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Trace- and Improved Data-Processing Inequalities for von Neumann Algebras

by

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Abstract

We prove a version of the data-processing inequality for the relative entropy for general von Neumann algebras with an explicit lower bound involving the measured relative entropy. The inequality, which generalizes previous work by Sutter et al. on finite-dimensional density matrices, yields a bound for how well a quantum state can be recovered after it has been passed through a channel. Some natural applications of our results are in quantum field theory where the von Neumann algebras are known to be of type III. Along the way we generalize various multi-trace inequalities to general von Neumann algebras.

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§1. Introduction

The relative entropy $S(\rho|\sigma) = \mathrm{Tr}(\rho \ln \rho - \rho \ln \sigma)$ is an important operationally defined measure for the distinguishability of two statistical operators ρ, σ. A fundamental property of S is that

$$S(\rho|\sigma) - S(T(\rho)|T(\sigma)) \geq 0 \tag{1}$$

for a quantum channel T, i.e. a completely positive linear trace preserving map. (In the body of the paper, we use the slightly different notation $\widetilde{T}$ for the action of a channel on a density matrix (Schrödinger picture), while T denotes the dual action (Heisenberg picture) of the channel on the observables.) The above difference

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represents the loss of distinguishability between σ, ρ if these are passed through the channel T.

An important general question that can be abstracted from concrete settings such as quantum communication or quantum error correction is to what extent the action of a quantum channel can be reversed, i.e. to what extent it may be possible to recover ρ from $T(\rho)$. It was understood a long time ago by Petz that the question of recoverability is intimately linked to the case of saturation of the data-processing inequality (DPI) (1); see e.g. [32]. As was understood by [18] – and has subsequently been generalized in various ways by [29, 39, 18, 9, 12, 25, 40, 46] – explicit lower bounds in the DPI or related information-theoretic inequalities can provide information on how well a channel may be reversed if the inequality is, e.g., nearly saturated.

Very interesting results in this direction have been obtained by Sutter et al. [39] and Junge at al. [29]. Both results consider an explicit recovery channel, and show that the recovered state is close to the original state ρ in a suitable information-theoretic measure, provided the difference in the DPI is also small. The recovery channel $\alpha_{\sigma,T}$ is called "explicit" because it is given by a concrete expression involving only the reference state σ and T (not the state ρ that is to be recovered), and always perfectly recovers σ, i.e. $\alpha_{\sigma,T}(T(\sigma)) = \sigma$. In fact, it is closely related – though not precisely equal – to the channel originally proposed by Petz [34, 35, 37, 32].

The above-mentioned works (other than [34, 35, 37, 32]) establish their results only for very special von Neumann algebras – for example [39] considers a finite-dimensional type I algebra (finite-dimensional Hilbert space), whereas [29] deals with possibly infinite-dimensional type I algebras. While this is well motivated by applications in quantum computing, there are cases of interest when the algebras are not of this type. A notable example is quantum-field-theoretic applications related to the "quantum null energy condition" (see e.g. [14]) where the algebras are of type III [11, 20]. With this application in mind we proved in [17] a generalization of [29] in the case when the channel T corresponds to an inclusion of general von Neumann algebras. This result has been generalized to arbitrary 2-positive channels T in [16], where the following improved DPI has been demonstrated:

$$S(\rho|\sigma) - S(T(\rho)|T(\sigma)) \geq \frac{1-s}{s} \int_{\mathbb{R}} \mathrm{d}t \, \beta_0(t) D_s(\alpha^t_{\sigma,T}(T(\rho))|\rho). \tag{2}$$

Here, $s \in [1/2, 1)$ and D_s are the so-called "sandwiched Rényi entropies" [31, 48], which for $s = 1/2$ become the negative log squared fidelity. The term $\beta_0(t)\,\mathrm{d}t$ is a certain explicit probability density and $\alpha^t_{\eta,T}$ is an explicit 1-parameter family of recovery channels that is a disintegration of $\alpha_{\eta,T}$ in the sense $\int \mathrm{d}t\, \beta_0(t)\alpha^t_{\eta,T} = \alpha_{\eta,T}$. Using the convexity of D_s and Jensen's inequality, the bound implies

$$S(\rho|\sigma) - S(T(\rho)|T(\sigma)) \geq \frac{1-s}{s} D_s(\alpha_{\sigma,T}(T(\rho))|\rho). \tag{3}$$

A qualitatively similar result has been proved for general von Neumann algebras by Junge and LaRacuente [28].[1] In their result, the sandwiched Rényi entropies are now replaced by some other information-theoretic quantity with an operational meaning. Both [16, 28] lead to the same inequality for $s = 1/2$. For type I algebras and $s = 1/2$, (2) is the result by [29], but the relation for general s is unclear to the author. We also mention recent results by Gao and Wilde [19] of a roughly similar flavor but different emphasis, which apply to von Neumann algebras with a trace though not type III.

In the present paper, we provide a generalization of [39] to arbitrary (sigma-finite) von Neumann algebras. This version of the improved DPI is qualitatively similar to (3). The definition of the recovery channel is in fact identical to that in (3), but we have yet another information-theoretic quantity on the right-hand side, namely (Theorem 2)

$$S(\rho|\sigma) - S(T(\rho)|T(\sigma)) \geq S_{\rm meas}(\alpha_{\sigma,T}(T(\rho))|\rho). \tag{4}$$

Here, $S_{\rm meas}$ is the "measured relative entropy", defined as the maximum possible value of the relative entropy restricted to a commutative von Neumann subalgebra. We show below (Proposition 1) that for $s = 1/2$, this inequality is sharper than (3) – though not in general the inequality (2) with the integral outside – for all ρ, σ. A conceptual advantage of (4) over both (2) and (3) is that it is saturated in the commutative case, as noted already by [39]. So in this respect (4) is sharp unlike its predecessors.

Our proof technique is similar in several respects to that in [39] and related antecedents such as [29] in that we also use interpolation arguments for L_p-norms. However, there are also some key differences requiring technical modifications: for instance, the operators $\ln\rho$ or $\ln\sigma$ no longer exist for general von Neumann algebras or the use of ordinary L_p (Schatten)-norms is prohibited since a general von Neumann algebra does not have a trace. As in our previous papers [17, 16] – referred to as papers I, II – our solution to the first problem is to work entirely with

[1] After a preprint of the present paper was released, the authors of [28] posted a substantially changed, new version of their original preprint. Their new version contained results similar to ours, such as trace inequalities and Theorem 2. We emphasize that their methods and results use rather different techniques and were found independently from ours.

Araki's relative modular operator, the log of which can roughly be viewed as a *difference* between $\ln \rho$ and $\ln \sigma$. Likewise, as in [17, 16], our solution to the second problem is to work with the Araki–Masuda non-commutative L_p-norms [8] which are very closely related to the sandwiched relative Rényi entropies.[2] For these norms, we require a complex interpolation theory, see Lemma 1, which generalizes a result in [17]. This result is then applied to a specially constructed analytic family of vectors and combined with certain cutoff techniques for appropriately extended domains of analyticity in a similar way to [17]. However, in [17, 16], such cutoff techniques were needed to control the limit of the Araki–Masuda norms as $p \to 2$, whereas in the present paper, it is the limit $p \to \infty$ which is relevant. The regularization is necessary here to apply the powerful technique of bounded perturbations of normal states of a von Neumann algebra, and a (somewhat modified) version of the Lie–Trotter product formula for von Neumann algebras [4]. These ideas go beyond [17, 16] and also yield various new "trace" inequalities for von Neumann algebras which could be of independent interest.

This paper is organized as follows. In Section 2 we review some prerequisite notions from the theory of von Neumann algebras. In Section 3 we establish an interpolation theorem for the Araki–Masuda L_p-norms, which we apply in Section 4 to obtain generalizations of various known multi-trace inequalities to von Neumann algebras. In Section 5 we establish our main result, Theorem 2. The definition of the L_p-norm and a summary of some of its properties relevant for this paper is relegated to the appendix.

§2. Von Neumann algebras and modular theory

Let $\mathcal{A} = M_n(\mathbb{C})$. The fundamental representation of this algebra is on $\mathbb{C}^n$, but one can also work in the "standard" Hilbert space ($\mathscr{H} \simeq M_n(\mathbb{C}) \simeq \mathbb{C}^n \otimes \mathbb{C}^n$). Vectors $|\zeta\rangle$ in $\mathscr{H}$ are thus identified with matrices $\zeta \in M_n(\mathbb{C})$. The space $\mathscr{H} \simeq M_n(\mathbb{C})$ is both a left and right module for $\mathcal{A}$,

$$l(a)|\zeta\rangle = |a\zeta\rangle, \quad r(b)|\zeta\rangle = |\zeta b\rangle, \tag{5}$$

and the inner product on $\mathscr{H}$ is the Hilbert–Schmidt inner product $\langle\zeta_1|\zeta_2\rangle = \mathrm{Tr}(\zeta_1^*\zeta_2)$. A mixed state, represented by a density matrix ω, gives rise to a linear functional on $\mathcal{A}$ by

$$\omega(a) = \mathrm{Tr}(\omega a), \tag{6}$$

[2] [28] use a somewhat different approach to L_p-norms to circumvent the absence of a tracial state in the general von Neumann algebra setting. Their approach appears to us less natural for the purposes of this paper.

where the functional and the state are denoted by the same symbol. These linear functionals are alternatively characterized by the properties $\omega(a^*a) \geq 0$, $\omega(1) = 1$.

A (σ-finite) von Neumann algebra in standard form $\mathcal{M}$ is an ultra-weakly closed linear subspace of the bounded operators on a Hilbert space $\mathscr{H}$ such that the following properties hold, and are assumed throughout this paper: $\mathcal{M}$ should contain 1, be closed under products and the $*$-operation, and it should have a cyclic and separating vector $|\psi\rangle \in \mathscr{H}$. Cyclic and separating means that $\mathcal{M}|\psi\rangle$ is dense in $\mathscr{H}$ and $m|\psi\rangle = 0$ implies $m = 0$. In the matrix example, ψ should therefore be invertible. The set of ultra-weakly continuous positive normalized linear functionals (thus satisfying $\omega(a^*a) \geq 0$, $\omega(1) = 1$) is called $\mathscr{S}(\mathcal{M})$. For a detailed account of von Neumann algebras see [41, 42, 43].

Associated with a von Neumann algebra in standard form[3] is a convex cone $\mathscr{P}^\natural_\mathcal{M} \subset \mathscr{H}$, called the "natural cone" and sometimes also denoted $\mathscr{P}^{1/4}_\mathcal{M}$ (see appendix item (4) for further explanation of this notation), and an anti-linear involution J, called "modular conjugation" leaving this cone elementwise invariant. A possible choice of this non-unique "natural cone" for $\mathcal{A} = M_n(\mathbb{C})$ is the subset of positive semi-definite matrices in $\mathscr{H}$, and in this case, $J|\zeta\rangle = |\zeta^*\rangle$. A general property of J, which is easily verified in this example, is that $J\mathcal{M}J = \mathcal{M}'$, the latter meaning the commutant of $\mathcal{M}$ on $\mathscr{H}$.

Going back to the case of a general von Neumann algebra $\mathcal{M}$ in standard form acting on $\mathscr{H}$, and given vectors $|\psi\rangle, |\eta\rangle, |\zeta\rangle \in \mathscr{P}^\natural_\mathcal{M}$ and $m \in \mathcal{M}$, one defines following Araki [5] a conjugate linear operator $S_{\eta,\psi}$ with domain $\mathscr{D}(S_{\eta,\psi}) = \mathcal{M}|\psi\rangle \oplus (1 - \pi^{\mathcal{M}'}(\psi))\mathscr{H}$ by

$$S_{\eta,\psi}\big(m|\psi\rangle + (1 - \pi^{\mathcal{M}'}(\psi))|\zeta\rangle\big) = \pi^{\mathcal{M}}(\psi)m^*|\eta\rangle; \tag{7}$$

see also [8, App. C] for many more details. Here, $\pi^{\mathcal{M}}(\psi) \in \mathcal{M}$ is the orthogonal projection onto the closure of the subspace $\mathcal{M}'|\psi\rangle$ and $\pi^{\mathcal{M}'}(\psi) \in \mathcal{M}'$ that onto the closure of $\mathcal{M}|\psi\rangle$. The definition is consistent because $m\pi^{\mathcal{M}}(\psi) = 0$ if $m|\psi\rangle = 0$. One shows that $S_{\eta,\psi}$ is a closable operator and denotes the closure by $\bar{S}_{\eta,\psi}$. By standard results in operator theory, such an operator has a polar decomposition, which for $|\psi\rangle, |\eta\rangle \in \mathscr{P}^\natural_\mathcal{M}$ is given by the first equality in

$$\bar{S}_{\eta,\psi} = J\Delta^{1/2}_{\eta,\psi}, \quad \bar{S}^*_{\eta,\psi}\bar{S}_{\eta,\psi} = \Delta_{\eta,\psi}. \tag{8}$$

One calls the self-adjoint, non-negative operator $\Delta^{1/2}_{\eta,\psi}$ (with domain $\mathscr{D}(\bar{S}_{\eta,\psi})$) the (square root of the) "relative modular operator". Its support is $s(\Delta_{\eta,\psi}) =$

[3] More precisely, standard form is actually defined by the combined structure $(\mathcal{M}, \mathscr{H}, \mathscr{P}^\natural_\mathcal{M}, J)$, which can be recovered if we have a cyclic and separating vector.

$\pi^{\mathcal{M}}(\eta)\pi^{\mathcal{M}'}(\psi)$ and complex powers $\Delta^z_{\eta,\psi}$ are understood as 0 on the orthogonal complement of this support. The term $\Delta_{\eta,\psi}$ depends on $|\eta\rangle$ only through its associated state functional ω_η, defined as in

$$\omega_\eta(m) = \langle \eta | m\eta\rangle \quad \forall\, m \in \mathcal{M}. \tag{9}$$

The modular conjugation and relative modular operators of $\mathcal{A} = M_n(\mathbb{C})$ with the above choice of natural cone are

$$J|\zeta\rangle = |\zeta^*\rangle, \quad \Delta_{\eta,\psi} = l(\omega_\eta) r(\omega_\psi^{-1}), \tag{10}$$

where we invert the density matrix ω_ψ on the range of $\pi^{\mathcal{M}}(\psi)$, which in the case at hand is the orthogonal projector onto the complement of the null space of ω_ψ.

For a general von Neumann algebra, every positive linear functional $\omega \in \mathscr{S}(\mathcal{M})$ corresponds to one and only one vector $|\xi_\omega\rangle$ in the natural cone $\mathscr{P}^\natural_{\mathcal{M}}$ such that $\omega(a) = \langle \xi_\omega | a\xi_\omega\rangle$. Vice versa, any vector $|\psi\rangle$ (in the natural cone or not) gives rise to a linear functional as in (9). For $\mathcal{A} = M_n(\mathbb{C})$, this linear functional is identified with the density matrix $\omega_\psi = \psi\psi^*$ and the natural cone vectors correspond to the unique positive square root of the corresponding density matrix, now thought of as pure states in the standard Hilbert space. So the vector representative of a density matrix ω in the natural cone is $|\xi_\omega\rangle = |\omega^{1/2}\rangle$. An important fact used implicitly in several places below is that if two linear functionals are close in norm, then the vectors in the natural cone are as well, and vice versa:

$$\|\xi_\psi - \xi_\eta\|^2 \le \|\omega_\eta - \omega_\xi\| \le \|\psi + \eta\|\,\|\psi - \eta\|, \tag{11}$$

where the norm of a linear functional is $\|\omega\| = \sup\{|\omega(m)| \colon m \in \mathcal{M},\ \|m\| = 1\}$. In the case $\mathcal{A} = M_n(\mathbb{C})$, the latter norm is $\|\omega\| = \mathrm{Tr}\,|\omega|$, so the first inequality in the above relation expresses the Powers–Størmer inequality between the trace norm and the Hilbert–Schmidt norm.

Let us finish this briefest of introductions to von Neumann algebras by summarizing (again) some of our notation.

Notation and conventions. Calligraphic letters $\mathcal{A}, \mathcal{M}, \ldots$ denote von Neumann algebras, always assumed σ-finite, i.e. they are assumed to have a normal faithful state. Calligraphic letters $\mathscr{H}, \mathscr{K}, \ldots$ denote complex Hilbert spaces, and $\mathscr{S}(\mathcal{M})$ denotes the set of all ultra-weakly continuous, positive, normalized linear functionals on $\mathcal{M}$ ("states"), which are in one-to-one correspondence with density matrices if $\mathcal{A} = M_n(\mathbb{C})$. Then $\mathcal{M}_+$ is the subset of all non-negative self-adjoint operators in $\mathcal{M}$, and $\mathcal{M}_{\text{s.a.}}$ the subset of all self-adjoint elements of the von Neumann algebra $\mathcal{M}$. We use the physicist's "ket"-notation $|\psi\rangle$ for vectors in a Hilbert space. The scalar product is written as

$$(|\psi\rangle, |\psi'\rangle)_{\mathscr{H}} =: \langle\psi|\psi'\rangle \tag{12}$$

and is anti-linear in the first entry. The norm of a vector is written simply as $\| \, |\psi\rangle \, \| =: \|\psi\|$. The action of a linear operator T on a ket is sometimes written as $T|\phi\rangle = |T\phi\rangle$. In this spirit, the norm of a bounded linear operator T on $\mathscr{H}$ is written as $\|T\| = \sup_{|\psi\rangle \,:\, \|\psi\|=1} \|T\psi\|$.

§3. Interpolation of non-commutative L_p-norms

For the algebra $\mathcal{A} = M_n(\mathbb{C})$, the standard Hilbert space $\mathscr{H} \cong M_n(\mathbb{C})$ on which $\mathcal{A}$ acts by left multiplication can be equipped with various norms. We have already mentioned that the 2-, or Hilbert–Schmidt, norm

$$\|\zeta\|_2 = (\mathrm{Tr}\, \zeta\zeta^*)^{1/2} \tag{13}$$

actually defines the Hilbert space norm on $\mathscr{H}$ (so the subscript "2" is generally omitted). For $p > 0$, one can generalize this to

$$\|\zeta\|_p = [\mathrm{Tr}(\zeta\zeta^*)^{p/2}]^{1/p}. \tag{14}$$

Given a faithful vector $|\psi\rangle \in \mathscr{H}$ with associated linear functional $\omega_\psi(a) = \langle\psi|a\psi\rangle = \mathrm{Tr}(a\omega_\psi)$ (Hilbert–Schmidt inner product), one can also define the yet more general norms

$$\|\zeta\|_{p,\psi} = [\mathrm{Tr}(\zeta\omega_\psi^{2/p-1}\zeta^*)^{p/2}]^{1/p}. \tag{15}$$

The faithful condition is relevant for $p > 2$ as it ensures that ω_ψ is invertible. The generalized L_p-norms $\|\zeta\|_{p,\psi}$ evidently reduce to a multiple of the usual L_p-norms if $\omega_\psi(a) = \mathrm{Tr}(a)/n$ is the tracial state. A general von Neumann algebra $\mathcal{M}$ in standard form need not have such a tracial state, but Araki and Masuda [8] have shown that one can still define analogs of the above "non-commuting L_p-norms" for $p \geq 1$ by variational expressions based on relative modular operators, involving a fixed cyclic and separating vector $|\psi\rangle$ in the natural cone of the von Neumann algebra $\mathcal{M}$ in standard form; see also [25, 26, 10]. Their basic definitions and properties used in this article are recalled for the convenience of the reader in the appendix. The following interpolation result for Araki–Masuda L_p-norms is one of the main workhorses of this article.

