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On Cohomology Vanishing with Polynomial Growth on Complex Manifolds with Pseudoconvex Boundary

Dedicated to Tetsuo Ueda on his seventieth birthday

by

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Abstract

The $\bar\partial$ cohomology groups with polynomial growth $H^{r,s}_{\mathrm{p.g.}}$ will be studied. It will be shown that, given a complex manifold M, a locally pseudoconvex bounded domain $\Omega \Subset M$ satisfying certain geometric boundary conditions and a holomorphic vector bundle $E \to M$, $H^{r,s}_{\mathrm{p.g.}}(\Omega, E) = 0$ holds for all $s \geq 1$ if E is Nakano positive and $r = \dim M$. It will also be shown that $H^{r,s}_{\mathrm{p.g.}}(\Omega, E) = 0$ for all r and s with $r+s > \dim M$ if, moreover, $\operatorname{rank} E = 1$. By the comparison theorem due to Deligne, Maltsiniotis (Astérisque **17** (1974), 141–160) and Sasakura (Inst. Math. Sci. **17** (1981), 371–552), it follows in particular that, for any smooth projective variety X, for any ample line bundle $L \to X$ and for any effective divisor D on X such that $[D]|_{|D|} \geq 0$, the algebraic cohomology $H^s_{\mathrm{alg}}(X \setminus |D|, \Omega^r_X(L))$ vanishes if $r + s > \dim X$.

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§0. Introduction

This is a continuation of the series of papers [Oh-2, Oh-3, Oh-4, Oh-5]. The motivation for [Oh-2] was to apply a method of Hörmander in [Ho, Prop. 3.4.5] to an extension problem on a compact complex manifold M with a holomorphic vector bundle $E \to M$ and an effective divisor D. To explain it more clearly, let us recall first that the original intention of [Ho] was to interpret the finiteness theorems of

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Andreotti and Grauert [A-G] in terms of the harmonic decomposition of L^2 differential forms so that one can eventually refine the solution of the Levi problem by Oka [Ok] in a quantitative way. In view of the remarkable development after [Ho], an extension of [Ho, Prop. 3.4.5] was applied in [Oh-1] and [N-R] to establish finiteness theorems on weakly 1-complete manifolds. After a while, it came into the author's mind that it might be worthwhile extending this type of application to the situation which was studied by Ueda [Ud] and recently by Koike [Kk] in connection with dynamical systems and foliations.

It was proved in [Oh-2] that the natural restriction map $H^0(M, \mathcal{O}(K_M \otimes E \otimes [D]^\mu)) \to H^0(|D|, \mathcal{O}_D(K_M \otimes E \otimes [D]^\mu))$ is surjective for sufficiently large μ if $E|_{|D|}$ is positive and $[D]$ is semipositive, where K_M denotes the canonical bundle of M. The proof is based on the isomorphisms between the L^2 cohomology groups $H^{n,s}_{(2)}(M, E, g, he^{-\mu\varphi})$ and $H^{n,s}_{(2)}(M, E, g, he^{-(\mu+1)\varphi})$ $(s \geq 1)$, where $n = \dim M$, g is a complete metric on M, h is a fiber metric of E and φ is a plurisubharmonic exhaustion function on $M \setminus |D|$ which is of logarithmic growth near $|D|$. An unconventional aspect of this extension theorem is that n is arbitrary, although the situation is quite analogous to the Bochner–Hartogs-type extension rather than that of Oka and Cartan. In this situation, it seems worthwhile generalizing the result by replacing the assumption $[D] \geq 0$ by $[D]|_{|D|} \geq 0$. We note that there are interesting cases where $[D] \ngeq 0$ but $[D]|_{|D|} \geq 0$ and that the neighborhoods of such D have been studied in detail when $n = 2$ and D is smooth by Ueda [Ud] and Koike [Kk]. In [Oh-3], it turned out that one can replace the condition $[D] \geq 0$ by $[D]|_{|D|} \geq 0$ by modifying Hörmander's technique in such a way that some nonplurisubharmonic function works as φ to establish $H^{n,s}_{(2)}(M, E, g, he^{-\mu\varphi}) \cong H^{n,s}_{(2)}(M, E, g, he^{-(\mu+1)\varphi})$ $(s \geq 1)$ for $\mu \gg 1$. By this refinement of Hörmander's method, a bundle-convexity theorem has been obtained in [Oh-4] for a class of locally pseudoconvex domains. An approximation theorem obtained in [Oh-3] was extended in [Oh-5] to a more restricted class of domains than in [Oh-4]. The purpose of the present article is to show in the same vein that the celebrated vanishing theorems of Akizuki–Kodaira–Nakano naturally extend to this latter class.

