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Generation and simplicity in the airplane rearrangement group

Matteo Tarocchi

Abstract. We study the group T_A of rearrangements of the airplane limit space introduced by Belk and Forrest (2019). We prove that T_A is generated by a copy of Thompson's group F and a copy of Thompson's group T, hence it is finitely generated. Then we study the commutator subgroup $[T_A, T_A]$, proving that the abelianization of T_A is isomorphic to $\mathbb{Z}$ and that $[T_A, T_A]$ is simple, finitely generated and acts 2-transitively on the so-called components of the airplane limit space. Moreover, we show that T_A is contained in T and contains a natural copy of the basilica rearrangement group T_B studied by Belk and Forrest (2015).

1. Introduction

In the work [2], Belk and Forrest introduced the basilica rearrangement group T_B of certain homeomorphisms of the basilica Julia set (depicted in Figure 11), a *Thompson-like* group that generalizes Thompson's groups F and T. These two famous groups are defined as certain groups of orientation-preserving homeomorphisms of $[0, 1]$ and S^1, respectively, but they have as many equivalent definitions as there are places in which they appear. The group T, introduced by Richard Thompson in the 1960's in connection with his work in logic, is the first example of a finitely presented infinite simple group, and it contains natural isomorphic copies of F. More about Thompson's groups can be read in [7].

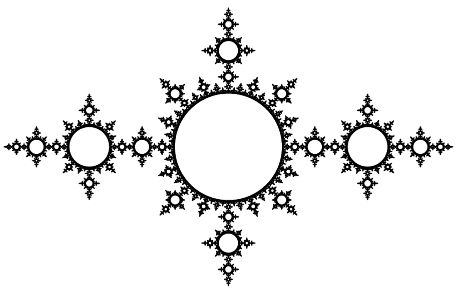

Figure 1. The airplane limit space.

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In the subsequent article [3], Belk and Forrest introduced the family of rearrangement groups of limit spaces, which includes Thompson's groups F, T and V and the basilica rearrangement group T_B. Each of these groups is associated to a certain fractal and consists of certain homeomorphisms of the fractal that permute the self-similar cells that it is made up of.

In this paper, we study the group T_A of rearrangements of the airplane limit space depicted in Figure 1, which is homeomorphic to a fractal known as the airplane Julia set. We prove that T_A is generated by natural copies of both Thompson's groups F and T, hence T_A is finitely generated. Then we focus our attention on the commutator subgroup of T_A, proving in particular that it is simple, finitely generated and infinite index in T_A. More precisely, we prove the results collected in the following theorem.

Main Theorem. *The group T_A is finitely generated and its commutator subgroup $[T_A, T_A]$ is simple and finitely generated. Moreover, $T_A \simeq [T_A, T_A] \rtimes \mathbb{Z}$.*

This result shows uncommon behavior in the world of generalized Thompson's groups, since T_A is a finitely generated group whose commutator subgroup is infinite index and finitely generated. In several known cases of finitely generated Thompson's groups whose commutator subgroup has been studied and shown to be infinite index, the commutator subgroups have also been proved to be infinitely generated: this is, in fact, true for Thompson's group F, the Cleary golden ratio group F_τ [6], generalized Thompson's groups F_n and more generally finitely generated Stein groups over the unit interval $[0, 1]$ (see [10]), since the commutator subgroup in all these groups has support bounded away from 0 and 1 and so the standard argument to show infinite generation works out (we will use this kind of argument in Proposition 9.1 for a special subgroup of $[T_A, T_A]$). One notable exception is given by certain topological full groups associated to irreducible shifts of finite type (the groups $[\![G_{\varphi_k}]\!]$ discussed at the end of [9]), although these groups are much more "V-like", whereas T_A is arguably more "T-like".

We also show that T contains an isomorphic copy of T_A, and we prove that T_A includes an unexpected natural copy of the basilica rearrangement group T_B studied in [3]. Moreover, we study an infinitely generated subgroup E of the commutator subgroup of T_A and investigate its transitivity properties, which then extend to both T_A and $[T_A, T_A]$.

This paper is organized as follows. Section 2 gives a brief introduction to Thompson's groups F and T. In Section 3, we recall the essential definitions of rearrangements and limit spaces from [3]. In Section 4, we define components, rays and component paths in the airplane limit space. In Section 5, we exhibit two important natural copies of F and T in T_A. In Section 6, we prove that T_A is finitely generated. Section 7 is all about the commutator subgroup $[T_A, T_A]$. In Section 8, we prove that T contains an isomorphic copy of T_A. Section 9 deals with a specific subgroup E of T_A defined by its action on the extremes of the airplane limit space. Finally, in Section 10, we exhibit a natural copy of T_B contained in T_A.

2. Thompson's groups F and T

In this section, we briefly introduce Thompson's groups F and T, giving their definitions and standard sets of generators. For more details about Thompson's group, we refer the reader to the introductory notes [7].

Consider those partitions of $[0, 1]$, such as the one depicted in Figure 2, which can be obtained by cutting the unit interval in half and obtaining $\{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$, then cutting one or both of these two intervals in half, and so on, a finite amount of times. These partitions are called *dyadic subdivisions* of $[0, 1]$, and they consist of intervals of the form

$$\left[\frac{a}{2^b}, \frac{a+1}{2^b}\right],$$

which are called *standard dyadic intervals*. Moreover, the extremes of these intervals are *dyadic points* of $[0, 1]$, which means that they belong to $\mathbb{Z}[\frac{1}{2}] \cap [0, 1]$.

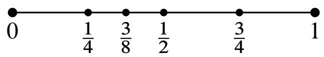


Figure 2. An example of dyadic subdivision of $[0, 1]$.

Consider the unit circle $S^1 := [0, 1]/\{0, 1\}$. By taking the quotient of dyadic subdivisions of $[0, 1]$ under the set $\{0, 1\}$, we obtain the dyadic subdivisions of S^1.

A *dyadic rearrangement* of the unit interval $[0, 1]$ (the unit circle S^1) is an orientation-preserving piecewise linear homeomorphism $f\colon [0, 1] \to [0, 1]$ ($S^1 \to S^1$) that maps linearly between the intervals of two dyadic subdivisions. *Thompson's groups* F and T are the groups under composition of the dyadic rearrangements of the unit interval $[0, 1]$ and the unit circle S^1, respectively.

The elements of Thompson's group F are specified by a pair of dyadic subdivisions of $[0, 1]$ with the same number of dyadic intervals, such as those in Figure 3. By this we mean that the n-th interval of the first subdivision is mapped linearly into the n-th interval of the second subdivision. In a similar fashion, the elements of Thompson's group T are specified by a pair of dyadic subdivisions of S^1 with the same number of dyadic intervals, along with certain colorations of the dyadic intervals for the two subdivisions, such as those in Figure 4. By this we mean that an interval of the first subdivision is mapped linearly to the interval of the second subdivision that has the same color (or, equivalently, the same letter). Note that a single color (or a single letter) would suffice since the elements of T are orientation-preserving.

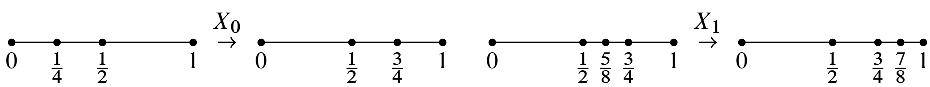


Figure 3. The generators X_0 and X_1 of Thompson's group F.

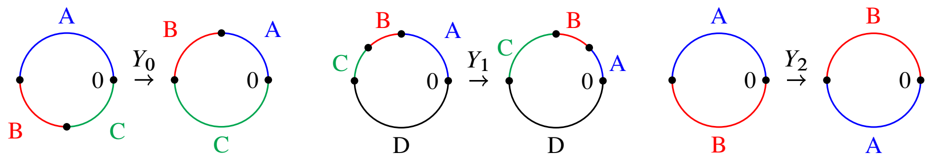


Figure 4. The three generators of Thompson's group T.

The group F is generated by the two elements X_0 and X_1 depicted in Figure 3, while T is generated by Y_0, Y_1 and Y_2 depicted in Figure 4. Further properties of F and T can be found in [1,5,7].

3. Limit spaces and rearrangements

Limit spaces of replacement systems and their rearrangements were introduced in [3], which goes in much more details than we will get to do. In this section, we briefly describe these notions, introducing the airplane limit space along the way.

3.1. Replacement systems and limit spaces

Essentially, a *replacement system* consists of a *base graph* Γ colored by the set of colors Col, along with a *replacement graph* R_c for each color $c \in$ Col. Figure 5 depicts the so-called airplane replacement system, denoted by $\mathcal{A}$. We can expand the base graph Γ by replacing one of its edges e by the replacement graph R_c indexed by the color c of e, as exemplified in Figure 6. The graph resulting from this process of replacing one edge by the appropriate replacement graph is called a *simple expansion*. Simple expansions can be repeated any finite amount of times, which generate the so-called *expansions* of the replacement system, such as the one in Figure 7.

Note that each edge of an expansion corresponds to the unique finite sequence of edges "converging" from the base graph. As an example, consider the leftmost edge in Figure 7. In order to identify this edge, one must first restrict their attention to the leftmost blue

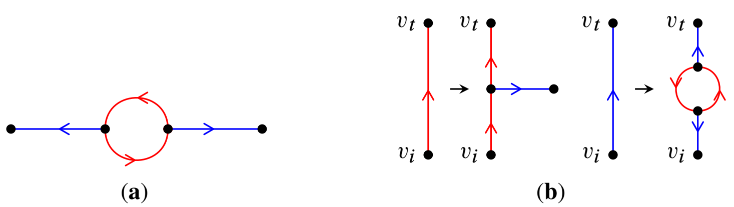


Figure 5. The airplane replacement system $\mathcal{A}$. (a) The base graph. (b) The two replacement rules: $e \to R_{\text{red}}$ if e is red, and $e \to R_{\text{blue}}$ if e is blue.

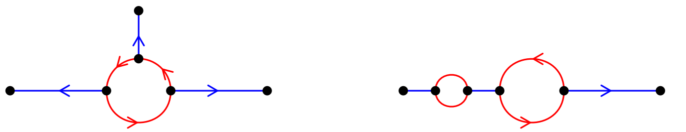


Figure 6. Two simple expansions of the base graph of the airplane replacement system $\mathcal{A}$.

edge e_0 of the base graph; then one expands this edge with the blue replacement graph and restrict their attention to the leftmost edge e_1 that has thus been generated; finally, one expands this edge and consider the leftmost edge e_2 generated by the last expansion, which is precisely the edge that we were looking for. In this sense, we say that the leftmost edge in Figure 7 corresponds to the sequence $e_0 e_1 e_2$. For more precise definitions, we refer the reader to [3, Section 1.1].

