Publ. RIMS Kyoto Univ. **59** (2023), 731–758
DOI 10.4171/PRIMS/59-4-2

On Galois Action on the Inertia Stack of Moduli Spaces of Curves

by

Benjamin Collas and Sylvain Maugeais

Abstract

We establish that the geometric action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the étale fundamental group of moduli spaces of curves induces a Galois action on its stack inertia subgroups, and that this action is given by *cyclotomy conjugacy.* This result extends the special case of inertia without étale factorisation previously established by the authors. It is here obtained in the general case by comparing deformations of Galois actions.

Since the *cyclic* stack inertia corresponds to the first level of the stack stratification of the space, this result, by analogy with the arithmetic of the Deligne–Mumford stratification, opens the way to a systematic Galois study of the stack inertia through the corresponding stratification of the moduli stack.

2020 Mathematics Subject Classification: Primary 11R32, 14H10; Secondary 14H30, 14H45.
Keywords: Algebraic fundamental group, inertia stack, special loci, deformation of curves, limits of Galois representation.

§1. Introduction

Let $\mathcal{M}_{g,[m]}$ denote the moduli space of curves of genus g with m unordered marked points endowed with its Deligne–Mumford stack structure over $\mathbb{Q}$, and assume that $2g-2+m \geqslant 0$. For a given geometric point $\bar{x}\colon \mathrm{Spec}(\overline{\mathbb{Q}}) \to \mathcal{M}_{g,[m]} \otimes \overline{\mathbb{Q}}$, the choice of a $\mathbb{Q}$-point s of $\mathcal{M}_{g,[m]}$, together with the choice of a path between s and $\bar{x}$, defines a geometric Galois representation

$$\rho_s\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}[\pi_1^{\mathrm{et}}(\mathcal{M}_{g,[m]} \otimes \overline{\mathbb{Q}}, \bar{x})], \tag{1.1}$$

Communicated by S. Mochizuki. Received November 10, 2018. Revised October 22, 2019; May 5, 2020; January 28, 2021; August 26, 2021.

B. Collas: Mathematisches Institut, Westfalische Wilhelms-Universität Münster, Einsteinstr. 62, 48149 Münster, Germany;
e-mail: b.collas@uni-muenster.de
S. Maugeais: Laboratoire manceau de Mathématiques, Université du Maine, Av. Olivier Messiaen, BP 535, 72017 Le Mans Cedex, France;
e-mail: sylvain.maugeais@univ-lemans.fr

whose description has been extensively studied following Grothendieck's seminal program [Gro97] in terms of the *Deligne–Mumford stratification* [Knu83] of its stable compactification $\overline{\mathcal{M}}_{g,[m]}$; see [DM69]. This approach deals essentially with the schematic structure of $\mathcal{M}_{g,[m]}$ by considering the Galois action on the divisorial inertia groups of the boundary of $\overline{\mathcal{M}}_{g,[m]}$ (see [Nak99, Nak97]); the arithmetic of the stratification resulted in the development of Grothendieck–Teichmüller theory as initiated by V. Drinfel'd and Y. Ihara (see for example [Dri90, NS00]).

Let $I_{\mathcal{M}}$ be the inertia stack of $\mathcal{M}_{g,[m]}$ that classifies the automorphism of curves, i.e. the stack whose objects over a $\mathbb{Q}$-scheme S consist of pairs (x, γ) with $x \in \mathcal{M}_{g,[m]}(S)$ and $\gamma \in \mathrm{Aut}_S(x)$. The fibre over an algebraically closed point $\bar{x}\colon \mathrm{Spec}(\overline{\mathbb{Q}}) \to \mathcal{M}_{g,[m]}$ gives a finite group $I_{\bar{x}} = I_{\mathcal{M}} \times_{\mathcal{M}_{g,[m]}} \mathrm{Spec}(\overline{\mathbb{Q}})$ called the *stack inertia group of* $\bar{x}$. This group is isomorphic to the automorphisms group of the curve $\bar{x}$. The fact that $I_{\bar{x}}$ injects into $\pi_1^{\mathrm{et}}(\mathcal{M}_{g,[m]} \otimes \overline{\mathbb{Q}}, \bar{x})$, proven in [Noo04], raises questions of the definition and the description of the global geometric $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action of equation (1.1) on the local stack inertia groups $I_{\bar{x}}$ of $\mathcal{M}_{g,[m]}$.

The $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on cyclic stack inertia groups gained some focus initially in genus 0 via Grothendieck–Teichmüller theory [LS97], then in higher genus with Galois considerations [NT03], [Col12a, §3].

The main result of the present article follows [Col12b, Col12a, CM15] and provides an answer to these questions in the case of cyclic stack inertias $I_{\bar{x}}$; see Theorem 4.8:

Theorem A. *For any cyclic stack inertia group $I = \langle \gamma \rangle$ of $\mathcal{M}_{g,[m]}$, there exists a geometric Galois representation $\rho_{\vec{s}}$ which induces a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on I given by χ-conjugacy, i.e. for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,*

$$\rho_{\vec{s}}(\sigma) \cdot \gamma = \delta_\sigma \gamma^{\chi(\sigma)} \delta_\sigma^{-1} \quad \textit{for } \delta_\sigma \textit{ some étale path in } \widetilde{\mathcal{M}}_{g,[m]} \otimes \overline{\mathbb{Q}},$$

where $\chi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathbb{Z}}^*$ denotes the cyclotomic character and $\widetilde{\mathcal{M}}_{g,[m]}$ is a certain partial compactification of $\mathcal{M}_{g,[m]}$.

The present approach follows that of [CM15] using irreducible components of special loci of the form $\mathcal{M}_{g,[m]}(G)$ – locus of points $x \in \mathcal{M}_{g,[m]}$ whose geometric stack inertia group $I_{\bar{x}}$ contains a subgroup isomorphic to G – which defines a *stack inertia stratification of* $\mathcal{M}_{g,[m]}$; see [Dou06]. The key ingredients are the Deligne–Mumford compactification and the arithmetic notion of a tangential base point: first to define a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action $\rho_{\vec{s}}^I\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(I)$ that is compatible with a tangential version $\rho_{\vec{s}}$ of equation (1.1), then to extend the cyclotomy result of [Dou06] from the case of stack inertia without étale factorisation (see Section 4.1.2 for the definition) to the general case.

The definition of an intrinsic *local Galois action on* $I_{\bar{x}}$ *within* $\pi_1^{\text{et}}(\mathcal{M}_{g,[m]}\otimes\overline{\mathbb{Q}})$ for general Deligne–Mumford stacks is indeed tedious: for K denoting the field of definition of $\bar{x}$, one obtains an outer action $\mathrm{Gal}(\overline{K}/K)\to\mathrm{Out}(I)$ *modulo a certain geometric monodromy group only*; see [LV18, Prop. 3.17]. In the case of the cyclic inertia of $\mathcal{M}_{g,[m]}$, we bypass this difficulty by the use of $\mathbb{Q}$-tangential base points (denoted by $\vec{s}$) – obtained from formal neighbourhoods of $\mathbb{Q}$-points of $\overline{\mathcal{M}}_{g,[m]}$ [IN97] (see Section 3.1.1 for a general stack definition) – and by explicit properties of deformation of G-curves [Eke95]; see Section 2.2.2. Following a stack version of Grothendieck–Murre formalism of the tame fundamental group [GM71] (see Section 3.1), this leads to some $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-tangential representations $\rho_{\vec{s}}$ that induce proper *stack inertia* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-actions* $\rho^I_{\vec{s},\bar{x}}\colon\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\mathrm{Aut}[I_{\bar{x}}]$; see Section 3.1.3.

The property that irreducible components of cyclic special loci are Deligne–Mumford stacks defined over $\mathbb{Q}$ (see [CM15, Prop. 3.12 and Thm. 4.3]) ensures their global $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariance by $\rho_{\vec{s}}$, hence the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stability of the conjugacy class of γ. While this property is sufficient to establish the cyclotomy result for the first non-trivial cases – when G is of prime order, or the G-action is without étale factorisation (see Proposition 4.3 then Corollary 4.7) – it is not for the general $\mathcal{M}_{g,[m]}(\mathbb{Z}/n\mathbb{Z})$. The extension of the cyclotomy result relies first on the construction of a specific G-deformation of a smooth curve to the boundary of $\overline{\mathcal{M}}_{g,[m]}$ (Theorem 2.6), then on the existence of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-tangential compatible Knudsen morphisms between $\overline{\mathcal{M}}_{g-1,[m]+2}$ and $\overline{\mathcal{M}}_{g,[m]}$ (Proposition 3.12), and finally on a specialisation result for stack inertia groups; see Section 4.2.1. This process takes the name *inertial limit Galois action* in Section 4.2.4.

The result of Theorem A, as well as the use of the inertial limit Galois action in the study of the stack inertia stratification of $\mathcal{M}_{g,[m]}$, strengthens the analogy between the arithmetic of the Deligne–Mumford stratification and of the stack inertia stratifications, which suggests further developments along this direction; see Section 4.2.4. Theorem A also supports a positive answer to the anabelian Question 8.5 of [Loc12]:

> If a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on $\pi_1^{\text{et}}(\mathcal{M}_{g,[m]}\otimes\overline{\mathbb{Q}})$ is given by χ-conjugacy on a protorsion element, is this element conjugate to a finite stack inertia one?

We refer to [Loc12] for further motivations and for the original formulation in terms of Dehn twists in the mapping class group $\widehat{\Gamma}_{g,[m]}\simeq\pi_1^{\text{et}}(\mathcal{M}_{g,[m]}\otimes\mathbb{Q})$, as well as to [Nak90, Thm. 3.4] for the divisorial analog for curves that motivates this question.

§2. Geometry at infinity of special loci

Let G be a finite group and $\mathcal{M}_{g,[m]}(G)$ the special loci associated to G, i.e. the Deligne–Mumford substack of $\mathcal{M}_{g,[m]}$ classifying families of smooth and proper marked curves of genus g whose automorphisms group admits a subgroup isomorphic to G; see [CM15, §2]. In the case that G is cyclic, a certain type of degeneracy of curves in $\mathcal{M}_{g,[m]}(G)$ to stable curves is sought in the boundary of $\mathcal{M}_{g,[m]}$: one of the main results is that any smooth G-curve admits a degeneracy to an irreducible singular curve with G-action whose normalisation is *without étale factorisation* (i.e. the associated G-cover does not factorise through a non-trivial étale cover).

§2.1. Deformation of stable G-curves

For curves endowed with a G-action, the analog of the stable curves [DM69] is given by *stable G-curves* [Eke95], i.e. stable curves endowed with an *admissible action*, whose definition is recalled below in the case of a *cyclic group* – see also [BR11, §4.1.1].

Definition 2.1 (Admissible action). Let G be a cyclic group acting faithfully on a semi-stable curve $\mathcal{C}/S$. For S the spectrum of an algebraically closed field, the action of G is said to be *admissible* if, for every singular point $P \in \mathcal{C}$ with stabiliser G_P, the two characters of G_P on the branches at P are each other's inverse. For S general, the action is admissible if it is so on every geometric fibre.

Denoting by $\Omega^1_{C/k}$ the sheaf of relative Kähler differentials, the cohomological theory that controls the G-equivariant deformation functor $\mathrm{Def}_{C,G}$ is given by the G-equivariant $\mathrm{Ext}^i_G(\Omega_{C/k}, \mathcal{O}_C)$; see [Eke95, Prop. 2.1]. It then follows from Schlessinger's theory [Sch06] that the equivariant deformation functor $\mathrm{Def}_{C,G}$ associated to a stable G-curve is pro-representable by a complete local ring $R_{C,G}$.

Theorem 2.2 ([Eke95, Props 2.1–2.2]). *Let C be a stable G-curve over a field k of characteristic 0 endowed with a G-admissible action, and let $R_{C,G}$ be its universal deformation ring with residue field k and field of fraction K. Then $R_{C,G}$ is formally smooth over k, and its generic point corresponds to a smooth curve over K.*

Following [Tuf93], there is no obstruction to infinitesimal lifting; one also obtains a *local–global principle* for a tame G-covering:

$$0 \to \mathrm{H}^1_G(C, \Theta_C) \to \mathrm{Ext}^1_G(\Omega^1_{C/k}, \mathcal{O}_C) \to \bigoplus_I \mathrm{Ext}^1_G(\widehat{\Omega}_{C/k,P_i}, \widehat{\mathcal{O}}_{C,P_i}) \to 0,$$

where Θ is the tangent sheaf and the direct sum is over a set of representatives $\{P_i\}_I$ of $\mathrm{sing}(C)/G$. The "local" contributions are given by the deformations of the P_i, each of them being of dimension 1.

