The Monodromy Map from Differential Systems to the Character Variety Is Generically Immersive

by

Indranil BISWAS and Sorin DUMITRESCU

Abstract

Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$ and $\mathfrak{g}$ its Lie algebra. We consider all pairs of the form (Y, D), where Y is a complex structure on a compact oriented C^∞ surface Σ, and D is a holomorphic connection on the trivial holomorphic principal G-bundle $Y \times G$ on Y; these are known as $\mathfrak{g}$-differential systems. We study the monodromy map from the space of $\mathfrak{g}$-differential systems to the character variety of G-representations of the fundamental group of Σ. If the complex dimension of G is at least three, and genus(Σ) ≥ 2, we show that the monodromy map is an immersion at the generic point.

§1. Introduction

Our aim here is to study the monodromy map, also called the Riemann–Hilbert map, from the differential systems over compact Riemann surfaces to the character varieties. In order to describe the framework, let us denote by Σ a given compact connected oriented C^∞ surface of genus $g \geq 2$ and by G a connected reductive affine algebraic group defined over $\mathbb{C}$.

Consider a complex structure X on Σ (it gives an element in the Teichmüller space for Σ) and a holomorphic (flat) connection ϕ on the trivial holomorphic principal G-bundle $X \times G$ over X. Recall that ϕ is determined by an element $\delta \in H^0(X, K_X) \otimes \mathfrak{g}$, where $\mathfrak{g}$ is the (complex) Lie algebra of G and K_X is the

I. Biswas: Department of Mathematics, Shiv Nadar University, NH91, Tehsil Dadri, Greater Noida, Uttar Pradesh 201314, India;
e-mail: `indranil.biswas@snu.edu.in`, `indranil29@gmail.com`
S. Dumitrescu: Université Côte d'Azur, CNRS, LJAD, France;
e-mail: `dumitres@unice.fr`

canonical line bundle of X. Fixing a base point $x_0 \in X$, consider the corresponding universal cover $\pi \colon \widetilde{X} \to X$ of X, and endow the trivial principal G-bundle $\widetilde{X} \times G$ over $\widetilde{X}$ with the pulled back holomorphic flat connection $\pi^* \phi$.

For any locally defined parallel section s of $X \times G$ for the connection ϕ, the pulled back local section $\pi^* s$ of $\widetilde{X} \times G$ extends to a $\pi^* \phi$-parallel section over the entire $\widetilde{X}$. This extension of $\pi^* s$ produces a holomorphic map $\widetilde{X} \to G$ which is $\pi_1(X, x_0)$-equivariant with respect to the natural action of $\pi_1(X, x_0)$ on $\widetilde{X}$ through deck transformations and the action of $\pi_1(X, x_0)$ on G through a group homomorphism $\pi_1(X, x_0) \to G$; this homomorphism $\pi_1(X, x_0) \to G$ is known as the *monodromy* of the flat connection ϕ.

Although the above-mentioned monodromy homomorphism depends on the choice of the holomorphic trivialization of the principal G-bundle, the element of the character variety of G-representations

$$\Xi := \operatorname{Hom}(\pi_1(X), G) /\!\!/ G$$

given by it is independent of both the trivialization of the principal G-bundle and the base point x_0 (and also of the choice of s). It should be clarified that the monodromy map is defined from the isomorphism classes of flat principal G-bundles on X to Ξ; to define this map, the underlying holomorphic principal G-bundle does not need to be trivial.

Recall that Ξ is a (singular) complex analytic space of dimension

$$2((g-1) \cdot \dim[G, G] + g \cdot (\dim G - \dim[G, G]));$$

see, for example, [Go], [Si, Prop. 49]. We will denote the dimension of the commutator group $\dim[G, G]$ by d and $\dim G - \dim[G, G]$ by c (it is the dimension of the center of G). With this notation, the complex dimension of Ξ is $2(g-1)d + 2gc$.

Let us adopt the notation of [CDHL] and denote by Syst the space of all pairs (X, ϕ), where X is an element of the Teichmüller space for Σ and ϕ is a holomorphic connection on the trivial principal G-bundle $X \times G$ over X; recall that a holomorphic connection on a holomorphic bundle over a Riemann surface is automatically flat. This space of all differential systems on Σ is a complex space of dimension $(g-1)(d+3) + gc$.

Sending a holomorphic connection to its monodromy representation, a holomorphic mapping

$$\widetilde{\operatorname{Mon}} \colon \operatorname{Syst} \longrightarrow \Xi$$

is obtained; in other words, $\widetilde{\operatorname{Mon}}$ is the restriction to Syst of the Riemann–Hilbert map.

Let us define the nonempty Zariski open subset $\mathrm{Syst}^{\mathrm{irred}}$ of Syst consisting of all pairs (X, ϕ) for which the connection ϕ is *irreducible*, meaning the monodromy homomorphism for ϕ does not factor through any proper parabolic subgroup of G. It is a connected complex orbifold of dimension $(g-1)(d+3)+gc$ (see Lemma 3.1). The image of $\mathrm{Syst}^{\mathrm{irred}}$ under the above map $\widetilde{\mathrm{Mon}}$, which sends a holomorphic connection to the corresponding monodromy representation, lies in the Zariski open subset

$$\Xi^{\mathrm{irred}} \subset \Xi$$

defined by the irreducible homomorphisms $\pi_1(X) \to G$ (i.e., the homomorphisms that do not factor through some proper parabolic subgroup of G). Although the complex space Ξ^{irred} is not smooth in general, its singularities are finite group quotients. Let

(1.1)
$$\mathrm{Mon}\colon \mathrm{Syst}^{\mathrm{irred}} \longrightarrow \Xi^{\mathrm{irred}}$$

be the holomorphic map between complex manifolds given by the restriction of $\widetilde{\mathrm{Mon}}$ to $\mathrm{Syst}^{\mathrm{irred}} \subset \mathrm{Syst}$.

The main result proved here is the following (see Theorem 5.3):

Theorem 1.1. *If the complex dimension of G is at least three, the monodromy map*

$$\mathrm{Mon}\colon \mathrm{Syst}^{\mathrm{irred}} \longrightarrow \Xi^{\mathrm{irred}}$$

in (1.1) *is an immersion at the generic point.*

Remark 5.4 explains that the assumption $\dim G \geq 3$ in Theorem 1.1 is necessary.

If $G = \mathrm{SL}(2, \mathbb{C})$, then the dimensions of $\mathrm{Syst}^{\mathrm{irred}}$ and Ξ^{irred} are both $6g-6$, and Theorem 1.1 implies that Mon in (1.1) is a local biholomorphism at the generic point (see Corollary 5.8). It should be mentioned that examples constructed in [CDHL] show that for $G = \mathrm{SL}(2, \mathbb{C})$ and Σ of genus $g \geq 3$, the monodromy map Mon is not always a local biholomorphism (over the entire $\mathrm{Syst}^{\mathrm{irred}}$).

When $G = \mathrm{SL}(2, \mathbb{C})$ and $g = 2$, the main result of [CDHL] says that the map Mon in (1.1) is a local biholomorphism over the entire $\mathrm{Syst}^{\mathrm{irred}}$. An alternative proof of this result of [CDHL] is given in Corollary 5.6. In this context it should be mentioned that our work was greatly influenced by [CDHL].