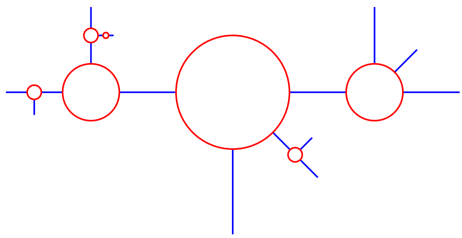


Figure 7. A generic expansion of the airplane replacement system $\mathcal{A}$.

Consider the *full expansion sequence*, which is the sequence of graphs obtained by replacing, at each step, every edge by the appropriate replacement graph, starting from the base graph. Figure 8 shows the first graphs (except for the base graph) of the full expansion sequence for the airplane replacement system $\mathcal{A}$. If a replacement system satisfies certain simple properties (which $\mathcal{A}$ satisfies), then we can define the *limit space*, which is essentially the limit of the full expansion sequence of the replacement system [3, Definition 1.8 and Proposition 1.9]. Keeping in mind that finite sequences of edges correspond to edges of expansions, we should consider the limit space as the set of infinite sequences of edges modulo an equivalence relation that "glues" certain sequences together. For example, if one expands the top red edge of the base graph and then keeps expanding (infinitely many times) the leftmost red edge, or if one expands the left blue edge of the base graph and

Figure 8. The beginning of the full expansion sequence for $\mathcal{A}$.

then keeps expanding the rightmost blue edge, then one has found the same point with two distinct sequences, which must then be glued together; intuitively, this point corresponds to the second vertex of the base graph, from the left. Again, for more details we refer to [3, Definitions 1.6 and 1.7]. In particular, the airplane limit space is the space depicted in Figure 1. By [3, Proposition 9], this is a compact and metrizable topological space.

It is worth mentioning that our airplane limit space (and limit spaces of replacement systems in general) is a *topological space*, whereas the airplane Julia set is a fractal embedded in the Euclidean plane. This Julia set is only one of the infinitely many quadratic Julia sets corresponding to the complex functions $f(z) = z^2 - p$ for p belonging to the same interior component of the Mandelbrot set as $p = 1.755$, and these are all homeomorphic as topological spaces, although they are different as Julia sets and have different metric properties. Here we will only be treating the airplane fractal as a topological space.

3.2. Cells and rearrangements

Intuitively, a cell $\chi(e)$ of a limit space corresponds to the edge e of some expansion, along with everything that appears from that edge in later expansions. More precisely, if the edge e corresponds to the finite sequence $e_0 \dots e_k$, then the cell $\chi(e)$ is the subset of a limit space consisting of those infinite sequences of edges that start with $e_0 \dots e_k$. For instance, Figure 9 shows examples of cells in $\mathcal{A}$. Moreover, we say that the cell $\chi(e)$ is colored by c if the edge e is colored by c.

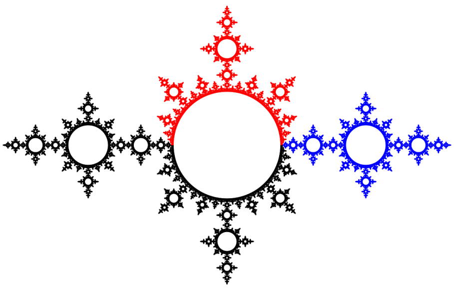


Figure 9. The two types of cells in the airplane replacement system $\mathcal{A}$, distinguished by the color of the generating edge.

There are different *types* of cells $\chi(e)$, distinguished by two aspects of the generating edge e: its color and whether it is a loop or not. It is not hard to see that there is a *canonical homeomorphism* between any two cells of the same type. More precisely, if the two cells correspond to the edges identified by the sequences $e_0 \dots e_k$ and $f_0 \dots f_l$, then the homeomorphism maps each infinite sequence $e_0 \dots e_k w$ to $f_0 \dots f_l w$. A canonical homeomorphism between two cells can essentially be thought of as a transformation that maps the first cell "rigidly" to the second. More details are given in [3, Section 1.3]. Note that there are only two types of cells in the airplane limit space, the red one and the blue one (depicted in Figure 9), because no edge of any expansion is ever a loop.

Definition 3.1. A *cellular partition* of the limit space X is a cover of X by finitely many cells whose interiors are disjoint.

Note that there is a natural bijection between the set of expansions of a replacement system and the set of cellular partitions, which consists of mapping each edge e of the expansion to the cell $\chi(e)$.

Definition 3.2 ([3, Definition 1.14 (2)]). A homeomorphism $f: X \to X$ is called a *rearrangement* of X if there exists a cellular partition $\mathcal{P}$ of X such that f restricts to a canonical homeomorphism on each cell of $\mathcal{P}$.

It can be proved that the rearrangements of a limit space X form a group under composition, called the *rearrangement group* of X [3, Proposition 1.16]. In particular, the rearrangement group of the airplane limit space is denoted by T_A.

Similarly to how dyadic rearrangements (the elements of Thompson's groups) are specified by certain pairs of dyadic subdivisions, rearrangements of a limit space are specified by certain graph isomorphisms between expansions of the replacement systems, called *graph pair diagrams* [3, Section 1.4]. For example, the rearrangement of the airplane limit space depicted in Figure 10 is specified by the graph isomorphism depicted in the same figure, where the colors mean that each edge of the domain graph is mapped to the edge of the same color in the range graph.

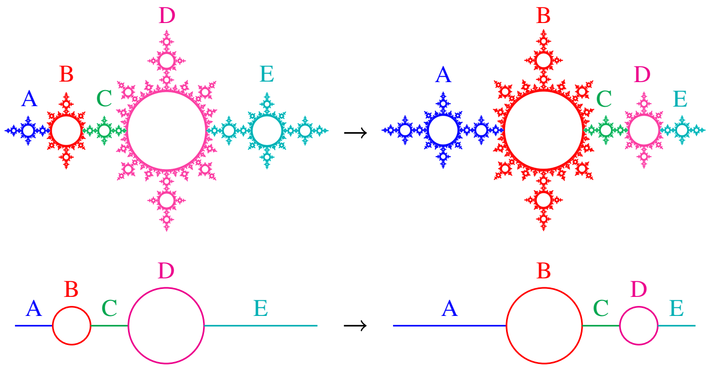


Figure 10. A rearrangement of the airplane limit space, along with a graph pair diagram that represents it.

Graph pair diagrams can be expanded by expanding an edge in the domain graph and its image in the range graph, resulting in a graph pair diagram that represents the same rearrangements. It is important to note that, for each rearrangement, there exists a unique *reduced* graph pair diagram, where reduced means that it is not the result of an expansion of any other graph pair diagram.

One may note that the graph isomorphism depicted in Figure 10 is not valid because the orientation of the green edge is reversed. In truth, it is not hard to see that expanding that edge in both domain and range graphs provides a valid graph pair diagram. This holds in general for blue edges, because there is a graph automorphism of the blue replacement graph that switches the initial and terminal vertices, so the expansion of a blue edge is independent of its orientation. In practice, this means that we do not need to keep track of the orientation of blue edges for graph pair diagrams. However, this is not true for red edges.

3.3. The rearrangement group of the basilica limit space

As another example of rearrangement group, consider the basilica limit space (Figure 11) resulting from the (monochromatic) basilica replacement system (Figure 12), whose rearrangement group T_B is the object of the study of Belk and Forrest in [2]. In particular, Belk and Forrest proved that T_B is generated by the four elements depicted in Figure 13, and that the commutator subgroup $[T_B, T_B]$ is simple. In Section 10, we will prove that T_A contains a natural copy of this group T_B.

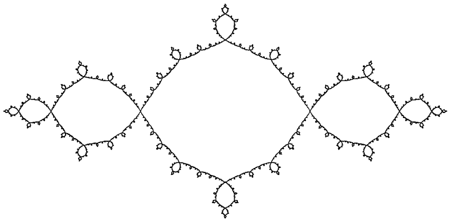


Figure 11. The basilica limit space B.

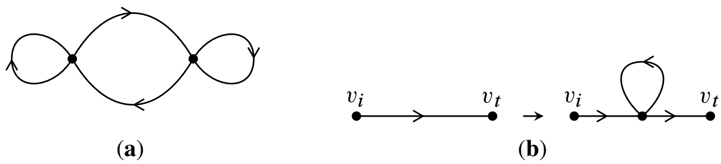


Figure 12. The basilica replacement system. (a) The base graph B_1. (b) The replacement rule $e \to R$.

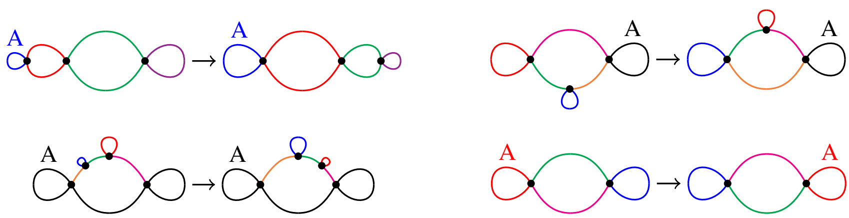


Figure 13. The four generators of the basilica rearrangement group T_B.

4. The airplane limit space and its rearrangements

In this section, we give a few definitions that are useful to work with the airplane limit space, and in Section 4.3 we exhibit five important elements of T_A that will later turn out to generate the entire group.

4.1. Components of the airplane limit space

Consider the airplane replacement system depicted in Figure 5. Note that each expansion of the base graph is a planar graph, thus it can be embedded in the Euclidean plane. When a blue edge is expanded, two new red edges are generated, and this pair of red edges encloses a connected region of the plane. These regions appear in each subsequent graph of the full expansion sequence, and then in the limit space, as exemplified in Figure 14. We call *component* any of these regions. The component colored in blue in Figure 14 is called *central component* and is denoted by C_0. It is not hard to see that elements of T_A map components to components.

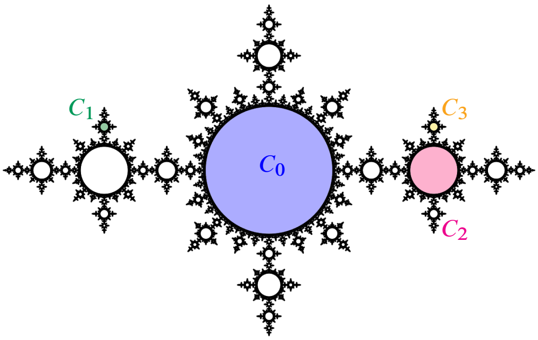


Figure 14. Examples of components of the airplane limit space.

Observe that the boundary ∂C of any component C is homeomorphic to

$$S^1 = [0, 2\pi]/\{0, 2\pi\},$$

and there is a natural homeomorphism under which the dyadic angles of S^1 (i.e., elements of $2\pi\mathbb{Z}[\frac{1}{2}]/\{0, 2\pi\} \subset S^1$) correspond to vertices between red edges. We will then refer to points of ∂C as *angles* of S^1 under this homeomorphism, under counterclockwise orientation.