Namely, we consider a locally pseudoconvex bounded domain Ω in a (not necessarily compact but connected) complex manifold M and a holomorphic vector bundle $E \to M$. In what follows we assume that $\partial\Omega$ has finitely many connected components and every connected component of $\partial\Omega$ is either a C^2-smooth real hypersurface or the support of an effective divisor. For simplicity we assume that each hypersurface component of $\partial\Omega$ separates M into two connected components, since one may take a double covering each time a nonseparating component of $\partial\Omega$ appears. In [Oh-4] we considered those Ω such that the divisorial components of

$\partial\Omega$ are supported on an effective divisor whose normal bundles are semipositive. Such Ω will be called *weakly pseudoconvex domains of regular type* and we set $\partial_{\mathrm{hyp}}\Omega := \partial\overline{\Omega}$ and $\partial_{\mathrm{div}}\Omega := \partial\Omega \setminus \partial_{\mathrm{hyp}}\Omega$. We shall restrict ourselves below to a smaller class of domains by imposing a condition on $\partial_{\mathrm{hyp}}\Omega$.

A weakly pseudoconvex domain of regular type Ω will be called a *weakly pseudoconvex domain of very regular type* if $\partial_{\mathrm{hyp}}\Omega$ is of class C^3 and the Levi form of the defining function, say ρ, of $\overline{\Omega}^\circ$ satisfies

$$\partial\bar{\partial}\rho \geq O(\rho^2)$$

along $\partial_{\mathrm{hyp}}\Omega$. Here, $\partial\bar{\partial}\rho \geq O(\rho^2)$ (at a point $x \in M$) is a short expression of the condition that

$$\sum \frac{\partial^2\rho}{\partial z_\alpha \partial \overline{z_\beta}} \xi_\alpha \overline{\xi_\beta} \geq c\rho^2 \|\xi\|^2 \quad (\xi \in \mathbb{C}^n)$$

holds for some $c \in \mathbb{R}$ around x, where c may depend on the choice of the local coordinate $(z_1, \ldots, z_n)$.

A weakly pseudoconvex domain of very regular type will also be called a *UBS-domain* in short, since the conditions on $\partial\Omega$ come from the works of Ueda [Ud] and Boas–Straube [B-S].

We shall prove the following.

Theorem 0.1. *In the above situation, assume that Ω is a UBS domain of dimension n and E is Nakano positive. Then*

$$H^{n,s}_{\mathrm{p.g.}}(\Omega, E) = 0 \quad \textit{for all } s \geq 1$$

(see Section 1 for the definition of $H^{r,s}_{\mathrm{p.g.}}(\Omega, E)$) and

$$H^{r,s}_{\mathrm{p.g.}}(\Omega, E) = 0 \quad \textit{for all } r \textit{ and } s \textit{ with } r + s > n \textit{ if } \operatorname{rank} E = 1.$$

By Deligne–Maltsiniotis–Sasakura's comparison theorem asserting the equivalence of the cohomology of polynomial growth and algebraic cohomology on quasi-projective varieties, one has the following in particular.