Just as for the authors of [CDHL], our main motivation came from a question of E. Ghys for $G = \mathrm{SL}(2, \mathbb{C})$, relating the monodromy of $\mathfrak{sl}(2, \mathbb{C})$-differential systems to the existence of holomorphic curves of genus $g > 1$ lying in compact

quotients of $\mathrm{SL}(2, \mathbb{C})$ by lattices Γ. Such compact quotients of $\mathrm{SL}(2, \mathbb{C})$ are non-Kähler manifolds. These non-Kähler manifolds do not admit any closed complex hypersurface [HM]. It is known that elliptic curves do exist in some of those manifolds, but the existence of holomorphic curves of genus $g > 1$ is still an open question. Ghys realized that constructing an irreducible $\mathfrak{sl}(2, \mathbb{C})$-differential system on a Riemann surface X with monodromy lying inside a cocompact lattice $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ would provide a nontrivial holomorphic map from X into the quotient of $\mathrm{SL}(2, \mathbb{C})/\Gamma$ (in fact the two problems are equivalent). While the question asked by Ghys is still open, the above Theorem 1.1 extends the results of [CDHL] and leads to an enhancement of the understanding of the monodromy of the differential systems.

The strategy of the proof of Theorem 1.1 and the organization of the paper are as follows. We consider the monodromy map, to the character variety, defined on the space of triples (X, E_G, ϕ), with X an element of the Teichmüller space for Σ, E_G a holomorphic principal G-bundle over X and ϕ a holomorphic connection on E_G. In Section 2 we define a two-term complex $\mathcal{C}_\bullet$ over X whose first hypercohomology gives the infinitesimal deformations of (X, E_G, ϕ) (see Theorem 2.2(2)). Moreover, the kernel of the differential of the monodromy map coincides with the image of the space of deformations of the complex structure $H^1(X, TX)$ through a certain homomorphism β_ϕ from $H^1(X, TX)$ to the 1-hypercohomology of $\mathcal{C}_\bullet$ (see Theorem 2.2(4)). In Section 3 we fix E_G to be the holomorphically trivial principal G-bundle over X and set ϕ to be an irreducible holomorphic connection on it. We show that the tangent space of $\mathrm{Syst}^{\mathrm{irred}}$ at (X, ϕ), which is naturally embedded in $\mathbb{H}^1(X, \mathcal{C}_\bullet)$, is transverse to the kernel of the monodromy map, provided ϕ satisfies a geometric criterion described in Proposition 3.2. In Section 4 we consider the special case of $G = \mathrm{SL}(2, \mathbb{C})$ and we prove that the criterion in Proposition 3.2 is satisfied at any point $(X, \phi) \in \mathrm{Syst}^{\mathrm{irred}}$ for surfaces Σ of genus two (see Proposition 4.1); the same holds for the generic point in $\mathrm{Syst}^{\mathrm{irred}}$ for surfaces Σ of genus three (see Lemma 4.2). The main result (Theorem 1.1) is obtained in Section 5, where Lemma 5.1 proves that the transversality criterion is satisfied at the generic point in $\mathrm{Syst}^{\mathrm{irred}}$. More precisely, the proof of Lemma 5.1 shows that the transversality criterion (in Proposition 3.2) is implied by the statement that for a nonhyperelliptic Riemann surface X, and a generic three-dimensional subspace $W \subset H^0(X, K_X)$, the natural homomorphism

$$\Theta_W \colon H^0(X, K_X) \otimes W \longrightarrow H^0(X, K_X^2)$$

is surjective. The above statement is precisely [Gi, p. 221, Thm. 1.1], where the proof of it is attributed to R. Lazarsfeld.

§2. Infinitesimal deformations of bundles and connections

In this section we introduce several infinitesimal deformation spaces and natural morphisms between them.

The holomorphic tangent bundle of a complex manifold Y will be denoted by TY.

Let X be a compact connected Riemann surface. The holomorphic cotangent bundle of X will be denoted by K_X. Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$. The Lie algebra of G will be denoted by $\mathfrak{g}$.

Take a holomorphic principal G-bundle over X,

$$(2.1) \qquad\qquad p \colon E_G \longrightarrow X.$$

So E_G is equipped with a holomorphic action of G on the right, which is both free and transitive on the fibers of p and, furthermore, $E_G/G = X$. Consider the holomorphic right action of G on the holomorphic tangent bundle TE_G given by the action of G on E_G. The quotient

$$\mathrm{At}(E_G) \coloneqq (TE_G)/G$$

is a holomorphic vector bundle over $E_G/G = X$; it is called the *Atiyah bundle* for E_G. The differential

$$dp \colon TE_G \longrightarrow p^* TX$$

of the projection p in (2.1) is G-equivariant for the trivial action of G on the fibers of $p^* TX$. The action of G on E_G produces a holomorphic homomorphism from the trivial holomorphic bundle

$$E_G \times \mathfrak{g} \longrightarrow \mathrm{kernel}(dp)$$

which is an isomorphism. Therefore, we have a short exact sequence of holomorphic vector bundles on E_G,

$$(2.2) \qquad 0 \longrightarrow \mathrm{kernel}(dp) = E_G \times \mathfrak{g} \longrightarrow \mathrm{At}(E_G) \xrightarrow{dp} p^* TX \longrightarrow 0,$$

in which all the homomorphisms are G-equivariant. The quotient $\mathrm{kernel}(dp)/G$ is the adjoint vector bundle $\mathrm{ad}(E_G) = E_G(\mathfrak{g})$, which is the holomorphic vector bundle over X associated to E_G for the adjoint action of G on $\mathfrak{g}$. Taking the quotient of the bundles in (2.2), by the actions of G, the following short exact sequence of holomorphic vector bundles on X is obtained:

$$(2.3) \qquad\qquad 0 \longrightarrow \mathrm{ad}(E_G) \xrightarrow{\iota} \mathrm{At}(E_G) \xrightarrow{d'p} TX \longrightarrow 0$$

[At]; it is known as the Atiyah exact sequence for E_G.

A holomorphic connection on E_G is a holomorphic homomorphism of vector bundles

$$\phi \colon TX \longrightarrow \mathrm{At}(E_G)$$

such that

(2.4) $(d'p) \circ \phi = \mathrm{Id}_{TX},$

where $d'p$ is the projection in (2.3) (see [At]). A holomorphic connection on a holomorphic bundle over X is automatically flat, because $\Omega_X^{2,0} = 0$. A holomorphic connection ϕ on E_G gives a holomorphic decomposition $\mathrm{At}(E_G) = TX \oplus \mathrm{ad}(E_G)$ into a direct sum of holomorphic vector bundles. This decomposition produces a holomorphic homomorphism

(2.5) $\phi' \colon \mathrm{At}(E_G) \longrightarrow \mathrm{ad}(E_G)$

such that $\phi' \circ \iota = \mathrm{Id}_{\mathrm{ad}(E_G)}$, where ι is the homomorphism in (2.3).

