^3, β^4 respectively.

The β-contributions are

$$
\begin{aligned}
&- \frac{1}{6}\nabla(h\nabla b\nabla(h^3\nabla\cdot w)) - \frac{1}{3}\nabla(h^3(\nabla b\cdot w)\nabla\cdot(\nabla h)) - \frac{1}{2}\nabla(h^3\nabla h\nabla(\nabla b\cdot w))\\
&\quad - \frac{1}{6}\nabla\big(h^4\nabla\cdot(\nabla(\nabla b\cdot w))\big) + \frac{1}{6}\nabla(h^3\nabla\cdot(h\nabla\cdot w\nabla b))
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{6}(\nabla h \nabla(h^3 \nabla \cdot w))\nabla b + \frac{1}{6} h \nabla \cdot (\nabla(h^3 \nabla \cdot w))\nabla b \\
&= T_1' + T_2' + \cdots + T_7'.
\end{aligned}
$$

The β^2-contributions are

$$
\begin{aligned}
&+ \frac{1}{4}\nabla\big(h\nabla b \nabla(h^2(\nabla b \cdot w))\big) - \frac{1}{4}\nabla(h^3 \nabla \cdot w(\nabla b \nabla b)) - \frac{1}{3}\nabla\big(h^3 \nabla \cdot ((\nabla b \cdot w)\nabla b)\big) \\
&\quad - \frac{1}{3}(\nabla b \nabla(h^3 \nabla \cdot w))\nabla b - \frac{1}{2} h^2 \nabla \cdot ((\nabla b \cdot w)\nabla h)\nabla b \\
&\quad - \frac{1}{4} h^2 \nabla \cdot (h \nabla(\nabla b \cdot w))\nabla b + \frac{1}{4} h^2 \nabla \cdot (h \nabla \cdot w \nabla b)\nabla b \\
&= P_1' + P_2' + \cdots + P_7'.
\end{aligned}
$$

The β^3-contributions are

$$
\begin{aligned}
&\frac{1}{2}\big(\nabla b \nabla(h^2(\nabla b \cdot w))\big)\nabla b - \frac{1}{2} h^2 \nabla \cdot w(\nabla b \nabla b)\nabla b \\
&\quad + \frac{1}{2}\nabla(h^2(\nabla b \nabla b)(\nabla b \cdot w)) - \frac{1}{2} h^2 \nabla \cdot ((\nabla b \cdot w)\nabla b)\nabla b.
\end{aligned}
$$

Lastly, the only term of order β^4 is $h(\nabla b \cdot w)(\nabla b \nabla b)\nabla b$.

Appendix A.3. Factorization of $\mathfrak{T} = \mathcal{T}(h^{-1}\mathcal{T}) - J_2$

As a result of the above two sections, the difference between the two operators is now simplified as follows. The non-topographical expression can be factorized in the term

$$
-\frac{1}{45}\nabla\big(\nabla \cdot (h^5 \nabla(\nabla \cdot v))\big).
$$

The purely topographical expressions are split into two categories where they are multiplied by β, β^2 respectively.

The β-contributions are

$$
\begin{aligned}
T_1' + \cdots + T_5' - T_1 - \cdots - T_5 &= \frac{1}{24}\nabla\big(\nabla \cdot (h^4 \nabla(\nabla b \cdot w))\big) \\
&\quad + \frac{1}{24}\nabla(\nabla \cdot w \nabla \cdot (h^4 \nabla b)), \\
T_6' + T_7' - T_6 - \cdots - T_9 &= -\frac{1}{24}\nabla \cdot (h^4 \nabla(\nabla \cdot w))\nabla b.
\end{aligned}
$$

The β^2-contributions are

$$
\begin{aligned}
P_1' + P_2' + P_3' - P_1 - \cdots - P_4 &= \frac{1}{12}\nabla(h^3 \nabla \cdot w(\nabla b \nabla b)) + \frac{1}{12}\nabla(h^3 \nabla b \nabla(\nabla b \cdot w)), \\
P_4' + \cdots + P_7' - P_5 - \cdots - P_{10} &= \frac{1}{12}\nabla \cdot (h^3 \nabla \cdot w \nabla b)\nabla b + \frac{1}{12}\nabla \cdot (h^3 \nabla(\nabla b \cdot w))\nabla b.
\end{aligned}
$$

One can see from above sections that the terms of size β^3 and β^4 will be eliminated.

