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On Characteristic Polynomials of Automorphisms of Enriques Surfaces

by

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Abstract

Let f be an automorphism of a complex Enriques surface Y and let p_f denote the characteristic polynomial of the isometry f^* of the numerical Néron–Severi lattice of Y induced by f. We combine a modification of McMullen's method with Borcherds' method to prove that the modulo-2 reduction $(p_f(x) \bmod 2)$ is a product of modulo-2 reductions of (some of) the five cyclotomic polynomials Φ_m, where $m \leq 9$ and m is odd. We study Enriques surfaces that realize modulo-2 reductions of Φ_7, Φ_9 and show that each of the five polynomials $(\Phi_m(x) \bmod 2)$ is a factor of the modulo-2 reduction $(p_f(x) \bmod 2)$ for a complex Enriques surface.

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§1. Introduction

The subject of this note is isometries of the numerical Néron–Severi lattices induced by automorphisms of Enriques surfaces. To state our results, let Y (resp. X) be a complex Enriques surface (resp. its K3 cover) and let $\mathrm{Num}(Y)$ be the numerical Néron–Severi lattice of Y (i.e. $\mathrm{Num}(Y) := \mathrm{NS}(Y)/\mathrm{Tors}$). Each automorphism $f \in \mathrm{Aut}(Y)$ induces an isometry $f^* \in \mathrm{O}(\mathrm{Num}(Y))$. Let $p_f(x)$ be its characteristic polynomial. As observed by Oguiso ([19, Lem. 4.1]), no degree-5 irreducible polynomials can appear in a factorization of the modulo-2 reduction $(p_f(x) \bmod 2)$.

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An attempt to characterize all factors of $(p_f(x) \bmod 2)$ was made in [13]. In this paper, we give a complete answer to the question of which factors do appear in the modulo-2 reduction $(p_f(x) \bmod 2)$ for an automorphism $f \in \mathrm{Aut}(Y)$, i.e. we prove the following theorem.

Theorem 1.1. *Let f be an automorphism of a complex Enriques surface Y and let p_f be the characteristic polynomial of the isometry $f^*\colon \mathrm{Num}(Y) \to \mathrm{Num}(Y)$.*

(a) *The modulo-2 reduction $(p_f(x) \bmod 2)$ is a product of (some of) the following polynomials:*

$$F_1(x) = x + 1, \quad F_3(x) = x^2 + x + 1, \quad F_5(x) = x^4 + x^3 + x^2 + x + 1,$$
$$F_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1, \quad F_9(x) = x^6 + x^3 + 1.$$

(b) *Each of the five polynomials F_1, F_3, F_5, F_7, F_9 appears in the factorization of the modulo-2 reduction $(p_f(x) \bmod 2)$ for an automorphism f of a complex Enriques surface. Any realization of F_9 is by a semi-symplectic automorphism.*

Recall that the proof of [13, Thm. 1.2] shows that each factor of $(p_f(x) \bmod 2)$ either equals one of the five polynomials listed in Theorem 1.1, or is the modulo-2 reduction F_{15} of the cyclotomic polynomial $\Phi_{15} \in \mathbb{Z}[x]$. Moreover, examples with factors F_1, F_3, F_5 were given in [7] (see also [13, Exa. 3.1]), whereas the question of whether F_7, F_9, and F_{15} can appear in the factorization of the modulo-2 reduction of p_f for an automorphism $f \in \mathrm{Aut}(Y)$ was left open (cf. [13, Exa. 3.1.b]).

To state the next theorem, we introduce some notation. Let us denote the covering involution of the double étale cover $\pi\colon X \to Y$ by ε. Moreover, we put $\tilde{f} \in \mathrm{Aut}(X)$ to denote a (non-unique) lift of an automorphism $f \in \mathrm{Aut}(Y)$. Let $N \coloneqq (H^2(X,\mathbb{Z})^{\varepsilon})^{\perp}$ be the orthogonal complement of the ε-invariant sublattice $H^2(X,\mathbb{Z})^{\varepsilon}$ in the lattice $H^2(X,\mathbb{Z})$. Recall that N is stable under the cohomological action $\tilde{f}^*$ and the restriction $f_N \coloneqq \tilde{f}^*|_N$ is of finite order. Using Theorem 1.1, we can sharpen [13, Thm. 1.1] as well.

Theorem 1.2. *Let Y be a complex Enriques surface and let f be an automorphism of Y. Then the order of f_N is a divisor of at least one of the following five integers:*

$$36, \quad 48, \quad 56, \quad 84, \quad 120.$$

Among the 28 numbers that satisfy the above condition, at least the 16 integers

$$1, \ldots, 10, \quad 12, \quad 14, \quad 15, \quad 20, \quad 18, \quad 30$$

are realized as orders.

Remark 1.3. We note that if the order of f_N is 7 or 9, then the cyclic subgroup generated by f_N is unique up to conjugacy in the orthogonal group $O(N)$. For the remaining 12 integers

$$16,\ 21,\ 24,\ 28,\ 36,\ 40,\ 42,\ 48,\ 56,\ 60,\ 84,\ 120,$$

we do not know whether they arise as orders of f_N for some $f \in \mathrm{Aut}(Y)$.

Originally, our interest in the subject of this note was motivated by the question of what constraints on the dynamical spectra of Enriques surfaces result from the existence of the double étale K3 cover (cf. [19, Thm. 1.2]). Indeed, Theorem 1.1(a) yields a new constraint on the Salem numbers that appear as the dynamical degrees of automorphisms of Enriques surfaces (e.g. it implies that none of the Salem numbers given as # 3, 13, 16, 34, 35 in the table in [13, App.] can be the dynamical degree of an automorphism of a complex Enriques surface), whereas Theorem 1.1(b) shows that the above constraint cannot be strengthened.

It should be mentioned that automorphism groups of Enriques surfaces remain a subject of intensive research. Much is known in the case of Enriques surfaces with finite automorphism groups (even in positive characteristic) and unnodal Enriques surfaces, but a general picture is still missing. In this context, both the constraints given by Theorem 1.2 and the geometry of the families of Enriques surfaces discussed in Propositions 5.3, 4.1, 4.7 are of separate interest. Still, such considerations exceed the scope of this paper.

We sketch our strategy for the proof of Theorem 1.1. The argument in [13] is based on criteria for a polynomial to be the characteristic polynomial of an isometry of a lattice. Unfortunately, all six polynomials $F_1, \ldots, F_9, F_{15}$ do appear as factors of modulo-2 reductions of characteristic polynomials of isometries of the lattice $U \oplus E_8(-1)$ and the lattice N. Thus we need to take Hodge structures and the ample cone into account as well. In this note we apply a modification of McMullen's method (see [14, 15]) to obtain constraints on automorphisms of Enriques surfaces that can realize the factors F_7, F_9, F_{15}. In particular, we can rule out the existence of the highest-degree factor F_{15} (Proposition 3.1), and derive properties of the K3 covers of Enriques surfaces which realize F_7 (Proposition 5.2) and F_9 (Section 4). To go further with McMullen's method, one has to fix the characteristic polynomial p_f. However, there are infinitely many possibilities for p_f. We provide an algorithmic solution based on Borcherds' method ([1, 2]) and the ideas from [26] and [3] which allow us to avoid fixing p_f. As a result we find abstract Enriques surfaces realizing F_7 and F_9. For the readers' convenience, the algorithm is presented in Section 6 in pseudocode.

Notation. In this note, we work over the field of complex numbers $\mathbb{C}$. Given a prime p, $\mathbb{Z}_p$ denotes the ring of p-adic integers. For a ring R, we denote by $R^\times$ its group of units. For a group G and a prime p, G_p is the p-Sylow subgroup of G.

§2. Preliminaries

§2.1. Basic notation

We maintain the notation of the previous section. In particular, $\pi\colon X \to Y$ is the K3 cover of Y and ε is the covering involution of π. Moreover, we have the finite index sublattice

$$M \oplus N \subseteq H^2(X, \mathbb{Z}), \tag{2.1}$$

where $M := H^2(X,\mathbb{Z})^\varepsilon$ coincides with the pullback of $H^2(Y,\mathbb{Z})$ by π and $N := M^\perp$ (see e.g. [18]). In particular, we have $M \simeq U(2) \oplus E_8(-2)$ and $N \simeq U \oplus U(2) \oplus E_8(-2)$, where U (resp. E_8) denotes the unimodular hyperbolic plane (resp. the unique even unimodular positive-definite lattice of rank 8). Let f be an automorphism of Y. The sublattices M and N are preserved by the isometry $\tilde{f}^* \in \mathrm{Aut}(H^2(X,\mathbb{Z}))$, so as in [13] we can define the maps

$$f_M := \tilde{f}^*|_M \quad \text{and} \quad f_N := \tilde{f}^*|_N$$

and let p_M, p_N (resp. μ_M, μ_N) denote their characteristic (resp. minimal) polynomials. Then (see [13, proof of Lem. 2.2(a)], [20, Lem. 6.3]) we have

$$p_M \equiv p_f \bmod 2 \quad \text{and} \quad (x+1)^2 \cdot p_M \equiv p_N \bmod 2. \tag{2.2}$$

As we have already mentioned, f_N is a map of finite order (see e.g. [19, Lem. 4.2]), so p_N is a product of cyclotomic polynomials.