Lemma 1. *Let $|\psi\rangle \in \mathscr{H}$ be a cyclic and separating vector in the natural cone of a von Neumann algebra $\mathcal{M}$ in standard form acting on $\mathscr{H}$. For $0 < \theta < 1/2$,*

$p_0, p_1 \in [1,2]$ or $p_0, p_1 \in [2,\infty]$, let

$$\frac{1}{p_\theta} = \frac{1-2\theta}{p_0} + \frac{2\theta}{p_1}. \tag{16}$$

Then, if $|G(z)\rangle$ is an $\mathscr{H}$-valued function holomorphic on the strip $\mathbb{S}_{1/2} = \{0 < \mathrm{Re} z < 1/2\}$ that is bounded and weakly continuous on the closure $\bar{\mathbb{S}}_{1/2}$ and such that

$$\sup_{t\in\mathbb{R}} \|G(it)\|_{p_0,\psi}, \sup_{t\in\mathbb{R}} \|G(1/2+it)\|_{p_1,\psi} < \infty, \tag{17}$$

then we have

$$\begin{aligned} \ln \|G(\theta)\|_{p_\theta,\psi} \le \int_{-\infty}^{\infty} \mathrm{d}t \, \big(&(1-2\theta)\alpha_\theta(t) \ln \|G(it)\|_{p_0,\psi} \\ &+ (2\theta)\beta_\theta(t) \ln \|G(1/2+it)\|_{p_1,\psi} \big), \end{aligned} \tag{18}$$

where

$$\begin{aligned} \alpha_\theta(t) &= \frac{\sin(2\pi\theta)}{(1-2\theta)(\cosh(2\pi t) - \cos(2\pi\theta))}, \\ \beta_\theta(t) &= \frac{\sin(2\pi\theta)}{2\theta(\cosh(2\pi t) + \cos(2\pi\theta))}. \end{aligned} \tag{19}$$

Remark. Regarding the statement of Lemma 1, let us further note the following:

(1) The condition that $\sup_{t\in\mathbb{R}} \|G(it)\|_{p_0,\psi}$, $\sup_{t\in\mathbb{R}} \|G(1/2+it)\|_{p_1,\psi}$ are both finite is redundant when $p_0, p_1 \in [1,2]$ because (see appendix item (8)) $\|\zeta\|_{p,\psi} \le \|\zeta\| \, \|\omega_\psi\|^{1/p-1/2}$ for all $|\zeta\rangle \in \mathscr{H}$, $p \in [1,2]$ (see e.g. [10, Lem. 8]), and $|G(z)\rangle$ is already assumed to be bounded in $\mathscr{H}$ on $\bar{\mathbb{S}}_{1/2}$.

(2) Bound (17) means that the integrand in (18) is bounded above, but can be equal to $-\infty$. Thus, the integral is definite in the Lebesgue sense, but can be equal to $-\infty$. In the latter case, the left-hand side of (18) is also $-\infty$.

Proof of Lemma 1. In this proof we implicitly use the cyclic and separating property of $|\psi\rangle$ in order to apply the results by [8].

(1) Assume that $p_0, p_1 \in [1,2]$. This part of the proof is taken from paper I ([17]) up to minor modifications and only included for convenience. Denote the dual of a Hölder index p by p', defined so that $1/p + 1/p' = 1$. The article [8] has shown that the non-commutative $L_p(\mathcal{M},\psi)$-norm of a vector $|\zeta\rangle \in \mathscr{H}$ relative to $|\psi\rangle$ can be characterized by

$$\|\zeta\|_{p,\psi} = \sup\big\{ |\langle\zeta|\zeta'\rangle| \colon |\zeta'\rangle \in L_{p'}(\mathcal{M},\psi), \ \|\zeta'\|_{p',\psi} \le 1 \big\}. \tag{20}$$

It has furthermore been shown ([8, Thm. 3]) that when $\infty > p' \geq 2$, any vector $|\zeta'\rangle \in L_{p'}(\mathcal{M}, \psi)$ has a unique generalized polar decomposition, i.e. can be written in the form $|\zeta'\rangle = u\Delta_{\phi,\psi}^{1/p'}|\psi\rangle$, where u is a unitary or partial isometry from $\mathcal{M}$ and $|\phi\rangle \in \mathscr{H}$. Furthermore, it has been shown that $\|\zeta'\|_{p',\psi} = \|\phi\|^{2/p'}$. Therefore, unless $p'_\theta = \infty$ (meaning $p_1 = p_0 = 1$), we may choose a u and a normalized $|\phi\rangle$, so that

$$\|G(\theta)\|_{p_\theta,\psi} = \langle u\Delta_{\phi,\psi}^{1/p'_\theta}\psi|G(\theta)\rangle + \varepsilon, \tag{21}$$

up to an arbitrarily small error $\varepsilon > 0$ which we will let go to zero at the end. Here we defined p_θ as in the statement, so that

$$\frac{1}{p'_\theta} = \frac{1-2\theta}{p'_0} + \frac{2\theta}{p'_1}. \tag{22}$$

Excluding for the moment the case $p_1 = p_0 = 1$ which is treated at the end, we can therefore define an auxiliary function $f(z)$ by

$$f(z) = \langle u\Delta_{\phi,\psi}^{2\bar{z}/p'_1+(1-2\bar{z})/p'_0}\psi|G(z)\rangle, \tag{23}$$

noting that

$$f(\theta) = \|G(\theta)\|_{p_\theta,\psi} - \varepsilon \tag{24}$$

by construction. By Tomita–Takesaki theory, $f(z)$ is holomorphic in $\mathbb{S}_{1/2}$ and bounded and continuous on the closure $\bar{\mathbb{S}}_{1/2}$. For the values at the boundary of the strip $\mathbb{S}_{1/2}$, we estimate

$$\begin{aligned} |f(it)| &= \big|\langle u\Delta_{\phi,\psi}^{-2it(1/p'_1-1/p'_0)}\Delta_{\phi,\psi}^{1/p'_0}\psi|G(it)\rangle\big| \\ &\leq \|u\Delta_{\phi,\psi}^{-2it(1/p'_1-1/p'_0)}\Delta_{\phi,\psi}^{1/p'_0}\psi\|_{p'_0,\psi}\|G(it)\|_{p_0,\psi} \\ &\leq \|\Delta_{\phi,\psi}^{-2it(1/p'_1-1/p'_0)}\Delta_{\phi,\psi}^{1/p'_0}\psi\|_{p'_0,\psi}\|G(it)\|_{p_0,\psi} \\ &\leq \|\phi\|^{2/p'_0}\|G(it)\|_{p_0,\psi} \\ &\leq \|G(it)\|_{p_0,\psi}. \end{aligned} \tag{25}$$

Here we used the version of Hölder's inequality proved by [8], we used $\|a^*\zeta\|_{p'_0,\psi} \leq \|a\|\,\|\zeta\|_{p'_0,\psi}$ for any $a \in \mathcal{A}$ (see [8, Lem. 4.4]), and we used

$$\|\Delta_{\phi,\psi}^{-2it(1/p'_1-1/p'_0)}\Delta_{\phi,\psi}^{1/p'_0}\psi\|_{p'_0,\psi} \leq \|\phi\|^{2/p'_0},$$

which we will prove momentarily. A similar chain of inequalities also gives

$$|f(1/2+it)| \leq \|G(1/2+it)\|_{p_1,\psi}. \tag{26}$$

To prove the remaining claim, let $|\zeta'\rangle = \Delta^z_{\phi,\psi}|\psi\rangle$ and $z = 1/p'_0 + 2it$. Then we have, using the variational characterization by [8] of the $L_{p'_0}(\mathcal{M},\psi)$-norm when $p'_0 \geq 2$,

$$\begin{aligned}
\|\zeta'\|_{p'_0,\psi} &= \sup\big\{\|\Delta^{1/2-1/p'_0}_{\chi,\psi}\Delta^z_{\phi,\psi}\psi\|\colon \|\chi\| = 1\big\} \\
&= \sup\big\{\|\Delta^{1/2-1/p'_0-2it}_{\chi,\psi}\Delta^{1/p'_0+2it}_{\phi,\psi}\psi\|\colon \|\chi\| = 1\big\} \\
&= \sup\big\{\|\Delta^{1/2-1/p'_0}_{\chi,\psi}[D\chi : D\phi]_{2t}\Delta^{1/p'_0}_{\phi,\psi}\psi\|\colon \|\chi\| = 1\big\} \\
&\leq \sup\big\{\|\Delta^{1/2-1/p'_0}_{\chi,\psi}a\Delta^{1/p'_0}_{\phi,\psi}\psi\|\colon \|\chi\| = 1,\ a\in\mathcal{M},\ \|a\| = 1\big\} \\
&\leq \sup\big\{\|a\Delta^{1/p'_0}_{\phi,\psi}\psi\|_{p'_0,\psi}\colon a\in\mathcal{M},\ \|a\| = 1\big\},
\end{aligned} \tag{27}$$

with $[D\chi : D\phi]_{2t}$ the Connes cocycle, which is isometric. Using [8, Lem. 4.4], we continue this estimation as

$$\leq \sup_{a\in\mathcal{M},\|a\|=1} \|a\|\,\|\Delta^{1/p'_0}_{\phi,\psi}\psi\|_{p'_0,\psi} = \|\phi\|^{2/p'_0}, \tag{28}$$

which gives the desired result.

To get (18), we use the Hirschman improvement of the Hadamard three lines theorem [23].

Lemma 2. *Let $g(z)$ be holomorphic on the strip $\mathbb{S}_{1/2}$, continuous and uniformly bounded on the closure $\bar{\mathbb{S}}_{1/2}$. Then, for $\theta \in (0, 1/2)$,*

$$\ln|g(\theta)| \leq \int_{-\infty}^{\infty} (\beta_\theta(t)\ln|g(1/2+it)|^{2\theta} + \alpha_\theta(t)\ln|g(it)|^{1-2\theta})\,\mathrm{d}t, \tag{29}$$

where $\alpha_\theta(t)$, $\beta_\theta(t)$ are as in Lemma 1.

Applying this to $g = f$ gives (18), with the left-hand side replaced by $\|G(\theta)\|_{p_\theta,\psi} - \varepsilon$. Since $\varepsilon > 0$ can be arbitrarily small, this proves the lemma for $p_0, p_1 \in [1, 2]$ except for the case $p_0 = p_1 = 1$ which we had left out for special consideration.

In that case, we first find $m \in \mathcal{M}$ such that $\|m\| \leq 1$ and such that

$$\|G(\theta)\|_{1,\psi} = \langle m\psi|G(\theta)\rangle + \varepsilon, \tag{30}$$

which is possible by the characterization of the L_1-norm, see e.g. paper I ([17, Lem. 3]) or [8]. Then we set $f(z) = \langle m\psi|G(z)\rangle$, and the rest of the argument is similar to before.

(2) Now we assume that $p_0, p_1 \in [2, \infty]$. Then [8] has shown that for any[4] $|\zeta'_+\rangle \in L^+_{p'}(\mathcal{M}, \psi) := L_{p'}$-closure of $\mathscr{P}^{1/(2p')}_{\mathcal{M}}$, $1 \le p' \le 2$, there is $|\phi\rangle \in \mathscr{H}$ such that for all $|\zeta\rangle \in L_p(\mathcal{M}, \psi)$ we have

$$\langle \zeta'_+|\zeta\rangle = \langle \Delta^{1/2}_{\phi,\psi}\psi|\Delta^{(1/p')-(1/2)}_{\phi,\psi}\zeta\rangle \tag{31}$$

and such that $\|\zeta'_+\|_{p',\psi} = \|\phi\|^{2/p'}$. Furthermore, by the non-commutative Hölder inequality proven in [8], there exists up to an arbitrarily small error $\varepsilon > 0$ an element $|\zeta'\rangle \in L_{p'_\theta}(\mathcal{M}, \psi)$ such that

$$\|G(\theta)\|_{p_\theta,\psi} = \langle \zeta'|G(\theta)\rangle + \varepsilon, \quad \|\zeta'\|_{p'_\theta,\psi} = 1. \tag{32}$$

Thus, since by [8, Thm. 3] we may write $|\zeta'\rangle = u|\zeta'_+\rangle$, $u \in \mathcal{M}$ with $u^*u \le 1$ and $|\zeta'_+\rangle \in L^+_{p'_\theta}(\mathcal{M}, \psi)$, we have

$$\begin{aligned} \|G(\theta)\|_{p_\theta,\psi} - \varepsilon &= \langle \Delta^{1/2}_{\phi,\psi}\psi|\Delta^{1/p'_\theta - 1/2}_{\phi,\psi}u^*G(\theta)\rangle \\ &= \langle \Delta^{1/2}_{\phi,\psi}\psi|\Delta^{(1-2\theta)/p'_0+(2\theta)/p'_1-1/2}_{\phi,\psi}u^*G(\theta)\rangle \end{aligned} \tag{33}$$

and $\|\phi\| = 1$. Similarly to the previous case we now consider the function

$$f(z) = \langle \Delta^{1/2}_{\phi,\psi}\psi|\Delta^{(1-2z)/p'_0+(2z)/p'_1-1/2}_{\phi,\psi}u^*G(z)\rangle, \tag{34}$$

where f is holomorphic for $z \in \mathbb{S}_{1/2}$ and continuous and bounded on the closure $\bar{\mathbb{S}}_{1/2}$, as can be seen by applying the following lemma, which is a slight generalization of [13, Lem. 2.1]:

Lemma 3. *Suppose $|F(z)\rangle$ is an $\mathscr{H}$-valued function which is analytic on $\mathbb{S}_{1/2}$ and bounded and weakly continuous on the closure $\bar{\mathbb{S}}_{1/2}$. Let $\alpha(z) = az + b$, $a, b \in \mathbb{R}$ be a linear map from $\bar{\mathbb{S}}_{1/2}$ to itself, and A a self-adjoint positive operator such that $\mathscr{D}(A^{1/2})$ is dense, and such that $C_0 := \sup_{t\in\mathbb{R}} \|A^{\alpha(it)}F(it)\|$, $C_1 := \sup_{t\in\mathbb{R}} \|A^{\alpha(1/2+it)}F(1/2+it)\|$ are both finite.*

Then $|A^{\alpha(z)}F(z)\rangle$ is an analytic function of $\mathbb{S}_{1/2}$ which is bounded and weakly continuous on the closure $\bar{\mathbb{S}}_{1/2}$.

Proof. First let $|\eta\rangle \in \mathscr{D}(A^{1/2})$. Then $z \mapsto \langle A^{\alpha(\bar z)}\eta|F(z)\rangle$ is analytic on $\mathbb{S}_{1/2}$ and bounded and continuous on the closure $\bar{\mathbb{S}}_{1/2}$ (because $A^{\alpha(z)}|\eta\rangle$ is strongly and $|F(z)\rangle$ is weakly continuous on $\bar{\mathbb{S}}_{1/2}$). On the upper and lower boundaries of the strip the assumptions give us $|\langle A^{\alpha(-it)}\eta|F(it)\rangle| \le C_0\|\eta\|$ and $|\langle A^{\alpha(1/2-it)}\eta|F(1/2+it)\rangle| \le C_1\|\eta\|$. Therefore, by the Hadamard three lines theorem,

$$\left|\langle A^{\alpha(\bar z)}\eta|F(z)\rangle\right| \le C\|\eta\| \quad (z \in \bar{\mathbb{S}}_{1/2},\ |\eta\rangle \in \mathscr{D}(A^{1/2}),\ C = \max(C_1, C_0)). \tag{35}$$

[4] The cone $\mathscr{P}^{1/(2p')}_{\mathcal{M}}$ is defined as the closure of $\Delta^{1/(2p')}_{\psi}\mathcal{M}_+|\psi\rangle$ and its properties are discussed in [8].

Since $\mathscr{D}(A^{1/2})$ is a core for $A^{\alpha(\bar{z})}$, we conclude that $|F(z)\rangle \in \mathscr{D}(A^{\alpha(z)})$, and moreover $\|A^{\alpha(z)}F(z)\| \le C$ on $\bar{\mathbb{S}}_{1/2}$. The analyticity of the function $z \mapsto \langle A^{\alpha(\bar{z})}\eta|F(z)\rangle$ when $|\eta\rangle \in \mathscr{D}(A^{1/2})$ implies that $A^{\alpha(z)}|F(z)\rangle$ is analytic in $\mathbb{S}_{1/2}$ as follows. For arbitrary $|\chi\rangle$, $z_0 \in \mathbb{S}_{1/2}$ we first have for sufficiently small $r > 0$,

$$\left|\langle\chi|A^{\alpha(z_0)}F(z_0)\rangle - \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \mathrm{d}z\,(z-z_0)^{-1}\langle\chi|A^{\alpha(z)}F(z)\rangle\right| \le 2C\|\chi-\eta\|, \tag{36}$$

where we have written $|\chi\rangle = |\eta\rangle + (|\chi\rangle - |\eta\rangle)$ for some $|\eta\rangle \in \mathscr{D}(A^{1/2})$ and used the analyticity of $\langle\eta|A^{\alpha(z)}F(z)\rangle$ and $\|A^{\alpha(z)}F(z)\| \le C$ on $\mathbb{S}_{1/2}$ to obtain the upper bound. Then, since $\mathscr{D}(A^{1/2})$ is dense, $\|\chi - \eta\|$ can be made arbitrarily small, hence showing that $\langle\chi|A^{\alpha(z)}F(z)\rangle$ fulfills Cauchy's integral formula, and is thus analytic on $\mathbb{S}_{1/2}$. Thus, $A^{\alpha(z)}|F(z)\rangle$ is weakly analytic, hence strongly analytic (by an application of the Banach–Steinhaus principle) on $\mathbb{S}_{1/2}$.

To prove continuity of $\langle\chi|A^{\alpha(z)}F(z)\rangle$ on $\bar{\mathbb{S}}_{1/2}$ (only continuity at the boundary is in question), let $z_0 \in \partial\mathbb{S}_{1/2}$, and let $|\chi\rangle \in \mathscr{H}$. For fixed $\varepsilon > 0$, we first pick $|\eta\rangle \in \mathscr{D}(A^{1/2})$ such that $\|\eta - \chi\| < \varepsilon$, and then

$$\begin{aligned}\limsup_{z\to z_0}&\big|\langle\chi|A^{\alpha(z)}F(z)\rangle - \langle\chi|A^{\alpha(z_0)}F(z_0)\rangle\big|\\ &\le 2C\|\eta-\chi\| + \lim_{z\to z_0}\big|\langle\eta|A^{\alpha(z)}F(z)\rangle - \langle\eta|A^{\alpha(z_0)}F(z_0)\rangle\big|\\ &\le 2C\varepsilon,\end{aligned} \tag{37}$$

using that we already know that $\langle\eta|A^{\alpha(z)}F(z)\rangle$ is continuous on $\bar{\mathbb{S}}_{1/2}$. Hence, the limit must vanish as $\varepsilon > 0$ was arbitrary. ☐

We apply this lemma to $|\chi\rangle = \Delta^{1/2}_{\phi,\psi}|\psi\rangle = J|\phi\rangle \in \mathscr{H}$, $\alpha(z) = (1-2z)/p_0' + (2z)/p_1' - 1/2 = 1/2 - (1-2z)/p_0 - (2z)/p_1$, $A = \Delta_{\phi,\psi}$, $|F(z)\rangle = u^*|G(z)\rangle$. Then we can estimate the boundary values as in $\|A^{\alpha(it)}F(it)\| \le \|G(it)\|_{p_0,\psi} =: C_0$ and $\|A^{\alpha(1/2+it)}F(1/2+it)\| \le \|G(1/2+it)\|_{p_1,\psi} =: C_1$ by a similar calculation to (38). This implies that the conditions of Lemma 3 are met, hence $f(z)$ is an analytic function of $z \in \mathbb{S}_{1/2}$ which is continuous and bounded on the closure $\bar{\mathbb{S}}_{1/2}$.