The universal deformation ring thus identifies with

$$R_{C,G} \simeq R_{\mathrm{glo}} \,\widehat{\otimes}\, k[\![q_1, \ldots, q_M]\!],$$

where

(i) M is the number of singular points of C and ramification points of the G-cover $C \to C/G$,

(ii) R_{glo} is a formally smooth k-algebra of finite dimension,

by [BR11, eq. (40)]. Denoting by g' the genus of C/G, one recovers that $\dim(R_{C,G}) = 3g' - 3 + b$, with b the degree of the branch point divisor.

In the G-stable compactification $\overline{\mathcal{M}}_{g,[m]}(G)$ of $\mathcal{M}_{g,[m]}(G)$, the choice of some deformation parameters $\mathbf{q} = \{q_1, \ldots, q_{3g'-3+b}\}$ of $R_{C,G}$ provides a formal neighbourhood $\mathrm{Spec}\, k[\![\mathbf{q}]\!] \to \overline{\mathcal{M}}_{g,[m]}(G)$ of the G-stable curve C.

§2.2. The case of G-curves without étale factorisation

In the following, denote by G a cyclic group of order $n \in \mathbb{N}$ and fix a generator γ of G. A degeneration result is established for G-covers in terms of their associated branch data $\underline{\mathbf{kr}}$ by building a specific stable marked G-curve and controlling the branching data through G-equivariant deformation. The process relies on a rigidification of Hurwitz data, the γ-type, first applied to the case of unmarked, then to marked curves.

2.2.1. Let C/k be a G-curve over a field k containing nth roots of unity. Then $C \to C/G$ is étale locally given by an equation of the form

$$y^n = \prod_I (x - \alpha_i)^{k_i},$$

and the order of the stabiliser group of α_i is given by $n/\gcd(n, k_i)$. For a given G-cover $C \to C/G$, let us denote by $\mathbf{k} = (k_1, \ldots, k_\nu) \in (\mathbb{Z}/n\mathbb{Z})^\nu$ the associated Hurwitz data, and write $\mathrm{ord}(k_i) = n/\gcd(n, k_i)$. The *$\gamma$-type* of a point of C gives a way to recover the Hurwitz data from local information.

Definition 2.3 (γ-type). Let k be an algebraically closed field of characteristic 0, let G be a cyclic group of order n, $\gamma \in G$ be a generator, and $\zeta \in k$ denote a primitive nth root of unity. Let C/k be a complete smooth curve endowed with a G-action and $P \in C$ be a closed point with non-trivial stabiliser under the action of G.

The γ-*type* of P, denoted $\operatorname{type}_\gamma(P)$, is the set of all $\zeta \in k$ such that, for a uniformising parameter u of C at P one has

$$\gamma^\ell(u) = \zeta^\ell u \bmod u^2 \quad \text{for all } \ell \in \mathbb{Z} \text{ such that } \gamma^\ell(P) = P.$$

The γ-type of a point is independent of the choice of the uniformising parameter u. The following lemma gives a link between the local γ-type and the Hurwitz data of the cover $C \to C/G$, and is used in the next section to build a stable G-curve by gluing two points of *inverse γ-types*.

Lemma 2.4. *Let C/k be a complete smooth curve endowed with a G-action, and denote by $\{P_i\}_I$ the ramification points of $C \to C/G$ with Hurwitz data $\mathbf{k} = \{k_i\}_I$. There exists a $\zeta \in k$ such that for all $i \in I$ one has $\zeta^{j_i \frac{n}{\operatorname{ord}(k_i)}} \in \operatorname{type}_\gamma(P_i)$, where j_i denotes the inverse of $k_i \frac{\operatorname{ord}(k_i)}{n}$ modulo $\operatorname{ord}(k_i)$.*

Note that for $a \in \mathbb{Z}/n\mathbb{Z}$, the element $a\frac{\operatorname{ord}(a)}{n}$ is well defined in $\mathbb{Z}/\operatorname{ord}(a)\mathbb{Z}$.

Proof of Lemma 2.4. By Kummer theory, the morphism $\pi\colon C \to C/G$ is given over the étale locus by an equation of the form $y^n = f(x)$ and the action of G is given by $\gamma(y) = \zeta y$. Let w be a uniformising parameter at $\pi(P_i)$ in C/G so that up to an nth power, $y^n = w^{k_i} t$ – where t is an invertible element.

Writing a decomposition $an + k_i j_i = \frac{n}{\operatorname{ord} k_i}$, the element

$$u = y^{j_i \frac{n}{\operatorname{ord} k_i}} w^a$$

is a uniformising parameter of C at P_i and we have

$$\begin{aligned} \gamma(u) &= \gamma(y)^{j_i \frac{n}{\operatorname{ord} k_i}} w^a \\ &= \zeta^{j_i \frac{n}{\operatorname{ord} k_i}} y^{j_i \frac{n}{\operatorname{ord} k_i}} w^a, \\ \gamma(u) &= \zeta^{j_i \frac{n}{\operatorname{ord} k_i}} u, \end{aligned}$$

hence the result on the γ-type of P_i. □

Remark 2.5. Once a generator γ of G and a primitive nth root ζ are fixed, the Hurwitz data $\mathbf{k}$ are read locally through the action of G on the tangent space of the ramification points. It is then possible to compute the genus of C/G using the Riemann–Hurwitz formula.

2.2.2. The degeneration of G-equivariant curves to the boundary of $\mathcal{M}_g$ is studied, first in the case of unmarked curves.

Theorem 2.6. *Let $g, g' \geq 1$ be integers, $G = \mathbb{Z}/n\mathbb{Z}$, $\mathbf{k} = (k_1, \ldots, k_\nu) \in (\mathbb{Z}/n\mathbb{Z})^\nu$ satisfying*

$$2g - 2 = n(2g' - 2) + \sum_i (\mathrm{ord}(k_i) - 1)\frac{n}{\mathrm{ord}(k_i)}, \tag{2.1}$$

$$\sum_i k_i = 0. \tag{2.2}$$

For all $\ell \in (\mathbb{Z}/n\mathbb{Z})^*$ there exists a singular curve C_ℓ/k of genus g endowed with a G-admissible action, with C_ℓ/G of genus g', and such that

(i) *the normalisation of C_ℓ is of genus $g - 1$ and its quotient by G has Hurwitz data $(k_1, \ldots, k_\nu, \ell, -\ell) \in (\mathbb{Z}/n\mathbb{Z})^{\nu+2}$;*
(ii) *the generic G-equivariant deformation of C_ℓ is smooth and has $\mathbf{k}$ for Hurwitz data.*

The construction below also illustrates the control of the Hurwitz data along G-equivariant deformations.

Proof of Theorem 2.6. Let E_0/k be a smooth curve of genus $g' - 1$ over an algebraically closed field k, and let $(\mathbf{k}, \ell, -\ell) \in (\mathbb{Z}/n\mathbb{Z})^{\nu+2}$. By [CM15, Prop. 3.7], there exists a G-equivariant cover $E_1 \to E_0$ with Hurwitz data $(\mathbf{k}, \ell, -\ell)$. Moreover, E_1 is of genus $g - 1$ since

$$\begin{aligned} n(2(g'-1) - 2) + \sum_i (\mathrm{ord}(k_i) - 1)\frac{n}{\mathrm{ord}(k_i)} + 2(n-1) &= 2g - 4 \\ &= 2(g-1) - 2. \end{aligned}$$

Let $\{P_1, \ldots, P_\nu\}$ denote the ramification points with Hurwitz data $\{k_1, \ldots, k_\nu\}$ and $\{P'_1, P'_2\}$ the points with data $\{\ell, -\ell\}$.

Following Lemma 2.4, since P'_1 and P'_2 are totally ramified by assumption, there exists $\zeta \in k$ such that

$$\zeta \in \mathrm{type}_\gamma(P'_1) \quad \text{and} \quad \zeta^{-1} \in \mathrm{type}_\gamma(P'_2). \tag{2.3}$$

Let C_ℓ be the curve obtained from E_1 by gluing P'_1 and P'_2 as a point P' as in Figure 1. As ℓ is prime to n, the points P'_1 and P'_2 are both fixed under G so that the curve C_ℓ is endowed with a G-action. Moreover, this action is admissible thanks to equation (2.3), and it satisfies property (i) of the theorem since E_1 is the normalisation of C_ℓ.

By Theorem 2.2, there exists a G-curve $\mathcal{C}$ over a complete local ring R of residue field k, with special fibre C_ℓ and a generic fibre that is smooth of genus g. For $P_i \in C_\ell$ a ramification point, let γ_i be a generator of $\mathrm{Stab}_G(P_i)$ and $\zeta \in k$

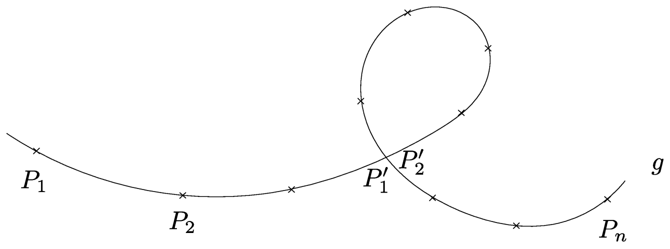


Figure 1: Curve C_ℓ obtained by gluing P'_1 and P'_2.

a primitive nth root of unity as given by Lemma 2.4. The action of γ_i on the completion of local ring $\widehat{\mathcal{O}}_{C_\ell,P_i} \simeq k[\![u_i]\!]$ is given by $\gamma(u_i) = \zeta^{j_i} u_i$ up to a change of parameter u_i. Moreover, there exists a lifting $\widetilde{P}_i$ of P_i such that the action of G on $\mathcal{C}$ is given in a formal neighbourhood of $\widetilde{P}_i$ by the same formula on the ring $R[\![u_i]\!]$, up to another change of parameter u_i. Following Remark 2.5, the Hurwitz data of the generic fibre of $\mathcal{C}/R$ is then $\mathbf{k}$ as it can be read on u_i through the γ-type. Finally, a vanishing cycle computation proves that no ramification point of the generic fibre of $\mathcal{C}/R$ specialises to P', so that C_ℓ satisfies property (ii). □

The theorem above is still valid with the assumption $\mathbf{k} = \emptyset$, so that a smooth curve with G-action of étale type can be built, which specialises to C_ℓ with only one singular point and whose normalisation has no étale factorisation.

2.2.3. In the case of curves *with m marked points*, i.e. endowed with a horizontal G-equivariant Cartier divisor D of degree m, the Hurwitz data $\mathbf{k}$ is replaced by the branch data $\underline{\mathbf{kr}}$: the *branch data of a curve $C \in \mathcal{M}_{g,[m]}(G)$* is the equivalence class of couples $\mathbf{kr} = (\mathbf{k}, \mathbf{r})$, where $\mathbf{k}$ is a Hurwitz data and $\mathbf{r} = (r_1, \dots, r_n)$ is an n-tuple given by

$$r_i = \#\big\{y \in D/G, \text{ the branching data at } y \text{ is equal to } i \bmod n\big\},$$

modulo the diagonal $\mathrm{Aut}(G)$-action whose role is to forget the choice of an nth root of unity – see [CM15, Def. 3.9].

Let m' denote the degree of the divisor D/G. In addition to equations (2.1) and (2.2), we assume that the branch data also satisfies

$$m = \sum_i r_i \gcd(i, n), \tag{2.4}$$

$$m' = \sum_i r_i. \tag{2.5}$$

We now state our G-deformation result in its complete form.

Corollary 2.7. *For any generic point $\eta \in \mathcal{M}_{g,[m]}(G)$, whose corresponding curve satisfies the branch data relations above, there exists a specialisation $z \in \overline{\mathcal{M}}_{g,[m]}(G)$ of η such that the normalisation of the curve corresponding to z has genus $g-1$ and is without étale factorisation.*

Proof. This is a direct consequence of the description of the set of irreducible components of $\mathcal{M}_{g,[m]}(G)$ by the set of the branch data $\underline{\mathbf{kr}}$ as given in [CM15], and of Theorem 2.6: for a given $\mathbf{kr}$, one constructs explicitly a G-equivariant marked curve as in the proof above. □

For an algebraic definition of $\underline{\mathbf{kr}}$ (resp. $\mathbf{k}$) for families of curves in terms of étale cohomology and for examples see [CM15, §3.2 (resp. §3.1)]. This reference also contains a discussion about the non-canonicity of $\mathbf{k}$ and $\mathbf{kr}$ relative to the choice of a generator γ of G and a primitive nth root of unity $\zeta \in \mu_n$.

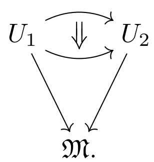
This G-deformation result is completed in Section 4.2.3 at the level of automorphism groups of curves. This is then a key ingredient to reduce the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action by cyclotomy to the case of stack inertia without étale factorisation.

