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Quantum Dilogarithm Identities Arising from the Product Formula for the Universal R-Matrix of Quantum Affine Algebras

by

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Abstract

In Dimofte, Gukov, and Soibelman (Lett. Math. Phys. **95** (2011), 1–25), four quantum dilogarithm identities containing infinitely many factors are proposed as wall-crossing formulas for the refined BPS invariant. We give an algebraic proof of these identities using the formula for the universal R-matrix of the quantum affine algebra developed by Ito (Hiroshima Math. J. **40** (2010), 133–183), which yields various product presentations of the universal R-matrix by choosing various convex orders on an affine root system. By the uniqueness of the universal R-matrix and appropriate degeneration, we can construct various quantum dilogarithm identities, including the ones proposed in Dimofte, Gukov, and Soibelman (Lett. Math. Phys. **95** (2011), 1–25), which turn out to correspond to convex orders of multiple row type.

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§1. Introduction

Dimofte, Gukov, and Soibelman proposed four remarkable identities with respect to quantum dilogarithm functions as the wall-crossing formulas for the refined BPS invariants, which they proposed in the study of type II string theory [2]. In [2], it is observed that the refined BPS invariants have very similar wall-crossing behavior to that of motivic Donaldson–Thomas invariants introduced by Kontsevich and Soibelman [10], and it is conjectured that the two invariants coincide under appropriate identification of variables.

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Let q, x_1, x_2 be indeterminate, satisfying the relations $qx_1 = x_1q$, $qx_2 = x_2q$, $x_1x_2 = q^2x_2x_1$, and let

$$\mathbb{E}(x) := \prod_{k=0}^{\infty} \frac{1}{1+q^{2k+1}x}, \quad \mathbf{U}_{m,n} := \mathbb{E}(q^{-mn}x_1^m x_2^n). \tag{1.1}$$

Then the identities they found are written as follows [3] (note that the parameter q in this paper corresponds to $q^{1/2}$ in [3]):

$$\begin{aligned} \mathbf{U}_{2,-1}\mathbf{U}_{0,1} &= \mathbf{U}_{0,1}\mathbf{U}_{2,1}\mathbf{U}_{4,1} \times \cdots \\ &\quad \times \mathbb{E}(-qx_1^2)^{-1}\mathbb{E}(-q^{-1}x_1^2)^{-1} \\ &\quad \times \cdots \times \mathbf{U}_{6,-1}\mathbf{U}_{4,-1}\mathbf{U}_{2,-1}, \end{aligned} \tag{1.2}$$

$$\begin{aligned} \mathbf{U}_{1,-1}\mathbf{U}_{1,0}\mathbf{U}_{0,1} &= \mathbf{U}_{0,1}\mathbf{U}_{1,1}\mathbf{U}_{2,1}\mathbf{U}_{3,1} \times \cdots \\ &\quad \times \mathbf{U}_{1,0}^2\mathbb{E}(-qx_1^2)^{-1}\mathbb{E}(-q^{-1}x_1^2)^{-1} \\ &\quad \times \cdots \times \mathbf{U}_{3,-1}\mathbf{U}_{2,-1}\mathbf{U}_{1,-1}, \end{aligned} \tag{1.3}$$

$$\begin{aligned} \mathbf{U}_{1,-1}^2\mathbf{U}_{0,1}^2 &= \mathbf{U}_{0,1}^2\mathbf{U}_{1,1}^2\mathbf{U}_{2,1}^2\mathbf{U}_{3,1}^2 \times \cdots \\ &\quad \times \mathbf{U}_{1,0}^4\mathbb{E}(-qx_1^2)^{-1}\mathbb{E}(-q^{-1}x_1^2)^{-1} \\ &\quad \times \cdots \times \mathbf{U}_{3,-1}^2\mathbf{U}_{2,-1}^2\mathbf{U}_{1,-1}^2, \end{aligned} \tag{1.4}$$

$$\begin{aligned} \mathbf{U}_{1,-2}\mathbf{U}_{0,1}^4 &= \mathbf{U}_{0,1}^4\mathbf{U}_{1,2}\mathbf{U}_{1,1}^4\mathbf{U}_{3,2}\mathbf{U}_{2,1}^4 \times \cdots \\ &\quad \times \mathbf{U}_{1,0}^6\mathbb{E}(-qx_1^2)^{-1}\mathbb{E}(-q^{-1}x_1^2)^{-1} \\ &\quad \times \cdots \times \mathbf{U}_{2,-1}^4\mathbf{U}_{3,-2}\mathbf{U}_{1,-1}^4\mathbf{U}_{1,-2}. \end{aligned} \tag{1.5}$$

The function $\mathbb{E}(x)$ is called the quantum dilogarithm since

$$\mathbb{E}(x) = \exp(\mathrm{Li}_{2,q^2}(-qx)), \quad \mathrm{Li}_{2,q}(x) := \sum_{n=1}^{\infty} \frac{x^n}{n(1-q^n)} \tag{1.6}$$

and $(1-q)\mathrm{Li}_{2,q}(x)$ degenerates to the classical dilogarithm as $q \to 1$.

These identities, however, are derived by physical insight, and mathematically rigorous proofs for them have not been given. In this paper we develop an algebraic construction of these identities, which eventually yields mathematical proofs of them as equalities of skew formal power series.

Kashaev and Nakanishi [9] established a systematic construction of quantum dilogarithm identities from periods of quantum cluster algebras. Their identities, however, involve only finitely many factors, while the four identities (1.2), (1.3), (1.4), (1.5) contain infinite products. Thus, these identities belong to an essentially new class of quantum dilogarithm identities.

On the other hand, K. Ito constructed the product formulas for the (quasi-) universal R-matrix of quantum affine algebras, which correspond to convex orders on an affine root system [6]. In the formulas, the factors corresponding to real roots are q-exponential functions, which are in fact written as $\exp_q(x) = \mathbb{E}((q - q^{-1})x)$. The resemblance between the wall-crossing formulas and product formulas for the universal R-matrix implies the existence of a connection between wall-crossing formulas and quantum groups.

By this observation we develop a systematic construction of quantum dilogarithm identities containing an infinite product, using the product formula for the universal R-matrix. As a result, we show that all four identities that Dimofte et al. found can be derived algebraically by our method. In Section 2 we review the general construction of convex orders on affine root systems, the concrete construction of PBW-type bases for the positive part U_q^+ of quantum affine algebra $U_q(\mathfrak{g})$ using convex order, and the explicit product formula for the quasi-universal R-matrix of $U_q(\mathfrak{g})$.

In Section 4 we show how to construct quantum dilogarithm identities using the quasi-universal R-matrix Θ of $U_q(\mathfrak{g})$. By virtue of the uniqueness of Θ, we can equate all the product presentations of Θ associated with convex orders. Thus, we have infinite product identities whose parts corresponding to real roots are q-exponential functions of root vectors. Next we construct a continuous projection of the completed quantum double algebra $U_q^+ \mathbin{\widehat{\otimes}} U_q^-$, which contains Θ, onto skew formal power series algebra $\mathcal{D}_Q$ associated with an affine Dynkin quiver Q. By this projection, some root vectors vanish and thus their q-exponentials become 1 in the image. If one chooses an appropriate convex order and Dynkin quiver Q, one can make infinitely many root vectors not vanish for the convex order, while only finitely many root vectors do not vanish in the image for the reversed convex order. Eventually one can obtain various quantum dilogarithm identities of the form "finite product = infinite product".

To obtain concrete identities, we have to compute the root vectors explicitly to determine whether they vanish by the projection. In Section 3 we show that every root vector can be written as a "q-commutator monomial", which is a finite application of a q-bracket on the Chevalley generators. We also developed a combinatorial algorithm for the computation of root vectors, which enables us to obtain concrete presentations of root vectors as q-commutator monomials. The computation is done as manipulations of binary trees.

We found appropriate convex orders and Dynkin quivers which produce identical identities to (1.2)–(1.5), which will be explicitly presented in Section 5. It is remarkable that the factor of Θ corresponding to imaginary roots becomes a q-exponential function by the projection, despite the factor itself not being a q-exponential function. We also note that the convex orders corresponding to (1.3), (1.4), and (1.5) are of multiple row type, which was newly found by Ito [4].

§2. Product formula for the universal R-matrix of quantum affine algebras

First we summarize Ito's works [4, 6], which provide explicit product presentations of the (quasi-) universal R-matrix of quantum affine algebras.

§2.1. Quantum algebra $U_q(\mathfrak{g})$

To begin with, we recall the quantum enveloping algebra $U_q(\mathfrak{g})$ corresponding to a symmetrizable Kac–Moody algebra $\mathfrak{g}$ of rank $\ell+1$, where q is an indeterminate (thus we work on the generic case). We use the following notation, as in [8]:

$$\begin{aligned}
e_i, f_i \in \mathfrak{g}&\colon \text{Chevalley generators},\\
\mathfrak{h} \subset \mathfrak{g}&\colon \text{Cartan subalgebra},\\
\check{\alpha}_i \in \mathfrak{h}&\colon \text{simple coroots},\\
\alpha_i \in \mathfrak{h}^*&\colon \text{simple roots},\\
s_i \in \operatorname{End}(\mathfrak{h}^*)&\colon \text{simple reflections } (i = 0, 1, \dots, \ell),\\
\Delta \subset \mathfrak{h}^*&\colon \text{set of all roots},\\
W := \langle s_0, s_1, \dots, s_\ell\rangle&\colon \text{Weyl group},\\
\Delta_+ \subset \Delta&\colon \text{set of all positive roots},\\
\Delta_- \subset \Delta&\colon \text{set of all negative roots},\\
\Delta^{\mathrm{re}} \subset \Delta&\colon \text{set of all real roots},\\
\Delta^{\mathrm{im}} := \Delta \setminus \Delta^{\mathrm{re}}&\colon \text{set of all imaginary roots}.
\end{aligned}$$

We also use the symbol $R_\pm := R \cap \Delta_\pm$ for every $R \subset \Delta$.

Definition 2.1. The quantum enveloping algebra $U_q(\mathfrak{g})$ is the associative $\mathbb{Q}(q)$-algebra defined by following generators and relations:

$$\text{generators}\colon E_i, F_i, K_\lambda \quad (i = 0, 1, \dots, \ell;\ \lambda \in P), \tag{2.1}$$

$$\text{relations}\colon K_\lambda K_\mu = K_{\lambda+\mu},\ K_0 = 1, \tag{2.2}$$

$$K_\lambda E_i K_\lambda^{-1} = q^{(\lambda,\alpha_i)} E_i, \quad K_\lambda F_i K_\lambda^{-1} = q^{-(\lambda,\alpha_i)} F_i, \tag{2.3}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad (i = 0, 1, \dots, \ell;\ \lambda, \mu \in P), \tag{2.4}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0, \tag{2.5}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0 \quad (i \neq j), \tag{2.6}$$

where $(\cdot,\cdot)$ is the invariant bilinear form on $\mathfrak{h}^*$,

$$P \coloneqq \{\lambda \in \mathfrak{h}^* \mid \langle \check{\alpha}_i, \lambda\rangle \in \mathbb{Z}\ (\forall\, i = 0, 1, \dots, \ell)\}$$

is the weight lattice, and let

$$a_{ij} \coloneqq \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} \in \mathbb{Z}, \quad q_i \coloneqq q^{\frac{1}{2}(\alpha_i,\alpha_i)},$$

$$[n]_q \coloneqq \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! \coloneqq [1]_q[2]_q \dots [n_q], \quad \begin{bmatrix} n \\ k \end{bmatrix}_q \coloneqq \frac{[n]_q!}{[k]_q![n-k]_q!} \in \mathbb{Z}[q, q^{-1}].$$

It is well known that $U_q(\mathfrak{g})$ becomes a Hopf algebra with the following coalgebra structure $(U_q(\mathfrak{g}), \Delta, \varepsilon)$ and antipode S:

$$\Delta(E_i) \coloneqq E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) \coloneqq F_i \otimes K_i^{-1} + 1 \otimes F_i,$$
$$\Delta(K_\lambda) \coloneqq K_\lambda \otimes K_\lambda, \quad \varepsilon(E_i) \coloneqq 0, \quad \varepsilon(F_i) \coloneqq 0, \quad \varepsilon(K_\lambda) \coloneqq 1,$$
$$S(E_i) \coloneqq -K_i^{-1}E_i, \quad S(F_i) \coloneqq -F_iK_i, \quad S(K_\lambda) \coloneqq K_\lambda^{-1}.$$

The terms $\Delta\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ and $\varepsilon\colon U_q(\mathfrak{g}) \to \mathbb{Q}(q)$ are uniquely extended as algebra homomorphisms, and $S\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is also extended as an anti-automorphism.

Several subalgebras of $U_q(\mathfrak{g})$ generated by standard generators are defined as usual:

$$U_q^+ \coloneqq \langle E_0, E_1, \dots, E_\ell\rangle, \quad U_q^0 \coloneqq \langle K_\lambda \mid \lambda \in P\rangle, \quad U_q^- \coloneqq \langle F_0, F_1, \dots, F_\ell\rangle.$$

Then we have a triangular decomposition of $U_q(\mathfrak{g})$ [11, Cor. 3.2.5].

$$U_q^- \otimes U_q^0 \otimes U_q^+ \cong U_q(\mathfrak{g}), \quad x \otimes y \otimes z \mapsto xyz. \tag{2.7}$$

Let $U_\mu \coloneqq \{x \in U_q(\mathfrak{g}) \mid K_\lambda x K_\lambda^{-1} = q^{(\lambda,\mu)}x\ (\forall\, \lambda \in P)\}$ be the weight space of weight $\mu \in P$. For convenience, let $V_\mu \coloneqq V \cap U_\mu$ for every subspace $V \subset U_q(\mathfrak{g})$. Then we also have the weight space decomposition

$$U_q(\mathfrak{g}) = \bigoplus_{\mu \in Q} U_\mu \quad \left(Q \coloneqq \textstyle\bigoplus_{i=0}^{\ell} \mathbb{Z}\alpha_i \subset P\text{: root lattice}\right), \tag{2.8}$$

and $U_q(\mathfrak{g})$ becomes a Q-graded algebra. Using this gradation, we introduce a *q-bracket* $[\cdot,\cdot]_q$, which is defined on each weight space as

$$[x, y]_q \coloneqq xy - q^{(\mu,\nu)}yx \quad (\mu, \nu \in Q;\ x \in U_\mu,\ y \in U_\nu). \tag{2.9}$$

§2.2. Convex orders on affine root systems

Next we introduce the definition and classification of convex orders on the set of positive roots [4]. We also prepare some notation on affine root systems.

Definition 2.2 ([6, Def. 3.3]). A total order $\leq$ on a set of positive roots $B \subset \Delta_+$ is called *convex* if it satisfies the following two conditions:

(1) For any pair of positive real roots $\beta, \gamma \in B \cap \Delta_+^{\mathrm{re}}$ satisfying $\beta < \gamma$ and $\beta + \gamma \in B$, the order relation $\beta < \beta + \gamma < \gamma$ holds.
(2) If $\beta \in B$, $\gamma \in \Delta_+ \setminus B$, and $\beta + \gamma \in B$, then $\beta < \beta + \gamma$.

Example 2.3. Set $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$. Then the following order on Δ_+ is convex:

$$\begin{aligned} &\delta - \alpha_1 < 2\delta - \alpha_1 < 3\delta - \alpha_1 < \cdots \\ &\qquad < \delta < 2\delta < 3\delta \\ &\qquad < \cdots < 2\delta + \alpha_1 < \delta + \alpha_1 < \alpha_1. \end{aligned} \tag{2.10}$$

Here, $\delta \coloneqq \alpha_0 + \alpha_1$ is null root.

When $\mathfrak{g}$ is of untwisted affine type, convex orders on Δ_+ have been classified by Ito [4]. To describe convex orders in general, we have to introduce numerous symbols on affine root systems. In the rest of this section, we restrict $\mathfrak{g}$ to be an untwisted affine Lie algebra of type $X_\ell^{(1)}$, where X is one of A, B, C, D, E, F, G and ℓ is a positive integer. We assign indices $0, 1, \ldots, \ell$ for each vertex of the Dynkin diagram corresponding to $\mathfrak{g}$ as in [7], so that the full subdiagram without the vertex 0 is of finite X_ℓ type.

First, let $\mathring{I} \coloneqq \{1, 2, \ldots, \ell\}$ be the set of indices other than 0, and $\mathring{\mathfrak{g}} \subset \mathfrak{g}$ be the Lie subalgebra generated by $\{e_i, f_i, \check{\alpha}_i \mid i \in \mathring{I}\}$. Then $\mathring{\mathfrak{g}}$ is isomorphic to the simple Lie algebra of type X_ℓ due to our assignment of indices, and $\mathring{\mathfrak{h}} \coloneqq \bigoplus_{i \in \mathring{I}} \mathbb{C}\check{\alpha}_i \subset \mathfrak{h}$ is a Cartan subalgebra of $\mathring{\mathfrak{g}}$. Let $\mathring{\Delta} \subset \mathring{\mathfrak{h}}^*$ be the set of all roots of $\mathring{\mathfrak{g}}$, and $\mathring{W} = \langle s_i \mid i \in \mathring{I} \rangle \subset W$ be the finite Weyl group.

Associated to each $J \subset \mathring{I}$, we introduce several symbols below [4]:

$$\begin{gathered} \mathring{\Pi}_J \coloneqq \{\alpha_j \mid j \in J\} \subset \mathfrak{h}^*, \quad \mathring{W}_J \coloneqq \langle s_j \mid j \in J \rangle \subset \mathring{W}, \\ \mathring{W}^J \coloneqq \{w \in \mathring{W} \mid w(\alpha_j) \in \mathring{\Delta}_+ \ (\forall j \in J)\}, \\ \mathring{\Delta}_J \coloneqq \mathring{W}_J(\mathring{\Pi}_J), \quad \mathring{\Delta}^J \coloneqq \mathring{\Delta} \setminus \mathring{\Delta}_J, \quad \mathring{\Delta}^J_\pm \coloneqq \mathring{\Delta}^J \cap \mathring{\Delta}_\pm, \\ \Delta^J(w, \pm) \coloneqq \{m\delta + \varepsilon \mid m \in \mathbb{Z}_{\geq 0},\ \varepsilon \in w\mathring{\Delta}^J_\pm\} \cap \Delta_+ \quad (w \in \mathring{W}), \\ \Delta_J(w, \pm) \coloneqq \{m\delta + \varepsilon \mid m \in \mathbb{Z}_{\geq 0},\ \varepsilon \in w\mathring{\Delta}_{J\pm}\} \cap \Delta_+ \quad (w \in \mathring{W}), \end{gathered}$$

where $\delta \in \Delta_+^{\mathrm{im}}$ is the null root. For every symbol X_J (resp. X^J), we omit the subscript (resp. superscript) J and write $X \coloneqq X_J$ (resp. $X \coloneqq X^J$) when $J = \mathring{I}$ (resp. $J = \emptyset$). Notice that $X^\emptyset = X_{\mathring{I}} = X$ for every symbol introduced above.

Since every proper full subdiagram of the affine Dynkin diagram is a finite direct sum of diagrams of finite type [8], so is the root subsystem $\mathring{\Delta}_J$. Thus we have the partition $J = \coprod_{c \in C} J_c$, where C is the set of connected components of the Dynkin diagram of $\mathring{\Delta}_J$ and $J_c \subset J$ is the set of vertices belonging to the connected component $c \in C$. Each component $\mathring{\Delta}_{J_c}$ is an irreducible root system of finite type, so that there exists a unique highest root $\theta_{J_c} \in \mathring{\Delta}_{J_c+}$.

Moreover, several symbols are defined for each connected component J_c:

$$\Pi_{J_c} \coloneqq \mathring{\Pi}_{J_c} \amalg \{\delta - \theta_{J_c}\}, \quad \Pi_J \coloneqq \coprod_{c \in C} \Pi_{J_c},$$
$$S_J \coloneqq \bigl\{ s_\alpha \mid \alpha \in \Pi_J \bigr\}, \quad W_J \coloneqq \langle S_J \rangle \subset W,$$
$$\Delta_J^{\mathrm{re}} \coloneqq W_J(\Pi_J), \quad \Delta_J \coloneqq \Delta_J^{\mathrm{re}} \amalg \Delta^{\mathrm{im}}.$$

Furthermore, we associate each $y \in W_J$ with a set of positive roots $\Phi_J(y) \coloneqq y\Delta_{J-} \cap \Delta_{J+}$. Also we set

$$\nabla(J, u, y) \coloneqq \Delta^J(u, -) \amalg u\Phi_J(y)$$

for each $u \in \mathring{W}^J$. These infinite sets of positive real roots $\nabla(J, u, y)$ have the biconvex property and play a crucial role in the classification of convex orders [4].

We also need to introduce a decomposition of elements of the Weyl group defined by the next lemma.

Lemma 2.4. *For every $w \in W$, there exists a unique decomposition $w = w^J w_J$, where $w^J \in W^J$, $w_J \in \mathring{W}_J$.*

By definition of W_J, each $w \in W_J$ can be written as a finite product of elements in S_J. An expression $w = t_1 t_2 \dots t_m$ ($t_i \in S_J$) is called *reduced* if the number m is smallest among all the expressions of w as a finite product of elements in S_J, and the smallest number m is called the *length* of w. Let $\ell_J(w)$ denote the length of w. An infinite sequence of elements $u_1, u_2, \dots$ in S_J is called an *infinite reduced word* when $\ell_J(u_1 u_2 \dots u_m) = m$ for all positive integers m. The set of all infinite reduced words is denoted by $\mathscr{W}_J^\infty$, and the kth factor of $\mathbf{s} \in \mathscr{W}_J^\infty$ is denoted by $\mathbf{s}(k) \in S_J$. We also use a function on positive integers $\phi_{\mathbf{s}} \colon \mathbb{Z}_{\geq 1} \to \Delta_J$, defined by $\phi_{\mathbf{s}}(k) \coloneqq \mathbf{s}(1)\mathbf{s}(2) \dots \mathbf{s}(k-1)(\beta_k)$, where $\beta_k \in \Pi_J$ is the positive root corresponding to $\mathbf{s}(k) = s_{\beta_k} \in S_J$. Note that

$$\Phi_J\bigl(\mathbf{s}(1)\mathbf{s}(2) \dots \mathbf{s}(k)\bigr) = \bigl\{\phi_{\mathbf{s}}(1), \phi_{\mathbf{s}}(2), \dots, \phi_{\mathbf{s}}(k)\bigr\} \quad (k \in \mathbb{Z}_{\geq 1}).$$

We associate each $\mathbf{s} \in \mathscr{W}_J^\infty$ with a infinite set of positive roots

$$\Phi_J(\mathbf{s}) \coloneqq \bigl\{\phi_{\mathbf{s}}(k) \mid k \in \mathbb{Z}_{\geq 1}\bigr\}.$$

Now we can state the general description of convex orders. To begin with, we pick an element $w \in \mathring{W}$. Then we have the decomposition

$$\Delta_+ = \Delta(w,-) \amalg \Delta_+^{\mathrm{im}} \amalg \Delta(w,+). \tag{2.11}$$

Note that $\Delta(w,+) = \Delta(ww_\circ,-)$ with the longest element $w_\circ \in \mathring{W}$, since $w_\circ$ reverses the sign of every root in $\mathring{\Delta}$. Thus, the set of positive real roots consists of two sets of the form $\Delta(w,-)$. We will construct convex orders on $\Delta(w,-)$ and connect them to construct the whole order.