Take a holomorphic connection

(2.6) $\phi \colon TX \longrightarrow \mathrm{At}(E_G)$

on E_G. Since

$$\mathrm{At}(E_G) = (TE_G)/G,$$

this homomorphism ϕ produces a G-equivariant holomorphic homomorphism of vector bundles

(2.7) $\hat{\phi} := p^*\phi \colon p^*TX \longrightarrow TE_G$

over E_G. Take any analytic open subset $U \subset X$. Let s be a holomorphic section of $\mathrm{At}(E_G)|_U$ over U. Since $\mathrm{At}(E_G) = (TE_G)/G$, we have

$$\hat{s} := p^*s \in H^0(p^{-1}(U), TE_G)^G \subset H^0(p^{-1}(U), TE_G).$$

For any holomorphic vector field $t \in H^0(U, TU)$, consider the Lie bracket

$$[\hat{\phi}(p^*t), \hat{s}] \in H^0(p^{-1}(U), TE_G),$$

where $\hat{\phi}$ is the homomorphism in (2.7). This vector field $[\hat{\phi}(p^*t), \hat{s}]$ on $p^{-1}(U)$ is G-invariant, because both $\hat{s}$ and $\hat{\phi}(p^*t)$ are so. Therefore, $[\hat{\phi}(p^*t), \hat{s}]$ produces a holomorphic section of $\mathrm{At}(E_G)$ over U; this section of $\mathrm{At}(E_G)|_U$ will be denoted by $A(t, s)$. Let

(2.8) $\phi'(A(t, s)) \in H^0(U, \mathrm{ad}(E_G))$

be the section of $\mathrm{ad}(E_G)|_U$, where ϕ' is the projection in (2.5).

Now, for any holomorphic function f defined on U, we have

$$(2.9) \qquad [\hat{\phi}(p^*(f \cdot t)), \hat{s}] = (f \circ p) \cdot [\hat{\phi}(p^*t), \hat{s}] - \hat{s}(f \circ p) \cdot \hat{\phi}(p^*t).$$

Since $\phi'(\hat{\phi}(p^*t)) = 0$, where ϕ' is constructed in (2.5), from (2.9) it follows immediately that

$$\phi'(A(f \cdot t, s)) = f \cdot \phi'(A(t, s));$$

$\phi'(A(t,s))$ is defined in (2.8). Let

$$(2.10) \qquad\qquad \Phi \colon \operatorname{At}(E_G) \longrightarrow \operatorname{ad}(E_G) \otimes K_X$$

be the homomorphism of sheaves defined by the equation

$$\langle \Phi(s), t \rangle = \phi'(A(t, s)) \in H^0(U, \operatorname{ad}(E_G)),$$

where s and t are holomorphic sections, over U, of $\operatorname{At}(E_G)$ and TX respectively, while $\langle -, - \rangle$ is the contraction of K_X by T_X.

Remark 2.1. It should be mentioned that the map Φ in (2.10) is an additive homomorphism and it is $\mathbb{C}$-linear, but it is not $\mathcal{O}_X$-linear. In fact, the composition of homomorphisms

$$\Phi \circ \iota \colon \operatorname{ad}(E_G) \longrightarrow \operatorname{ad}(E_G) \otimes K_X,$$

where ι is the inclusion map in (2.3), satisfies the Leibniz identity. This map $\Phi \circ \iota$ is the connection on $\operatorname{ad}(E_G)$ induced by the connection ϕ on E_G.

The composition

$$\Phi \circ \phi \colon TX \longrightarrow \operatorname{ad}(E_G) \otimes K_X$$

coincides with the curvature of the connection ϕ. Since ϕ is flat, we have

$$(2.11) \qquad\qquad\qquad \Phi \circ \phi = 0.$$

Let $\mathcal{C}_\bullet$ be the two-term complex

$$\mathcal{C}_\bullet \colon \mathcal{C}_0 := \operatorname{At}(E_G) \xrightarrow{\Phi} \mathcal{C}_1 := \operatorname{ad}(E_G) \otimes K_X,$$

where $\mathcal{C}_i$ is at the ith position and Φ is the $\mathbb{C}$-linear additive homomorphism constructed in (2.10). Using (2.11) we have the following commutative diagram of

homomorphisms of complexes of sheaves on X:

$$
(2.12) \qquad
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
TX & \longrightarrow & 0 \\
\downarrow{\scriptstyle \phi} & & \downarrow \\
\mathcal{C}_\bullet: \quad \mathcal{C}_0 & \xrightarrow{\ \Phi\ } & \mathcal{C}_1 \\
\downarrow{\scriptstyle =} & & \downarrow \\
\mathrm{At}(E_G) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}
$$

It should be clarified that this is not a complex of complexes of sheaves – the composition map does not vanish, because $\phi \neq 0$. Let

$$
(2.13) \qquad H^1(X, TX) \xrightarrow{\ \beta_\phi\ } \mathbb{H}^1(X, \mathcal{C}_\bullet) \xrightarrow{\ \gamma\ } H^1(X, \mathrm{At}(E_G))
$$

be the homomorphisms of (hyper)cohomologies associated to the homomorphisms in (2.12), where $\mathbb{H}^i$ denotes the ith hypercohomology. It should be clarified that $\gamma \circ \beta_\phi$ does not vanish. Indeed, the composition of $\gamma \circ \beta_\phi$ with the homomorphism

$$
(d'p)_* \colon H^1(X, \mathrm{At}(E_G)) \longrightarrow H^1(X, TX),
$$

where $d'p$ is the projection in (2.3), coincides with the identity map of $H^1(X, TX)$.

The following known theorem will be used (see [Ch2, Do, In, BHH]).

Theorem 2.2. *The following statements hold:*

(1) *The infinitesimal deformations of the pair (X, E_G) are parametrized by the elements of the cohomology $H^1(X, \mathrm{At}(E_G))$.*

(2) *The infinitesimal deformations of the triple (X, E_G, ϕ) are parametrized by the elements of the hypercohomology $\mathbb{H}^1(X, \mathcal{C}_\bullet)$.*

(3) *The forgetful map from the infinitesimal deformations of the triple (X, E_G, ϕ) to the infinitesimal deformations of the pair (X, E_G), which forgets the connection ϕ, is the homomorphism γ in (2.13).*

(4) *The infinitesimal isomonodromy map, from the infinitesimal deformations of X to the infinitesimal deformations of the triple (X, E_G, ϕ), coincides with the homomorphism β_ϕ in (2.13).*

For the proof of Theorem 2.2 the reader is referred to [Ch2, p. 1413, Prop. 4.3] (for a proof of Theorem 2.2(1)), [Ch2, p. 1415, Prop. 4.4] (for a proof of

Theorem 2.2(2)) and [Ch2, p. 1417, Prop. 5.1] (for a proof of Theorem 2.2(4)); see also [Ch1]. Theorem 2.2(3) is evident.

The Atiyah exact sequence in (2.3) produces a long exact sequence of cohomologies

$$H^1(X, \mathrm{ad}(E_G)) \xrightarrow{\iota_*} H^1(X, \mathrm{At}(E_G)) \xrightarrow{(d'p)_*} H^1(X, TX) \longrightarrow 0.$$

We note that the infinitesimal deformations of E_G (keeping the Riemann surface X fixed) are parametrized by $H^1(X, \mathrm{ad}(E_G))$ (see [Do]), and the above homomorphism ι_* coincides with the natural homomorphism of infinitesimal deformations. The above projection $(d'p)_*$ is the forgetful map that sends the infinitesimal deformations of the pair (X, E_G) to the infinitesimal deformations of X that forgets the principal G-bundle.

§3. Infinitesimal deformations of connections on the trivial bundle

Recall the moduli space of differential systems Syst and its subset of irreducible differential systems $\mathrm{Syst}^{\mathrm{irred}}$, both defined in Section 1.

In this section we realize the tangent space to $(X, \phi) \in \mathrm{Syst}^{\mathrm{irred}}$ as a subspace embedded in $\mathbb{H}^1(X, \mathcal{C}_\bullet)$, the space of infinitesimal deformations of triples (X, E_G, ϕ) (see below), and we prove a criterion for transversality to the kernel of the monodromy map (Proposition 3.2).