For the lower boundary value we next calculate

$$\begin{aligned}|f(it)| &= \big|\langle\Delta^{1/2}_{\phi,\psi}\psi|\Delta^{-2it(1/p_0'-1/p_1')}_{\phi,\psi}\Delta^{1/p_0'-1/2}_{\phi,\psi}u^*G(it)\rangle\big|\\ &\le \|\Delta^{1/2}_{\phi,\psi}\psi\|\,\|\Delta^{1/p_0'-1/2}_{\phi,\psi}u^*G(it)\|\\ &= \|\phi\|\,\|\Delta^{1/2-1/p_0}_{\phi,\psi}u^*G(it)\|\\ &\le \sup\big\{\|\Delta^{1/2-1/p_0}_{\chi,\psi}u^*G(it)\| \colon \|\chi\| = 1\big\}\\ &= \|u^*G(it)\|_{p_0,\psi} \le \|u^*\|\,\|G(it)\|_{p_0,\psi} = \|G(it)\|_{p_0,\psi},\end{aligned} \tag{38}$$

using in the last line the variational characterization of the L_p-norms and [8, Lem. 4.4]. A similar chain of inequalities also gives $|f(1/2+it)| \le \|G(1/2+it)\|_{p_1,\psi}$. Claim (18) with the left-hand side replaced by $\|G(\theta)\|_{p_\theta,\psi} - \varepsilon$ then follows from Hirschman's improvement as in the previous case (1). Since $\varepsilon > 0$ can be arbitrarily small, the claim of the lemma follows. $\square$

We would now like to remove the cyclic and separating (i.e. faithful) condition on the state vector $|\psi\rangle$ used for the definition of the L_p-norms. For σ-finite $\mathcal{M}$, there exists some cyclic and separating vector $|\eta\rangle$ for $\mathcal{M}$ and we put

$$\omega_{\psi_\varepsilon} = (1-\varepsilon)\omega_\psi + \varepsilon\omega_\eta \tag{39}$$

so that $|\psi_\varepsilon\rangle \in \mathscr{P}^\natural_{\mathcal{M}}$ is now faithful for $\mathcal{M}$ and $\mathcal{M}'$, hence cyclic and separating. Thus, Lemma 1 holds for $|\psi_\varepsilon\rangle$ and the obvious idea is to prove the analog of Lemma 1 for $|\psi\rangle$ from this by taking the limit $\varepsilon \to 0$ in some way.

A lemma that we use to control this limit is the following.

Lemma 4. *Let $\omega_\psi, \omega_\eta, \omega_{\psi_n}, \omega_{\eta_n} \in \mathscr{S}(\mathcal{M})$ be such that $\lim_n \|\omega_\psi - \omega_{\psi_n}\| = 0$, $\lim_n \|\omega_\eta - \omega_{\eta_n}\| = 0$ and such that $\omega_{\eta_n} \le C\omega_\eta$, $\omega_\psi \le C\omega_{\psi_n}$ for some $C < \infty$ and all n. Then*

$$\lim_n \|(\Delta^{\alpha/2}_{\eta,\psi} - \Delta^{\alpha/2}_{\eta_n,\psi_n})\zeta\| = 0 \tag{40}$$

for any $\alpha \in (0,1), |\zeta\rangle \in s(\Delta_{\eta,\psi})\mathscr{H} \cap \mathscr{D}(\Delta^{\alpha/2}_{\eta,\psi})$ (where $|\psi\rangle, |\psi_n\rangle$ are taken in the natural cone).

Proof. In this proof, we use the shorthands $\Delta = \Delta_{\eta,\psi}$, $\Delta_n = \Delta_{\eta_n,\psi_n}$ and use that $|\psi\rangle$, $|\psi_n\rangle$ are in the natural cone implicitly when referring to known properties of the modular operators. Combining the domination conditions $\omega_{\eta_n} \le C\omega_\eta$, $\omega_\psi \le C\omega_{\psi_n}$ with the definition of the modular operators and their standard properties gives by the same argument as in the proof of Lemma 9 in the appendix that $s(\Delta)\Delta_n^\alpha s(\Delta) \le C^{2\alpha}\Delta^\alpha$, $\alpha \in [0,1]$ on $\mathscr{D}(\Delta^{\alpha/2})$ and in fact $s(\Delta)\colon \mathscr{D}(\Delta^{\alpha/2}) \to \mathscr{D}(\Delta_n^{\alpha/2})$. In particular, vectors in the domain of a power $\Delta^{\alpha/2}$ intersected with $s(\Delta)\mathscr{H}$ are always in the domain of the corresponding power $\Delta_n^{\alpha/2}$

We begin by defining $|\zeta_N\rangle = E_\Delta([0,N])|\zeta\rangle$ in terms of the spectral decomposition E_Δ of the non-negative self-adjoint operator Δ, and then we write

$$\|(\Delta^{\alpha/2} - \Delta_n^{\alpha/2})\zeta\| \le \|(\Delta^{\alpha/2} - \Delta_n^{\alpha/2})\zeta_N\| + 2C^\alpha\|\Delta^{\alpha/2}(\zeta - \zeta_N)\|, \tag{41}$$

using the triangle inequality and the aforementioned fact that $s(\Delta)\Delta_n^\alpha s(\Delta) \le C^{2\alpha}\Delta^\alpha$ on $\mathscr{D}(\Delta^{\alpha/2})$ and that both $|\zeta\rangle$, $|\zeta_N\rangle$ are in the range of $s(\Delta)$. By choosing N sufficiently large, we can then achieve that the last term on the right-hand side of (41) is $< \varepsilon$ for all n, so

$$\|(\Delta^{\alpha/2} - \Delta_n^{\alpha/2})\zeta\| \leq \|(\Delta^{\alpha/2} - \Delta_n^{\alpha/2})\zeta_N\| + \varepsilon \tag{42}$$

for all n.

To deal with the powers in the statement of the lemma, we employ the standard formula

$$X^\alpha = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \mathrm{d}\lambda\, \lambda^\alpha [\lambda^{-1} - (\lambda + X)^{-1}] \tag{43}$$

for $\alpha \in (0,1)$, $X \geq 0$, which holds on the domain of X^α. So we can use (43) on the first term on the right-hand side of (42) with $X = \Delta^{1/2}$ and $= \Delta_n^{1/2}$ on $|\zeta_N\rangle \in s(\Delta)\mathscr{H} \cap \mathscr{D}(\Delta^{\alpha/2})$, giving us that

$$\begin{aligned} &\|(\Delta^{\alpha/2} - \Delta_n^{\alpha/2})\zeta_N\| \\ &\quad \leq c \int_0^\infty \mathrm{d}\lambda\, \lambda^{\alpha-1} \big\|[(1+\lambda\Delta^{-1/2})^{-1} - (1+\lambda\Delta_n^{-1/2})^{-1}]\zeta_N\big\|. \end{aligned} \tag{44}$$

Here, and in the rest of this proof, we denote by c any constant depending possibly on α, C, N, but not n. We split the integration domain into three parts: $(0,\delta)$, (δ, L), (L,∞).

(i) Range $(0,\delta)$: In this range, we use

$$\begin{aligned} &\int_0^\delta \mathrm{d}\lambda\, \lambda^{\alpha-1} \big\|[(1+\lambda\Delta^{-1/2})^{-1} - (1+\lambda\Delta_n^{-1/2})^{-1}]\zeta_N\big\| \\ &\quad = \int_0^\delta \mathrm{d}\lambda\, \lambda^{\alpha} \big\|[(\lambda+\Delta^{1/2})^{-1} - (\lambda+\Delta_n^{1/2})^{-1}]\zeta_N\big\| \\ &\quad \leq \int_0^\delta \mathrm{d}\lambda\, \lambda^{\alpha} \{\|(\lambda+\Delta^{1/2})^{-1}\zeta_N\| + \|(\lambda+\Delta_n^{1/2})^{-1}\zeta_N\|\} \\ &\quad \leq 2\|\zeta_N\| \int_0^\delta \mathrm{d}\lambda\, \lambda^{\alpha-1} \leq c\|\zeta_N\|\delta^\alpha \end{aligned} \tag{45}$$

using that $\Delta, \Delta_n \geq 0$.

(ii) Range (δ, L): By [7, Lem. 4.1],

$$\big\|[(\lambda+\Delta^{1/2})^{-1} - (\lambda+\Delta_n^{1/2})^{-1}]\zeta_N\big\| \to 0 \quad \text{as } n \to \infty \tag{46}$$

uniformly for λ in the compact set $[\delta, L]$.

(iii) Range (L,∞): Recall that the domination assumption gives $s(\Delta)\Delta_n s(\Delta) \leq C^2\Delta$. The function $f\colon \mathbb{R}_+ \ni x \mapsto (\lambda + x^{-1/2})^{-2}$ is bounded and operator monotone by a standard characterization of such functions (see e.g. [21]), thus $f(s(\Delta)\Delta_n s(\Delta)) \leq f(C^2\Delta)$. Then [33, Thm. C] furthermore gives $f(s(\Delta)\Delta_n s(\Delta)) \geq s(\Delta)f(\Delta_n)s(\Delta)$, hence for $|\zeta\rangle \in s(\Delta)\mathscr{H}$, we get

$$\begin{aligned}\|(1+\lambda\Delta_n^{-1/2})^{-1}\zeta_N\| &= \langle\zeta_N|(1+\lambda\Delta_n^{-1/2})^{-2}\zeta_N\rangle^{1/2}\\ &\leq \langle\zeta_N|(1+\lambda C^{-1}\Delta^{-1/2})^{-2}\zeta_N\rangle^{1/2}.\end{aligned}\tag{47}$$

Since $C \geq 1$, we also have trivially

$$\begin{aligned}\|(1+\lambda\Delta^{-1/2})^{-1}\zeta_N\| &= \langle\zeta_N|(1+\lambda\Delta^{-1/2})^{-2}\zeta_N\rangle^{1/2}\\ &\leq \langle\zeta_N|(1+\lambda C^{-1}\Delta^{-1/2})^{-2}\zeta_N\rangle^{1/2}.\end{aligned}\tag{48}$$

Using these inequalities under the integral (44) gives

$$\begin{aligned}&\int_L^\infty \mathrm{d}\lambda\,\lambda^{\alpha-1}\big\|[(1+\lambda\Delta^{-1/2})^{-1}-(1+\lambda\Delta_n^{-1/2})^{-1}]\zeta_N\big\|\\ &\quad\leq \int_L^\infty \mathrm{d}\lambda\,\lambda^{\alpha-1}\big\{\|(1+\lambda\Delta^{-1/2})^{-1}\zeta_N\|+\|(1+\lambda\Delta_n^{-1/2})^{-1}\zeta_N\|\big\}\\ &\quad\leq 2\int_L^\infty \mathrm{d}\lambda\,\lambda^{\alpha-1}\langle\zeta_N|(1+\lambda C^{-1}\Delta^{-1/2})^{-2}\zeta_N\rangle^{1/2}\\ &\quad\leq 2CN^{1/2}\|\zeta_N\|\int_L^\infty \mathrm{d}\lambda\,\lambda^{\alpha-2} \leq c\|\zeta_N\|L^{\alpha-1}.\end{aligned}\tag{49}$$

Now for our given N, we first choose δ, L so small/large that the contributions to (44) from (i), (iii), cf. (45), (49), are $< \varepsilon$ each (independently of n) and then n so large that the contribution (ii) from (δ, L) is $< \varepsilon$. Then $\|(\Delta^{\alpha/2}-\Delta_n^{\alpha/2})\zeta\|$ is $< 4\varepsilon$ by (i), (ii), (iii), and (42), and the proof is complete. $\square$

The next lemma is a consequence of Lemma 4.

Lemma 5. *Let $\varepsilon \in (0,1)$, $p \in [1,\infty)$, $|\psi\rangle \in \mathscr{P}^\natural_{\mathcal{M}}$, $|\eta\rangle \in \mathscr{H}$, and let $|\psi_\varepsilon\rangle \in \mathscr{P}^\natural_{\mathcal{M}}$ be the unique vector such that $\omega_{\psi_\varepsilon} = (1-\varepsilon)\omega_\psi + \varepsilon\omega_\eta$. Then $\lim_{\varepsilon\to 0+}\|\zeta\|_{p,\psi_\varepsilon} = \|\zeta\|_{p,\psi}$ for $|\zeta\rangle \in \pi^{\mathcal{M}'}(\psi)\mathscr{H} \cap L_p(\mathcal{M},\psi)$ when $p \in [2,\infty)$ and $|\zeta\rangle \in \mathscr{H}$ when $p \in [1,2)$.*

Remark. As the referee has pointed out to us, the lemma follows alternatively from the general (lower semi-) continuity properties of the L_p-norms; see item (10) in the appendix and references therein. However, the proof of these goes through a relatively non-trivial identification of the Araki–Masuda L_p-norms with L_p-norms defined in a different framework; see [26, 27, 22]. We therefore think that it is still useful to have a direct proof.

Proof of Lemma 5. **Case $p \in (2, \infty)$.** In combination with $\omega_\psi \le (1-\varepsilon)^{-1}\omega_{\psi_\varepsilon}$, Lemma 9 gives

$$\limsup_{\varepsilon \to 0} \|\zeta\|_{p,\psi_\varepsilon} \le \|\zeta\|_{p,\psi}. \tag{50}$$

We need to get a similar relation for lim inf. Let $\delta > 0$ be small but fixed. We can pick a unit vector $|\phi\rangle$ such that $\|\zeta\|_{\psi,p} \le \|\Delta_{\phi,\psi}^{(1/2)-(1/p)}\zeta\| + \delta/2$ by the variational definition of the L_p-norm. The condition that $|\zeta\rangle \in L_p(\mathcal{M}, \psi)$ means in particular that $s(\Delta_{\phi,\psi})|\zeta\rangle \in \mathscr{D}(\Delta_{\phi,\psi}^{(1/2)-(1/p)})$. We also note that $\|\psi_\varepsilon - \psi\|^2 \le \|\omega_{\psi_\varepsilon} - \omega_\psi\| \le 2\varepsilon \to 0$, implying that the conditions of Lemma 4 are met under the replacements $|\psi_n\rangle \to |\psi_\varepsilon\rangle$, $|\eta_n\rangle \to |\phi\rangle$, $|\zeta\rangle \to s(\Delta_{\phi,\psi})|\zeta\rangle$, $(\alpha/2) \to (1/2) - (1/p)$ (using that $p \in (2, \infty)$). Lemma 4 and the triangle inequality therefore show that there is an $\varepsilon > 0$ such that

$$\begin{aligned}
\|\zeta\|_{p,\psi} &\le \|\Delta_{\phi,\psi}^{(1/2)-(1/p)}\zeta\| + \delta/2 \\
&\le \|\Delta_{\phi,\psi_\varepsilon}^{(1/2)-(1/p)}\zeta\| + \|(\Delta_{\phi,\psi}^{(1/2)-(1/p)} - \Delta_{\phi,\psi_\varepsilon}^{(1/2)-(1/p)})\zeta\| + \delta/2 \\
&= \|\Delta_{\phi,\psi_\varepsilon}^{(1/2)-(1/p)}\zeta\| + \|(\Delta_{\phi,\psi}^{(1/2)-(1/p)} - \Delta_{\phi,\psi_\varepsilon}^{(1/2)-(1/p)})s(\Delta_{\phi,\psi})\zeta\| + \delta/2 \\
&\le \|\zeta\|_{p,\psi_\varepsilon} + \delta/2 + \delta/2,
\end{aligned} \tag{51}$$

meaning that

$$\liminf_{\varepsilon \to 0} \|\zeta\|_{p,\psi_\varepsilon} \ge \|\zeta\|_{p,\psi}. \tag{52}$$

The proof of Lemma 5 is complete by (50), (52).

Case $p \in (1, 2)$. (1) We first show $\limsup_{\varepsilon \to 0} \|\zeta\|_{p,\psi_\varepsilon} \le \|\zeta\|_{p,\psi}$. By the properties of the L_p-norms (see appendix item (10)), we may assume without loss of generality that $|\zeta\rangle$ is in the natural cone. Consider a unit vector $|\phi\rangle$ in the natural cone such that, for a given $\delta > 0$, $\|\zeta\|_{p,\psi} \ge \|\Delta_{\phi,\psi}^{-\alpha}\zeta\| - \delta$ and $\pi^{\mathcal{M}}(\phi) \ge \pi^{\mathcal{M}}(\zeta)$ (here $\alpha = 1/p - 1/2 \in (0, 1/2)$). Such a vector must exist by the variational definition of the L_p-norm; see the appendix. We set ($\delta \ge 0$)

$$\omega_{\phi_\delta} = \omega_\phi + \delta\omega_\zeta, \tag{53}$$

with $|\phi_\delta\rangle$ in the natural cone, implying that $\pi^{\mathcal{M}}(\phi_\delta) = \pi^{\mathcal{M}}(\phi)$ and $\omega_\phi \le \omega_{\phi_\delta}$. Standard properties of the natural cone also imply that $\lim_{\delta\to 0} \|\phi_\delta - \phi\| = 0$. Furthermore, the following relations follow directly from the definitions of the modular operators and their basic properties, combined with $\omega_\psi \le (1-\varepsilon)^{-1}\omega_{\psi_\varepsilon}$ and with the operator monotonicity of the function $x^{2\alpha}$: $\Delta_{\phi_\delta,\psi}^{-2\alpha} \le \Delta_{\phi,\psi}^{-2\alpha} \le (1-\varepsilon)^{-2\alpha}\Delta_{\phi,\psi_\varepsilon}^{-2\alpha}$. So we get $\|\zeta\|_{p,\psi} \ge \|\Delta_{\phi_\delta,\psi}^{-\alpha}\zeta\| - \delta$, for example. We can also show $|\zeta\rangle \in \mathscr{D}(\Delta_{\phi_\delta,\psi_\varepsilon}^{-\alpha})$ for $\delta > 0$ and $\varepsilon \ge 0$ in the following manner: By [13, Lem. 2.1], it suffices to show that $\|\Delta_{\phi_\delta,\psi_\varepsilon}^{-1/2}\zeta\| < \infty$. Now, $\delta^{-1}\omega_{\phi_\delta} \geq \omega_\zeta$ implies that $|\zeta\rangle = m'|\phi_\delta\rangle$ for some $m' \in \mathcal{M}'$ such that $\|m'\| \leq \delta^{-1/2}$. Then

$$\|\Delta_{\phi_\delta,\psi_\varepsilon}^{-1/2}\zeta\| = \|(\Delta'_{\psi_\varepsilon,\phi_\delta})^{1/2}m'\phi_\delta\| = \|(m')^*\psi_\varepsilon\| \leq \delta^{-1/2}\|\psi_\varepsilon\|. \tag{54}$$

We next claim that, for $\delta > 0$,

$$\lim_{\varepsilon\to 0}\|\Delta_{\phi_\delta,\psi_\varepsilon}^{-\alpha}\zeta\| = \|\Delta_{\phi_\delta,\psi}^{-\alpha}\zeta\|. \tag{55}$$

Assuming that this is the case, we get

$$\|\zeta\|_{p,\psi} \geq \lim_{\varepsilon\to 0}\|\Delta_{\phi_\delta,\psi_\varepsilon}^{-\alpha}\zeta\| - \delta \geq (1+\delta)^{-\alpha}\limsup_{\varepsilon\to 0}\|\zeta\|_{p,\psi_\varepsilon} - \delta \tag{56}$$

and the claim is proven since $\delta > 0$ can be taken arbitrarily small. To demonstrate (55), we use the integral representation (again $\delta > 0$)

$$\infty > \|\Delta_{\phi_\delta,\psi_\varepsilon}^{-\alpha}\zeta\|^2 = \frac{\sin(2\pi\alpha)}{\pi}\int_{0^+}^{\infty}\lambda^{2\alpha}\langle\zeta|[\lambda^{-1} - (\lambda + \Delta_{\phi_\delta,\psi_\varepsilon}^{-1})^{-1}]|\zeta\rangle\, \mathrm{d}\lambda, \tag{57}$$

which holds since $|\zeta\rangle \in \mathscr{D}(\Delta_{\phi_\delta,\psi_\varepsilon}^{-\alpha})$. The integrand is pointwise (for $\lambda > 0$) non-negative, decreasing as $\varepsilon \to 0$ and in fact convergent to $\langle\zeta|(\lambda^{-1} - (\lambda + \Delta_{\phi_\delta,\psi}^{-1})^{-1})|\zeta\rangle$, by an easy adaption of [6, Thm. 4.1]. (To see this, write $(\lambda + \Delta_{\phi_\delta,\psi_\varepsilon}^{-1})^{-1}\pi^{\mathcal{M}}(\phi) = J(i\lambda^{1/2} + \Delta_{\psi_\varepsilon,\phi_\delta}^{1/2})^{-1}(-i\lambda^{1/2} + \Delta_{\psi_\varepsilon,\phi_\delta}^{1/2})^{-1}\pi^{\mathcal{M}'}(\phi)J$ and note that the proof of [6, Thm. 4.1] goes through for $r = \pm i\lambda^{1/2}$.) This demonstrates (55).