Recall that (see [21, Prop. 2.2], [11, Thm. 1.1])

$$N \cap \mathrm{NS}(X) \text{ contains no vectors of square } (-2). \tag{2.3}$$

For an automorphism f and an integer $k \in \mathbb{N}$ we define two lattices

$$N_k := \ker(\Phi_k(f_N)) \quad \text{and} \quad M_k := \ker(\Phi_k(f_M)), \tag{2.4}$$

where $\Phi_k(x)$ stands for the kth cyclotomic polynomial. Finally, to simplify our notation we put

$$F_k(x) := (\Phi_k(x) \bmod 2).$$

An automorphism f of an Enriques surface is called *semi-symplectic* if it acts trivially on the global sections $H^0(Y, K_Y^{\otimes 2})$ of the bi-canonical bundle. This is the case if and only if both lifts $\tilde{f}$ and $\tilde{f} \circ \varepsilon$ of f act on $H^0(X, \Omega_X^2)$ as ± 1. We denote by $\mathrm{Aut}_s(Y)$ the subgroup of semi-symplectic automorphisms.

§2.2. Lattice

Let $R \in \{\mathbb{Z}, \mathbb{Z}_p\}$ and K be the fraction field of R. An *R-lattice* is a finitely generated free R-module equipped with a non-degenerate symmetric K-valued bilinear form b. If the form is R valued, we call the lattice *integral*. If, further, $b(x,x) \in 2R$ for every $x \in L$, the lattice is called *even*. The *dual lattice* of L is

$$L^\vee = \big\{x \in L \mid b(x, L) \subseteq R\big\}.$$

If L is integral, then $L \subseteq L^\vee$ and we call the quotient $L^\vee/L$ the *discriminant group* of L. For $r \in R$, an R-lattice L is called *r-modular* if $rL^\vee = L$. If $r = 1$, we call the lattice *unimodular*. The Gram matrix $G = (G_{ij})$ with respect to an R-basis $(e_1, \ldots, e_n)$ of L is defined by $G_{ij} = b(e_i, e_j)$. The determinant $\det L \in R/R^{\times 2}$ of L is the determinant of any Gram matrix. For $R = \mathbb{Z}$ we have $|L^\vee/L| = |\det L|$. The discriminant group carries the discriminant bilinear form induced by $b(x,y)$ mod R for $x, y \in L^\vee$. If L is an even lattice, its discriminant group moreover carries a torsion quadratic form induced by $x \mapsto b(x,x) \mod 2R$, called the *discriminant form*. We say that two R-lattices (L,b), (L',b') are isomorphic if there is an R-linear isomorphism $\phi\colon L \to L'$ such that $b(x,x) = b'(\phi(x), \phi(x))$. For $r \in R$ we denote by $L(r)$ the lattice with the same underlying free module as L, but with bilinear form rb.

Let L, L', L'' be lattices. The orthogonal direct sum of two lattices is denoted by $L \oplus L'$. A sublattice $L' \subseteq L$ is called *primitive* if L/L' is torsion-free. This is equivalent to $(L' \otimes K) \cap L = L'$. We call

$$L' \oplus L'' \subseteq L$$

a *primitive extension* if L', L'' are primitive sublattices of L and $\operatorname{rank} L' + \operatorname{rank} L'' = \operatorname{rank} L$. The finite group $L''/(L \oplus L')$ is the *glue* of the primitive extension. For any prime p dividing its order, we say that L and L' are glued above/over p. The signature (pair) (s_+, s_-) of a $\mathbb{Z}$-lattice L is the signature of $L \otimes \mathbb{R}$, where s_+ is the number of positive and s_- is the number of negative eigenvalues of a Gram matrix. We denote by U the even unimodular lattice of signature $(1,1)$. Moreover, A_n ($n \in \mathbb{N}$), (resp. D_n ($n \geq 4$), E_6, E_7, E_8) stands for the positive-definite root lattice with the respective Dynkin diagram.

§2.3. Genus

Two $\mathbb{Z}$-lattices L and L' are in the same *genus* if $L \otimes \mathbb{R} \cong L' \otimes \mathbb{R}$ and for all prime numbers p we have $L \otimes \mathbb{Z}_p \cong L' \otimes \mathbb{Z}_p$. The genus is an effectively computable invariant and has a compact description in terms of the so-called *genus symbols* introduced by Conway and Sloane (see [6, Chap. 15]).

Definition 2.1. A 2-adic lattice, all of whose Jordan constituents are even, is called *completely even*.

We denote by n_q the rank of a q-modular Jordan constituent and by $\epsilon_q \in \{\pm 1\}$ its unit square class. Two completely even lattices are isomorphic if and only if they have the same symbols $q^{\epsilon_q n_q}$ for all prime powers q. If the lattices in question are not completely even, the symbol involves an additional quantity called the oddity. However, in this note (almost) all lattices considered are completely even.

Note that Conway and Sloane give necessary and sufficient conditions on when a collection of local symbols defines a non-empty genus [6, Thm. 15.11 on p. 383].

Remark 2.2. The genus symbols and their relation with discriminant forms are implemented in `sageMath` [23] by the first author. For instance, the function `sage.quadratic_forms.genera.genus.all_genera_by_det` returns all (valid) genus symbols of a given signature, determinant, and level. This allows us to avoid checking the existence conditions for a genus symbol by hand.

It is possible to compute all classes in a definite genus using Kneser's neighboring algorithm [24] and Siegel's mass formula. An indefinite lattice is usually unique in its genus. Similarly, roots can be found using short vector enumerators [5, §.2.7.3]. We used the implementation provided by `PARI` [22] via `sageMath`.

For later reference we state (without proofs) two immediate lemmas which relate the genus symbols with primitive extensions and isometries.

Lemma 2.3. *Let L and L' be completely even p-adic lattices with symbols $(\epsilon_q, n_q)_q$, respectively $(\epsilon'_q, n'_q)_q$; then $L \oplus L'$ has symbol $(\epsilon_q\epsilon'_q, n_q + n'_q)$.*

Lemma 2.4. *Let L and L' be completely even p-adic lattices with symbols $(\epsilon_q, n_q)_q$ and $(\epsilon'_q, n'_q)_q$. Then there is a primitive extension $L \oplus L' \subseteq L''$ with L'' unimodular if and only if, for all $q > 1$, $n'_q = n_q$ and $\epsilon'_q = \delta^{n_q}\epsilon_q$, where*

$$\delta = \begin{cases} 1 & \text{for } p \equiv 1, 2 \mod 4, \\ -1 & \text{for } p \equiv 3 \mod 4. \end{cases}$$

In the sequel we will apply the following lemma.

Lemma 2.5. *Let L be a $\mathbb{Z}$-lattice and let $g \in \mathrm{O}(L)$ be an isometry with minimal polynomial Φ_3. Then L is completely even and the 2-adic symbols of the genus of L are of the form*

$$q_i^{\epsilon_i n_i}, \quad \text{where } q_i = 2^i,\ n_i \text{ is even and } \epsilon_i = (-1)^{n_i/2}.$$

Proof. This is a special case of [10, Prop. 2.17, Kor. 2.36]. □

In particular, when L is a rank-2 (resp. rank-4) lattice of discriminant at most 4 (resp. 16) its 2-adic symbols are 1^{-2}, 2^{-2} (resp. 1^4, $1^{-2}2^{-2}$, 2^4, $1^{-2}4^{-2}$).

§2.4. Φ_n-lattices

In the sequel we need the notion of a Φ_n-lattice.

The reader can consult [9], [15, §5] for a concise and more general exposition of the facts we briefly sketch below.

Recall that a Φ_n-lattice is defined to be a pair (L, f), where L is an integral lattice and $f \in \mathrm{O}(L)$ is an isometry with characteristic polynomial Φ_n.

