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Connectedness of a space of branched coverings with a periodic cycle

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Abstract. We prove the connectedness of the following locus: the space of degree-d branched self-coverings of S^2 with two critical points of order d, one of which is n-periodic.

Equivalently, all branched self-coverings of S^2 with two critical points of order d, one of which is n-periodic, are combinatorially equivalent.

1. Introduction

Consider the space $\mathcal{M}_d$ of degree-d branched self-coverings of the sphere S^2. Much is known about its topology [13]; for example, it is connected, and its fundamental group is $\mathbb{Z}/2d$. Interesting subspaces arise by imposing dynamical conditions; the one we will focus on in this article is

$$\mathcal{P}_{d,n} = \{f : S^2 \circlearrowleft : f \text{ has two critical points of order } d, \text{ one of which has period exactly } n\}.$$

Theorem A. *The space $\mathcal{P}_{d,n}$ is path connected.*

An equivalent statement is that any two maps in $\mathcal{P}_{d,n}$ are isotopic through maps within $\mathcal{P}_{d,n}$; yet equivalently, *for any $f_0, f_1 \in \mathcal{P}_n$, there are $h_0, h_1 \in$ Homeo(S^2) with $f_1 \circ h_0 = h_1 \circ f_0$ and h_0 is isotopic to h_1 rel, the marked n-cycle of f_0.*

This purely topological statement has a complex analytic avatar: endow S^2 with its complex structure, now written as $\mathbb{P}^1$. In this manner,

$$\mathbb{Rat}_d = \{f \in \mathbb{C}(z) : \deg(f) = d\}$$

embeds into $\mathcal{M}_d$, and Teichmüller theory implies that $\mathbb{Rat}_d$ is a deformation retract of $\mathcal{M}_d$. The locus $\mathcal{P}_{d,n} \cap \mathbb{Rat}_d$ is an important “slice” of parameter space, whose connectedness was asked by Milnor [9]. In fact, the group of Möbius transformations acts on $\mathbb{Rat}_d$ by conjugation, preserving the locus $\mathcal{P}_{d,n} \cap \mathbb{Rat}_d$, so this question may be studied in the quotient space. Milnor proves that

$$\{f \in \mathbb{Rat}_d : f \text{ has two critical points of order } d\} / \operatorname{PSL}_2(\mathbb{C})$$

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is isomorphic to $\mathbb{C}^2$, for instance, by identifying the map $(\alpha z^d + \beta)/(\gamma z^d + \delta)$ with the pair $(\beta\gamma/(\alpha\delta - \beta\gamma), (\alpha^{d+1}\beta^{d-1} + \gamma^{d-1}\delta^{d+1})/(\alpha\delta - \beta\gamma)^{2d})$. Then, the image of $\mathcal{P}_{d,n} \cap \mathbb{Rat}_d$ in $\mathbb{C}^2$ is an algebraic curve called $\mathbb{Per}_n^d(0)$, whose connectedness is a tantalizing open problem. Theorem A should be seen as a solution to this problem in the topological context.

Note that $\mathbb{Per}_n^d(0)$ is definitely *not* homotopy equivalent to $\mathcal{P}_{d,n}$; the former is a punctured Riemann surface, so its fundamental group is free, while the homotopy type of $\mathcal{P}_{d,n}$ is determined quite explicitly in Proposition D, and its fundamental group contains a copy of $\mathbb{Z}^2$, following the arguments in [5, §8].

1.1. Spaces of marked maps

The general setting is a structure $\Pi = \langle A, B, C, \Phi, \deg\rangle$ consisting of sets A, B, C with $A \subset B$ and $A \subset C$ and maps $\Phi\colon C \to B$ and $\deg\colon C \to \{1, 2, \dots\}$, called a *portrait*. The corresponding space of marked maps is

$$\begin{aligned}\mathcal{P}_\Pi := \big\{(f, b, c) : {}& b\colon B \hookrightarrow S^2,\ c\colon C \hookrightarrow S^2,\\ & f\colon (S^2, c(C)) \to (S^2, b(B)) \text{ branched covering},\ f \circ c = b \circ \Phi,\\ & \deg_{c(x)} f = \deg(x),\ b \restriction A = c \restriction A,\ f \text{ ramifies only above } b(B)\big\}.\end{aligned}$$