4.2. Rays of the airplane limit space

We have already seen what blue cells look like in Figure 9. We say that a blue cell is a *primary blue cell* if it is maximal according to the set inclusion order. Note that these cells are adjacent to the central component, and that every other blue cell is included in exactly one of the primary blue cells.

Now, consider a blue edge e. The corresponding cell must depart from a certain component C, and it identifies a segment departing perpendicularly at some dyadic angle of ∂C. When expanding e, the segment is divided into three parts: the central one is split into two red edges that make up the boundary of a component, and the two other parts result in new blue edges that lie on the segment. By expanding these blue edges, each half of the segment is broken again in a similar fashion. Expanding the red edges generates new blue edges departing from the associated component. Then the blue cell $\chi(e)$ is made up of the segment identified by e, after it has been split into the boundaries of the infinitely many components that lie on that segment, along with the infinitely many other blue cells departing from each of those components.

Consider the blue cell $\chi(e)$ and subtract from it its two extremes and all of the blue cells that it includes and that do not lie on the segment. The resulting set is what we call an *interval*, and the subset denoted by I in Figure 10 is an example. Note that intervals are quite similar to blue cells, but not the same: each interval corresponds to the segment associated to some blue edge e, after it has been split into the boundaries of the infinitely many components that lie on it, but it does not include anything else that departs from these components. In this sense, intervals follow a unique direction, while blue cells also expand from many points of the boundaries of the included components into new rays.

What we are really interested in are those intervals that are maximal with respect to the set inclusion order, which means that they represent an entire segment departing from a component. We call these *rays*, and if a ray departs from the central component, we call it a *central ray*. Figure 15 shows two examples of rays, R_1 and R_2, of which R_1 is a central ray. Note that, differently from intervals, each ray is uniquely determined by the component it departs from along with the angle it departs at. This idea will be explored further in Section 4.4.

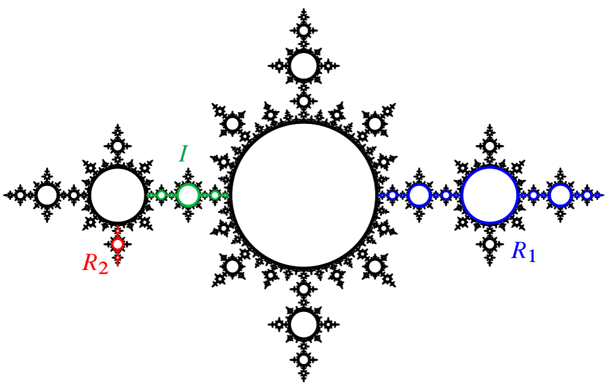


Figure 15. Examples of an interval, highlighted in green, and two rays. The blue ray is central, while the red one is not.

We also define the *right* and *left horizontal rays* as the central rays departing from the central component C_0 with angles 0 and π, respectively, and denote the right one by R_0. We then define Hor (which stands for *horizon*) as the union of the two horizontal rays and the boundary of the central component. Hor and C_0 will have great importance in the study of T_A, as described in the next section.

Finally, given two components, we say that they are *related* if one lies on a ray departing from the other. For example, in Figure 14 the component C_2 is related to both components C_3 (small on the right) and C_0, but these last two are not themselves related. Instead, the component C_1 (small on the left) is not related to any of the components colored in this picture.

4.3. The elements of T_A

Figure 16 depicts five specific elements of T_A: α, β, γ, δ and ε. In this figure, gray dotted lines show the action of these rearrangements. In Section 6, we will show that these elements generate the entire group T_A.

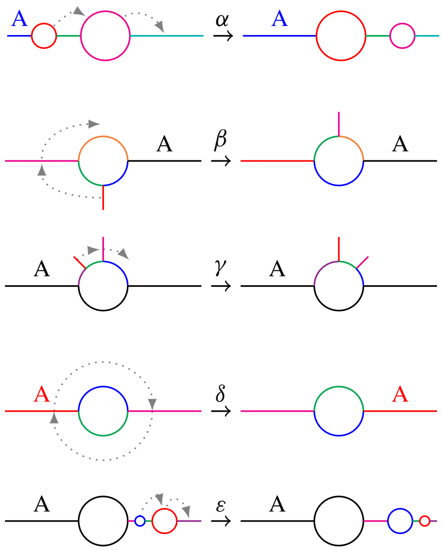


Figure 16. The five generators of T_A.

Those who are familiar with the basilica rearrangement group T_B studied in [2] might have noted that the first four of the five rearrangements we just introduced are very similar to the four rearrangements of the basilica depicted in Figure 13, which generate the entire T_B; this idea will be discussed in Section 10. The element ε instead does not correspond to any element of T_B; we will see in Section 5.2 that it allows T_A to act on rays as Thompson's group F does on $(0, 1)$.

4.4. Component paths

Consider the airplane replacement system $\mathcal{A}$ (Figure 5). We ignore the direction of the edges in the replacement system and give a new orientation as described below. Also, we identify each angle $2\pi k$ with the number k, which is the point of $S^1 = [0,1]/\{0,1\}$ that corresponds to the angle.

Note that at each step of the full expansion sequence, rays are generated by halving red edges and components are generated by halving blue edges. This means that rays departing from a fixed component correspond to certain dyadic numbers, and the same holds for components lying on a fixed ray. In particular, we identify each ray with the unit interval $(0,1)$, where 0 corresponds to the inner extreme, and we identify the border of each component with S^1, where the angle 0 corresponds to the direction that needs to be taken to travel back towards the central component (or the right direction if the component is the central one itself).

Let C be the component colored in green in Figure 17. We can build a "path" of a finite amount of pairwise related components starting from the central one and ending at C, and it is unique if we assume that it is minimal. This path is depicted in red in Figure 17, and it is specified by the following indications:

(1) departing from the central component at angle $\frac{1}{2}$, which identifies a ray;

(2) traveling on that ray for $\frac{1}{2}$ of its length, which identifies a component;

(3) departing from that component at angle $\frac{3}{4}$, which identifies a new ray;

(4) traveling on that ray for $\frac{1}{2}$ of its length, which identifies the component C.

We then say that this path is identified by $((\frac{1}{2},\frac{1}{2}),(\frac{3}{4},\frac{1}{2}))$.

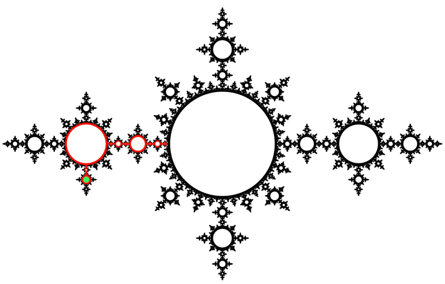


Figure 17. The component path $((\frac{1}{2},\frac{1}{2}),(\frac{3}{4},\frac{1}{2}))$, highlighted in red, and the component that it identifies, highlighted in green.

For each component C, there is a unique minimal path, built like the one we just described, that goes from the central component C_0 to C. This path is identified by the list of components at which the path takes a turn, which is why we call such paths by the name *component paths*. Note that the empty path represents the central component.

When citing a component path, we refer to the component of A that it identifies. We can represent the component path as the list $(C_1, C_2, \dots, C_n)$ of components at which the component path takes a turn. We can also use certain finite sequences of pairs of dyadic numbers: if $((\theta_1, l_1), (\theta_2, l_2), \dots, (\theta_n, l_n))$ is such a sequence, then θ_1 identifies the ray that departs from the central component at angle $2\pi\theta_1$, while l_1 identifies the component lying on the ray previously found that corresponds to the dyadic number l_1; the following pairs then work in the same way starting from the new component.

We also define the *depth* of a component in the airplane limit space to be the number of non-central components in its component path, including itself unless it is the central component. Note that if $(C_1, C_2, \dots, C_n)$ or $((\theta_1, l_1), (\theta_2, l_2), \dots, (\theta_n, l_n))$ is a component path for the component $C = C_n$, then n equals its depth.

5. Copies of Thompson's groups F and T in T_A

In this section, we exhibit two natural copies of Thompson's groups T and F contained in T_A that will have great importance in studying the action of T_A.

We say that a rearrangement f *extends canonically* on a cell $\chi(e)$ if it restricts to a canonical homeomorphism on $\chi(e)$, which intuitively means that f maps $\chi(e)$ "rigidly" to some other cell of the same type.

5.1. A copy of Thompson's group T in T_A

Definition 5.1. The *rigid stabilizer* of C_0, denoted by $\mathrm{rist}(C_0)$, is the group of all elements of T_A that map the component C_0 to itself and that extend canonically on the blue cells that depart from C_0.

It is clear that this is a subgroup of T_A. The name "rigid" comes from the fact that an element of $\mathrm{rist}(C_0)$ is determined entirely by its action on ∂C_0. Also note that we can equivalently define $\mathrm{rist}(C_0)$ as the group of all elements of T_A whose reduced graph pair diagrams have no component other that C_0.

Theorem 5.2. *We have* $\mathrm{rist}(C_0) = \langle \beta, \gamma, \delta \rangle \simeq T$*, and* $\mathrm{rist}(C_0)$ *acts on* ∂C_0 *as* T *does on* S^1*. In particular, its action is* 2*-transitive on the set of central rays.*

The proof of this theorem is essentially the same as the ones of [2, Theorem 6.3 and Corollary 6.4], with the proper definitions. It is important to note that β, γ and δ (Figure 16) act on ∂C_0 exactly as the three generators Y_0, Y_1 and Y_2 of Thompson's group T (Figure 4) act on S^1.

5.2. A copy of Thompson's group F in T_A

Definition 5.3. The *rigid stabilizer* of Hor, denoted by $\mathrm{rist}(\mathrm{Hor})$, is the group of all elements of T_A that map the horizon Hor to itself and that extend canonically on the red cells that make up components lying on Hor.

It is clear that this is a subgroup of T_A. The name "rigid" comes from the fact that an element of rist(Hor) is determined entirely by its action on Hor. Also note that we can equivalently define rist(Hor) as the group of all elements of T_A whose reduced graph pair diagrams have no ray other than the two horizontal ones.

Theorem 5.4. *We have* rist(Hor) $= \langle \alpha, \varepsilon \rangle \simeq F$*, and* rist(Hor) *acts on* Hor *as* F *does on* $[0, 1]$. *In particular, its action is transitive on the set of components lying on the horizon.*

The proof of this theorem is very similar to the one of Theorem 5.2, itself similar to the ones of [2, Theorem 6.3 and Corollary 6.4]. Note that the elements α and ε (Figure 16) act on Hor exactly as the two generators X_0 and X_1 of Thompson's group F (Figure 3) act on $[0, 1]$.

In a similar manner, we can study the subgroup $\langle \varepsilon, \varepsilon^{\alpha^{-1}} \rangle$ of rist(Hor), which we denote by rist(R_0). The two generators act on the right horizontal ray R_0 as the generators X_0 and X_1 of Thompson's group F act on $[0, 1]$, so we obtain the following assertion.