Convex orders on $\Delta(w,-)$ are constructed by the following procedure:

(1) Select a positive integer n and a filtration of indices $\mathring{I} = J_0 \supsetneqq J_1 \supsetneqq J_2 \supsetneqq \cdots \supsetneqq J_n = \emptyset$.

(2) Select elements $y_1 \in W_{J_1}, y_2 \in W_{J_2}, \ldots, y_n \in W_{J_n}$ and infinite reduced words $\mathbf{s}_0 \in \mathscr{W}^\infty_{J_0}, \mathbf{s}_1 \in \mathscr{W}^\infty_{J_1}, \ldots, \mathbf{s}_{n-1} \in \mathscr{W}^\infty_{J_{n-1}}$ satisfying the conditions

$$\begin{aligned}\emptyset = \nabla(J_0, w^{J_0}, 1_W) &\subsetneqq \nabla(J_1, w^{J_1}, y_1)\\ &\subsetneqq \cdots \subsetneqq \nabla(J_n, w^{J_n}, y_n) = \Delta(w,-),\end{aligned} \tag{2.12}$$

$$\begin{aligned}\nabla(J_i, w^{J_i}, y_i) &= \nabla(J_{i-1}, w^{J_{i-1}}, y_{i-1})\\ &\times \amalg w^{J_{i-1}} y_{i-1} \Phi_{J_{i-1}}(\mathbf{s}_{i-1}) \quad (i = 1, 2, \ldots, n).\end{aligned} \tag{2.13}$$

(3) Then every root $\alpha \in \Delta(w,-)$ can be uniquely written as

$$\alpha = w^{J_{k-1}} y_{k-1} \phi_{\mathbf{s}_{k-1}}(p) \quad (1 \le k \le n,\ p \in \mathbb{Z}_{\ge 1}). \tag{2.14}$$

Using this expression we define a total order $\le$ on $\Delta(w,-)$ by

$$\begin{aligned}&w^{J_{k-1}} y_{k-1}(\phi_{\mathbf{s}_{k-1}}(p)) \le w^{J_{l-1}} y_{l-1}(\phi_{\mathbf{s}_{l-1}}(q))\\ &\overset{\mathrm{def}}{\Leftrightarrow}\ (k < l) \text{ or } (k = l,\ p \le q) \quad (k, l, p, q \in \mathbb{Z}_{\ge 1};\ k, l \le n).\end{aligned} \tag{2.15}$$

Then $\le$ is well ordered and its ordinal number is $n\omega$, so that this well-ordered $\le$ is called *n-low type.*

Using this procedure, we construct two convex orders $\le_-$, $\le_+$ on $\Delta(w,-)$, $\Delta(w,+) = \Delta(ww_\circ,-)$ respectively. The parameters used in the procedure can be chosen independently between $\le_-$ and $\le_+$. We also set a total order $\le_0$ on Δ_+^{im} arbitrarily. Then we define a total order $\le$ on the whole Δ_+ as follows:

$$\alpha \in \Delta(w,-),\ \beta \in \Delta_+^{\mathrm{im}},\ \gamma \in \Delta(w,+) \Rightarrow \alpha < \beta < \gamma;$$

$$\alpha \le \alpha' \overset{\mathrm{def}}{\Leftrightarrow} \alpha \le_- \alpha' \quad (\alpha, \alpha' \in \Delta(w,-)); \qquad \beta \le \beta' \overset{\mathrm{def}}{\Leftrightarrow} \beta \le_0 \beta' \quad (\beta, \beta' \in \Delta_+^{\mathrm{im}});$$

$$\gamma \le \gamma' \overset{\mathrm{def}}{\Leftrightarrow} \gamma' \le_+ \gamma \quad (\gamma, \gamma' \in \Delta(w,+)).$$

Notice that $\le_+$ needs to be reversed, and therefore the whole $\le$ is not well ordered.

Theorem 2.5 ([4, Thm. 7.9, Cor. 7.10]). *The total order $\leq$ on Δ_+ constructed above is convex, and any convex order on Δ_+ can be constructed by the above procedure.*

§2.3. Convex bases of U_q^+ constructed by convex orders

When $\mathfrak{g}$ is of finite type, it is known that U_q^+ has canonical bases, which can be described concretely by using the braid group action on $U_q(\mathfrak{g})$ and correspond to each reduced expression of the longest element $w_\circ$ of the Weyl group W [11]. In the affine-type case, however, a couple of difficulties arise in constructing a basis of U_q^+ due to the absence of a longest element of W and the existence of imaginary roots. These problems are solved by constructing certain elements corresponding to imaginary roots, using the extended braid group action on $U_q(\mathfrak{g})$, which is proposed by Beck [1]. Then Ito generalized this construction to general convex orders [6]. In this subsection we summarize the construction of PBW-type bases of U_q^+ from convex orders. We first introduce the notion of a convex basis, which is a PBW-type basis with a convexity property.

Definition 2.6. Let U be a $\mathbb{Q}(q)$-algebra, $\Lambda \subset U$ be a subset, and $\leq$ be a total order on Λ. For every subset $\Sigma \subset \Lambda$, the set of increasing monomials consisting of the elements in Σ is denoted by

$$\mathscr{E}_<(\Sigma) := \left\{E_{\lambda_1}E_{\lambda_2}\dots E_{\lambda_m} \mid E_{\lambda_k} \in \Sigma,\ E_{\lambda_1} \leq E_{\lambda_2} \leq \dots \leq E_{\lambda_m}\right\} \subset U.$$

We call a subset $I \subset \Lambda$ an *interval* if $I = \Lambda$, or I coincides with one of $(x, *)$, $[x, *)$, $(*, y)$, $(*, y]$, (x, y), $[x, y)$, $(x, y]$, or $[x, y]$ for some $x, y \in \Lambda$, where $(x, *) := \{\lambda \in \Lambda \mid x < \lambda\}$, $[x, y) := \{\lambda \in \Lambda \mid x \leq \lambda < y\}$, and so on.

The term $\mathscr{E}_<(\Lambda)$ is called a *convex basis* of U if it has following properties:

(1) $\mathscr{E}_<(\Lambda)$ is a basis of U as $\mathbb{Q}(q)$-linear space.

(2) For every interval $I \subset \Lambda$ with respect to a given order $\leq$, let U_I denote the $\mathbb{Q}(q)$-subalgebra of U generated by I. Then $\mathscr{E}_<(I)$ is a basis of U_I as a $\mathbb{Q}(q)$-linear space.

It is known that one can construct convex bases for a quantum algebra $U_q(\mathfrak{g})$ by using the braid group action on $U_q(\mathfrak{g})$, which is given explicitly by the following fundamental result.

Theorem 2.7 ([11, Chaps 37, 39]). *There exists a unique $\mathbb{Q}(q)$-algebra automorphism $T_i \in \operatorname{Aut} U_q(\mathfrak{g})$ $(i = 0, 1, \ldots, \ell)$ satisfying*

$$T_i(E_i) = -F_i K_i, \quad T_i(F_i) = -K_i^{-1} E_i, \quad T_i(K_\lambda) = K_{s_i(\lambda)} \quad (\lambda \in P), \tag{2.16}$$

$$T_i(E_j) = \frac{1}{[-a_{ij}]_{q_i}!} \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} \begin{bmatrix} -a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{-a_{ij}-k} E_j E_i^k, \tag{2.17}$$

$$T_i(F_j) = \frac{1}{[-a_{ij}]_{q_i}!} \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{k} \begin{bmatrix} -a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{k} F_j F_i^{-a_{ij}-k} \quad (j \neq i). \tag{2.18}$$

Moreover, the automorphisms T_i satisfy the braid relation

$$\overbrace{T_i T_j T_i \ldots}^{m(i,j)} = \overbrace{T_j T_i T_j \ldots}^{m(i,j)} \quad (i \neq j,\ m(i,j) \neq \infty), \tag{2.19}$$

where $m(i,j) \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ is the order of $s_i s_j$ in the Weyl group.

Recall that the braid group $\mathcal{B}$ associated with the Weyl group W is defined by generators T_i and relation (2.19). It is well known that $\mathcal{B}$ has the following property with respect to reduced expressions in W.

Proposition 2.8. *Let $w = s_{i_1} s_{i_2} \ldots s_{i_q} = s_{j_1} s_{j_2} \ldots s_{j_q}$ be two reduced expressions of $w \in W$. Then $T_{i_1} T_{i_2} \ldots T_{i_q} = T_{j_1} T_{j_2} \ldots T_{j_q} \in \mathcal{B}$. Therefore a map*

$$\begin{aligned} &f\colon W \to \mathcal{B}; \\ &f(w) \coloneqq T_{i_1} T_{i_2} \ldots T_{i_{q-1}}(E_{i_q}) \in \mathcal{B} \quad (w = s_{i_1} s_{i_2} \ldots s_{i_q}\colon \text{reduced}) \end{aligned} \tag{2.20}$$

is well defined and a section of canonical surjection $\pi\colon \mathcal{B} \to W$; $T_i \mapsto s_i$.

Thus we can define the action of $w \in W$ on $U_q(\mathfrak{g})$ by $T_w \coloneqq T_{f(w)}$, where T_b denotes the action of $b \in \mathcal{B}$ on $U_q(\mathfrak{g})$.

When $\mathfrak{g}$ is of finite type, set $X_q \coloneqq T_{i_1} T_{i_2} \ldots T_{i_{q-1}}(E_{i_q})$ $(q = 1, 2, \ldots, N)$, where $w_\circ = s_{i_1} s_{i_2} \ldots s_{i_N}$ is a reduced expression of the longest element. Then it is known that increasing monomials $X_1^{k_1} X_2^{k_2} \ldots X_N^{k_N}$ constitute a convex basis of U_q^+. The term X_q has weight $\beta_q \coloneqq s_{i_1} s_{i_2} \ldots s_{i_{q-1}}(\alpha_{i_q})$ and is called a *root vector* associated to the root β_q. Root vectors depend on the reduced expression of $w_\circ$.

The reduced expression of $w_\circ = s_{i_1} s_{i_2} \ldots s_{i_N}$ induces a convex order $\beta_1 < \beta_2 < \cdots < \beta_N$. This is because if $\alpha_{i_k} + s_{i_k} s_{i_{k+1}} \ldots s_{i_{l-1}}(\alpha_{i_l}) = s_{i_k} \ldots s_{i_{m-1}}(\alpha_{i_m})$ and suppose that $l < m$, then applying $s_{i_{m-1}} s_{i_{m-2}} \ldots s_{i_k}$ to both sides yields $\alpha_{i_m} \in \Delta_-$, which is absurd. Thus the given order has the convexity property. Conversely, let $w \in W$ and $\beta_1 < \beta_2 < \cdots < \beta_k$ be a convex order on $\Phi(w) \coloneqq w\Delta_- \cap \Delta_+$. Then β_1 must be a simple root α_{i_1}. To see this, we suppose that β_1 is not simple. Then β_1

can be written as the sum of two positive roots, and at least one of them belongs to $\Phi(w)$ due to the biconvexity of $\Phi(w)$. This contradicts the minimality of β_1, and now we conclude $\beta_1 = \alpha_{i_1}$. Since the action of W preserves the addition of roots, $s_{i_1}(\beta_2) < s_{i_1}(\beta_3) < \cdots < s_{i_1}(\beta_N)$ is a convex order on $\Phi(s_{i_1}w)$. By induction on the length of w, we can construct a reduced expression of w from a given convex order. Therefore, each convex order on Δ_+ generates a reduced expression of $w_\circ$. These correspondences between convex orders and reduced expressions of $w_\circ$ are clearly inverses of each other; the correspondences are one-to-one.

Using the correspondences above, we want to construct bases for U_q^+ from convex orders on Δ_+. To extend the construction for the affine case, we have to deal with several problems, such as the definition of root vectors when a given convex order has multiple lows, the existence of imaginary roots, which are unreachable from simple roots by only using the braid group action. These problems have already been solved by Beck [1] and Ito [6].

Before introducing their construction, we need to extend the affine Weyl group properly. We now return to consider the case when $\mathfrak{g}$ is an untwisted affine Lie algebra of type $X_\ell^{(1)}$. The linear map $t_\lambda \in \operatorname{End}\mathfrak{h}^*$, called the translation by $\lambda \in \mathring{\mathfrak{h}}^*$, is defined by

$$t_\lambda(\mu) \coloneqq \mu + (\mu,\delta)\lambda - \Big\{\frac{1}{2}(\lambda,\lambda)(\mu,\delta) + (\mu,\lambda)\Big\}\delta \quad (\mu \in \mathfrak{h}^*), \tag{2.21}$$

where $\delta \in \Delta_+^{\mathrm{im}}$ is the null root. Let $T \coloneqq \{t_{\nu(\check\alpha)} \mid \check\alpha \in \mathring{\check{Q}}\}$ be the group of translations, where $\nu\colon \mathfrak{h} \to \mathfrak{h}^*$ is the canonical isometry and $\mathring{\check{Q}} \subset \mathfrak{h}$ is the coroot lattice. Then it is well known that $W = \mathring{W} \ltimes T$ [8]. Note that in general T does not contain the translation t_{ε_i} by fundamental coweight $\varepsilon_i \in \mathring{\mathfrak{h}}^*$, which is characterized by $(\varepsilon_i, \alpha_j) = \delta_{ij}$ for $i, j = 1, \dots, \ell$. We extend the affine Weyl group W by appending the translations t_{ε_i}. Let $\widehat{W}$ denote the subgroup of $\mathrm{GL}(\mathfrak{h}'^*)$ generated by W and $t_{\varepsilon_i}|_{\mathfrak{h}'^*}$ $(i \in \mathring{I})$, where $\mathfrak{h}'^* \coloneqq \bigoplus_{i=0}^{\ell} \mathbb{C}\alpha_i \subset \mathfrak{h}^*$. In fact, the extended Weyl group $\widehat{W}$ coincides with a semidirect product of W and a subgroup of the Dynkin automorphism group.

Proposition 2.9 ([7], [6, Prop. 2.1]). *Let $\mathring{I}_* \coloneqq \{j \in \mathring{I} \mid (\varepsilon_j, \theta_{\mathring{I}}) = 1\}$ and $\rho_{\mathring{I}j} \coloneqq t_{\varepsilon_j} w_{\circ j} w_\circ$ for each $j \in \mathring{I}_*$, where $w_\circ$, $w_{\circ j}$ are the longest elements of $\mathring{W}$, $\mathring{W}_{\mathring{I}\setminus\{j\}}$ respectively. Then there exists an automorphism ρ of the Dynkin diagram of type $X_\ell^{(1)}$ such that $\rho_{\mathring{I}j}(\alpha_i) = \alpha_{\rho(i)}$ for all $i = 0, 1, \dots, \ell$. The correspondence $j \mapsto \rho$ is one-to-one. Moreover, $\Omega \coloneqq \{\rho_{\mathring{I}j} \mid j \in \mathring{I}_*\} \amalg \{\mathrm{id}_{\mathfrak{h}'^*}\}$ forms a subgroup of $\mathrm{GL}(\mathfrak{h}'^*)$, and*

$$\widehat{W} = \Omega \ltimes W, \tag{2.22}$$

where $\rho_{\mathring{I}j} \in \Omega$ acts on W by $\rho_{\mathring{I}j}.s_i \coloneqq s_{\rho(i)}$.

We define the length of $w \in \widehat{W}$ by

$$\ell(w) := \ell(u) = \ell_{\mathring{I}}(u), \tag{2.23}$$

where we use the decomposition $w = \rho u$ ($\rho \in \Omega$, $u \in W$) given by (2.22). Recall that $W = W_{\mathring{I}}$. We can also consider reduced expressions of $w \in \widehat{W}$. Namely, we call an expression $w = t_1 t_2 \dots t_m \in \widehat{W}$, $t_i \in S \amalg \Omega$ *reduced* if the sequence of integers $\ell(t_1), \ell(t_1 t_2), \dots, \ell(w)$ is increasing. Thus every element of Ω has length 0, and reduced expressions of $w \in \widehat{W}$ may have a different number of factors, but the number of factors which belong to S must coincide with the length of w.

The Dynkin automorphism ρ acts on the subalgebra $U'_q(\mathfrak{g}) := \langle E_i, F_i, K_{\alpha_i}^{\pm 1}\rangle \subset U_q(\mathfrak{g})$ as an algebra automorphism by permuting indices: $E_i \mapsto E_{\rho(i)}$, $F_i \mapsto F_{\rho(i)}$, $K_{\alpha_i} \mapsto K_{\alpha_{\rho(i)}}$. Thus we have an action of the *extended braid group* $\widehat{\mathcal{B}} := \Omega \ltimes \mathcal{B}$ on $U'_q(\mathfrak{g})$ by extending the braid group action of Theorem 2.7. Proposition 2.8 also holds for $\widehat{W}$ and $\widehat{\mathcal{B}}$, and therefore every $w \in \widehat{W}$ has an action T_w on $U'_q(\mathfrak{g})$.

We will define the root vectors associated to real roots by lifting the expression (2.14) to the quantum algebra $U_q(\mathfrak{g})$, in which process simple reflection s_i is replaced by T_i and simple root α_i is replaced by E_i. In this lifting process, we also have to specify appropriate alternatives for $\delta - \theta_{J_c} \in \Pi_{J_c}$ and $s_{\delta-\theta_{J_c}} \in S_{J_c}$, where $J_c \subset \mathring{I}$ is a connected subdiagram. The simple root vector $E_{\delta-\theta_{J_c}}$ is in fact uniquely determined due to the following lemma.

Lemma 2.10 ([6, Lem. 5.1]). *Let $\varepsilon \in \mathring{\Delta}_+$ and suppose that $\mathbf{s} := s_{i_1} s_{i_2} \dots s_{i_m}$ is a reduced expression in W satisfying $\delta - \varepsilon = s_{i_1} s_{i_2} \dots s_{i_{m-1}}(\alpha_{i_m})$ and $\Phi(\mathbf{s}) \subset \Delta(1, -)$. Such an $\mathbf{s}$ exists and*

$$E_{\delta-\varepsilon} := T_{i_1} T_{i_2} \dots T_{i_{m-1}}(E_{i_m}) \in U_q^+ \tag{2.24}$$

is independent of the choice of $\mathbf{s}$.

The appropriate alternative for $s_{\delta-\theta_{J_c}}$ is given in a somewhat technical manner.

Definition 2.11 ([6, Def. 3.4]). First we fix an index $j_c \in J_c$ satisfying $(\varepsilon_{j_c}, \theta_{J_c}) = 1$ for every nonempty connected subdiagram $J_c \subset \mathring{I}$. Then we define a map $\widehat{\cdot}\colon S_J \to \widehat{W}$ by

$$\widehat{s_j} := s_j \quad (j \in J), \quad \widehat{s_{\delta-\theta_{J_c}}} := (t_{\varepsilon_{j_c}})^{J_c} s_{\overline{j_c}} (t_{\varepsilon_{\overline{j_c}}})^{J_c}, \tag{2.25}$$

where $\overline{j_c} \in \mathring{I}$ is the unique index which satisfies $w_\circ(\alpha_{\overline{j_c}}) = -\alpha_{j_c}$. We also define the extended map $\widehat{\cdot}\colon W_J \to \widehat{W}$ simply by $\widehat{w} := \widehat{t_1}\widehat{t_2} \dots \widehat{t_m}$ when $w = t_1 t_2 \dots t_m$ is a reduced expression in W_J.

Now we can describe the construction of root vectors for the affine case.

Definition 2.12 ([6, Thm. 8.4]). Suppose that $\leq$ is a convex order on Δ_+. Let $\alpha = w^{J_{k-1}} y_{k-1} \phi_{\mathbf{s}_{k-1}}(p)$ be expression (2.14) of a positive real root α determined by $\leq$. Then the root vector $E_{\leq,\alpha} \in U_\alpha$ associated to α is defined by

$$E_{\leq,\alpha} \coloneqq \begin{cases} T_{w^{J_{k-1}}} T_{\widehat{y_{k-1}}} T_{\widehat{\mathbf{s}_{k-1}(1)}} T_{\widehat{\mathbf{s}_{k-1}(2)}} \\ \quad \dots T_{\widehat{\mathbf{s}_{k-1}(p-1)}}(E_{\mathbf{s}_{k-1}(p)}) & (\alpha \in \Delta(w,-)), \\ \Psi T_{w^{J_{k-1}}} T_{\widehat{y_{k-1}}} T_{\widehat{\mathbf{s}_{k-1}(1)}} T_{\widehat{\mathbf{s}_{k-1}(2)}} \\ \quad \dots T_{\widehat{\mathbf{s}_{k-1}(p-1)}}(E_{\mathbf{s}_{k-1}(p)}) & (\alpha \in \Delta(w,+)), \end{cases} \tag{2.26}$$

where $E_{s_i} \coloneqq E_i$, and $\Psi \colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the anti-automorphism of the $\mathbb{Q}(q)$-algebra defined by $\Psi(E_i) \coloneqq E_i$, $\Psi(F_i) \coloneqq F_i$, $\Psi(K_\lambda) \coloneqq K_\lambda^{-1}$.

The root vectors for imaginary roots are constructed using the action of the extended braid group, which contains a coweight lattice [1]. Since each imaginary root has multiplicity ℓ in the affine Lie algebra $\mathfrak{g}$ of type $X_\ell^{(1)}$, we will construct as many root vectors as the multiplicity. The construction is rather technical and we proceed step by step.