Corollary 0.1. *For any n-dimensional smooth projective variety X, for any Nakano positive vector bundle $E \to X$ and for any effective divisor D on X such that $[D]|_{|D|} \geq 0$, the algebraic cohomology group $H^s_{\mathrm{alg}}(X \setminus |D|, \Omega^n_X(E))$ vanishes for $s \geq 1$, where Ω^r_X denotes the sheaf of holomorphic r-forms. If moreover* $\operatorname{rank} E = 1$, *$H^s_{\mathrm{alg}}(X \setminus |D|, \Omega^r_X(E)) = 0$ for $r + s > n$.*

Here, by $[D]|_{|D|} \geq 0$ we mean that the line bundle $[D]$ admits a fiber metric whose curvature form is semipositive when it is restricted to the Zariski tangent spaces of $|D|$ (and no further semipositivity is assumed on the finite neighborhoods of $|D|$).

We note that an analogous vanishing for $H^s(X, \Omega_X^r(\log D) \otimes L)$ was proved by Norimatsu [Nr] by combining Akizuki–Nakano's vanishing theorem and Deligne's filtration of $\Omega_X^r(\log D)$ in [D-1] by assuming that D is a divisor of simple normal crossings. Recently, a vanishing theorem for $H^s(X, \Omega_X^r(\log D))$ was obtained by Liu, Wan and Yang [L-W-Y] by combining a vanishing for H_{alg} with [D-2] while the Norimatsu vanishing for $H^s(X, \Omega_X^r(\log D) \otimes L)$ was extended by Liu, Rao and Wan [L-R-W] by the standard L^2 method. In contrast to these beautiful results, the assumption $[D]|_{|D|} \geq 0$ in Corollary 0.1 cannot be removed because H^1_{alg} are infinite-dimensional if $X \setminus |D| \cong \mathbb{P}^2 \setminus \{x\}$ ($x \in \mathbb{P}^2$) and $\operatorname{rank} E > 0$.

§1. Cohomology with polynomial growth

After recalling the basic notation, $\bar\partial$ cohomology groups with polynomial growth will be defined.

Let M be a connected complex manifold equipped with a Hermitian metric g and let $E \to M$ be a holomorphic Hermitian vector bundle with a C^∞ fiber metric h. For any continuous function $\varphi\colon M \to \mathbb{R}$, we denote by $L^{r,s}_{(2)}(M, E, g, he^{-\varphi})$ the set of square integrable E-valued (r, s)-forms with respect to $(g, he^{-\varphi})$. For simplicity we shall often abbreviate $L^{r,s}_{(2)}(M, E, g, he^{-\varphi})$ as $L^{r,s}_{(2),\varphi}(M, E)$. By $C^{r,s}(M, E)$ we denote the set of C^∞ (r, s)-forms on M with values in E and by

$$\bar\partial\colon C^{r,s}(M, E) \to C^{r,s+1}(M, E)$$

the complex exterior derivative of type $(0, 1)$. We put

$$C_0^{r,s}(M, E) = \left\{u \in C^{r,s}(M, E); \operatorname{supp} u \Subset M\right\}$$

and denote also by $\bar\partial$ the maximal closed extension of $\bar\partial|_{C_0^{r,s}(M,E)}$ as a linear operator from $L^{r,s}_{(2),\varphi}(M, E)$ to $L^{r,s+1}_{(2),\varphi}(M, E)$. Namely, the domain of the operator $\bar\partial\colon L^{r,s}_{(2),\varphi}(M, E) \to L^{r,s+1}_{(2),\varphi}(M, E)$ is defined as

$$\left\{u \in L^{r,s}_{(2),\varphi}(M, E); \bar\partial u \in L^{r,s+1}_{(2),\varphi}(M, E)\right\},$$

where $\bar\partial u$ is defined in the sense of distributions.