Proposition 5.5. *We have* rist(R_0) $\simeq F$*, and* rist(R_0) *acts on* R_0 *as* F *does on* $[0, 1]$. *In particular, its action is transitive on the set of components lying on* R_0.

We will use the subgroup rist(R_0) in the proofs of Lemma 6.1 and Theorem 7.12.

6. Generators of T_A

In this section, we prove that α, β, γ, δ and ε (depicted in Figure 16) generate T_A.

Lemma 6.1. *The group* $\langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$ *acts transitively on the set of components of* A.

Proof. Let C_n be any component of depth n. We will show that C_n can be mapped to the central component by an element of $\langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$ by induction on n.

If $n = 0$, then C_n is itself the central component, and we are done. Otherwise, C_n is connected to the central component C_0 by a component path $(C_0, C_1, \dots, C_n)$. Consider the component C_1, which is related to the central component, thus it lies on some central ray R. Because of Theorem 5.2, there exists an element $f \in \langle \beta, \gamma, \delta \rangle$ for which $f(R)$ is the right horizontal ray. Then $f(C_1)$ must be a component lying on the right horizontal ray and so, because of Proposition 5.5, there is a $g \in \langle \varepsilon, \varepsilon^{\alpha^{-1}} \rangle$ such that $(g \circ f)(C_1)$ is the component in the middle of the right horizontal ray. Then clearly $(\alpha^{-1} \circ g \circ f)(C_1)$ is the central component C_0.

It is easy to see that $((\alpha^{-1} \circ g \circ f)(C_1), \dots, (\alpha^{-1} \circ g \circ f)(C_n))$ is a component path, hence $(\alpha^{-1} \circ g \circ f)(C_n)$ has depth $n - 1$. By our induction hypothesis, $(\alpha^{-1} \circ g \circ f)(C_n)$ can be mapped to C_0 by $\langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$, thus C_n can as well. ∎

We will now prove that these five elements generate the entire T_A. Recall that if f and g are rearrangements, then in order to compute their composition $f \circ g$, we need to expand both their graph pair diagrams so that the range graph for g is the same as the domain graph for f.

Theorem 6.2. *The group T_A is generated by the elements $\{\alpha, \beta, \gamma, \delta, \varepsilon\}$.*

Proof. Let $f \in T_A$. By the previous lemma, up to replacing f by $g \circ f$ for a suitable $g \in \langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$, we may assume that $f(C_0) = C_0$. We must show that $f \in \langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$. We proceed by induction on the number n of non-central components in the reduced graph pair diagram for f.

The base graph for the airplane does not contain any non-central component, so the base case is $n = 0$. If $n = 0$, then $f \in \mathrm{rist}(C_0)$, which is generated by β, γ and δ because of Theorem 5.2; therefore, f belongs to $\langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$.

Suppose that $n \geq 1$. Since $f(C_0) = C_0$, the action of f permutes the points of departure of central rays from C_0, which correspond precisely to dyadic points of S^1. Because of Theorem 5.2, there exists an element $g \in \langle \beta, \gamma, \delta \rangle$ that permutes the points of departure of the central rays in the same way as f. Then the composition $h := g^{-1} \circ f$ fixes each of these points. It suffices to prove that $h \in \langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$.

Since g^{-1} does not have any non-central component in the domain graph for its reduced graph pair diagram, h has exactly n non-central components in its reduced graph pair diagram. Now, call $p_1, \ldots, p_m$ the points of departure of the central rays from C_0 in the reduced graph pair diagram for h. The right horizontal ray R_0 appears in every expansion of $\mathcal{A}$, so we can assume that p_1 is the point of adjacency between C_0 and R_0. Note that the component path for each non-central component travels through one and only one of these p_i. We now distinguish two cases.

Case 1: Suppose that the reduced graph pair diagram for h has non-central components with component paths that travel through more than one p_i (as in Figure 18). Then we can express h as a composition $h_1 \circ \cdots \circ h_m$, where each h_i has a graph pair diagram obtained from the reduced graph pair diagram for h by removing all non-central components except for those with component paths that travel through p_i (along with every ray that departs from the components removed this way). For example, Figure 19 depicts h_1 and h_2 for the h in Figure 18. Then each h_i must have fewer than n non-central components. By induction, it follows that each $h_i \in \langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$, and therefore $h \in \langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$ as well.

Case 2: Suppose that all non-central components in the reduced graph pair diagram for h have component paths that travel through the same p_i. Because of Theorem 5.2, there exists a $\sigma \in \langle \beta, \gamma, \delta \rangle$ such that $\sigma(p_1) = p_i$. Let $k := h^\sigma$, and note that we can conjugate h by σ without expanding the blue cells involved in the action of h, so k has at most n

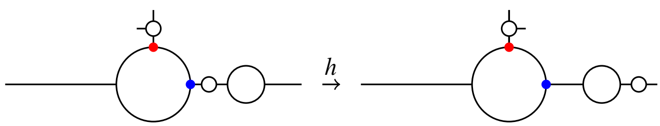


Figure 18. An example for case 1 of the proof of Theorem 6.2. The two colored points represent p_1 and p_2.

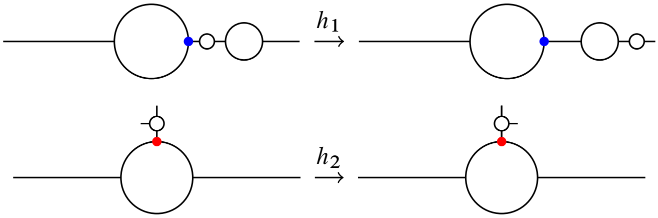


Figure 19. The element h in Figure 18 can be written as $h_1 \circ h_2$ since it has two center-adjacent components.

non-central components. Moreover, all the non-central components of the reduced graph pair diagram for k have component paths that travel through p_1. It suffices to prove that $k \in \langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$.

Consider the reduced graph pair diagram for k, which is sketched in Figure 20. Both the domain and the range graphs only have non-central components in the right primary blue cell. Let C_D be the component on the right horizontal ray that is closest to C_0 in the domain graph, and let C_R be the component in the range graph that is the closest to C_0, which must be $k(C_D)$. Note that $C_D = ((0, \frac{1}{2^d}))$ and $C_R = ((0, \frac{1}{2^r}))$ for some $d \geq 1$ and $r \geq 1$. Therefore, $\varepsilon^{d-1}(C_D) = ((0, \frac{1}{2}))$ and $\varepsilon^{r-1}(C_R) = ((0, \frac{1}{2}))$, so both the domain and the range graphs of the reduced graph pair diagram for $l := \varepsilon^{r-1} \circ k \circ \varepsilon^{-d+1}$ have $((0, \frac{1}{2}))$ as the closest component to C_0 on the right horizontal ray. Hence both domain and range graphs in the reduced graph pair diagram for l are such that no component lies on the inner half of R_0, as shown in Figure 20, and the number of non-central components in the reduced graph pair diagram for l is the same as the one for k. It suffices to prove that $l \in \langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$.

We can now conjugate l by α without performing any expansion of the cells involved in the action of l. The resulting graph pair diagram for $l^\alpha = \alpha^{-1} \circ l \circ \alpha$ is shown in Figure 20: it is similar to the one of l, but the action is shifted to the left, and the central

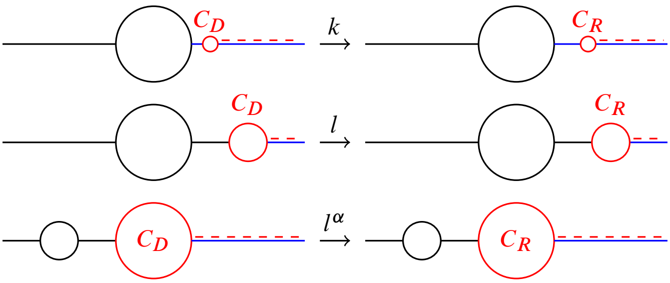


Figure 20. Sketches of k, l and l^α for case 2 of the proof of Theorem 6.2. Everything drawn in black is fixed pointwise, while everything colored may not be; rays may depart from red components, and components may lie on dashed red lines.

component in both the domain and range graphs for l ends up in $((\frac{1}{2}, \frac{1}{2}))$ in the graphs for l^α, with no ray departing from it, so it can then be reduced. Then l^α has $n-1$ or less non-central components, so by induction we have that l^α belongs to $\langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$. Thus l does too. ■

Remark 6.3. Recall from Theorems 5.2 and 5.4 that $\mathrm{rist}(C_0) = \langle \beta, \gamma, \delta \rangle$ and $\mathrm{rist}(\mathrm{Hor}) = \langle \alpha, \varepsilon \rangle$. Therefore, as a consequence of Theorem 6.2, we have that

$$T_A = \langle \mathrm{rist}(C_0), \mathrm{rist}(\mathrm{Hor}) \rangle,$$

where $\mathrm{rist}(C_0)$ and $\mathrm{rist}(\mathrm{Hor})$ are arguably the two "most natural" copies of Thompson's groups T and F in T_A, respectively.

Question 6.4. We have just proved that T_A is finitely generated. Is T_A also finitely presented? We recall that T_B is not, as proved by Witzel and Zaremsky in [11].

7. The commutator subgroup of T_A

In this section, we study the commutator subgroup of T_A. We first find a characterization of the rearrangements of $[T_A, T_A]$ in terms of their action on the extremities of the airplane, and we find an infinite generating set along the way. Then we prove that $[T_A, T_A]$ is simple, and finally we find a finite generating set for $[T_A, T_A]$.

Remark 7.1. It is easy to see that $[T_A, T_A]$ is quite large. Indeed,

(1) $[T_A, T_A] \geq \mathrm{rist}(C_0)$ since $\mathrm{rist}(C_0) \simeq T$ (Theorem 5.2) and $T = [T, T]$.

(2) $[T_A, T_A] \geq [\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})]$, which is the group of those rearrangements of $\mathrm{rist}(\mathrm{Hor})$ that act trivially at the left and right extremes of Hor, since $\mathrm{rist}(\mathrm{Hor}) \simeq F$ (Theorem 5.4).

Since $\mathrm{rist}(C_0) = \langle \beta, \gamma, \delta \rangle$, we have that $\beta, \gamma, \delta \in [T_A, T_A]$. A direct computation shows that $\alpha = [\varepsilon, \delta] \circ [\varepsilon^{-1}, \alpha^{-2}]$, so the commutator subgroup of T_A also contains α. Hence, we already know that $[T_A, T_A]$ contains four out of five generators of T_A. We will see along the way that it does not contain ε.

7.1. A characterization of the commutator subgroup

Let $\mathcal{E}$ be the set of all external extremes of rays in the airplane limit space. Note that all rearrangements $f \in T_A$ act on the set $\mathcal{E}$ by permutation. We now define a concept of derivative of a rearrangement of A around an extreme, which gives an idea of how much the rearrangement dilates or shrinks the extremities of A compared to the length of the respective ray. We then study the product of all these derivatives, which gives a characterization for the commutator subgroup of T_A.