§3. Galois action at infinity

Generalising [Zoo01] and [Nak99], we give the definition of Galois actions at infinity attached to a normal crossing divisor of a generic Deligne–Mumford algebraic stack, and then discuss their compatibility through Knudsen morphisms. We clarify this result in the case of $\mathcal{M}_{g,[m]}$, leading to non-canonical comparisons of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations of the fundamental groups of $\mathcal{M}_{g-1,[m]+2}$ and $\mathcal{M}_{g,[m]}$.

§3.1. Tangential base points

Adapting [Zoo01] to the case of a non-geometric base point, we define a tangential base point for a Deligne–Mumford algebraic stack $\mathfrak{M}$ in terms of the *tame fundamental group* of $\mathfrak{M}$ along a normal crossing divisor, which is adapted from the case of the scheme in [GM71]: for a normal crossing divisor $\mathcal{D} \to \mathfrak{M}$, the category $\mathfrak{Rev}^{\mathcal{D}}(\mathfrak{M})$ of tamely ramified covers of $\mathfrak{M}$ along $\mathcal{D}$ is defined via the scheme category $\mathfrak{Rev}^{\mathcal{D}\times_{\mathfrak{M}} X}(X)$ by pull-back along a presentation $X \to \mathfrak{M}$ – see [Zoo01, §3]. In $\mathfrak{Rev}^{\mathcal{D}}(\mathfrak{M})$, morphisms of between covers are defined up to 2-morphisms, as in the diagram



While based on Zoonekynd's approach for *geometric* tangential points, the present one is here developed for rational base points and as such implies the definition of a Galois action.

3.1.1. Let k be a field of characteristic zero and $\mathfrak{M}$ a Deligne–Mumford algebraic k-stack. A point $x \in \mathfrak{M}(\operatorname{Spec} k)$ is said to have a *Nisnevich neighbourhood* if there exist an étale morphism $f\colon V \to \mathfrak{M}$ with V a scheme, and a point $v \in f^{-1}(x)$ with residue field k. For a point x having a Nisnevich neighbourhood, we define the *local ring at x in $\mathfrak{M}$*, denoted by $\mathcal{O}^h_{\mathfrak{M},x}$, as

$$\mathcal{O}^h_{\mathfrak{M},x} = \varinjlim_{(v,V)} \mathcal{O}_{V,v},$$

where the limit is taken over the couples (v, V) as above, and $\mathcal{O}^h_{\mathfrak{M},x}$ is the Henselisation $\mathcal{O}^h_{V,v}$ for any Nisnevich neighbourhood. Note that for a stack $\mathfrak{M}$ over k and a smooth point x, the completion of $\mathcal{O}^h_{\mathfrak{M},x}$ can be identified with $k[\![t_1, \dots, t_n]\!]$ via the choice of a system of parameters $t_1, \dots, t_n \in \mathcal{O}^h_{\mathfrak{M},x}$.

Let $\mathfrak{M}$ be a k-stack, $x \in \mathfrak{M}(\operatorname{Spec} k)$ a smooth point which is supposed to have a Nisnevich neighbourhood and $\mathbf{t} = \{t_1, \dots, t_n\}$ be a system of parameters of $\mathcal{O}^h_{\mathfrak{M},x}$. We define the *Puiseux ring* of $\mathfrak{M}$ at x with respect to $\mathbf{t}$ as the ring

$$\widetilde{\mathcal{O}}^{\mathbf{t}}_{\mathfrak{M},x} = \varinjlim_{\ell \in \mathbb{N}^\times} \big(\widehat{\mathcal{O}^h_{\mathfrak{M},x}} \,\widehat{\otimes}_k\, \bar{k}\big)[t_1^{1/\ell}, \dots, t_n^{1/\ell}].$$

Remark 3.1. In the case of a *geometric point* x, the local ring $\mathcal{O}^h_{\mathfrak{M},x}$ in the construction of the Puiseux ring above is replaced by the *strict Henselisation* $\mathcal{O}^{sh}_{\mathfrak{M},x}$ which exists without condition – see [LMB00, Rem. 6.2.1].

The following gives a class of points with Nisnevich neighbourhood, which includes the schematic points of $\mathfrak{M}$.

Proposition 3.2. *Let $\mathfrak{M}$ be a Deligne–Mumford k-algebraic stack and consider $x \in \mathfrak{M}(\operatorname{Spec} k)$. If $\operatorname{Aut}_k(x)$ is a constant group scheme, then x admits a Nisnevich neighbourhood.*

Proof. Consider the functor $F\colon (k-\mathrm{Art}) \to \mathfrak{Set}$ defined on the Artinian k-algebras by the isomorphism classes of objects of $\mathfrak{M}$ which are deformations of points whose images are equal to that of x in $\mathfrak{M}$. As $\operatorname{Aut}_k(x)$ is a constant group-scheme and the diagonal of $\mathfrak{M}$ is unramified, the functor F is actually a sheaf for the étale topology on $(k-\mathrm{Art})$. Then [LMB00, Thm. 10.10] gives an étale presentation with a k-point above x, thus a Nisnevich neighbourhood of x. □

Consider a normal crossing divisor $\mathcal{D}$ on $\mathfrak{M}$ whose support contains x, and let $t_D = \{t_1, \dots, t_n\}$ be a system of parameters of $\mathfrak{M}$ at x such that $\mathcal{D}$ is given in an étale neighbourhood of x by $t_1 \cdots t_m = 0$. A *k-rational tangential base point* on $\mathfrak{M} \setminus \mathcal{D}$ at x is then defined as a fibre functor in terms of the *Puiseux ring*:

Definition 3.3. Let $\mathfrak{M}$ be a Deligne–Mumford k-stack, x be a smooth k-point of $\mathfrak{M}$ having a Nisnevich neighbourhood, and $\mathcal{D}$ be a normal crossing divisor on $\mathfrak{M}$, with $t_{\mathcal{D}}$ a system of parameters of $\mathcal{D}$ at x. The *k-rational tangential base point* associated to $\vec{s} = (x, t_{\mathcal{D}})$ is defined as the functor

$$\begin{aligned} F_x^{t_{\mathcal{D}}} : \mathfrak{Rev}^{\mathcal{D}}(\mathfrak{M}) &\to \mathfrak{Set}, \\ Y &\mapsto \mathrm{Hom}_{\mathrm{Frac}(\mathfrak{M})}(\mathrm{Frac}\, Y, \mathrm{Frac}(\widetilde{\mathcal{O}}^{t_{\mathcal{D}}}_{\mathfrak{M},x})), \end{aligned}$$

where Frac denotes the ring of fraction.

Unlike the classical Grothendieck–Murre theory, the base point is here supposed to belong to the normal crossing divisor. Following [Zoo01, §3] to which we refer for details, one obtains the following theorem:

Theorem 3.4. *Let $\mathfrak{M}$ be a Deligne–Mumford k-stack, x be a smooth k-point of $\mathfrak{M}$ having a Nisnevich neighbourhood, $\mathcal{D}$ be a normal crossing divisor on $\mathfrak{M}$, and $t_{\mathcal{D}}$ a system of parameters of $\mathcal{D}$ at x. Then the* tangential base point *functor $F_x^{t_{\mathcal{D}}}$ is a* fibre functor*.*

The proof follows essentially that of [Zoo01, Thm. 3.7]: the functor is isomorphic to a functor defined by a geometric point $x' \in \mathfrak{M}$, which by the theory of the étale fundamental group is then a fibre functor on the Galois category $\mathfrak{Rev}^{\mathcal{D}}(\mathfrak{M})$. Here x' is given by the generic point of $\mathrm{Frac}(\widetilde{\mathcal{O}}^{t_{\mathcal{D}}}_{\mathfrak{M},x})$, which is an algebraic closure of $\mathrm{Frac}(\mathcal{O}^h_{\mathfrak{M},x})$ by the Puiseux theorem, since $\mathrm{char}(k) = 0$.

In particular, this defines the *arithmetic tame fundamental group based at a k-rational tangential base point* $\pi_1^{\mathcal{D}}(\mathfrak{M}; t_{\mathcal{D}}, x) = \mathrm{Aut}(F_x^{t_{\mathcal{D}}})$ as the automorphism group of the tangential fibre functor.

3.1.2. Consider the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ of k, as well as the canonical projection $\pi_1^{\mathcal{D}}(\mathfrak{M}; t_{\mathcal{D}}, x) \xrightarrow{\text{can}} \mathrm{Gal}(\bar{k}/k)$ given by the corresponding restriction of the isomorphism $\mathrm{Frac}(\widetilde{\mathcal{O}}^{t_{\mathcal{D}}}_{\mathfrak{M},x}) \xrightarrow{\sim} \mathrm{Frac}(\widetilde{\mathcal{O}}^{t_{\mathcal{D}}}_{\mathfrak{M},x})$. Denoting by $\pi_1^{\mathcal{D}\otimes\bar{k}}(\mathfrak{M} \otimes \bar{k}; t_{\mathcal{D}\otimes\bar{k}}, x)$ the kernel of can, we obtain the following proposition:

Proposition 3.5. *Let $\mathfrak{M}$ be a k-algebraic stack. The choice of a k-rational tangential base point $\vec{s} = (x, t_{\mathcal{D}})$ of $\mathfrak{M}$ defines a* $\mathrm{Gal}(\bar{k}/k)$*-action*

$$\rho_{\vec{s}} \colon \mathrm{Gal}(\bar{k}/k) \longrightarrow \mathrm{Aut}[\pi_1^{\mathcal{D}\otimes\bar{k}}(\mathfrak{M} \otimes \bar{k}; t_{\mathcal{D}\otimes\bar{k}}, x)]$$

which is given by conjugacy and is called a tangential $\mathrm{Gal}(\bar{k}/k)$-representation.

This follows directly and formally from the construction: the action of $\mathrm{Gal}(\bar{k}/k)$ on $\widehat{\mathcal{O}^h_{\mathfrak{M},x}} \mathbin{\widehat{\otimes}} \bar{k}$ via the second component defines first a splitting $\iota_{\vec{s}} \colon \mathrm{Gal}(\bar{k}/k) \to \pi_1^{\mathcal{D}}(\mathfrak{M}; t_{\mathcal{D}}, x)$ of can, then a $\mathrm{Gal}(\bar{k}/k)$-action on $\pi_1^{\mathcal{D}\otimes\bar{k}}(\mathfrak{M}\otimes\bar{k}; t_{\mathcal{D}\otimes\bar{k}}, x)$. The latter is group theoretically given by conjugacy

$$\rho_{\vec{s}}(\sigma) \colon h \to \iota_{\vec{s}}(\sigma)\cdot h \cdot \iota_{\vec{s}}(\sigma)^{-1}, \quad h \in \pi_1^{\mathcal{D}\otimes\bar{k}}(\mathfrak{M}\otimes\bar{k}; t_{\mathcal{D}\otimes\bar{k}}, x), \tag{3.1}$$

with $\iota_{\vec{s}}(\sigma) \in \pi_1^{\mathcal{D}}(\mathfrak{M}; t_{\mathcal{D}}, x)$ the lift of $\sigma \in \mathrm{Gal}(\bar{k}/k)$ via $\vec{s} = (x, t_{\mathcal{D}})$.

Remark 3.6. Consider the context of Section 3.2.1 of the moduli $\mathbb{Q}$-stack of stable curves $\overline{\mathcal{M}}_{g,[m]}$, where $\vec{s}$ is defined via a maximally degenerated curve x supported in a singular locus divisor $\mathcal{D}$. Grothendieck–Murre theory as in [IN97] identifies the generators of $\pi_1^{\mathrm{et}}(\mathcal{M}_{g,[m]} \otimes_k \overline{\mathbb{Q}}, *)$ as Tate generators of $\widehat{\mathbb{Z}}(1)$s in $\pi_1^{\mathcal{D}\otimes\overline{\mathbb{Q}}}(\overline{\mathcal{M}}_{g,[m]} \otimes \overline{\mathbb{Q}}; t_{\mathcal{D}\otimes\overline{\mathbb{Q}}}, x)$, which allows the explicit computation of the corresponding tangential $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation $\rho_{\vec{s}}$. For $\mathcal{M}_{0,[m]}$, the splitting of equation (3.1) defined by the tangential base point $\vec{b}$ of [Nak97], gives, for example,

$$\rho_{\vec{b}}(\sigma)(\sigma_i) = f_\sigma(y_i, \sigma_i^2)^{-1} \cdot \sigma_i^{\chi_\sigma} \cdot f_\sigma(y_i, \sigma_i^2) \quad \text{with } y_i = \sigma_{i-1}\ldots\sigma_1\sigma_1\ldots\sigma_{i-1}, \tag{3.2}$$

where σ_i are the classical generators of $\pi_1^{\mathrm{et}}(\mathcal{M}_{0,[m]} \otimes \overline{\mathbb{Q}}, *)$.