Let Y be a compact connected Riemann surface, and let ψ be a holomorphic connection on the holomorphically trivial principal G-bundle $Y \times G \to Y$ over Y. Let

$$\tag{3.1} \mathcal{T}(Y, \psi)$$

denote the infinitesimal deformations of the pair (Y, ψ) (keeping the underlying holomorphic principal G-bundle as the trivial principal G-bundle on the moving Riemann surface).

Henceforth, we assume that $\mathrm{genus}(X) = g \geq 2$.

Let $Z_G \subset G$ be the center of G. As before, $d := \dim[G, G]$ and $c := \dim Z_G$.

Lemma 3.1. *The moduli space $\mathrm{Syst}^{\mathrm{irred}}$ in Section 1 is a connected (smooth) complex orbifold. The complex dimension of Syst is $(g-1)(d+3) + gc$.*

Proof. Let $\mathbb{T}_g$ denote the Teichmüller space for genus g Riemann surfaces. We have a universal family of genus g Riemann surfaces

$$\varphi \colon \mathbf{C}_g \longrightarrow \mathbb{T}_g.$$

Let $\Omega_\varphi \to \mathbf{C}_g$ be the relative holomorphic cotangent bundle for the projection φ. Consider the direct image

$$\mathcal{W} := \varphi_* \Omega_\varphi \longrightarrow \mathbb{T}_g.$$

So $\mathcal{W}$ is a holomorphic vector bundle over $\mathbb{T}_g$ whose fiber over any given Riemann surface $Y \in \mathbb{T}_g$ is $H^0(Y, K_Y)$. Now define the holomorphic vector bundle $\mathcal{W}(G)$ over $\mathbb{T}_g$,

$$\mathcal{W}(G) := \mathcal{W} \otimes_{\mathbb{C}} \mathfrak{g}.$$

The adjoint action of the group G on $\mathfrak{g}$ and the trivial action of G on $\mathcal{W}$ together produce an action of G on $\mathcal{W}(G)$. Note that this action of G on $\mathcal{W}$ factors through the quotient G/Z_G of G. Let

$$\widehat{\mathcal{W}}(G) \subset \mathcal{W}(G)$$

be the open subset consisting of all points $(Y, \theta) \in \mathcal{W}(G)$, where $Y \in \mathbb{T}_g$ and

$$\theta \in \mathcal{W}(G)_Y = H^0(Y, K_Y) \otimes \mathfrak{g},$$

such that θ is not contained in $H^0(Y, K_Y) \otimes \mathfrak{p}$ for some parabolic subgroup $\mathfrak{p} \subsetneq \mathfrak{g}$. For any $(Y, \theta) \in \widehat{\mathcal{W}}(G)$, consider the holomorphic connection $D_0^Y + \theta$, where D_0^Y is the trivial connection on the trivial holomorphic principal G-bundle $Y \times G \to Y$. Let

$$\Psi_\theta \colon \pi_1(Y, y_0) \longrightarrow G$$

be the monodromy representation for this flat connection $D_0^Y + \theta$. The definition of $\widehat{\mathcal{W}}(G)$ ensures that $\Psi_\theta(\pi_1(Y, y_0))$ is not contained in some proper parabolic subgroup of G.

The action of G/Z_G on $\mathcal{W}(G)$ evidently preserves $\widehat{\mathcal{W}}(G)$. For any

$$(Y, \theta) \in \widehat{\mathcal{W}}(G),$$

the isotropy subgroup of (Y, θ), for the action of G/Z_G on $\mathcal{W}(G)$, is

$$N(\Psi_\theta(\pi_1(Y, y_0))')/\Psi_\theta(\pi_1(Y, y_0))',$$

where $\Psi_\theta(\pi_1(Y, y_0))'$ is the image of $\Psi_\theta(\pi_1(Y, y_0))$ in G/Z_G and

$$N(\Psi_\theta(\pi_1(Y, y_0))') \subset G/Z_G$$

is its normalizer. From the definition of $\widehat{\mathcal{W}}(G)$ it follows that this isotropy subgroup is finite.

For any Riemann surface $Y \in \mathbb{T}_g$, the space of all holomorphic connections on the trivial holomorphic principal G-bundle $Y \times G \to Y$ is an affine space for the vector space $\mathcal{W}(G)_Y$, where $\mathcal{W}(G)_Y$ is the fiber of $\mathcal{W}(G)$ over the point Y

(see (3.5)); note that $\mathrm{ad}(Y \times G) = Y \times \mathfrak{g}$. Consequently, we have a biholomorphism from the quotient space $\widehat{\mathcal{W}}(G)/G$,

$$\eta \colon \widehat{\mathcal{W}}(G)/G \xrightarrow{\sim} \mathrm{Syst}^{\mathrm{irred}}$$

that sends any $\theta \in \widehat{\mathcal{W}}(G)_Y$, $Y \in \mathbb{T}_g$, to the holomorphic connection $D_0^Y + \theta$, where D_0^Y as before is the trivial connection on the trivial holomorphic principal G-bundle $Y \times G \to Y$; note that D_0^Y does not depend on the choice of the holomorphic trivialization of the principal G-bundle (see the paragraph following (3.2)). Also, this map evidently factors through the quotient $\widehat{\mathcal{W}}(G)/G$. In view of the biholomorphism η we conclude that $\mathrm{Syst}^{\mathrm{irred}}$ is a connected complex orbifold.

The complex dimension of $\mathrm{Syst}^{\mathrm{irred}}$ is $(g-1)(d+3) + gc$, because the complex dimension of the total space of $\mathcal{W}(G)$ is $3(g-1) + dg + gc$, and hence the complex dimension of the total space of $\widehat{\mathcal{W}}(G)$ is $(g-1)(d+3) + gc$. $\qquad\square$

Now take E_G in (2.1) to be the holomorphically trivial principal G-bundle $X \times G$ on X. As in (2.6), take a holomorphic connection ϕ on $E_G = X \times G$. Since the infinitesimal deformations of the triple (X, E_G, ϕ) are parametrized by $\mathbb{H}^1(X, \mathcal{C}_\bullet)$ (see Theorem 2.2(2)), we have a natural homomorphism

$$\mathcal{T}(X, \phi) \longrightarrow \mathbb{H}^1(X, \mathcal{C}_\bullet),$$

where $\mathcal{T}(X, \phi)$ is defined in (3.1). Let

$$(3.2) \qquad\qquad \mathcal{S}(X, \phi) \subseteq \mathbb{H}^1(X, \mathcal{C}_\bullet)$$

be the image of this homomorphism from $\mathcal{T}(X, \phi)$.

The trivial holomorphic principal G-bundle $E_G = X \times G$ has a unique holomorphic connection whose monodromy is the trivial representation. Once we fix an isomorphism of E_G with $X \times G$, the trivial holomorphic connection on $X \times G$ induces a holomorphic connection on E_G using the chosen isomorphism. However, this induced connection on E_G does not depend on the choice of the trivialization of E_G; this unique connection on E_G will be called the trivial connection. The monodromy of the trivial connection is evidently trivial.

Using the trivial connection on E_G, we have a canonical holomorphic decomposition

$$(3.3) \qquad\qquad \mathrm{At}(E_G) = \mathrm{ad}(E_G) \oplus TX.$$

Using (3.3), the holomorphic connections on E_G are identified with holomorphic homomorphisms from TX to $\mathrm{ad}(E_G)$. More precisely, to any holomorphic homomorphism

$$(3.4) \qquad\qquad \rho \colon TX \longrightarrow \mathrm{ad}(E_G)$$

we assign the corresponding homomorphism

$$(3.5) \qquad \hat{\rho} \colon TX \longrightarrow \mathrm{ad}(E_G) \oplus TX = \mathrm{At}(E_G), \quad v \longmapsto (\rho(v), v);$$

clearly $\hat{\rho}$ satisfies the equation in (2.4). Note that the decomposition in (3.3) is used in the construction of $\hat{\rho}$ in (3.5).