(2) We next show $\liminf_{\varepsilon\to 0}\|\zeta\|_{p,\psi_\varepsilon} \geq \|\zeta\|_{p,\psi}$. For given $\delta > 0$ we can find a cyclic and separating unit vector $|\chi_\delta\rangle$ from the natural cone such that $\|\zeta\|_{p,\psi_\varepsilon} \geq \|\Delta_{\chi_\delta,\psi_\varepsilon}^{-\alpha}\zeta\| - \delta$ from the variational definition of the L_p-norm. By using $\Delta_{\chi_\delta,\psi_\varepsilon}^{-2\alpha} \geq (1-\varepsilon)^{2\alpha}\Delta_{\chi_\delta,\psi}^{-2\alpha}$, the right-hand side is greater than or equal to $(1-\varepsilon)^{\alpha}\|\zeta\|_{p,\psi} - \delta$, again by the variational definition. The claim is proven since $\delta > 0$ can be taken arbitrarily small.

Case $p \in \{1, 2\}$. These cases follow from the well-known continuity properties of the fidelity (for $p = 1$) (see e.g. paper I ([17, Lem. 9])) and the relationship of the L_2-norm with the projected Hilbert space norm (for $p = 2$); see appendix item (1). □

We can now prove the following theorem:

Theorem 1. *The conclusion of Lemma 1 continues to hold for arbitrary vectors $|\psi\rangle$ in the natural cone and $\theta \in (0, 1/2)$ such that $\|G(\theta)\|_{p_\theta,\psi} < \infty$.*

Proof. **Case** $p_0, p_1 \in [2, \infty]$**.** Take a small $\varepsilon > 0$, a faithful normal state ω_η, and define the vectors $|\psi_\varepsilon\rangle$ in the natural cone as in (39). Then $|\psi_\varepsilon\rangle$ is cyclic and separating for $\mathcal{M}$. The condition $\omega_\psi \le (1-\varepsilon)^{-1}\omega_{\psi_\varepsilon}$ and Lemma 9 in the appendix give

$$\begin{aligned} \|\pi^{\mathcal{M}'}(\psi)G(1/2+it)\|_{p_1,\psi_\varepsilon} &\le (1-\varepsilon)^{(1/p_1)-(1/2)}\|G(1/2+it)\|_{p_1,\psi}, \\ \|\pi^{\mathcal{M}'}(\psi)G(it)\|_{p_0,\psi_\varepsilon} &\le (1-\varepsilon)^{(1/p_0)-(1/2)}\|G(it)\|_{p_0,\psi} \end{aligned} \tag{58}$$

for all $t \in \mathbb{R}$. It follows that the assumptions of Lemma 1 apply to the vector-valued function $\pi^{\mathcal{M}'}(\psi)|G(z)\rangle$ and the vectors $|\psi_\varepsilon\rangle$. This gives

$$\limsup_{\varepsilon\to 0} \ln \|\pi^{\mathcal{M}'}(\psi)G(\theta)\|_{p_\theta,\psi_\varepsilon} \le \text{right-hand side of equation (18).} \tag{59}$$

The lim sup is equal to $\ln\|\pi^{\mathcal{M}'}(\psi)G(\theta)\|_{p_\theta,\psi} = \ln\|G(\theta)\|_{p_\theta,\psi}$ by Lemma 5 provided that $p_\theta \in [2,\infty)$, which is the case unless $p_1 = p_0 = \infty$ (so $p_\theta = \infty$).

For $p_1 = p_0 = \infty$, (59) still holds but Lemma 5 as stated does not control the lim sup. However, we get convergence of $\lim_{\varepsilon\to 0}\|\pi^{\mathcal{M}'}(\psi)G(\theta)\|_{p,\psi_\varepsilon} = \|G(\theta)\|_{p,\psi} < \infty$ for any Hölder index $2 \le p < \infty$. By the monotonicity of the L_p-norms (in particular $L_p(\mathcal{M},\psi) \supset L_\infty(\mathcal{M},\psi)$) as stated in [10, Lem. 8] or appendix item (8), this implies that

$$\frac{p}{p-2}\ln\|G(\theta)\|_{p,\psi} \le \text{right-hand side of equation (18) for } p_0 = p_1 = \infty. \tag{60}$$

Taking the limit $p \to \infty$ of the expression on the left-hand side and using the limit in appendix item (8) (also using $\|G(\theta)\|_{\infty,\psi} < \infty$), the statement of the theorem also follows for $p = \infty$.

Case $p_0, p_1 \in [1, 2]$**.** We define $|\psi_\varepsilon\rangle$ in the same manner as before and wish to apply Lemma 1 to this vector and to the vector-valued function $\pi^{\mathcal{M}'}(\psi)|G(z)\rangle$. Standard properties of the L_q-norms (see appendix item (8)) in the range $q \in [1,2]$ give $\|\zeta\|_{q,\psi} \le \|\zeta\|\,\|\omega_\psi\|^{1/q-1/2}$, so ($t \in \mathbb{R}$)

$$\begin{aligned} \|\pi^{\mathcal{M}'}(\psi)G(1/2+it)\|_{p_1,\psi_\varepsilon} &\le \|G(1/2+it)\| < C_1, \\ \|\pi^{\mathcal{M}'}(\psi)G(it)\|_{p_0,\psi_\varepsilon} &\le \|G(it)\| < C_0, \end{aligned} \tag{61}$$

with $C_0, C_1 < \infty$. Thus we can apply Lemma 1, and this gives, with Lemma 5,

$$\ln\|G(\theta)\|_{p_\theta,\psi} \le \limsup_{\varepsilon\to 0}\begin{pmatrix}\text{right-hand side of equation (18)} \\ \text{for } \psi \to \psi_\varepsilon,\ G(z) \to \pi^{\mathcal{M}'}(\psi)G(z)\end{pmatrix}. \tag{62}$$

As the integrand on the right-hand side of (62) is bounded from above as a function of $t \in \mathbb{R}$, we can apply the Fatou lemma to pull the lim sup inside the integral, and then we can use Lemma 5 again. The result follows. □

§4. Multi-trace inequalities for von Neumann algebras

As applications of Lemma 1 we now prove various inequalities that reduce to "multi-trace inequalities" in the case of finite type I factors. To apply this and other results from the literature, it will be assumed in this section that ω_ψ is a faithful normalized state on the von Neumann algebra $\mathcal{M}$ in standard form, meaning $\omega_\psi(m^*m) = 0$ implies $m = 0$ for all $m \in \mathcal{M}$. We will always take $|\psi\rangle$ to be the vector representative in the natural cone, which is hence cyclic and separating. However, we mention without proof that with some extra work all results of this section could be suitably generalized to arbitrary, non-faithful ω_ψ, e.g. using Theorem 1 and the approximation trick used in its proof.

Corollary 1. *Let $a_1, \dots, a_n \in \mathcal{M}_+$, $r \in (0,1]$, $p \geq 2$. Then*

$$\frac{1}{r} \ln \|a_1^r \cdots a_n^r \psi\|_{p/r,\psi} \leq \int_{\mathbb{R}} \mathrm{d}t\, \beta_{r/2}(t) \ln \|a_1^{1+it} \cdots a_n^{1+it} \psi\|_{p,\psi}. \tag{63}$$

Proof. We want to choose $p_1 = p$, $p_0 = \infty$, $\theta = r/2$ and

$$|G(z)\rangle = a_1^{2z} \cdots a_n^{2z} |\psi\rangle \tag{64}$$

in Lemma 1. Note that $|G(z)\rangle$ is a holomorphic on $\mathbb{S}_{1/2}$ and bounded as well as continuous on the closure $\bar{\mathbb{S}}_{1/2}$. An elementary calculation shows $p_\theta = p/r$. At the lower boundary of the strip,

$$\|G(it)\|_{p_0,\psi} = \|a_1^{2it} \cdots a_n^{2it} \psi\|_{\infty,\psi} = \|a_1^{2it} \cdots a_n^{2it}\| \leq 1 \tag{65}$$

because a_k^{2it} are isometries, and because we have the isometric identification of $L_\infty(\mathcal{M}, \psi) \ni a|\psi\rangle \mapsto a \in \mathcal{M}$ proven in [8]. The term from the upper boundary satisfies

$$\begin{aligned} \|G(1/2 + it)\|_{p_1,\psi} &= \|a_1^{1+it} \cdots a_n^{1+it} \psi\|_{p,\psi} \\ &\leq \prod_k \|a_k\| \, \|\psi\|_{p,\psi} \\ &= \prod_k \|a_k\| \, \|\psi\|^{2/p}. \end{aligned} \tag{66}$$

So Lemma 1 is applicable, the term from the lower boundary does not contribute, and we obtain the statement. □

Another corollary of a similar nature is the following:

Corollary 2 (Araki–Lieb–Thirring inequality). *For $r \geq 2$, $|\zeta\rangle \in \mathscr{H}$ there holds*

$$\|\zeta\|_{r,\psi}^2 \leq \|\Delta_{\zeta,\psi}^{r/4} \psi\|^{4/r}. \tag{67}$$

Proof. A proof for this was given first by [30] (in a somewhat different setting) and later (essentially in our setting) in [10, Thm. 12]. So the only point is to show an alternative proof. We may assume that $\|\Delta_{\zeta,\psi}^{r/4}\psi\| < \infty$ for otherwise the statement is trivial. Also, we may assume without loss of generality that $|\zeta\rangle$ is in the natural cone. In Lemma 1, we take $|G(z)\rangle = \Delta_{\zeta,\psi}^{rz/2}|\psi\rangle$, $p_1 = 2$, $p_0 = \infty$, $\theta = 1/r$, so $p_\theta = r$. Then $|G(z)\rangle$ is holomorphic on $\mathbb{S}_{1/2}$ and bounded and weakly continuous on $\bar{\mathbb{S}}_{1/2}$; see e.g. [4, Lem. 3] or apply Lemma 3.

We compute at the lower boundary of the strip,

$$\begin{aligned} \|G(it)\|_{p_0,\psi} &= \|\Delta_{\zeta,\psi}^{irt/2}\psi\|_{\infty,\psi} \\ &= \|\Delta_{\zeta,\psi}^{irt/2}\Delta_{\psi,\psi}^{-irt/2}\psi\|_{\infty,\psi} \\ &= \|u(rt/2)\psi\|_{\infty,\psi} \\ &= \|u(rt/2)\| = 1. \end{aligned} \tag{68}$$

Here, $u(t) = \Delta_{\zeta,\psi}^{it}\Delta_{\psi,\psi}^{-it}$ is the Connes cocycle which is an isometry from $\mathcal{M}$ and we used again the isometric identification of $L_\infty(\mathcal{M},\psi) \ni a|\psi\rangle \mapsto a \in \mathcal{M}$ proven in [8]. At the upper boundary of the strip,

$$\|G(1/2+it)\|_{p_1,\psi} = \|\Delta_{\zeta,\psi}^{irt/2+r/4}\psi\|_{2,\psi} = \|\Delta_{\zeta,\psi}^{r/4}\psi\| < \infty, \tag{69}$$

which no longer depends upon t, using that the L_2-norm is equal to the Hilbert space norm [8] and that $\Delta_{\zeta,\psi}^{it}$ is a unitary operator on its support. On the left-hand side of Lemma 1, which is applicable, we obtain $\ln\|\Delta_{\zeta,\psi}^{1/2}\psi\|_{r,\psi} = \ln\|\zeta\|_{r,\psi}$. On the right-hand side, the term from the lower boundary does not contribute due to $\ln(1) = 0$. Since $\int \mathrm{d}t\, \beta_\theta(t) = 1$ we obtain the statement. ☐

Let h be a self-adjoint element of $\mathcal{M}$. Following Araki [3], the non-normalized perturbed state $|\psi^h\rangle$ is defined by the absolutely convergent series

$$|\psi^h\rangle = \sum_{n=0}^{\infty}\int_0^{1/2}\mathrm{d}s_1\int_0^{s_1}\mathrm{d}s_2\cdots\int_0^{s_{n-1}}\mathrm{d}s_n\,\Delta_\psi^{s_n}h\Delta_\psi^{s_{n-1}-s_n}h\cdots\Delta_\psi^{s_1-s_2}h|\psi\rangle, \tag{70}$$

which can also be written as $e^{(\ln\Delta_\psi+h)/2}|\psi\rangle$ [4]. This technique of perturbations has been generalized to semi-bounded – instead of bounded – operators by [15]; see also [32, Sect. 12]. The perturbations, h, that would normally be in $\mathcal{M}_{\text{s.a.}}$, are in this framework generalized to so-called “extended-valued upper bounded self-adjoint operators affiliated with $\mathcal{M}$”, the space of which is called $\mathcal{M}_{\text{ext}}$. More precisely, $h \in \mathcal{M}_{\text{ext}}$ if[5]

[5] Our conventions differ from the literature such as [15] in the sense that $-h$ would be in $\mathcal{M}_{\text{ext}}$ as defined there.

(i) it is an affine, upper semi-continuous map $\mathscr{S}(\mathcal{M}) \ni \sigma \mapsto \sigma(h) \in \mathbb{R} \cup \{-\infty\}$, and
(ii) the set $\{\sigma(h) \colon \sigma \in \mathscr{S}(\mathcal{M})\}$ is bounded from above.

For any "operator" $h \in \mathcal{M}_{\rm ext}$, one shows that it is consistent to make the following definition:

Definition 1 (See [15, Thm. 3.1]). If $h \in \mathcal{M}_{\rm ext}$, the perturbed state σ^h of a normal state $\sigma \in \mathscr{S}(\mathcal{M})$, is given by the unique extremizer of the convex variational problem[6]

$$c(\sigma, h) = \sup\big\{\rho(h) - S(\rho|\sigma) \colon \rho \in \mathscr{S}(\mathcal{M})\big\} \tag{71}$$

provided the sup is not $-\infty$.

The condition $c(\sigma, h) > -\infty$ holds for example if $h \in \mathcal{M}_{\rm s.a.}$ is an ordinary self-adjoint element of the von Neumann algebra $\mathcal{M}$, and in this case the above "thermodynamic" definition of the perturbed state is up to normalizations equivalent to Araki's "perturbative" definition (70):

$$c(\sigma, h) = \ln \|\eta^h\|^2, \quad \sigma^h(m) = \langle \eta^h | m | \eta^h \rangle / \|\eta^h\|^2, \tag{72}$$

wherein $|\eta\rangle$ is a vector representer of the state σ; see [15, Ex. 3.3]. Furthermore, $h \in \mathcal{M}_{\rm ext}$ has the spectral decomposition [15, Prop. 2.13(B)]

$$h = \int_{-\infty}^{c} \lambda E_h(\mathrm{d}\lambda) - \infty \cdot q \equiv h_c - \infty \cdot q. \tag{73}$$

Here, $q \in \mathcal{M}$ is the projector onto the subspace where h is $-\infty$, and the spectral measure $E_h(\mathrm{d}\lambda)$ of h_c takes values in the projections in $(1-q)\mathcal{M}(1-q)$, so it commutes with q. The term h_c is a self-adjoint operator affiliated with $\mathcal{M}$ such that $\sigma(h) = \sigma(h_c)$ whenever σ is a normal linear functional on $\mathcal{M}$ having support $s(\sigma) \le (1-q)$.

Corollary 3 (Generalized Golden–Thompson inequality). *For $h_i \in \mathcal{M}_{\rm ext}$, $\|\psi\| = 1$ there holds*

$$\ln \|\psi^{h_1 + \dots + h_k}\|^2 \le \int_{\mathbb{R}} \mathrm{d}t \, \beta_0(t) \ln\left\{ \left\| \prod_{j=1}^{k} e^{(1/2+it)h_j} \psi \right\| \left\| \prod_{j=k}^{1} e^{(1/2-it)h_j} \psi \right\| \right\}. \tag{74}$$

Remark. By (73) and standard theorems for self-adjoint operators (e.g. [38, Thm. VIII.7(b)]), the functions $t \mapsto \prod_j e^{(1/2+it)h_j} |\psi\rangle$ (any ordering of the factors) are

[6] $S(\rho|\sigma)$ is the relative entropy defined in (92).

strongly continuous, and they are also bounded uniformly in $t \in \mathbb{R}$ in norm due to the semi-bounded nature of h_j expressed in (73). Thus the integral in (74) is well defined and definite (but could be $-\infty$) in the Lebesgue sense.

Proof of Corollary 3. **Case I.** First we assume each $h_j \in \mathcal{M}_{\text{s.a.}}$, i.e. it is bounded. We let

$$|G(z)\rangle = \Delta_\psi^{z/2} e^{zh_1} \cdots e^{zh_k} |\psi\rangle. \tag{75}$$

This family of vectors is analytic on $\mathbb{S}_{1/2}$ and uniformly bounded and weakly continuous on the closure, for instance using Lemma 3. In Lemma 1, we would like to use this with $p_1 = 2$, $p_0 = \infty$, $\theta = 1/n$, $n \in 2\mathbb{N}$, so $p_\theta = n$. At the lower boundary of $\mathbb{S}_{1/2}$, we get

$$\|G(it)\|_{\infty,\psi} = \|\varsigma_\psi^t(e^{ith_1} \cdots e^{ith_k})|\psi\rangle\|_{\infty,\psi} = \|\varsigma_\psi^t(e^{ith_1} \cdots e^{ith_k})\| = 1, \tag{76}$$

where we used the isometric identification of $L_\infty(\mathcal{M}, \psi) \ni a|\psi\rangle \mapsto a \in \mathcal{M}$ proven in [8], and where $\varsigma_\psi^t = \mathrm{Ad}\Delta_\psi^{it}$ is the modular automorphism. At the upper boundary of $\mathbb{S}_{1/2}$, we get

$$\|G(1/2 + it)\|_{2,\psi} = \|\Delta_\psi^{1/4} e^{(1/2+it)h_1} \cdots e^{(1/2+it)h_k} \psi\|, \tag{77}$$

using this time that the L_2-norm is equal to the Hilbert space norm as proven in [8]. By an application of the Hadamard three lines theorem (84), the right-hand side is uniformly bounded in t.