For example, setting $A = \{a_0, \dots, a_{n-1}\}$ and $B = A \cup \{v\}$ and $C = A \cup \{c\}$ with $\deg(a_0) = \deg(c) = d$ and $\Phi(a_i) = a_{i+1 \bmod n}$ and $\Phi(c) = b$ specifies a structure $\Pi_{d,n}$ yielding precisely the space $\mathcal{P}_{d,n}$ introduced above. The method of this article should serve to solve a variety of connectedness problems about loci $\mathcal{P}_\Pi$.

Thurston's theory of iteration of rational maps involves considering their topological counterparts. A branched covering f of the sphere S^2 is *critically finite* if the forward orbit P_f of its critical points is finite. Setting $A = B = C = P_f$ and $\Phi = f \restriction P_f$ recovers the space of all maps with the same post-critical behavior as f, and his fundamental result implies that the connected component of f is contractible and contains at most one holomorphic representative (with a combinatorial criterion to determine whether there is one), unless the map is double-covered by a homothety on the torus. At the other extreme, $A = \emptyset$ amounts to considering branched coverings with no dynamical constraint and subsumes the classical Hurwitz theory of coverings of surfaces, which we briefly recall:

Theorem 1.1 (Hurwitz [8]). *There is a bijection between, on the one hand, equivalence classes of degree-d branched coverings $\Sigma \to S^2$ with n ordered critical values and, on the other hand, orbits of n-tuples of permutations in $\mathrm{Sym}(d)$ with product equal to 1. Two branched coverings $f_0, f_1\colon \Sigma \to S^2$ are equivalent if there are $h \in \mathrm{Homeo}(\Sigma)$ and $h' \in \mathrm{Homeo}(S^2)$ such that $h \circ f_0 = f_1 \circ h'$, and orbits of n-tuples are considered with respect to the diagonal action of $\mathrm{Sym}(d)$ by conjugation and the mapping class group action generated by all $(\dots, \pi_i, \pi_{i+1}, \dots) \mapsto (\dots, \pi_i^{\pi_i\pi_{i+1}}, \pi_{i+1}^{\pi_i\pi_{i+1}}, \dots)$. The degrees of the preimages of the ith critical value are given by the cycle lengths $\lambda_{i,j}$ of π_i. The surface Σ is connected if and only if $\langle \pi_1, \dots, \pi_n\rangle$ is transitive on $\{1, \dots, d\}$, and then the genus of Σ is $2d - 2 - \sum_{i,j}(\lambda_{i,j} - 1)$.* ∎

Let us also briefly recall *combinatorial equivalence*: two maps $f_0, f_1 \in \mathcal{P}_\Pi$ are combinatorially equivalent if there are $h \in \mathrm{Homeo}(S^2, B)$, $h' \in \mathrm{Homeo}(S^2, C)$ with $h \circ f_0 = f_1 \circ h'$ and h isotopic to h' rel A. By expressing h as a motion of B in S^2, this is the same as saying that there is a path from f_0 to f_1 in $\mathcal{P}_\Pi$; thus, combinatorial equivalence is the same as being in the same path component. We deduce that Milnor's connectivity question cannot be addressed with topology alone.

Corollary B. *Any two maps $f, g \in \mathbb{P}\mathrm{er}_n^d(0)$ are combinatorially equivalent.*

We also note that $\mathcal{P}_\Pi$ may be disconnected even if $A = \emptyset$; this holds, e.g., in degree 5, with $B = \{b_1, b_2, b_3\}$ and $C = \{c_1, c_1', c_2, c_3\}$ and $\Phi(c_i) = \Phi(c_i') = b_i$ and $\deg(c_1) = \deg(c_1') = 2$, $\deg(c_2) = \deg(c_3) = 4$, since then by Theorem 1.1 there are two equivalence classes. In degree 2 with $A = \emptyset$, the space $\mathcal{P}_\Pi$ is connected, but additional constraints may disconnect it; e.g., with $A = B = C = \{a_0, a_1, a_2, a_\infty\}$ and $\Phi(a_i) = a_{i+1\%3}$ and $\deg(a_0) = \deg(a_\infty) = 2$, there are three connected components, the "rabbit", "corabbit", and "airplane" (represented by the three polynomials $z^2 + c$ with $c^3 + 2c^2 + c + 1 = 0$). Thus, connectedness is by far not a ubiquitous phenomenon.