Proposition 3.2. *Let ϕ be a holomorphic connection on $E_G = X \times G$ such that the homomorphism ρ in (3.4) corresponding to ϕ satisfies the following condition: the homomorphism of first cohomologies corresponding to ρ, namely the homomorphism*

$$\rho_* \colon H^1(X, TX) \longrightarrow H^1(X, \mathrm{ad}(E_G)),$$

is injective. Then

$$\mathbb{H}^1(X, \mathcal{C}_\bullet) \supset \mathcal{S}(X, \phi) \cap \beta_\phi(H^1(X, TX)) = 0,$$

where $\mathcal{S}(X, \phi)$ is the subspace constructed in (3.2) and β_ϕ is the homomorphism in (2.13).

Proof. Let

$$q \colon \mathrm{At}(E_G) = \mathrm{ad}(E_G) \oplus TX \longrightarrow TX$$

be the projection constructed using the decomposition in (3.3). Note that q coincides with the projection $d'p$ in (2.3); indeed, this follows from the construction of the decomposition in (3.3). Let

$$(3.6) \qquad q_* \colon H^1(X, \mathrm{At}(E_G)) = H^1(X, \mathrm{ad}(E_G)) \oplus H^1(X, TX) \longrightarrow H^1(X, TX)$$

be the homomorphism of first cohomologies induced by q. From the equation

$$q \circ \phi = \mathrm{Id}_{TX}$$

(see (2.4)), it can be deduced that

$$(3.7) \qquad\qquad\qquad q_* \circ \gamma \circ \beta_\phi = \mathrm{Id}_{H^1(X, TX)},$$

where β_ϕ, γ are the homomorphisms in (2.13) and q_* is constructed in (3.6). To prove (3.7), just note that $\gamma \circ \beta_\phi$ coincides with the homomorphism of cohomologies induced by the homomorphism ϕ.

Consider the two subspaces

$$(3.8) \qquad\qquad \gamma(\mathcal{S}(X, \phi)), \ \gamma(\beta_\phi(H^1(X, TX))) \subset H^1(X, \mathrm{At}(E_G))$$

of $H^1(X, \mathrm{At}(E_G))$, where β_ϕ, γ are the homomorphisms in (2.13). From (3.7) it follows immediately that the homomorphism $\gamma \circ \beta_\phi$ is injective.

Consequently, to prove the proposition it suffices to show that

$$(3.9) \qquad \gamma(\mathcal{S}(X,\phi)) \cap (\gamma(\beta_\phi(H^1(X,TX)))) = 0,$$

where $\gamma(\mathcal{S}(X,\phi))$ and $\gamma(\beta_\phi(H^1(X,TX)))$ are the subspaces in (3.8). At this point it might be helpful to have a look at Remark 3.3.

The isomonodromic deformation of the trivial connection on $E_G \to X$ is evidently the trivial connection on the trivial principal G-bundle over the moving Riemann surface. Since the decomposition in (3.3) is given by the trivial connection on E_G, from Theorem 2.2(4) and Theorem 2.2(3) it follows that the subspace

$$\gamma(\mathcal{S}(X,\phi)) \subset H^1(X, \mathrm{At}(E_G))$$

coincides with the natural subspace

$$(3.10) \qquad \begin{aligned} H^1(X,TX) &\subset H^1(X,\mathrm{ad}(E_G)) \oplus H^1(X,TX) \\ &= H^1(X,\mathrm{ad}(E_G) \oplus TX) = H^1(X,\mathrm{At}(E_G)) \end{aligned}$$

corresponding to the decomposition in (3.3) (given by the trivial connection); the subspace $H^1(X,TX) \subset H^1(X,\mathrm{ad}(E_G)) \oplus H^1(X,TX)$ in (3.10) consists of all $(0,v)$ with $v \in H^1(X,TX)$. The above statement follows from the fact that for the isomonodromic deformation of the trivial connection on $E_G \to X$, the underlying principal G-bundle remains trivial (recall that the isomonodromic deformation of the trivial connection on $E_G \to X$ is the trivial connection on the trivial principal G-bundle over the moving Riemann surface).

Therefore, using the construction of the homomorphism $\hat{\rho}$ from ρ (see (3.5) – note that $\phi = \hat{\rho}$ by the definition of ρ given in the statement of the proposition) it follows that the given condition – that

$$\rho_* \colon H^1(X,TX) \longrightarrow H^1(X,\mathrm{ad}(E_G))$$

is injective – implies that (3.9) holds. As observed before, (3.9) completes the proof of the proposition. $\square$

Remark 3.3. Since Proposition 3.2 is the key tool here, we would make some clarifying comments on the proof of it. Let A, B be finite-dimensional vector spaces and

$$H \colon A \longrightarrow B$$

a linear map. Let S_1, S_2 be subspaces of A such that the homomorphism

$$H|_{S_2} \colon S_2 \longrightarrow B$$

is injective (in other words, $S_2 \cap \mathrm{kernel}(H) = 0$). Now, if $H(S_1) \cap H(S_2) = 0$, then it is straightforward to check that $S_1 \cap S_2 = 0$. In the proof of Proposition 3.2, set

$$A = \mathbb{H}^1(X, \mathcal{C}_\bullet), \quad B = H^1(X, \mathrm{At}(E_G)), \quad S_1 = \mathcal{S}(X, \phi),$$
$$S_2 = \beta_\phi(H^1(X, TX)), \quad H = \gamma.$$

The above condition that $H(S_1) \cap H(S_2) = 0$ coincides with (3.9). The above statement that $S_1 \cap S_2 = 0$ actually coincides with the statement of Proposition 3.2.

§4. Some examples with $G = \mathrm{SL}(2, \mathbb{C})$

This section focuses on the case $G = \mathrm{SL}(2, \mathbb{C})$. In this case we prove that the criterion in Proposition 3.2 is satisfied at any point $(X, \phi) \in \mathrm{Syst}^{\mathrm{irred}}$ for $g = 2$ (see Proposition 4.1) and at the generic point in $\mathrm{Syst}^{\mathrm{irred}}$ for $g = 3$ (see Lemma 4.2).

Proposition 4.1. *Let X be a compact connected Riemann surface of genus two. Set $G = \mathrm{SL}(2, \mathbb{C})$. Let ϕ be an irreducible holomorphic connection on the trivial holomorphic principal G-bundle $E_G = X \times G \to X$. Then the homomorphism ρ as in (3.4) corresponding to ϕ satisfies the following condition: the homomorphism of cohomologies corresponding to ρ, namely the homomorphism*

$$\rho_* \colon H^1(X, TX) \longrightarrow H^1(X, \mathrm{ad}(E_G)),$$

is injective.