Thus, the assumptions of Lemma 1 are met. Only the term from the upper boundary makes a contribution in the integral, so we have

$$\begin{aligned} &\ln \|\Delta_\psi^{1/(2n)} e^{h_1/n} \cdots e^{h_k/n} \psi\|_{n,\psi}^n \\ &\qquad \le \int_{\mathbb{R}} \mathrm{d}t\, \beta_{1/n}(t) \ln \|\Delta_\psi^{1/4} e^{(1/2+it)h_1} \cdots e^{(1/2+it)h_k} \psi\|^2. \end{aligned} \tag{78}$$

Now we consider the left-hand side of this inequality, putting $a_n = e^{h_1/n} \cdots e^{h_k/n}$. By [8, Thm. 3(4)] and [24, Lem. 1], there exists $|\phi_n\rangle \in \mathscr{H}$ such that

$$\Delta_{\phi_n,\psi}^{1/n} |\psi\rangle = \Delta_\psi^{1/(2n)} a_n |\psi\rangle, \quad \|\phi_n\|^2 = \|\Delta_\psi^{1/(2n)} a_n \psi\|_{n,\psi}^n. \tag{79}$$

It follows that

$$|\phi_n\rangle = J\Delta_{\phi_n,\psi}^{1/2} |\psi\rangle = J(\Delta_\psi^{1/(2n)} a_n \Delta_\psi^{1/(2n)})^{n/2} |\psi\rangle \tag{80}$$

by a straightforward repeated application of [8, Lem. 7.7(2)]; for the details see e.g. [24, Lem. 1]. Combining (78), (79), (80), we arrive at

$$
\begin{aligned}
&\ln \|(\Delta_\psi^{1/(2n)} e^{h_1/n} \cdots e^{h_k/n} \Delta_\psi^{1/(2n)})^{n/2}\psi\|^2 \\
&\quad \le \int_{\mathbb{R}} \mathrm{d}t\, \beta_{1/n}(t) \ln \|\Delta_\psi^{1/4} e^{(1/2+it)h_1} \cdots e^{(1/2+it)h_k}\psi\|^2. \qquad (81)
\end{aligned}
$$

We now take the limit $n \to \infty$ on the left-hand side. Araki's version of the Lie–Trotter formula (suitably generalized to k operators $h_1, \dots, h_k$, using that $e^{h_1/n} \cdots e^{h_k/n} = 1 + n^{-1}(h_1 + \cdots + h_k) + O(n^{-2})$ where $\|O(n^{-2})\| \le Cn^{-2}$ for all $n > 0$), see [4, Rems 1 and 2], establishes that

$$
\begin{aligned}
\operatorname*{s\text{-}lim}_n (\Delta_\psi^{1/(2n)} e^{h_1/n} \cdots e^{h_k/n} \Delta_\psi^{1/(2n)})^{n/2}|\psi\rangle &= |\psi^{h_1+\cdots+h_k}\rangle \\
&= e^{(\ln \Delta_\psi + h_1 + \cdots + h_k)/2}|\psi\rangle, \qquad (82)
\end{aligned}
$$

so we get

$$
\ln \|\psi^{h_1+\cdots+h_k}\|^2 \le \int_{\mathbb{R}} \mathrm{d}t\, \beta_0(t) \ln \|\Delta_\psi^{1/4} e^{(1/2+it)h_1} \cdots e^{(1/2+it)h_k}\psi\|^2. \qquad (83)
$$

On the integrand we finally use the following well-known application of the Hadamard three lines theorem ($0 \le \alpha < 1/2$, $m \in \mathcal{M}$),

$$
\|\Delta_\psi^\alpha m\psi\| \le \|\Delta_\psi^{1/2} m\psi\|^{2\alpha} \|m\psi\|^{1-2\alpha} = \|m^*\psi\|^{2\alpha}\|m\psi\|^{1-2\alpha} \qquad (84)
$$

using that $z \mapsto \ln \|\Delta_\psi^z m\psi\|$ is subharmonic on $\mathbb{S}_{1/2}$. Using this with $\alpha = 1/4$, $m = e^{(1/2+it)h_1} \cdots e^{(1/2+it)h_k}$ gives the statement of the corollary.

Case II. The proof can be generalized to the case when $h_j \in \mathcal{M}_{\text{ext}}$ by reducing to case I via an approximation argument: elements $k \in \mathcal{M}_{\text{ext}}$ can be approximated by bounded self-adjoint elements $k_n \in \mathcal{M}_{\text{s.a.}}$ by introducing a cutoff in the spectral decomposition (73), as in

$$
k_n = \int_{-n}^{c} \lambda E_k(\mathrm{d}\lambda) - n \cdot q; \qquad (85)
$$

in fact one shows that $|\psi^{k_n}\rangle \to |\psi^k\rangle$ strongly (see [15, Prop. 3.15]). We perform this cutoff for every h_j obtaining an $h_{j,n}$.

Since the desired inequality holds for $h_{j,n}$ by case I, the proof is completed by the fact that $e^{(1/2+it)h_{j,n}} \to e^{(1/2+it)h_j}$ as $n \to \infty$ strongly pointwise in $t \in \mathbb{R}$ (and even uniformly in t on finite intervals of $\mathbb{R}$ by (73), (85) and standard results such as [38, p. 314, Exs 20, 21]) and Fatou's lemma, noting that $t \mapsto -\ln \|\prod_j e^{(1/2+it)h_{j,n}}\psi\|$ (any ordering of the factors) is continuous in $t \in \mathbb{R}$ and bounded below uniformly in n. □

Examples. The following examples are illustrative:

(1) In the previous corollary we take $k = 1$, $h_1 = h$. Then the norm in the integrand no longer depends upon t and we can use that $\int \mathrm{d}t\, \beta_0(t) = 1$ to get

$$\|\psi^h\| \le \|e^{h/2}\psi\|, \tag{86}$$

as shown previously by [4].

(2) Finite-dimensional type I algebras. Let $\mathcal{A} = M_n(\mathbb{C})$. We will work in the standard Hilbert space ($\mathscr{H} \simeq M_n(\mathbb{C}) \simeq \mathbb{C}^n \otimes (\mathbb{C}^n)^*$) and identify state functionals such as ω_ψ with density matrices via $\omega_\psi(a) = \mathrm{Tr}(a\omega_\psi)$. Vectors $|\zeta\rangle$ in $\mathscr{H}$ are thus identified with matrices $\zeta \in M_n(\mathbb{C})$. We have already mentioned that the $L_p(\mathcal{A}, \psi)$-norms can be computed using the expression [10] $\|\zeta\|_{p,\psi}^p = \mathrm{Tr}(\zeta\omega_\psi^{2/p-1}\zeta^*)^{p/2}$, where $|\zeta\rangle \in \mathscr{H}$ is identified with a matrix $\zeta \in M_n(\mathbb{C})$ as described. Let a_i be non-negative matrices. The multi-matrix inequality in Corollary 1 then reads, when ω_ψ is the normalized tracial state $\omega_\psi(a) = \mathrm{Tr}(a)/n$,

$$\ln \mathrm{Tr}\, |a_1^r \cdots a_k^r|^{p/r} \le \int_{\mathbb{R}} \mathrm{d}t\, \beta_{r/2}(t) \ln \mathrm{Tr}\, |a_1 a_2^{1+it} \cdots a_{k-1}^{1+it} a_k|^p, \tag{87}$$

which generalizes the Araki–Lieb–Thirring inequality (corresponding to $k = 2$). This was derived previously in [47, 39], so our result can be seen as a generalization of these results to arbitrary von Neumann algebras. Corollary 2 is another generalization of this inequality which gives nothing new in the present case. Corollary 3 gives the following inequality. Under the above identification of vectors $|\psi\rangle \in \mathscr{H}$ and matrices, the perturbed vector is

$$|\psi^h\rangle = |e^{\ln\psi + h/2}\rangle \tag{88}$$

(assuming $|\psi\rangle$ to be in the natural cone, i.e. self-adjoint and non-negative), and then choosing $|\psi = 1_n/\sqrt{n}\rangle$ as the vector representing the tracial state on $\mathcal{A}$, we have

$$\begin{aligned} &\ln \mathrm{Tr}\, e^{h_1 + \cdots + h_k} \\ &\quad \le \int_{\mathbb{R}} \mathrm{d}t\, \beta_0(t) \ln \mathrm{Tr}\, |e^{(1/2)h_1} e^{(1/2+it)h_2} \cdots e^{(1/2+it)h_{k-1}} e^{(1/2)h_k}|^2, \end{aligned} \tag{89}$$

for any hermitian matrices h_i. This reduces to the Golden–Thompson inequality for $k = 2$,

$$\mathrm{Tr}\, e^{h_1 + h_2} \le \mathrm{Tr}(e^{h_1} e^{h_2}), \tag{90}$$

using that the trace in the integrand no longer depends on t and $\int \mathrm{d}t\, \beta_0(t) = 1$. For an arbitrary number of matrices this is due to [39], who also explain the relation with Lieb's triple matrix inequality (corresponding to the case $k = 3$).

§5. Improved DPI and recovery channels

§5.1. Relative entropy and measured relative entropy

For the von Neumann algebra $\mathcal{A} = M_n(\mathbb{C})$, the relative entropy between two states (density matrices) ω_ψ, ω_η is defined by

$$S(\omega_\psi|\omega_\eta) = \mathrm{Tr}(\omega_\psi \ln \omega_\psi - \omega_\psi \ln \omega_\eta). \tag{91}$$

This may be expressed in terms of the logarithm of the relative modular operator in (10), and this observation is the basis for Araki's approach [5, 6, 7] to relative entropy for general von Neumann algebras. The main technical difference in the general case is that the individual terms in the above expression, such as the von Neumann entropy $-\mathrm{Tr}(\omega_\psi \ln \omega_\psi)$, are usually infinite. Thus, from a mathematical viewpoint, the relative but not the absolute entropy is the primary concept.

Let $(\mathcal{M}, J, \mathscr{P}^\natural_\mathcal{M}, \mathscr{H})$ be a von Neumann algebra in standard form acting on a Hilbert space $\mathscr{H}$, with natural cone $\mathscr{P}^\natural_\mathcal{M}$ and modular conjugation J, and let ω_ψ, ω_η be normal state functionals with vector representatives $|\psi\rangle$, $|\eta\rangle$ in the natural cone. According to [5, 6, 7], if $\pi^\mathcal{M}(\eta) \geq \pi^\mathcal{M}(\psi)$, the relative entropy may be defined in terms of them by[7]

$$S(\omega_\psi|\omega_\eta) = -\lim_{\alpha\to 0^+} \frac{\langle\psi|\Delta^\alpha_{\eta,\psi}\psi\rangle - 1}{\alpha}, \tag{92}$$

otherwise it is by definition infinite. Araki's definition of $S(\omega_\psi|\omega_\eta)$ is independent of the choice of natural cone $\mathscr{P}^\natural_\mathcal{M}$ and it still satisfies the DPI (1) ([45]) along with many other properties; see e.g. [32]. For generalizations of S in various directions in the setting of von Neumann algebras, see e.g. [22].

For $t \in \mathbb{R}$, the Connes cocycle $[D\psi : D\eta]_t$ is the isometric operator from $\mathcal{M}$ satisfying

$$[D\psi : D\eta]_{-t} \pi^{\mathcal{M}'}(\psi) = \Delta^{it}_{\psi,\psi} \Delta^{-it}_{\eta,\psi}. \tag{93}$$

In terms of the Connes cocycle, the relative entropy (92) may also be defined as

$$S(\omega_\psi|\omega_\eta) = -i \frac{\mathrm{d}}{\mathrm{d}t} \omega_\psi([D\eta : D\psi]_t)|_{t=0}. \tag{94}$$

The derivative exists whenever $S(\omega_\psi|\omega_\eta) < \infty$ [32, Thm. 5.7].

Later we will use the following variational expression for the relative entropy [36, Thm. 9]:

$$S(\omega_\psi|\omega_\eta) = \sup_{h\in\mathcal{M}_{\mathrm{s.a.}}} \{\omega_\psi(h) - \ln \|\eta^h\|^2\}, \tag{95}$$

[7] The limit exists under this condition but may be equal to $+\infty$.

with $\mathcal{M}_{\text{s.a.}}$ is the set of self-adjoint elements of $\mathcal{M}$. A related variational quantity is the "measured relative entropy" S_{meas}, defined as (see, e.g. [32, Prop. 7.13])

$$S_{\text{meas}}(\omega_\psi|\omega_\eta) = \sup_{h \in \mathcal{M}_{\text{s.a.}}} \{\omega_\psi(h) - \ln \|e^{h/2}\eta\|^2\}. \tag{96}$$

From the Golden–Thompson inequality (86) we find

$$S_{\text{meas}}(\omega_\psi|\omega_\eta) \leq S(\omega_\psi|\omega_\eta). \tag{97}$$

The measured relative entropy S_{meas} can also be written in terms of the classical relative entropy $S(\mu|\nu)$ (Kullback–Leibler divergence) of two probability measures:

$$S(\mu|\nu) = \int \mathrm{d}\mu \, \ln \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \tag{98}$$

as follows. Let $a \in \mathcal{M}_{\text{s.a.}}$ be a self-adjoint element of $\mathcal{M}$. Then it has a spectral decomposition

$$a = \int \lambda E_a(\mathrm{d}\lambda) \tag{99}$$

with an $\mathcal{M}$-valued projection measure $E_a(\mathrm{d}\lambda)$. Given $|\psi\rangle, |\eta\rangle \in \mathscr{H}$, we get Borel measures $\mathrm{d}\mu_{\psi,a} = \langle\psi|E_a(\mathrm{d}\lambda)\psi\rangle$, and likewise for $|\eta\rangle$. Physically, these correspond to the probability distributions for measurement outcomes of a in the states $|\psi\rangle$ resp. $|\eta\rangle$. The relative entropy between these measures is defined (but can be $+\infty$) if $\operatorname{supp}\mu_{\eta,a} \subset \operatorname{supp}\mu_{\psi,a}$, wherein $\mathrm{d}\mu_{\psi,a}/\mathrm{d}\mu_{\eta,a}$ means the Radon–Nikodym derivative between the measures. We may perform the maximization in (96) over $f(h)$ with[8] $f \in L^\infty(\mathbb{R};\mathbb{R})$ and $h \in \mathcal{M}_{\text{s.a.}}$ because $f(h) \in \mathcal{M}_{\text{s.a.}}$. Maximizing first for fixed h over f and using (= equation (95) in the commutative case)

$$\sup\left\{\textstyle\int f \, \mathrm{d}\mu - \ln \int e^f \, \mathrm{d}\nu \colon f \in L^\infty(\mathbb{R};\mathbb{R})\right\} = S(\mu|\nu), \tag{100}$$

we can write the measured relative entropy in the following way:

$$\begin{aligned} &S_{\text{meas}}(\omega_\psi|\omega_\eta) \\ &\quad = \sup\left\{S(\mu_{\psi,h}|\mu_{\eta,h}) \colon h \in \mathcal{M}_{\text{s.a.}}\right\} \\ &\quad = \sup\left\{S(\omega_{\psi|\mathcal{C}}|\omega_{\eta|\mathcal{C}}) \colon \mathcal{C} \subset \mathcal{M} \text{a commutative von Neumann subalgebra}\right\}. \end{aligned} \tag{101}$$

This motivates the name "measured relative entropy". The second equality holds by [32, Prop. 7.13].

[8] More precisely, the space L^∞ is defined relative to the measure $\mu_{h,\psi}$ relative to some faithful normal state $\psi \in \mathscr{S}(\mathcal{M})$. Depending on the nature of this measure, "L^∞" means either $\ell^\infty(\{1,\dots,n\})$, $\ell^\infty(\mathbb{N})$, or $L^\infty(\mathbb{R})$, or a combination thereof, wherein the counting measure is understood in the first two cases, whereas the Lebesgue measure is understood in the last case.

For later, we would like to know the relationship between S_{meas} and the fidelity, F. According to [44, 1, 2], the fidelity between two states $\omega_\eta, \omega_\psi \in \mathscr{S}(\mathcal{M})$ on a von Neumann algebra $\mathcal{M}$ in standard form may be defined as

$$F(\omega_\psi|\omega_\eta) = \sup\big\{|\langle\eta|u'\psi\rangle| \colon u' \in \mathcal{M}',\ \|u'\| = 1\big\}. \tag{102}$$

It is related to the L_1-norm relative to $\mathcal{M}'$ by $F(\omega_\psi|\omega_\eta) = \|\eta\|_{1,\psi,\mathcal{M}'}$; see e.g. paper I ([17, Lem. 3(1)]). We make the following claim:

Proposition 1. *If $\omega_\eta \in \mathscr{S}(\mathcal{M})$ is a faithful state on the von Neumann algebra $\mathcal{M}$, then $S_{\mathrm{meas}}(\omega_\psi|\omega_\eta) \geq -\ln F(\omega_\psi|\omega_\eta)^2$.*

Proof. We may assume that $|\eta\rangle$ is in the natural cone $\mathscr{P}^\sharp_{\mathcal{M}}$, hence cyclic and separating. Consider in $L_1(\mathcal{M}', \eta)$ the polar decomposition $|\psi_+\rangle = u'|\psi\rangle$ into a $u' \in \mathcal{M}'$ such that $u'^*u' = \pi^{\mathcal{M}'}(\psi) \leq 1$ and $|\psi_+\rangle \in L_1^+(\mathcal{M}', \eta) \cap \mathscr{H} = \mathscr{P}^{1/2}_{\mathcal{M}'}$; see [8, Thm. 3]. By definition, the cone $\mathscr{P}^{1/2}_{\mathcal{M}'}$ is the closure of $\Delta'^{1/2}_\eta \mathcal{M}'_+|\eta\rangle$ (in the topology of $\mathscr{H}$), which equals the closure of $\mathcal{M}_+|\eta\rangle$, since $J\Delta'^{1/2}_\eta a'|\eta\rangle = a'|\eta\rangle$ for $a' \in \mathcal{M}'_+$, $J|\eta\rangle = |\eta\rangle$, and $J\mathcal{M}'J = \mathcal{M}$. Thus, there exists a sequence $\{a_n\} \subset \mathcal{M}_+$ such that $\lim_n a_n|\eta\rangle = u'|\psi\rangle$ in the topology of $\mathscr{H}$, so

$$\lim_n \langle\eta|a_n\eta\rangle = \langle\eta|u'\psi\rangle \in \mathbb{R}_+. \tag{103}$$

Let $E_{a_n}(\mathrm{d}\lambda)$ be the spectral decomposition of a_n and $\mathrm{d}\mu_{a_n,\psi} = \langle\psi|E_{a_n}(\mathrm{d}\lambda)\psi\rangle$, $\mathrm{d}\mu_{a_n,\eta} = \langle\eta|E_{a_n}(\mathrm{d}\lambda)\eta\rangle$. Applying [10, Thm. 13 and Lem. 9] to the commutative case gives

$$\begin{aligned} S_{\mathrm{meas}}(\omega_\psi|\omega_\eta) \geq S(\mu_{a_n,\psi}|\mu_{a_n,\eta}) &\geq -2\ln F(\mu_{a_n,\psi}|\mu_{a_n,\eta}) \\ &= -2\ln \int \Big(\frac{\mathrm{d}\mu_{a_n,\psi}}{\mathrm{d}\mu_{a_n,\eta}}\Big)^{1/2} \mathrm{d}\mu_{a_n,\eta} \end{aligned} \tag{104}$$