By abuse of notation, we will think of both B and C as abstract sets (used to define Π) and as variable subsets of S^2; and from now on, we abbreviate

$$\mathcal{P}_\Pi = \{f\colon (S^2, C) \to (S^2, B) : f \restriction A = \Phi \restriction A, \deg_f = \deg\}.$$

We quickly note in passing that $\mathrm{Homeo}(S^2)$ naturally acts on $\mathcal{P}_\Pi$, by conjugation on f and post-composition on A, B, C. We avoid any discussion on the quotient, preferring to remember this action in the background.

1.2. Teichmüller theory

Even though our main result is phrased in purely topological terms, it has profound connections to complex dynamics. Let us first add a complex structure to the spheres in the definition of $\mathcal{P}_\Pi$, *not* requiring f to be conformal:

$$\mathcal{P}'_\Pi = \{f\colon (\mathbb{P}^1, C) \to (\mathbb{P}^1, B) : f \restriction A = \Phi \restriction A, \deg_f = \deg\}.$$

We may then deformation-retract $\mathcal{P}'_\Pi$ to its subspace of maps with minimal quasiconformal distortion; by Teichmüller's theorem (see, e.g., [7, Theorem 5.3.12]), every connected component with fixed B, C contains a single such extremal map.

Kahn, Firsova, and Selinger [5] consider, following Mary Rees [12], a space of maps modeled on Teichmüller space, which we can express as follows in our setting. Recall that the Teichmüller space of a marked sphere consists of all its complex structures; for $B \subset S^2$, we may define it as

$$\mathcal{T}_B = \{\tau\colon S^2 \to \mathbb{P}^1 \text{ homeomorphism}\}/\sim,$$

with $m \circ \tau \circ h \sim \tau$ for every Möbius transformation m and every homeomorphism h of (S^2, B) that is isotopic to the identity rel B, henceforth written as $h \in \mathrm{Homeo}_0(S^2, B)$. In analogy, *Rees space* is the quotient $\mathcal{R}_\Pi := \mathcal{P}'_\Pi/\sim$, with $m \circ f \sim f \circ h$ for every Möbius transformation m and every homeomorphism h of $\mathbb{P}^1$ that is isotopic to m rel A.

Every choice of $f\colon (S^2, C) \to (S^2, B)$ in $\mathcal{P}_\Pi$ yields a map

$$\mathcal{T}_B \to \mathcal{R}_\Pi,$$

by $[\tau] \mapsto [\tau \circ f \circ \tau^{-1}]$, which by [5, Lemma 4.1] is a covering, the group of deck transformations being the centralizer

$$Z_A(f) = \{h \in \mathrm{Mod}(B) : h \circ f = f \circ h \text{ rel } A\}$$

of f. Theorem A can be rephrased as the following statement.

Corollary C. *For $\Pi = \Pi_{d,n}$ marking two order-d critical points, one of which is n-periodic, the space $\mathcal{R}_\Pi$ is a quotient of Teichmüller space $\mathcal{T}_B$ by a subgroup of* $\mathrm{Mod}(B)$*, so in particular it is a $K(\pi, 1)$ space for some $\pi = Z_A(f) \leq \mathrm{Mod}(B)$.*

We return to a general Π. There is a fibration $\mathcal{P}_\Pi \to \mathcal{Q}$, the space of configurations of $B \cup_A C$ in S^2. Let $\mathcal{W}$ denote a fiber, thought of as a space of branched coverings with specified marked points.