A tangential base point is functorial through base change over k, while it is not for general k-stack morphisms. Indeed, there are some extra data and assumptions required to define a *stack morphism between tangential base points*. Consider a representable morphism $f \colon \mathfrak{N} \to \mathfrak{M}$ between Deligne–Mumford k-stacks, $\mathcal{D}$ a normal crossing divisor on $\mathfrak{M}$, and $x \in \mathcal{D}(\mathrm{Spec}\, k)$, with $y \in f^{-1}(x)$ a k-rational point, and suppose that x and y are both smooth and admit a Nisnevich neighbourhood.

Following [Zoo01, Lem. 2.9] there is an identification

$$\mathrm{Hom}_{\mathrm{Frac}(\mathfrak{M})}(\mathrm{Frac}\, Y, \mathrm{Frac}(\widetilde{\mathcal{O}}^{t_{\mathcal{D}}}_{\mathfrak{M},x})) = \mathrm{Hom}_{\mathfrak{M}}(\mathrm{Spec}\, \widetilde{\mathcal{O}}^{t_{\mathcal{D}}}_{\mathfrak{M},x}, Y),$$

so that it is possible to define a tangential base point in terms of rings instead of fields. Suppose moreover that there are two systems of parameters $t_{f^*\mathcal{D}} = \{t'_1, \ldots, t'_n\}$ and $t_{\mathcal{D}} = \{t_1, \ldots, t_\ell\}$ respectively of $f^*\mathcal{D}$ at y and of $\mathcal{D}$ at x such that the induced morphism $f^h \colon \mathcal{O}^h_{\mathfrak{M},x} \to \mathcal{O}^h_{\mathfrak{N},y}$ sends t'_j to t_j or 0. Then the $\mathrm{Gal}(\bar{k}/k)$-equivariant natural transformation of functors

$$F_f \colon F_x^{t_{\mathcal{D}}} \to F_y^{t_{f^*\mathcal{D}}}$$

is obtained using the $\mathrm{Gal}(\bar{k}/k)$-equivariant morphism

$$\tilde{f} \colon \widetilde{\mathcal{O}}^{t_{\mathcal{D}}}_{\mathfrak{M},x} \to \widetilde{\mathcal{O}}^{t_{f^*\mathcal{D}}}_{\mathfrak{N},y}.$$

In the two important cases of unramified and smooth morphisms, an extension property guarantees the following nearly "functorial" result.

Proposition 3.7. *Let $f\colon \mathfrak{N} \to \mathfrak{M}$ be a morphism of Deligne–Mumford k-stacks, $\mathcal{D}$ be a normal crossing divisor on $\mathfrak{M}$ such that $f^*\mathcal{D}$ is a normal crossing divisor on $\mathfrak{N}$, and let $x \in \mathcal{D}(\operatorname{Spec} k)$ and $y \in f^{-1}(x)$ be k-rational points such that both x and y are smooth and have Nisnevich neighbourhoods.*

If f is either smooth or unramified, then there exist regular systems of parameters $t_{f^*\mathcal{D}}$ and $t_{\mathcal{D}}$ of $f^*\mathcal{D}$ at y and $\mathcal{D}$ at x, and a Galois equivariant morphism

$$\pi_1^{f^*\mathcal{D}}(\mathfrak{N}; t_{f^*\mathcal{D}}, y) \to \pi_1^{\mathcal{D}}(\mathfrak{M}; t_{\mathcal{D}}, x).$$

Proof. Following the discussion above, it is sufficient to establish that there exist two systems of parameters $t_{\mathcal{D}}$ for $\mathcal{D}$ at x and $t_{f^*\mathcal{D}}$ for $f^*\mathcal{D}$ at y such that f induces a morphism $f^h\colon \mathcal{O}^h_{\mathfrak{M},x} \to \mathcal{O}^h_{\mathfrak{N},y}$ that sends an element of $t_{\mathcal{D}}$ to an element of $t_{f^*\mathcal{D}}$ or 0.

If f is unramified, the morphism $f^\#\colon \widehat{\mathcal{O}}^h_{\mathfrak{M},x} \to \widehat{\mathcal{O}}^h_{\mathfrak{N},y}$ is a surjection. Consider $t_1, \dots, t_n$ a system of parameters of $\mathcal{D}$ in $\widehat{\mathcal{O}}^h_{\mathfrak{M},x}$. As $f^*\mathcal{D}$ is a divisor by assumption, it is possible to extract from $f^\#(t_1), \dots, f^\#(t_n)$ a system of generators of $\widehat{\mathcal{O}}^h_{\mathfrak{N},y}$, because $f^\#$ is formally unramified and induces an injection on tangent spaces.

If f is smooth, then the morphism $f^\#$ is injective and a system of parameters $t_1, \dots, t_n$ for $\mathcal{D}$ in $\widehat{\mathcal{O}}^h_{\mathfrak{M},x}$ can be completed into a system of parameters $t_1, \dots, t_{n'}$ of $\widehat{\mathcal{O}}^h_{\mathfrak{N},y}$ by picking up vectors in the tangent space. $\square$

The lack of functoriality comes from the fact that *there is no obvious choice of parameters*, which has the important consequence below.

Remark 3.8. A k-rational change of parameters $t_{\mathcal{D}}$ to $t'_{\mathcal{D}}$ – or *infinitesimal homotopic transformation* – leads to two *k-homotopically* equivalent k-rational base points $F_x^{t_{\mathcal{D}}} \simeq F_x^{t'_{\mathcal{D}}}$, but *not to equivalent* $\operatorname{Gal}(\bar{k}/k)$*-actions* on the fundamental groups: the action on Puiseux series makes some Kummer character appear from the Nth roots of the involved rational coefficients.

3.1.3. Let us fix a Galois representation defined by the choice of a k-tangential base point $\vec{s} = (x, t_{\mathcal{D}})$ on $\mathfrak{M}$ as in Proposition 3.5:

$$\rho_{\vec{s}}\colon \operatorname{Gal}(\bar{k}/k) \to \operatorname{Aut}[\pi_1^{\mathcal{D}\otimes\bar{k}}(\mathfrak{M}\otimes\bar{k}; \vec{s})].$$

We now provide some additional assumptions under which this tangential Galois action $\rho_{\vec{s}}$ induces a $\operatorname{Gal}(\bar{k}/k)$-action $\rho^I_{\vec{s},\bar{z}}$ on the stack inertia group $I_{\bar{z}}$.

Consider a geometric point $w\colon \operatorname{Spec}(\bar{k}) \to \mathfrak{M}$ and let $I_{\mathfrak{M},w} = \operatorname{Spec}(\bar{k})\,{}_w{\times} I_{\mathfrak{M}}$ denote its stack inertia group of 2-transformations, with $I_{\mathfrak{M}} = \mathfrak{M} \times_{\mathfrak{M}\times\mathfrak{M}} \mathfrak{M}$ denoting the inertia stack of $\mathcal{M}$. Since $\gamma \in I_w = I_{\mathfrak{M},w}$ induces a transformation of the fibre functor F_w, this defines a morphism $\omega_w\colon I_w \to \pi_1(\mathfrak{M}\otimes\bar{k}, w)$ that we assume is injective in the rest of this section – see also Remark 4.4. The group I_w is also called the group of *hidden paths* of the étale fundamental group; cf. [Noo04, §4].

Let $\bar{z}\colon \operatorname{Spec}\overline{K} \to \mathfrak{M}$ be a geometric point, let K denote its field of definition, and choose an injection $\bar{k} \subset \overline{K}$. As any $\sigma \in \operatorname{Gal}(\bar{k}/k)$ can be extended to a k-automorphism $\tilde{\sigma}$ of $\overline{K}$, fix one such $\tilde{\sigma}$ and define $\tilde{\sigma}(\bar{z})$ by base change.

Let us furthermore fix an étale path $\bar{z} \rightsquigarrow x$ (resp. $\tilde{\sigma}(\bar{z}) \rightsquigarrow x$) from $\bar{z}$ (resp. $\tilde{\sigma}(\bar{z})$) to $\vec{s} = (x, t_{\mathcal{D}})$, which defines a morphism

$$\phi_{\bar{z}\rightsquigarrow x}\colon \pi_1(\mathfrak{M}, \bar{z}) \longrightarrow \pi_1^{\mathcal{D}}(\mathfrak{M}; \vec{s}) \quad (\text{resp. } \phi_{\tilde{\sigma}(\bar{z})\rightsquigarrow x}\colon \pi_1(\mathfrak{M}, \tilde{\sigma}(\bar{z})) \longrightarrow \pi_1^{\mathcal{D}}(\mathfrak{M}; \vec{s})),$$

and all together, thanks to the compatibility between σ and $\tilde{\sigma}$, induce a diagram

$$\begin{array}{ccccc} I_{\bar{z}} & & \xrightarrow{\tau\mapsto\tilde{\sigma}^{-1}\tau\tilde{\sigma}} & & I_{\tilde{\sigma}(\bar{z})} \\ \downarrow{\scriptstyle \omega_{\bar{z}}} & & & & \downarrow{\scriptstyle \omega_{\tilde{\sigma}(\bar{z})}} \\ \pi_1(\mathfrak{M}\otimes\bar{k}, \bar{z}) & & & & \pi_1(\mathfrak{M}\otimes\bar{k}, \tilde{\sigma}(\bar{z})) \\ & \searrow^{\phi_{\bar{z}\rightsquigarrow x}} & & \nearrow^{\phi^{-1}_{\tilde{\sigma}(\bar{z})\rightsquigarrow x}} & \\ & \pi_1^{\mathcal{D}\otimes\bar{k}}(\mathfrak{M}\otimes\bar{k}; \vec{s}) & \underset{\sigma}{\xrightarrow{\sim}} & \pi_1^{\mathcal{D}\otimes\bar{k}}(\mathfrak{M}\otimes\bar{k}; \vec{s}), & \end{array} \tag{3.3}$$

where the bottom line is the action by conjugacy defined by the tangential $\operatorname{Gal}(\bar{k}/k)$-action $\rho_{\vec{s}}$ on $\pi_1^{\mathcal{D}\otimes\bar{k}}(\mathfrak{M}\otimes\bar{k}, \bar{z})$ of equation (3.1). This diagram is a priori commutative *up to conjugacy by a hidden path* ε from $\bar{z}$ to $\tilde{\sigma}(\bar{z})$, i.e. a 2-transformation

$$\operatorname{Spec}(\overline{K}) \underset{\tilde{\sigma}(\bar{z})}{\overset{\bar{z}}{\rightrightarrows}} \mathfrak{M}, \quad \Downarrow\varepsilon .$$

Let us further assume that K is linearly disjoint from $\bar{k}$ over k. In this case, $\bar{z}$ is then stable under $\operatorname{Gal}(\bar{k}/k)$ so that we can choose $\phi_{\tilde{\sigma}(\bar{z})\rightsquigarrow x} = \phi_{\bar{z}\rightsquigarrow x}$ for all $\sigma \in \operatorname{Gal}(\bar{k}/k)$, and diagram (3.3) becomes strictly commutative, i.e. without any hidden path conjugacy by $\varepsilon\colon \bar{z} \Rightarrow \sigma(\bar{z})$. It has the following consequences:

(1) Since $\rho_{\vec{s}}$ sends the image $\omega_{\bar{z}}(I_{\bar{z}})$ into itself in the first line of diagram (3.3), the tangential $\operatorname{Gal}(\bar{k}/k)$-action $\rho_{\vec{s}}$ defines a $\operatorname{Gal}(\bar{k}/k)$-*stack inertia Galois action*

$$\rho^{I}_{\vec{s},\bar{z}}\colon \operatorname{Gal}(\bar{k}/k) \longrightarrow \operatorname{Aut}[I_{\bar{z}}] \tag{3.4}$$

that is defined by $\rho^{I}_{\vec{s},\bar{z}}(\sigma) = \phi^{-1}_{\bar{z}\rightsquigarrow x} \circ \rho_{\vec{s}} \circ \phi_{\bar{z}\rightsquigarrow x}$.