Appendix A.4. Invertibility of operator J

The proof of the invertibility of $J\colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$, defined by $J\phi = \phi - \frac{\mu^2}{45}\partial_x(h^5\partial_x\phi)$, is a direct application of the Lax–Milgram theorem (see [16] for a similar operator). Concerning (3.5), one can show that for a fixed $\mu \in (0,1)$, if $\phi \in H^1_\mu(\mathbb{R})$ solves $J\phi = q + \mu\partial_x g$ for any $q, g \in L^2(\mathbb{R})$, then using the coercivity condition and integration by parts one has

$$|\phi|_{H^1_\mu} \lesssim |q|_{L^2} + |g|_{L^2}.$$

The estimates follow by taking $q = 0$ once and $g = 0$ for a second time.

Acknowledgements

The authors would like to thank the anonymous referee for valuable remarks, suggestions, and careful reading. B.K gratefully acknowledges partial support from the LAU-GSR research program. S.I gratefully acknowledges partial support from the Lebanese University research program.

References

[1] S. Alinhac and P. Gérard, *Opérateurs pseudo-différentiels et théorème de Nash–Moser*, Savoirs Actuels, InterEditions, Paris; Éditions du Centre National de la Recherche Scientifique (CNRS), Meudon, 1991. Zbl 0791.47044 MR 1172111

[2] B. Alvarez-Samaniego and D. Lannes, A Nash-Moser theorem for singular evolution equations. Application to the Serre and Green–Naghdi equations, Indiana Univ. Math. J. **57** (2008), 97–131. Zbl 1144.35007 MR 2400253

[3] D. M. Ambrose and N. Masmoudi, The zero surface tension limit of two-dimensional water waves, Comm. Pure Appl. Math. **58** (2005), 1287–1315. Zbl 1086.76004 MR 2162781

[4] D. M. Ambrose and N. Masmoudi, The zero surface tension limit of three-dimensional water waves, Indiana Univ. Math. J. **58** (2009), 479–521. Zbl 1172.35058 MR 2514378

[5] W. Craig, An existence theory for water waves and the Boussinesq and Korteweg–de Vries scaling limits, Comm. Partial Differential Equations **10** (1985), 787–1003. Zbl 0577.76030 MR 795808

[6] A. B. de Saint-Venant, Sur la houle et le clapotis, C. R. Acad. Sc. Paris **73** (1871) 521–528, 589–593. Zbl 03.0484.02

[7] A. B. de Saint-Venant, Théorie du mouvement non permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leur lit, C. R. Acad. Sc. Paris **73** (1871) 147–154. Zbl 03.0482.04

[8] V. Duchêne and T. Iguchi, A Hamiltonian structure of the Isobe–Kakinuma model for water waves, Water Waves **3** (2021), 193–211. Zbl 1481.76044 MR 4246393

[9] V. Duchêne, S. Israwi, and R. Talhouk, A new fully justified asymptotic model for the propagation of internal waves in the Camassa–Holm regime, SIAM J. Math. Anal. **47** (2015), 240–290. Zbl 1317.76027 MR 3296608

[10] H. Fujiwara and T. Iguchi, A shallow water approximation for water waves over a moving bottom, in *Nonlinear dynamics in partial differential equations*, Advanced Studies in

Pure Mathematics 64, Mathematical Society of Japan, Tokyo, 2015, 77–88. Zbl 1335.35194 MR 3381193

[11] A. E. Green and P. M. Naghdi, A derivation of equations for wave propagation in water of variable depth, J. Fluid Mech. **78** (1976), 237–246. Zbl 0351.76014

[12] T. Iguchi, A shallow water approximation for water waves, J. Math. Kyoto Univ. **49** (2009), 13–55. Zbl 1421.76020 MR 2531127

[13] T. Iguchi, Isobe–Kakinuma model for water waves as a higher order shallow water approximation, J. Differential Equations **265** (2018), 935–962. Zbl 1390.35272 MR 3788631

[14] T. Iguchi, A mathematical justification of the Isobe–Kakinuma model for water waves with and without bottom topography, J. Math. Fluid Mech. **20** (2018), 1985–2018. Zbl 1419.76087 MR 3877504