Then we put

$$H^{r,s}_{(2)}(M, E, g, he^{-\varphi}) = \frac{\operatorname{Ker} \bar\partial \cap L^{r,s}_{(2)}(M, E, g, he^{-\varphi})}{\bar\partial(L^{r,s-1}_{(2)}(M, E, g, he^{-\varphi})) \cap L^{r,s}_{(2)}(M, E, g, he^{-\varphi})}.$$

Given a bounded domain $\Omega \Subset M$ we put

$$\delta(z) = \delta_\Omega(z) := \mathrm{dist}_g(z, M \setminus \Omega) \quad (z \in \Omega).$$

Here, $\mathrm{dist}_g(A, B)$ denotes the distance between A and B with respect to g. Then $H^{r,s}_{\mathrm{p.g.}}(\Omega, E)$, the E-valued $\bar\partial$ cohomology group of Ω of type (r, s) with polynomial growth, is defined as the inductive limit of $H^{r,s}_{(2)}(\Omega, E, g, h\delta^\mu)$ as $\mu \to \infty$. Clearly $H^{r,s}_{(2)}(\Omega, E, g, h\delta^\mu)$ and $H^{r,s}_{\mathrm{p.g.}}(\Omega, E)$ do not depend on the choices of g and h.

The most basic fact on $H^{r,s}_{\mathrm{p.g.}}(\Omega, E)$ is the following, which is a direct consequence of the combination of [Ho, Thm. 2.2.3] with Oka's lemma asserting the plurisubharmonicity of $\log \frac{1}{\delta_\Omega}$ for any pseudoconvex domain $\Omega \subset \mathbb{C}^n$, with respect to the Euclidean metric. Although [Ho, Thm. 2.2.3] is only stated when E is the trivial bundle, the proof of the general case is similar.

Theorem 1.1. *For any bounded pseudoconvex domain Ω in $\mathbb{C}^n$ and for any holomorphic Hermitian vector bundle E on a neighborhood of $\overline{\Omega}$,*

$$H^{r,s}_{\mathrm{p.g.}}(\Omega, E) = 0 \quad \textit{for all } r \geq 0 \textit{ and } s \geq 1.$$

Based on Theorem 1.1, combining Cauchy's estimate with the canonical equivalence between Dolbeault and Čech cohomology, one has the following.

Theorem 1.2. *For any smooth projective algebraic variety X, for any algebraic vector bundle $E \to X$ and for any analytic set $D \subset X$ of codimension one, $H^{r,s}_{\mathrm{p.g}}(X \setminus D, E)$ is canonically isomorphic to the corresponding algebraic cohomology group $H^s_{\mathrm{alg}}(X \setminus D, \Omega^r_X(E))$ for any r and s.*

We note that Theorem 1.2 is naturally extended to the equivalence between the cohomology groups $H_{\mathrm{p.g.}}$ and H_{alg} with coefficients in coherent algebraic sheaves over quasi-projective algebraic varieties (cf. [Ss]).

§2. Vanishing of $H^{r,s}_{(2)}$

Let us recall a general vanishing theorem for those $H^{r,s}_{(2)}$ which arise in the circumstance of Theorem 0.1.

Let (E, h) be a holomorphic Hermitian vector bundle over a complex manifold M. Let Θ_h denote the curvature form of h. Recall that Θ_h is naturally identified with a Hermitian form along the fibers of $E \otimes T^{1,0}_M$, where $T^{1,0}_M$ denotes the holomorphic tangent bundle of M, and that (E, h) is said to be Nakano positive if $\Theta_h > 0$ as such a Hermitian form (cf. [N-1]). If (E, h) is Nakano positive and $\mathrm{rank}\, E = 1$, this positivity notion is first due to Kodaira [Kd]. For the proof of Theorem 0.1 we shall apply the following generalization of Nakano's vanishing theorem (cf. [N-1]) and Akizuki–Nakano's vanishing theorem (cf. [A-N]).