(2) It follows from the assumption on K that $\pi\colon \operatorname{Gal}(\overline{K}/K) \twoheadrightarrow \operatorname{Gal}(K\otimes_k \bar{k}/K) \simeq \operatorname{Gal}(\bar{k}/k)$. The action $\rho^{I}_{\vec{s},\bar{z}}$ is thus induced by the canonical local $\operatorname{Gal}(\overline{K}/K)$-

action $\rho^I_{\bar{z}}$, via the commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(\overline{K}/K) & \xrightarrow{\rho^I_{\bar{z}}} & \mathrm{Aut}[I_{\bar{z}}] \\ {\scriptstyle\pi}\downarrow & & \| \\ \mathrm{Gal}(\bar{k}/k) & \xrightarrow{\rho^I_{\vec{s},\bar{z}}} & \mathrm{Aut}[I_{\bar{z}}]. \end{array} \tag{3.5}$$

Proposition 3.9. *Let us fix a k-rational tangential base point $\vec{s}$ of $\mathfrak{M}$, a K-point $z\colon \mathrm{Spec}(K) \to \mathfrak{M}$ of $\mathfrak{M}$, a geometric point $\bar{z}\colon \mathrm{Spec}(\overline{K}) \to \mathfrak{M}$ above z, and an étale path $\phi_{\bar{z}\rightsquigarrow x}$ from $\bar{z}$ to $\vec{s}$. Suppose that K is linearly disjoint from $\bar{k}$ over k.*

Then the tangential Galois representation $\rho_{\vec{s}}$: $\mathrm{Gal}(\bar{k}/k) \to \mathrm{Aut}[\pi_1^{\mathcal{D}\otimes\bar{k}}(\mathfrak{M}\otimes\bar{k};\vec{s})]$ defined by $\vec{s}$ defines a $\mathrm{Gal}(\bar{k}/k)$-action $\rho^I_{\vec{s},\bar{z}}$ on the stack inertia $I_{\bar{z}} \to \pi_1(\mathfrak{M}\otimes\bar{k},\bar{z})$. This action coincides furthermore with the action $\rho^I_{\bar{z}}$ of $\mathrm{Gal}(\overline{K}/K)$ on $I_{\bar{z}}$ that is given by conjugacy.

Proof. This follows from the discussion above. Since K is linearly disjoint from $\bar{k}$, the $\bar{k}$-image of $\bar{z}\colon \mathrm{Spec}(\overline{K}) \to \mathfrak{M}$ is stable under the $\mathrm{Gal}(\bar{k}/k)$-action. The tangential $\mathrm{Gal}(\bar{k}/k)$-action $\rho_{\vec{s}}$ on $\pi_1(\mathfrak{M}\otimes\bar{k},\bar{z})$ then sends the image $I_{\bar{z}}$ into itself as given by equation (3.4), and so induces an action $\rho^I_{\vec{s},\bar{z}}$ of $\mathrm{Gal}(\bar{k}/k)$ on $I_{\bar{z}}$ according to the commutativity of the diagram (3.3).

This proves furthermore that $\rho^I_{\bar{z}}$ and $\rho^I_{\vec{s},\bar{z}}$, seen as an action of $\mathrm{Gal}(\overline{K}/K)$ through the surjection $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(\bar{k}/k)$, define the same action on $I_{\bar{z}}$. □

Remark 3.10. The following are of general arithmetic interest:

(i) In the case where $\mathfrak{M} = \mathcal{M}_{g,[m]}$ and $I_{\mathfrak{M},w} = \langle\gamma\rangle$ is cyclic, one shows that K above can be taken as the field of moduli-definition of the generic point of an irreducible component of the special loci $\mathcal{M}_{g,[m]}(\gamma)$ – see [CM15, Lem. 5.2 and Cor. 4.2].

(ii) The action $\rho^I_{\vec{s},\bar{z}}$ of equation (3.4) depends on the choice of the étale path $\phi_{\bar{z}\rightsquigarrow x}$ between $\bar{z}$ and $\vec{s}$. See Remark 4.9(iii) for the example $(\mathfrak{M},\vec{s},z) = (\mathcal{M}_{0,[4]},\vec{01},1/2)$, and [LS97] which, via $r = \phi_{\vec{01}\rightsquigarrow 1/2}$, relates the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the 2-torsion of $\mathcal{M}_{0,[4]}$ to the étale path f_σ of equation (3.2).

(iii) Consider a given tangential morphism $f\colon \mathfrak{N} \to \mathfrak{M}$ as in Proposition 3.7 – i.e. a point $y \in \mathfrak{N}(\mathrm{Spec}\,\overline{K})$ with image $x \in \mathfrak{M}(\mathrm{Spec}\,\overline{K})$ with all the compatible data. The compatible $\mathrm{Gal}(\bar{k}/k)$-representations in $\pi_1^{f^*\mathcal{D}}(\mathfrak{N};t_{f^*\mathcal{D}},y)$ and $\pi_1^{\mathcal{D}}(\mathfrak{M};t_{\mathcal{D}},x)$ then induce compatible $\mathrm{Gal}(\bar{k}/k)$-actions $\rho^I_{\vec{s},\bar{x}}$ and $\rho^I_{\vec{s}',\bar{y}}$ on the respective inertia groups I_x and I_y by the commutativity of diagram (3.3).

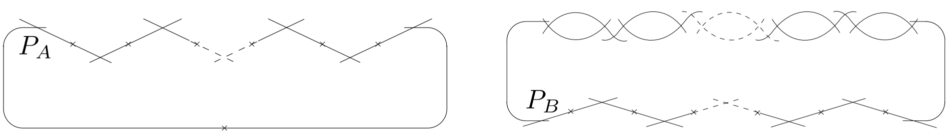
Proposition 3.9 and the compatibility through the Knudsen morphism are applied in various situations to the moduli spaces of curves in Section 4.

§3.2. Tangential Galois action and clutching morphisms

The tools built in the previous section are now applied to describe more explicitly the tangential $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action in the case of the Deligne–Mumford $\mathbb{Q}$-stack of the moduli space of stable curves $\overline{\mathcal{M}}_{g,[m]}$ and their link through Knudsen morphisms. Since the case of the stack inertia groups requires an additional specific result, it is dealt with in Sections 4.1.1 and 4.1.2.

3.2.1. The first step is the choice of a tangential base point in $\mathcal{M}_{g,[m]}$. Let $x \in \overline{\mathcal{M}}_{g,[m]}(\operatorname{Spec}\mathbb{Q})$ be a maximally degenerated $\mathbb{Q}$-curve defined as a graph of $\mathbb{P}^1$ such that marked points and singular points are rational, so that x has only rational automorphisms. Then by Proposition 3.2, $x \in \overline{\mathcal{M}}_{g,[m]}(\mathbb{Q})$ admits a Nisnevich neighbourhood.

Examples of such curves are given by [IN97, Figs $(\mathrm{ii})_n$, $(\mathrm{iii})'_{k,n}$] and are reproduced in Figure 2 for $g \geqslant 1$. Let us denote them by X_A and X_B.



(a) Curve X_A – type $g = 1$, $n \geqslant 1$ (b) Curve X_B – type $g \geqslant 2$, $n \geqslant 1$

Figure 2: Maximally degenerated curves of type (g, n)

Remark 3.11. The present construction in Section 3.1.1 is complementary to the original approach of [IN97] by *tangential base point on* $\mathcal{M}_{g,m}$, where a maximally degenerated curve X is defined by a $\mathbb{P}^1_{0,1,\infty}$-diagram and a canonical choice of a set of coordinates $\mathbf{q}$ of the universal deformation ring $\mathcal{O}_X^{\mathrm{def}}$ of X is fixed. This corresponds exactly to the choice of a system of parameters which led to Definition 3.3.

3.2.2. Let us consider Knudsen's clutching morphism between moduli spaces of stable curves,

$$\overline{\mathcal{M}}_{g-1,m+2} \longrightarrow \overline{\mathcal{M}}_{g,m}$$

that is defined by gluing the last two marked points as in [Knu83, §3]. Let us denote by $\overline{\mathcal{M}}_{g-1,[m]+2}$ the quotient of $\overline{\mathcal{M}}_{g-1,m+2}$ under the action of the permutation group $\mathfrak{S}_m$ on the m marked points $\{1, \ldots, m\}$. The quotient of the clutching morphism by this action defines a morphism

$$\beta\colon \overline{\mathcal{M}}_{g-1,[m]+2} \longrightarrow \overline{\mathcal{M}}_{g,[m]}.$$

Let $\mathcal{S}_{g,m} = \overline{\mathcal{M}}_{g,[m]} \backslash \mathcal{M}_{g,[m]}$, and let $\mathcal{E}$ be the closure of $\mathcal{S}_{g,m} \backslash \mathrm{Im}(\beta)$ in $\overline{\mathcal{M}}_{g,[m]}$. Then $\mathcal{E}$ is a normal crossing divisor in $\overline{\mathcal{M}}_{g,[m]}$ and $\beta^*(\mathcal{E})$ is a normal crossing divisor in $\overline{\mathcal{M}}_{g-1,[m]+2}$ that is equal to $\mathcal{S}_{g-1,m+2} = \overline{\mathcal{M}}_{g-1,[m]+2} \setminus \mathcal{M}_{g-1,[m]+2}$. Similarly to [Nak96, §(3.5)], one defines a partial compactification $\widetilde{\mathcal{M}}_{g,[m]} = \overline{\mathcal{M}}_{g,[m]} \setminus \mathcal{E}$ whose locus of singular curves is exactly the image through β of the locus of smooth curves. Note that β sends a maximally degenerated stable curve of $\mathcal{S}_{g-1,m+2}$ to a maximally degenerated stable curve of higher arithmetic genus in $\mathcal{E}$. In particular, the partial normalisation X'_A (resp. X'_B) of X_A (resp. X_B) at the singular point P_A (resp. P_B), see Figure 2, pointed at the preimages of P_A (resp. P_B), is a curve supported in $\beta^*(\mathcal{E}) \subset \mathcal{S}_{g-1,m+2} \subset \overline{\mathcal{M}}_{g-1,[m]+2}$ that is sent to X_A (resp. X_B) by β.

In the rest of the paper, we consider some tangential base point $\vec{s} = (x, t_{\mathcal{E}})$ on $\widetilde{\mathcal{M}}_{g,[m]}$ (resp. $\vec{s}\,' = (x', t_{\beta^*(\mathcal{E})})$ on $\mathcal{M}_{g-1,[m]+2}$) with $x \in \{X_A, X_B\}$ (resp. $x' \in \{X'_A, X'_B\}$ given by β-compatibility). We denote the associated fundamental group $\pi_1^{\mathcal{E}}(\overline{\mathcal{M}}_{g,[m]}; \vec{s})$ of Theorem 3.4 by $\pi_1(\widetilde{\mathcal{M}}_{g,[m]}; \vec{s})$, and in the same way $\pi_1^{\beta^*(\mathcal{E})}(\overline{\mathcal{M}}_{g-1,[m]+2}; \vec{s}\,')$ is denoted $\pi_1(\mathcal{M}_{g-1,[m]+2}; \vec{s}\,')$. Their geometric variants are denoted accordingly.

3.2.3. Recall that by Artin–Mazur étale homotopy type theory applied to the $\mathbb{Q}$-stack $\mathcal{M}_{g,[m]}$ as in [Oda97], the étale fundamental group associated to a geometric point $\bar{x}\colon \mathrm{Spec}(\overline{\mathbb{Q}}) \to \mathcal{M}_{g,[m]}$ yields an arithmetic–geometric (short) exact sequence:

$$1 \to \pi_1^{\mathrm{et}}(\mathcal{M}_{g,[m]} \otimes \overline{\mathbb{Q}}, \bar{x}) \to \pi_1^{\mathrm{et}}(\mathcal{M}_{g,[m]}, \bar{x}) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1, \tag{3.6}$$

and a rational point $s\colon \mathrm{Spec}\,\mathbb{Q} \to \mathcal{M}_{g,[m]}$ induces a geometric Galois representation ρ_s as in equation (1.1). For $\mathcal{E}$, $\widetilde{\mathcal{M}}_{g,[m]}$, and $\vec{s} = (x, t_{\mathcal{E}})$ a $\mathbb{Q}$-tangential base point as in Section 3.2.2, Proposition 3.5 yields a tangential $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation

$$\rho_{\vec{s}}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}[\pi_1(\widetilde{\mathcal{M}}_{g,[m]} \otimes \overline{\mathbb{Q}}; \vec{s})], \tag{3.7}$$

where $\pi_1(\widetilde{\mathcal{M}}_{g,[m]} \otimes \overline{\mathbb{Q}}; \vec{s})$ denotes $\pi_1^{\mathcal{E}\otimes\overline{\mathbb{Q}}}(\overline{\mathcal{M}}_{g,[m]} \otimes \overline{\mathbb{Q}}; \vec{s})$. As recalled in Remark 3.6, the tangential $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation gives a splitting of equation (3.6) by computing ρ_s explicitly on some geometric generators of $\pi_1^{\mathrm{et}}(\mathcal{M}_{g,[m]} \otimes \overline{\mathbb{Q}}, \bar{x})$.