Proposition D. *The space $\mathcal{W}$ is homotopy equivalent to a discrete set. Every connected component of $\mathcal{P}_\Pi$ is homotopy equivalent to a principal* SO_3*-bundle over a $K(Z_A(f), 1)$ space for any f in that component.*

Thus, the most mysterious part of the topology of $\mathcal{P}_\Pi$ is its number of connected components.

Proof. The group $\mathrm{Homeo}_0(S^2, C)$ is contractible and acts freely on $\mathcal{W}$ by pre-composition with discrete quotient; this implies the first claim. The group $\mathrm{Homeo}(S^2)$ acts freely on $\mathcal{P}_{d,n}$ by conjugation and has the homotopy type of SO_3, giving $\mathcal{P}_{d,n}$ the structure of a principal bundle. The base deformation retracts to $\mathcal{R}_\Pi$. ∎

Every $(\tau, f) \in \mathcal{R}_\Pi$ yields two complex structures on (S^2, A): one locally given by τ and one by $\tau \circ f$ and denoted as $\sigma_f(\tau)$. The *rational Rees space* $\mathcal{R}_\Pi^{\mathbb{R}\mathrm{at}}$ is the locus at which these complex structures agree; so for $(\tau, f) \in \mathcal{R}_\Pi^{\mathbb{R}\mathrm{at}}$, we have a commutative diagram

$$\begin{array}{ccc} (S^2, C) & \xrightarrow{\tau} & \mathbb{P}^1 \\ {\scriptstyle f}\downarrow & \ulcorner\sim & \downarrow{\scriptstyle r_{\tau,f}} \\ (S^2, B) & \xrightarrow{\tau} & \mathbb{P}^1 \end{array}$$

that commutes up to homotopy, for some rational map $r_{\tau,f} \in \mathbb{R}\mathrm{at}_d$, unique up to conjugation by a Möbius transformation.

Let **MRat** denote the quotient of $\mathbb{R}\mathrm{at} = \mathbb{C}(z)$ by $\mathrm{PSL}_2(\mathbb{C})$ acting under conjugation; and consider the locus

$$\mathbf{MRat}_\Pi = (\mathbb{R}\mathrm{at} \cap \mathcal{P}_\Pi)/\,\mathrm{PSL}_2(\mathbb{C}).$$

Note that every $f \in \mathbf{MRat}_\Pi$ has attached subsets $A, B, C \subset \mathbb{P}^1$, well defined up to a Möbius transformation. Every choice of $\tau \in \mathcal{T}_B$ gives a natural map $\mathcal{P}_\Pi \to \mathcal{R}_\Pi$, by $f \mapsto (\tau, f)$. The natural map $\mathcal{R}_\Pi^{\mathbb{R}\mathrm{at}} \to \mathbf{MRat}_\Pi$, given by $(\tau, f) \mapsto r_{\tau,f}$, is an isomorphism. There is finally a map $\mathbf{MRat}_\Pi \to \mathcal{M}_B$, the moduli space of (S^2, B), given by

$$r_{\tau,f} \mapsto \tau \upharpoonright B.$$

Consider, to complete the picture, Epstein's equalizer space: it is the space

$$\mathcal{D}\!ef_{\Pi,f} = \{\tau \in \mathcal{T}_B : \tau \text{ and } \sigma_f(\tau) \text{ have the same image in } \mathcal{T}_A\}.$$

Epstein's transversality theory [4] shows that $\mathcal{D}\!ef_{\Pi,f}$ is a submanifold of dimension $\#B - \#A - 3$, unless f is a flexible Lattès map, and A contains the post-critical set of f. In summary, we have

$$\begin{array}{ccc}
\mathcal{D}\!ef_{\Pi,f} & \hookrightarrow & \mathcal{T}_B \\
{\scriptstyle \tau\mapsto(\tau,f)}\downarrow & & \downarrow \\
\mathbf{MRat}_\Pi = \mathcal{R}_\Pi^{\mathbb{R}\mathrm{at}} & \hookrightarrow & \mathcal{R}_\Pi \\
{\scriptstyle (\tau,f)\mapsto\tau\upharpoonright B}\downarrow & & \downarrow \\
\mathcal{M}_\Pi & \hookrightarrow & \mathcal{M}_B
\end{array}$$

and our result shows that, for $\Pi = \Pi_{d,n}$, the top vertical maps are onto. The bottom left vertical map is a finite-to-one map [5, Lemma 4.5] and in many cases, in particular that of $\Pi_{d,n}$, is actually a bijection [5, Lemma 4.7].