First we introduce weight vectors $\mathcal{E}_{n\delta-\alpha_i}$ $(i \in \mathring{I})$, which are independent of the convex order:

$$\mathcal{E}_{n\delta-\alpha_i} \coloneqq T_{\varepsilon_i}^n T_i^{-1}(E_i) \quad (n \in \mathbb{Z}_{\geq 1},\ i \in \mathring{I}), \tag{2.27}$$

where $T_{\varepsilon_i} \coloneqq T_{t_{\varepsilon_i}} \in \operatorname{Aut} U_q'(\mathfrak{g})$ was defined via the extended braid group action and lifting a reduced expression of $t_{\varepsilon_i} \in \widehat{W}$ to $\widehat{\mathcal{B}}$ by (2.20). Then we set

$$\varphi_{i,n} \coloneqq [\mathcal{E}_{n\delta-\alpha_i}, E_i]_q = \mathcal{E}_{n\delta-\alpha_i} E_i - q_i^{-2} E_i \mathcal{E}_{n\delta-\alpha_i} \quad (n \in \mathbb{Z}_{\geq 1},\ i \in \mathring{I}). \tag{2.28}$$

Despite these $\varphi_{i,n}$ having weight $n\delta \in \Delta_+^{\mathrm{im}}$, the $\varphi_{i,n}$ are not yet suitable for imaginary root vectors. The genuine imaginary root vectors are constructed by modifying $\varphi_{i,n}$ through the following technical procedure. For every $i \in \mathring{I}$, let

$$\varphi_i(z) \coloneqq (q_i - q_i^{-1}) \sum_{n=1}^{\infty} \varphi_{i,n} z^n \in U_q^+[\![z]\!] \tag{2.29}$$

be the generating function of $\varphi_{i,n}$. The term $U_q^+[\![z]\!]$ has a topological algebra structure by declaring that z is central and $U_q^+[\![z]\!]$ has z-adic topology. Then *imaginary root vectors* $I_{i,n} \in U_q^+$ are defined as the coefficients of the function

$$I_i(z) \coloneqq \log(1 + \varphi_i(z)) = (q_i - q_i^{-1}) \sum_{n=1}^{\infty} I_{i,n} z^n, \tag{2.30}$$

where the logarithm is defined by

$$\log(1+x) \coloneqq \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} x^m.$$

It is shown that these root vectors constitute convex bases for the positive part of the quantum affine algebra U_q^+.

Theorem 2.13 ([6, Thm. 8.6]). *Let $\leq$ be a convex order on positive roots Δ_+ of an untwisted affine root system, and let $w \in \mathring{W}$ be the parameter determined by decomposition (2.11) of Δ_+ in accordance with the given convex order $\leq$. Let*

$$\Lambda \coloneqq \left\{E_{\leq,\alpha} \mid \alpha \in \Delta_+^{\mathrm{re}}\right\} \amalg \left\{T_w(I_{i,m}) \mid m \in \mathbb{Z}_{\geq 1};\ i = 1, 2, \ldots, \ell\right\} \tag{2.31}$$

denote the set of root vectors constructed above, and we set the order on Λ by using given order $\leq$ and

$$T_w(I_{i,m}) \leq T_w(I_{j,m'}) \iff (m \leq m') \text{ or } (m = m', i \leq j).$$

Then increasing monomials $\mathscr{E}_<(\Lambda)$ constitute a convex basis of U_q^+.

Once a convex basis of U_q^+ is constructed, we also obtain the one for U_q^- through Chevalley involution $\Omega\colon U_q^+ \to U_q^-;\ E_i \mapsto F_i,\ q \mapsto q^{-1}$, which is an anti-automorphism of a $\mathbb{Q}$-algebra.

§2.4. Product formula for the quasi-universal R-matrix

The convex bases for a quantum affine algebra enable explicit construction of the quasi-universal R-matrix. By applying Drinfeld's quantum double construction, Ito obtained the product formula for the quasi-universal R-matrix [6]. Since the quasi-universal R-matrix does not lie in the algebraic tensor product $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, we have to give an appropriate topology on $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ and complete it.

First we set the gradation of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ by

$$\begin{aligned}(U_q \otimes U_q)_h \coloneqq &\bigoplus_{\substack{\mu,\nu \in Q_+ \\ \mathrm{ht}(\mu+\nu)=h}} (U_q^- U_q^0 \otimes U_q^- U_q^0) \cdot U_\mu^+ \otimes U_\nu^+ \\ &\subset U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \quad (h \in \mathbb{Z}_{\geq 0}),\end{aligned} \tag{2.32}$$

that is, we only count the weight of positive part with respect to the triangular decomposition (2.7). Then we set a topology which is generated by subsets of the form

$$x + \bigoplus_{h=k}^{\infty} (U_q \otimes U_q)_h \quad (x \in U_q(\mathfrak{g}),\ k \in \mathbb{Z}_{\geq 0}). \tag{2.33}$$

In short, we give $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ linear topology. Let

$$\widehat{U_q} \mathbin{\widehat{\otimes}} \widehat{U_q} := \operatorname*{proj\,lim}_{k \geq 0} \left(U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \Big/ \bigoplus_{h=k}^{\infty} (U_q \otimes U_q)_h \right) \tag{2.34}$$

be the completion of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, and $U_q^+ \mathbin{\widehat{\otimes}} U_q^- \subset \widehat{U_q} \mathbin{\widehat{\otimes}} \widehat{U_q}$ denote the closure of $U_q^+ \otimes U_q^-$. The algebra structure of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ extends uniquely onto $\widehat{U_q} \mathbin{\widehat{\otimes}} \widehat{U_q}$.

Definition 2.14 ([11, Thm. 4.1.2]). Let $\Upsilon \in \operatorname{Aut} U_q(\mathfrak{g})$ be the $\mathbb{Q}$-algebra automorphism determined by

$$\Upsilon(E_i) := E_i, \quad \Upsilon(F_i) := F_i, \quad \Upsilon(K_\lambda) := K_\lambda^{-1}, \quad \Upsilon(q) := q^{-1},$$

and set $\bar{\Delta} := (\Upsilon \otimes \Upsilon) \circ \Delta \circ \Upsilon$. The *quasi-universal R-matrix* of $U_q(\mathfrak{g})$ is the *unique* element $\Theta \in \widehat{U_q} \mathbin{\widehat{\otimes}} \widehat{U_q}$ satisfying

(1) $\Theta \cdot \bar{\Delta}^{\mathrm{op}}(u) = \Delta^{\mathrm{op}}(u) \cdot \Theta \quad (\forall\, u \in U_q(\mathfrak{g}))$,

(2) $\Theta_0 = 1 \otimes 1$,

where $\Theta_0 \in (U_q \otimes U_q)_0$ is the image of Θ by the canonical projection

$$\widehat{U_q} \mathbin{\widehat{\otimes}} \widehat{U_q} \twoheadrightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \Big/ \bigoplus_{h=1}^{\infty} (U_q \otimes U_q)_h \cong (U_q \otimes U_q)_0,$$

and $f^{\mathrm{op}}(u) := \sum y_i \otimes x_i$ if $f(u) = \sum x_i \otimes y_i$.

The uniqueness of Θ will be the core of the proofs of the identities. Finally, we introduce the product formula for the quasi-universal R-matrix.

Theorem 2.15 ([5, 6]). *Let $\leq$ be a convex order on Δ_+ of an affine root system, and $E_{\leq,\alpha}$, $I_{i,n}$ denote the root vectors constructed above. For every $i, j \in \mathring{I}$ and positive integer n, let*

$$b_{i,j;n} := \operatorname{sgn}(a_{ij})^n \frac{[a_{ij} n]_{q_i}}{n(q_j^{-1} - q_j)}, \quad \operatorname{sgn}(x) := \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0. \end{cases} \tag{2.35}$$

Let $(c_{i,j;n})_{i,j=1}^{\ell} \in \operatorname{Mat}(\mathbb{Q}(q), \ell)$ denote the inverse matrix of $(b_{i,j;n})_{i,j=1}^{\ell}$.

We also set

$$F_{\leq,\alpha} := \Omega(E_{\leq,\alpha}) \in U_{-\alpha} \quad (\alpha \in \Delta_+^{\mathrm{re}}), \quad J_{i,n} := \Omega(I_{i,n}) \in U_{-n\delta},$$

$$\exp_q(x) := \sum_{n=0}^{\infty} \frac{q^{-\frac{1}{2}n(n-1)}}{[n]_q!} x^n, \quad q_\alpha := q^{\frac{1}{2}(\alpha,\alpha)} \quad (\alpha \in \Delta),$$

$$(2.36)\qquad S_n := \sum_{i,j\in \mathring{I}} c_{j,i;n} I_{i,n} \otimes J_{j,n} \quad \in U_q^+ \otimes U_q^-,$$

$$\Theta_{\le,\alpha} := \begin{cases} \exp_{q_\alpha}\{(q_\alpha^{-1} - q_\alpha) E_{\le,\alpha} \otimes F_{\le,\alpha}\}, & \alpha \in \Delta_+^{\mathrm{re}}, \\ \exp\{T_w \otimes T_w(S_n)\}, & \alpha = n\delta \quad (n = 1, 2, \dots). \end{cases}$$

Then the quasi-universal R-matrix Θ *has the product presentation*

$$(2.37)\qquad \Theta = \prod_{\alpha\in\Delta_+}^{>} \Theta_{\le,\alpha} \quad \in U_q^+ \mathbin{\widehat{\otimes}} U_q^-,$$

where $\prod_{\alpha\in\Delta_+}^{>} X_\alpha$ *means that if* $\alpha < \beta$*, the order of multiplication is* $X_\beta X_\alpha$*. In short, the order of multiplication is the reverse of the given convex order.*

§3. Explicit presentation of root vectors using the q-bracket

We will construct quantum dilogarithm identities by using various presentations (2.37) of the quasi-universal R-matrix Θ, taking advantage of the uniqueness of Θ. However, to obtain specific identities, we have to calculate root vectors explicitly, which is described by the braid group action (Theorem 2.7). In this section we show that in the general quantum algebra $U_q(\mathfrak{g})$ of the symmetrizable Kac–Moody algebra $\mathfrak{g}$, the element $T_w(E_i) \in U_q^+$ $(w \in W)$ can be written as a "q-commutator monomial", that is, a finite application of the q-bracket on the generators E_i. We also construct a concrete algorithm for getting an explicit presentation of $T_w(E_i)$ as a q-commutator monomial, which enables us to perform direct computation.

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra of rank n.

Definition 3.1. For every subset $A, B \subset U_q(\mathfrak{g})$, let

$$[A, B]_q := \{[x, y]_q \mid x \in A,\ y \in B\} \subset U_q(\mathfrak{g}).$$

We define subsets $P_k \subset U_q(\mathfrak{g})$ inductively by

$$P_0 := \{E_1, E_2, \dots, E_n\}, \quad P_{k+1} := \bigcup_{i,j=0}^{k} [P_i, P_j]_q \quad (k \in \mathbb{Z}_{\ge 0}).$$

We call elements of the form $cM \in U_q(\mathfrak{g})$ for some $c \in \mathbb{Q}(q)$, $M \in \bigcup_{k=0}^{\infty} P_k$ a *q-commutator monomial.*

Our claim is that $T_w(E_i) \in U_q^+$ is q-commutator monomial for all $i = 1, \dots, n$ and $w \in W$. To prove it, several formulas have to be prepared. First we recall

$$(3.1)\qquad T_i(E_j) = \frac{1}{[-a_{ij}]_{q_i}!} \overbrace{[E_i, [E_i, \dots, [E_i,}^{-a_{ij}} E_j]_q]_q \dots]_q \quad (i \ne j),$$

by definition of the braid group action, and

$$T_i([x,y]_q) = [T_i(x), T_i(y)]_q \quad (x, y \in U_q(\mathfrak{g}),\ i = 1, 2, \dots, n), \tag{3.2}$$

since the Weyl group action preserves the invariant bilinear form. The basic process of calculation for $T_w(E_j)$ ($w \in W$, $1 \leq j \leq n$) is as follows: choose a reduced expression $w = s_{i_1} s_{i_2} \dots s_{i_m}$, and expand every T_{i_k} of $T_w = T_{i_1} \dots T_{i_m}$ from the tail using (3.1) and (3.2). However, there is a problem in that $T_k(E_k) = -F_k K_k$ may appear in the process of expansion. To resolve it, we use the following formula.

Lemma 3.2. *For every $1 \leq i \neq j \leq n$ and positive integer m,*

$$\left[\left(\overrightarrow{\mathrm{ad}}\, E_i\right)^m (E_j), T_i(E_i)\right]_q = [m]_{q_i} [1 - a_{ij} - m]_{q_i} \left(\overrightarrow{\mathrm{ad}}\, E_i\right)^{m-1} (E_j), \tag{3.3}$$

where $\overrightarrow{\mathrm{ad}}\, x(y) \coloneqq [x,y]_q$.

Proof. By the defining relation of $U_q(\mathfrak{g})$,

$$E_i K_i = q_i^{-2} K_i E_i, \quad E_j K_i = q_i^{-a_{ij}} K_i E_j,$$
$$E_i F_i = F_i E_i + \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad E_j F_i = F_i E_j.$$

Thus, the commutation relations of $F \coloneqq T_i(E_i) = -F_i K_i$ and E_i, E_j are

$$E_i F = q_i^{-2} F E_i - \frac{K_i^2 - 1}{q_i - q_i^{-1}}, \quad E_j F = q_i^{-a_{ij}} F E_j.$$

Since the weight of $F = -F_i K_i$ is $-\alpha_i$, we have

$$[E_j, F]_q = E_j F - q_i^{-a_{ij}} F E_j = q_i^{-a_{ij}} F E_j - q_i^{-a_{ij}} F E_j = 0. \tag{3.4}$$

Now we begin the proof by induction on m. Suppose that (3.3) holds for some positive integer m. Let

$$C_m \coloneqq [m]_{q_i} [1 - a_{ij} - m]_{q_i}, \quad X_m \coloneqq \left(\overrightarrow{\mathrm{ad}}\, E_i\right)^m (E_j).$$

Since the weight of X_m is $m\alpha_i + \alpha_j$, we have $X_m K_i = q_i^{-2m - a_{ij}} K_i X_m$. Then by the induction hypothesis,

$$X_m F = q_i^{-2m - a_{ij}} F X_m + C_m X_{m-1}.$$

Using these commutation relations, we obtain the equation for the case $m+1$:

$$\begin{aligned} [X_{m+1}, F]_q &= [[E_i, X_m]_q, F]_q \\ &= [E_i X_m - q_i^{2m + a_{ij}} X_m E_i, F]_q \end{aligned}$$

$$
\begin{aligned}
&= E_iX_mF - q_i^{-2(m+1)-a_{ij}}FE_iX_m \\
&\quad - q_i^{2m+a_{ij}}\{X_mE_iF - q_i^{-2(m+1)-a_{ij}}FX_mE_i\} \\
&= E_i\{q_i^{-2m-a_{ij}}FX_m + C_mX_{m-1}\} - q_i^{-2(m+1)-a_{ij}}FE_iX_m \\
&\quad - q_i^{2m+a_{ij}}\left\{X_m\left(q_i^{-2}FE_i - \frac{K_i^2-1}{q_i-q_i^{-1}}\right) - q_i^{-2(m+1)-a_{ij}}FX_mE_i\right\} \\
&= q_i^{-2m-a_{ij}}\left(q_i^{-2}FE_i - \frac{K_i^2-1}{q_i-q_i^{-1}}\right)X_m + C_mE_iX_{m-1} \\
&\quad - q_i^{2(m+1)+a_{ij}}FE_iX_m - q_i^{2m+a_{ij}}q_i^{-2}(q_i^{-2m-a_{ij}}FX_m + C_mX_{m-1})E_i \\
&\quad + q_i^{2m+a_{ij}}X_m\frac{K_i^2-1}{q_i-q_i^{-1}} + q_i^{-2}FX_mE_i \\
&= -q_i^{-2m-a_{ij}}\frac{K_i^2-1}{q_i-q_i^{-1}}X_m + C_mE_iX_{m-1} \\
&\quad - q_i^{2(m-1)+a_{ij}}C_mX_{m-1}E_i + q_i^{2m+a_{ij}}\frac{q_i^{2(-2m-a_{ij})}K_i^2-1}{q_i-q_i^{-1}}X_m \\
&= \frac{q_i^{-2m-a_{ij}} - q_i^{2m+a_{ij}}}{q_i-q_i^{-1}}X_m + C_m[E_i, X_{m-1}]_q \\
&= ([-a_{ij}-2m]_{q_i} + C_m)X_m.
\end{aligned}
$$

Thus we obtain the recursion formula

$$C_{m+1} = C_m + [-a_{ij}-2m]_{q_i} \quad (m \geq 1).$$

It is easy to verify that $C_m := [m]_{q_i}[1-a_{ij}-m]_{q_i}$ satisfies this recurrence relation. Therefore, (3.3) holds for $m+1$.

For the case $m = 1$, the above calculation works if one uses (3.4) in place of the induction hypothesis and lets $C_0 := 0$. □

Proposition 3.3. *Suppose that the root subsystem spanned by α_i, α_j ($i \neq j$) is of finite type and $w = s_is_js_is_j\dots$ is a reduced expression. Then $T_w(E_k)$ ($k = i$ if $\ell(w)$ is even, $k = j$ otherwise) is a q-commutator monomial consisting of E_i and E_j.*

Proof. Since the length of the reduced expression of the form $s_is_js_is_j\dots$ is at most 5 when we have the finite-type case, our task is just to compute $T_w(E_k)$ directly for all cases. Using formula (3.3), the computation is easily accomplished. For example, when $a_{ij} = a_{ji} = -1$ we have

$$
\begin{aligned}
T_iT_j(E_i) &\overset{(3.1)}{=} T_i([E_j, E_i]_q) \overset{(3.2)}{=} [T_i(E_j), T_i(E_i)]_q \overset{(3.1)}{=} [[E_i, E_j]_q, T_i(E_j)]_q \\
&\underset{\text{Lemma 3.2}}{=} [1]_{q_i}[1-(-1)-1]_{q_i}E_j = E_j.
\end{aligned}
$$

Thus we have a reduction formula

$$T_iT_j(E_i) = E_j \quad \text{if } (a_{ij}, a_{ji}) = (-1,-1). \tag{3.5}$$ □

For the infinite-type case, we use the following formulas, which can be verified by direct computation using (3.3):

Lemma 3.4. *For indices i, j $(i \neq j)$ and nonnegative integers p, k, let*

$$\mathbf{s}_{i,j;p} := \overbrace{\dots s_j s_i s_j s_i s_j}^{p}, \tag{3.6}$$

$$V^{(k)}_{i,j;p} := T_{\mathbf{s}_{i,j;p}}\big((\overleftarrow{\operatorname{ad}} E_j)^k(E_i)\big) \tag{3.7}$$

$$= \overbrace{\dots T_iT_jT_iT_j}^{p}([[\dots[E_i, \overbrace{E_j]_q, E_j]_q, \dots, E_j]_q}^{k}). \tag{3.8}$$

These $V^{(k)}_{i,j;p}$ satisfy the following recurrence relations:

$$V^{(1)}_{i,j;p+1} = \frac{1}{[-a_{ji}-1]_{q_j}!}\big(\overrightarrow{\operatorname{ad}} E_{\mathbf{s}_{i,j;p+1}}\big)^{-a_{ji}-2}(V^{(1)}_{j,i;p}) \quad (a_{ji} \leq -2), \tag{3.9}$$

$$V^{(1)}_{i,j;p+2} = \frac{1}{[-a_{ji}-1]_{q_j}!}\big(\overrightarrow{\operatorname{ad}} V^{(1)}_{i,j;p}\big)^{-a_{ji}-3}(V^{(2)}_{i,j;p}) \quad (a_{ij} = -1,\ a_{ji} \leq -3), \tag{3.10}$$

$$V^{(2)}_{i,j;p+2} = \frac{[2]_{q_j}}{[-a_{ji}-2]_{q_j}!}\big(\overrightarrow{\operatorname{ad}} V^{(1)}_{i,j;p}\big)^{-a_{ji}-4}(V^{(2)}_{i,j;p}) \quad (a_{ij} = -1,\ a_{ji} \leq -4). \tag{3.11}$$

Theorem 3.5. *For every $w \in W$ and index j satisfying $w(\alpha_j) \in \Delta_+$, $T_w(E_j) \in U_q(\mathfrak{g})$ is a q-commutator monomial.*

Proof. The proof is by induction on $\ell(w)$. The case $\ell(w) = 1$ is immediate by (3.1). Suppose that there exists an integer $m \geq 2$ such that $T_w(E_j)$ is a q-commutator monomial if $w(\alpha_j) \in \Delta_+$ and $\ell(w) < m$. Let $w \in W$ satisfy $w(\alpha_j) \in \Delta_+$ and $\ell(w) = m$. Take a reduced expression of w and let s_i be its suffix. Then $i \neq j$ due to the assumption. Let $w^{\{i,j\}} \in W$ be the shortest element satisfying $w = w^{\{i,j\}} \dots s_j s_i s_j s_i$. Then $w^{\{i,j\}}(\alpha_i), w^{\{i,j\}}(\alpha_j) \in \Delta_+$ due to the minimality of $w^{\{i,j\}}$. By the induction hypothesis, $T_{w^{\{i,j\}}}(E_i)$ and $T_{w^{\{i,j\}}}(E_j)$ are q-commutator monomials. Thus, using (3.2), the proof completes if $\dots T_jT_i(E_j)$ turns out to be a q-commutator monomial consisting of only E_i and E_j.

Let $\mathscr{F}_{ij}$ $(i \neq j)$ denote the set of q-commutator monomials consisting of only E_i and E_j. We are going to prove that if $\mathbf{s}_{i,j;p}$ $(p \in \mathbb{Z}_{\geq 1})$ is a reduced expression, then $E_{\mathbf{s}_{i,j;p}} := \dots T_jT_i(E_j) \in \mathscr{F}_{ij}$ by induction on p. The cases when $p = 1, 2$ are immediate by (3.1). When α_i and α_j span a finite root system, $\mathbf{s}_{i,j;p}$ is reduced only for finitely many $p \in \mathbb{Z}_{\geq 1}$. Thus, when $(a_{ij}, a_{ji}) = (-1,-1), (-1,-2)$,

$(-1,-3)$, $(-2,-1)$, $(-3,-1)$, we can verify $E_{\mathbf{s}_{i,j;p}} \in \mathscr{F}_{ij}$ by direct computation since there exist only finitely many cases. The computation is easily accomplished using formula (3.3). For example, when $(a_{ij}, a_{ji}) = (-1,-1)$ and $p = 3$, we have

$$\begin{aligned} E_{\mathbf{s}_{i,j;3}} &:= T_j T_i(E_j) \overset{(3.1)}{=} T_j([E_i, E_j]_q) \overset{(3.2)}{=} [T_j(E_i), T_j(E_j)]_q \\ &\overset{(3.1)}{=} [[E_j, E_i]_q, T_j(E_j)]_q \overset{\text{Lemma 3.2}}{=} [1]_{q_j}[1-(-1)-1]_{q_j} E_i = E_i. \end{aligned}$$

When α_i and α_j span an infinite root system, then $a_{ij}a_{ji} \geq 4$. Now we suppose that $p \geq 2$ and $E_{\mathbf{s}_{i,j;r}} \in \mathscr{F}_{ij}$ for all $r \leq p$. First, $E_{\mathbf{s}_{i,j;p+1}}$ can be written as

$$E_{\mathbf{s}_{i,j;p+1}} = \frac{1}{[-a_{ij}]_{q_i}!} (\overrightarrow{\mathrm{ad}}\, E_{\mathbf{s}_{j,i;p}})^{-a_{ij}-1} (V^{(1)}_{i,j;p-1}). \tag{3.12}$$

Since $E_{\mathbf{s}_{j,i;p}} \in \mathscr{F}_{ij}$, which is the induction hypothesis, we are reduced to verifying $V^{(1)}_{i,j;p-1} \in \mathscr{F}_{ij}$. When $a_{ij} \leq -2$, $a_{ji} \leq -2$, the fact $V^{(1)}_{i,j;1} \in \mathscr{F}_{ij}$ derived from (3.3) and inductive use of formula (3.9) show $V^{(1)}_{i,j;p-1} \in \mathscr{F}_{ij}$. When $a_{ij} = -1$, $a_{ji} \leq -4$, we can directly verify $V^{(1)}_{i,j;0}$, $V^{(1)}_{i,j;1}$, $V^{(2)}_{i,j;0}$, $V^{(2)}_{i,j;1} \in \mathscr{F}_{ij}$ using (3.3), and the recurrence formulas (3.10), (3.11) show $V^{(1)}_{i,j;p-1}, V^{(2)}_{i,j;p-1} \in \mathscr{F}_{ij}$ for all $p \geq 1$.