[15] S. Israwi, Derivation and analysis of a new 2D Green–Naghdi system, Nonlinearity **23** (2010), 2889–2904. Zbl 1202.35125 MR 2727175

[16] S. Israwi, Large time existence for 1D Green–Naghdi equations, Nonlinear Anal. **74** (2011), 81–93. Zbl 1381.86012 MR 2734979

[17] T. Kano and T. Nishida, Sur les ondes de surface de l'eau avec une justification mathématique des équations des ondes en eau peu profonde, J. Math. Kyoto Univ. **19** (1979), 335–370. Zbl 0419.76013 MR 545714

[18] B. Khorbatly, A remark on the well-posedness of the classical Green–Naghdi system, Math. Methods Appl. Sci. **44** (2021), 14545–14555. Zbl 1497.35374 MR 4342853

[19] B. Khorbatly and S. Israwi, A conditional local existence result for the generalized nonlinear Kawahara equation, Math. Methods Appl. Sci. **43** (2020), 5522–5531. Zbl 1446.35126 MR 4088949

[20] B. Khorbatly, I. Zaiter, and S. Israwi, Derivation and well-posedness of the extended Green–Naghdi equations for flat bottoms with surface tension, J. Math. Phys. **59** (2018), art. no. 071501, 20 pp. Zbl 1394.76027 MR 3819952

[21] D. Lannes, Well-posedness of the water-waves equations, J. Amer. Math. Soc. **18** (2005), 605–654. Zbl 1069.35056 MR 2138139

[22] D. Lannes, *The water waves problem*, Mathematical Surveys and Monographs 188, American Mathematical Society, Providence, RI, 2013. Zbl 1410.35003 MR 3060183

[23] D. Lannes and P. Bonneton, Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation, Phys. Fluids **21** (2009), art. no. 016601, 9 pp. Zbl 1183.76294

[24] D. Lannes and F. Marche, A new class of fully nonlinear and weakly dispersive Green–Naghdi models for efficient 2D simulations, J. Comput. Phys. **282** (2015), 238–268. Zbl 1351.76114 MR 3291451

[25] Y. A. Li, A shallow-water approximation to the full water wave problem, Comm. Pure Appl. Math. **59** (2006), 1225–1285. Zbl 1169.76012 MR 2237287

[26] Y. Matsuno, Hamiltonian formulation of the extended Green–Naghdi equations, Phys. D **301/302** (2015), 1–7. Zbl 1364.76031 MR 3343808

[27] Y. Matsuno, Hamiltonian structure for two-dimensional extended Green–Naghdi equations, Proc. R. Soc. A. **472** (2016), art. no. 20160127, 24 pp. Zbl 1371.76032 MR 3522536

[28] J. Miles and R. Salmon, Weakly dispersive nonlinear gravity waves, J. Fluid Mech. **157** (1985), 519–531. Zbl 0583.76013 MR 808127

[29] M. Ming and Z. Zhang, Well-posedness of the water-wave problem with surface tension, J. Math. Pures Appl. (9) **92** (2009), 429–455. Zbl 1190.35186 MR 2558419

[30] V. I. Nalimov, The Cauchy–Poisson problem, Dinamika Splošn. Sredy Vyp. 18 Dinamika Židkost. so Svobod. Granicami, **254** (1974), 104–210. MR 0609882

[31] L. V. Ovsjannikov, Cauchy problem in a scale of Banach spaces and its application to the shallow water theory justification, in *Applications of methods of functional analysis to problems in mechanics (Joint Sympos., IUTAM/IMU, Marseille, 1975)*, Lecture Notes in Mathematics 503, Springer, Berlin, 1976, 426–437. MR 0670760

[32] M. E. Taylor, *Partial differential equations. II*, Applied Mathematical Sciences 116, Springer, New York, 1996. MR 1395149

[33] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 2-D, Invent. Math. **130** (1997), 39–72. Zbl 0892.76009 MR 1471885

[34] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 3-D, J. Amer. Math. Soc. **12** (1999), 445–495. Zbl 0921.76017 MR 1641609

[35] H. Yosihara, Gravity waves on the free surface of an incompressible perfect fluid of finite depth. Publ. Res. Inst. Math. Sci. **18** (1982), 49–96. Zbl 0493.76018 MR 660822

[36] V. E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, J. Appl. Mech. Tech. Phys. **9** (1968), 190–194.