Let $n > 2$; the principal Φ_n-lattice $(L_0, \langle\cdot,\cdot\rangle_0, f_0)$ is defined as the $\mathbb{Z}$-module $L_0 \coloneqq \mathbb{Z}[\zeta_n]$ equipped with the scalar product

$$\langle g_1, g_2\rangle_0 = \mathrm{Tr}_{\mathbb{Q}}^{\mathbb{Q}[\zeta_n]}\left(\frac{g_1\overline{g_2}}{r_n'(\zeta_n + \zeta_n^{-1})}\right)$$

where ζ_n is a primitive nth root of unity, Tr is the field trace of $\mathbb{Q}[\zeta_n]/\mathbb{Q}$, $r_n \in \mathbb{Q}[x]$ is the minimal polynomial of $(\zeta_n + \zeta_n^{-1})$, and r_n' is its derivative. Finally, $f_0\colon L_0 \to L_0$, $x \mapsto \zeta_n \cdot x$, is an isometry with minimal polynomial Φ_n. One can show that L_0 is an even lattice and

$$\det(L_0) = |\Phi_n(1)\Phi_n(-1)|. \tag{2.5}$$

Given a pair (L, f) as above and an element $a \in \mathbb{Z}[f + f^{-1}] \subset \mathrm{End}(L)$ one can define another inner product on L by the formula $\langle g_1, g_2\rangle_a \coloneqq \langle ag_1, g_2\rangle_0$. We say that the resulting lattice is the twist of L by a and denote it by $L(a)$. Recall that for $2 < n$ with $\deg(\Phi_n) \leq 20$ the class number of $\mathbb{Q}(\zeta_n)$ is 1. Thus, if $\deg(\Phi_n) \leq 20$, then

(2.6) any even Φ_n-lattice is a twist of the principal lattice $(L_0, \langle\cdot,\cdot\rangle_0, f_0)$,

by [15, Thm. 5.2], [9, §4]. The genus symbols of Φ_n-lattices are computed in [10, Satz 2.57], though in practice we used a computer to construct the lattice and compute its symbol.

§2.5. Equivariant gluing

We note the following well-known lemma for later use.

Lemma 2.6. *If $A \oplus B \subseteq C$ is a primitive extension, then*

$$\det A \det B = [C : A \oplus B]^2 \cdot \det C$$

and

$$\det A \mid [C : A \oplus B] \cdot \det C.$$

Moreover, if p is a prime such that $p \nmid [C : A \oplus B]$, then

$$C \otimes \mathbb{Z}_p = (A \otimes \mathbb{Z}_p) \oplus (B \otimes \mathbb{Z}_p).$$

Let $a \in \mathrm{O}(A)$, $b \in \mathrm{O}(B)$, $c \in \mathrm{O}(C)$ be isometries. We call $(A, a) \oplus (B, b) \subseteq (C, c)$ an equivariant primitive extension if the restriction $c|_{A \oplus B} = a \oplus b$.

Lemma 2.7. *Let $(A, a) \oplus (B, b) \hookrightarrow (C, c)$ be an equivariant primitive extension with characteristic polynomials p_A, p_B. Then any prime dividing the index $[C : A \oplus B]$ divides the resultant $\mathrm{res}(p_A, p_B)$.*

Proof. Apply [15, Prop. 4.2] to $G = C/(A \oplus B)$. □

Lemma 2.8. *Let $(A, a) \oplus (B, b) \hookrightarrow (C, c)$ be an equivariant primitive extension. Suppose that the characteristic polynomial p_a of a is $\Phi_n(x)$. Then the glue $G = C/(A \oplus B)$ satisfies*

$$|G| \mid \mathrm{res}(\Phi_n, \mu),$$

where $\mu = \mu_b$ is the minimal polynomial of b.

Proof. Let G_A denote the orthogonal projection of G to $A^\vee/A$ and $\bar{a}$ the automorphism on G_A induced by a. Since $A^\vee$ and A are $\mathbb{Z}[\zeta_n]$-modules of rank 1, they are isomorphic to fractional ideals of $\mathbb{Z}[\zeta_n]$. Thus we have $G_A = \mathbb{Z}[\zeta_n]/I$, where I is the kernel of the map $\mathbb{Z}[\zeta_n] \mapsto \mathrm{End}\, G_A$ that sends the root of unity ζ_n to $\bar{a}$. This yields

$$\mu(\bar{a}) = 0, \quad \text{thus } \mu(\zeta_n) \in I$$

and

$$|G| = |G_A| = |\mathcal{O}_K/I| = N(I) \mid N(\mu(\zeta_n)) = \prod_{(k,n)=1} \mu(\zeta_n^k) = \mathrm{res}(\phi_n, \mu_b),$$

where $N(I)$ is the norm of the ideal I. □

The following lemma is elementary. For the convenience of the reader, we give a proof below.

Lemma 2.9. *If L is a lattice of rank 2 and $g \in O(L)$ is an isometry of spectral radius 0, then g is of finite order.*

Proof. By Kronecker's theorem, the characteristic polynomial of g is a product of cyclotomic polynomials. Moreover, it suffices to prove the claim for a power of g, so we can assume that the characteristic polynomial of g is $(x-1)^2$.

Let $v \in L$ be an eigenvector of g. If v is anisotropic, then we have $(\mathbb{Z}v)^\perp \neq \mathbb{Z}v$ and $(\mathbb{Z}v)^\perp$ consists of eigenvectors of g. Thus $g = \mathrm{id}$ and we are done.

If v is isotropic, we find $w \in L$ with $\langle w, v\rangle \neq 0$. Then $g(w) = av + bw$ for some $a, b \in \mathbb{Q}$. From $\langle w, v\rangle = \langle g(w), g(v)\rangle$ we infer $b = 1$. Finally, $\langle w, w\rangle = \langle g(w), g(w)\rangle$ yields $a = 0$. Thus $g(w) = w$ and the proof is complete. □

§3. Ruling out the factor F_{15}

The main aim of this section is to prove the following proposition.

Proposition 3.1. *Let f be an automorphism of an Enriques surface Y and let p_f be the minimal polynomial of the map $f^*\colon \mathrm{Num}(Y) \to \mathrm{Num}(Y)$. Then the modulo-2 reduction $(p_f(x) \bmod 2)$ is never divisible by the polynomial*

$$F_{15} = x^8 + x^7 + x^5 + x^4 + x^3 + x + 1,$$

i.e. by the modulo-2 reduction of the cyclotomic polynomial $\Phi_{15}(x) \in \mathbb{Z}[x]$.

Recall (see e.g. [4]) that p_f is a product of cyclotomic polynomials and at most one Salem factor. Since p_f is reciprocal, $(p_f(x) \bmod 2)$ is divisible by an irreducible factor of F_{15} if and only if it is divisible by the whole F_{15} (cf. [13]).

Proof of Proposition 3.1. Assume that $F_{15} \mid (p_f \bmod 2)$. Combined with [13, Rem. 2.4], this implies that

$$(p_M \bmod 2) = F_{15} \cdot F_1^2 \quad \text{and} \quad (F_{15} \cdot F_1^4) = (p_N \bmod 2). \tag{3.1}$$

By [13, Lems. 2.1 and 2.5], the characteristic polynomial p_N is a product of cyclotomic polynomials of degree at most 8. Computing modulo-2 reductions of all such cyclotomic polynomials, one infers that either $\Phi_{15} \mid p_N$ or $\Phi_{30} \mid p_N$. Replacing $\tilde{f}$ by a power coprime to 15 we can assume that p_N is a product of the Φ_k for $k \in \{1, 3, 5, 15\}$. Together with (3.1) this leaves us with

$$p_N = \Phi_{15} \cdot \Phi_1^4.$$

We consider the (primitive) f_N-invariant sublattices N_{15} and N_1 (see (2.4)). Since $\Phi_{15}(x)$ has no real roots, the signature of N_{15} is of the form $(2k, 2(4-k))$ with $k \in \{0, 1, 2, 3, 4\}$. Recall that N is of signature $(2, 10)$ and contains N_{15}. Thus the signature of N_{15} is either $(0, 8)$ or $(2, 6)$.