Proof. Fix a holomorphic trivialization of E_G. Then the adjoint vector bundle $\mathrm{ad}(E_G)$ is the trivial holomorphic vector bundle $X \times \mathfrak{sl}(2, \mathbb{C})$ over X, where $\mathfrak{sl}(2, \mathbb{C})$ is the Lie algebra consisting of 2×2 complex matrices of trace zero. So ρ as in (3.4) corresponding to ϕ,

$$\rho \colon TX \longrightarrow \mathrm{ad}(E_G) = X \times \mathfrak{sl}(2, \mathbb{C}),$$

sends any $v \in T_x X$ to

$$\left(x, \begin{pmatrix} \omega_1(x)(v) & \omega_2(x)(v) \\ \omega_3(x)(v) & -\omega_1(x)(v) \end{pmatrix} \right) \in X \times \mathfrak{sl}(2, \mathbb{C}),$$

where $\omega_j \in H^0(X, K_X)$, $1 \leq j \leq 3$.

Let

(4.1) $$V \subset H^0(X, K_X)$$

be the linear span of $\{\omega_1, \omega_2, \omega_3\}$. The given condition that the connection ϕ is irreducible implies that $\dim V > 1$. Indeed, if we assume by contradiction that $\dim V \leq 1$, then

$$\rho = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & -\omega_1 \end{pmatrix} = \omega \otimes B,$$

where $\omega \in H^0(X, K_X)$ and $B \in \mathfrak{sl}(2, \mathbb{C})$ is a fixed element. Since the standard action on $\mathbb{C}^2$ of B is reducible, the connection ϕ is reducible: a contradiction. So we have $\dim V > 1$, and, since $\dim H^0(X, K_X) = 2$, we conclude that

$$(4.2) \qquad\qquad V = H^0(X, K_X).$$

Let

$$(4.3) \qquad \Theta \colon H^1(X, TX) \otimes H^0(X, K_X) \longrightarrow H^1(X, TX \otimes K_X) = H^1(X, \mathcal{O}_X)$$

be the natural homomorphism. For any $\mathbb{C}$-linear map

$$(4.4) \qquad\qquad h \colon \mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathbb{C},$$

let

$$(4.5) \qquad\qquad \tilde{h}_* \colon H^1(X, TX) \longrightarrow H^1(X, \mathcal{O}_X)$$

be the homomorphism induced by the composition

$$TX \xrightarrow{\rho} \mathrm{ad}(E_G) = \mathcal{O}_X \times_{\mathbb{C}} \mathfrak{sl}(2, \mathbb{C}) \xrightarrow{\mathrm{Id} \otimes h} \mathcal{O}_X.$$

For any

$$\mu \in \mathrm{kernel}(\rho_*) \subset H^1(X, TX),$$

we evidently have

$$(4.6) \qquad\qquad \tilde{h}_*(\mu) = 0,$$

because $\tilde{h}_* = (\mathrm{Id} \otimes h)_* \circ \rho_*$, where $(\mathrm{Id} \otimes h)_*$ is the homomorphism of cohomologies induced by $\mathrm{Id} \otimes h$.

From (4.6) it follows that $\Theta(\mu \otimes V) = 0$ for all $\mu \in \mathrm{kernel}(\rho_*)$, where Θ is the homomorphism in (4.3) and V is the subspace in (4.1). Now, since $V = H^0(X, K_X)$ (see (4.2)), we conclude that

$$(4.7) \qquad\qquad \Theta(\mu \otimes \omega) = 0$$

for all $\mu \in \mathrm{kernel}(\rho_*)$ and $\omega \in H^0(X, K_X)$.

To complete the proof of the proposition we need to show that there is no nonzero cohomology class $\mu \in H^1(X, TX)$ that satisfies (4.7) for all $\omega \in H^0(X, K_X)$.

Using Serre duality, it suffices to prove that the tensor product homomorphism

(4.8) $$\Theta'\colon H^0(X, K_X) \otimes H^0(X, K_X) \longrightarrow H^0(X, K_X^2)$$

is surjective; note that Θ' is given by the dual of Θ.

It is known that for a genus two Riemann surface X, the homomorphism Θ' in (4.8) is indeed surjective. To be somewhat self-contained, we give the outline of an argument for it. Consider the short exact sequence of sheaves on $X \times X$,

(4.9) $\quad 0 \to (p_1^* K_X) \otimes (p_2^* K_X) \otimes \mathcal{O}_{X \times X}(-\Delta) \to (p_1^* K_X) \otimes (p_2^* K_X) \to i_* K_X^2 \to 0,$

where p_j is the projection of $X \times X$ to the jth factor for $j = 1, 2$, and i is the inclusion map of the diagonal $\Delta \subset X \times X$. Using the short exact sequence

$$0 \longrightarrow (p_2^* K_X) \otimes \mathcal{O}_{X \times X}(-\Delta) \longrightarrow p_2^* K_X \longrightarrow i_* K_X \longrightarrow 0,$$

we have

$$p_{1*}((p_2^* K_X) \otimes \mathcal{O}_{X \times X}(-\Delta)) = \left(\bigwedge^2 (p_{1*} p_2^* K_X)\right) \otimes (p_{1*}(i_* K_X))^* = TX;$$

this is because the homomorphism $p_{1*} p_2^* K_X \to p_{1*}(i_* K_X)$ is surjective as K_X is base-point free. Therefore, the projection formula gives that

$$p_{1*}((p_1^* K_X) \otimes (p_2^* K_X) \otimes \mathcal{O}_{X \times X}(-\Delta)) = \mathcal{O}_X.$$

Hence we have

(4.10) $$H^0(X \times X, (p_1^* K_X) \otimes (p_2^* K_X) \otimes \mathcal{O}_{X \times X}(-\Delta)) = H^0(X, \mathcal{O}_X).$$

Consider the long exact sequence of cohomologies for the short exact sequence of sheaves in (4.9),

$$0 \longrightarrow H^0(X \times X, (p_1^* K_X) \otimes (p_2^* K_X) \otimes \mathcal{O}_{X \times X}(-\Delta))$$

(4.11) $$\longrightarrow H^0(X \times X, (p_1^* K_X) \otimes (p_2^* K_X)) = H^0(X, K_X)^{\otimes 2} \xrightarrow{\Theta'} H^0(X, K_X^2),$$

where Θ' is the homomorphism in (4.8). Since

$$\dim H^0(X, K_X)^{\otimes 2} = 4 = \dim H^0(X, K_X^2) + 1,$$

from (4.10) and (4.11) it follows that Θ' is surjective. $\qquad \square$

Lemma 4.2. *Let X be a compact connected Riemann surface of genus three which is not hyperelliptic. Set $G = \mathrm{SL}(2, \mathbb{C})$. Then there is a nonempty Zariski open subset $\mathcal{U}$ of the space of all holomorphic connections on the trivial holomorphic principal G-bundle $E_G = X \times G \to X$ such that for any $\phi \in \mathcal{U}$, the homomorphism*

ρ in (3.4) *corresponding to ϕ satisfies the following condition: the homomorphism of cohomologies corresponding to ρ, namely the homomorphism*

$$\rho_* \colon H^1(X, TX) \longrightarrow H^1(X, \mathrm{ad}(E_G)),$$

is injective.