When $a_{ij} \leq -4$, $a_{ji} = -1$, we need to continue the calculation of (3.12) slightly. Using formula (3.3), we have

$$T_j([E_i, [E_i, E_j]_q]_q) = [[E_j, E_i]_q, E_i]_q.$$

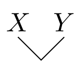
Thus $E_{\mathbf{s}_{i,j;p+1}}$ can be written as

$$E_{\mathbf{s}_{i,j;p+1}} = \frac{1}{[-a_{ij}]_{q_i}!} (\overrightarrow{\mathrm{ad}}\, E_{\mathbf{s}_{j,i;p}})^{-a_{ij}-2} (V^{(2)}_{j,i;p-2}).$$

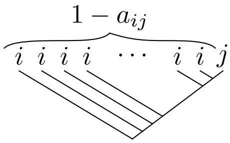
Since $V^{(2)}_{j,i;p-2} \in \mathscr{F}_{ij}$ due to the discussion of the case $a_{ij} = -1$, $a_{ji} \leq -4$, we conclude $E_{\mathbf{s}_{i,j;p+1}} \in \mathscr{F}_{ij}$. □

By the proof of Theorem 3.5, we can easily construct an algorithm for describing $T_w(E_j)$ as a concrete q-commutator monomial once formulas for elements of the form $\ldots T_i T_j T_i(E_j)$ are prepared. In particular, for the simply laced case, we have a simple graphical algorithm for the calculation of $T_w(E_j)$, which we describe below.

First we introduce graphical notation for the q-bracket, which is convenient for writing down q-commutator monomials:

(3.13)  $:= [X, Y]_q \quad (X, Y \in U_q(\mathfrak{g}))$.

We also abbreviate E_i to i in the schematic notation. For example, the q-Serre relation (2.5) can be written as the following binary tree:



$$= \left(\overrightarrow{\mathrm{ad}}\, E_i\right)^{1-a_{ij}}(E_j) = 0 \quad (i \neq j).$$

Using this notation, we can describe every q-commutator monomial as a binary tree, each node of which represents a q-bracket and each leaf of which denotes a Chevalley generator E_i. Now we can describe the algorithm for the simply laced case.

Proposition 3.6. *Let $\mathfrak{g}$ be a simply laced Kac–Moody algebra, $w \in W$, and j be an index satisfying $w(\alpha_j) \in \Delta_+$. Let $w = s_{i_1}s_{i_2}\dots s_{i_m}$ be a reduced expression. Then the binary tree constructed by the following procedure represents a q-commutator monomial equal to $T_w(E_j)$:*

(1) *In this procedure, we manipulate a binary tree, each leaf of which holds a pair of a reduced expression $s_{j_1}s_{j_2}\dots s_{j_k}$ and an index p such that $s_{j_1}s_{j_2}\dots s_{j_k}(\alpha_p) \in \Delta_+$.*

(2) *At the beginning we have a binary tree consisting of only the root, whose reduced expression is $s_{i_1}s_{i_2}\dots s_{i_m}$ and whose index is j. The procedure terminates immediately when $m = 0$.*

(3) *For each leaf of the binary tree, the following manipulations are applied recursively. Let $s_{j_1}s_{j_2}\dots s_{j_k}$ and p be the reduced expression and the index of the leaf we are working on respectively:*

   (a) *We are done for the leaf if $k = 0$.*

   (b) *If $k \geq 1$, then $j_k \neq p$. If $a_{j_k p} = 0$, then delete the factor s_{j_k} in the reduced expression since $T_{j_k}(E_p) = E_p$. Repeat this deletion until $a_{j_k p} = -1$.*

   (c) *If $s_{j_1}s_{j_2}\dots s_{j_{k-1}}(\alpha_p) \in \Delta_-$, then there exists a number l such that*

$$s_{j_1}s_{j_2}\dots s_{j_{k-1}} = s_{j_1}s_{j_2}\dots s_{j_{l-1}}s_{j_{l+1}}\dots s_{j_{k-1}}s_p,$$

   due to the exchange condition [8]. By $a_{j_k p} = a_{p j_k} = -1$ and (3.5), we have

$$\begin{aligned} T_{j_1}T_{j_2}\dots T_{j_{k-1}}T_{j_k}(E_p) &= T_{j_1}T_{j_2}\dots T_{j_{l-1}}T_{j_{l+1}}\dots T_{j_{k-1}}T_pT_{j_k}(E_p) \\ &= T_{j_1}T_{j_2}\dots T_{j_{l-1}}T_{j_{l+1}}\dots T_{j_{k-1}}(E_{j_k}). \end{aligned}$$

According to this calculation, replace the reduced expression with $s_{j_1}s_{j_2}\ldots s_{j_{l-1}}s_{j_{l+1}}\ldots s_{j_{k-1}}$ *and replace the index with* j_k*. Repeat this replacement until* $s_{j_1}s_{j_2}\ldots s_{j_{k-1}}(\alpha_p)\in\Delta_+$.

(d) *Finally, when* $s_{j_1}s_{j_2}\ldots s_{j_{k-1}}(\alpha_p)\in\Delta_+$*, then* $a_{j_kp}=a_{pj_k}=-1$ *by the manipulations so far. Thus* $T_{j_k}(E_p)=[E_{j_k},E_p]_q$ *by (3.1), and* $s_{j_1}s_{j_2}\ldots s_{j_{k-1}}s_{j_k}$, $s_{j_1}s_{j_2}\ldots s_{j_{k-1}}s_p$ *are reduced. Therefore, create a new branch at the current leaf and generate two leaves as in Figure 1, where* $\mathbf{s}':=s_{j_1}s_{j_2}\ldots s_{j_{k-1}}$ *and* $\mathbf{s}[p]$ *denotes the reduced expression* $\mathbf{s}$ *and index* p*. The two new leaves have indexes* j_k, p *respectively, and both reduced expressions are* $\mathbf{s}'$.

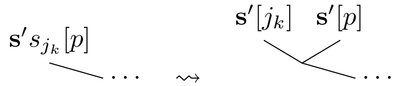


Figure 1. Branching rule

(4) *Repeat the above procedure until all reduced expressions in the leaves have length 0. This algorithm terminates within a finite number of steps because each manipulation shortens the length of the reduced expression of the target leaf.*

§4. Construction of quantum dilogarithm identities

In this section we show how to construct quantum dilogarithm identities using the product formula (2.37) of the quasi-universal R-matrix Θ.

First we introduce certain projections of the algebra $U_q^+\,\widehat{\otimes}\,U_q^-$ onto skew formal power series algebras determined by Dynkin quivers. Through the projections, most of the elements of the form $\Theta_{\leq,\alpha}\in U_q^+\,\widehat{\otimes}\,U_q^-$ $(\alpha\in\Delta_+^{\mathrm{re}})$ become the unit of image, while several factors survive and retain their form as q-exponential functions, which can also be seen as quantum dilogarithm functions. Moreover, under appropriate setting of parameters, the product of factors in Θ associated with imaginary roots can be written using quantum dilogarithm functions in the image of the projection. Thus the image of Θ will be written as a certain product of quantum dilogarithm functions. Choosing various convex orders, one can obtain various product presentations of the image of Θ, which have finitely or infinitely many factors depending on the selected order. Eventually, we can construct quantum dilogarithm identities of the form “finite product = infinite product”, which will exactly coincide with the identities proposed in [3] after a suitable change of variables.

§4.1. Projections of $U_q^+ \mathbin{\widehat{\otimes}} U_q^-$ onto skew formal power series algebras

Let $\mathfrak{g}$ be symmetrizable Kac–Moody algebra of rank n and $A = (a_{ij})_{i,j=1}^n$ be its Cartan matrix. Let $d_1, d_2, \dots, d_n$ be coprime positive integers such that $c_{ij} := d_i a_{ij} = d_j a_{ji}$ $(1 \le i, j \le n)$. Then $C := (c_{ij})_{i,j=1}^n$ is a symmetrized matrix of A. We normalize the invariant bilinear form $(\cdot,\cdot)$ so that $(\alpha_i, \alpha_j) = c_{ij}$ for $1 \le i, j \le n$. Choose $\sigma_{ij} \in \{\pm 1\}$ for each pair of indices $i < j$ such that $a_{ij} \neq 0$, and set

$$b_{ij} := \begin{cases} \sigma_{ij} c_{ij}, & i < j, \\ 0, & i = j, \\ -\sigma_{ij} c_{ij}, & i > j. \end{cases} \tag{4.1}$$

Then the matrix $B = (b_{ij})_{i,j=1}^n$ is a skew-symmetric matrix, and this data can be interpreted as the Dynkin quiver which has an arrow from i to j if $\sigma_{ij} = +1$. Let $\{\cdot,\cdot\}_B \colon Q \times Q \to \mathbb{Z}$ be the skew-symmetric form satisfying $\{\alpha_i, \alpha_j\}_B = b_{ij}$ for all indices i, j.

Let $\mathcal{P}_B$ be a $\mathbb{Q}(q)$-algebra defined by the generators and relations below:

$$\text{generators: } e_1, e_2, \dots, e_n,$$
$$\text{relations: } e_i e_j = q^{b_{ij}} e_j e_i \quad (i, j = 1, 2, \dots, n).$$

The term $\mathcal{P}_B$ has a natural Q-graded algebra structure if each e_i is supposed to have weight α_i, hence the q-bracket makes sense on $\mathcal{P}_B$. Each weight space of $\mathcal{P}_B$ is a one-dimensional subspace spanned by a monomial of the form $e_1^{k_1} e_2^{k_2} \dots e_n^{k_n}$. The degree of each monomial

$$\deg e_1^{k_1} e_2^{k_2} \dots e_n^{k_n} := k_1 + k_2 + \dots + k_n \tag{4.2}$$

coincides with the height of its weight. Let $\mathcal{P}_{B,m}$ be the subspace of $\mathcal{P}_B$ spanned by monomials of degree m. Also note that

$$[e_i, e_j]_q = e_i e_j - q^{(\alpha_i,\alpha_j)} e_j e_i = (q^{b_{ij}} - q^{c_{ij}}) e_j e_i = 0 \quad \text{if } \sigma_{ij} = +1. \tag{4.3}$$

Due to the following well-known fact, $\mathcal{P}_B$ turns out to be a quotient of $U_q^+ \subset U_q(\mathfrak{g})$.

Proposition 4.1 ([11, Thm. 33.1.3]). *The positive part of the quantum enveloping algebra U_q^+ is isomorphic to the $\mathbb{Q}(q)$-algebra whose generators are $E_1, E_2, \dots, E_n$ and whose relation is given by the quantum Serre relation (2.5).*

Proposition 4.2. *There exists a unique Q-graded algebra surjection*

$$\pi_B \colon U_q^+ \to \mathcal{P}_B \tag{4.4}$$

such that $\pi_B(E_i) = e_i$ for all $i = 1, \dots, n$.

Let $\mathcal{P}_B^+ := \mathcal{P}_B$ and $\mathcal{P}_B^-$ be copies of $\mathcal{P}_B$ but with generators e_i replaced by f_i. Recall that U_q^- is isomorphic to U_q^+ as an algebra [11, Cor. 3.2.6]. Let $\pi_B^+ := \pi_B\colon U_q^+ \to \mathcal{P}_B^+$, $\pi_B^-\colon U_q^- \to \mathcal{P}_B^-$ ($\pi_B^-(F_i) := f_i$) be Q-graded algebra surjections given by Proposition 4.2. Then we have an algebra surjection

$$\pi_B^+ \otimes \pi_B^-\colon U_q^+ \otimes U_q^- \twoheadrightarrow \mathcal{P}_B^+ \otimes \mathcal{P}_B^-. \tag{4.5}$$

We want to construct a completion of this surjection to define the image of $\Theta \in U_q^+ \mathbin{\widehat{\otimes}} U_q^-$. To define the completion, we give a topology on $\mathcal{D}_B := \mathcal{P}_B^+ \otimes \mathcal{P}_B^-$ so that the surjection $\pi_B^+ \otimes \pi_B^-$ becomes continuous.

For every nonnegative integer m, set

$$\mathcal{D}_m := \bigoplus_{k=m}^{\infty} \mathcal{P}_{B,m}^+ \otimes \mathcal{P}_B^- \quad \subset \mathcal{D}_B, \tag{4.6}$$

and define the completion of $\mathcal{D}_B$ by

$$\widehat{\mathcal{D}_B} := \operatorname{proj}\lim_{m\geq 0} \mathcal{D}_B/\mathcal{D}_m. \tag{4.7}$$

Recalling definition (2.34) of $\widehat{U_q} \mathbin{\widehat{\otimes}} \widehat{U_q}$, the composition of the surjection $\pi_B^+ \otimes \pi_B^-\colon U_q^+ \otimes U_q^- \twoheadrightarrow \mathcal{D}_B$ and inclusion $\iota\colon \mathcal{D}_B \hookrightarrow \widehat{\mathcal{D}_B}$ is continuous with respect to the relative topology on $U_q^+ \otimes U_q^- \subset \widehat{U_q} \mathbin{\widehat{\otimes}} \widehat{U_q}$. Hence, this map induces a unique continuous map

$$\pi_B^+ \mathbin{\widehat{\otimes}} \pi_B^-\colon U_q^+ \mathbin{\widehat{\otimes}} U_q^- \to \widehat{\mathcal{D}_B} \tag{4.8}$$

due to the completeness of $\widehat{\mathcal{D}_B}$.

§4.2. Skew formal power series algebras

Let $y_i := (q_i - q_i^{-1})e_i \otimes (q_i^{-1} - q_i)f_i \in \widehat{\mathcal{D}_B}$, $\mathcal{S}_B \subset \widehat{\mathcal{D}_B}$ be a $\mathbb{Q}(q)$-subalgebra generated by $y_1, y_2, \dots, y_n$, and $\widehat{\mathcal{S}_B} \subset \widehat{\mathcal{D}_B}$ be its closure. Since the increasing monomials

$$e_1^{m_1} e_2^{m_2} \dots e_n^{m_n} \otimes f_1^{m_1'} f_2^{m_2'} \dots f_n^{m_n'} \in \widehat{\mathcal{D}_B}$$

form a topological basis of $\widehat{\mathcal{D}_B}$, the increasing monomials $y_1^{m_1} y_2^{m_2} \dots y_n^{m_n}$ form a topological basis of $\widehat{\mathcal{S}_B}$. Therefore, $\widehat{\mathcal{S}_B}$ is isomorphic to the formal power series algebra $\mathbb{Q}(q)[\![y_1, y_2, \dots, y_n]\!]$ as a $\mathbb{Q}(q)$-linear space. This isomorphism endows the $\mathbb{Q}(q)$-linear space $\mathbb{Q}(q)[\![y_1, y_2, \dots, y_n]\!]$ with a complete topological $\mathbb{Q}(q)$-algebra structure, whose multiplication is uniquely determined by the commutation relations $y_i y_j = q^{2b_{ij}} y_j y_i$.

In the same way, the skew Laurent polynomial algebra $\mathcal{L}_B$ can be defined. Namely, $\mathcal{L}_B$ is a Laurent polynomial algebra $\mathbb{Q}(q)[y_1^{\pm 1}, y_2^{\pm 2}, \dots, y_n^{\pm 1}]$ as a $\mathbb{Q}(q)$-linear space, and multiplication in $\mathcal{L}_B$ is uniquely defined by the commutation relations $y_i y_j = q^{2b_{ij}} y_j y_i$. The term $\mathcal{S}_B$ can be naturally considered as a subalgebra of $\mathcal{L}_B$.

Let L be the lower triangular part of B. Since B is skew symmetric, $B = L - {}^tL$. We define a normal ordered product in $\mathcal{L}_B$ by

$$:y^{\mathbf{m}}: := q^{{}^t\mathbf{m}L\mathbf{m}} y^{\mathbf{m}} \quad (\mathbf{m} = {}^t(m_1, m_2, \dots, m_n) \in \mathbb{Z}^n), \tag{4.9}$$

where $y^{\mathbf{m}} := y_1^{m_1} y_2^{m_2} \dots y_n^{m_n}$.

Let $B' = (b'_{kl})_{k,l=1}^{n'} \in M_{n'}(\mathbb{Z})$ be another skew-symmetric matrix. We will consider an algebra homomorphism $\psi_R \colon \mathcal{L}_B \to \mathcal{L}_{B'} \cong \mathbb{Q}(q)[y_1'^{\pm 1}, y_2'^{\pm 1}, \dots, y_{n'}'^{\pm 1}]$ which is determined by an $n' \times n$-matrix $R \in M_{n',n}(\mathbb{Z})$ and

$$\psi_R(y_i) := :y'^{Rv_i}:, \tag{4.10}$$

where $v_i \in \mathbb{Z}^n$ is the ith unit vector. The term ψ_R is well defined if and only if it preserves the commutation relation $y_i y_j = q^{2b_{ij}} y_j y_i$ for all $i, j = 1, 2, \dots, n$; in other words,

$$:y'^{Rv_i}::y'^{Rv_j}: = q^{2\,{}^tv_i B v_j} :y'^{Rv_j}::y'^{Rv_i}: \quad (i, j = 1, 2, \dots, n).$$

On the other hand,

$$y'^{Rv_i} y'^{Rv_j} = q^{2\,{}^t(Rv_i)B'Rv_j} y'^{Rv_j} y'^{Rv_i} \quad \in \mathcal{L}_{B'}.$$

Thus ψ_R is well defined if and only if ${}^tv_i\,{}^tRB'Rv_j = {}^tv_i B v_j$ for all i, j. This shows the following proposition:

Proposition 4.3. *Let $B \in M_n(\mathbb{Z})$, $B' \in M_{n'}(\mathbb{Z})$ be skew-symmetric matrices, and R be an integer-valued $n' \times n$-matrix. There exists a unique algebra homomorphism $\psi_R \colon \mathcal{L}_B \to \mathcal{L}_{B'}$ satisfying $\psi_R(y_i) = :y'^{Rv_i}:$ $(i = 1, 2, \dots, n)$ if and only if*

$${}^tRB'R = B. \tag{4.11}$$

Moreover, ψ_R preserves the normal ordered product.

Proposition 4.4. *Suppose that R satisfies (4.11). Then*

$$\psi_R(:y^{\mathbf{m}}:) = :y'^{R\mathbf{m}}: \quad (\mathbf{m} \in \mathbb{Z}^n). \tag{4.12}$$

Proof. We prove the proposition by induction on $\deg \mathbf{m} := |m_1| + |m_2| + \dots + |m_n|$. The case $\deg \mathbf{m} = 1$ is trivial since ${}^tv_i L v_i = 0$ for any unit vector v_i. Suppose that (4.12) holds if $\deg \mathbf{m} < N$ for some integer $N \geq 2$. Let $\mathbf{m}$ be of degree N.

Choose $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{Z}^n$ so that $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2$ and $\deg \mathbf{m}_k < N$ $(k = 1, 2)$. Since $y^{\mathbf{m}_1} y^{\mathbf{m}_2} = q^{2\,{}^t\mathbf{m}_1 L \mathbf{m}_2} y^{\mathbf{m}_1+\mathbf{m}_2}$, we have

$$\begin{aligned} {:}y^{\mathbf{m}_1+\mathbf{m}_2}{:} &= q^{{}^t(\mathbf{m}_1+\mathbf{m}_2)L(\mathbf{m}_1+\mathbf{m}_2)} y^{\mathbf{m}_1+\mathbf{m}_2} \\ &= q^{{}^t\mathbf{m}_2 L \mathbf{m}_1 - {}^t\mathbf{m}_1 L \mathbf{m}_2} {:}y^{\mathbf{m}_1}{:}\,{:}y^{\mathbf{m}_2}{:} \\ &= q^{{}^t\mathbf{m}_2 B \mathbf{m}_1} {:}y^{\mathbf{m}_1}{:}\,{:}y^{\mathbf{m}_2}{:}. \end{aligned}$$

Then, by the induction hypothesis,

$$\begin{aligned} \psi_R({:}y^{\mathbf{m}_1+\mathbf{m}_2}{:}) &= q^{{}^t\mathbf{m}_2 B \mathbf{m}_1} \psi_R({:}y^{\mathbf{m}_1}{:}\,{:}y^{\mathbf{m}_2}{:}) \\ &= q^{{}^t\mathbf{m}_2 B \mathbf{m}_1} {:}y'^{R\mathbf{m}_1}{:}\,{:}y'^{R\mathbf{m}_2}{:} \\ &= q^{{}^t\mathbf{m}_2 B \mathbf{m}_1 - {}^t(R\mathbf{m}_2)B' R\mathbf{m}_1} {:}y'^{\mathbf{m}_1+\mathbf{m}_2}{:} \\ &= {:}y'^{\mathbf{m}_1+\mathbf{m}_2}{:} \quad (\because (4.11)). \end{aligned}$$

Thus (4.12) holds for arbitrary $\mathbf{m}$ of degree N. □

When all the components of matrix R satisfying (4.11) are nonnegative, we have the restricted homomorphism $\psi_R \colon \mathcal{S}_B \to \mathcal{S}_{B'}$.

Proposition 4.5. *Suppose that $R \in M_{n',n}(\mathbb{Z}_{\geq 0})$ satisfies (4.11). The algebra homomorphism $\psi_R \colon \mathcal{S}_B \to \mathcal{S}_{B'}$ is continuous with respect to the relative topology in $\widehat{\mathcal{S}_B}$ and $\widehat{\mathcal{S}_{B'}}$ if and only if each column of R contains a nonzero component.*

Proof. By the definition of ψ_R, it is continuous if and only if

> for any $N \in \mathbb{Z}_{\geq 0}$, there exists some $M \in \mathbb{Z}_{\geq 0}$ such that $\mathbf{m} = {}^t(m_1, \dots, m_n) \in \mathbb{Z}^n_{\geq 0}$ of total degree $\deg \mathbf{m} \geq M \Rightarrow \deg R\mathbf{m} \geq N$.