Theorem 2.1 (Cf. [A-V]; see also [N-2, N-3, Kz]). *Let (E,h) be a Nakano positive vector bundle over a complete Kähler manifold (M,g) of dimension n. If $\Theta_h - \mathrm{Id}_E \otimes g \geq 0$, then $H^{n,s}_{(2)}(M,E,g,h) = 0$ holds for $s \geq 1$. If $\operatorname{rank} E = 1$ and $\Theta_h = g$, one has $H^{r,s}_{(2)}(M,E,g,h) = 0$ for $r+s>n$.*

We note that Theorem 2.1 was applied in [N-2, N-3, Kz] to show the vanishing of ordinary $\bar\partial$ cohomology groups on weakly 1-complete manifolds. For related materials see also [Kb].

§3. Proof of Theorem 0.1

Let Ω be a weakly pseudoconvex domain of very regular type in a complex manifold M and let (E,h) be a Hermitian holomorphic vector bundle over M whose curvature form Θ_h is Nakano positive. Since each component of $\partial\Omega$ is either a C^2 real hypersurface or a divisor, there exist a function $\psi\colon M \to [0,\infty)$ of class C^2 with $\psi^{-1}(0) = \partial\Omega$ and a positive number A such that $-\partial\bar\partial \log\psi + A\Theta_{\det h} > 0$ holds on Ω, where $\partial\bar\partial\rho$ for a real-valued C^2 function ρ is identified with the complex Hessian by an abuse of notation.

For such a function ψ, one may take $|s|^2$ on a neighborhood of $\partial_{\mathrm{div}}\Omega$ for a canonical section s of $[D]$, for any effective divisor D supported on $\partial_{\mathrm{div}}\Omega$, and take ρ^2 on a neighborhood of $\partial_{\mathrm{hyp}}\Omega$ for a C^2 defining function ρ of $\partial_{\mathrm{hyp}}\Omega$.

We note that the UBS condition imposes a condition on ρ somewhat stronger than that $\partial\bar\partial\rho$ is semipositive along $\partial_{\mathrm{hyp}}\Omega$, which was required in [B-S] to study the $\bar\partial$-Neumann problem.

Accordingly one has more than $-\partial\bar\partial\log\psi + A\Theta_{\det h} > 0$ if Ω is UBS, as follows.

First of all, it is easy to see that ψ can be chosen for any UBS domain Ω in such a way that, given any $\varepsilon > 0$, one can find a neighborhood U of $\partial_{\mathrm{div}}\Omega$ such that

$$-\partial\bar\partial\log\psi + \varepsilon\Theta_{\det h} > 0$$

holds on $U \setminus \partial\Omega$. Moreover, we are allowed to modify the metric $-\partial\bar\partial\log\psi + \varepsilon\Theta_{\det h}$ near $\partial_{\mathrm{div}}\Omega$ by adding a term $\partial\bar\partial\frac{1}{\log(-\log\psi)}$ so that it becomes complete near $\partial_{\mathrm{div}}\Omega$. This can be verified by a straightforward computation.

Furthermore, for any $\varepsilon > 0$, one can also find a neighborhood $V \supset \partial_{\mathrm{hyp}}\Omega$ such that $-\partial\bar\partial\log\psi + \varepsilon\Theta_{\det h}$ is a metric on $V \cap \Omega$ which is complete near $\partial_{\mathrm{hyp}}\Omega$. This follows immediately from the following lemma.

Lemma 3.1. *Let Ω be a UBS domain in a Hermitian manifold (M,g) and let $\rho\colon M \to \mathbb{R}$ be a C^3 function satisfying $\overline{\Omega}^\circ = \{z; \rho(z) < 0\}$ and $d\rho|_{\partial_{\mathrm{hyp}}\Omega} \neq 0$. Then,* *for any* $\varepsilon > 0$*, there exists a neighborhood* $U \supset \partial_{\mathrm{hyp}}\Omega$ *such that* $-\rho^{-1}\partial\bar{\partial}\rho + \varepsilon(g + \rho^{-2}\partial\rho\bar{\partial}\rho) > 0$ *holds on* $U \cap \Omega$.