Proof. As in the proof of Proposition 4.1, fixing a holomorphic trivialization of E_G identifies $\mathrm{ad}(E_G)$ with the trivial holomorphic vector bundle $X \times \mathfrak{sl}(2, \mathbb{C})$ over X. Let $\{\omega_j\}_{j=1}^3$ be a basis of $H^0(X, K_X)$ (recall that $H^0(X, K_X)$ has dimension three). Define the homomorphism ρ,

$$\rho \colon TX \longrightarrow \mathrm{ad}(E_G) = X \times \mathfrak{sl}(2, \mathbb{C}),$$

that sends any $v \in T_x X$ to

$$\left(x, \begin{pmatrix} \omega_1(x)(v) & \omega_2(x)(v) \\ \omega_3(x)(v) & -\omega_1(x)(v) \end{pmatrix} \right) \in X \times \mathfrak{sl}(2, \mathbb{C}).$$

As in (4.3), let

$$\Theta \colon H^1(X, TX) \otimes H^0(X, K_X) \longrightarrow H^1(X, \mathcal{O}_X)$$

be the natural homomorphism. For any h as in (4.4), the homomorphism $\tilde{h}_*$ as in (4.5) vanishes. Therefore, from the above construction of ρ it follows immediately that every

$$\mu \in \mathrm{kernel}(\rho_*) \subset H^1(X, TX)$$

satisfies the equation

$$(4.12) \qquad\qquad\qquad \Theta(\mu \otimes \omega) = 0$$

for all $\omega \in H^0(X, K_X)$; recall that $\{\omega_j\}_{j=1}^3$ is a basis of $H^0(X, K_X)$.

We will show that there is no nonzero cohomology class $\mu \in H^1(X, TX)$ that satisfies (4.12) for all $\omega \in H^0(X, K_X)$.

Using Serre duality, it suffices to prove that the tensor product homomorphism

$$\Theta' \colon H^0(X, K_X) \otimes H^0(X, K_X) \longrightarrow H^0(X, K_X^2)$$

is surjective. Now, Max Noether's theorem says that the homomorphism Θ' is surjective because X is not hyperelliptic [ACGH, p. 117]. Hence $\mathrm{kernel}(\rho_*) = 0$.

The condition on a homomorphism $\rho \colon TX \to \mathrm{ad}(E_G)$ that

$$\rho_* \colon H^1(X, TX) \longrightarrow H^1(X, \mathrm{ad}(E_G))$$

is injective, is Zariski open (in the space of all holomorphic homomorphisms). This completes the proof of the lemma. $\qquad\square$

§5. Holomorphic connections on the trivial bundle

In this section we prove the main result of the article (Theorem 5.3) and deduce several consequences (Corollaries 5.6 and 5.8). Corollary 5.6 is the main result in [CDHL]. Corollary 5.8 answers positively a question of B. Deroin.

As before, G is a connected reductive affine algebraic group defined over $\mathbb{C}$. In this section we further assume that $\dim G \geq 3$. As before, the Lie algebra of G will be denoted by $\mathfrak{g}$.

As before, X is a compact connected Riemann surface of genus g, with $g \geq 2$. Given an element

$$\delta \in H^0(X, K_X) \otimes \mathfrak{g},$$

we have an $\mathcal{O}_X$-linear homomorphism

$$(5.1) \qquad\qquad M(\delta)\colon TX \longrightarrow \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}$$

that sends any $v \in T_x X$ to the contraction $\langle \delta(x), v \rangle \in \mathfrak{g}$. Let

$$(5.2) \qquad M(\delta)_*\colon H^1(X, TX) \longrightarrow H^1(X, \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}) = H^1(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \mathfrak{g}$$

be the homomorphism of first cohomologies induced by the homomorphism $M(\delta)$ in (5.1).

Notice that the homomorphism $M(\delta)$ in (5.1) is similar to ρ in (3.4).

Lemma 5.1. *Let X be a compact connected Riemann surface of genus $g \geq 2$ such that one of the following two conditions holds:*

(1) *X is nonhyperelliptic;*

(2) *$g = 2$.*

Then there is a nonempty Zariski open subset $\mathcal{U} \subset H^0(X, K_X) \otimes \mathfrak{g}$ such that for every $\delta \in \mathcal{U}$, the homomorphism $M(\delta)_*$ constructed in (5.2) is injective.

Proof. First assume that X is nonhyperelliptic. Under this assumption, [Gi, p. 221, Thm. 1.1] says that for a generic three-dimensional subspace $W \subset H^0(X, K_X)$, the natural homomorphism

$$\Theta_W\colon H^0(X, K_X) \otimes W \longrightarrow H^0(X, K_X^2)$$

is surjective; in [Gi], the proof of this theorem is attributed to R. Lazarsfeld (see the sentence in [Gi] just after Theorem 1.1). The dual homomorphism for it,

$$\Theta_W^*\colon H^1(X, TX) \longrightarrow H^1(X, \mathcal{O}_X) \otimes W^*,$$

obtained using Serre duality, is injective if Θ_W is surjective.

Take any W as above such that Θ_W^* is injective. Set

$$\delta \in H^0(X, K_X) \otimes \mathfrak{g}$$

to be such that the image of the homomorphism $\mathfrak{g}^* \to H^0(X, K_X)$ corresponding to δ contains W; note that the given condition that $\dim G \geq 3$ ensures that such a δ exists. Then, from the injectivity of Θ_W^*, it follows immediately that the homomorphism $M(\delta)_*$ constructed in (5.2) is injective.

Since the condition on $\delta \in H^0(X, K_X) \otimes \mathfrak{g}$ that $M(\delta)_*$ is injective is actually Zariski open (in $H^0(X, K_X) \otimes \mathfrak{g}$), the proof of the lemma is complete under the assumption that X is nonhyperelliptic.

Next assume that $g = 2$. This case is actually covered in the proof of Proposition 4.1. More precisely, the proof of Proposition 4.1 shows that as long as δ is not of the form $\omega \otimes B$, where $B \in \mathfrak{g}$ and $\omega \in H^0(X, K_X)$ are fixed elements, the homomorphism $M(\delta)_*$ is injective. $\square$

Corollary 5.2. *Let X be a compact connected Riemann surface of genus $g \geq 2$ such that one of the following two conditions holds:*

(1) *X is nonhyperelliptic;*

(2) *$g = 2$.*

Then, for the generic holomorphic connection ϕ on the holomorphically trivial principal G-bundle $E_G = X \times G \to X$,

$$\mathbb{H}^1(X, \mathcal{C}_\bullet) \supset \mathcal{S}(X, \phi) \cap \beta_\phi(H^1(X, TX)) = 0,$$

where $\mathcal{S}(X, \phi)$ is the subspace constructed in (3.2) and β_ϕ is the homomorphism in (2.13).

Proof. This follows from the combination of Proposition 3.2 and Lemma 5.1. $\square$

For any holomorphic connection ϕ on a holomorphic principal G-bundle E_G (it need not be trivial) on X, consider the monodromy representation for ϕ. Let $\mathbb{L}(\phi)$ be the $\mathbb{C}$-local system on X for the flat connection on $\mathrm{ad}(E_G)$ induced by ϕ. The infinitesimal deformations of the monodromy representation are parametrized by the elements of $H^1(X, \mathbb{L}(\phi))$ [Go]. The differential of the monodromy map is a homomorphism

$$(5.3) \qquad \mathcal{H}(\phi) \colon \mathbb{H}^1(X, \mathcal{C}_\bullet) \longrightarrow H^1(X, \mathbb{L}(\phi))$$

(see Theorem 2.2(2)).

Theorem 5.3. *Let X be a compact connected Riemann surface of genus $g \geq 2$ such that one of the following two conditions holds:*

(1) *X is nonhyperelliptic;*

(2) *$g = 2$.*

Then, for the generic holomorphic connection ϕ on the holomorphically trivial principal G-bundle $E_G = X \times G \to X$, the restriction of the homomorphism $\mathcal{H}(\phi)$ in (5.3) to the subspace $\mathcal{S}(X, \phi) \subset \mathbb{H}^1(X, \mathcal{C}_\bullet)$ in (3.2) is injective.