With the notation of Section 3.2.2, since β is *unramified* by [Knu83, Cor. 3.9], we construct some β-compatible tangential $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations:

Proposition 3.12. *There exists a choice of $\mathbb{Q}$-tangential base points $\vec{s}$ and $\vec{s}\,'$ of type X_A or X_B, respectively on $\widetilde{\mathcal{M}}_{g,[m]}$ and on $\overline{\mathcal{M}}_{g-1,[m]+2}$ which induces a morphism*

$$\pi_1(\mathcal{M}_{g-1,[m]+2} \otimes \overline{\mathbb{Q}}; \vec{s}\,') \longrightarrow \pi_1(\widetilde{\mathcal{M}}_{g,[m]} \otimes \overline{\mathbb{Q}}; \vec{s})$$

and is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant with respect to $\rho_{\vec{s}}$ and $\rho_{\vec{s}\,'}$.

Proof. Let us first consider the case of ordered marked points $\overline{\mathcal{M}}_{g,m}$. As the morphism $\overline{\mathcal{M}}_{g-1,m+2} \to \overline{\mathcal{M}}_{g,m}$ is unramified by [Knu83, Cor. 3.9], Proposition 3.7 ensures the existence of $\mathbb{Q}$-tangential base points based at curves of types X_A and X_B that are compatible with β. Since a tangential representation is defined by the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the parameters $t_{\mathcal{E}}$ and on the $\overline{\mathbb{Q}}$-coefficients of $\vec{s}$, the result follows directly from [Knu83, Cor. 3.9] applied to the arithmetic fundamental groups $\pi_1(\mathcal{M}_{g-1,m+2}; \vec{s}\,')$ and $\pi_1(\widetilde{\mathcal{M}}_{g,m}; \vec{s})$.

For the unordered case, we consider the cartesian diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g-1,m+2} & \longrightarrow & \overline{\mathcal{M}}_{g,m} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \overline{\mathcal{M}}_{g-1,[m]+2} & \longrightarrow & \overline{\mathcal{M}}_{g,[m]}, \end{array}$$

where vertical morphisms are étale surjective since the marked points are supposed distinct. By the descent property, to be unramified is local at the source for the étale topology, thus the bottom morphism is unramified and the result follows from the unordered case and the same arguments. □

We insist on the fact that Knudsen morphisms do not lead to *canonical* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions – see Remark 3.8. The comparison of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action by change of parameters illustrates the non-$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ invariance of *analytic continuation*, and is indeed the *core of the arithmetic geometry of moduli spaces of curves*, as illustrated by the role of Deligne's *droit chemin* p *from* $\vec{01}$ *to* $\vec{10}$ in $\mathcal{M}_{0,4}$ as in [Iha91].

Remark 3.13. As a special case and as another general application of Proposition 3.7, we signal the following:

(i) The above construction of β-compatible $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations is the algebraic generalisation of the topological approach of [Col12a] where a *mapping class group* morphism $\Gamma^2_{0,[m]} \to \Gamma_{1,[m]}$ is defined to deal with the étale-type inertia in genus 1.

(ii) The Knudsen clutching morphisms

$$\beta_{g_1,g_2} \colon \overline{\mathcal{M}}_{g_1,m_1} \times \overline{\mathcal{M}}_{g_2,m_2} \longrightarrow \overline{\mathcal{M}}_{g,m},$$

being closed immersions, the approach above readily applies to the study of various β_{g_1,g_2}-compatible $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations.

§4. Galois action on inertia

This section presents the main result of this paper: the description of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the cyclic stack inertia of $\mathcal{M}_{g,[m]}$ defined by a tangential $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation. First, the approach using irreducible components of *special loci* initiated in [CM15] is recalled, and is shown to provide a favourable context for applying Section 3. The behaviour of the inertia Galois action under specialisation is then established and used to prove the main theorem as a result of the previous sections.

§4.1. Special loci and inertia groups

Let w be a geometric point of $\mathcal{M}_{g,[m]}$ and $G < I_w$ its stack inertia group. We consider the *special loci* $\mathcal{M}_{g,[m]}(G)$ associated to G, i.e. the loci of curves of $\mathcal{M}_{g,[m]}$ that admit a G-action – see [CM15, §2.1]. By the residual finiteness property of the orbifold fundamental group of $\mathcal{M}_{g,[m]}(\mathbb{C})^{an}$, the morphism of Section 3.1.3 turns into an embedding $\omega_w\colon I_w \hookrightarrow \pi_1(\mathcal{M}_{g,[m]} \otimes \overline{\mathbb{Q}}, w)$ – see also Remark 4.4.

Let $\rho_s\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}[\pi_1(\widetilde{\mathcal{M}}_{g,[m]} \otimes \overline{\mathbb{Q}}; \vec{s})]$ be a Galois tangential representation defined by a $\mathbb{Q}$-tangential base point $\vec{s}$ with support in the compactification $\overline{\mathcal{M}}_{g,[m]}$ as in Section 3.2.3. In the case that w is stable under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, this defines a stack inertia $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action $\rho^I_{\vec{s}}$ on I_w (cf. equation (3.4)), that we further describe when $G = \langle\gamma\rangle$ is cyclic.

4.1.1. The study of the action $\rho^I_{\vec{s}}$ is linked to the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of irreducible components of $\mathcal{M}_{g,[m]}(G)\otimes\overline{\mathbb{Q}}$ (cf. [CM15, §2.2]), so it is fundamental to first have a good description of it. One then proves that the irreducible components of $\mathcal{M}_{g,[m]}(\gamma)$ are geometrically irreducible (see [CM15, Cor. 3.12 and Thm. 4.3]): denoting by $\underline{\mathbf{kr}}$ the *algebraic branch data* of Section 2.2.3, it is shown that such an irreducible component is of the form $\mathcal{M}_{g,[m],\underline{\mathbf{kr}}}(\gamma)$, composed of curves with given $\underline{\mathbf{kr}}$ data; see [CM15].

Furthermore, one obtains the following lemma:

Lemma 4.1 ([CM15, Lem. 5.2]). *Let $\mathcal{Z}$ be an irreducible component of $\mathcal{M}_{g,[m]}(\gamma)$, and let $m \gg 0$ so that $\mathcal{M}_{g,m}$ is representable. Then the residue field $K = \kappa(\zeta)$ of a generic point ζ of $\mathcal{M}_{g,m} \times_{\mathcal{M}_{g,[m]}} \mathcal{Z}$ provides a point*

$$z\colon \operatorname{Spec} K \to \mathcal{M}_{g,m} \times_{\mathcal{M}_{g,[m]}} \mathcal{Z} \to \mathcal{Z} \to \mathcal{M}_{g,[m]}(\gamma)$$

of $\mathcal{M}_{g,[m],\underline{\mathbf{kr}}}(\gamma)$ such that K is linearly disjoint from $\overline{\mathbb{Q}}$ over $\mathbb{Q}$.

Let $G < I$ be the generic stack inertia group of an irreducible component of the special loci $\mathcal{M}_{g,[m]}(G)$. Lemma 4.1 gives a K-point z of the component whose geometric inertia $I_{\bar{z}}$ contains G, and it follows from Proposition 3.9 that there exists a stack inertia Galois action

$$\rho^I_{\vec{s},\bar{z}} \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}[I_{\bar{z}}] \tag{4.1}$$

which is defined by conjugacy and is induced by the local $\mathrm{Gal}(\overline{K}/K)$-action $\rho^I_{\bar{z}}$ on $I_{\bar{z}}$.

The definition of $\rho^I_{\vec{s},\bar{z}}$ relies indeed on many choices, such as fixing an algebraic closure of K or choosing a specialisation morphism $\phi_{\bar{z}\rightsquigarrow\vec{s}}$ from $\bar{z}$ to the boundary of $\overline{\mathcal{M}}_{g,[m]}$. This results in the identification of $\rho^I_{\bar{z}}$ to the stack inertia $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action $\rho^I_{\vec{s},\bar{z}}$ of equation (3.4), which is induced by the given tangential $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action $\rho_{\vec{s}}$ – see the discussion above Proposition 3.9.

Remark 4.2. When the irreducible component admits a dense open subset with trivial automorphism group, the field K of Lemma 4.1 can be chosen to be the residue field of the generic point of the component. In the case that $G = \langle\gamma\rangle$ is cyclic, see also Remark 3.10(i).

4.1.2. For a general curve $C \in \mathcal{M}_{g,[m]}(G)$, recall that the associated G-cover $C \to C/G$ factorises as below with the properties

(i) the group $H < G$ is generated by the stabilisers of ramification points of the G-cover $C \to C/G$;

$$\begin{array}{ccc} C & & \\ \downarrow & \searrow & \\ & & C/H \\ \downarrow & \swarrow & \\ C/G; & & \end{array}$$

(ii) the cover $C/H \to C/G$ is étale.

When $H = G$, the action of G on C is said to be *without étale factorisation*. In this case, the stack inertia $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action of equation (4.1) is given by the proposition below, which also plays a key role in the final proof of the general case.

Proposition 4.3. *Let $G = \langle\gamma\rangle$, and let $\eta\colon \mathrm{Spec}\, K \to \mathcal{M}_{g,[m]}(G)$ be a morphism with value into a field K linearly disjoint from $\overline{\mathbb{Q}}$. If the curve η is without étale factorisation, then the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action $\rho^I_{\vec{s}}$ on $G < I_{\bar{\eta}}$ is given by cyclotomy i.e. for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\gamma \in G$ we have $\sigma \cdot \gamma = \gamma^{\chi(\sigma)}$.*

The proposition is actually [CM15, Thm. 5.4] to which we refer for details. The idea of the proof goes as follows: Let $C\colon \mathrm{Spec}\, K \to \mathcal{M}_{g,[m],\underline{\mathbf{kr}}}(G)$ be a morphism as in Lemma 4.1, and suppose that the action of G on the curve C is without étale factorisation. Since the stabilisers of a ramification point are generating subgroups of the stack inertia group, the *branch cycle argument* implies that the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action is given by cyclotomy *on a generator* γ *of the inertia group.*

The following section establishes a similar result for tangential $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action $\rho_{\vec{s}}$ on the fundamental group *for curves with possible étale factorisation* using the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-compatibility of the Knudsen morphism of Section 3.2.3.

§4.2. Inertial limit Galois action and cyclotomy

We describe the tangential $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action $\rho_{\vec{s}}$ on cyclic stack inertia, first for curves without étale factorisation, then in the general case. Note that the results of the first section readily extend to any Deligne–Mumford stack.

4.2.1. Let $\widetilde{\mathcal{M}}_{g,[m]}$ be the partial compactification of Section 3.2.2. We establish the behaviour under specialisation of the stack inertia groups within $\pi_1^{\mathrm{et}}(\widetilde{\mathcal{M}}_{g,[m]})$. More precisely, the goal of this section is to compare different Galois actions on the étale fundamental group based on different points or tangential points.

Let $\vec{s}$ be a tangential base point of $\widetilde{\mathcal{M}}_{g,[m]}$, $\eta \in \widetilde{\mathcal{M}}_{g,[m]}$ be a point above the generic point of $\vec{s}$, and $z \in \mathcal{M}_{g,[m],\underline{\mathbf{kr}}}(G)$ be a specialisation of η. More precisely, let R be a valuation ring with algebraically closed fraction field K and residue field k, endowed with a morphism $T\colon \operatorname{Spec} R \to \mathcal{M}_{g,[m]}(G)$ which sends the generic point of $\operatorname{Spec} R$ onto the image of the generic point of $\vec{s}$ – thus defining two geometric points $\bar{\eta}$ and $\bar{z}$. Also let $\bar{\eta} \rightsquigarrow \vec{s}$ be an étale path from $\bar{\eta}$ to $\vec{s}$ as given by a change of base point in $\widetilde{\mathcal{M}}_{g,[m]}$. Since étale coverings are proper morphisms, the choice of T defines an étale path $\bar{\eta} \rightsquigarrow \bar{z}$ from $\bar{\eta}$ to $\bar{z}$, and one obtains the diagram

$$
\begin{array}{ccccc}
I_{\bar{z}} & & & & I_{\bar{\eta}} \\
\Big\downarrow{\scriptstyle \omega_{\bar{z}}} & & & & \Big\downarrow{\scriptstyle \omega_{\bar{\eta}}} \\
\pi_1(\widetilde{\mathcal{M}}_{g,[m]}, \bar{z}) & & \xrightarrow{\ \phi^{-1}_{\bar{\eta}\rightsquigarrow\bar{z}}\ } & & \pi_1(\widetilde{\mathcal{M}}_{g,[m]}, \bar{\eta}), \\
& \searrow^{\phi_{\bar{z}}} & & \swarrow_{\phi_{\bar{\eta}\rightsquigarrow\vec{s}}} & \\
& & \pi_1(\widetilde{\mathcal{M}}_{g,[m]}; \vec{s}) & &
\end{array}
\tag{4.2}
$$

with $\phi_{\bar{z}} = \phi_{\bar{\eta}\rightsquigarrow\vec{s}} \circ \phi^{-1}_{\bar{\eta}\rightsquigarrow\bar{z}}$.