Let e be a blue edge of an expansion of the airplane replacement system. Note that e lies on a unique ray R, and it corresponds to a portion of R: it can be half of it, a quarter, an

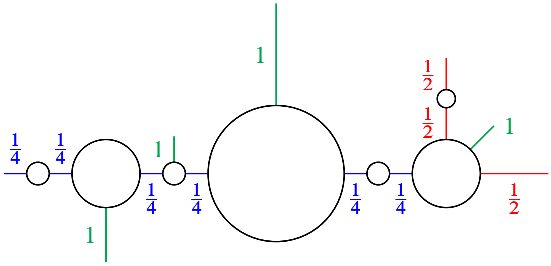


Figure 21. Examples of lengths of blue edges: green, red and blue ones have lengths 1, $\frac{1}{2}$ and $\frac{1}{4}$, respectively.

eighth, etc. More precisely, it can be $\frac{1}{2^k}$ for any $k \in \mathbb{N}$. We call *length* of e this number $\frac{1}{2^k}$. Figure 21 depicts a few examples. Additionally, if $\chi(e)$ is the blue cell generated by the blue edge e, we define the *length* of $\chi(e)$ to be the same as the length of e.

Let $p \in \mathcal{E}$ and $f \in T_A$. Consider the reduced graph pair diagram for f. If p does not appear in the domain graph, we define $D_p(f)$ to be 1. Otherwise, p corresponds to the extremity of some blue edge e_p in the graph. Then $f(e_p)$ is the blue edge that appears in the range graph whose extreme is $f(p)$. Let $\frac{1}{2^d}$ be the length of e_p and $\frac{1}{2^r}$ the length of $f(e_p)$. We define the *extremal derivative* of f in p as the ratio of the length of $f(e_p)$ to the length of e_p, which is

$$D_p(f) := 2^{r-d}.$$

This represents how much the action of f dilates or shrinks the extremity around p when compared to the ray of which p and $f(p)$ are the external extremes.

Finally, we define the *global extremal derivative* of an element f of T_A as the product of all its extremal derivatives, which is

$$D(f) := \prod_{p \in \mathcal{E}} D_p(f).$$

Note that $D_p(f)$ equals 1 for each extreme p that does not appear in the domain graph of the reduced graph pair diagram for f, so this product only has a finite amount of non-trivial factors.

Now, note that $D_p(f \circ g) = D_{g(p)}(f) \cdot D_p(g)$ for all $f, g \in T_A$ and for all $p \in \mathcal{E}$. Therefore, since g permutes the set $\mathcal{E}$, we have that

$$D(f \circ g) = D(f) \cdot D(g).$$

Hence, the map $D \colon T_A \to \langle 2 \rangle_{\mathbb{Q}^*}$ is a group morphism, where $\langle 2 \rangle_{\mathbb{Q}^*}$ is the multiplicative group of all the integer powers of 2, which is an infinite cyclic group.

Theorem 7.2. *We have* $[T_A, T_A] = \mathrm{Ker}(D)$.

Proof. Note that $D(\varepsilon^{-k}) = 2^k$ for each integer k, so D is surjective. Hence,

$$T_A / \operatorname{Ker}(D) \simeq \langle 2 \rangle_{\mathbb{Q}^*},$$

which is abelian, and so $[T_A, T_A] \leq \operatorname{Ker}(D)$. We only need to prove that $\operatorname{Ker}(D) \leq [T_A, T_A]$.

Let $f \in \operatorname{Ker}(D)$. Since $T_A = \langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$, the rearrangement f can be written as $f_1 \circ f_2 \ldots f_k$, where each f_i is a generator or an inverse of one. Since $D(f) = 1$ and $D(\alpha) = D(\beta) = D(\gamma) = D(\delta) = 1$, the number of ε's among the f_i must be equal to the number of ε^{-1}'s. Then f is the product of elements chosen from the set $\{\alpha^{\varepsilon^j}, \beta^{\varepsilon^j}, \gamma^{\varepsilon^j}, \delta^{\varepsilon^j} \mid j \in \mathbb{Z}\}$. Since α, β, γ and δ all belong to the commutator subgroup (Remark 7.1) and $[T_A, T_A]$ is normal in T_A, these elements all belong to $[T_A, T_A]$. Therefore, $f \in [T_A, T_A]$, and so $\operatorname{Ker}(D) = [T_A, T_A]$. ∎

As a consequence, we can immediately find an infinite generating set.

Corollary 7.3. *We have* $[T_A, T_A] = \langle \beta, \gamma, \alpha^{\varepsilon^k}, \delta^{\varepsilon^k} \mid k \in \mathbb{Z} \rangle$, *which is the normal closure of the subgroup* $H := \langle \alpha, \beta, \gamma, \delta \rangle$ *in* T_A.

Proof. As seen in the proof of the previous theorem, $\operatorname{Ker}(D) = \langle \alpha^{\varepsilon^j}, \beta^{\varepsilon^j}, \gamma^{\varepsilon^j}, \delta^{\varepsilon^j} \mid j \in \mathbb{Z} \rangle$, which is clearly the normal closure of H.

Since the supports of both β and γ have empty intersections with the support of ε, we have that $\beta^\varepsilon = \beta$ and $\gamma^\varepsilon = \gamma$, so $[T_A, T_A] = \operatorname{Ker}(D) = \langle \beta, \gamma, \alpha^{\varepsilon^k}, \delta^{\varepsilon^k} \mid k \in \mathbb{Z} \rangle$. ∎

Since $D \colon T_A \to \langle 2 \rangle_{\mathbb{Q}^*}$ is surjective and $\langle 2 \rangle_{\mathbb{Q}^*}$ is an infinite cyclic group, applying the first isomorphism theorem to D immediately gives us the following result.

Corollary 7.4. *The abelianization of* T_A *is an infinite cyclic group. In particular, the index of* $[T_A, T_A]$ *in* T_A *is infinite.*

Finally, note that $[T_A, T_A] \cap \langle \varepsilon \rangle = \emptyset$ because $D(\varepsilon^k) = 1 \Leftrightarrow k = 0$ while $[T_A, T_A] = \operatorname{Ker}(D)$, and that $[T_A, T_A]\langle \varepsilon \rangle$ contains the five generators of T_A. Then we obtain the following assertion.

Corollary 7.5. *We have* $T_A = [T_A, T_A] \rtimes \langle \varepsilon \rangle$.

7.2. The simplicity of the commutator subgroup

We start by noting certain transitivity properties of $[T_A, T_A]$ that we will use later.

Lemma 7.6. *The commutator subgroup* $[T_A, T_A]$ *acts transitively on the set of components of* A.

The proof of this follows the same outline of the proof of Lemma 6.1. By induction on the depth n of a component C_n whose component path is $(C_0, C_1, \ldots, C_n)$, we find $f \in \langle \beta, \gamma, \delta \rangle \leq [T_A, T_A]$ such that $f(C_1)$ lies on the right horizontal ray R_0. Then, since $\operatorname{rist}(\mathrm{Hor}) \simeq F$ and $[F, F]$ is transitive on the set of dyadic points of $(0, 1)$, we find $g \in [\operatorname{rist}(\mathrm{Hor}), \operatorname{rist}(\mathrm{Hor})] \leq [T_A, T_A]$ such that $g \circ f(C_1) = C_0$, and our induction hypothesis does the rest.

As a consequence of this lemma, we have the following assertion.

Corollary 7.7. *The commutator subgroup* $[T_A, T_A]$ *acts transitively on the set of adjacency points between a red and a blue cell.*

The proof consists of showing that an adjacency point between a red and a blue cell can be mapped to such a point lying on ∂C_0, which is easy using the previous lemma and the transitivity of $\mathrm{rist}(C_0) \leq [T_A, T_A]$ (Theorem 5.2).

We will find additional transitivity properties of $[T_A, T_A]$ in Section 9. Now, with the aid of this corollary, we are ready to prove the simplicity of $[T_A, T_A]$.

Theorem 7.8. *The commutator subgroup* $[T_A, T_A]$ *is simple.*

This follows immediately from the following more general result.

Proposition 7.9. *The commutator subgroup* $[T_A, T_A]$ *is the only non-trivial proper subgroup of* T_A *that is normalized by* $[T_A, T_A]$.

Proof. This proof shares the overall structure with the proof of [2, Theorem 8.4], which itself follows the basic outline for the proof of the simplicity of T, which in turn is based on the work [8] of Epstein on the simplicity of groups of diffeomorphisms. In particular, we use Epstein's double commutator trick, together with an argument that T_A is generated by elements of small support.

Let N be a non-trivial subgroup of T_A that is normalized by $[T_A, T_A]$. We wish to prove that $N = [T_A, T_A]$, and we will do so by proving that N contains each of the generators of the commutator subgroup exhibited in Corollary 7.3.

Let f be a non-trivial element of N. There must exist a small enough cell Δ of the airplane limit space such that Δ and $f(\Delta)$ are disjoint. Then, for all subsets I of Δ, we have that I and $f(I)$ are disjoint. We will now prove a property that holds for every such I; we denote this property by ($\star$), and we will later use it twice.

Consider any two elements g and h of $[T_A, T_A]$ with support contained in I. The conjugate $f \circ g^{-1} \circ f^{-1}$ has support in $f(I)$, so $[g, f] = g \circ f \circ g^{-1} \circ f^{-1}$ has support in $I \cup f(I)$ and agrees with g on I. Since h has support in I, it follows that

$$[[g, f], h] = [g, h].$$

Since N is normalized by $[T_A, T_A]$ by hypothesis, the double commutator on the left must be an element of N, so we have proved that

$$[g, h] \in N \quad \text{for all } g, h \in [T_A, T_A] \text{ with support in } I. \tag{$\star$}$$

We will use property ($\star$) for two distinct choices of the subset I of Δ. For each of these choices, we will conjugate ($\star$) by some properly chosen $k^{-1} \in [T_A, T_A]$; then, since N is normalized by $[T_A, T_A]$, we will have that

$$[g, h] \in N \quad \text{for all } g, h \in [T_A, T_A] \text{ with support in } k(I).$$

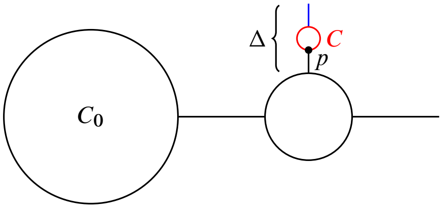


Figure 22. An example of the first choice of I, which is the union of the blue cell and the two red ones. The figure is "zoomed in" on the right horizontal ray for better clarity.

First choice of $I \subseteq \Delta$. Consider a component C that is contained in Δ. Let I be the union of the blue cell that departs from C at angle $\frac{1}{2}$ and the two maximal red cells that make up the boundary of C. Call p the point of ∂C located at angle 0. Figure 22 shows an example of such I and p.