Proof. From Theorem 2.2(4) we know that the kernel of the homomorphism $\mathcal{H}(\phi)$ is the image of the homomorphism β_ϕ in (2.13). In view of this, the theorem is an immediate consequence of Corollary 5.2. $\square$

Remark 5.4. The assumption in Theorem 1.1 that $\dim_{\mathbb{C}} G \geq 3$ is essential. Otherwise (i.e., if $\dim_{\mathbb{C}} G \leq 2$), the map Mon in Theorem 1.1 fails to be an immersion for dimensional reasons. To illustrate this, set G to be the two-dimensional affine algebraic torus $\mathbb{C}^* \times \mathbb{C}^*$. Then

$$\dim_{\mathbb{C}} \Xi^{\mathrm{irred}} = 4g,$$

and from Lemma 3.1 we know that

$$\dim_{\mathbb{C}} \mathrm{Syst}^{\mathrm{irred}} = 5g - 3.$$

Therefore, in this case, the map Mon in Theorem 1.1 is nowhere an immersion.

Remark 5.5. The map $\mathcal{H}(\phi)$ in (5.3) is a homomorphism of tangent spaces. So the generic injectivity statement in Theorem 5.3 is unrelated to whether the points of the moduli spaces, to which the tangent spaces are associated, are genuinely orbifold points or not. On the other hand, the injectivity statement in Theorem 5.3 is only for the generic point. Therefore, Theorem 5.3 does not say whether the injectivity statement holds at an orbifold point or not.

The following result was first proved in [CDHL]:

Corollary 5.6. *Let X be a compact connected Riemann surface of genus two. Set $G = \mathrm{SL}(2, \mathbb{C})$. Let ϕ be an irreducible holomorphic connection on the trivial holomorphic principal G-bundle $E_G = X \times G \to X$. Then the restriction of the homomorphism $\mathcal{H}(\phi)$ in (5.3) to the subspace $\mathcal{S}(X, \phi)$ in (3.2) is an isomorphism between $\mathcal{S}(X, \phi)$ and $H^1(X, \mathbb{L}(\phi))$.*

Proof. Since $\mathrm{kernel}(\mathcal{H}(\phi)) = \beta_\phi(H^1(X, TX))$ (see Theorem 2.2(4)), it follows from the combination of Propositions 4.1 and 3.2 that the restriction of $\mathcal{H}(\phi)$ to $\mathcal{S}(X, \phi)$

is injective. Since the complex dimensions agree, i.e.,

$$\dim \mathcal{S}(X, \phi) = 6 = \dim H^1(X, \mathbb{L}(\phi)),$$

injectivity implies isomorphism. □

Remark 5.7. Regarding Remark 5.5, we note that Corollary 5.6 holds for every point $(X, \phi) \in \mathrm{Syst}^{\mathrm{irred}}$ (under the assumptions that $g = 2$ and $G = \mathrm{SL}(2, \mathbb{C})$). In particular, Corollary 5.6 holds for the orbifold points of $\mathrm{Syst}^{\mathrm{irred}}$.

For Riemann surfaces of higher genus we have the following:

Corollary 5.8. *Let X be a compact connected Riemann surface of genus $g \geq 3$ which is not hyperelliptic. Set $G = \mathrm{SL}(2, \mathbb{C})$.*

Then, for the generic holomorphic connection ϕ on the holomorphically trivial principal G-bundle $E_G = X \times G \to X$, the restriction of the homomorphism $\mathcal{H}(\phi)$ in (5.3) to the subspace $\mathcal{S}(X, \phi) \subset \mathbb{H}^1(X, \mathcal{C}_\bullet)$ is an isomorphism between $\mathcal{S}(X, \phi)$ and $H^1(X, \mathbb{L}(\phi))$.

Proof. This follows directly from the injectivity statement obtained in Theorem 5.3. Indeed, here the complex dimensions agree:

$$\dim \mathcal{S}(X, \phi) = 6g - 6 = \dim H^1(X, \mathbb{L}(\phi)).$$

On the other hand, Theorem 5.3 says that for the generic holomorphic connection ϕ on the holomorphically trivial principal G-bundle $E_G = X \times G \to X$, the restriction of the homomorphism $\mathcal{H}(\phi)$ in (5.3) to the subspace $\mathcal{S}(X, \phi) \subset \mathbb{H}^1(X, \mathcal{C}_\bullet)$ is injective. Hence this restriction of the homomorphism $\mathcal{H}(\phi)$ is an isomorphism between $\mathcal{S}(X, \phi)$ and $H^1(X, \mathbb{L}(\phi))$. □

In [CDHL] the following question was asked: Is there a compact Riemann surface X, and a holomorphic connection D on $\mathcal{O}_X^{\oplus 2}$, such that D is irreducible and the image of the monodromy homomorphism for D is contained in $\mathrm{SL}(2, \mathbb{R})$ [CDHL, p. 161]? Such pairs (X, D) were constructed in [BDH].

Acknowledgements

References

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of algebraic curves. Vol. I*, Grundlehren der Mathematischen Wissenschaften 267, Springer, New York, 1985. Zbl 0559.14017   MR 770932

[At]   M. F. Atiyah, Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc. **85** (1957), 181–207. Zbl 0078.16002   MR 86359

[BDH]   I. Biswas, S. Dumitrescu and S. Heller, Irreducible flat $SL(2, \mathbb{R})$-connections on the trivial holomorphic bundle, J. Math. Pures Appl. (9) **149** (2021), 28–46. Zbl 1462.30079   MR 4238995

[BHH]   I. Biswas, V. Heu and J. Hurtubise, Isomonodromic deformations of logarithmic connections and stability, Math. Ann. **366** (2016), 121–140. Zbl 1349.14125   MR 3552235

[CDHL]   G. Calsamiglia, B. Deroin, V. Heu and F. Loray, The Riemann–Hilbert mapping for $\mathfrak{sl}_2$ systems over genus two curves, Bull. Soc. Math. France **147** (2019), 159–195. Zbl 1422.30063   MR 3943741

[Ch1]   T. Chen, The associated map of the nonabelian Gauss–Manin connection, PhD thesis, University of Pennsylvania, 2012.

[Ch2]   T. Chen, The associated map of the nonabelian Gauss–Manin connection, Cent. Eur. J. Math. **10** (2012), 1407–1421. Zbl 1282.14059   MR 2925612

[Do]   I. F. Donin, Construction of a versal family of deformations for holomorphic principal bundles over a compact complex space, Mat. Sb. (N.S.) **94(136)** (1974), 430–443. Zbl 0325.32008   MR 393581

[Gi]   D. Gieseker, A lattice version of the KP equation, Acta Math. **168** (1992), 219–248. Zbl 0809.14015   MR 1161266

[Go]   W. M. Goldman, The symplectic nature of fundamental groups of surfaces, Adv. in Math. **54** (1984), 200–225. Zbl 0574.32032   MR 762512

[HM]   A. T. Huckleberry and G. A. Margulis, Invariant analytic hypersurfaces, Invent. Math. **71** (1983), 235–240. Zbl 0507.32024   MR 688266

[In]   M.-A. Inaba, Moduli of parabolic connections on curves and the Riemann–Hilbert correspondence, J. Algebraic Geom. **22** (2013), 407–480. Zbl 1314.14063   MR 3048542

[Si]   A. S. Sikora, Character varieties, Trans. Amer. Math. Soc. **364** (2012), 5173–5208. Zbl 1291.14022   MR 2931326