Remark 4.4. The injectivity of $\omega_{\bar{z}}\colon I_{\bar{z}} \hookrightarrow \pi_1(\widetilde{\mathcal{M}}_{g,[m]}, \bar{z})$ follows from [Noo04, Thm. 6.2], where the Deligne–Mumford stack $\widetilde{\mathcal{M}}_{g,[m]}$ is uniformisable as a global algebraic space quotient by a finite group, as given for example via Looijenga level structures – see [Loo94] and [Bog14, Thm. 3.10] for its generalisation to $\overline{\mathcal{M}}_{g,[m]}$.

The following lemma is an analog of Grothendieck's specialisation theorem for the fundamental group in the case of stack inertia groups.

Lemma 4.5. *Let η and z be two points in $\widetilde{\mathcal{M}}_{g,[m]}$ such that z is a specialisation of η. Let R be a valuation ring and $T \in \widetilde{\mathcal{M}}_{g,[m]}(\operatorname{Spec} R)$ whose generic fibre is a geometric point $\bar{\eta}$ above η and whose special fibre is a geometric point $\bar{z}$ above z. Then the choice of T induces an étale path $\bar{\eta} \rightsquigarrow \bar{z}$ which sends the stack inertia $I_{\bar{\eta}}$ into $I_{\bar{z}}$:*

$$\begin{array}{ccc} \operatorname{Aut}(C_{\bar{\eta}}) = I_{\bar{\eta}} & \hookrightarrow & \pi_1(\widetilde{\mathcal{M}}_{g,[m]}, \bar{\eta}) \\ \big\downarrow \phi & & \big\downarrow \phi_{\bar{\eta} \rightsquigarrow \bar{z}} \\ \operatorname{Aut}(C_{\bar{z}}) = I_{\bar{z}} & \hookrightarrow & \pi_1(\widetilde{\mathcal{M}}_{g,[m]}, \bar{z}). \end{array} \tag{4.3}$$

Proof. Since étale coverings are proper morphisms, the choice of T defines an étale path $\bar{\eta} \rightsquigarrow \bar{z}$ from $\bar{\eta}$ to $\bar{z}$ by using the valuative criterion for properness. This choice is by definition compatible with specialisation.

Consider the curves $C_{\bar{\eta}}$ and $C_{\bar{z}}$ with their respective automorphism groups $\operatorname{Aut}(C_{\bar{\eta}})$ and $\operatorname{Aut}(C_{\bar{z}})$. The stable reduction process induces a morphism between automorphism groups $\phi\colon \operatorname{Aut}(C_{\bar{\eta}}) \to \operatorname{Aut}(C_{\bar{z}})$, where ϕ is injective thanks to the non-ramification of the diagonal of $\overline{\mathcal{M}}_{g,[m]}$. The lemma follows from the commutativity of diagram (4.3). ☐

This result should be read in relation with Theorem 2.6 on the generic G-deformation of a smooth curve to the boundary of $\overline{\mathcal{M}}_{g,[m]}$. Notice that the subgroups $\phi_{\bar{z}}(I_{\bar{z}})$ and $\phi_{\bar{\eta} \rightsquigarrow \bar{z}}(I_{\bar{\eta}})$ can be seen as subgroups of $\pi_1(\widetilde{\mathcal{M}}_{g,[m]}; \vec{s})$.

4.2.2. When the $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action $\rho^I_{\vec{s}}$ on the stack inertia group of the generic point $I_{\bar{\eta}}$ is given by cyclotomy, one obtains the following description of the tangential $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action $\rho_{\vec{s}}$ on $\pi_1(\widetilde{\mathcal{M}}_{g,[m]} \otimes \overline{\mathbb{Q}}; \vec{s})$ on the stack inertia group $I_{\bar{z}}$ of the specialisation@

Lemma 4.6. *Let $\rho_{\vec{s}}$ be given a tangential $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action as before. Let η be the generic point of an irreducible component of the special loci $\mathcal{M}_{g,[m]}(G)$, and z a specialisation of η in $\widetilde{\mathcal{M}}_{g,[m]}$. If $\rho_{\vec{s}}$ induces a $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action $\rho^I_{\vec{s}}$ on $G < I_{\bar{\eta}}$ that is given by cyclotomy, then there exists for $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ an étale path δ_σ of $\pi_1(\widetilde{\mathcal{M}}_{g,[m]} \otimes \overline{\mathbb{Q}}; \vec{s})$ such that for any $\gamma \in G < I_{\bar{z}}$,*

$$\rho_{\vec{s}}(\sigma) \cdot \gamma = \delta_\sigma \cdot \gamma^{\chi(\sigma)} \cdot \delta_\sigma^{-1}.$$

In the following, we say that such a tangential $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on a stack inertia element *is given by χ-conjugacy.*

Proof of Lemma 4.6. Let τ be an element of $I_{\bar{\eta}}$ and write

$$\gamma = \phi_{\bar{\eta}\rightsquigarrow\bar{z}} \circ \tau \circ \phi^{-1}_{\bar{\eta}\rightsquigarrow\bar{z}}.$$

For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the discussion above and the compatibility of $\rho^{I}_{\vec{s},\bar{z}}$ and $\rho_{\vec{s}}$ of Proposition 3.9 give

$$\begin{aligned}\rho_{\vec{s}}(\sigma)\cdot\gamma &= \rho_{\vec{s}}(\sigma)(\phi_{\bar{\eta}\rightsquigarrow\bar{z}})\cdot\rho_{\vec{s}}(\sigma)(\tau)\cdot\rho_{\vec{s}}(\sigma)(\phi^{-1}_{\bar{\eta}\rightsquigarrow\bar{z}})\\ &= \delta_\sigma\cdot\gamma^{\chi(\sigma)}\cdot\delta_\sigma^{-1},\end{aligned}$$

where $\delta_\sigma = \rho_{\vec{s}}(\sigma)(\phi_{\bar{\eta}\rightsquigarrow\bar{z}})\circ\phi^{-1}_{\bar{\eta}\rightsquigarrow\bar{z}}$ is an étale path in $\pi_1(\widetilde{\mathcal{M}}_{g,[m]}\otimes\overline{\mathbb{Q}};\vec{s})$. □

For curves without étale factorisation, the lemma above and Proposition 4.3 give, in particular, the following corollary:

Corollary 4.7. *Let $\vec{s}$ be a tangential base point of $\widetilde{\mathcal{M}}_{g,[m]}$, denote by $\rho_{\vec{s}}$ the tangential* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-representation induced by $\vec{s}$, and let $G = \langle\gamma\rangle$ be a cyclic stack inertia group of $\widetilde{\mathcal{M}}_{g,[m]}$. If G satisfies the non-étale factorisation property, then $\rho_{\vec{s}}$ induces a* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-action on G given by χ-conjugacy.*

4.2.3. We now establish the main result of the article, which follows from all the results collected in the previous sections: the compatibility of some local, stack inertia, and tangential Galois actions (resp. $\rho^{I}_{\bar{z}}$, $\rho^{I}_{\vec{s},\bar{z}}$, and $\rho_{\vec{s}}$ in Section 3.1.3), the specific action by cyclotomy conjugacy of Section 4.2.2, the Galois-invariant tangential morphisms of Section 3.2.3, and the generic degeneracy of G-covers of Section 2.2.2.

Theorem 4.8. *Let $I = \langle\gamma\rangle$ be a cyclic stack inertia group of $\mathcal{M}_{g,[m]}$. Then the tangential* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-actions on $\pi_1(\widetilde{\mathcal{M}}_{g,[m]}\otimes\overline{\mathbb{Q}},\vec{s})$ are given by χ-conjugacy on I:*

$$\rho_{\vec{s}}(\sigma)\cdot\gamma = \delta_\sigma\gamma^{\chi(\sigma)}\delta_\sigma^{-1},$$

where δ_σ is an étale path of $\pi_1(\widetilde{\mathcal{M}}_{g,[m]}\otimes\overline{\mathbb{Q}},\vec{s})$.

Recall that such a tangential $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation can be explicitly given by a curve of type X_A ($g \geqslant 1$) or X_B ($g \geqslant 2$) of Figure 2, that the tangential $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action $\rho_{\vec{s}}$ on I is defined via a stack inertia Galois action $\rho^{I}_{\vec{s},\bar{z}}$ as in equation (4.1), and that one can work with generic points if up to conjugacy.

Proof of Theorem 4.8. Let η be the generic point of an irreducible component $\mathcal{M}_{g,[m],\underline{\mathbf{kr}}}(\gamma)$ of the special loci $\mathcal{M}_{g,[m]}(\gamma)$, so that $I < I_{\bar{\eta}}$. The case where η has no étale factorisation is dealt with by Corollary 4.7. Since the automorphisms of curves of genus 0 are without étale factorisation, it can be assumed that $g \geqslant 1$.

Let $z \in \widetilde{\mathcal{M}}_{g,[m]}(\gamma)$ be a specialisation of η as given by Corollary 2.7 and consider $\beta^{-1}(z)$ – the preimage by the Knudsen morphism β as in Section 3.2.2 – which belongs to an irreducible component $\mathcal{M}_{g-1,[m]+2,\underline{\mathbf{kr}}'}(\gamma) \subset \mathcal{M}_{g-1,[m]+2}(\gamma)$ of generic point η'. One obtains furthermore a specialisation $\xi = \beta(\eta') \in \widetilde{\mathcal{M}}_{g,[m]}(\gamma)$ of η whose normalisation has genus $g-1$ and a $\langle\gamma\rangle$-action without étale factorisation, and such that $I_{\bar{\eta}} < I_{\bar{\xi}}$ by the property of injectivity under the specialisation of Lemma 4.5.

Following Proposition 3.12, let us consider two tangential $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions $\rho_{\vec{s}}$ on $\pi_1(\widetilde{\mathcal{M}}_{g,[m]} \otimes \overline{\mathbb{Q}}; \vec{s})$ and $\rho_{\vec{s}'}$ on $\pi_1(\mathcal{M}_{g-1,[m]+2} \otimes \overline{\mathbb{Q}}; \vec{s}')$ that are compatible with the Knudsen morphism

$$\pi_1(\mathcal{M}_{g-1,[m]+2} \otimes \overline{\mathbb{Q}}, \vec{s}') \xrightarrow{\beta} \pi_1(\widetilde{\mathcal{M}}_{g,[m]} \otimes \overline{\mathbb{Q}}, \vec{s}).$$

Considering the two étale paths $\eta \rightsquigarrow \vec{s}$ and $\eta' \rightsquigarrow \vec{s}'$ that are defined by specialisation of the curves $z \in \widetilde{\mathcal{M}}_{g,[m]}$ and $\beta^{-1}(z) \in \mathcal{M}_{g-1,[m]+2}$, the actions $\rho_{\vec{s}}$ and $\rho_{\vec{s}'}$ then define by Proposition 3.9 and Lemma 4.1 two $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions on the generic stack inertia groups $I_{\bar{\eta}}$ and $I_{\bar{\eta}'}$ that respectively are $\rho^I_{\vec{s}} = \phi^{-1}_{\eta\rightsquigarrow\vec{s}} \circ \rho_{\vec{s}} \circ \phi_{\eta\rightsquigarrow\vec{s}}$ and $\rho^I_{\vec{s}'} = \phi^{-1}_{\eta\rightsquigarrow\vec{s}'} \circ \rho_{\vec{s}'} \circ \phi_{\eta\rightsquigarrow\vec{s}'}$.

Since curves of $\mathcal{M}_{g-1,[m]+2,\underline{\mathbf{kr}}'}(\gamma)$ satisfy the without étale factorisation property, the action $\rho^I_{\vec{s}'}$ on $I \lesssim I_{\bar{\eta}'}$ is given by χ-conjugacy by Corollary 4.7 and via $\bar{\eta}' \rightsquigarrow \vec{s}'$. Denote by x and x' the respective supports of $\vec{s}$ and $\vec{s}'$. From the β-compatibility of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions $\rho^I_{\vec{s},\bar{x}}$ and $\rho^I_{\vec{s}',\bar{x}'}$ – i.e. $\beta \circ \rho^I_{\vec{s}',\bar{x}'} = \rho^I_{\vec{s},\bar{x}}$ (see Remark 3.10(iii)) – and finally from

$$\rho^I_{\vec{s}} = (\phi^{-1}_{\bar{\eta}\rightsquigarrow\vec{s}} \circ \phi_{\bar{\xi}\rightsquigarrow\vec{s}}) \circ \rho^I_{\vec{s},\bar{\xi}} \circ (\phi^{-1}_{\bar{\eta}\rightsquigarrow\vec{s}} \circ \phi_{\bar{\xi}\rightsquigarrow\vec{s}})^{-1}$$

– where the χ-conjugacy action of $\rho^I_{\vec{s},\bar{x}}$ has been similarly transported to $\rho^I_{\vec{s},\bar{\xi}}$ – it follows that the generic action $\rho^I_{\vec{s}}$ is given by χ-conjugacy, first on $I_{\bar{\xi}}$, then on $I < I_{\bar{\eta}} < I_{\bar{\xi}}$.