(where the Radon–Nikodym derivative is defined since $|\eta\rangle$ is faithful). As functionals on $\mathcal{M}$, we have $\omega_\psi = \omega_{u'\psi}$ because $u'^*u' = \pi^{\mathcal{M}'}(\psi)$ and u' commutes with $\mathcal{M}$. Let $\mathcal{C}_n = \{a_n\}''$ be the commutative von Neumann subalgebra of $\mathcal{M}$ generated by a_n. It can be identified canonically with $L^\infty(\mathbb{R}, \mathrm{d}\mu_{a_n,\eta})$ via the spectral theorem. Denoting by $\|f\|_1$ the norm of a linear functional $f \colon L^\infty(\mathbb{R}, \mathrm{d}\mu_{\eta,a_n}) \to \mathbb{C}$, then if f is the restriction of some normal functional ω on $\mathcal{M}$ to $\mathcal{C}_n$, we obviously have $\|f\|_1 \leq \|\omega\|$. Therefore, we have

$$\begin{aligned} \|\mu_{\psi,a_n} - \mu_{a_n\eta,a_n}\|_1 &= \|\omega_\psi|_{\mathcal{C}_n} - \omega_{a_n\eta}|_{\mathcal{C}_n}\| \\ &\leq \|\omega_\psi - \omega_{a_n\eta}\| \\ &= \|\omega_{u'\psi} - \omega_{a_n\eta}\| \\ &\leq \|u'\psi + a_n\eta\|\,\|u'\psi - a_n\eta\|, \end{aligned} \tag{105}$$

using (11) in the last step. Since $\lim_n a_n|\eta\rangle = u'|\psi\rangle$ in the topology of $\mathscr{H}$, we thus get $\lim_n \|\mu_{\psi,a_n} - \mu_{a_n\eta,a_n}\|_1 = 0$. By paper I ([17, Lem. 11]) applied to the commutative von Neumann algebra $\mathcal{C}_n$, we thereby get

$$|F(\mu_{\psi,a_n}|\mu_{\eta,a_n}) - F(\mu_{a_n\eta,a_n}|\mu_{\eta,a_n})| \le \|\mu_{\psi,a_n} - \mu_{a_n\eta,a_n}\|_1^{1/2} \to 0. \tag{106}$$

By definition,

$$\left(\frac{\mathrm{d}\mu_{a_n\eta,a_n}(\lambda)}{\mathrm{d}\mu_{\eta,a_n}(\lambda)}\right)^{1/2} = \lambda \quad \text{for } \lambda \in \mathbb{R}_+, \tag{107}$$

hence using (106) in the $\lim_n$ of (104) gives

$$\begin{aligned} S_{\mathrm{meas}}(\omega_\psi|\omega_\eta) &\ge -2\ln\lim_n \int \lambda\,\mathrm{d}\mu_{\eta,a_n} \\ &= -2\ln\lim_n \int \lambda\langle\eta|E_{a_n}(\mathrm{d}\lambda)\eta\rangle \\ &= -2\ln\lim_n \langle\eta|a_n\eta\rangle \\ &= -2\ln\langle\eta|u'\psi\rangle \\ &= -2\ln|\langle\eta|u'\psi\rangle|. \end{aligned} \tag{108}$$

The right-hand side is by definition $\ge -\ln F(\omega_\psi|\omega_\eta)^2$ as $\|u'\| = 1$, $u' \in \mathcal{M}'$, which concludes the proof. $\square$

Remark. The unknown referee has pointed out the following alternative proof. By [10, Thm. 13 and Lem. 9], $S_{\mathrm{meas}}(\omega_\psi|\omega_\eta) \ge -2\ln F_{\mathrm{meas}}(\omega_\psi|\omega_\eta)$, where F_{meas} is the measured version of the fidelity defined analogously to S_{meas} by restricting to all possible commutative subalgebras. But it has also been shown that $F_{\mathrm{meas}}(\omega_\psi|\omega_\eta) = F(\omega_\psi|\omega_\eta)$; see [22, eqn. (5.26)].

§5.2. Petz recovery map

We now recall the definition of the Petz map in the case of general von Neumann algebras, discussed in more detail in [32, Sect. 8]. Let $T\colon \mathcal{B} \to \mathcal{A}$ be a normal (ultraweakly continuous) *-preserving linear map between two von Neumann algebras $\mathcal{A}$, $\mathcal{B}$ in standard form acting on Hilbert spaces $\mathscr{H}$, $\mathscr{K}$. If

$$\Big(\langle\zeta_1|\,\langle\zeta_2|\Big)\, T\left(\begin{bmatrix} a\ b \\ c\ d\end{bmatrix}\begin{bmatrix} a^*\ c^* \\ b^*\ d^*\end{bmatrix}\right)\begin{pmatrix}|\zeta_1\rangle \\ |\zeta_2\rangle\end{pmatrix} \ge 0 \quad \forall\,|\zeta_i\rangle \in \mathscr{H},\ T(1_{\mathcal{B}}) = 1_{\mathcal{A}}, \tag{109}$$

and for all $a, b, c, d \in \mathcal{B}$, then T is called 2-positive and unital. In the matrix inequality, we mean T applied to each matrix element. By duality between $\mathcal{A}$ and

$\mathscr{S}(\mathcal{A})$, $T\colon \mathcal{B} \to \mathcal{A}$ gives a corresponding map $\widetilde{T}\colon \mathscr{S}(\mathcal{A}) \to \mathscr{S}(\mathcal{B})$ by $\omega \mapsto \widetilde{T}(\omega) \coloneqq \omega \circ T$. For finite-dimensional von Neumann algebras $\mathcal{A}$, $\mathcal{B}$, where state functionals are identified with density matrices through $\omega(a) = \mathrm{Tr}(\omega a)$, we can think of $\widetilde{T}$ as the linear operator on density matrices defined by

$$\mathrm{Tr}\, \omega T(b) = \mathrm{Tr}\, \widetilde{T}(\omega) b \quad \forall\, b \in \mathcal{B}. \tag{110}$$

This operator $\widetilde{T}$ is 2-positive and trace preserving. The quantum DPI [45] states that

$$S(\omega_\psi | \omega_\eta) \geq S(\omega_\psi \circ T | \omega_\eta \circ T), \tag{111}$$

where the right-hand side could also be written as $S(\widetilde{T}(\omega_\psi)|\widetilde{T}(\omega_\eta))$.

We recall the definition of the Petz map. Let $|\eta_\mathcal{A}\rangle$ be a cyclic and separating vector in the natural cone of a von Neumann algebra $\mathcal{A}$ in standard form. Then the "KMS" scalar product on $\mathcal{A}$ is defined as

$$\langle a_1, a_2 \rangle_\eta = \langle \eta_\mathcal{A} | a_1^* \Delta_\eta^{1/2} a_2 \eta_\mathcal{A} \rangle. \tag{112}$$

Let ω_η be the faithful normal state functional on $\mathcal{A}$ associated with $|\eta_\mathcal{A}\rangle$. Then its pull-back $\omega_\eta \circ T$ to $\mathcal{B}$, which is also assumed to be faithful (for simplicity), has a cyclic and separating vector representative $|\eta_\mathcal{B}\rangle \in \mathscr{K}$ in the natural cone. So

$$\omega_\eta(a) = \langle \eta_\mathcal{A} | a \eta_\mathcal{A} \rangle, \quad \omega_\eta \circ T(b) = \langle \eta_\mathcal{B} | b \eta_\mathcal{B} \rangle. \tag{113}$$

The terms $|\eta_\mathcal{A}\rangle$ resp. $|\eta_\mathcal{B}\rangle$ give KMS scalar products for $\mathcal{A}$ resp. $\mathcal{B}$, which we can use to define the adjoint $T^+\colon \mathcal{A} \to \mathcal{B}$ (depending on the choices of these vectors) of the normal, unital, and 2-positive $T\colon \mathcal{B} \to \mathcal{A}$, which is again normal, unital, and 2-positive; see [32, Prop. 8.3]. For finite-dimensional matrix algebras, T^+ corresponds dually to the linear operator $\widetilde{T}^+$ acting on density matrices ρ for $\mathcal{B}$ given by

$$\widetilde{T}^+(\rho) = \sigma_\mathcal{A}^{1/2} T\big(\sigma_\mathcal{B}^{-1/2} \rho \sigma_\mathcal{B}^{-1/2}\big) \sigma_\mathcal{A}^{1/2}, \tag{114}$$

wherein $\sigma_\mathcal{A}$ is the density matrix of $|\eta_\mathcal{A}\rangle$ and $\sigma_\mathcal{B} = \widetilde{T}(\sigma_\mathcal{A})$ for $|\eta_\mathcal{B}\rangle$. The rotated Petz map, which we call $\alpha^t_{\eta,T}\colon \mathcal{A} \to \mathcal{B}$, is defined by conjugating this with the respective modular flows, i.e.

$$\alpha^t_{\eta,T} = \varsigma^t_{\eta,\mathcal{B}} \circ T^+ \circ \varsigma^{-t}_{\eta,\mathcal{A}}, \tag{115}$$

where $\varsigma^t_{\eta,\mathcal{A}} = \mathrm{Ad}\Delta^{it}_{\eta,\mathcal{A}}$ is the modular flow for $\mathcal{A}, |\eta_\mathcal{A}\rangle$ etc. For finite-dimensional matrix algebras, $\alpha^t_{\eta,T}$ gives by duality a linear operator $\tilde{\alpha}^t_{\eta,T}$ acting on density matrices ρ for $\mathcal{B}$, which is

$$\tilde{\alpha}^t_{\eta,T}(\rho) = \sigma_\mathcal{A}^{1/2-it} T\big(\sigma_\mathcal{B}^{-1/2+it} \rho \sigma_\mathcal{B}^{-1/2-it}\big) \sigma_\mathcal{A}^{1/2+it}. \tag{116}$$

The following is an equivalent definition of the rotated Petz map:

Definition 2. Let $T\colon \mathcal{B} \to \mathcal{A}$ be a unital, normal, and 2-positive linear map and ω_η a normal state on $\mathcal{A}$ with ω_η, $\omega_\eta \circ T$ faithful. Then the rotated Petz map $\alpha^t_{\eta,T}\colon \mathcal{A} \to \mathcal{B}$ is defined implicitly by the identity

$$\langle b\eta_{\mathcal{B}}|J_{\mathcal{B}}\Delta^{it}_{\eta_{\mathcal{B}}}\alpha^t_{\eta,T}(a)\eta_{\mathcal{B}}\rangle = \langle T(b)\eta_{\mathcal{A}}|J_{\mathcal{A}}\Delta^{it}_{\eta_{\mathcal{A}}}a\eta_{\mathcal{A}}\rangle \tag{117}$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$.

Closely related to the Petz map is the linear map [33, 35] $V_\psi\colon \mathscr{K} \to \mathscr{H}$ associated with T and a vector $|\psi_{\mathcal{A}}\rangle$ in the natural cone of $\mathcal{A}$. Let ω_ψ be the associated state functional on $\mathcal{A}$, $\omega_\psi \circ T$ its pull-back to $\mathcal{B}$ with vector representative $|\psi_{\mathcal{B}}\rangle$ in the natural cone of $\mathcal{B}$. If $|\psi_{\mathcal{B}}\rangle$ is separating (hence cyclic), V_ψ is defined by[9]

$$V_\psi b|\psi_{\mathcal{B}}\rangle := T(b)|\psi_{\mathcal{A}}\rangle \quad (b \in \mathcal{B}). \tag{118}$$

It follows from Kadison's property $T(a^*a) \geq T(a^*)T(a)$ (which is a consequence of (109)) that V_ψ is a contraction $\|V_\psi\| \leq 1$; see e.g. [33, proof of Thm. 4].

As in paper II ([16]), we introduce a vector-valued function

$$z \mapsto |\Gamma_\psi(z)\rangle := \Delta^z_{\eta_{\mathcal{A}},\psi_{\mathcal{A}}} V_\psi \Delta^{-z}_{\eta_{\mathcal{B}},\psi_{\mathcal{B}}}|\psi_{\mathcal{B}}\rangle \quad (z \in \bar{\mathbb{S}}_{1/2}), \tag{119}$$

the existence and properties of which are established in paper II ([16, Lems 3, 4]). In particular, $|\Gamma_\psi(z)\rangle$ is holomorphic inside the strip $\mathbb{S}_{1/2}$ and bounded in the closure $\bar{\mathbb{S}}_{1/2}$ in norm by 1. The relation to the Petz map is as follows (paper II, [16, Lem. 2]):

$$\langle\Gamma_\psi(1/2+it)|a\Gamma_\psi(1/2+it)\rangle \leq \omega_\psi \circ T \circ \alpha^t_{\eta,T}(a) \quad t \in \mathbb{R},\ a \in \mathcal{A}_+. \tag{120}$$

§5.3. Improved DPI

Our main theorem is the following:

Theorem 2. *Let $T\colon \mathcal{B} \to \mathcal{A}$ be a 2-positive, unital (in the sense of (109)) linear map between two von Neumann algebras, and let ω_ψ, ω_η be normal states on $\mathcal{A}$, with $\omega_\eta, \omega_\eta \circ T$ faithful. Then*

$$S(\omega_\psi|\omega_\eta) - S(\omega_\psi \circ T|\omega_\eta \circ T) \geq S_{\mathrm{meas}}(\omega_\psi|\omega_\psi \circ T \circ \alpha_{T,\eta}), \tag{121}$$

[9] In the general case, one can define [33] instead

$$V_\psi(b|\psi_{\mathcal{B}}\rangle + |\zeta\rangle) := T(b)|\psi_{\mathcal{A}}\rangle \quad (b \in \mathcal{B},\ \pi^{\mathcal{B}'}(\psi)|\zeta\rangle = 0).$$

with the recovery channel

$$\alpha_{T,\eta} \equiv \int_{\mathbb{R}} \mathrm{d}t\, \beta_0(t) \alpha^t_{T,\eta}. \tag{122}$$

When both $S(\omega_\psi|\omega_\eta), S(\omega_\psi \circ T|\omega_\eta \circ T) = +\infty$, *we agree that the left-hand side of* (121) *should be considered as* $+\infty$.

Remark. The following points may help the reader appreciate the scope of Theorem 2:

(1) The theorem should generalize to non-faithful states by applying appropriate support projections in a similar way to paper I ([17, Lem. 1]).

(2) For finite-dimensional type I von Neumann algebras, i.e. matrices, our result is due to [39]. The recovery channel is given explicitly by (116) in this case as an operator on density matrices, where $\sigma_{\mathcal{A}}$, $\sigma_{\mathcal{B}}$ are the density matrices corresponding to ω_η, $\omega_\eta \circ T$ respectively.

(3) By Proposition 1, our bound implies that given in our previous paper II ([16, Thm. 2]) for the fidelity; in fact it is stronger in many cases. However, it is not stronger than the version of the theorem in paper II ([16, Thm. 1]) with the integral outside.

(I) Proof of Theorem 2 under a majorization condition. First we consider the special case where there exists $\infty > c \geq 1$ such that

$$c^{-1}\omega_\eta \leq \omega_\psi \leq c\omega_\eta. \tag{123}$$

Since ω_η is faithful, it follows that so is ω_ψ. We choose the vector representatives $|\eta\rangle$, $|\psi\rangle$ (alternatively called $|\eta_{\mathcal{A}}\rangle$, $|\psi_{\mathcal{A}}\rangle$, depending on the context, to make their relation to the algebra $\mathcal{A}$ clear) for ω_η, ω_ψ in the natural cone, which are then cyclic and separating. Note that (123) implies $c^{-1}\omega_\eta \circ T \leq \omega_\psi \circ T \leq c\omega_\eta \circ T$ as T is positive. Again, since $\omega_\eta \circ T$ is faithful, it follows that so is $\omega_\psi \circ T$. We choose the vector representatives $|\eta_{\mathcal{B}}\rangle$, $|\psi_{\mathcal{B}}\rangle$ for $\omega_\eta \circ T$, $\omega_\psi \circ T$ in the natural cone, which are then cyclic and separating.

By [32, Thm. 12.11] (due to Araki), there exists an $h = h^* \in \mathcal{A}$ such that $|\psi_{\mathcal{A}}\rangle = |\eta^h_{\mathcal{A}}\rangle / \|\eta^h_{\mathcal{A}}\|$ with $\|h\| \leq \ln c$, and vice versa. As is well known, this furthermore implies that the Connes cocycle $[D\eta_{\mathcal{A}} : D\psi_{\mathcal{A}}]_{iz}$ is holomorphic in the two-sided strip $\{z \in \mathbb{C} \colon |\mathrm{Re}(z)| < 1/2\}$, weakly continuous and bounded in norm (by $c^{\mathrm{Re}(z)}$) on the closure of this strip (see e.g. paper II [16, Lem. 5]), and similar statements hold for $[D\eta_{\mathcal{B}} : D\psi_{\mathcal{B}}]_{iz}$.

By (94), we thereby conclude that $S(\omega_\psi|\omega_\eta), S(\omega_\psi \circ T|\omega_\eta \circ T) < \infty$, and near $z = 0$, we have an absolutely convergent (in the operator norm) power series expansion

$$[D\eta_{\mathcal{B}} : D\psi_{\mathcal{B}}]_{iz} = 1 + \sum_{l=1}^{\infty} z^l k_l, \tag{124}$$

with bounded operators $k_l \in \mathcal{B}$ such that $\|k_l\| \le C^l$. We set

$$k := \frac{\mathrm{d}}{i\,\mathrm{d}t} T([D\eta_{\mathcal{B}} : D\psi_{\mathcal{B}}]_t)|_{t=0} \in \mathcal{A}_{\text{s.a.}}. \tag{125}$$

Using [32, Thm. 5.7] and the definition of the relative entropy in terms of the Connes cocycle (94),

$$\begin{aligned} S_{\mathcal{A}}(\omega_\psi|\omega_{\eta^k}) &= S_{\mathcal{A}}(\omega_\psi|\omega_\eta) - \omega_\psi(k) \\ &= S_{\mathcal{A}}(\omega_\psi|\omega_\eta) - \left\langle \psi_{\mathcal{A}} \middle| \frac{\mathrm{d}}{i\,\mathrm{d}t} T([D\eta_{\mathcal{B}} : D\psi_{\mathcal{B}}]_t)\psi_{\mathcal{A}} \right\rangle \Big|_{t=0} \\ &= S_{\mathcal{A}}(\omega_\psi|\omega_\eta) - S_{\mathcal{B}}(\omega_\psi|\omega_\eta), \end{aligned} \tag{126}$$

which is one side of the inequality that we would like to prove. The variational expression (95) then gives

$$S_{\mathcal{A}}(\omega_\psi|\omega_\eta) - S_{\mathcal{B}}(\omega_\psi|\omega_\eta) = \sup_{h \in \mathcal{A}_{\text{s.a.}}} \{\omega_\psi(h) - \ln \|\eta^{k+h}\|^2\}, \tag{127}$$

where we used $|(\eta^k)^h\rangle = |\eta^{k+h}\rangle$; see [32, Thm. 12.10]. To get the desired DPI we will establish an upper bound on $\ln \|\eta^{k+h}\|^2$.

We want to use Lemma 1 with (here $|\Gamma_\psi(z)\rangle$ is as in (119))

$$|G(z)\rangle = e^{zh}|\Gamma_\psi(z)\rangle, \tag{128}$$

with the cyclic and separating vector $|\psi\rangle$, and with $p_0 = \infty$, $p_1 = 2$, $\theta = 1/n$, where $n \in 4\mathbb{N}$ and $h = h^* \in \mathcal{A}$. We have the representation $|\Gamma_\psi(z)\rangle = \Delta^z_{\eta_{\mathcal{A}},\psi_{\mathcal{A}}} T([D\eta_{\mathcal{B}} : D\psi_{\mathcal{B}}]_{-iz})|\psi_{\mathcal{B}}\rangle$, and $T([D\eta_{\mathcal{B}} : D\psi_{\mathcal{B}}]_{-iz})|\psi_{\mathcal{B}}\rangle$ is holomorphic on $\mathbb{S}_{1/2}$, and bounded and weakly continuous on $\bar{\mathbb{S}}_{1/2}$. Then applying Lemma 3 twice proves that the same is true for $|\Gamma_\psi(z)\rangle$ and for $|G(z)\rangle$.