By Lemma 2.8 the glue between N_{15} and $N_{15}^{\perp}$ is trivial, i.e.

$$N_{15} \oplus N_{15}^{\perp} = N \in \mathrm{II}_{(2,10)} 2^{10}.$$

Let (ϵ_q, n_q) be the 2-adic genus symbol of N_{15} and (ϵ'_q, n'_q) the symbol of $N_{15}^{\perp}$. From Lemma 2.3 we infer that $10 = n_2 + n'_2$. Further, $n'_2 \leq \operatorname{rank} N_{15}^{\perp} = 4$ and $n_2 \leq \operatorname{rank} N_{15} = 8$. Thus we obtain $6 \leq n_2 \leq 8$. Since N_{15} is a Φ_{15}-lattice, we can calculate all Φ_{15}-lattices matching this condition. There is exactly one such lattice up to isometry:

$$N_{15} \cong E_8(-2) \in \mathrm{II}_{(0,8)} 2^8. \tag{3.2}$$

Using Lemma 2.3 once more, we calculate the genus symbol of $N_1 = N_{15}^{\perp}$ from those of N and N_{15} and see that

$$N_1 \cong U \oplus U(2) \in \mathrm{II}_{(2,2)} 2^2 \tag{3.3}$$

is the unique class in its genus. From (3.2), (3.3), and [20, Lem. 7.7] we infer that the spectral radius of f_M is 1 (i.e. f has trivial entropy). Thus p_M is not divisible by a Salem polynomial and must be a product of cyclotomic polynomials. A direct computation of modulo-2 reductions of all cyclotomic polynomials of degree at most 8 shows that either Φ_{30} or Φ_{15} divides p_M. By replacing $\tilde{f}$ with its iteration (i.e. by $\tilde{f}^2$ or $\tilde{f}^4$) we can assume that

$$p_M = \Phi_{15} \cdot \Phi_1^2.$$

We consider the equivariant orthogonal decomposition $M = M_{15}^{\perp} \oplus M_{15}$ into the rank-2 lattice $M_{15}^{\perp}$ and the rank-8 lattice M_{15} (see (2.4)). Being a Φ_{15}-lattice, M_{15} has signature $(2k, 2(4-k))$ for some k. But M is of signature $(1, 9)$, so M_{15} is definite and $f_M|M_{15}$ is of finite order. Since $M_{15}^{\perp}$ is of rank 2 and $f_M|M_{15}^{\perp}$ has spectral radius 0, it is of finite order (cf. Lemma 2.9). Thus a power of f is an automorphism of a complex Enriques surface of order 15. However, no such automorphisms exist (by [17, Prop. 4.5 and Cor. 4.7]; see also [16, Props. 1.1 and 3.14]). □

§4. The factor F_9

In this section we maintain the notation of previous sections and prove Theorems 1.1, 1.2. We assume that $f \in \mathrm{Aut}(Y)$ satisfies the condition

$$F_9 \mid (p_f \bmod 2).$$

After replacing $\tilde{f}$ by some power coprime to 3, we assume that f_N is of order 9. Since $F_9 F_1^2$ divides p_N, we can rule out $p_N = \Phi_9^2$. Furthermore, by [13, Rem. 2.4], we have $(p_M \bmod 2) \neq F_3^2 F_9$, which rules out $p_N = \Phi_1^2 \Phi_3^2 \Phi_9$. This leaves us with the two possibilities

$$p_N = \Phi_9 \Phi_1^6 \quad \text{or} \quad p_N = \Phi_9 \Phi_3 \Phi_1^4.$$

As usual we set $N_9 := \ker(\Phi_9(f_N))$ and denote by $N_9^\perp$ the orthogonal complement of N_9 in $N \in \mathrm{II}_{(2,10)}2^{10}$. By Lemma 2.8, $\det N_9 \mid 2^6 \operatorname{res}(\Phi_9, \Phi_3\Phi_1) = 2^6 \cdot 3^3$. Using the description of N_9 as a Φ_9-lattice, we enumerate the possibilities for N_9. This yields 4 cases and with Lemmas 2.3 and 2.4 we calculate the corresponding genus of $N_9^\perp$:

$$N_9 \in \mathrm{II}_{(0,6)}2^{-6}3^1 \quad \text{and} \quad N_9^\perp \in \mathrm{II}_{(2,4)}2^{-4}3^{-1}, \tag{4.1}$$

$$N_9 \in \mathrm{II}_{(0,6)}2^{-6}3^{-3} \quad \text{and} \quad N_9^\perp \in \mathrm{II}_{(2,4)}2^{-4}3^{3}, \tag{4.2}$$

$$N_9 \in \mathrm{II}_{(2,4)}2^{-6}3^{-1} \quad \text{and} \quad N_9^\perp \in \mathrm{II}_{(0,6)}2^{-4}3^{1}, \tag{4.3}$$

$$N_9 \in \mathrm{II}_{(2,4)}2^{-6}3^{3} \quad \text{and} \quad N_9^\perp \in \mathrm{II}_{(0,6)}2^{-4}3^{-3}. \tag{4.4}$$

We can rule out cases (4.3) and (4.4) since in each case the genus of $N_9^\perp$ consists of a single class (see Remark 2.2), which contains roots. We continue by determining the characteristic polynomial. If $p_N = \Phi_9\Phi_1^6$, then we must be in case (4.1) and $N_9^\perp = N_1$. Since the signature of N_1 is $(2,4)$, it contains the transcendental lattice. In particular, f is semi-symplectic. Choosing the covering K3 surface general enough, we may assume that N_1 is its transcendental lattice. This situation is analyzed in the next proposition.

Proposition 4.1. *Let Y be an Enriques surface such that its covering K3 surface X has transcendental lattice*

$$T(X) \cong U \oplus U(2) \oplus A_2(-2) \in \mathrm{II}_{(2,4)}2^{-4}3^{-1}$$

and satisfies the condition

$$N \cap \mathrm{NS}(X) \cong E_6(-2) \in \mathrm{II}_{(0,6)}2^{-6}3^{1}.$$

Then the image of $\mathrm{Aut}_s(Y) \to \mathrm{O}(\mathrm{Num}(Y)) \otimes \mathbb{F}_2$ generates a group isomorphic to $\mathcal{S}_5$.

Proof. The image of $\mathrm{Aut}_s(Y) \to \mathrm{O}(\mathrm{Num}(Y))$ can be calculated with Algorithm 6.6. It is generated by 64 explicit matrices (see [28]). Their modulo-2 reductions generate a group isomorphic to $\mathcal{S}_5$. The latter can be checked with the help of [8]. □

Since $\mathcal{S}_5$ does not contain an element of order 9, we are left with

$$p_N = \Phi_9\Phi_3\Phi_1^4.$$

We derive further restrictions.

Lemma 4.2. *Let $g \in \mathrm{O}(N)$ be an isometry with characteristic polynomial*

$$p_N = \Phi_9\Phi_3\Phi_1^4.$$

Then $N_3 = A_2(n)$ with $n \in \{\pm 2, \pm 6\}$.

Proof. One can easily see that A_2 is the principal Φ_3-lattice. By (2.6), $N_3 = A_2(n)$ for some $n \in \mathbb{Z}$. In the following we show that $n \in \{\pm 2, \pm 6\}$ by bounding the determinant of N_3. By Lemma 2.8 we have

$$\det N_3 \mid 2^2 \operatorname{res}(\Phi_3, \Phi_9\Phi_1) = 2^2 3^3.$$

By Lemma 2.5, the 2-adic symbol of N_3 is either 1^{-2} or 2^{-2}. The first one is not a direct summand of $N_9^\perp \otimes \mathbb{Z}_2$ (see Lemma 2.3), so we are left with the second. Hence $|n| \neq 1, 3$. □

Lemma 4.3. *Let $f \in \operatorname{Aut}(Y)$ be an automorphism of an Enriques surface such that $p_N = \Phi_9\Phi_3^1\Phi_1^4$ and (4.1) holds. Then $N_3 \cong A_2(-2)$ and $N_1 \cong U(2) \oplus U$.*

Proof. By assumption (4.1), $\det N_9^\perp = 2^4 3$, and Lemma 2.8 yields $\det N_3 \mid 2^2 9$. Thus, by Lemma 4.2, we are left with $N_3 = A_2(\pm 2)$. We see that $\det N_1 \mid 2^2 3^2$. Suppose that $N_3 = A_2(2) \in \mathrm{II}_{(2,0)} 2^{-2} 3^1$. There is a single genus of signature $(0, 4)$, 2-adic symbol $1^2 2^2$, and determinant dividing $2^2 3^2$, namely $N_1 \in \mathrm{II}_{(0,4)} 2^2 3^2$. It consists of a single class which has roots. Thus $N_3 \cong A_2(-2)$. We calculate the possible genus symbols of N_1 as $\mathrm{II}_{(2,2)} 2^2$ and $\mathrm{II}_{(2,2)} 2^2 9^{\pm 1}$. In the second case, N_1 and N_3 must be glued non-trivially over 3. This is impossible, as the only possibilities for the glue groups are $(N_3^\vee/N_3)_3$, whose discriminant form is non-degenerate, and $3(N_1^\vee/N_1)_3$, whose discriminant form is degenerate. Thus $N_1 \in \mathrm{II}_{(2,2)} 2^2$, which implies $N_1 \cong U(2) \oplus U$ since it is unique in this genus. □

If the transcendental lattice is $U \oplus U(2)$, then as before we see that the spectral radius of $\tilde{f}$ is 1. Since $M_1' = \ker(f_M - 1)^2$ is of rank 2 and $f_M|M_1'$ has spectral radius 0, it is of finite order (cf. Lemma 2.9) and $M_1 = M_1'$. Since $M_1^\perp$ is definite, f_M is of finite order there as well. Thus f is an automorphism of order 9 on a complex Enriques surface. However, no such automorphism exists (cf. [17]). We are left with case (4.2) and $p_N = \Phi_9\Phi_3\Phi_1^4$.