Since $\deg R\mathbf{m} = \sum_{i=1}^n r_i m_i$, where r_i is the sum of components in ith column of R, this condition holds if and only if all the r_i are positive. □

Thus, when R satisfies these conditions, ψ_R uniquely extends to the continuous algebra homomorphism $\widehat{\psi_R} \colon \widehat{\mathcal{S}_B} \to \widehat{\mathcal{S}_{B'}}$.

§4.3. Several formulas related to the quantum dilogarithm

To compute the image of the quasi-universal R-matrix Θ, we briefly prepare for a couple of formulas related to the quantum dilogarithm function $\mathrm{Li}_{2,q}(x)$. Let

$$\log(1-x) := -\sum_{n=1}^{\infty} \frac{x^n}{n}, \tag{4.13}$$

$$\mathrm{Li}_{2,q}(x) := \sum_{n=1}^{\infty} \frac{x^n}{n(1-q^n)}, \tag{4.14}$$

$$\mathbb{E}(x) := \exp(\mathrm{Li}_{2,q^2}(-qx)), \tag{4.15}$$

$$(x;q)_\infty := \prod_{n=0}^{\infty}(1-q^n x). \tag{4.16}$$

In this paper we consider these functions just as formal power series. The function $(x;q)_\infty$ is characterized by the recurrence relation

$$(1-x)(qx;q)_\infty = (x;q)_\infty. \tag{4.17}$$

Since $-\mathrm{Li}_{2,q}(qx) + \log(1-x) = -\mathrm{Li}_{2,q}(x)$, $\exp(-\mathrm{Li}_{2,q}(x))$ satisfies the recurrence relation. Therefore,

$$\exp \mathrm{Li}_{2,q}(x) = (x;q)_\infty^{-1}, \tag{4.18}$$

$$\mathbb{E}(x) = (-qx;q^2)_\infty^{-1}, \tag{4.19}$$

which coincides with the product presentation of $\mathbb{E}(x)$ in the introduction. By this presentation, $\mathbb{E}(x)$ is characterized by the recurrence relation

$$(1+qx)\mathbb{E}(x) = \mathbb{E}(q^2x). \tag{4.20}$$

Recall that the q-exponential function was defined by

$$\exp_q(x) := \sum_{n=0}^{\infty} \frac{q^{-\frac{1}{2}n(n-1)}}{[n]_q!} x^n. \tag{4.21}$$

Then it can be directly verified that $\exp_q(x)$ satisfies

$$(1+q(q-q^{-1})x)\exp_q(x) = \exp_q(q^2x) \tag{4.22}$$

and we conclude that

$$\exp_q(x) = \mathbb{E}((q-q^{-1})x). \tag{4.23}$$

In the same way, we can also prove another presentation of $(x;q)_\infty$ [12], which we will use in the computation of imaginary root vectors:

$$\exp\left(\sum_{m=1}^{\infty} -\frac{1}{m(1-q^m)}x^m\right) = (x;q)_\infty. \tag{4.24}$$

§4.4. Computation of the image of the quasi-universal R-matrix Θ

Now we suppose that $\mathfrak{g}$ is an untwisted affine Lie algebra. We will compute $\pi_B^+ \widehat{\otimes} \pi_B^-(\Theta) \in \widehat{\mathcal{D}_B}$ for various product presentations of Θ (2.37) and equate them to obtain concrete identities.

First we remark that q-commutator monomials degenerate to ordinary monomials. Let $X_\alpha, X_\beta \in \mathcal{P}_B^+$ have weights $\alpha, \beta \in Q$ respectively. Then by definition of $\mathcal{P}_B^+$, X_α is a linear combination of monomials $e_{i_1}e_{i_2}\dots e_{i_m}$, where $\alpha_{i_1}+\alpha_{i_2}+\dots+\alpha_{i_m}=\alpha$. Hence $X_\alpha X_\beta = q^{\{\alpha,\beta\}_B}X_\beta X_\alpha$ and we have

$$[X_\alpha, X_\beta]_q \coloneqq X_\alpha X_\beta - q^{(\alpha,\beta)}X_\beta X_\alpha = (1-q^{(\beta,\alpha)+\{\beta,\alpha\}_B})X_\alpha X_\beta. \tag{4.25}$$

It is convenient to introduce the bilinear form $\langle\alpha,\beta\rangle_B \coloneqq (\alpha,\beta)-\{\alpha,\beta\}_B$. Then the values of the bilinear form $\langle\cdot,\cdot\rangle_B$ are even integers, since

$$\langle\alpha_i,\alpha_j\rangle_B = c_{ij}-b_{ij} = \begin{cases}(1\pm 1)c_{ij}, & i\neq j,\\ 2d_i, & i=j.\end{cases}$$

Formula (4.25) shows that $[X_\alpha, X_\beta]_q$ vanishes if and only if $\langle\alpha,\beta\rangle_B=0$, otherwise it is a nonzero multiple of $X_\alpha X_\beta$. Therefore, we have the following vanishing criteria for q-commutator monomials.

Proposition 4.6. *A q-commutator monomial $M\in U_q^+$ lies in the kernel of π_B^+ if and only if there exists an application of the q-bracket $[E_\alpha, E_\beta]_q$ in M for some $E_\alpha, E_\beta\in U_q^+$ of weight $\alpha,\beta\in Q$ satisfying $\langle\alpha,\beta\rangle_B=0$. If there are no such applications of q-bracket in M, $\pi_B^+(M)\in\mathcal{P}_B^+$ is a nonzero monomial.*

Recall that the root vectors for U_q^- were defined by $F_{\le,\alpha}\coloneqq\Omega(E_{\le,\alpha})$, where $\alpha\in\Delta_+^{\mathrm{re}}$ and $\Omega\colon U_q^+\to U_q^-$; $\Omega(E_i)\coloneqq F_i$, $\Omega(q)\coloneqq q^{-1}$ was the Chevalley involution, which is anti-automorphism of a $\mathbb{Q}$-algebra. Notice that Ω preserves the q-bracket, except for multiple of a power of q:

$$\Omega([E_\alpha,E_\beta]_q) = -q^{-(\alpha,\beta)}[F_\alpha,F_\beta]_q. \tag{4.26}$$

Thus, the $F_{\le,\alpha}$ are also q-commutator monomials, and they coincide with $E_{\le,\alpha}$ except for multiples of $\pm q^k$ and replacing E_i with F_i.

There exists a unique anti-isomorphism of a $\mathbb{Q}$-algebra $\overline{\Omega_B}\colon\mathcal{P}_B^+\to\mathcal{P}_B^-$ which sends e_i to f_i and q to q^{-1}. This is useful to compute $\pi_B^-(F_{\le,\alpha})$ because

$$\overline{\Omega_B}\circ\pi_B^+ = \pi_B^-\circ\Omega. \tag{4.27}$$

Since there are $\operatorname{ht}\alpha-1$ occurrences of q-brackets in $E_{\le,\alpha}$ for $\alpha\in\Delta_+^{\mathrm{re}}$, its image takes the form

$$\pi_B^+(E_{\le,\alpha}) = Cq^u\left(\prod_{i=1}^{\operatorname{ht}\alpha-1}(1-q^{k_i})\right)e_0^{m_0}e_1^{m_1}\dots e_n^{m_n}, \tag{4.28}$$

where $C\in\mathbb{Q}(q)$ is the coefficient of $E_{\le,\alpha}$ as a q-commutator monomial, $\alpha=\sum_{i=0}^n m_i\alpha_i$ and $u,k_i\in\mathbb{Z}$. In the simply laced case, $C=1$ because no nontrivial scalar multiples occur in the algorithm of Section 3 (Proposition 3.6). We also note that each k_i is a value of the bilinear form $\langle\cdot,\cdot\rangle_B$ and thus an even integer. Using $\overline{\Omega_B}$, the image of $F_{\leq,\alpha}$ is

$$\begin{aligned}\pi_B^-(F_{\leq,\alpha}) &= \overline{\Omega_B}\circ\pi_B^+(E_{\leq,\alpha})\\ &= \Omega(C)q^{-u}\left(\prod_{i=1}^{\operatorname{ht}\alpha-1}(1-q^{-k_i})\right)f_n^{m_n}f_{n-1}^{m_{n-1}}\dots f_0^{m_0}.\end{aligned}\tag{4.29}$$

Recall that the subalgebra $\mathcal{S}_B\subset\mathcal{D}_B$, which is generated by $y_i := (q_i-q_i^{-1})\times(q_i^{-1}-q_i)e_i\otimes f_i$ for $i=0,1,\dots,n$. The normal ordered product of the monomial $y^{\mathbf{m}} := y_0^{m_0}y_1^{m_1}\dots y_n^{m_n}$ ($\mathbf{m}={}^t(m_0,m_1,\dots,m_n)\in\mathbb{Z}_{\geq 0}^{n+1}$) was defined as $:y^{\mathbf{m}}: := q^{{}^t\mathbf{m}L\mathbf{m}}y^{\mathbf{m}}$, where L was the lower triangular part of B.

Using this notation, in the simply laced case we have

$$\begin{aligned}&\pi_B^+\otimes\pi_B^-((q-q^{-1})(q^{-1}-)E_{\leq,\alpha}\otimes F_{\leq,\alpha})\\ &\qquad=\left(\prod_{i=1}^{\operatorname{ht}\alpha-1}\left[\frac{k_i}{2}\right]_q^2\right):y_0^{m_0}y_1^{m_1}\dots y_n^{m_n}:.\end{aligned}\tag{4.30}$$

Recall that $\exp_q(x)=\mathbb{E}((q-q^{-1})x)$ (4.23). Therefore, if $k_i=\pm 2$ for all i, we have a simple description of the image of $\Theta_{\leq,\alpha}$ for a positive real root α.

Proposition 4.7. *Let Δ be a simply laced affine root system, $\alpha=\sum_{i=0}^n m_i\alpha_i\in\Delta_+^{\mathrm{re}}$, and $\leq$ be a convex order. We suppose that when a presentation of $E_{\leq,\alpha}$ as a q-commutator monomial is given, $\langle\alpha,\beta\rangle_B=\pm 2$ for each application of the q-bracket $[X_\alpha,X_\beta]_q$, where $\alpha,\beta\in\Delta^+$ and $X_\alpha\in U_\alpha^+$, $X_\beta\in U_\beta^+$. Then*

$$\pi_B^+\mathbin{\widehat{\otimes}}\pi_B^-(\Theta_{\leq,\alpha})=\mathbb{E}(:y_0^{m_0}y_1^{m_1}\dots y_n^{m_n}:).\tag{4.31}$$

While we have the general simple description of the images of real root vectors, the computation of the images of imaginary root vectors requires some ingenuity. Recall that $\varphi_{i,n}\in U_{n\delta}^+$ were defined as

$$\varphi_{i,n} := [T_{\varepsilon_i}^nT_i^{-1}(E_i),E_i]_q\quad(n\in\mathbb{Z}_{\geq 1},\ i\in\mathring{I}),\tag{4.32}$$

and imaginary root vectors $I_{i,n}$ were polynomials consisting of $\varphi_{i,n}$. Since $T_{\varepsilon_i}^nT_i^{-1}(E_i)$ can be written as a q-commutator monomial by using the algorithm in Section 3, the $\varphi_{i,n}$ themselves are q-commutator monomials. But we need to compute $T_w(I_{i,n})$ ($w\in\mathring{W}$) for a general convex order, and we cannot apply the algorithm to $T_w(\varphi_{i,n})$ when $w(\alpha_i)\in\Delta_-$ because $T_w(E_i)$ no longer lies in U_q^+.

First we compute $T_i(\varphi_{i,n})$ using the following fact.

Proposition 4.8 ([5]). *For every $i,j\in\mathring{I}$ and positive integer n,*

$$T_{\varepsilon_j}(\varphi_{i,n})=\varphi_{i,n}.\tag{4.33}$$

We will also use the property that for $u, v \in \widehat{W}$, $T_{uv} = T_u T_v$ if $\ell(uv) = \ell(u) + \ell(v)$.

Since $t_{\varepsilon_i}(\alpha_i) = -\delta + \alpha_i \in \Delta_-$, the length of $u := t_{\varepsilon_i} s_i$ in $\widehat{W}$ is $\ell(u) = \ell(t_{\varepsilon_i}) - 1$. Thus $T_{\varepsilon_i} = T_u T_i$ and we have

$$T_i(\varphi_{i,n}) = T_u^{-1}(\varphi_{i,n}) = [T_i T_{\varepsilon_i}^{n-1} T_i^{-1}(E_i), T_u^{-1}(E_i)]_q. \tag{4.34}$$

Now we can compute an arbitrary $T_w(\varphi_{i,n})$ for $w \in \mathring{W}$. When $w(\alpha_i) \in \Delta_+$, we have

$$T_w(\varphi_{i,n}) = [T_w T_{\varepsilon_i}^n T_i^{-1}(E_i), T_w(E_i)]_q \tag{4.35}$$

and thus simply applying the algorithm to $T_w T_{\varepsilon_i}^{n-1} T_u(E_i)$ and $T_w(E_i)$ yields an explicit presentation of $T_w(\varphi_{i,n})$ as a q-commutator monomial.

When $w(\alpha_i) \in \Delta_-$, let $w' := w s_i$. Then $T_w = T_{w'} T_i$ and using (4.34) we have

$$T_w(\varphi_{i,n}) = T_{w'}[T_i T_{\varepsilon_i}^{n-1} T_i^{-1}(E_i), T_u^{-1}(E_i)]_q. \tag{4.36}$$

Let $\lambda := \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_\ell \in \mathring{\mathfrak{h}}^*$, which is a strictly dominant weight. Then $T_\lambda := T_{t_\lambda} = T_{\varepsilon_1} T_{\varepsilon_2} \dots T_{\varepsilon_\ell}$ and $t_\lambda(\alpha_i) = -\delta + \alpha_i \in \Delta_-$ $(i \in \mathring{I})$. Thus t_λ inverts every positive root in $\mathring{\Delta}_+$, which implies that there is an expression $t_\lambda = v w_\circ$, where $v \in \widehat{W}$ and $w_\circ \in \mathring{W}$ is the longest element satisfying $\ell(t_\lambda) = \ell(v) + \ell(w_\circ)$. Notice that

$$T_{w'} T_u^{-1}(E_i) = T_w T_i^{-1} T_u^{-1}(E_i) = T_w T_{\varepsilon_i}^{-1}(E_i),$$

and $T_{\varepsilon_i}^{-1}(E_i) = T_\lambda^{-1}(E_i)$ since $T_{\varepsilon_j}(E_i) = E_i$ if $i \neq j$. We also note that $T_{w_\circ} = T_{w_\circ w^{-1}} T_w$ for every $w \in \mathring{W}$ due to the maximality of $w_\circ$.

Recall the anti-automorphism of the $\mathbb{Q}(q)$-algebra $\Psi \colon U_q^+ \to U_q^+$ defined by $\Psi(E_i) := E_i$. Since Ψ preserves weights, it reverses q-brackets:

$$\Psi([x, y]_q) = [\Psi(y), \Psi(x)]_q \quad (x, y \in U_q^+). \tag{4.37}$$

It is also easy to verify that $\Psi T_i = T_i^{-1} \Psi$ for $i \in I$.

Let $t_\lambda = \tau v' w_\circ$ $(\tau \in \Omega,\ v' \in W)$ and $w_\circ = \widetilde{w} w$. Then

$$\begin{aligned} T_w T_\lambda^{-1}(E_i) &= T_w T_w^{-1} T_{\widetilde{w}}^{-1} T_{v'}^{-1} T_\tau^{-1}(E_i) \\ &= T_{\widetilde{w}}^{-1} T_{v'}^{-1} T_\tau^{-1} \Psi(E_i) \\ &= \Psi T_{(v'\widetilde{w})^{-1}}(E_{\tau^{-1}(i)}). \end{aligned}$$

Finally, we have

$$T_w(\varphi_{i,n}) = [T_w T_{\varepsilon_i}^{n-1} T_i^{-1}(E_i), \Psi T_{(v'\widetilde{w})^{-1}}(E_{\tau^{-1}(i)})]_q. \tag{4.38}$$

We can apply the algorithm to $T_{\widetilde{w}v'}(E_{\tau^{-1}(i)})$. Since Ψ just reverses the directions of the q-brackets, $\Psi T_{\widetilde{w}v'}(E_{\tau^{-1}(i)})$ is a q-commutator monomial.

Proposition 4.9. *$T_w(\varphi_{i,n})$ is a q-commutator monomial for every $w \in \mathring{W}$, $i \in \mathring{I}$, and positive integer n.*

§5. Examples of quantum dilogarithm identities

In this final section, we give specific convex orders and Dynkin quivers, which eventually induce the identities proposed in [3].

Recall that the affine positive root system Δ_+ is decomposed as

$$\Delta_+ = \Delta(w,-) \amalg \Delta_+^{\mathrm{im}} \amalg \Delta(w,+),$$

and convex orders on Δ_+ consist of convex orders on each $\Delta(w,\pm)$ (the order on Δ_+^{im} is not significant since any total order can be chosen). The convex order on $\Delta(w,-)$ was determined by the following parameters with several restrictions (2.12), (2.13):

(1) a positive integer n and a filtration of indices $\mathring{I} = J_0 \supsetneqq J_1 \supsetneqq J_2 \supsetneqq \cdots \supsetneqq J_n = \emptyset$;
(2) $y_1 \in W_{J_1}, y_2 \in W_{J_2}, \dots, y_n \in W_{J_n}$;
(3) infinite reduced words $\mathbf{s}_0 \in \mathscr{W}_{J_0}^{\infty}, \mathbf{s}_1 \in \mathscr{W}_{J_1}^{\infty}, \dots, \mathbf{s}_{n-1} \in \mathscr{W}_{J_{n-1}}^{\infty}$.

We have to specify the parameters not only for $\Delta(w,-)$, but also for $\Delta(w,+) = \Delta(ww_\circ,-)$ to construct the whole convex order on Δ_+. In the examples below, let $\check{\cdot}$ denote the parameters for $\Delta(w,+)$. For instance, $\check{w} = ww_\circ$.

Fortunately, the parameters y_i are all 1 in our examples below, so we omit the value of y_i in the examples. Also, the numbers of rows n are the same for both $\Delta(w,-)$ and $\Delta(w,+)$; in other words, $\check{n} := n$ for all the examples below.

§5.1. Type $A_1^{(1)}$

Let $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$ be an affine algebra of type $A_1^{(1)}$. In this case, there are only two convex orders on Δ_+ except for the order on Δ_+^{im}, and one of them is just the reverse of the other one. The corresponding parameters are

$$\begin{aligned} w &:= 1, \quad n := 1; \\ \mathring{I} &= \{1\} = J_0 \supsetneqq J_1 = \emptyset, \\ \mathbf{s}_0 &:= (s_0 s_1)^{\infty}, \\ \mathring{I} &= \{1\} = \check{J}_0 \supsetneqq \check{J}_1 = \emptyset, \\ \check{\mathbf{s}}_0 &:= (s_1 s_0)^{\infty}, \end{aligned}$$

where $(\mathbf{s})^{\infty} := \mathbf{sss}\dots$ denotes infinite repetition of $\mathbf{s}$. Then the corresponding convex order $\preceq$ turns out to coincide with (2.10).

Next we compute the root vectors from this convex order. Since $A_1^{(1)}$ is not simply laced, we cannot use the algorithm of Section 3. But the following formula is sufficient to accomplish the computation:

$$T_1([E_0, E_1]_q) = [E_1, E_0]_q, \quad T_0([E_1, E_0]_q) = [E_0, E_1]_q. \tag{5.1}$$

By the definition of root vectors and the chosen order, one can verify

$$\begin{aligned} E_{\leq,(2n+1)\delta-\alpha_1} &= (T_0T_1)^n(E_0) \\ &= \frac{1}{[2]_q^{2n}}\big(\big(\overleftarrow{\mathrm{ad}}[E_0, E_1]_q\big)^{2n}(E_0)\big), \end{aligned} \tag{5.2}$$

$$\begin{aligned} E_{\leq,(2n+2)\delta-\alpha_1} &= (T_0T_1)^n E_0(E_1) \\ &= \frac{1}{[2]_q^{2n+1}}\big(\big(\overleftarrow{\mathrm{ad}}[E_0, E_1]_q\big)^{2n+1}(E_0)\big), \end{aligned} \tag{5.3}$$

$$\begin{aligned} E_{\leq,2n\delta+\alpha_1} &= \Psi(T_1T_0)^n(E_1) \\ &= \frac{1}{[2]_q^{2n}}\big(\big(\overrightarrow{\mathrm{ad}}[E_0, E_1]_q\big)^{2n}(E_1)\big), \end{aligned} \tag{5.4}$$

$$\begin{aligned} E_{\leq,(2n+1)\delta+\alpha_1} &= \Psi(T_1T_0)^n T_1(E_0) \\ &= \frac{1}{[2]_q^{2n+1}}\big(\big(\overrightarrow{\mathrm{ad}}[E_0, E_1]_q\big)^{2n+1}(E_1)\big), \end{aligned} \tag{5.5}$$

for all $n = 0, 1, 2, \ldots$.

Using the reduced expression $t_{\varepsilon_1} = \rho s_1$, where $\rho \in \Omega$ is the transposition of 0 and 1, one can show that for any positive integer m,

$$\begin{aligned} \varphi_{1,m} &= [(T_\rho T_1)^{m-1}(E_0), E_1]_q = [E_{\leq,m\delta-\alpha_1}, E_1]_q, \\ T_1(\varphi_{1,m}) &= T_{\rho^{-1}}(\varphi_{1,m}) = [\Psi E_{\leq,(m-1)\delta+\alpha_1}, E_0]_q. \end{aligned} \tag{5.6}$$

We set the projection of Section 4.1 by $\sigma_{01} := +1$. Then the corresponding skew-symmetric matrix is $B = \left(\begin{smallmatrix} 0 & -2 \\ 2 & 0 \end{smallmatrix}\right)$, and the matrix presentation of the bilinear form $\langle\cdot,\cdot\rangle_B$ is $(\langle\alpha_i, \alpha_j\rangle_B)_{i,j=0}^1 = \left(\begin{smallmatrix} 2 & 0 \\ -4 & 2 \end{smallmatrix}\right)$. Since $\langle\alpha_0, \alpha_1\rangle_B = 0$, the projection $\pi_B^+ \colon U_q^+ \to \mathcal{P}_B^+$ annihilates $[E_0, E_1]_q$. Thus all the root vectors vanish in $\mathcal{P}_B^+$ except for simple root vectors $E_{\leq,\alpha_i} = E_i$ $(i = 0, 1)$. Therefore, the image of the quasi-universal R-matrix Θ is

$$\pi_B^+ \mathbin{\widehat{\otimes}} \pi_B^-(\Theta) = \mathbb{E}(y_1)\mathbb{E}(y_0) \quad \in \widehat{\mathcal{D}_B}. \tag{5.7}$$

Beware that the order of the product is the reverse of the given convex order (2.37).