At the lower boundary of $\mathbb{S}_{1/2}$ we have, with $u_{\mathcal{B}}(t) := [D\eta_{\mathcal{B}} : D\psi_{\mathcal{B}}]_t \in \mathcal{B}$, $u_{\mathcal{A}}(t) := [D\eta_{\mathcal{A}} : D\psi_{\mathcal{A}}]_t \in \mathcal{A}$ the unitary Connes cocycles,

$$\begin{aligned} \|G(it)\|_{p_0,\psi} &= \|e^{ith}\Delta^{it}_{\eta_{\mathcal{A}},\psi_{\mathcal{A}}} V_\psi \Delta^{-it}_{\eta_{\mathcal{B}},\psi_{\mathcal{B}}} \psi_{\mathcal{B}}\|_{\infty,\psi} \\ &= \|e^{ith}\Delta^{it}_{\eta_{\mathcal{A}},\psi_{\mathcal{A}}} T(u_{\mathcal{B}}(t))\psi\|_{\infty,\psi} \\ &= \|e^{ith}\varsigma^t_\eta[T(u_{\mathcal{B}}(t))]u_{\mathcal{A}}(-t)\psi\|_{\infty,\psi} \\ &= \|e^{ith}\varsigma^t_\eta[T(u_{\mathcal{B}}(t))]u_{\mathcal{A}}(-t)\| \\ &= \|\varsigma^t_\eta[T(u_{\mathcal{B}}(t))]\| \\ &= \|T(u_{\mathcal{B}}(t))\| \le \|u_{\mathcal{B}}(t)\| = 1, \end{aligned} \tag{129}$$

where we used $\|\varsigma_\eta^t[a]\| = \|a\|$ for all $a \in \mathcal{A}$ (since $\varsigma_\eta^t = \mathrm{Ad}\Delta_{\eta_\mathcal{A}}^{it}$), $\|T(b)\| \le \|b\|$ for all $b \in \mathcal{B}$ (since T is 2-positive and unital[10]), and the isometric identification of $L_\infty(\mathcal{A}, \psi) \ni a|\psi\rangle \mapsto a \in \mathcal{A}$ proven in [8]. At the upper boundary of $\mathbb{S}_{1/2}$ we have

$$\|G(1/2 + it)\|_{p_1,\psi} = \|e^{h/2}\Gamma_\psi(1/2 + it)\| \tag{130}$$

using the isometric identification of $L_2(\mathcal{A}, \psi)$ and $\mathscr{H}$ proven in [8]. We have already argued that the right-hand side is uniformly bounded in t.

Thus, we can apply Lemma 1, and since $p_\theta = n$ and $\ln \|G(it)\|_{p_0,\psi} \le 0$ as just shown, we get from this lemma that

$$\begin{aligned}
\ln \|e^{h/n}\Gamma_\psi(1/n)\|_{n,\psi}^n &\le \int_\mathbb{R} \mathrm{d}t\, \beta_{1/n}(t) \ln \|G(1/2 + it)\|_{p_1,\psi}^2 \\
&= \int_\mathbb{R} \mathrm{d}t\, \beta_{1/n}(t) \ln \|e^{h/2}\Gamma_\psi(1/2 + it)\|^2 \\
&\le \ln \int_\mathbb{R} \mathrm{d}t\, \beta_{1/n}(t) \|e^{h/2}\Gamma_\psi(1/2 + it)\|^2 \\
&\le \ln \int_\mathbb{R} \mathrm{d}t\, \beta_{1/n}(t) \omega_\psi \circ T \circ \alpha_{\eta,T}^t(e^h),
\end{aligned} \tag{131}$$

using (120) in the fourth line, Jensen's inequality in the third (noting that $\|e^{h/2}\Gamma_\psi(1/2 + it)\|^2$ is uniformly bounded in $t \in \mathbb{R}$), and (130) in the second. Taking the $\limsup_{n\to\infty}$, we get, using the definition of the recovery channel $\alpha_{T,\eta}$,

$$\limsup_n \ln \|e^{h/n}\Gamma_\psi(1/n)\|_{n,\psi}^n \le \omega_\psi \circ T \circ \alpha_{\eta,T}(e^h). \tag{132}$$

This is our first main intermediate result. The next lemmas give an expression for the lim sup:

Lemma 6. *We have* $\|e^{h/n}\Gamma_\psi(1/n)\|_{n,\psi}^n = \|(e^{h/n}\Delta_{\eta,\psi}^{1/n} a_n \Delta_{\eta,\psi}^{1/n} e^{h/n})^{n/4}\psi\|^2$, *where*

$$a_n = T([D\eta_\mathcal{B} : D\psi_\mathcal{B}]_{-i/n}) T([D\eta_\mathcal{B} : D\psi_\mathcal{B}]_{-i/n})^* \in \mathcal{A}_+. \tag{133}$$

Lemma 7. *We have* $\lim_n \|(e^{h/n}\Delta_{\eta,\psi}^{1/n} a_n \Delta_{\eta,\psi}^{1/n} e^{h/n})^{n/4}\psi\|^2 = \|\eta^{k+h}\|^2$.

Combining the two lemmas with equations (127), (132) gives the statement of the theorem:

$$\begin{aligned}
S_\mathcal{A}(\omega_\psi|\omega_\eta) - S_\mathcal{B}(\omega_\psi|\omega_\eta) &\ge \sup_{h \in \mathcal{A}_{\mathrm{s.a.}}} \{\omega_\psi(h) - \ln \omega_\psi \circ T \circ \alpha_{\eta,T}(e^h)\} \\
&= S_{\mathrm{meas}}(\omega_\psi|\omega_\psi \circ T \circ \alpha_{T,\eta}),
\end{aligned} \tag{134}$$

using the variational definition (96) of S_{meas} in the last step.

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[10]Indeed, we have $\|T(b)\|^2 = \|T(b)T(b)^*\| = \|T(b)T(b^*)\| \le \|T(bb^*)\| \le \|bb^*\| = \|b\|^2$ using the properties of the norm, the $*$-preserving property of T, Kadison's inequality, and the unital property of T.

Proof of Lemma 7. Since (124) is an absolutely convergent power series in the operator norm, it follows from (125) that $a_n = 1 + 2n^{-1}k + O(n^{-2})$, where $O(n^\alpha)$ denotes a family of operators such that $\|O(n^\alpha)\| \le cn^\alpha$ for all $n > 0$. Since h is bounded, we also have $e^{h/n} = 1 + n^{-1}h + O(n^{-2})$. Replacing $n \to 2n$ to simplify some expressions we trivially get

$$e^{h/(2n)}\Delta_{\eta,\psi}^{1/(2n)} a_{2n} \Delta_{\eta,\psi}^{1/(2n)} e^{h/(2n)} = \Delta_{\eta,\psi}^{1/n} + n^{-1}X_n + n^{-2}Y_n, \tag{135}$$

where X_n, Y_n are finite sums of terms of the form $x_0\Delta_{\eta,\psi}^{s_1}x_1 \cdots x_l\Delta_{\eta,\psi}^{s_l}x_l$, wherein $\sum s_j = 1/n$, $s_j \ge 0$ and $\|x_j\| \le c$ uniformly in n. Then X_n is given explicitly by

$$X_n = \tfrac{1}{2}h\Delta_{\eta,\psi}^{1/n} + \tfrac{1}{2}\Delta_{\eta,\psi}^{1/n}h + \Delta_{\eta,\psi}^{1/(2n)}k\Delta_{\eta,\psi}^{1/(2n)}. \tag{136}$$

By [5, proof of Thm. 3.1], the functions

$$F(z) := x_1\Delta_{\eta,\psi}^{z_1}x_2 \cdots x_j\Delta_{\eta,\psi}^{z_j}x_{j+1}|\psi\rangle, \quad z \in \bar{\mathbb{S}}_{1/2}^j \tag{137}$$

defined for given $x_j \in \mathcal{A}$ are analytic in the domain $\mathbb{S}_{1/2}^j := \{(z_1, \dots, z_j) \in \mathbb{C}^j : 0 < \mathrm{Re}(z_i),\ \sum \mathrm{Re}(z_i) < 1/2\}$ and strongly continuous on the closure. Subharmonic analysis as in [5, proof of Thm. 3.1] or [8] furthermore gives the bound

$$\|F(z)\| \le \prod_i \|x_i\| \quad \forall\, z \in \bar{\mathbb{S}}_{1/2}^j. \tag{138}$$

This bound, and the elementary formula

$$(A + tB)^N = \sum_{j=0}^N t^j \sum_{\substack{m_0+\cdots+m_j=N-j \\ m_j\in\mathbb{N}_0}} A^{m_0}B \cdots A^{m_{j-1}}BA^{m_j}, \tag{139}$$

show that the difference

$$\begin{aligned} |\zeta_n\rangle = {} & (e^{h/(2n)}\Delta_{\eta,\psi}^{1/(2n)} a_{2n} \Delta_{\eta,\psi}^{1/(2n)} e^{h/(2n)})^{n/2}|\psi\rangle \\ & - \sum_{j=0}^{n/2} n^{-j} \sum_{\substack{m_0+\cdots+m_j=n/2-j \\ m_j\in\mathbb{N}_0}} \Delta_{\eta,\psi}^{m_0/n}X_n \cdots \Delta_{\eta,\psi}^{m_{j-1}/n}X_n\Delta_{\eta,\psi}^{m_j/n}|\psi\rangle \end{aligned} \tag{140}$$

is bounded in norm by

$$\|\zeta_n\| \le (1 + n^{-1}(\|h\| + \|k\|) + n^{-2}c)^{n/2} - (1 + n^{-1}(\|h\| + \|k\|))^{n/2} \tag{141}$$

for some $c < \infty$, hence it tends to zero in norm as $n \to \infty$. Now setting

$$|\phi_{n,j}\rangle = n^{-j} \sum_{\substack{m_0+\cdots+m_j=n/2-j \\ m_j\in\mathbb{N}_0}} \Delta_{\eta,\psi}^{m_0/n}X_n \cdots \Delta_{\eta,\psi}^{m_{j-1}/n}X_n\Delta_{\eta,\psi}^{m_j/n}|\psi\rangle, \tag{142}$$

the strong continuity of the functions F and the usual definition of the Riemann integral imply

$$
\begin{aligned}
|\phi_j\rangle &:= \lim_n |\phi_{n,j}\rangle \\
&= \int_0^{1/2} \mathrm{d}s_0 \cdots \int_0^{s_{j-1}} \mathrm{d}s_j \, \Delta_{\eta,\psi}^{s_0-s_1}(h+k)\Delta_{\eta,\psi}^{s_1-s_2}(h+k) \\
&\qquad\qquad \cdots \Delta_{\eta,\psi}^{s_{j-1}-s_j}(h+k)\Delta_{\eta,\psi}^{s_j}|\psi\rangle,
\end{aligned}
\tag{143}
$$

and the usual perturbation theory by bounded operators as in [3, Prop. 16] or [5] gives $\sum_{j=0}^{\infty}|\phi_j\rangle = e^{(\ln\Delta_{\eta,\psi}+h+k)/2}|\psi\rangle$. Hence,

$$
\lim_n (e^{h/(2n)}\Delta_{\eta,\psi}^{1/(2n)} a_{2n} \Delta_{\eta,\psi}^{1/(2n)} e^{h/(2n)})^{n/2}|\psi\rangle = e^{(\ln\Delta_{\eta,\psi}+h+k)/2}|\psi\rangle \tag{144}
$$

strongly, as was argued more carefully in [4, proof of Lem. 5]. We have $e^{(\ln\Delta_{\eta,\psi}+h+k)/2}|\psi\rangle = e^{(\ln\Delta_{\eta,\psi}+p'h+p'k)/2}|\psi\rangle$ (here $p' = \pi^{\mathcal{A}'}(\psi) \in \mathcal{A}'$). Also, using [32, Thm. 12.6], we have $\ln\Delta_{\eta,\psi} + p'h + p'k = \ln\Delta_{\eta^{h+k},\psi}$, and this gives $|\eta^{h+k}\rangle = J|\eta^{h+k}\rangle = e^{(\ln\Delta_{\eta,\psi}+h+k)/2}|\psi\rangle$ by relative modular theory. This completes the proof. □

Proof of Lemma 6. From the definitions,

$$
\begin{aligned}
e^{h/n}|\Gamma_\psi(1/n)\rangle &= e^{h/n}\Delta_{\eta_{\mathcal{A}},\psi_{\mathcal{A}}}^{1/n} V_\psi \Delta_{\eta_{\mathcal{B}},\psi_{\mathcal{B}}}^{-1/n}\Delta_{\psi_{\mathcal{B}}}^{1/n}|\psi_{\mathcal{B}}\rangle \\
&= e^{h/n}\Delta_{\eta_{\mathcal{A}},\psi_{\mathcal{A}}}^{1/n} T([D\eta_{\mathcal{B}} : D\psi_{\mathcal{B}}]_{-i/n})|\psi_{\mathcal{A}}\rangle,
\end{aligned}
\tag{145}
$$

using the definition of the Connes cocycle and the fact that $[D\eta_{\mathcal{B}} : D\psi_{\mathcal{B}}]_{-i/n} \in \mathcal{B}$ under our assumption (123); see paper II ([16, proof of Lem. 4]). In the following, let $b = e^{h/n}$, $a = T([D\eta_{\mathcal{B}} : D\psi_{\mathcal{B}}]_{-i/n}) \in \mathcal{A}$ and $|\psi_{\mathcal{A}}\rangle = |\psi\rangle$, $|\eta_{\mathcal{A}}\rangle = |\eta\rangle$ etc., so $\|e^{h/n}\Gamma_\psi(1/n)\|_{n,\psi}^n = \|b\Delta_{\eta,\psi}^{1/n} a\psi\|_{n,\psi}^n$.

By the results of [8] (which hold in the present context since ω_ψ is faithful), the vector $b\Delta_{\eta,\psi}^{1/n}a|\psi\rangle \in L_n(\mathcal{A},\psi)$ has a polar decomposition $b\Delta_{\eta,\psi}^{1/n}a|\psi\rangle = u\Delta_{\phi_n,\psi}^{1/n}|\psi\rangle$, where $\|b\Delta_{\eta,\psi}^{1/n}a\psi\|_{n,\psi}^n = \|\phi_n\|^2$ and where $u \in \mathcal{A}$ is a partial isometry. To get an expression for $|\phi_n\rangle$, we use the formalism of "script" $\mathscr{L}_p$-spaces of [8, notation 7.6]: as a vector space, $\mathscr{L}_p^*(\mathcal{A},\psi)$, $p \geq 1$ consists of all formal linear combinations of formal expressions of the form

$$
A = x_1\Delta_{\zeta_1,\psi}^{z_1}x_2\cdots x_n\Delta_{\zeta_n,\psi}^{z_n}x_{n+1}, \tag{146}
$$

wherein $\mathrm{Re}(z_i) \geq 0$, $\sum_i \mathrm{Re}(z_i) \leq 1 - 1/p$, $x_i \in \mathcal{A}$, $\zeta_i \in \mathscr{H}$, the formal adjoint of which is defined to be

$$
A^* = x_{n+1}^*\Delta_{\zeta_n,\psi}^{\bar{z}_n}x_n^*\cdots x_2^*\Delta_{\zeta_1,\psi}^{\bar{z}_1}x_1^*. \tag{147}
$$

The notation $\mathscr{L}^*_{p,0}(\mathcal{A},\psi)$ is reserved for formal elements A such that, in addition to all other conditions, $\sum_i \mathrm{Re}(z_i) = 1-1/p$. It is then clear that $\mathscr{L}^*_{p,0}(\mathcal{A},\psi)\mathscr{L}^*_{q,0}(\mathcal{M},\psi) = \mathscr{L}^*_{r,0}(\mathcal{M},\psi)$ as formal products where $1/r' = 1/p' + 1/q'$ with $1/p' = 1 - 1/p$ as usual. By [8, Lem. 7.3], if $1 \le p \le 2$, any element $A \in \mathscr{L}^*_p(\mathcal{A},\psi)$ can be viewed as an element of $L_{p'}(\mathcal{A},\psi)$ in the sense that $|\psi\rangle \in \mathscr{D}(A)$ and $A|\psi\rangle \in L_{p'}(\mathcal{A},\psi)$.[11] Furthermore, by [8, Lem. 7.7(2)], if $A_1, A_2 \in \mathscr{L}^*_{p,0}(\mathcal{A},\psi)$ correspond to the same element under this identification, then so do A_1^*, A_2^* or A_1B, A_2B or BA_1, BA_2 if $B \in \mathscr{L}^*_{q,0}(\mathcal{A},\psi)$ (as long as $1/p' + 1/q' \le 1/2$, for example).

We now start with the trivial statement that $u\Delta^{1/n}_{\phi_n,\psi} = b\Delta^{1/n}_{\eta,\psi}a$ in the sense that these elements of $\mathscr{L}^*_{n',0}(\mathcal{A},\psi)$ are identified with the same element of $L_n(\mathcal{A},\psi)$. Then repeated application of [8, Lem. 7.7(2)] and the definition of adjoint gives

$$u\Delta^{2/n}_{\phi_n,\psi}u^* = b\Delta^{1/n}_{\eta,\psi}aa^*\Delta^{1/n}_{\eta,\psi}b^* \quad \text{in } \mathscr{L}^*_{n/(n-2),0}(\mathcal{A},\psi). \tag{148}$$

Successively forming $n/4$ products of this equality and applying [8, Lem. 7.7(2)] in each step, we find that

$$u\Delta^{1/2}_{\phi_n,\psi}u^* = (b\Delta^{1/n}_{\eta,\psi}aa^*\Delta^{1/n}_{\eta,\psi}b^*)^{n/4} \quad \text{in } \mathscr{L}^*_{2,0}(\mathcal{A},\psi), \tag{149}$$

meaning that both sides are equal as elements of $\mathscr{H} = L_2(\mathcal{A},\psi)$ after we apply them to $|\psi\rangle$. Thus,

$$\|(b\Delta^{1/n}_{\eta,\psi}aa^*\Delta^{1/n}_{\eta,\psi}b^*)^{n/4}\psi\|^2 = \|u\Delta^{1/2}_{\phi_n,\psi}u^*\psi\|^2 = \|uJu\phi_n\|^2 = \|\phi_n\|^2 \tag{150}$$

using modular theory in the penultimate step. Therefore,

$$\|(b\Delta^{1/n}_{\eta,\psi}aa^*\Delta^{1/n}_{\eta,\psi}b^*)^{n/4}\psi\|^2 = \|b\Delta^{1/n}_{\eta,\psi}a\psi\|^n_{n,\psi}, \tag{151}$$

and the proof of the lemma is complete. □

(II) Proof of Theorem 2 in the general case. We will now remove the majorization condition (123). This condition has been used in an essential way in most of the arguments so far. For example, without it, the operator k in (125) is unbounded and thus not an element of $\mathcal{A}$. For unbounded operators the Araki–Trotter product formula and the L_p-techniques are not available in the form in which we used them and it seems non-trivial to extend them to an unbounded framework. We will therefore proceed in a different way and define a regularization of ω_ψ such that the majorization condition (123) holds and such that, at the same time, the desired entropy inequality can be obtained in a limit wherein the regulator is removed. However, it is clear that this regularization must be carefully chosen

[11] In fact, $\|A\psi\|_{p',\psi} \le \|x_{n+1}\| \prod_{i=1}^n (\|x_i\|\,\|\zeta_i\|^{\mathrm{Re}(z_i)})$.

because the relative entropy is not continuous but only lower semi-continuous. By itself the latter is insufficient for our purposes since the desired inequality (121) has both signs of the relative entropy.