Proof. For simplicity we assume that $\dim M = 2$, since the proof is similar for the general case. Let $x \in \partial_{\mathrm{hyp}}\Omega$ and let (z, w) be a local coordinate around x such that the Taylor expansion of ρ at x is given by

$$\rho = \operatorname{Re} w + \rho_2 + \rho_3 + o(3),$$

where ρ_k are homogeneous polynomials in $(z, w, \bar{z}, \bar{w})$ of degree k. By the assumption that Ω is UBS, $\partial\bar{\partial}\rho_2 \geq 0$.

We put

$$\partial\bar{\partial}\rho_2 = adzd\bar{z} + bdzd\bar{w} + \bar{b}dwd\bar{z} + cdwd\bar{w}.$$

Then $a \geq 0, c \geq 0$ and $ac - |b|^2 \geq 0$. If $a > 0$, it is easy to see that, for any $\varepsilon > 0$ one can find a neighborhood $V \ni x$ such that $-\rho^{-1}\partial\bar{\partial}\rho + \varepsilon(g + \rho^{-2}\partial\rho\bar{\partial}\rho) > 0$ holds on $V \cap \Omega$.

Let us assume that $a = 0$. Then it follows from $\partial\bar{\partial}\rho(0,0) \geq 0$ that $b = 0$ and $c \geq 0$. Therefore, in this case, by letting

$$\partial\bar{\partial}\rho = Adzd\bar{z} + Bdzd\bar{w} + \bar{B}dwd\bar{z} + Cdwd\bar{w} \quad (\bar{A} = A, \bar{C} = C),$$

one sees that $A_z(0,0) = 0$ and $A_w(0,0) = 0$ should hold, since otherwise a contradiction with the assumption $\partial\bar{\partial}\rho \geq O(\rho^2)$ would arise.

Hence, for any $\varepsilon > 0$ one can find a neighborhood $V \ni x$ such that $-\rho^{-1}\partial\bar{\partial}\rho + \varepsilon(g + \rho^{-2}\partial\rho\bar{\partial}\rho) > 0$ holds on $V \cap \Omega$ along the inner normal of $\partial_{\mathrm{hyp}}\Omega$ at x in the coordinate neighborhood. Clearly, the choice of V can be made uniformly in x. Hence, by the compactness of $\partial_{\mathrm{hyp}}\Omega$ we have the desired conclusion. □

Combining Lemma 3.1 with the preceding argument, we obtain the following, which is crucial for the L^2 estimate needed for the proof of Theorem 0.1.

Lemma 3.2. *If* Ω *is a UBS domain, one can choose the above* ψ *in such a way that for any* $\varepsilon > 0$ *there exists a neighborhood* $U \supset \partial\Omega$ *such that*

$$\mathrm{Id}_E \otimes \partial\bar{\partial}\Big(-\log\psi + \frac{\varepsilon}{\log(-\log\psi)}\Big) + \varepsilon\Theta_h > 0$$

holds on $U \cap \Omega$.

Proof of Theorem 0.1. Let ψ be as in Lemma 3.2. Then there exists an increasing sequence $(m_\mu) \in \mathbb{R}^{\mathbb{N}}$ such that

$$\mathrm{Id}_E \otimes \partial\bar{\partial}\Big(-\mu\log\psi + \frac{1}{\log(-\log\psi)}\Big) + \Theta_h > 0$$

holds on $\{x \in \Omega; \psi(x) \leq e^{-m_\mu}\}$. We may assume that ψ is C^∞ on Ω.