Now we consider the reversed order $\leq'$, which is in fact obtained by just swapping $\Delta(w, -)$ for $\Delta(w, +)$. The corresponding parameters are also just swapping every parameter $\cdot$ for $\because$. Thus $E_{\leq',\alpha} = \Psi E_{\leq,\alpha}$ for every real root $\alpha \in \Delta_+^{\mathrm{re}}$. One can verify that all the real root vectors for $\leq'$ satisfy the condition of Proposition 4.7 and thus do not vanish.

Since $\breve{w} = s_1$, $T_1(I_{1,m})$ $(m \geq 1)$ are used as imaginary root vectors. Using (5.6) we have

$$\pi_B^+(T_1\varphi_{1,m}) = [m+1]_q (q-q^{-1})^{2m-1}(e_0e_1)^m. \tag{5.8}$$

Let $D := (q-q^{-1})^2 e_0 e_1$. Then the image of the generating function $T_1\varphi_1(z) \in U_q^+[\![z]\!]$ is

$$\pi_B^+(T_1(1+\varphi_1(z))) = \sum_{m=0}^{\infty}[m+1]_q(Dz)^m = \frac{1}{(1-qDz)(1-q^{-1}Dz)}, \tag{5.9}$$

where $\pi_B^+ \colon U_q^+[\![z]\!] \to \mathcal{P}_B^+[\![z]\!]$ is defined degreewise. Recall that $I_1(z) = (q-q^{-1}) \times \sum_{m=1}^{\infty} I_{1,m} z^m := \log(1+\varphi_1(z))$. Since $\log(1+x) = -\sum_{m=1}^{\infty}(-1)^m x^m/m$ (4.13),

$$\pi_B^+(T_1 I_1(z)) = \sum_{m=1}^{\infty} \frac{q^m + q^{-m}}{m} D^m z^m \tag{5.10}$$

and therefore

$$\pi_B^+(I_{1,m}) = \frac{q^m+q^{-m}}{m(q-q^{-1})} D^m. \tag{5.11}$$

Now we compute the image of $S'_m := T_1 \otimes T_1(S_m)$ (2.36). By definition, $b_{1,1;m} = [2m]_q/(m(q^{-1}-q))$ and $c_{1,1;m} = b_{1,1;m}^{-1}$. Thus

$$S_m = c_{1,1;m} I_{1,m} \otimes J_{1,m} = \frac{m(q^{-1}-q)}{[2m]_q} I_{1,m} \otimes \Omega I_{1,m} \quad \in U_q^+ \otimes U_q^-.$$

Let $D' := \overline{\Omega_B}(D) = (q^{-1}-q)^2 f_1 f_0$. By virtue of (4.27) and $\Omega T_i = T_i \Omega$ $(i = 0, 1, \ldots, \ell)$, we can compute as follows:

$$\begin{aligned}\pi_B^+ \otimes \pi_B^-(S'_m) &= \frac{m(q^{-1}-q)}{[2m]_q} \frac{q^m+q^{-m}}{m(q-q^{-1})} \frac{q^m+q^{-m}}{m(q^{-1}-q)} D^m \otimes D'^m \\ &= \frac{1}{m} \frac{q^m+q^{-m}}{q^m-q^{-m}} (D\otimes D')^m = -\frac{q^m(q^m+q^{-m})}{m(1-q^{2m})}(D\otimes D')^m.\end{aligned}$$

By (4.24), we obtain the image of $\Theta_{\mathrm{im}} := \prod_{m=1}^{\infty} \Theta_{m\delta}$:

$$\begin{aligned}\pi_B^+ \widehat{\otimes} \pi_B^-(\Theta_{\mathrm{im}}) &= \mathbb{E}(-qD\otimes D')^{-1}\mathbb{E}(-q^{-1}D\otimes D')^{-1} \\ &= \mathbb{E}(-q{:}y_0y_1{:})^{-1}\mathbb{E}(-q^{-1}{:}y_0y_1{:})^{-1}.\end{aligned} \tag{5.12}$$

Finally, we have

$$\begin{aligned}\pi_B^+ \widehat{\otimes} \pi_B^-(\Theta) = {} & \mathbb{E}(:y_0:)\mathbb{E}(:y_0^2y_1:)\mathbb{E}(:y_0^3y_1^2:) \times \cdots \\ & \times \mathbb{E}(-q:y_0y_1:)^{-1}\mathbb{E}(-q^{-1}:y_0y_1:)^{-1} \\ & \times \cdots \times \mathbb{E}(:y_0^2y_1^3:)\mathbb{E}(:y_0y_1^2:)\mathbb{E}(:y_1:).\end{aligned}$$

Comparing with (5.7), we eventually obtain the following quantum dilogarithm identity, which was first found by Terasaki [12].

Theorem 5.1 ([12]). *Let y_0, y_1 be indeterminate. Then the following identity holds in the skew formal power series algebra $\widehat{\mathcal{S}_1} \coloneqq \mathbb{Q}(q)[\![y_0, y_1]\!]$ with commutation relation $y_0y_1 = q^{-4}y_1y_0$:*

$$\begin{aligned}\mathbb{E}(:y_1:)\mathbb{E}(:y_0:) = {} & \mathbb{E}(:y_0:)\mathbb{E}(:y_0^2y_1:)\mathbb{E}(:y_0^3y_1^2:) \times \cdots \\ & \times \mathbb{E}(-q:y_0y_1:)^{-1}\mathbb{E}(-q^{-1}:y_0y_1:)^{-1} \\ & \times \cdots \times \mathbb{E}(:y_0^2y_1^3:)\mathbb{E}(:y_0y_1^2:)\mathbb{E}(:y_1:), \end{aligned} \tag{5.13}$$

where $:y_0^{m_0}y_1^{m_1}: = q^{2m_0m_1}y_0^{m_0}y_1^{m_1}$.

Let $B' = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\widehat{\mathcal{S}} \coloneqq \widehat{\mathcal{S}_{B'}} \cong \mathbb{Q}(q)[\![x_1, x_2]\!]$. Then $x_1x_2 = q^2x_2x_1$, which coincides with the commutation relation in the introduction. If we set $R \coloneqq \left(\begin{smallmatrix} 0 & 2 \\ 1 & 3 \end{smallmatrix}\right)$, R satisfies ${}^tRB'R = B$. Thus by Propositions 4.3 and 4.5, there exists a unique continuous algebra homomorphism $\widehat{\psi_1}\colon \widehat{\mathcal{S}_1} \to \widehat{\mathcal{S}}$ satisfying

$$\widehat{\psi_1}(y_0) = x_2, \quad \widehat{\psi_1}(y_1) = :x_1^2x_2^3: = q^{-6}x_1^2x_2^3. \tag{5.14}$$

Let $\mathcal{L} \coloneqq \mathcal{L}_{B'} \cong \mathbb{Q}(q)[x_1^{\pm 1}, x_2^{\pm 1}]$. Since $S \coloneqq \left(\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right)$ satisfies ${}^tSB'S = B'$, we have the algebra automorphism $\psi_S \in \operatorname{Aut}\mathcal{L}$. The term ψ_S transforms variables x_1, x_2 as $\psi_S(x_1) = :x_1x_2^{-2}: = q^2x_1x_2^{-2}$, $\psi_S(x_2) = x_2$.

Applying $\widehat{\psi_1}$ on (5.13) and transforming the variables x_1, x_2 by ψ_S, we obtain (1.2) in $\mathbb{Q}(q)[\![\frac{x_1}{x_2^2}, x_2]\!]$. Recall that $\widehat{\psi_1}$ and ψ_S preserve the normal ordered product (Proposition 4.4).

Corollary 5.2. *Identity (1.2) holds in $\mathbb{Q}(q)[\![\frac{x_1}{x_2^2}, x_2]\!]$.*

Remark. We will call a group homomorphism $Z\colon Q \to \mathbb{C}$ a central charge. When Z is injective and $Z(\Delta_+)$ lies in the (closure of) the upper half-plane $\overline{\mathcal{H}} \coloneqq \{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0\}$,

$$\alpha \leq_Z \beta \overset{\text{def}}{\Leftrightarrow} \arg Z(\alpha) \leq \arg Z(\beta) \quad (\alpha, \beta \in \Delta_+^{\text{re}}) \tag{5.15}$$

defines a convex order on positive real roots, where we choose the principal value of the argument so that $0 \leq \arg z < 2\pi$.

Setting $Z(\alpha_0) := 1$, $Z(\alpha_1) := 1 + \sqrt{-1}$ yields the convex order (2.10). Notice that $\le_Z$ *yields only the convex order of a single row*, because every root is of the form $m\delta + \alpha$ ($m \in \mathbb{Z}$, $\alpha \in \mathring{\Delta} \cup \{0\}$) and thus $Z(\Delta)$ lies in a finite number of lines parallel to $Z(\delta)$.

§5.2. Type $A_2^{(1)}$

Let $\mathfrak{g} = \widehat{\mathfrak{sl}}_3$ be an affine algebra of type $A_2^{(1)}$. We choose a convex order by setting

$$\begin{aligned} &w := s_1, \quad n := 2;\\ &\mathring{I} = \{1,2\} = J_0 \supsetneqq J_1 := \{1\} \supsetneqq J_2 = \emptyset,\\ &\mathbf{s}_0 := (s_0 s_1 s_2)^\infty, \quad \mathbf{s}_1 := (s_1 s_{\delta-\alpha_1})^\infty,\\ &\mathring{I} = \check{J}_0 \supsetneqq \check{J}_1 := \{2\} \supsetneqq \check{J}_2 = \emptyset,\\ &\check{\mathbf{s}}_0 := (s_2 s_1 s_0)^\infty, \quad \check{\mathbf{s}}_1 := (s_{\delta-\alpha_2} s_2)^\infty. \end{aligned}$$

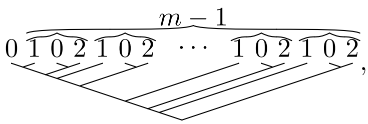
Then the corresponding convex order $\le$ is

$$\begin{aligned} \delta - \alpha_1 - \alpha_2 < \delta - \alpha_2 < 2\delta - \alpha_1 - \alpha_2 < 2\delta - \alpha_2 < \cdots&\\ < \alpha_1 < \delta + \alpha_1 < 2\delta + \alpha_1 < 3\delta + \alpha_1 < \cdots&\\ < \delta < 2\delta < 3\delta < 4\delta&\\ < \cdots < 3\delta - \alpha_1 < 2\delta - \alpha_1 < \delta - \alpha_1&\\ < \cdots < \delta + \alpha_1 + \alpha_2 < \delta + \alpha_2 < \alpha_1 + \alpha_2 < \alpha_2,& \end{aligned}$$

where the null root $\delta = \alpha_0 + \alpha_1 + \alpha_2$.

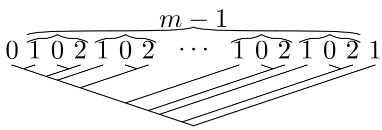
Using the algorithm of Proposition 3.6 and notation (3.13), real root vectors in the first row of $\Delta(w, -)$ are computed as

(5.16) $\quad E_{\le, m\delta - \alpha_1 - \alpha_2} =$



,

(5.17) $\quad E_{\le, m\delta - \alpha_2} =$



$(m \ge 1)$.

This can be directly proven by induction on m, noting that

$$T_0 T_1 T_2([E_0, E_2]_q) = E_1, \quad T_0 T_1 T_2(E_1) = [E_0, E_2]_q, \tag{5.18}$$

and $T_0 T_1 T_2([E_0, E_1]_q) = E_{\le, 3\delta - \alpha_1 - \alpha_2}$, which means that applying $T_0 T_1 T_2$ on $[E_0, E_1]_q$ adds three branches from the right.

The computation of real root vectors in the second row requires more preparation. First we have to compute $E_{\delta-\alpha_1}$ of (2.24). Since $\Phi(s_0 s_2) = \{\alpha_0, \alpha_0 + \alpha_2\} \subset \Delta(1,-)$, we have $E_{\delta-\alpha_1} = T_0(E_2) = [E_0, E_2]_q$. Next we need to compute $\widehat{s_{\delta-\alpha_1}} \in \widehat{W}$, which is an appropriate extension of $s_{\delta-\alpha_1} \in W_{J_1}$. By definition (2.25), $\widehat{s_{\delta-\alpha_1}} = t^{J_1}_{\varepsilon_1} s_1 t^{J_1}_{\varepsilon_1}$. To compute this, we require a reduced expression of $t_{\varepsilon_1} \in \widehat{W}$. Let $\rho \in \Omega$ denote the Dynkin automorphism which acts on the indices as $\rho(0) = 1$, $\rho(1) = 2$, $\rho(2) = 0$. Then

$$t_{\varepsilon_1} = \rho s_2 s_1, \quad t_{\varepsilon_2} = \rho^2 s_1 s_2 \quad \in \widehat{W} \subset \mathrm{GL}(\mathfrak{h}'^*) \tag{5.19}$$

are reduced expressions. Note that the length of $w \in \widehat{W}$ defined by (2.23) coincides with the number of positive roots $\alpha \in \Delta_+$ such that $w(\alpha) \in \Delta_-$. More generally, we have the following proposition:

Proposition 5.3. *Let $\mathfrak{g} = \widehat{\mathfrak{sl}_{\ell+1}}$ be an affine algebra of type $A^{(1)}_\ell$. We set the indices $0, 1, \dots, \ell$ so that $a_{ii+1} \neq 0$ for $i = 0, 1, \dots, \ell-1$. Let $\rho \in \Omega$ be the Dynkin automorphism which acts on indices as $\rho(i) = i+1$ $(0 \leq i \leq \ell-1)$, $\rho(\ell) = 0$. Then*

$$t_{\varepsilon_i} = (\rho^{-1} s_1 s_2 \dots s_i)^{\ell+1-i} \quad (i = 1, 2, \dots, \ell) \tag{5.20}$$

are reduced expressions in $\widehat{W}$.

Proof. First we have to check the equality. It is enough to compare the action of both sides on simple roots since $\widehat{W} \subset \mathrm{GL}(\mathfrak{h}'^*)$. Moreover, since every element of $\widehat{W}$ fixes the null root $\delta = \alpha_0 + \alpha_1 + \dots + \alpha_\ell$, it is enough to check the action on α_j for $j = 1, 2, \dots, \ell$. On the one hand, $t_{\varepsilon_i}(\alpha_j) = \alpha_j - \delta_{ij}\delta$. On the other hand, let $R_i \coloneqq \rho^{-1} s_1 s_2 \dots s_i$. When $\ell = 1$, $R_1(\alpha_1) = \rho^{-1}(-\alpha_1) = -\alpha_0 = \alpha_1 - \delta$ and hence (5.20) holds. Next we assume $\ell \geq 2$. Recall that for $1 \leq i, j \leq \ell$,

$$s_i(\alpha_j) = \begin{cases} \alpha_j, & |i-j| > 1, \\ \alpha_i + \alpha_j, & j = i \pm 1, \\ -\alpha_i, & j = i, \end{cases} \tag{5.21}$$

in the root system of type A_ℓ. By direct calculation we have

$$R_i(\alpha_j) = \begin{cases} \alpha_j, & 1 \leq j \leq i-1, \\ -\delta + \alpha_i + \alpha_{i+1} + \dots + \alpha_\ell, & j = i, \\ \delta - \alpha_{i+1} - \alpha_{i+2} - \dots - \alpha_\ell, & j = i+1, \\ \alpha_{j-1}, & j > i+1. \end{cases} \tag{5.22}$$

When $1 \leq j < i$, it is clear that $R_i^{\ell+1-i}(\alpha_j) = \alpha_j$.

When $j = i$, the case $i = \ell$ is the above formula. When $i < \ell$, notice that $R_i(\alpha_i + \alpha_{i+1}) = \alpha_i$. Using this inductively,

$$\begin{aligned} R_i^{\ell+1-i}(\alpha_i) &= R_i^{\ell-i}(-\delta + \alpha_i + \alpha_{i+1} + \cdots + \alpha_\ell) \\ &= R_i^{\ell-i-1}(-\delta + \alpha_i + \alpha_{i+1} + \cdots + \alpha_{\ell-1}) \\ &\cdots \\ &= R_i(-\delta + \alpha_i + \alpha_{i+1}) = -\delta + \alpha_i \end{aligned}$$

and thus $R_i^{\ell+1-i}(\alpha_i) = -\delta + \alpha_i$.

When $j = i + 1$, notice that $R_i^2(\alpha_{i+1}) = \alpha_\ell$. Thus

$$R_i^{\ell+1-i}(\alpha_{i+1}) = R_i^{\ell-i-1}(\alpha_\ell) = \alpha_{i+1}.$$

When $j > i + 1$, $R_i^{\ell+1-i}(\alpha_j) = R_i^{\ell-j+2}(\alpha_{i+1}) = R_i^{\ell-j}(\alpha_\ell) = \alpha_j$.

By the above calculation, we conclude that $t_{\varepsilon_i} = R_i^{\ell+1-i}$.

To verify that $R_i^{\ell+1-i}$ is a reduced expression, it is enough to show that the length of $t_{\varepsilon_i} \in \widehat{W}$ is $i(\ell + 1 - i)$. The length of t_{ε_i} coincides with the number of positive roots which t_{ε_i} sends to negative roots. Recall that

$$\mathring{\Delta} = \left\{ \pm(\alpha_i + \alpha_{i+1} + \cdots + \alpha_j) \mid 1 \le i \le j \le \ell \right\} \tag{5.23}$$

in a finite root system of type A_ℓ. The term t_{ε_i} translates the roots containing $\pm\alpha_i$ by $\mp\delta$. Thus, if $\alpha = m\delta + \varepsilon \in \Delta_+$ ($m \in \mathbb{Z}_{\ge 0}$, $\varepsilon \in \mathring{\Delta}$) satisfies $t_{\varepsilon_i}(\alpha) \in \Delta_-$, then $m = 0$ and ε must be a positive root containing α_i. Such an ε takes the form $\alpha_j + \alpha_{j+1} + \cdots + \alpha_k$ ($j \le i \le k$), and the number of such (j, k) is $i(\ell - i + 1)$. This shows that the length of t_{ε_i} is $i(\ell + 1 - i)$. □

By Proposition 5.3, $t_{\varepsilon_1}^{J_1} = (\rho^{-1} s_1 \rho^{-1} s_1)^{J_1} = (\rho s_2 s_1)^{J_1} = \rho s_2$ and we have

$$\widehat{s_{\delta-\alpha_1}} = \rho s_2 s_1 \rho s_2. \tag{5.24}$$

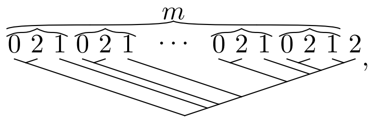
Now we can compute real root vectors in the second row. Since $w^{J_1} = s_1^{J_1} = 1$ and $E_{\delta-\alpha_1} = T_0(E_1) = T_\rho T_2(E_1)$,

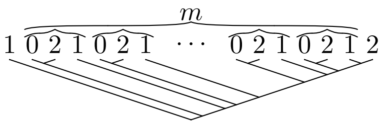
$$\begin{aligned} E_{\le, m\delta+\alpha_1} &= \begin{cases} (T_1 T_{\widehat{s_{\delta-\alpha_1}}})^{m/2}(E_1), & m\text{: even}, \\ (T_1 T_{\widehat{s_{\delta-\alpha_1}}})^{(m-1)/2} T_1(E_{\delta-\alpha_1}), & m\text{: odd}, \end{cases} \\ &= (T_1 T_\rho T_2)^m (E_1) \quad (m \ge 0). \end{aligned}$$

Observing that $T_1 T_\rho T_2(E_0) = E_0$ and $T_1 T_\rho T_2([E_1, E_2]_q) = [E_1, E_2]_q$, we have

$$E_{\le, m\delta+\alpha_1} = 1\, \overbrace{\widehat{0\,1\,2}\;\widehat{0\,1\,2}\;\cdots\;\widehat{0\,1\,2}\;\widehat{0\,1\,2}}^{m} \quad (m \ge 0). \tag{5.25}$$

Similarly, root vectors for real roots in $\Delta(w,+)=\Delta(s_2s_1,-)$ are computed as follows. Comparing $\mathbf{s}_0=(s_0s_1s_2)^\infty$ and $\check{\mathbf{s}}_0=(s_2s_1s_0)^\infty$, $E_{\leq,m\delta+\alpha_2}$ and $E_{\leq,m\delta+\alpha_1+\alpha_2}$ are obtained by swapping all the indexes 0 for 2 in $E_{\leq,m\delta-\alpha_1-\alpha_2}$, $E_{\leq,m\delta-\alpha_2}$ and applying Ψ, which just reverses all the directions of the q-bracket. As a result,

(5.26) $E_{\leq,m\delta+\alpha_2}=$  ,

(5.27) $E_{\leq,m\delta+\alpha_1+\alpha_2}=$  $(m\geq 0)$.