Lemma 4.4. *Let $f \in \operatorname{Aut}(Y)$ be an automorphism of an Enriques surface such that $p_N = \Phi_9\Phi_3^1\Phi_1^4$ and (4.2) holds. Then $N_3 \cong A_2(-6)$ and $N_1 \in \mathrm{II}_{(2,2)} 2^{-2} 9^1$. Moreover, $N_1^\perp \cong A_8(-2)$.*

Proof. Recall that $\zeta_9 \cdot x := g(x)$ defines a $\mathbb{Z}[\zeta_9]$-module structure on N_9 and its discriminant group. Thus $N_9^\vee/N_9 \cong \mathbb{Z}[\zeta_9]/I$ for some ideal I. Since we are in case (4.2), I is of norm $\det N_9 = 2^6 3^3$. There is only one such ideal, namely $2(1-\zeta_9)^3$ (since (2) is inert and (3) completely ramified in $\mathbb{Z}[\zeta_9]$). We see that the action of g on the 3-primary part $(N_9^\vee/N_9)_3 \cong \mathbb{Z}[\zeta_9]/(1-\zeta_9)^3$ has minimal polynomial $(x-1)^3 = x^3 - 1$. In particular, it has order 3. Thus the order of g on

$$(N_9^{\perp\vee}/N_9^\perp)_3 \cong (N_9^\vee/N_9)_3$$

is 3 as well. This is only possible if the order of g on $(N_3^\vee/N_3)_3 \cong \mathbb{Z}[\zeta_3]/(1-\zeta_3)^i$ is 3 (this group is a subquotient of $(N_3\oplus N_1)^\vee/(N_3\oplus N_1)$). This implies that $i \geq 2$, i.e. that $\det N_3$ is divisible by 9. From Lemma 4.2 we see that $N_3 = A_2(\pm 6)$. Now that we know the determinants of N_3 and $N_9^\perp$, we can estimate that of N_1 to be a divisor of $2^2 3^2$. Since N_3 has a 3-adic Jordan component of scale 9 and $N_9^\perp$ not, N_3 cannot be a direct summand of $N_9^\perp$. Thus N_3 and N_1 are glued non-trivially over 3. Consequently, the determinant of N_1 is $2^2 3^2$.

Suppose that $N_3 \cong A_2(6)$; then the signature of N_1 is $(0,4)$. There is only one genus with 2-adic genus symbol $1^2 2^2$, signature $(0,4)$, and determinant $2^2 3^2$: $\mathrm{II}_{(0,4)} 2^2 3^2$ which consists of a single class that has roots.

Suppose now that $N_3 \cong A_2(-6)$. Then we obtain three possibilities for the genus of N_1:

(1) $\mathrm{II}_{(2,2)} 2^2 3^{-2}$: There is only one possibility to glue N_3 and N_1 equivariantly over 3 (up to isomorphism). It results in $\mathrm{II}_{(2,4)} 2^{-4} 3^1 9^1$, which is not what we need.

(2) $\mathrm{II}_{(2,2)} 2^2 9^{-1}$: The full 3-adic symbol is $1^{-3} 9^{-1}$. But that has the wrong sign at scale 1.

(3) $\mathrm{II}_{(2,2)} 2^2 9^1$: Indeed, there is a unique possibility to glue N_3 and N_1 equivariantly over 3. It yields the correct result. □

Corollary 4.5. *If F_9 divides $(p_f \bmod 2)$, then $F_1^2 F_3 F_9 = (p_f \bmod 2)$.*

Proof. If we replace f by some power f^k with k coprime to 3, then the previous considerations apply and lead us to $p_N = \Phi_9 \Phi_3 \Phi_1^4$. By Lemma 4.4, $(N_3^\vee/N_3)_2 \cong \mathbb{F}_2^2$. Hence F_3 divides $p_N \mod 2$. Since $F_1^2(p_f \bmod 2) = p_N \mod 2 = F_9 F_3 F_1^4$, the corollary is proven for f^k. If k is a power of 2, then the characteristic polynomials of f_M and f_M^k coincide and we are done. If k is not a power of 2 then $(p_f \bmod 2)$ must be divisible by one of F_5, F_7, F_{15}, which is absurd. For instance, if $(p_f \bmod 2) = F_9 F_{15}$, then $(p_{f^5} \bmod 2) = F_9 F_3^2 \neq F_1^2 F_3 F_9$. □

After these preparations, we can prove the following lemma that we will need for the proof of Theorem 1.1(b).

Lemma 4.6. *If F_9 divides $(p_f \bmod 2)$, then f is semi-symplectic.*

Proof. By Corollary 4.5 we have $(p_N \mod 2) = F_9 F_3 F_1^4$. Thus the order of f_N is $2^k 9$ for some k. Set $L = \ker(f^{2^k} - 1) \in \mathrm{II}_{(2,2)} 2^{-2} 9^1$ and note that the transcendental lattice T is contained in L. We suppose that f is not semi-symplectic. Then the order of $\tilde{f}$ on $H^2(X, \Omega_X^2)$ is 2^l for some $l > 1$. After replacing $\tilde{f}$ by $\tilde{f}^{l-2}$ we may and will assume that $l = 2$, i.e. the order of $f_N|T$ is 4.

Set $L_4 = \ker \Phi_4(f_N|L)$. Suppose that $\operatorname{rank} L_4 = 4$, i.e. $L = L_4$. Then the discriminant group of L is a $\mathbb{Z}[\zeta_4]$ module. But since 3 is prime in $\mathbb{Z}[\zeta_4]$, there is no $\mathbb{Z}[\zeta_4]$ module isomorphic to $\mathbb{Z}/9\mathbb{Z}$. Thus $\operatorname{rank} L_4 = 2$ and $3 \nmid \det L_4$. Since $\det L_4 \mid 2^2 \operatorname{res}(\Phi_4, \Phi_2\Phi_1) = 2^4$ and L_4 is a Φ_4-lattice, either $L_4 \cong [2] \oplus [2]$ or $L_4 \cong [4] \oplus [4]$ holds. In both cases $L_4^\perp \subseteq L$ has 3-adic symbol 9^{-1} and determinant 36. The only such lattice is $[-2] \oplus [-18]$, which contains roots. □

At this point we have determined the Néron–Severi lattice of the K3 cover of a generic Enriques surface admitting an automorphism with F_9 dividing $p_f \mod 2$. This allows us to compute the semi-symplectic part of the automorphism group and locate f in there.

Proposition 4.7. *Let Y be an Enriques surface such that its K3 cover X satisfies the condition*

$$\mathrm{NS}(X) \cap N \cong A_8(-2) \in \mathrm{II}_{(0,8)} 2^8 9^1$$

and has transcendental lattice given by

$$N_1 \in \mathrm{II}_{(2,2)} 2^{-2} 9^1.$$

Then the image of $\mathrm{Aut}_s(Y) \to \mathrm{O}(\mathrm{Num}(Y) \otimes \mathbb{F}_2)$ generates a group isomorphic to $\mathcal{S}_9$.

In particular, the polynomials F_7 and F_9 do appear as factors of modulo-2 reductions of characteristic polynomials of isometries induced by some automorphisms of the Enriques surface Y.

Proof. The proof is a direct computation with the help of Algorithm 6.6 (cf. the proof of Proposition 4.1). The existence of the factors F_7 and F_9 follows since the symmetric group $\mathcal{S}_9$ has elements of order 7 and 9. □

Finally, we can give the proofs of the main results of this note.

Proof of Theorem 1.1.

(a) One can repeat verbatim the proof of [13, Thm. 1.2] to see that the modulo-2 reduction $(p_N(x) \bmod 2)$ is the product of some of the polynomials F_1, F_3, F_5, F_7, F_9, F_{15}. By (2.2) the same holds for $(p_f(x) \bmod 2)$. The claim follows from Proposition 3.1.

(b) The existence of the automorphisms with required properties follows from Proposition 4.7. Lemma 4.6 implies the second claim. □

Proof of Theorem 1.2. In view of [13, Thm. 1.1] it suffices to rule out the possibility that the order of the map f_N is one of the integers 90, 45, 72. Suppose to the contrary that the order of f_N is 90, 45, 72. Then F_9 divides ($p_N \mod 2$). Thus, by (2.2), F_9 divides ($p_f \mod 2$) and we can apply Corollary 4.5 to show that ($p_N \mod 2$) is divisible by $F_1^2 F_3 F_9$.