Now, because of Corollary 7.7, there exists an element $k \in [T_A, T_A]$ that maps p to the point of adjacency between the central component and the left horizontal ray. Then $k(I)$ must be the union of the right horizontal blue cell and the two maximal red cells that make up the boundary of C_0. We denote this set by K.

Note that I is included in Δ, so the property ($\star$) holds. Then, conjugating ($\star$) by k^{-1}, we find that

$$[g, h] \in N \quad \text{for all } g, h \in [T_A, T_A] \text{ with support in } K.$$

Next recall that $\mathrm{rist}(C_0) \leq [T_A, T_A]$ (Remark 7.1), so $[g, h] \in N$ for all $g, h \in \mathrm{rist}(C_0)$ with support in K. Also recall from Theorem 5.2 that $\mathrm{rist}(C_0)$ is isomorphic to Thompson's group T; under this isomorphism, the set of those elements of $\mathrm{rist}(C_0)$ whose support is included in K corresponds to the stabilizer of $\frac{1}{2}$ in T, which is a copy of Thompson's group F. Since F is not abelian, there exist at least two elements $g, h \in \mathrm{rist}(C_0)$ with support in K for which $[g, h]$ is non-trivial. Then $[g, h]$ belongs to both N and $\mathrm{rist}(C_0)$, so the intersection $N \cap \mathrm{rist}(C_0)$ is a non-trivial normal subgroup of $\mathrm{rist}(C_0)$. But $\mathrm{rist}(C_0) \simeq T$ and T is simple, so $N \cap \mathrm{rist}(C_0) = \mathrm{rist}(C_0)$, and therefore $\mathrm{rist}(C_0) \leq N$.

Second choice of $I \subseteq \Delta$. Let C be a component that is contained in Δ. Let R_1 and R_2 be distinct rays that depart from C, and let p_1 be the point of adjacency between C and R_1, and p_2 between C and R_2. Call I the union of ∂C, $\chi(R_1)$ and $\chi(R_2)$, where $\chi(R_i)$ are the blue cells associated with the rays R_i. Figure 23 shows an example of I, p_1 and p_2.

Because of Corollary 7.7, there exists an element $k_1 \in [T_A, T_A]$ that maps p_1 to the point of adjacency between the central component and the right horizontal ray. Then $k_1(\partial C) = \partial C_0$, $k_1(\chi(R_1)) = \chi(R_0)$, $k_1(\chi(R_2))$ is some blue cell adjacent to C_0, and so $k_1(I)$ is the union of these three sets.

Now, since T acts 2-transitively on the set of dyadic points of S^1 and because of Theorem 5.2, there exists an element $k_2 \in \mathrm{rist}(C_0)$ that fixes $k_1(p_1)$ and maps $k_1(p_2)$ to

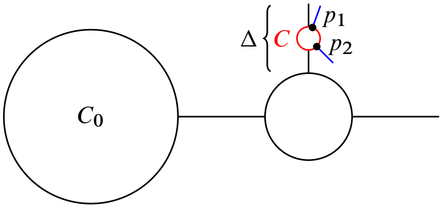


Figure 23. An example of the second choice of I, which is the union of ∂C and two blue cells departing from C. The figure is "zoomed in" on the right horizontal ray for better clarity.

the point of adjacency between the central component and the left horizontal ray. Then $k_2 \circ k_1(I)$ is the union of ∂C_0 and the two horizontal central blue cells, and we denote this set by K. Also, since $\mathrm{rist}(C_0) \leq [T_A, T_A]$ (Remark 7.1), we have that $k_2 \in [T_A, T_A]$, therefore $k_2 \circ k_1 \in [T_A, T_A]$.

Note that I is included in Δ, so the property ($\star$) holds. Then, conjugating ($\star$) by $k_2 \circ k_1$, we find that

$$[g, h] \in N \quad \text{for all } g, h \in [T_A, T_A] \text{ with support in } K.$$

Next, recall from Theorem 5.4 that the group rist(Hor) is isomorphic to Thompson's group F. Under this isomorphism, the set of those elements of rist(Hor) whose support is included in K corresponds to the stabilizer of $\frac{1}{2}$. Since $\frac{1}{2}$ corresponds to the central component, this means exactly that

$$[g, h] \in N \quad \text{for all } g, h \in S(C_0), \tag{†}$$

where $S(C_0) := \mathrm{stab}_{[\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})]}(C_0)$ is the group consisting of those elements of $[\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})]$ that fix the component C_0.

Now, since $[F, F]$ is transitive on the set of dyadic points of $(0, 1)$, for each component C' lying on Hor, there exists an element l of $[\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})] \leq [T_A, T_A]$ such that $l(C') = C_0$. Conjugating the group $S(C_0)$ by l, we clearly obtain exactly the group $S(C') := \mathrm{stab}_{[\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})]}(C')$ consisting of those elements of $[\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})]$ that fix C'. Since we can do this for each component C' lying on Hor, conjugating (†) by each l found this way, we have that

$$[g, h] \in N \quad \text{for all } g, h \in S(C'), \text{ for all } C' \text{ lying on Hor}. \tag{\#}$$

We now prove that $[\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})] \leq N$. First note that, since $\mathrm{rist}(\mathrm{Hor}) \simeq F$ and $[F, F] = F''$ (where F'' denotes the group $[[F, F], [F, F]]$), it suffices to prove that $\mathrm{rist}(\mathrm{Hor})'' \leq N$. By definition, $\mathrm{rist}(\mathrm{Hor})''$ is generated by the elements $[g, h]$ for $g, h \in [\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})]$, so we only need to prove that these elements belong to N. If $g, h \in [\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})]$, then they both act trivially around the extremes of Hor, so the intersection of their supports cannot be the entire Hor and there must be some "external enough" component C' lying on Hor that is fixed by both g and h, which means that $g, h \in S(C')$. Therefore, because of (#), their commutator $[g, h]$ belongs to N. So $[\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})] \leq N$.

So far we have shown that

(1) $\mathrm{rist}(C_0) = \langle \beta, \gamma, \delta \rangle \leq N$;

(2) $[\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})] = [\langle \alpha, \varepsilon \rangle, \langle \alpha, \varepsilon \rangle] \leq N$.

With this in mind, we now show that N contains each element of the set $\{\beta, \gamma, \alpha^{\varepsilon^k}, \delta^{\varepsilon^k}\}$, which generates $[T_A, T_A]$ as seen in Corollary 7.3.

Since $\mathrm{rist}(C_0) \leq N$, we have that β, γ and δ belong to N. Consider δ^{ε^k} for any $k \in \mathbb{Z}$: with a direct computation, we note that $\delta \circ \beta$ has order three and $\delta = \delta^{-1}$, so $\delta = \delta^{-1} = \beta \circ \delta \circ \beta \circ \delta \circ \beta$, and then $\delta^{\varepsilon^k} = (\beta \circ \delta \circ \beta \circ \delta \circ \beta)^{\varepsilon^k}$. Since β and ε^k have disjoint supports, we have that $\beta^{\varepsilon^k} = \beta$. Therefore, $\delta^{\varepsilon^k} = \beta \circ \delta^{\varepsilon^k} \circ \beta \circ \delta^{\varepsilon^k} \circ \beta = \beta \circ \beta^{\delta^{\varepsilon^k}} \circ \beta$, which belongs to N because $\beta \in N$ and $\delta^{\varepsilon^k} \in [T_A, T_A] \trianglerighteq N$. Then $\delta^{\varepsilon^k} \in N$ for all $k \in \mathbb{Z}$, and we only need to prove that $\alpha^{\varepsilon^k} \in N$.

Recall that, as noted in Remark 7.1, $\alpha = [\varepsilon, \delta] \circ [\varepsilon^{-1}, \alpha^{-2}]$, which can be seen with a direct computation. Note that

$$[\varepsilon, \delta] = \delta^{\varepsilon^{-1}} \circ \delta \in N \quad \text{and} \quad [\varepsilon^{-1}, \alpha^{-2}] \in [\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})] \leq N,$$

and so $\alpha \in N$. Finally, note that $\alpha^{\varepsilon^k} = \alpha \circ [\alpha^{-1}, \varepsilon^{-k}] \in \alpha\, [\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})] \leq N$ for all $k \in \mathbb{Z}$.

Therefore, β, γ, δ^{ε^k} and α^{ε^k} belong to N for all $k \in \mathbb{Z}$, so $[T_A, T_A] \leq N$, which is what we needed to prove. ∎

Question 7.10. Having just proved that $[T_A, T_A]$ is finitely generated, it is natural to ask if it is finitely presented or if it is not, as is the case for T_B and $[T_B, T_B]$ (see [11]). We have not investigated this question when writing this paper.

7.3. The commutator subgroup is finitely generated

In this subsection, we exhibit a finite set of generators for the commutator subgroup $[T_A, T_A]$.

Remark 7.11. It is easy to prove that $[\delta, \varepsilon]^k = [\delta, \varepsilon^k]$ using the fact that ε^{-1} and ε^{δ} have disjoint supports.

Theorem 7.12. *We have* $[T_A, T_A] = \langle \alpha, \beta, \gamma, \delta, [\delta, \varepsilon], [\varepsilon^{-1}, \varepsilon^{-1} \circ \alpha] \rangle$.

Proof. Consider $G := \langle \alpha, \beta, \gamma, \delta, [\delta, \varepsilon], [\varepsilon^{-1}, \varepsilon^{-1} \circ \alpha] \rangle$, which is clearly a subgroup of $[T_A, T_A]$. We will show that G is the entire commutator subgroup of T_A by proving that it contains the infinite generating set $\{\beta, \gamma, \alpha^{\varepsilon^k}, \delta^{\varepsilon^k}\}$ of Corollary 7.3.

Clearly, both β and γ belong to G. Also note that thanks to Remark 7.11, we have $\delta^{\varepsilon^k} = \delta \circ [\delta, \varepsilon^{-k}] = \delta \circ [\delta, \varepsilon]^{-k}$, which belongs to G. Hence, we only need to prove that $\alpha^{\varepsilon^k} \in G$ for all $k \in \mathbb{Z}$.

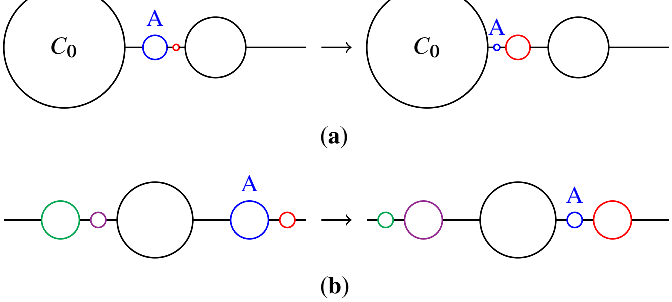


Figure 24. The two generators of F_0. (a) The element $[\varepsilon^{-1}, \varepsilon^{-1} \circ \alpha]$. The left horizontal ray is fixed, hence it is omitted. (b) The element $[\delta, \varepsilon]$.