Since $\check{w}^{\check{J}_1}=s_2s_1$ and $\check{\mathbf{s}}_1=(s_{\delta-\alpha_2}s_2)^\infty$, root vectors for $m\delta-\alpha_1$ are defined as

$$E_{\leq,m\delta-\alpha_1}=\begin{cases}\Psi T_2T_1(T_{\widehat{s_{\delta-\alpha_2}}}T_2)^{(m-1)/2}(E_{\delta-\alpha_2}), & m\text{: odd},\\ \Psi T_2T_1(T_{\widehat{s_{\delta-\alpha_2}}}T_2)^{(m-2)/2}T_{\widehat{s_{\delta-\alpha_2}}},(E_2) & m\text{: even}\end{cases}\quad (m\geq 1), \tag{5.28}$$

where $E_{\delta-\alpha_2}=T_0(E_1)$. By Proposition 5.3, $t_{\varepsilon_2}^{\check{J}_1}=(\rho^{-1}s_1s_2)^{\check{J}_1}=\rho^2s_1$ and thus $\widehat{s_{\delta-\alpha_2}}:=t_{\varepsilon_2}^{\check{J}_1}s_2t_{\varepsilon_2}^{\check{J}_1}=\rho^2s_1s_2\rho^2s_1$. Rewriting $E_{\delta-\alpha_2}=T_0(E_1)=T_{\rho^2}T_1(E_2)$,

$$E_{\leq,m\delta-\alpha_1}=\Psi T_2T_1(T_{\rho^2}T_1T_2)^{m-1}T_{\rho^2}T_1(E_2)\quad (m\geq 1). \tag{5.29}$$

Moreover, one can deform this presentation to the form $\Psi(T_u)^mT_v(E_i)$ by realizing $T_{\varepsilon_1}^{-1}(E_2)=E_2$. Since $T_{\varepsilon_1}^{-1}=T_1^{-1}T_2^{-1}T_\rho^{-1}$, we can replace E_2 with $T_1^{-1}T_2^{-1}T_\rho^{-1}(E_2)$. In the extended braid group $\widehat{B}$,

$$T_{\rho^2}T_1T_{\varepsilon_1}^{-1}=T_1^{-1}T_\rho,$$
$$T_{\rho^2}T_1T_2\cdot T_1^{-1}T_\rho=T_1^{-1}T_\rho\cdot T_{\rho^2}T_0T_1,$$

where we used $T_1^{-1}T_{\rho^2}=T_{\rho^2}T_2^{-1}$ and $T_1T_2T_1^{-1}=T_2^{-1}T_1T_2$. Therefore, we have

$$T_2T_1(T_{\rho^2}T_1T_2)^{m-1}T_{\rho^2}T_1(E_2)=T_2T_\rho(T_{\rho^2}T_0T_1)^{m-1}(E_2).$$

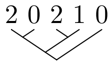
Finally, rewriting $E_2=T_\rho^{-1}(E_0)$ yields

$$\begin{aligned}T_2T_\rho(T_{\rho^2}T_0T_1)^{m-1}(E_2)&=T_2T_\rho(T_{\rho^2}T_0T_1)^{m-1}T_\rho^{-1}(E_0)\\&=T_2(T_{\rho^2}T_1T_2)^{m-1}(E_0)\\&=(T_2T_{\rho^2}T_1)^{m-1}T_2(E_0).\end{aligned}$$

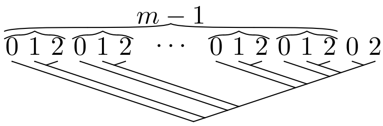
As a result, we obtain

$$E_{\leq,m\delta-\alpha_1}=\Psi(T_2T_{\rho^2}T_1)^{m-1}T_2(E_0)\quad (m\geq 1). \tag{5.30}$$

The advantage of this presentation is that inductive computation becomes easy. In fact,

(5.31) $$T_2 T_{\rho^2} T_1 T_2(E_0) =$$ 

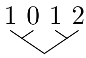
and thus applying $T_2 T_{\rho^2} T_1$ to $[E_2, E_0]_q$ adds two branches from the right. By virtue of $[E_2, E_1]_q$ and E_0 being invariant by $T_2 T_{\rho^2} T_1$, finally we have

(5.32) $$E_{\leq, m\delta-\alpha_1} =$$ 

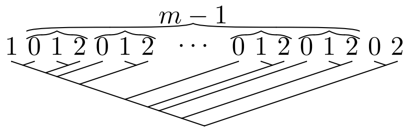
$(m \geq 1)$.

Next we compute imaginary root vectors. Since $w = s_1$, $T_1(I_{i,m})$ ($i = 1, 2$, $m \in \mathbb{Z}_{\geq 1}$) are used as imaginary root vectors. We use (4.34) to compute $T_1(\varphi_{1,m})$. Since $t_{\varepsilon_1} = \rho s_2 s_1 = u s_1$, $u = \rho s_2$. Thus

$$\begin{aligned} T_1(\varphi_{1,m}) &= [T_1(T_\rho T_2 T_1)^{m-1} T_1^{-1}(E_1), T_u^{-1}(E_1)]_q \\ &= [(T_1 T_\rho T_2)^{m-1}(E_1), \Psi T_2 T_\rho^{-1}(E_1)]_q. \end{aligned}$$

Since $T_1 T_\rho T_2(E_1) =$ 

and $T_1 T_\rho T_2$ fixes E_0 and $[E_1, E_2]_q$,

(5.33) $$T_1(\varphi_{1,m}) =$$ 

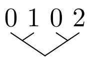
$(m \geq 1)$.

The computation of $T_1(\varphi_{2,m})$ is easier. By definition,

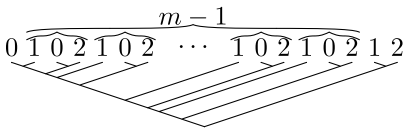
$$T_1(\varphi_{2,m}) = [T_1 T_{\varepsilon_2}^m T_2^{-1}(E_2), T_1(E_2)]_q,$$

and in the computation of $E_{\leq, m\delta-\alpha_1}$ we have already computed

$$T_1 T_{\varepsilon_2}^m T_2^{-1}(E_2) = T_\rho (T_{\rho^2} T_0 T_1)^{m-1}(E_2) = (T_{\rho^2} T_1 T_2)^{m-1}(E_0) = T_{\varepsilon_2}^{m-1}(E_0).$$

Since $T_{\rho^2} T_1 T_2(E_0) =$ 

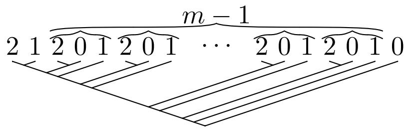
and T_{ε_2} fixes E_1 and $[E_0, E_2]_q$,

(5.34) $$T_1(\varphi_{2,m}) =$$ 

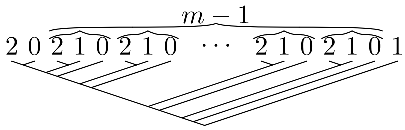
$(m \geq 1)$.

We also require $T_{\check{w}}(\varphi_{i,m}) = T_2 T_1(\varphi_{i,m})$ for the reversed order $\leq'$. Fortunately, in this case we can simply apply T_2 on every leaf of the presentation of $T_1(\varphi_{1,m})$.

As a result,

(5.35) $$T_2T_1(\varphi_{1,m}) =$$ 

,

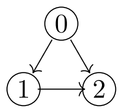
(5.36) $$T_2T_1(\varphi_{2,m}) =$$ 

$(m \geq 1)$.

Set the projection $\pi_B^+\colon U_q^+ \to \mathcal{P}_B^+$ of Section 4.1 by $\sigma_{01} \coloneqq +1$, $\sigma_{02} \coloneqq +1$, $\sigma_{12} \coloneqq +1$. Then corresponding skew-symmetric matrix is

$$B = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix},$$

which corresponds to the Dynkin quiver



.

The matrix presentation of the bilinear form $\langle \cdot, \cdot \rangle_B$ is

$$(\langle \alpha_i, \alpha_j \rangle_B)_{i,j=0}^2 = \begin{pmatrix} 2 & 0 & 0 \\ -2 & 2 & 0 \\ -2 & -2 & 2 \end{pmatrix}.$$

Thus $[E_0, E_1]_q$, $[E_0, E_2]_q$, and $[E_1, E_2]_q$ lie in the kernel of π_B^+. Examining the computed presentations of root vectors above, all the root vectors except for simple root vectors vanish by π_B^+. Therefore, the image of the quasi-universal R-matrix Θ is

(5.37) $$\pi_B^+ \mathbin{\widehat{\otimes}} \pi_B^-(\Theta) = \mathbb{E}(y_2)\mathbb{E}(y_1)\mathbb{E}(y_0) \quad \in \widehat{\mathcal{D}_B}.$$

Now we reverse the given order. Recall that the real root vectors for the reversed order $\leq'$ are obtained by just reversing all the direction of the q-bracket. Then one can verify that the real root vectors in the first row, namely, $E_{\leq', m\delta-\alpha_1} = \Psi(E_{\leq, m\delta-\alpha_1})$, $E_{\leq', m\delta-\alpha_1-\alpha_2} = \Psi(E_{\leq, m\delta-\alpha_1-\alpha_2})$ $(m \geq 1)$ satisfy the condition of Proposition 4.7.

However, by contrast, real root vectors in the second row behave differently. The term $E_{\leq', \delta-\alpha_1} = \Psi(E_{\leq, \delta-\alpha_1}) = [E_2, E_0]_q$ does not vanish by π_B^+ since

$\langle \alpha_2, \alpha_0 \rangle_B = -2 \neq 0$ and satisfies the condition of Proposition 4.7. In the same way $E_{\leq', \alpha_1} = E_1$ does not vanish. But $E_{\leq', m\delta - \alpha_1}$ and $E_{\leq', (m-1)\delta + \alpha_1}$ vanish for $m > 1$ because $\langle \alpha_0 + \alpha_2, \alpha_1 + \alpha_2 \rangle_B = \langle \alpha_0, \alpha_1 \rangle_B = 0$. Thus, the real root vectors in the second row vanish except for $E_{\leq', \delta - \alpha_1}$ and $E_{\leq', \alpha_1}$.

Next we have to compute the images of imaginary root vectors $T_2 T_1(I_{i,m})$. Using (4.25) and the presentations (5.35), (5.36), we have

$$\begin{aligned} \pi_B^+ T_2 T_1(\varphi_{1,m}) &= (1 - q^{-2})^{3m-2}(1 - q^{-2(m+1)}) e_2 e_1 (e_2 e_0 e_1)^{m-1} e_0 \\ &= (q - q^{-1})^{3m-1} [m+1]_q (e_0 e_1 e_2)^m, \\ \pi_B^+ T_2 T_1(\varphi_{2,m}) &= \begin{cases} (q - q^{-1}) e_0 e_2, & m = 1, \\ 0, & m > 1, \end{cases} \end{aligned}$$

since $\langle \alpha_2 + \alpha_0, \alpha_2 + \alpha_1 \rangle_B = 0$. The second equality is due to the following calculation. Recall that the commutation relations in $\mathcal{P}_B^+$ become $e_1 e_0 = q e_0 e_1$, $e_2 e_0 = q e_0 e_2$, $e_2 e_1 = q e_1 e_2$. Thus

$$e_2 e_1 (e_2 e_0 e_1)^{m-1} e_0 = q e_1 e_2 (q^2 e_0 e_1 e_2)^{m-1} e_0 = q^{4m-1} (e_0 e_1 e_2)^m \quad (m \geq 1).$$

Let $D := (q - q^{-1})^3 e_0 e_1 e_2$. Then by definition of the generating function $\varphi_i(z) \in U_q^+[\![z]\!]$,

$$\begin{aligned} \pi_B^+(T_2 T_1 \varphi_1(z)) &= \sum_{m=1}^{\infty} [m+1]_q D^m z^m = \frac{1}{(1 - qDz)(1 - q^{-1}Dz)} \quad \in \mathcal{P}_B^+[\![z]\!], \\ \pi_B^+(T_2 T_1 \varphi_2(z)) &= Dz. \end{aligned}$$

Thus, the images of imaginary root vectors are

$$\pi_B^+(T_2 T_1 I_{1,m}) = \frac{q^m + q^{-m}}{m(q - q^{-1})} D^m, \tag{5.38}$$

$$\pi_B^+(T_2 T_1 I_{2,m}) = \frac{(-1)^{m-1}}{m(q - q^{-1})} D^m \quad (m \geq 1). \tag{5.39}$$

Finally, we compute the image of $S'_m := (T_2 T_1 \otimes T_2 T_1)(S_m)$ $(m \geq 1)$ in (2.36). By definition,

$$\begin{aligned} \begin{bmatrix} b_{1,1;m} & b_{1,2;m} \\ b_{2,1;m} & b_{2,2;m} \end{bmatrix} &= \frac{1}{m(q^{-1} - q)} \begin{bmatrix} [2m]_q & (-1)^{m-1}[m]_q \\ (-1)^{m-1}[m]_q & [2m]_q \end{bmatrix}, \\ \begin{bmatrix} c_{1,1;m} & c_{1,2;m} \\ c_{2,1;m} & c_{2,2;m} \end{bmatrix} &= \frac{m(q^{-1} - q)}{[2m]_q^2 - [m]_q^2} \begin{bmatrix} [2m]_q & (-1)^m [m]_q \\ (-1)^m [m]_q & [2m]_q \end{bmatrix} \end{aligned}$$

and thus S_m is written down as

$$S_m = \frac{m(q^{-1}-q)}{[2m]_q^2-[m]_q^2}\big\{[2m]_q(I_{1,m}\otimes J_{1,m}+I_{2,m}\otimes J_{2,m}) \\ +(-1)^m[m]_q(I_{1,m}\otimes J_{2,m}+I_{2,m}\otimes J_{1,m})\big\}. \tag{5.40}$$

Let $D' := \overline{\Omega_B}(D) = (q^{-1}-q)^2 f_2 f_1 f_0$. Then

$$\pi_B^-(T_2T_1J_{1,m}) = \frac{q^m+q^{-m}}{m(q^{-1}-q)}D'^m, \tag{5.41}$$

$$\pi_B^-(T_2T_1J_{2,m}) = \frac{(-1)^{m-1}}{m(q^{-1}-q)}D'^m \quad (m\geq 1). \tag{5.42}$$

Therefore the image of S'_m is computed as

$$\begin{aligned}
&(\pi_B^+\otimes\pi_B^-)(S'_m)\\
&= \frac{m(q^{-1}-q)}{[2m]_q^2-[m]_q^2}\Big\{[2m]_q\Big(-\frac{(q^m-q^{-m})^2}{m^2(q-q^{-1})^2}-\frac{1}{m^2(q-q^{-1})^2}\Big)D^m\otimes D'^m\\
&\qquad\qquad +(-1)^m[m]_q\cdot 2\frac{(-1)^m(q^m+q^{-m})}{m^2(q-q^{-1})^2}D^m\otimes D'^m\Big\}\\
&= \frac{m(q^{-1}-q)}{[2m]_q^2-[m]_q^2}\cdot\frac{-[2m]_q(1+(q^m+q^{-m})^2)+2[m]_q(q^m+q^{-m})}{m^2(q-q^{-1})^2}(D\otimes D')^m\\
&= \frac{[2m]_q(q^{2m}+3+q^{-2m})-2[m]_q(q^m+q^{-m})}{([2m]_q^2-[m]_q^2)\cdot m(q-q^{-1})}(D\otimes D')^m\\
&= \frac{1}{m}\frac{(q^{2m}-q^{-2m})(q^{2m}+3+q^{-2m})-2(q^m-q^{-m})(q^m+q^{-m})}{(q^{2m}-q^{-2m})^2-(q^m-q^{-m})^2}(D\otimes D')^m\\
&= \frac{1}{m}\frac{(q^m-q^{-m})(q^m+q^{-m})(q^{2m}+1+q^{-2m})}{(q^m-q^{-m})^2(q^{2m}+1+q^{-2m})}(D\otimes D')^m\\
&= -\frac{1}{m}\frac{q^m(q^m+q^{-m})}{1-q^{2m}}(D\otimes D')^m.
\end{aligned}$$

This result coincides with the case of type $A_1^{(1)}$. Therefore,

$$\begin{aligned}\pi_B^+\,\widehat{\otimes}\,\pi_B^-(\Theta_{\mathrm{im}}) &= \mathbb{E}(-qD\otimes D')^{-1}\mathbb{E}(-q^{-1}D\otimes D')^{-1}\\ &= \mathbb{E}(-q{:}y_0y_1y_2{:})^{-1}\mathbb{E}(-q^{-1}{:}y_0y_1y_2{:})^{-1}.\end{aligned} \tag{5.43}$$

Comparing (5.37), finally we attain the following identity.

Theorem 5.4. *Let y_0, y_1, y_2 be indeterminate with commutation relations $y_0y_1 = q^{-2}y_1y_0$, $y_0y_2 = q^{-2}y_2y_0$, $y_1y_2 = q^{-2}y_2y_1$. Then the following identity holds in the* *skew formal power series algebra* $\widehat{\mathcal{S}_2} \coloneqq \mathbb{Q}(q)[\![y_0, y_1, y_2]\!]$*:*

$$\begin{aligned}\mathbb{E}(:y_2:)\mathbb{E}(:y_1:)\mathbb{E}(:y_0:) = &\left\{ \overrightarrow{\prod_{m\geq 0}} \mathbb{E}(:y_0^{m+1}y_1^m y_2^m:)\mathbb{E}(:y_0^{m+1}y_1^{m+1}y_2^m:)\right\}\mathbb{E}(:y_0y_2:)\\ &\times \mathbb{E}(-q{:}y_0y_1y_2{:})^{-1}\mathbb{E}(-q^{-1}{:}y_0y_1y_2{:})^{-1}\\ &\times \mathbb{E}(:y_1:)\left\{ \overleftarrow{\prod_{m\geq 0}} \mathbb{E}(:y_0^m y_1^{m+1}y_2^{m+1}:)\mathbb{E}(:y_0^m y_1^m y_2^{m+1}:)\right\},\end{aligned} \tag{5.44}$$

where $\overrightarrow{\prod}_{m\geq 0} a_m \coloneqq a_0a_1a_2\dots$, $\overleftarrow{\prod}_{m\geq 0} a_m \coloneqq \dots a_2a_1a_0$, *and the normal ordered product is* $:y_0^{m_0}y_1^{m_1}y_2^{m_2}: = q^{m_0m_1+m_0m_2+m_1m_2}y_0^{m_0}y_1^{m_1}y_2^{m_2}$.

Set $R \coloneqq \left(\begin{smallmatrix}0&1&1\\1&2&1\end{smallmatrix}\right)$. Then the condition of Proposition 4.3 holds for $B' = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. Thus we have a continuous algebra homomorphism $\widehat{\psi_2}\colon \widehat{\mathcal{S}_2} \to \widehat{\mathcal{S}}$ which satisfies

$$\widehat{\psi_2}(y_0) = x_2, \quad \widehat{\psi_2}(y_1) = {:}x_1x_2^2{:}, \quad \widehat{\psi_2}(y_2) = {:}x_1x_2{:}. \tag{5.45}$$

Applying $\widehat{\psi_2}$ on (5.44) and transforming variables using ψ_S, we obtain (1.3). This proves that identity (1.3) holds in $\mathbb{Q}(q)[\![\frac{x_1}{x_2^2}, x_2]\!]$.

Remark. To derive (1.3), the convex order of multiple rows is mandatory because the factor $\mathbf{U}_{1,0}^2$ in the middle cannot appear by only using convex orders of a single row. We also note that convex orders of multiple rows never appear in the form $\leq_Z$ determined by the central charge Z (5.15).

§5.3. Type $A_3^{(1)}$

Let $\mathfrak{g} = \widehat{\mathfrak{sl}_4}$ be an affine algebra of type $A_3^{(1)}$. We set a convex order by

$$\begin{aligned}&w \coloneqq s_2s_1s_3, \quad n \coloneqq 3;\\ &\mathring{I} = J_0 \supsetneqq J_1 \coloneqq \{1,3\} \supsetneqq J_2 \coloneqq \{3\} \supsetneqq J_3 = \emptyset,\\ &\mathbf{s}_0 \coloneqq (s_0s_2s_1s_3s_2s_0s_3s_1)^\infty, \quad \mathbf{s}_1 \coloneqq (s_1s_{\delta-\alpha_1})^\infty, \quad \mathbf{s}_2 \coloneqq (s_3s_{\delta-\alpha_3})^\infty,\\ &\check{w} = ww_\circ = s_1s_3s_2;\\ &\mathring{I} = \check{J}_0 \supsetneqq \check{J}_1 \coloneqq \{1,3\} \supsetneqq \check{J}_2 \coloneqq \{1\} \supsetneqq \check{J}_3 = \emptyset,\\ &\check{\mathbf{s}}_0 \coloneqq (s_1s_3s_2s_0s_3s_1s_0s_2)^\infty, \quad \check{\mathbf{s}}_1 \coloneqq (s_{\delta-\alpha_3}s_3)^\infty, \quad \check{\mathbf{s}}_2 \coloneqq (s_{\delta-\alpha_1}s_1)^\infty.\end{aligned}$$

Then the corresponding convex order $\leq$ is

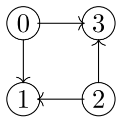
$$\begin{aligned}\delta - \alpha_1 - \alpha_2 - \alpha_3 < \alpha_2 < \delta - \alpha_3 < \delta - \alpha_1&\\ < 2\delta - \alpha_1 - \alpha_2 - \alpha_3 < \delta + \alpha_2 < 2\delta - \alpha_3 < 2\delta - \alpha_1&\\ \cdots&\\ < \alpha_1 + \alpha_2 < \delta + \alpha_1 + \alpha_2 < 2\delta + \alpha_1 + \alpha_2 < 3\delta + \alpha_1 + \alpha_2 < \cdots&\end{aligned}$$

$$
\begin{aligned}
&< \alpha_2+\alpha_3 < \delta+\alpha_2+\alpha_3 < 2\delta+\alpha_2+\alpha_3 < 3\delta+\alpha_2+\alpha_3 < \cdots \\
&< \delta < 2\delta < 3\delta < 4\delta < \cdots \\
&< 3\delta-\alpha_2-\alpha_3 < 2\delta-\alpha_2-\alpha_3 < \delta-\alpha_2-\alpha_3 < \cdots \\
&< 3\delta-\alpha_1-\alpha_2 < 2\delta-\alpha_1-\alpha_2 < \delta-\alpha_1-\alpha_2 \\
&\cdots \\
&< 2\delta-\alpha_2 < \delta+\alpha_1+\alpha_2+\alpha_3 < \delta+\alpha_3 < \delta+\alpha_1 \\
&< \delta-\alpha_2 < \alpha_1+\alpha_2+\alpha_3 < \alpha_3 < \alpha_1,
\end{aligned}
$$

where the null root $\delta=\alpha_0+\alpha_1+\alpha_2+\alpha_3$.

Set

$$
B := \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \left(\text{with corresponding quiver} \right). \tag{5.46}
$$



The matrix presentation of the bilinear form $\langle\cdot,\cdot\rangle_B$ is

$$
(\langle \alpha_i, \alpha_j\rangle_B)_{i,j=0}^3 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -2 & 2 & -2 & 0 \\ 0 & 0 & 2 & 0 \\ -2 & 0 & -2 & 2 \end{pmatrix}.
$$

In the same way as in the examples so far, one can verify that all the root vectors except for simple root vectors vanish by the projection π_B^+. On the other hand, real root vectors for the reversed order $\leq'$, namely, $E_{\leq',m\delta+\alpha}$ for $\alpha = \pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm(\alpha_1+\alpha_2+\alpha_3)$ ($m \geq 0$ if $\alpha \in \Delta_+$, $m \geq 1$ if $\alpha \in \Delta_-$) do not vanish by π_B^+ and satisfy the condition of Proposition 4.7. The real root vectors in the second and third rows $E_{\leq',m\delta+\alpha_1+\alpha_2}$, $E_{\leq',m\delta+\alpha_2+\alpha_3}$, $E_{\leq',(m+1)\delta-\alpha_1-\alpha_2}$, $E_{\leq',(m+1)\delta-\alpha_2-\alpha_3}$ vanish if and only if $m > 0$, and satisfy the condition of Proposition 4.7 when $m = 0$. This behavior in the second and third rows resembles that of the case of type $A_2^{(1)}$.