For a general $\gamma \in I_{\bar{z}}$, one deduces the same result for $\rho_{\vec{s}}$ after conjugacy by a factor $\delta_\sigma = \rho_{\vec{s}}(\sigma)(\phi_{\bar{\eta}\rightsquigarrow\bar{z}}) \circ \phi^{-1}_{\bar{\eta}\rightsquigarrow\bar{z}}$ as in Lemma 4.6, and this concludes the proof. □

Remark 4.9. The following make precise the role of the conjugacy factors with respect to the compactification:

(i) For any point $x \in \mathcal{M}_{g,[m]}$ and $\bar{x}$ a geometric point above x, we have a Galois-equivariant commutative diagram

$$\begin{array}{ccc} I_{\bar{x}} & \hookrightarrow & \pi_1(\mathcal{M}_{g,[m]}, \bar{x}) \\ \| & & \downarrow \\ I_{\bar{x}} & \hookrightarrow & \pi_1(\widetilde{\mathcal{M}}_{g,[m]}, \bar{x}). \end{array}$$

In particular, a Galois action given by χ-conjugacy on the bottom row induces a χ-conjugacy on the top row. This statement is still true for a tangential action by using an étale path $\pi_1(\widetilde{\mathcal{M}}_{g,[m]},\bar{x}) \to \pi_1(\widetilde{\mathcal{M}}_{g,[m]};\vec{s})$.

(ii) The proof above refines the compatibility under the Knudsen morphism β of the Galois actions $\rho^I_{\vec{s},\bar{z}}$ and $\rho^I_{\vec{s}',\bar{z}'}$ in terms of the various conjugacy factors.

(iii) In $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 \setminus \{0,1,\infty\}$, the *droit chemin* p from the tangential base point $\vec{01} = \operatorname{Spec}\mathbb{Q}[\![t]\!]$ to $\vec{10} = \operatorname{Spec}\mathbb{Q}[\![-t]\!]$ – see [Iha91] – admits a factorisation by the path r from $\vec{01}$ to $1/2 \in \mathcal{M}_{0,4}$. The $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on p gives a factor f_σ while r gives a factor g_σ – see [LS97]. Since the point $1/2 \in \mathcal{M}_{0,4}$ represents a point in $\mathcal{M}_{0,[4]}$ with (reduced) cyclic inertia $\mathbb{Z}/2\mathbb{Z}$, the factor g_σ plays the role of the conjugacy factor in the stack inertia $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action.

4.2.4. By analogy with the Deligne–Mumford stratification, the results and methods of this article encourage further studies of the arithmetic of the stack inertia stratification (see [Dou06] for a description), either by describing the Galois action for higher non-cyclic stack inertia strata, or by describing the conjugacy factors δ_σ in the χ-conjugacy action of Theorem 4.8.

For the Deligne–Mumford stratification, Grothendieck–Murre theory implies that the tangential $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on $\pi_1^{\text{et}}(\mathcal{M}_{g,[m]}\otimes\overline{\mathbb{Q}})$ is given by χ-conjugacy on the divisorial inertia groups I_D, $D \in \partial\overline{\mathcal{M}}_{g,[m]}$, while the *stratification by topological type* (g,m) – given by the Knudsen morphisms – reduces the description of these actions to the 4 strata of modular dimension 1 and 2 only. A finer description of the conjugacy factor is obtained by comparing $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions on different topological strata via Knudsen clutching morphisms; see for example [Nak96] in the case of the clutching $\overline{\mathcal{M}}_{g_1,m_1} \times \overline{\mathcal{M}}_{g_2,m_2} \to \overline{\mathcal{M}}_{g_1+g_2,m_1+m_2-2}$.

The stack inertia stratification is given by the decreasing dimensional locus $\mathcal{M}_{g,[m]}(G_i) \subset \mathcal{M}_{g,[m]}(G_{i-1})$, where $G_i > G_{i+1}$ and $G_0 = \{\mathrm{Id}\}$; see [LMB00, Thm. 11.5]. We show that the *cyclic stack inertia stratification is given by the branch data* $\underline{\mathbf{kr}}$ *of Section* 2.2.3, thus the corresponding strata $\mathcal{M}_{g,[m],\underline{\mathbf{kr}}}(\mathbb{Z}/n\mathbb{Z})$; in Section 4.2.3, we compare two stack inertia $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions

$$\rho^I_{\vec{s},\bar{x}}\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{Aut}[I_{\bar{x}}] \quad \text{and} \quad \rho^I_{\vec{s}',\bar{y}}\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{Aut}[I_{\bar{y}}]$$

on the automorphism groups $I_{\bar{x}} \simeq I_{\bar{y}}$ of curves $x \in \mathcal{M}_{g,[m],\underline{\mathbf{kr}}}(\mathbb{Z}/n\mathbb{Z})$ and $y \in \mathcal{M}_{g-1,[m]+2,\underline{\mathbf{kr}}'}(\mathbb{Z}/n\mathbb{Z})$ *of stack strata of different types* – i.e. with distinct $\mathbb{Z}/n\mathbb{Z}$-invariants (genus of the quotients $g'_x \neq g'_y$, and branch data $\underline{\mathbf{kr}} \neq \underline{\mathbf{kr}}'$). This process deserves the name of *inertial limit Galois action*. Developing a combinatorial description of the geometry of the cyclic stack inertia stratification should lead to a finer description of these inertial limit Galois actions, for example by *comparing the conjugacy factors of the prime and general cyclic strata*.

In another direction, and following a long, geometric Galois action tradition, this χ-conjugacy Galois action also motivates the search for *new stack inertia* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-equations in higher genus* – see [NT03] and [Sch06] in genus 0.

Acknowledgements

B. Collas thanks Prof. H. Nakamura for his encouragement at the beginning of the development of this manuscript. The first author has benefited from many fruitful environments during the preparation of this paper, and accordingly thanks Prof. Moshe Jarden and Tel Aviv University, Prof. Hossein Movasati and the IMPA, and the Isaac Newton Institute for Mathematical Sciences Cambridge and the *Grothendieck–Teichmüller groups, deformation and operads* semester's organisers for their hospitality. The authors are also grateful to the referee for his thoughtful reading and valuable comments.

Supported by Prof. Michael S. Weiss' Humboldt Grant Professorship, DFG programme DE 1442/5-1 Bayreuth, and DFG SPP 1786.

References

[BR11] J. Bertin and M. Romagny, Champs de Hurwitz, arXiv:math/0701680v1 (2007).

[Bog14] M. Boggi, Galois coverings of moduli spaces of curves and loci of curves with symmetry, Geom. Dedicata **168** (2014), 113–142. Zbl 1286.14042 MR 3158034

[Col12a] B. Collas, Action of a Grothendieck–Teichmüller group on torsion elements of full Teichmüller modular groups of genus one, Int. J. Number Theory **8** (2012), 763–787. Zbl 1288.14015 MR 2904929

[Col12b] B. Collas, Action of the Grothendieck–Teichmüller group on torsion elements of full Teichmüller modular groups in genus zero, J. Théor. Nombres Bordeaux **24** (2012), 605–622. Zbl 1278.14040 MR 3010631

[CM15] B. Collas and S. Maugeais, Composantes irréductibles de lieux spéciaux d'espaces de modules de courbes, action galoisienne en genre quelconque, Ann. Inst. Fourier (Grenoble) **65** (2015), 245–276. Zbl 1326.11069 MR 3449153

[DM69] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, Inst. Hautes Études Sci. Publ. Math. (1969), 75–109. Zbl 0181.48803 MR 262240

[Dou06] J.-C. Douai, Étude des gerbes apparaissant dans la stratification du champ des modules des courbes de genre g, Publication IRMA **65** (2006), 2–16.

[Dri90] V. G. Drinfel'd, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Algebra i Analiz **2** (1990), 149–181. Zbl 0718.16034 MR 1080203

[Eke95] T. Ekedahl, Boundary behaviour of Hurwitz schemes, in *The moduli space of curves (Texel Island, 1994)*, Progress in Mathematics 129, Birkhäuser Boston, Boston, MA, 1995, 173–198. Zbl 0862.14018 MR 1363057

[Gro97] A. Grothendieck, Esquisse d'un programme, in *Geometric Galois actions, 1*, London Mathematical Society Lecture Note Series 242, Cambridge University Press, Cambridge, 1997, 5–48. Zbl 0901.14001 MR 1483107

[GM71] A. Grothendieck and J. P. Murre, The tame fundamental group of a formal neighbourhood of a divisor with normal crossings on a scheme, in *The tame fundamental group of a formal neighbourhood of an irreducible divisor*, Lecture Notes in Mathematics 208, Springer, Berlin-New York, 1971. Zbl 0216.33001 MR 316453

[Iha91] Y. Ihara, Braids, Galois groups, and some arithmetic functions, in *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, Mathematical Society of Japan, Tokyo, 1991, 99–120. Zbl 0757.20007 MR 1159208

[IN97] Y. Ihara and H. Nakamura, On deformation of maximally degenerate stable marked curves and Oda's problem, J. reine angew. Math. **487** (1997), 125–151. Zbl 0910.14010 MR 1454262

[Knu83] F. F. Knudsen, The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$, Math. Scand. **52** (1983), 161–199. Zbl 0544.14020 MR 702953

[LMB00] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3. Folge) 39, Springer, Berlin, 2000. Zbl 0945.14005 MR 1771927

[Loc12] P. Lochak, Results and conjectures in profinite Teichmüller theory, in *Galois–Teichmüller theory and arithmetic geometry*, Advanced Studies in Pure Mathematics 63, Mathematical Society of Japan, Tokyo, 2012, 263–335. Zbl 1321.11113 MR 3051247

[LS97] P. Lochak and L. Schneps, A cohomological interpretation of the Grothendieck–Teichmüller group, Invent. Math. **127** (1997), 571–600. Zbl 0883.20016 MR 1431139

[LV18] P. Lochak and M. Vaquié, Groupe fondamental des champs algébriques, inertie et action galoisienne, Ann. Fac. Sci. Toulouse Math. (6) **27** (2018), 199–264. Zbl 1423.14175 MR 3771543

[Loo94] E. Looijenga, Smooth Deligne–Mumford compactifications by means of Prym level structures, J. Algebraic Geom. **3** (1994), 283–293. Zbl 0814.14030 MR 1257324

[Nak90] H. Nakamura, Galois rigidity of the étale fundamental groups of punctured projective lines, J. reine angew. Math. **411** (1990), 205–216. Zbl 0702.14024 MR 1072981

[Nak96] H. Nakamura, Coupling of universal monodromy representations of Galois–Teichmüller modular groups, Math. Ann. **304** (1996), 99–119. Zbl 0835.14007 MR 1367885

[Nak97] H. Nakamura, Galois representations in the profinite Teichmüller modular groups, in *Geometric Galois actions, 1*, London Mathematical Society Lecture Note Series 242, Cambridge University Press, Cambridge, 1997, 159–173. Zbl 0911.14014 MR 1483116

[Nak99] H. Nakamura, Limits of Galois representations in fundamental groups along maximal degeneration of marked curves I, Amer. J. Math. **121** (1999), 315–358. Zbl 1006.12001 MR 1680325

[NS00] H. Nakamura and L. Schneps, On a subgroup of the Grothendieck–Teichmüller group acting on the tower of profinite Teichmüller modular groups, Invent. Math. **141** (2000), 503–560. Zbl 1077.14030 MR 1779619

[NT03] H. Nakamura and H. Tsunogai, Harmonic and equianharmonic equations in the Grothendieck–Teichmüller group, Forum Math. **15** (2003), 877–892. Zbl 1054.14038 MR 2010283

[Noo04] B. Noohi, Fundamental groups of algebraic stacks, J. Inst. Math. Jussieu **3** (2004), 69–103. Zbl 1052.14001 MR 2036598

[Oda97] T. Oda, Etale homotopy type of the moduli spaces of algebraic curves, in *Geometric Galois actions, 1*, London Mathematical Society Lecture Note Series 242, Cambridge University Press, Cambridge, 1997, 85–95. Zbl 0902.14019 MR 1483111

[Sch06] L. Schneps, Automorphisms of curves and their role in Grothendieck-Teichmüller theory Math. Nachr. **279** (2006), 656–671. Zbl 1101.14039 MR 2213599

[Tuf93] S. Tufféry, Déformations de courbes avec action de groupe, Forum Math. **5** (1993), 243–259. Zbl 0809.14005 MR 1216034

[Zoo01] V. Zoonekynd, Tangencial base point on algebraic stacks, arXiv:math/0111072 (2001).