Let $F_0 := \langle [\delta, \varepsilon], [\varepsilon^{-1}, \varepsilon^{-1} \circ \alpha] \rangle \leq G$. The generators are depicted in Figure 24. It is not hard to see that F_0 acts on R_0 as Thompson's group F does on $[0, 1]$, which is also the same way $\mathrm{rist}(R_0)$ acts on R_0 (Proposition 5.5). We remark that the overall action of F_0 on the airplane limit space is not trivial outside of R_0.

Let $r, l \in \mathcal{E}$ be the external extremes of the right and the left horizontal rays, respectively, and let f be an element of F_0 such that $D_r(f) = 1$. Then $D_l(f) = 1$ as well, because $f \in [T_A, T_A]$ and every other extremal derivative is clearly trivial. Now, since $f \in F_0$, there exists a finite product of elements chosen among $\{[\delta, \varepsilon], [\varepsilon^{-1}, \varepsilon^{-1} \circ \alpha]\}$ that equals f. Since both these elements fix r and l, and since all extremal derivatives of $[\varepsilon^{-1}, \varepsilon^{-1} \circ \alpha]$ are trivial, the value of $D_l(f)$ only depends on the total sum of the exponents of $[\delta, \varepsilon]$ in that product, so that total sum must be zero. Now, since $[\varepsilon^{-1}, \varepsilon^{-1} \circ \alpha]$ acts trivially on the left horizontal ray, the action of f on that ray only depends on the total sum of exponents of $[\delta, \varepsilon]$. Then f must act trivially on the left horizontal ray, which means that f only acts on R_0. This proves that all elements $f \in F_0$ such that $D_r(f) = 1$ belong to $\mathrm{rist}(R_0)$.

Now consider an element g of $\mathrm{rist}(R_0) \cap [T_A, T_A]$: since F_0 acts on R_0 as $\mathrm{rist}(R_0)$ does, there exists an $f \in F_0$ such that f acts on R_0 as g does. Then note that $D_r(f) = D_r(g)$ (because r is the external extreme of R_0), and $D_r(g) = 1$ because $g \in [T_A, T_A]$, so $D_r(f) = 1$. Then, as noted right above, f belongs to $\mathrm{rist}(R_0)$, and therefore it acts exactly as g does on the entire Hor. Since both f and g are elements of $\mathrm{rist}(\mathrm{Hor})$, they are entirely determined by their action on Hor, so $f = g$. Therefore, we have that $\mathrm{rist}(R_0) \cap [T_A, T_A] \leq F_0$.

Finally, let us prove that $\alpha^{\varepsilon^k} \in G$. Since $\alpha \in G$ and $\alpha^{\varepsilon^k} = [\varepsilon^{-k}, \alpha] \circ \alpha$, it suffices to prove that $[\varepsilon^{-k}, \alpha] \in G$. Note that $[\varepsilon^{-k}, \alpha]$ acts trivially on the left horizontal ray, and so it belongs to $\mathrm{rist}(R_0)$. Then

$$[\varepsilon^{-k}, \alpha] \in [T_A, T_A] \cap \mathrm{rist}(R_0) \leq F_0 \leq G,$$

and we are done. ■

8. Thompson's group T contains a copy of T_A

In this section, we show that T contains an isomorphic copy of T_A.

Let $\mathcal{C}(\mathcal{A})$ be the replacement system whose set of colors is {blue, red} and having base graph and replacement graphs as depicted in Figure 25. It is easy to see that this replacement system is such that its limit space exists (see [3, Proposition 1.22]), hence its rearrangement group exists, and we denote it by C_A. We will now prove that T contains a copy of C_A and then that C_A contains a copy of T_A.

A replacement system is said to be *circular* if its base graph is a closed path and each of its replacement graphs consists solely of a path. Note that each expansion of a circular replacement system is always a closed path. Then it is not hard to prove that Thompson's group T contains an isomorphic copy of any rearrangement group of a circular replacement system. It is clear that $\mathcal{C}(\mathcal{A})$ is a circular replacement system, so T contains an isomorphic copy of C_A.

Then we only need to prove that C_A contains an isomorphic copy of T_A. We will first define an injective map $\Phi\colon \mathbb{E}_{\mathcal{A}} \to \mathbb{E}_{\mathcal{C}(\mathcal{A})}$, where $\mathbb{E}_{\mathcal{A}}$ is the set of all expansions of the airplane replacement system $\mathcal{A}$ and $\mathbb{E}_{\mathcal{C}(\mathcal{A})}$ is the set of all expansions of $\mathcal{C}(\mathcal{A})$. Then we will use Φ to build an injective group morphism $\phi\colon T_A \to C_A$.

Note that the base graph Γ_C of $\mathcal{C}(\mathcal{A})$ can be obtained from the base graph of $\mathcal{A}$ by splitting each vertex into two except for the two extremes, as shown in Figure 26. We define $\Phi(\Gamma) := \Gamma_C$. Each red edge of Γ is mapped to a red edge of Γ_C, while each blue edge is split into a pair of blue edges. Conversely, each red edge of Γ_C descends from

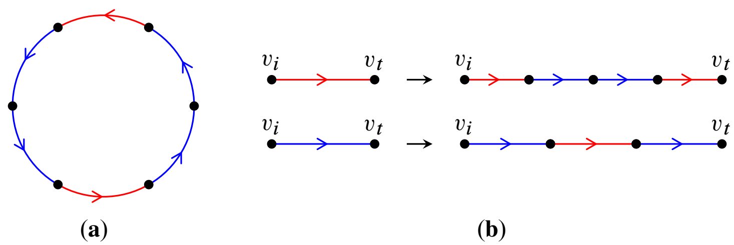


Figure 25. The replacement system $\mathcal{C}(\mathcal{A})$. (a) The base graph Γ_C. (b) The replacement rules.

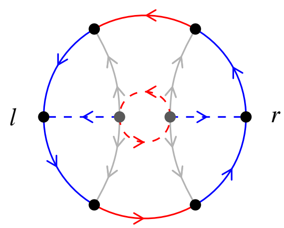


Figure 26. The base graph Γ_C of $\mathcal{C}(\mathcal{A})$ obtained from the base graph Γ of $\mathcal{A}$ (drawn with dashed lines).

a unique red edge of Γ, while certain pairs of blue edges of Γ_C descend from a unique blue edge of Γ. Informally, this gives a correspondence between the edges of Γ and those of Γ_C that is one-to-one between red edges and one-to-two between blue edges.

We now extend this correspondence to any expansion of the airplane replacement system $\mathcal{A}$. If E is an expansion of $\mathcal{A}$ and e is one of its edges, we denote by $E \lhd e$ the simple expansion obtained by replacing e in E. Since each expansion E of $\mathcal{A}$ is a finite sequence of simple expansions, it suffices to define $\Phi(E \lhd e)$ starting from $\Phi(E)$. We do this in the following way, depending on the color of the edge e:

- if e is red, then $\Phi(E \lhd e)$ comes from $\Phi(E)$ by replacing its red edge corresponding to e according to the red replacement rule of $\mathcal{C}(\mathcal{A})$ (Figure 27);
- if e is blue, then $\Phi(E \lhd e)$ comes from $\Phi(E)$ by replacing the blue edges corresponding to e according to the blue replacement rule of $\mathcal{C}(\mathcal{A})$ (Figure 28).

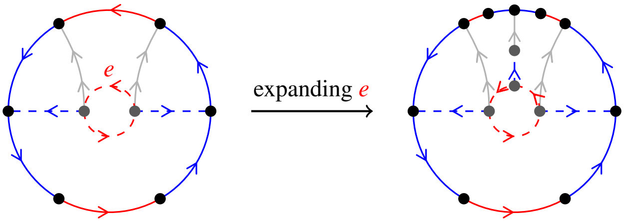


Figure 27. An example of $\Phi(E \lhd e)$ from $\Phi(E)$, where e is a red edge.

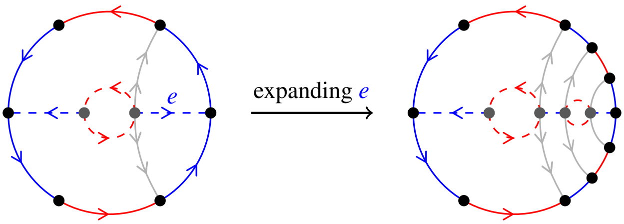


Figure 28. An example of $\Phi(E \lhd e)$ from $\Phi(E)$, where e is a blue edge.

Note that we still have a correspondence between edges of E and edges of $\Phi(E)$ that is one-to-one between red edges and "one-to-two" between blue edges. It is not hard to see that the order of simple expansions does not matter (that is, if $e_1, e_2 \in E^1$, then $\Phi(E \lhd e_1 \lhd e_2) = \Phi(E \lhd e_2 \lhd e_1)$), and that Φ is injective.

Now consider the map $\phi\colon T_A \to C_A$ defined in the following way. If f is an element of T_A, let $D \to R$ be a graph pair diagram for f. Then $\phi(f)$ is the element of C_A that is represented by the graph pair diagram $\Phi(D) \to \Phi(R)$, where the graph isomorphism is defined by the correspondence between red edges of D (resp. R) and red edges of $\Phi(D)$ (resp. $\Phi(R)$), as well as between blue edges of D (resp. R) and pairs of blue edges of $\Phi(D)$ (resp. $\Phi(R)$), that is, if $D \to R$ maps the edge e_D to the edge e_R, then $\Phi(D) \to \Phi(R)$ maps the edge(s) corresponding to e_D to the edge(s) corresponding to e_R. This definition does not depend on the graph pair diagram chosen to represent f, since expansions in $\mathcal{A}$ correspond to expansions in $\mathcal{C}_A$ (one-to-one or "one-to-two", depending on the color of the edge).

Now, let f and g be rearrangements of the airplane limit space, and consider their respective graph pair diagrams $D_f \to R_f$ and $D_g \to R_g$ such that $D_f = R_g$. Then their composition $f \circ g$ is represented by the graph pair diagram $D_g \to R_f$, so $\phi(f \circ g)$ is represented by $\Phi(D_g) \to \Phi(R_f)$. We also have that $\Phi(D_f) = \Phi(R_g)$, which means that the domain graph of $\Phi(D_f) \to \Phi(R_f)$ is the same as the range graph of $\Phi(D_g) \to \Phi(R_g)$. Hence, their composition $\phi(f) \circ \phi(g)$ is represented by $\Phi(D_g) \to \Phi(R_f)$. It is easy to see that these graph pair diagrams share the same graph isomorphism, thus ϕ is a group morphism.

Also, the kernel of ϕ is trivial. Indeed, if $f \in T_A$ is such that $\phi(f)$ is trivial, then consider a graph pair diagram for f and let e be a red edge of the domain graph: the corresponding edge in the domain graph of $\phi(f)$ must be fixed by $\phi(f)$, hence e is fixed by f. The same holds for blue edges. Then f must be trivial too.