Although the computation of imaginary root vectors also has resemblance to the previous examples and in fact we will obtain an identical presentation of $\pi_B^+ \mathbin{\widehat{\otimes}} \pi_B^-(\Theta_{\mathrm{im}})$, the process of computation is far from obvious. After a somewhat lengthy computation (we used $T_{\varepsilon_2}^{-1}(E_i) = E_i$ for $i = 1, 3$ in the process), one will obtain

$$
\begin{aligned}
T_{\breve{w}}(\varphi_{1,m}) &= [(T_1T_\rho T_3T_2)^{m-1}T_1(E_0), [E_3, E_2]_q]_q, \\
T_{\breve{w}}(\varphi_{2,m}) &= [(T_1T_3T_2T_{\rho^2}T_2)^{m-1}T_3T_1(E_2), E_0]_q,
\end{aligned}
$$

$$T_{\breve{w}}(\varphi_{3,m}) = [(T_3T_{\rho^3}T_1T_2)^{m-1}T_3(E_0), [E_1, E_2]_q]_q \quad (m \geq 1),$$

$$\pi_B^+T_{\breve{w}}(\varphi_{1,m}) = \pi_B^+T_{\breve{w}}(\varphi_{3,m}) = \begin{cases} (q-q^{-1})e_0e_2e_1e_3, & m=1, \\ 0, & m>1, \end{cases}$$

$$\pi_B^+T_{\breve{w}}(\varphi_{2,m}) = (q-q^{-1})^{4m-1}[m+1]_q(e_0e_2e_1e_3)^m \quad (m \geq 1).$$

Thus the images of the imaginary root vectors are

$$\pi_B^+T_{\breve{w}}(I_{1,m}) = \pi_B^+T_{\breve{w}}(I_{3,m}) = \frac{(-1)^{m-1}}{m(q-q^{-1})}D^m,$$

$$\pi_B^+T_{\breve{w}}(I_{2,m}) = \frac{q^m+q^{-m}}{m(q-q^{-1})}D^m \quad (m \geq 1),$$

where $D := (q-q^{-1})^4e_0e_2e_1e_3$.

Our last task is to compute the image of $S'_m := (T_{\breve{w}} \otimes T_{\breve{w}})(S_m)$ $(m \geq 1)$. Definition (2.35) reads

$$(b_{i,j;m})_{i,j=1}^3 = \frac{1}{m(q^{-1}-q)}\begin{bmatrix} M_2 & (-1)^{m-1}M_1 & 0 \\ (-1)^{m-1}M_1 & M_2 & (-1)^{m-1}M_1 \\ 0 & (-1)^{m-1}M_1 & M_2 \end{bmatrix},$$

$$(c_{i,j;m})_{i,j=1}^3 = \frac{m(q^{-1}-q)}{M_2^3-2M_1^2M_2}\begin{bmatrix} M_2^2-M_1^2 & (-1)^mM_1M_2 & M_1^2 \\ (-1)^mM_1M_2 & M_2^2 & (-1)^mM_1M_2 \\ M_1^2 & (-1)^mM_1M_2 & M_2^2-M_1^2 \end{bmatrix},$$

where $M_k := [km]_q$ for $k=1,2$. Thus the image of S'_m is computed as

$$\begin{aligned}(\pi_B^+ \otimes \pi_B^-)(S'_m) &= \frac{m(q^{-1}-q)}{[2m]_q^3 - 2[2m]_q[m]_q^2} \\ &\quad \times \Bigg(2([2m]_q^2-[m]_q^2)\frac{1}{m(q-q^{-1})m(q^{-1}-q)} \\ &\qquad + 4(-1)^m[2m]_q[m]_q\frac{(-1)^{m-1}(q^m+q^{-m})}{m(q-q^{-1})m(q^{-1}-q)} \\ &\qquad + 2[m]_q^2\frac{1}{m(q-q^{-1})m(q^{-1}-q)} \\ &\qquad + [2m]_q^2\frac{(q^m+q^{-m})^2}{m(q-q^{-1})m(q^{-1}-q)}\Bigg) \\ &\quad \times D^m \otimes D'^m \\ &= -\frac{1}{m}\frac{q^m(q^m+q^{-m})}{1-q^{2m}}(D \otimes D')^m,\end{aligned}$$

where $D' := \overline{\Omega_B}(D)$. This is identical to the previous examples and we conclude that $\pi_B^+ \widehat{\otimes} \pi_B^-(\Theta_{\mathrm{im}}) = \mathbb{E}(-qD \otimes D')^{-1}\mathbb{E}(-q^{-1}D \otimes D')^{-1}$. As a result, we have the following theorem:

Theorem 5.5. *The following identity holds in the skew formal power series algebra* $\widehat{\mathcal{S}_3} := \widehat{\mathcal{S}_B} \cong \mathbb{Q}(q)[\![y_0, y_1, y_2, y_3]\!]$*:*

$$\begin{aligned} \mathbb{E}(:y_1:)\mathbb{E}(:y_3:)\mathbb{E}(:y_2:)\mathbb{E}(:y_0:) = {} & \Big\{ \overrightarrow{\prod_{m\geq 0}} X_m \Big\} \mathbb{E}(:y_1y_2:)\mathbb{E}(:y_2y_3:) \\ & \times \mathbb{E}(-q:y_0y_1y_2y_3:)^{-1}\mathbb{E}(-q^{-1}:y_0y_1y_2y_3:)^{-1} \\ & \times \mathbb{E}(:y_0y_1:)\mathbb{E}(:y_0y_3:)\Big\{ \overleftarrow{\prod_{m\geq 0}} Y_m \Big\}, \end{aligned} \tag{5.47}$$

where

$$\begin{aligned} X_m &= \mathbb{E}(:y_0^{m+1}y_1^m y_2^m y_3^m:)\mathbb{E}(:y_0^m y_1^m y_2^{m+1} y_3^m:) \\ &\quad \times \mathbb{E}(:y_0^{m+1}y_1^{m+1}y_2^{m+1}y_3^m:)\mathbb{E}(:y_0^{m+1}y_1^m y_2^{m+1}y_3^{m+1}:), \\ Y_m &= \mathbb{E}(:y_0^{m+1}y_1^{m+1}y_2^m y_3^{m+1}:)\mathbb{E}(:y_0^m y_1^{m+1}y_2^{m+1}y_3^{m+1}:) \\ &\quad \times \mathbb{E}(:y_0^m y_1^m y_2^m y_3^{m+1}:)\mathbb{E}(:y_0^m y_1^{m+1} y_2^m y_3^m:). \end{aligned}$$

Set $R := \left(\begin{smallmatrix} 0&1&0&1 \\ 1&1&1&1 \end{smallmatrix}\right)$. Then the condition of Proposition 4.3 holds for the same B'. Thus we have a continuous algebra homomorphism $\widehat{\psi_3}\colon \widehat{\mathcal{S}_3} \to \widehat{\mathcal{S}}$ which satisfies

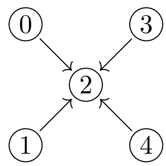
$$\widehat{\psi_3}(y_0) = \widehat{\psi_3}(y_2) = x_2, \quad \widehat{\psi_3}(y_1) = \widehat{\psi_3}(y_3) = :x_1x_2:. \tag{5.48}$$

Applying $\widehat{\psi_3}$ on (5.47) and transforming x_1, x_2 by ψ_S yields (1.4). This proves the identity (1.4).

§5.4. Type $D_4^{(1)}$

Let $\mathfrak{g} = \widehat{\mathfrak{so}_8}$ be an affine algebra of type $D_4^{(1)}$. Let

$$B := \begin{pmatrix} 0&0&-1&0&0 \\ 0&0&-1&0&0 \\ 1&1&0&1&1 \\ 0&0&-1&0&0 \\ 0&0&-1&0&0 \end{pmatrix} \quad \left(\text{with corresponding quiver}\right). \tag{5.49}$$



Using the assignment of indices in the Dynkin quiver above, we set a convex order by

$$\begin{aligned} &w := s_1s_3s_4s_2s_1s_3s_4, \quad n := 4; \\ &\mathring{I} = J_0 \supsetneq J_1 := \{1,3,4\} \supsetneq J_2 := \{3,4\} \supsetneq J_3 := \{4\} \supsetneq J_4 = \emptyset, \end{aligned}$$

$$\mathbf{s}_0 := (s_0 s_1 s_3 s_4 s_2)^\infty, \quad \mathbf{s}_1 := (s_1 s_{\delta-\alpha_1})^\infty,$$
$$\mathbf{s}_2 := (s_3 s_{\delta-\alpha_3})^\infty, \quad \mathbf{s}_3 := (s_4 s_{\delta-\alpha_4})^\infty,$$

$$\check{w} = w w_\circ = s_2 s_1 s_3 s_4 s_2;$$
$$\mathring{I} = \check{J}_0 \supsetneqq \check{J}_1 := \{1,3,4\} \supsetneqq \check{J}_2 := \{3,4\} \supsetneqq \check{J}_3 := \{3\} \supsetneqq \check{J}_4 = \emptyset,$$
$$\check{\mathbf{s}}_0 := (s_2 s_1 s_3 s_4 s_0)^\infty, \quad \check{\mathbf{s}}_1 := (s_{\delta-\alpha_1} s_1)^\infty,$$
$$\check{\mathbf{s}}_2 := (s_{\delta-\alpha_3} s_3)^\infty, \quad \check{\mathbf{s}}_3 := (s_{\delta-\alpha_4} s_4)^\infty.$$

Then the corresponding convex order $\leq$ is

$$\begin{aligned}
&\delta - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 < \alpha_1 < \alpha_3 < \alpha_4 \\
&\quad < \delta - \alpha_2 < \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < \delta - \alpha_1 - \alpha_2 \\
&\quad < \delta - \alpha_2 - \alpha_3 < \delta - \alpha_2 - \alpha_4 < 2\delta - \alpha_2 \\
&\quad < 2\delta - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 < \delta + \alpha_1 < \delta + \alpha_3 < \delta + \alpha_4 \\
&\quad < 3\delta - \alpha_2 < \delta + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 2\delta - \alpha_1 - \alpha_2 \\
&\quad < 2\delta - \alpha_2 - \alpha_3 < 2\delta - \alpha_2 - \alpha_4 < 4\delta - \alpha_2 \\
&\quad \cdots \\
&\quad < \alpha_2 + \alpha_3 + \alpha_4 < \delta + \alpha_2 + \alpha_3 + \alpha_4 < 2\delta + \alpha_2 + \alpha_3 + \alpha_4 < \cdots \\
&\quad < \alpha_1 + \alpha_2 + \alpha_4 < \delta + \alpha_1 + \alpha_2 + \alpha_4 < 2\delta + \alpha_1 + \alpha_2 + \alpha_4 < \cdots \\
&\quad < \alpha_1 + \alpha_2 + \alpha_3 < \delta + \alpha_1 + \alpha_2 + \alpha_3 < 2\delta + \alpha_1 + \alpha_2 + \alpha_3 < \cdots \\
&\quad < \delta < 2\delta < 3\delta < 4\delta < \cdots \\
&\quad < 3\delta - \alpha_1 - \alpha_2 - \alpha_3 < 2\delta - \alpha_1 - \alpha_2 - \alpha_3 < \delta - \alpha_1 - \alpha_2 - \alpha_3 < \cdots \\
&\quad < 3\delta - \alpha_1 - \alpha_2 - \alpha_4 < 2\delta - \alpha_1 - \alpha_2 - \alpha_4 < \delta - \alpha_1 - \alpha_2 - \alpha_4 < \cdots \\
&\quad < 3\delta - \alpha_2 - \alpha_3 - \alpha_4 < 2\delta - \alpha_2 - \alpha_3 - \alpha_4 < \delta - \alpha_2 - \alpha_3 - \alpha_4 \\
&\quad \cdots \\
&\quad < \delta + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 < 2\delta - \alpha_4 < 2\delta - \alpha_3 < 2\delta - \alpha_1 \\
&\quad < 3\delta + \alpha_2 < 2\delta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 < \delta + \alpha_2 + \alpha_4 \\
&\quad < \delta + \alpha_2 + \alpha_3 < \delta + \alpha_1 + \alpha_2 < 2\delta + \alpha_2 \\
&\quad < \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 < \delta - \alpha_4 < \delta - \alpha_3 < \delta - \alpha_1 \\
&\quad < \delta + \alpha_2 < \delta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 < \alpha_2 + \alpha_4 \\
&\quad < \alpha_2 + \alpha_3 < \alpha_1 + \alpha_2 < \alpha_2,
\end{aligned}$$

where the null root $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$.

To compute root vectors, reduced expressions of the fundamental translations $t_{\varepsilon_i} \in \widehat{W}$ $(i = 1,2,3,4)$ are required. Let $\tau := \left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 4 & 3 \end{smallmatrix}\right)$, $\tau' := \left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 0 & 1 \end{smallmatrix}\right)$ be Dynkin automorphisms. In the same way as the proof of Proposition 5.3, one can verify that

$$t_{\varepsilon_1} = \tau s_1 s_2 s_3 s_4 s_2 s_1, \quad t_{\varepsilon_2} = s_0 s_2 s_3 s_4 s_2 s_1 s_2 s_3 s_4 s_2,$$
$$t_{\varepsilon_3} = \tau' s_3 s_2 s_1 s_4 s_2 s_3, \quad t_{\varepsilon_4} = \tau\tau' s_4 s_2 s_1 s_3 s_2 s_4 \tag{5.50}$$

are reduced expressions in $\widehat{W}$. Using these formulas, one can compute all the root vectors and verify that all of them except for simple root vectors vanish by π_B^+. For the reversed order $\leq'$, real root vectors in the first row satisfy the condition of Proposition 4.7. On the other hand, real root vectors in the second, third, and fourth rows, namely,

$$E_{\leq', m\delta+\alpha_1+\alpha_2+\alpha_3}, \quad E_{\leq', m\delta+\alpha_1+\alpha_2+\alpha_4}, \quad E_{\leq', m\delta+\alpha_2+\alpha_3+\alpha_4},$$
$$E_{\leq', (m+1)\delta-\alpha_1-\alpha_2-\alpha_3}, \quad E_{\leq', (m+1)\delta-\alpha_1-\alpha_2-\alpha_4}, \quad E_{\leq', (m+1)\delta-\alpha_2-\alpha_3-\alpha_4}$$

vanish for $m > 0$, and satisfy the condition of Proposition 4.7 when $m = 0$.

The images of $T_{\breve{w}}(\varphi_{i,m})$ are computed as

$$\begin{aligned} \pi_B^+ T_{\breve{w}}(\varphi_{1,m}) = \pi_B^+ T_{\breve{w}}(\varphi_{3,m}) &= \pi_B^+ T_{\breve{w}}(\varphi_{4,m}) \\ &= \begin{cases} q(q-q^{-1})^5 e_0 e_1 e_3 e_4 e_2^2, & m = 1, \\ 0, & m > 1, \end{cases} \\ \pi_B^+ T_{\breve{w}}(\varphi_{2,m}) &= q^m (q-q^{-1})^{6m-1} [m+1]_q (e_0 e_1 e_3 e_4 e_2^2)^m. \end{aligned} \tag{5.51}$$

Note that $e_i e_j = e_j e_i$ and $e_2 e_i = q e_i e_2$ in $\mathcal{P}_B^+$ for $i, j = 0, 1, 3, 4$. Thus,

$$\begin{aligned} \pi_B^+ T_{\breve{w}}(I_{i,m}) &= \frac{(-1)^{m-1}}{m(q-q^{-1})} (qD)^m \quad (i = 0, 1, 3, 4), \\ \pi_B^+ T_{\breve{w}}(I_{2,m}) &= \frac{q^m + q^{-m}}{m(q-q^{-1})} (qD)^m \quad (m \geq 1), \end{aligned} \tag{5.52}$$

where $D \coloneqq (q-q^{-1})^6 e_0 e_1 e_3 e_4 e_2^2$ and $D' \coloneqq \overline{\Omega_B}(D)$. Definition (2.35) reads

$$\begin{aligned} b_{i,j;m} &= \frac{1}{m(q^{-1}-q)} \times \begin{cases} s, & i = j, \\ t, & i \neq j;\ 2 \in \{i, j\}, \\ 0, & i \neq j;\ i, j \neq 2, \end{cases} \\ c_{i,j;m} &= \frac{m(q^{-1}-q)}{s^2(s^2-3t^2)} \times \begin{cases} s^3, & i = j = 2, \\ s(s^2-2t^2), & i = j \neq 2, \\ -s^2 t, & i \neq j;\ 2 \in \{i, j\}, \\ s t^2, & i \neq j;\ i, j \neq 2, \end{cases} \end{aligned} \tag{5.53}$$

where $s \coloneqq [2m]_q$ and $t \coloneqq (-1)^{m-1}[m]_q$. Finally, the image of $S'_m \coloneqq (T_{\breve{w}} \otimes T_{\breve{w}})(S_m)$ is computed as

$$\begin{aligned}(\pi_B^+ \otimes \pi_B^-)(S'_m) &= \frac{m(q^{-1}-q)}{s^2(s^2-3t^2)} \frac{1}{m(q-q^{-1})m(q^{-1}-q)} \\ &\quad \times \big[\{3 \cdot s(s^2-2t^2) + 6 \cdot st^2\} \cdot (-1)^{m-1} \cdot (-1)^{m-1} \\ &\qquad + 6 \cdot (-s^2 t) \cdot (-1)^{m-1} \cdot (q^m + q^{-m}) \\ &\qquad + (q^m + q^{-m})^2 s^3\big] \\ &\quad \times (qD)^m \otimes (q^{-1}D')^m \\ &= \frac{1}{m(q-q^{-1})s^2(s^2-3t^2)} \cdot (q^{2m} - 1 + q^{-2m}) s^3 \cdot D^m \otimes D'^m \\ &= -\frac{1}{m} \frac{q^m(q^m+q^{-m})}{1-q^{2m}} (D \otimes D')^m.\end{aligned}$$

Therefore, surprisingly, the image of S'_m is identical to that in the examples of type $A_\ell^{(1)}$, and we have $\pi_B^+ \mathbin{\widehat{\otimes}} \pi_B^-(\Theta_{\mathrm{im}}) = \mathbb{E}(-qD \otimes D')^{-1}\mathbb{E}(-q^{-1}D \otimes D')^{-1}$. As a result, we have the following theorem:

Theorem 5.6. *The following identity holds in the skew formal power series algebra* $\widehat{\mathcal{S}_4} \coloneqq \widehat{\mathcal{S}_B} \cong \mathbb{Q}(q)[\![y_0, y_1, y_2, y_3, y_4]\!]$*:*

$$\begin{aligned}&\mathbb{E}({:}y_2{:})\mathbb{E}({:}y_4{:})\mathbb{E}({:}y_3{:})\mathbb{E}({:}y_1{:})\mathbb{E}({:}y_0{:}) \\ &\quad = \Big\{ \prod_{m \geq 0}^{\rightarrow} X_m \Big\} \mathbb{E}({:}y_2y_3y_4{:})\mathbb{E}({:}y_1y_2y_4{:})\mathbb{E}({:}y_1y_2y_3{:}) \\ &\qquad \times \mathbb{E}(-q{:}y_0y_1y_2^2y_3y_4{:})^{-1}\mathbb{E}(-q^{-1}{:}y_0y_1y_2^2y_3y_4{:})^{-1} \\ &\qquad \times \mathbb{E}({:}y_0y_2y_4{:})\mathbb{E}({:}y_0y_2y_3{:})\mathbb{E}({:}y_0y_1y_2{:}) \Big\{ \prod_{m \geq 0}^{\leftarrow} Y_m \Big\},\end{aligned} \tag{5.54}$$

where

$$\begin{aligned}X_m &= \mathbb{E}({:}y_0^{m+1}y_1^m y_2^{2m} y_3^m y_4^m{:})\mathbb{E}({:}y_0^m y_1^{m+1} y_2^{2m} y_3^m y_4^m{:}) \\ &\quad \times \mathbb{E}({:}y_0^m y_1^m y_2^{2m} y_3^{m+1} y_4^m{:})\mathbb{E}({:}y_0^m y_1^m y_2^{2m} y_3^m y_4^{m+1}{:}) \\ &\quad \times \mathbb{E}({:}y_0^{2m+1} y_1^{2m+1} y_2^{4m+1} y_3^{2m+1} y_4^{2m+1}{:})\mathbb{E}({:}y_0^m y_1^{m+1} y_2^{2m+1} y_3^{m+1} y_4^{m+1}{:}) \\ &\quad \times \mathbb{E}({:}y_0^{m+1} y_1^m y_2^{2m+1} y_3^{m+1} y_4^{m+1}{:})\mathbb{E}({:}y_0^{m+1} y_1^{m+1} y_2^{2m+1} y_3^m y_4^{m+1}{:}) \\ &\quad \times \mathbb{E}({:}y_0^{m+1} y_1^{m+1} y_2^{2m+1} y_3^{m+1} y_4^m{:})\mathbb{E}({:}y_0^{2m+2} y_1^{2m+2} y_2^{4m+3} y_3^{2m+2} y_4^{2m+2}{:}),\end{aligned}$$

$$\begin{aligned} Y_m = {} & \mathbb{E}(:y_0^m y_1^{m+1} y_2^{2m+2} y_3^{m+1} y_4^{m+1}:)\mathbb{E}(:y_0^{m+1} y_1^{m+1} y_2^{2m+2} y_3^{m+1} y_4^m:) \\ & \times \mathbb{E}(:y_0^{m+1} y_1^{m+1} y_2^{2m+2} y_3^m y_4^{m+1}:)\mathbb{E}(:y_0^m y_1^{m+1} y_2^{2m+2} y_3^{m+1} y_4^{m+1}:) \\ & \times \mathbb{E}(:y_0^{2m+1} y_1^{2m+1} y_2^{4m+3} y_3^{2m+1} y_4^{2m+1}:)\mathbb{E}(:y_0^{m+1} y_1^m y_2^{2m+1} y_3^m y_4^m:) \\ & \times \mathbb{E}(:y_0^m y_1^m y_2^{2m+1} y_3^m y_4^{m+1}:)\mathbb{E}(:y_0^m y_1^m y_2^{2m+1} y_3^{m+1} y_4^m:) \\ & \times \mathbb{E}(:y_0^m y_1^{m+1} y_2^{2m+1} y_3^m y_4^m:)\mathbb{E}(:y_0^{2m} y_1^{2m} y_2^{4m+1} y_3^{2m} y_4^{2m}:). \end{aligned}$$

Set $R := \left(\begin{smallmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 \end{smallmatrix}\right)$. Then the condition of Proposition 4.3 holds. Thus we have the continuous algebra homomorphism $\widehat{\psi_4}\colon \widehat{\mathcal{S}_4} \to \widehat{\mathcal{S}}$ which satisfies

$$\widehat{\psi_4}(y_0) = \widehat{\psi_4}(y_1) = \widehat{\psi_4}(y_3) = \widehat{\psi_4}(y_4) = x_2, \quad \widehat{\psi_4}(y_2) = {:}x_1 x_2^2{:}. \tag{5.55}$$

Applying ψ_4 on (5.54) and transforming variables using ψ_S yields (1.5). This proves the identity (1.5).

Main Theorem 5.7. *The four identities* (1.2), (1.3), (1.4), (1.5) *hold in the skew formal power series algebra* $\mathbb{Q}(q)[\![\frac{x_1}{x_2^2}, x_2]\!]$.

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