In particular, p_N (of degree 12) cannot be divisible by Φ_5 as well. This excludes orders 45 and 90. Suppose that the map f_N is of order 72. Then its characteristic polynomial p_N cannot be divisible by Φ_{72} and $\Phi_{24}\Phi_9$ for they have the wrong degree. Thus Φ_9 or Φ_{18} must divide p_N. In particular, F_9 divides ($p_N \mod 2$) and Corollary 4.5 implies that ($p_N \mod 2$) is divisible by $F_1^2 F_3 F_9$. This leaves us with $p_N = \Phi_8 \Phi_{3a} \Phi_{9b}$, where $a, b \in \{1, 2\}$. From Lemma 4.4 (applied to $\tilde{f}^8$) we know that $N_8 \in \mathrm{II}_{(2,2)} 2^{-2} 9^1$. This is impossible, as can be seen using the description of N_8 as a twist of the principal Φ_8-lattice. Indeed, 3 splits into two primes of degree 2 in $\mathbb{Z}[\zeta_8]$. □

§5. The factor F_7

The main aim of this section is to study Enriques surfaces Y with an automorphism $f \in \mathrm{Aut}(Y)$ such that

$$F_7 \mid (p_f \mod 2). \tag{5.1}$$

The existence of such surfaces follows from Proposition 4.7. Here we derive a lattice-theoretic constraint given by (5.1) and show that it indeed defines Enriques surfaces with the desired property. We maintain the notation of the previous sections. Recall (see (2.1)) that

$$N \in \mathrm{II}_{(2,10)} 2^{10}.$$

In the sequel we will need the following lemma.

Lemma 5.1. *Let $g \in \mathrm{O}(N)$ be an isometry of finite order such that its characteristic polynomial is the product $\Phi_7(x)\Phi_1(x)^6$. Then there are two possibilities for the genera of the lattices $N_7 := \ker \Phi_7(g)$ and $N_1 := \ker \Phi_1(g)$: either*

$$N_7 \in \mathrm{II}_{(2,4)} 2^6 7^{-1} \quad \text{and} \quad N_1 \in \mathrm{II}_{(0,6)} 2^4 7^1$$

or

$$N_7 \in \mathrm{II}_{(0,6)} 2^6 7^1 \quad \text{and} \quad N_1 \in \mathrm{II}_{(2,4)} 2^4 7^{-1}.$$

In either case the genus of N_1 contains a single class. In the first case the class of N_1 has roots.

Proof. Observe that we assumed g to be of finite order, so it is semi-simple, and $\mathrm{rank}(N_1) = 6$. Since $\mathrm{res}(\Phi_1, \Phi_7) = 7$, Lemma 2.8 implies that the index

$[N : N_7 \oplus N_1]$ divides 7. But in any case $7 = |\Phi_7(1)\Phi_7(-1)|$ divides $\det N_7$ (see (2.5) and (2.6)). Thus we obtain

$$[N : N_7 \oplus N_1] = 7.$$

Consequently, for all $p \neq 7$, $N \otimes \mathbb{Z}_p = (N_7 \otimes \mathbb{Z}_p) \oplus (N_1 \otimes \mathbb{Z}_p)$, and in particular for $p = 2$. Using the description of N_7 as a twist of the principal Φ_7-lattice we compute the two possibilities for the genus of N_7 (see Remark 2.2).

It remains to determine the genus of N_1. Since we have

$$N \otimes \mathbb{Z}_2 = (N_7 \otimes \mathbb{Z}_2) \oplus (N_1 \otimes \mathbb{Z}_2),$$

the 2-adic symbol of N_1 must be 2^4. To compute the 7-adic symbol note that $N \otimes \mathbb{Z}_7$ is unimodular, thus Lemma 2.4 applies. As (-1) is a non-square in $\mathbb{Z}_7$ this means that the signs ϵ_7 of the 7-modular Jordan constituents of N_7 and N_1 must be different. The claim that N_1 is unique in its genus in the first case is checked with a computer algebra system (see Remark 2.2). In the second case, N_1 is indefinite and we can use [6, Thm. 15.19]. □

Recall that X (resp. $\tilde{f} \in \mathrm{Aut}(X)$) stands for the K3 cover of an Enriques surface Y (resp. for a lift of an automorphism $f \in \mathrm{Aut}(Y)$).

Proposition 5.2. *Let Y be an Enriques surface with automorphism $f \in \mathrm{Aut}(Y)$ such that (5.1) holds. Then $\mathrm{NS}(X)$ contains a primitive $\tilde{f}^*$-invariant sublattice which belongs to the genus $\mathrm{II}_{(1,15)}2^4 7^1$ and $N \cap \mathrm{NS}(X)$ contains the $\tilde{f}^*$-invariant sublattice $A_6(-2) \cong N_7 \in \mathrm{II}_{(0,6)}2^6 7^1$ primitively.*

Proof. Since F_7 divides p_f, (2.2) implies that the characteristic polynomial p_N is divisible by the cyclotomic polynomial Φ_7. Moreover, after replacing f by f^k with $k \in \mathbb{N}$ coprime to 7, we may assume that

$$p_N = \Phi_7(x)\Phi_1(x)^6.$$

Now we can apply Lemma 5.1. The first case is impossible as then N_1 is contained in $\mathrm{NS}(X) \cap N$ and contains roots (see (2.3)). Thus we are left with the second case. Since $N_1 \subseteq N$ is of signature $(2,4)$ it must contain the transcendental lattice (and f is semi-symplectic). Thus the orthogonal complement of N_1 in $H^2(X, \mathbb{Z})$ is the sought for $\tilde{f}^*$-invariant sublattice of $\mathrm{NS}(X)$. □

Finally, we apply Algorithm 6.6 to check that the condition of Proposition 5.2 indeed gives Enriques surfaces such that (5.1) holds.

Proposition 5.3. *If the K3 cover X of an Enriques surface Y satisfies the conditions*

(a) $\mathrm{NS}(X) \in \mathrm{II}_{(1,15)}2^4 7^1$ *and*
(b) $N \cap \mathrm{NS}(X) \cong A_6(-2) \in \mathrm{II}_{(0,6)}2^6 7^1$,

then the image of $\mathrm{Aut}_s(Y) \to \mathrm{O}(\mathrm{Num}(Y)) \otimes \mathbb{F}_2$ *generates a group isomorphic to* $\mathcal{S}_7$*. In particular, the Enriques surface* Y *admits an automorphism* $f \in \mathrm{Aut}(Y)$ *such that the modulo-2 reduction* $(p_f(x) \bmod 2)$ *is divisible by the polynomial* F_7*.*

Proof. Apply Algorithm 6.6 and [8] as in the proof of Proposition 4.1. □

§6. An algorithm to calculate generators

In this section we present an algorithm to calculate a finite generating set of the image of the natural homomorphism from the automorphism group of an Enriques surface to the orthogonal group of the numerical Néron–Severi lattice of the Enriques surface. Our algorithm is based on Borcherds' method [1, 2] with the result in [3].

§6.1. Borcherds' method

We use the notation and terminologies in [3]. In particular, we denote by Y an Enriques surface, $\pi\colon X \to Y$ the universal covering of Y, and S_X and S_Y the numerical Néron–Severi lattices of X and of Y, respectively (that is, $S_X = \mathrm{NS}(X)$ and $S_Y = \mathrm{Num}(Y)$ in the notation of previous sections.) Let $\mathcal{P}_X$ (resp. $\mathcal{P}_Y$) be the positive cone of $S_X \otimes \mathbb{R}$ (resp. $S_Y \otimes \mathbb{R}$) containing an ample class. Let N_X (resp. N_Y) be the cone consisting of all $x \in \mathcal{P}_X$ (resp. all $x \in \mathcal{P}_Y$) such that $\langle x, [\Gamma]\rangle \geq 0$ for any curve Γ on X (resp. on Y). We let the orthogonal group $\mathrm{O}(L)$ of a $\mathbb{Z}$-lattice L act on the lattice from the *right*. Suppose that L is even. A vector $r \in L$ is a (-2)-*vector* if $\langle r, r\rangle = -2$. Let $W(L)$ denote the subgroup of $\mathrm{O}(L)$ generated by the reflections $s_r\colon x \mapsto x + \langle x, r\rangle r$ with respect to (-2)-vectors r of L. For a subset A of $L \otimes \mathbb{R}$, we denote by A^g the image of A under the action of $g \in \mathrm{O}(L)$ (*not* the fixed locus of g in A), and put

$$\mathrm{O}(L, A) := \{g \in \mathrm{O}(L) \mid A = A^g\}.$$

We have natural homomorphisms

$$\mathrm{Aut}(X) \to \mathrm{O}(S_X, \mathcal{P}_X), \quad \mathrm{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y).$$

We denote by $\mathrm{aut}(X)$ and $\mathrm{aut}(Y)$ the images of these homomorphisms. Recall that $\mathrm{Aut}_s(Y)$ consists of the semi-symplectic automorphisms, i.e. those that act trivially on $H^0(Y, \omega_Y^{\otimes 2})$. We denote by $\mathrm{Aut}_s(X)$ the subgroup consisting of those automorphisms acting as ± 1 on $H^0(X, \Omega_X^2) \cong H^{2,0}(X)$. The subgroups

$\mathrm{aut}_s(X) \subseteq \mathrm{aut}(X)$ and $\mathrm{aut}_s(Y) \subseteq \mathrm{aut}(Y)$ are defined as the respective images. Our goal is to calculate a finite generating set of $\mathrm{aut}_s(Y)$.