Our regularization combines a trick invented in paper I ([17]) with the convexity of the relative entropy. As in paper I ([17]), we consider a function $f(t)$, $t \in \mathbb{R}$ with the following properties:

(A) The Fourier transform of f,

$$\tilde{f}(p) = \int_{-\infty}^{\infty} e^{-itp} f(t)\, \mathrm{d}t, \tag{152}$$

exists as a real-valued and non-negative Schwarz-space function. This implies that the original function f is Schwarz and has finite $L_1(\mathbb{R})$-norm, $\|f\|_1 < \infty$.

(B) $f(t)$ has an analytic continuation to the upper complex half-plane such that the $L_1(\mathbb{R})$-norm of the shifted function has $\|f(\cdot + i\theta)\|_1 < \infty$ for $0 < \theta < \infty$.

Such functions certainly exist (e.g. Gaussians). We also let $f_P(t) = Pf(tP)$ for our regulator $P > 0$, and we define a regulated version of $|\psi\rangle$ by

$$|\psi_P\rangle = \frac{\tilde{f}_P(\ln \Delta_{\eta,\psi})|\psi\rangle}{\|\tilde{f}_P(\ln \Delta_{\eta,\psi})\psi\|}. \tag{153}$$

As shown in paper I ([17]), some key properties of the regulated vectors are

(P1) $\omega_{\psi_P} \le c_P \omega_\eta$ for some $c_P > 0$ which may diverge as $P \to \infty$,

(P2) $\text{s-lim}_{P\to\infty} |\psi_P\rangle = |\psi\rangle$ (strong convergence),

(P3) $-2\ln(\|f\|_1/\|\tilde{f}\|_\infty) + \limsup_{P\to\infty} S(\psi_P|\eta) \le S(\psi|\eta)$,

where the first item gives at least "half" of the domination condition (123), the second states in which sense $|\psi_P\rangle$ approximates $|\psi\rangle$, and the third gives us an upper semi-continuity property of the relative entropy opposite to the usual lower semi-continuity property which holds for generic approximations. We define, for small $\varepsilon > 0$,

$$\sigma(a) = \langle \eta | a\eta \rangle, \quad \rho_{P,\varepsilon}(a) = (1-\varepsilon)\langle \psi_P | a\psi_P \rangle + \varepsilon \langle \eta | a\eta \rangle. \tag{154}$$

Thus, by (P1), the relative majorization condition (123) holds, e.g. with $c = \max(c_P, \varepsilon^{-1})$, between $\rho_{P,\varepsilon}$ and σ. By (P2), $\lim_{P\to\infty} \lim_{\varepsilon\to 0} \|\rho - \rho_{P,\varepsilon}\| = 0$. In (P3), we choose a function f such that $\|f\|_1/\|\tilde{f}\|_\infty = 1$ (which must be Gaussian). The well-known convexity of the relative entropy gives, together with the definition of $\rho_{P,\varepsilon}$ that ($\rho_P = \langle \psi_P | \cdot \psi_P \rangle$)

$$S(\rho_{P,\varepsilon}|\sigma) \le (1-\varepsilon)S(\rho_P|\sigma) + \varepsilon S(\sigma|\sigma) = (1-\varepsilon)S(\rho_P|\sigma). \tag{155}$$

Combining this with (P3), we get

$$\limsup_{P\to\infty}\limsup_{\varepsilon\to 0} S(\rho_{P,\varepsilon}|\sigma) \le S(\rho|\sigma). \tag{156}$$

The norm convergence $\lim_P \lim_\varepsilon \rho_{P,\varepsilon}\circ T = \rho\circ T$ by (P2) also gives, in combination with the usual lower semi-continuity of the relative entropy ([7, Thm. 3.7(2)]) that

$$\liminf_{P\to\infty}\liminf_{\varepsilon\to 0} S(\rho_{P,\varepsilon}\circ T|\sigma\circ T) \ge S(\rho\circ T|\sigma\circ T). \tag{157}$$

Now we combine equations (156), (157) with part (I) of the proof applied to the states $\rho_{P,\varepsilon}$ and σ, which obey the relative majorization condition. If $S(\rho\circ T|\sigma\circ T) < \infty$, the difference $S(\rho|\sigma) - S(\rho\circ T|\sigma\circ T)$ is meaningful (possibly $=\infty$) and we get from (156) and (157) that

$$S(\rho|\sigma) - S(\rho\circ T|\sigma\circ T) \ge \limsup_{P\to\infty}\limsup_{\varepsilon\to 0} S_{\rm meas}(\rho_{P,\varepsilon}|\rho_{P,\varepsilon}\circ T\circ\alpha_{T,\sigma}). \tag{158}$$

If $S(\rho\circ T|\sigma\circ T) = \infty$ and hence $S(\rho|\sigma)=\infty$, the statement of the theorem is vacuous and there is nothing to prove. The proof of part (II) is then finished by proving lower semi-continuity for the measured relative entropy:

Lemma 8. *If $\mu_n, \nu_n, \mu, \nu \in \mathscr{S}(\mathcal{A})$ are such that $\lim_n \mu_n = \mu$ and $\lim_n \nu_n = \nu$ in the norm sense, then $S_{\rm meas}(\mu|\nu) \le \liminf_n S_{\rm meas}(\mu_n|\nu_n)$.*

Proof. This is a straightforward consequence of the variational definition (96) of $S_{\rm meas}$, choosing a near optimal h. □

Appendix A. Araki–Masuda L_p-spaces [8, 10, 26, 27, 22]

The weighted L_p-spaces that we use in this paper were defined by [8] relative to a fixed vector $|\psi\rangle \in \mathscr{H}$ in the natural cone of a standard representation of a von Neumann algebra $\mathcal{M}$. For $p\ge 2$, the space $L_p(\mathcal{M},\psi)$ is defined as

$$L_p(\mathcal{M},\psi) = \big\{|\zeta\rangle \in \textstyle\bigcap_{|\phi\rangle\in\mathscr{H}} \mathscr{D}(\Delta_{\phi,\psi}^{(1/2)-(1/p)}),\ \|\zeta\|_{p,\psi}<\infty\big\}. \tag{159}$$

Here, the norm is

$$\|\zeta\|_{p,\psi} = \sup_{\|\phi\|=1} \|\Delta_{\phi,\psi}^{(1/2)-(1/p)}\zeta\|. \tag{160}$$

For $1\le p<2$, $L_p(\mathcal{M},\psi)$ is defined as the completion of $\mathscr{H}$ with respect to the norm

$$\|\zeta\|_{p,\psi} = \inf\big\{\|\Delta_{\phi,\psi}^{(1/2)-(1/p)}\zeta\| \colon \|\phi\|=1,\ \pi^{\mathcal{M}}(\phi)\ge\pi^{\mathcal{M}}(\zeta)\big\}. \tag{161}$$

In [8], it is assumed for most results that $|\psi\rangle$ is cyclic and separating. When using such results in the main text, we will be in that situation. An equivalent approach replacing the relative modular operator by the Connes spatial derivative and containing also several new results is laid out some detail in [10]. A different approach to L_p-norms is taken in [26, 27, 22]. However, these definitions are eventually shown to be equivalent to those used by [8, 10]; see [26, Thm. 3.3] and [27, Prop. 2.4 and Thm. 3.1]. Thus, we may use the results on L_p-norms in their setting in our setting too; see e.g. item (10) below.

For the convenience of the reader, we list here some of the properties of L_p-norms that we refer to in the main text:

(1) From the definition, we have $\|\zeta\|_{2,\psi} = \|\pi^{\mathcal{M}'}(\psi)\zeta\|$, so for cyclic and separating vectors $|\psi\rangle$ we have $L_2(\mathscr{H},\psi) = \mathscr{H}$.

(2) ([8, Thm. 4]) The map $\mathcal{M} \ni m \mapsto m|\psi\rangle \in L_\infty(\mathcal{M},\psi)$ is an isomorphism of Banach spaces (for cyclic and separating vectors $|\psi\rangle$; in the general case one can see from the definitions that $\|m\psi\|_{\infty,\psi} = \|\pi^{\mathcal{M}}(\psi)m\pi^{\mathcal{M}}(\psi)\|$).

(3) From the definitions, $L_p(\mathcal{M},\psi) \supset \mathscr{H}$ for $p \in [1,2]$ and $L_p(\mathcal{M},\psi) \subset \mathscr{H}$ for $p \in [2,\infty]$ (for cyclic and separating vectors $|\psi\rangle$).

(4) ([8, Thm. 3]) Any $|\zeta\rangle \in L_p(\mathcal{M},\psi)$ has the unique polar decomposition $u|\zeta_+\rangle$, where u is a partial isometry of $\mathcal{M}$ satisfying $uu^* = \pi^{\mathcal{M}}(\zeta)$, where $|\zeta_+\rangle \in L_p^+(\mathcal{M},\psi)$, and where

$$L_p^+(\mathcal{M},\psi) = \begin{cases} L_p(\mathcal{M},\psi) \cap \mathscr{P}_{\mathcal{M}}^{1/(2p)} & \text{for } p \in [2,\infty], \\ L_p(\mathcal{M},\psi) - \text{closure of } \mathscr{P}_{\mathcal{M}}^{1/(2p)} & \text{for } p \in [1,2]. \end{cases} \tag{162}$$

Here, $|\psi\rangle$ is assumed cyclic and separating and $\mathscr{P}^\alpha_{\mathcal{M}} =$ closure of $\Delta^\alpha_\psi \mathcal{M}_+|\psi\rangle$ for $\alpha \in [0,1/2]$.

(5) ([8, Thm. 3]) If $|\zeta\rangle \in L_p(\mathcal{M},\psi)$ there exists a unique $\omega_\phi \in \mathscr{S}(\mathcal{M})$ such that $|\zeta\rangle = \Delta^{1/p}_{\phi,\psi}|\psi\rangle$ if $p \in [2,\infty)$ and $\langle\zeta|\zeta'\rangle = \langle\Delta^{1/2}_{\phi,\psi}\psi|\Delta^{(1/p)-(1/2)}_{\phi,\psi}\zeta'\rangle$ for all $|\zeta'\rangle \in L_{p'}(\mathcal{M},\psi)$, $1/p+1/p' = 1$ if $p \in [1,2]$. For this state, $\|\omega_\phi\|^{1/p} = \|\zeta\|_{p,\psi}$ (for cyclic and separating vectors $|\psi\rangle$).

(6) ([8, Thm. 1]) Let p' be the dual Hölder index of $p \in (1,\infty)$, $1/p + 1/p' = 1$. Then $\|\zeta\|_{p,\psi} = \sup\{|\langle\zeta|\zeta'\rangle| \colon \|\zeta'\|_{p',\psi} \le 1,\ |\zeta'\rangle \in L_{p'}(\mathcal{M},\psi) \cap \mathscr{H}\}$ for $|\zeta\rangle \in \mathscr{H} \cap L_p(\mathcal{M},\psi)$ (for cyclic and separating vectors $|\psi\rangle$).

(7) (See Lemma 9.) Let $\omega_{\psi_1} \le \omega_{\psi_2}$, and let $|\psi_1\rangle$, $|\psi_2\rangle$ be the representers in the natural cone. For $p \in [2,\infty]$ and $|\zeta\rangle \in \pi^{\mathcal{M}'}(\psi_1) \cap L_p(\mathcal{M},\psi_1)$, we have $\|\zeta\|_{p,\psi_2} \le \|\zeta\|_{p,\psi_1}$. For $p \in [1,2)$ and $|\zeta\rangle \in \mathscr{H}$, we have $\|\zeta\|_{p,\psi_2} \ge \|\zeta\|_{p,\psi_1}$.

(8) ([10, Lems 8, 9] and [26, Prop. 3.8]) The function $p \mapsto (p/2-1)^{-1}\ln\|\zeta\|^p_{p,\psi}$ is continuous and monotonically increasing for $p \in [1,2) \cup (2,\infty]$. In particular

for $\infty \geq p > q > 2$, we have $L_p(\mathcal{M}, \psi) \subset L_q(\mathcal{M}, \psi)$, while for $2 > q > p \geq 1$, we have $L_q(\mathcal{M}, \psi) \subset L_p(\mathcal{M}, \psi)$. If $\|\zeta\|_{\infty,\psi} < \infty$, we have $\lim_{p\to\infty} \|\zeta\|_{p,\psi} = \|\zeta\|_{\infty,\psi}$, and in fact

$$\|\zeta\|^2_{\infty,\psi} = \inf\{\lambda > 0 \colon \omega_\zeta \leq \lambda\omega_\psi\}. \tag{163}$$

Furthermore, $\|\zeta\|_{p,\psi} \leq \|\zeta\| \, \|\omega_\psi\|^{1/p-1/2}$ for all $|\zeta\rangle \in \mathscr{H}$, $p \in [1, 2]$; see e.g. [10, Lem. 8].

(9) (See e.g. paper I ([17, Lem. 3]).) There holds $F_{\mathcal{M}'}(\omega'_\zeta|\omega'_\psi) = \|\zeta\|_{\infty,\psi}$, where $F_{\mathcal{M}'}$ is Uhlmann's fidelity relative to $\mathcal{M}'$ ($|\psi\rangle$ in the natural cone).

(10) ([22, Thm. 3.16(3)] and [26, Prop. 3.10]) For $p \in [1, 2]$ the map $(\omega'_\psi, \omega'_\zeta) \mapsto \|\zeta\|^p_{p,\psi}$ is continuous and for $p \in (2, \infty]$ the map $(\omega'_\psi, \omega'_\zeta) \mapsto \|\zeta\|^p_{p,\psi}$ is lower semi-continuous in the norm topology. Here, it is used implicitly that the L_p-norms with respect to $\mathcal{M}$ are invariant when sending $|\zeta\rangle \to v|\zeta\rangle = |\zeta'\rangle$ provided e.g. $v \in \mathcal{M}$, $v^*v = \pi^{\mathcal{M}}(\zeta)$, $vv^* = \pi^{\mathcal{M}}(\zeta')$, meaning that they only depend on ω'_ζ, the state functional induced on $\mathcal{M}'$. Furthermore, it is understood that $\omega'_\psi \mapsto |\psi\rangle$ is the map that associates the unique vector representative in the natural cone with a state functional on $\mathcal{M}'$.

Lemma 9. *Let $\omega_{\psi_1} \leq \omega_{\psi_2}$, and let $|\psi_1\rangle$, $|\psi_2\rangle$ be the representers in the natural cone. For $p \in [2, \infty]$ and $|\zeta\rangle \in \pi^{\mathcal{M}'}(\psi_1)\mathscr{H} \cap L_p(\mathcal{M}, \psi_1)$, we have $\|\zeta\|_{p,\psi_2} \leq \|\zeta\|_{p,\psi_1}$. For $p \in [1, 2)$ and $|\zeta\rangle \in \mathscr{H}$, we have*[12] $\|\zeta\|_{p,\psi_2} \geq \|\zeta\|_{p,\psi_1}$.

Remark. As the referee has pointed out to us, this lemma can also be seen from [26, Prop. 3.9] and [22, Thm. 3.16(7)].

Proof of Lemma 9. In this proof, we use the shorthands $\Delta_i = \Delta_{\phi,\psi_i}$, $i = 1, 2$ and use that $|\psi_1\rangle$, $|\psi_2\rangle$ are in the natural cone implicitly when referring to known properties of the modular operators.

Case $p \in [2, \infty]$**.** Our assumption implies $\|\zeta\|_{p,\psi_1} < \infty$. Combining the domination condition $\omega_{\psi_1} \leq \omega_{\psi_2}$ with the definition of the modular operators and their standard properties (see e.g. [8, Thm. C.1]) gives $s(\Delta_1)\Delta_2 s(\Delta_1) \leq \Delta_1$ for example in the quadratic form sense on the domain $\mathscr{D}(\Delta_1^{1/2})$, where $s(\Delta_1) = \pi^{\mathcal{M}}(\phi)\pi^{\mathcal{M}'}(\psi_1)$ is the support. Then $s(\Delta_1)$ is a projection mapping $\mathscr{D}(\Delta_1^{1/2}) \to \mathscr{D}(\Delta_2^{1/2})$, and [32, Lem. 5.2] gives, for $\gamma \in [0, 1]$, that $s(\Delta_1)\Delta_2^\gamma s(\Delta_1) \leq \Delta_1^\gamma$, at first on $\mathscr{D}(\Delta_1^{1/2})$. If $|\zeta\rangle \in \mathscr{D}(\Delta_1^{\gamma/2})$ for some $\gamma \in [0, 1]$, then approximating it with $|\zeta_N\rangle \in E_{\Delta_1}([0, N])|\zeta\rangle$ using the spectral measure E_{Δ_1} of Δ_1, we can easily see that $s(\Delta_1)\Delta_2^\gamma s(\Delta_1) \leq \Delta_1^\gamma$ also holds on $\mathscr{D}(\Delta_1^{\gamma/2})$ and in fact $s(\Delta_1) \colon \mathscr{D}(\Delta_1^{\gamma/2}) \to \mathscr{D}(\Delta_2^{\gamma/2})$. In particular, vectors in the domain of a power $\Delta_1^{\gamma/2}$,

[12] This case is not actually used in the body of the paper.

$\gamma \in [0,1]$ intersected with $s(\Delta_1)\mathscr{H}$ are always in the domain of the corresponding power $\Delta_2^{\gamma/2}$.

By the variational definition of the L_p-norm, there is for any $\varepsilon > 0$ a unit vector $|\phi\rangle$ such that $|\zeta\rangle \in \mathscr{D}(\Delta_{\phi,\psi_2}^{1/2-1/p})$ (because of $\|\zeta\|_{p,\psi_1} < \infty$ and (165), $|\zeta\rangle$ is in the intersection of domains $\bigcap_{|\phi\rangle\in\mathscr{H}} \mathscr{D}(\Delta_{\phi,\psi_2}^{1/2-1/p})$) and such that

$$\|\Delta_{\phi,\psi_2}^{1/2-1/p}\zeta\| \geq \|\zeta\|_{p,\psi_2} - \varepsilon. \tag{164}$$

On the other hand, using $\pi^{\mathcal{M}'}(\psi_1)|\zeta\rangle = |\zeta\rangle$,

$$\begin{aligned}
\|\Delta_{\phi,\psi_2}^{1/2-1/p}\zeta\| &= \|\Delta_{\phi,\psi_2}^{1/2-1/p}\pi^{\mathcal{M}}(\phi)\pi^{\mathcal{M}'}(\psi_1)\zeta\| \\
&= \|\Delta_{\phi,\psi_2}^{1/2-1/p}s(\Delta_{\phi,\psi_1})\zeta\| \\
&\leq \|\Delta_{\phi,\psi_1}^{1/2-1/p}\zeta\| \\
&\leq \|\zeta\|_{p,\psi_1}.
\end{aligned} \tag{165}$$

Combining (165) and (164) the claim follows since $\varepsilon > 0$ can be arbitrarily small.

Case $p \in [1,2)$. The proof is rather similar to the previous case and follows again from the definition of the modular operators and their standard properties (see e.g. [8, Thm. C.1]), and the variational definition of the L_p-norms. So we leave it to the reader. □

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