Now, since ϕ is an injective morphism, we have found that C_A contains an isomorphic copy of T_A. Since we have previously seen that T contains an isomorphic copy of C_A, we can finally conclude the following.

Theorem 8.1. *Thompson's group T contains an isomorphic copy of T_A.*

The replacement system $\mathcal{C}(\mathcal{A})$ (Figure 25) and the way in which it is related to the airplane limit space (as depicted in Figures 26, 27 and 28) are inspired by the original study of the basilica rearrangement group T_B in [2]: there, the group is not defined by the action on the basilica Julia set, but instead by its action on the lamination of the fractal. The lamination is essentially the "explosion" of the basilica on S^1, and it expresses the canonical way in which the basilica Julia set is a quotient of the circle. Here we essentially did the same for the airplane, and it would be interesting to see how and when this can be replicated to other fractals. It must be noted, however, that there are fractals for which similar arguments would not provide an embedding into T such as the Vicsek fractal [3, Example 1.12], for its rearrangement group contains finite subgroups that are not cyclic.

9. Rearrangements with trivial extremal derivatives

In this section, we study the following subgroup of $[T_A, T_A]$:

$$E := \{f \in T_A \mid D_p(f) = 1 \; \forall p \in \mathcal{E}\}.$$

We show that E is not finitely generated, and then we study its transitivity properties. In particular, we will see that its action is 2-transitive on the set of components, and so is the action of both T_A and $[T_A, T_A]$.

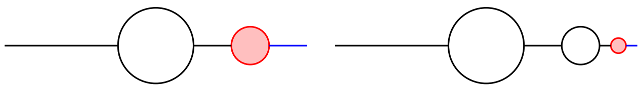
9.1. The subgroup E is not finitely generated

We say that a blue edge is *external* if one of its two vertices corresponds to an element of $\mathcal{E}$ in the limit space. With this in mind, we say that a blue cell is *external* if it is generated by an external blue edge.

Recall the definition of length of blue edges and cells given at the beginning of Section 7.1. Recall from Section 4.4 that each component is uniquely identified on the ray on which it lies by a dyadic number in $(0, 1)$. For each $n \in \mathbb{N} \setminus \{0\}$, let $C(n)$ be the set of those components that lie at position $1 - \frac{1}{2^n}$ on some ray. For each $n \in \mathbb{N} \setminus \{0\}$, we define E_n to be the subset of T_A consisting of all rearrangements f such that

- f acts by permutation on the set $C(n)$;
- f acts canonically on the red cells corresponding to the boundaries of components in $C(n)$;
- f acts canonically on the external blue cells of length $1 - \frac{1}{2^n}$.

Intuitively, these rearrangements are the ones that act "rigidly" on anything that lies "beyond" a component of $C(n)$, while they act without further constraints on the inner part of the airplane delimited by the components of $C(n)$. Figure 29, for $n = 1$ and $n = 2$, respectively, exhibits in red the component lying on the right horizontal ray that belongs to $C(n)$ and in blue the corresponding set on which E_n acts canonically. Keep in mind that E_n acts canonically on similar sets on every ray.



(a) **(b)**

Figure 29. The component of $C(n)$ that lies on the right horizontal ray, for $n = 1, 2$. The groups E_n act "rigidly" on the entire colored subset, and the same happens for each ray of the airplane. (a) $n = 1$. (b) $n = 2$.

Note that all E_n are groups, and they are all proper subgroups of E because they preserve lengths around each extreme. Also, it is clear that $E_n \leq E_{n+1}$ for all $n \in \mathbb{N} \setminus \{0\}$. Moreover, it is easy to see that

$$\bigcup_{n>0} E_n = E.$$

Then, since E is an ascending union of proper subgroups, we have the following.

Proposition 9.1. *The group E is not finitely generated.*

Recall that a subgroup H of a finitely generated group G is also finitely generated if the index $|G : H|$ is finite (see [4, p. 70, Corollary 9.2]). Then we immediately have the following consequence of the previous proposition.

Corollary 9.2. *The index of E in $[T_A, T_A]$ is infinite.*

9.2. Transitivity properties of E

If C and C' are two distinct components, we can define, similarly to what we have done in Section 4.4, a unique component path that reaches C' starting from C instead of C_0. Let $n \geq 2$ and let $C_1, C_2, \ldots, C_n$ be distinct components of the airplane limit space. We say that these components are *aligned* if there exist two of them, C_i and C_j, such that the component path between C_i and C_j travels through all the components. We then say that C_i and C_j are the *extremes* of these aligned components. It is not hard to see that rearrangements of the airplane limit space preserve alignment, that is, if $f \in T_A$, then $C_1, C_2, \ldots, C_n$ are aligned if and only if $f(C_1), f(C_2), \ldots, f(C_n)$ are.

Note that if $C_1, C_2, \ldots, C_n$ are aligned, we can rename them so that C_1 and C_n are the extremes and the components C_i are ordered from the first to the last traveled from C_1 to C_n. Once we have renamed them in this fashion, we say that $(C_1, C_2, \ldots, C_n)$ is an *ordered n-tuple of aligned components*.

It is clear that E contains both $\mathrm{rist}(C_0)$ and $[\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})]$. Recall that $\mathrm{rist}(C_0)$ acts transitively on the set of central rays (Theorem 5.2). Also, it is well known that $[F, F]$ acts transitively on the set of ordered n-tuples of dyadic points of $(0, 1)$. Hence, because of Theorem 5.4, the group $[\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})]$ acts transitively on the set of ordered n-tuples of aligned components that lie on Hor. Using these transitivity properties of $\mathrm{rist}(C_0)$ and $[\mathrm{rist}(\mathrm{Hor}), \mathrm{rist}(\mathrm{Hor})]$, it is not hard to prove the following result.

Proposition 9.3. *For all natural $n > 0$, the group E acts transitively on the set of ordered n-tuples of aligned components.*

Since any two components are aligned, the case $n = 2$ is interesting on its own.

Corollary 9.4. *The group E acts 2-transitively on the set of components. In particular, the groups $[T_A, T_A]$ and T_A act 2-transitively on the set of components.*

Remark 9.5. The group T_A does *not* act 3-transitively on the set of components, therefore neither $[T_A, T_A]$ nor E do. Indeed, consider three components that are not aligned (for example, any three components lying on three distinct central rays) and three components that are aligned (for example, any three components that lie on Hor): since rearrangements preserve alignment, there cannot exist any rearrangement that maps those first three components to the others.

10. A copy of T_B in T_A

In this section, we exhibit an isomorphic copy of the basilica rearrangement group T_B that is contained in T_A. Note that it is already clear that T_A contains an isomorphic copy of T_B: indeed, the subgroup $\mathrm{rist}(C_0)$ of T_A is isomorphic to T (Theorem 5.2), and T contains a copy of T_B (proved in [2]). This reasoning, however, does not tell us much about the nature of this copy of T_B in T_A. Here we instead find a natural copy of T_B in T_A

by specifying its generators. In particular, this section focuses on proving the following result.

Theorem 10.1. *We have* $T_B \simeq \langle \alpha, \beta, \gamma, \delta \rangle \leq T_A$.

We start by defining a non-directed graph Γ_B in the following way:

- its vertices are the components of B;
- its edges link pairs of adjacent components.

Note then that Γ_B is a rooted regular tree of infinite degree, where the root is the central component. Since rearrangements of the basilica limit space permute components of B and preserve their adjacency, T_B acts faithfully on Γ_B by graph automorphisms. This means that there exists an injective morphism

$$\phi_B \colon T_B \to G,$$

where $G := \mathrm{Aut}(\Gamma_B)$.

We now define a similar tree for the airplane limit space by considering the components whose component paths have the following form:

$$\left(\left(\theta_1, \frac{2^{k_1} - 1}{2^{k_1}} \right), \ldots, \left(\theta_n, \frac{2^{k_n} - 1}{2^{k_n}} \right) \right)$$

for some naturals $n > 0$ and $k_i > 0$, and for some dyadic $\theta_i \in [0, 1)$. We denote by $\mathfrak{C}$ the set consisting of all these components, along with the central component. Figure 30 depicts examples of components that belong to $\mathfrak{C}$ (in blue) and that do not (in red). Note that these components are the ones that can be obtained by halving any central ray (which locates certain central components), then halving the external remaining part of that ray or halving any new ray departing from the located component, and so on until one stops at the newfound component.

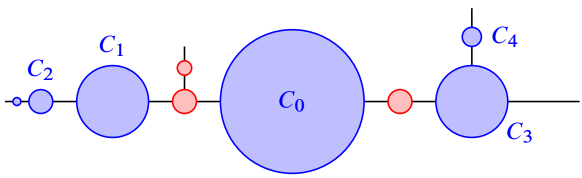


Figure 30. The components colored in blue belong to $\mathfrak{C}$, while those in red do not.

Note that these components have a natural concept of adjacency: the component path representing a component C that belongs to $\mathfrak{C}$ travels through a finite amount of other components in $\mathfrak{C}$, the last of which is said to be $\mathfrak{C}$-adjacent to C. For example, in Figure 30 the component C_1 is adjacent to both C_2 and C_0, but it is not adjacent to C_3 nor C_4 (which are themselves adjacent to each other); moreover, C_2 and C_0 are not adjacent (despite being related).

Now, let Γ_A be the graph defined in the following way:

- its vertices are the elements of $\mathfrak{C}$;
- its edges link pairs of $\mathfrak{C}$-adjacent components.

Note that Γ_A is a rooted regular tree of infinite degree, hence it is isomorphic to Γ_B, and so the previously defined group $G = \mathrm{Aut}(\Gamma_B)$ is isomorphic to $\mathrm{Aut}(\Gamma_A)$. We can then refer to G as the automorphism group of both Γ_B and Γ_A.

Let $H := \langle \alpha, \beta, \gamma, \delta \rangle \leq T_A$. It is easy to see that the action of H maps elements of $\mathfrak{C}$ in elements of $\mathfrak{C}$ and preserves $\mathfrak{C}$-adjacency, hence it is by graph automorphisms on Γ_A. It can also be shown that the action is faithful: it suffices to prove that if $h \in H$ fixes each component in $\mathfrak{C}$, then it must fix each component C of the airplane. Now, since the action of H on Γ_A is faithful and it is by graph automorphisms, there exists an injective group morphism $\phi_A \colon H \to G$. Finally, it is easy to note that the four generators of the group T_B (depicted in Figure 13) act on Γ_B exactly as α, β, γ and δ do on Γ_A, which proves that $T_B \simeq \langle \alpha, \beta, \gamma, \delta \rangle \leq T_A$.

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Matteo Tarocchi
Dipartimento di Matematica e Informatica "Ullise Dini", Universitá degli Studi di Firenze, Viale Morgagni 67A, 50134 Firenze; Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Edificio U5, via Roberto Cozzi 55, 20125 Milano, Italy;
matteo.tarocchi.math@gmail.com