Remark 6.1. We note that $\mathrm{Aut}_s(Y)$ is of finite index in $\mathrm{Aut}(Y)$. This index is 1 if the only isometries of T_X that preserve $H^{2,0}(X) \subset T_X \otimes \mathbb{C}$ are ± 1, where T_X is the transcendental lattice of X.

We have the primitive embedding

$$\pi^* \colon S_Y(2) \hookrightarrow S_X,$$

which induces $\mathcal{P}_Y \hookrightarrow \mathcal{P}_X$. We regard S_Y as a submodule of S_X, and $\mathcal{P}_Y$ as a subspace of $\mathcal{P}_X$ by π^*. Then we have

$$N_Y = N_X \cap \mathcal{P}_Y. \tag{6.1}$$

If $\alpha \in S_Y$ is ample on Y, then $\pi^*(\alpha)$ is ample on X. Hence we have $N_Y^\circ = N_X^\circ \cap \mathcal{P}_Y$, where N_Y° and N_X° are the interiors of N_Y and N_X, respectively. Let Q denote the orthogonal complement of the sublattice $S_Y(2)$ in S_X. Since Q is negative definite, the group $\mathrm{O}(Q)$ is finite. We consider the following assumptions for an element g of $\mathrm{O}(S_Y, \mathcal{P}_Y)$:

- (i) There exists an isometry $h \in \mathrm{O}(Q)$ such that the action of $g \oplus h$ on $S_Y(2) \oplus Q$ preserves the overlattice S_X of $S_Y(2) \oplus Q$ and the action of $(g \oplus h)|S_X$ on the discriminant group $S_X^\vee/S_X$ of S_X is ± 1.
- (ii-a) There exists an ample class $\alpha \in S_Y$ of Y such that there exist no vectors $r \in S_X$ with $\langle r, r\rangle = -2$ satisfying $\langle \pi^*(\alpha), r\rangle > 0$ and $\langle \pi^*(\alpha^g), r\rangle < 0$.
- (ii-b) For an arbitrary ample class $\alpha \in S_Y$ of Y, there exist no vectors $r \in S_X$ with $\langle r, r\rangle = -2$ satisfying $\langle \pi^*(\alpha), r\rangle > 0$ and $\langle \pi^*(\alpha^g), r\rangle < 0$.

Proposition 6.2. *Let g be an element of* $\mathrm{O}(S_Y, \mathcal{P}_Y)$. *Then g is in* $\mathrm{aut}_s(Y)$ *if* (i) *and* (ii-a) *hold. If g is in* $\mathrm{aut}_s(Y)$, *then* (i) *and* (ii-b) *hold.*

Proof. An element g of $\mathrm{O}(S_Y, \mathcal{P}_Y)$ is in $\mathrm{aut}_s(Y)$ if and only if there exists an element $\tilde{g} \in \mathrm{aut}_s(X)$ that preserves $S_Y \subset S_X$ and satisfies $\tilde{g}|S_Y = g$. By the Torelli theorem, we see that an element $\tilde{g}'$ of $\mathrm{O}(S_X, \mathcal{P}_X)$ is in $\mathrm{aut}_s(X)$ if and only if the action of $\tilde{g}'$ on $S_X^\vee/S_X$ is ± 1 and $\tilde{g}'$ preserves N_X. Since N_X is a standard fundamental domain of the action of $W(S_X)$ on $\mathcal{P}_X$ (see [3, Exa. 1.5]), we have

$$N_X^\circ \cap N_X^h \neq \emptyset \implies N_X = N_X^h$$

for any $h \in \mathrm{O}(S_X, \mathcal{P}_X)$. Therefore, both (ii-a) and (ii-b) are equivalent to the condition that $N_X^{\tilde{g}} = N_X$ for any $\tilde{g} \in \mathrm{O}(S_X, \mathcal{P}_X)$ satisfying $S_Y^{\tilde{g}} = S_Y$ and $\tilde{g}|S_Y = g$. □

Suppose that we have a primitive embedding

$$\iota_X\colon S_X \hookrightarrow L_{26},$$

where L_{26} is an even unimodular hyperbolic lattice of rank 26, which is unique up to isomorphism. (More standard notation is $\mathrm{II}_{1,25}$.) Composing π^* and ι_X, we obtain a primitive embedding

$$\iota_Y\colon S_Y(2) \hookrightarrow L_{26}.$$

Let $\mathcal{P}_{26}$ be the positive cone of L_{26} into which $\mathcal{P}_Y$ is mapped. We regard S_Y as a primitive submodule of L_{26}, and $\mathcal{P}_Y$ as a subspace of $\mathcal{P}_{26}$ by ι_Y. Recall from [3] that a Conway chamber is a standard fundamental domain of the action of $W(L_{26})$ on $\mathcal{P}_{26}$. The tessellation of $\mathcal{P}_{26}$ by Conway chambers induces a tessellation of $\mathcal{P}_Y$ by induced chambers.

Proposition 6.3. *The action of* $\mathrm{aut}_s(Y)$ *on* $\mathcal{P}_Y$ *preserves the tessellation of* $\mathcal{P}_Y$ *by induced chambers.*

Proof. Let g be an element of $\mathrm{aut}_s(Y)$. By the proof of Proposition 6.2, there exists an isometry $\tilde{g} \in \mathrm{O}(S_X, \mathcal{P}_X)$ such that $S_Y^{\tilde{g}} = S_Y$, $\tilde{g}|S_Y = g$ and the action of $\tilde{g}$ on $S_X^\vee/S_X$ is ± 1. By the last condition, we see that $\tilde{g}$ further extends to an isometry $g_{26} \in \mathrm{O}(L_{26}, \mathcal{P}_{26})$. Since the action of g_{26} on $\mathcal{P}_{26}$ preserves the tessellation by Conway chambers, the action of g on $\mathcal{P}_Y$ preserves the tessellation by induced chambers. □

Let L_{10} be an even unimodular hyperbolic lattice of rank 10, which is unique up to isomorphism. In [3] we classified all primitive embeddings of $S_Y(2) \cong L_{10}(2)$ into L_{26}, and studied the tessellation of $\mathcal{P}_Y$ by induced chambers. It turns out that, up to the actions of $\mathrm{O}(L_{10})$ and $\mathrm{O}(L_{26})$, there exist exactly 17 primitive embeddings $L_{10}(2) \hookrightarrow L_{26}$, and except for one primitive embedding named "infty", the associated tessellation of $\mathcal{P}_Y$ by induced chambers has the following properties:

- Each induced chamber D is bounded by a finite number of walls, and each wall is defined by a (-2)-vector.
- If a (-2)-vector r defines a wall $w = D \cap (r)^\perp$ of an induced chamber D, then the reflection $s_r\colon x \mapsto x + \langle x, r\rangle r$ into the mirror $(r)^\perp$ maps D to the induced chamber adjacent to D across the wall w.

In particular, the tessellation of $\mathcal{P}_Y$ by induced chambers is *simple* in the sense of [27].

§6.2. Main algorithm

Suppose that the primitive embedding ι_Y is not of type "infty". Suppose also that we have calculated the walls of an induced chamber $D_0 \subset \mathcal{P}_Y$ contained in N_Y.

Before starting the main algorithm, we calculate the finite groups $\mathrm{O}(Q)$ and $\mathrm{O}(S_Y, D_0)$. We also fix an ample class α that is contained in the interior of D_0. In the following, an induced chamber D is expressed as an element $\tau_D \in \mathrm{O}(S_Y, \mathcal{P}_Y)$ such that $D = {D_0}^{\tau_D}$. Note that τ_D is uniquely determined by D up to left multiplications of elements of $\mathrm{O}(S_Y, D_0)$.

Then we have the following auxiliary algorithms.

Algorithm 6.4. Given an induced chamber D, we can determine whether $D \subset N_Y$ or not. Indeed, by (6.1), we have $D \subset N_Y$ if and only if there exist no (-2)-vectors r of S_X such that $\langle \pi^*(\alpha), r\rangle > 0$ and $\langle \pi^*(\alpha^{\tau_D}), r\rangle < 0$. The set of such (-2)-vectors can be calculated by the algorithm in [25, Sect. 3.3].