Therefore, one can find positive numbers a and C and an increasing sequence of C^∞ convex increasing functions $\lambda_\mu : \mathbb{R} \to \mathbb{R}$ ($\mu \in \mathbb{N}$) such that $\lambda_\mu(t) = 0$ if $t < m_1$, $\lambda'_\mu(t) = \mu$ if $t > m_{\mu+1}$, $\lambda_\mu(t) = \lambda_{\mu+1}(t)$ if $t < m_{\mu+1}$ and

$$\mathrm{Id}_E \otimes \Big(\partial\bar{\partial}\Big(\lambda_\mu(-\log\psi) + \frac{a}{\log(-\log\psi + C)}\Big)\Big) + \Theta_h > 0$$

on Ω.

Hence, for every $\mu \in \mathbb{N}$ one can find positive numbers ε_μ and δ_μ such that

$$g_{\varepsilon_\mu,\delta_\mu} := \varepsilon_\mu \partial\bar{\partial}\Big(\lambda_\mu(-\log\psi) + \frac{a}{\log(-\log\psi + C)}\Big) + \delta_\mu \Theta_{\det h}$$

is a complete Kähler metric on Ω satisfying

$$\mathrm{Id}_E \otimes \Big(\partial\bar{\partial}\Big(\lambda_\mu(-\log\psi) + \frac{a}{\log(-\log\psi + C)}\Big)\Big) + \Theta_h > \mathrm{Id}_E \otimes g_{\varepsilon_\mu,\delta_\mu}.$$

Therefore, by Theorem 2.1 we obtain

$$H^{n,s}_{(2)}(\Omega, E, g_{\varepsilon_\mu,\delta_\mu}, h\psi^\mu) \;\; (\cong H^{n,s}_{(2)}(\Omega, E, g_{\varepsilon_\mu,\delta_\mu}, he^{-\lambda_\mu(-\log\psi)})) \;\; = 0 \quad \text{for } s \geq 1,$$

since $\frac{1}{\log(-\log\psi + C)}$ is bounded.

Now let $\mu \in \mathbb{N}$ and let v be any representative of an element of $H^{n,s}_{(2)}(\Omega, E, \Theta_{\det h, h\delta^\mu})$ ($s \geq 1$). Then it is clear that one can find $\nu \geq \mu$ such that $v \in L^{n,s}_{(2)}(\Omega, E, g_{\varepsilon_\nu,\delta_\nu}, h\psi^\nu)$, so that by the above vanishing of $H^{n,s}_{(2)}(\Omega, E, g_{\varepsilon_\nu,\delta_\nu}, h\psi^\nu)$, $\bar{\partial}u = v$ holds for some $u \in L^{n,s-1}_{(2)}(\Omega, E, g_{\varepsilon_\nu,\delta_\nu}, h\psi^\nu)$. Since

$$L^{r,s}_{(2)}(\Omega, E, g_{\varepsilon_\nu,\delta_\nu}, h\psi^\nu) \subset \bigcup_{\kappa=1}^{\infty} L^{r,s}_{(2)}(\Omega, E, \Theta_{\det h}, h\delta^\kappa),$$

it follows that v represents zero in $H^{n,s}_{\mathrm{p.g.}}(\Omega, E)$.

Similarly one has $H^{r,s}_{\mathrm{p.g.}}(\Omega, E) = 0$ if $\operatorname{rank} E = 1$ and $r + s > n$. $\square$

Remark 3.1. If a complex manifold M is mapped onto a Stein space V by a holomorphic map f and (E, h) is a Nakano positive Hermitian holomorphic vector bundle over M, Theorem 0.1 can be generalized to a vanishing theorem on a locally pseudoconvex domain $\Omega \subset M$ such that $\partial\Omega$ consists of real hypersurfaces and divisors in such a way that the restriction of f to them is proper, where the UBS condition is imposed similarly to the case of bounded domains. As a corollary, one has the corresponding vanishing for the direct images of relatively algebraic sheaves. In the case that Ω is a smooth family over V with respect to f equipped with a divisor D for which $f|_D$ is proper, it may be an interesting question to extend the theorems in [L-R-W] and [L-W-Y] to this situation.

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