Suppose that $D \subset N_Y$. A wall $D \cap (r)^\perp$ of D is said to be *inner* if the induced chamber D^{s_r} adjacent to D across $D \cap (r)^\perp$ is contained in N_Y. Otherwise, we say that $D \cap (r)^\perp$ is *outer*.

Algorithm 6.5.
Input: An embedding $S_Y(2) \hookrightarrow S_X \hookrightarrow L_{26}$, the groups $\mathrm{O}(S_Y, D_0)$, $\mathrm{O}(Q)$, and two induced chambers D, $D' \subset N_Y$ represented by τ_D, $\tau_{D'}$.

Output: The set $\{\gamma \in \mathrm{aut}_s(Y) \mid D' = D^\gamma\}$.

1: Compute $\mathrm{Isom}(D, D') := \tau_D^{-1}\mathrm{O}(S_Y, D_0)\tau_{D'}$.
   This is the set of all isometries $g \in \mathrm{O}(S_Y, \mathcal{P}_Y)$ that satisfy $D' = D^g$.
2: Initialize $\mathcal{I} := \{\}$.
3: **for** $g \in \mathrm{Isom}(D, D')$ **do**
      Use $\mathrm{O}(Q)$ and Proposition 6.2 to check
4:    **if** $g \in \mathrm{aut}_s(Y)$ **then**
5:       add g to $\mathcal{I}$.
6: Return $\mathcal{I}$.

Note that since both D and D' are contained in N_Y, condition (ii-a) of Proposition 6.2 is always satisfied in line 4. For $D = D'$, Algorithm 6.5 calculates the group

$$\mathrm{aut}_s(Y, D) := \mathrm{O}(S_Y, D) \cap \mathrm{aut}_s(Y).$$

Two induced chambers D and D' in N_Y are said to be $\mathrm{aut}_s(Y)$-*equivalent* if there exists an element $\gamma \in \mathrm{aut}_s(Y)$ such that $D' = D^\gamma$.

Algorithm 6.6.
Input: An embedding $S_Y(2) \hookrightarrow S_X \hookrightarrow L_{26}$ and an induced chamber $D_0 \subset N_Y$.

Output: A list $\mathcal{R}$ of representatives of $\mathrm{aut}_s(Y)$-equivalence classes of induced chambers contained in N_Y and a generating set $\mathcal{G}$ of $\mathrm{aut}_s(Y)$.

Initialize $\mathcal{R} := [D_0]$, $\mathcal{G} := \{\}$, and $i := 0$.
while $i \leq |\mathcal{R}|$ **do**
  Let D_i be the $(i+1)$st element of $\mathcal{R}$.
  Replace $\mathcal{G}$ by $\mathcal{G} \cup \mathrm{aut}_s(Y, D_i)$.
  Let $\mathcal{W}$ be the set of walls of D_i.
  Compute orbit representatives of $\mathcal{W}$ under the action of $\mathrm{aut}_s(Y, D_i)$.
  for each representative wall w of $\mathcal{W}/\,\mathrm{aut}_s(Y, D_i)$ **do**
    Let r be the (-2)-vector of S_Y defining the wall $w = D \cap (r)^\perp$.
    Let s_r be the reflection $x \mapsto x + \langle x, r\rangle r$.
    Let $D_w = D_i^{s_r}$ be the induced chamber adjacent to D_i across w.
    Set $\tau_{D_w} := \tau_{D_i} s_r$.
    if $D_w \not\subset N_Y$ **then**
      Continue with the next representative wall.
    Set $f :=$ true.
    for each $D \in \mathcal{R}$ **do**
      if D is $\mathrm{aut}_s(Y)$-equivalent to D_w **then**
        Let $\gamma \in \mathrm{aut}_s(Y)$ be an element such that $D_w = D^\gamma$.
        Add γ to $\mathcal{G}$.
        Replace f by false.
        Break the for loop.
    if $f =$ true **then**
      Add D_w to $\mathcal{R}$.
  Increment i.
Return $\mathcal{R}$ and $\mathcal{G}$.

Proof. This algorithm is proved in the same way as [26, Prop. 6.3]. □

Remark 6.7. The termination of Algorithm 6.6 follows, in the same way as in [26, proof of Thm. 3.7], from the fact that the subgroup of $\mathrm{O}(S_Y, \mathcal{P}_Y)$ consisting of isometries g that extends to an isometry of $H^2(X, \mathbb{Z})$ preserving the sublattice $S_X \subset H^2(X, \mathbb{Z})$ is of finite index, and its membership can be decided by the action of g on the discriminant form of $S_Y(2)$. This algorithm provides us with an effective version of the cone theorem for Enriques surfaces ([18, 29]).

§6.3. Examples

The details of the following computations are available at [28].

6.3.1. The Enriques surface in Proposition 5.3. The Picard number of the covering $K3$ surface is 16, and the orthogonal complement Q of $S_Y(2)$ in S_X is $A_6(-2)$. Therefore $\mathrm{O}(Q)$ is of order 10080. The ADE-type of (-2)-vectors in the orthogonal complement P of $S_Y(2)$ in L_{26} is $8A_1 + 2D_4$. Hence the embedding ι_Y is of type 40B in the notation of [3]. The number $|\mathcal{R}|$ of $\mathrm{aut}_s(Y)$-equivalence classes of induced chambers in N_Y is 2. Let D_0 and D_1 be the representatives of $\mathrm{aut}_s(Y)$-equivalence classes. For $i = 0, 1$, the group $\mathrm{aut}_s(Y, D_i)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the 40 walls of D_i are decomposed into 10 orbits under the action of $\mathrm{aut}_s(Y, D_i)$. Among the 40 walls, exactly $3 \times 4 = 12$ walls are outer walls. For each inner wall w, the two induced chambers containing w are not $\mathrm{aut}_s(Y)$-equivalent, that is, one is $\mathrm{aut}_s(Y)$-equivalent to D_0 and the other is $\mathrm{aut}_s(Y)$-equivalent to D_1.

6.3.2. The Enriques surface in Proposition 4.1. The Picard number of the covering $K3$ surface is 16, and the orthogonal complement Q of $S_Y(2)$ in S_X is $E_6(-2)$. Therefore $\mathrm{O}(Q)$ is of order 103680. The ADE-type of (-2)-vectors in the orthogonal complement P of $S_Y(2)$ in L_{26} is D_4+D_5. Hence the embedding ι_Y is of type 20A, which means that D_0 is bounded by walls defined by (-2)-vectors that form the dual graph of Nikulin–Kōndo type V [12]. The number $|\mathcal{R}|$ of $\mathrm{aut}_s(Y)$-equivalence classes of induced chambers in N_Y is 20. They are decomposed into the following three types:

Type	$\lvert\mathrm{aut}_s(Y, D)\rvert$	Outer walls	Inner walls	Number
a	1	1×7	1×13	2
b	1	1×5	1×15	6
c	2	$1 \times 2 + 2 \times 2$	$1 \times 2 + 2 \times 6$	12

For example, there exist 12 $\mathrm{aut}_s(Y)$-equivalence classes of type c. If D is an induced chamber of type c, then $\mathrm{aut}_s(Y, D)$ is $\mathbb{Z}/2\mathbb{Z}$, and D has 6 outer walls and 14 inner walls. Under the action of $\mathrm{aut}_s(Y, D)$, the 6 outer walls are decomposed into 4 orbits of sizes 1, 1, 2, 2, and the 14 inner walls are decomposed into 8 orbits of sizes 1, 1, $2, \dots, 2$.

6.3.3. The Enriques surface in Proposition 4.7. The Picard number of the covering $K3$ surface is 18, and the orthogonal complement Q of $S_Y(2)$ in S_X is $A_8(-2)$. Therefore $\mathrm{O}(Q)$ is of order 725760. The ADE-type of (-2)-vectors in the orthogonal complement P of $S_Y(2)$ in L_{26} is A_3+A_4. Hence the embedding ι_Y is of type 20D, which means that D_0 is bounded by walls defined by (-2)-vectors that form the dual graph of Nikulin–Kōndo type VII [12]. The number $|\mathcal{R}|$ of $\mathrm{aut}_s(Y)$-equivalence classes of induced chambers in N_Y is 1. The group $\mathrm{aut}_s(Y, D_0)$ is isomorphic to $\mathfrak{S}_3$, and the 20 walls of D_0 are decomposed into 6 orbits, each of which consists of

$$6 \text{ outer}, \;\; 3 \text{ outer}, \;\; 3 \text{ outer}, \qquad 3 \text{ inner}, \;\; 3 \text{ inner}, \;\; 2 \text{ inner}.$$

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