Previous literature concentrated on connectivity and contractibility of $\mathcal{D}\!ef_\Pi$. The arguments in [5, 6], proving that $\mathcal{D}\!ef_\Pi$ is disconnected or at least not contractible, rely on showing that the fundamental group of the punctured Riemann surface $\mathbf{MRat}_\Pi$ is not isomorphic to a centralizer $Z_A(f)$.

1.3. Sketch of proof

We reduce Theorem A to a discrete problem by means of isotopy. From now on, we write $\Pi = \Pi_{d,n}$.

(1) Consider the fibration $\mathcal{P}_\Pi \to \mathcal{Q}$, and let W be the collection of connected components of a fiber. In our concrete situation, we imagine that sets A, B, C of respective sizes $n, n+1, n+1$ are frozen on S^2, and we consider the collection W of isotopy classes of maps

$$(S^2, C) \to (S^2, B)$$

that permute A cyclically. Thus, W is the collection of isotopy classes of maps $f \in \mathcal{P}_{d,n}$ with these fixed A, B, C.

(2) There are two commuting actions on W by pure mapping class groups, respectively, $\mathrm{Mod}(B)$ acting by post-composition and $\mathrm{Mod}(C)$ acting by pre-composition. The action of $\mathrm{Mod}(C)$ is free with d^{n-2} orbits.

(3) There is a single degree-d covering with two order-d marked critical points; this is the "Hurwitz problem" for the data (d, d); see Theorem 1.1. In fact, by [14, Théorème (1)], this holds more generally for any degree-d covering with a critical point of order d. It follows that there is a single $(\mathrm{Mod}(B) \times \mathrm{Mod}(C))$-orbit on W.

(4) Denote, respectively, by ε_B, ε_C the natural restriction maps

$$\mathrm{Mod}(B) \to \mathrm{Mod}(A), \mathrm{Mod}(C) \to \mathrm{Mod}(A).$$

The subgroup $E = \{(g, h) \in \mathrm{Mod}(B) \times \mathrm{Mod}(C) : \varepsilon_B(g) = \varepsilon_C(h)\}$ acts on W, and elements of E can be realized as continuous deformations of S^2 that preserve the cycle A. The connectedness of $\mathcal{P}_{d,n}$ is therefore equivalent to the transitivity of E on W.

(5) To prove the transitivity of E, fix a polynomial map $f \in \mathcal{P}_{d,n}$, thereby assuming $B = C$. We consider two subgroups of E: the diagonal $\Delta = \{(g, g) : g \in \mathrm{Mod}(B)\}$, which acts on W by conjugation, and $P = \ker(\varepsilon_B)$, the "point pushes" of the critical value, which act on W by post-composition. It suffices to show that every element of W may be written as $pgfg^{-1}$ with $g \in \mathrm{Mod}(B)$ and $p \in P$.

(6) Compute the "lifting" operation: let $L \leq \mathrm{Mod}(B)$ denote the index-d^{n-2} subgroup of liftable classes; namely, all $h \in \mathrm{Mod}(B)$ such that there exists $\phi(h) \in \mathrm{Mod}(C)$ with $h \circ f \cong f \circ \phi(h)$. Enough images under the homomorphism $\phi\colon L \to \mathrm{Mod}(C)$ may be written explicitly to show that all generators of $\mathrm{Mod}(B)$ may be obtained from P via ϕ and conjugation.

2. Dynamical bisets and mapping class bisets

Let us consider the polynomial $f(z) = z^d + c$ for which the supporting rays have angles $\{1, 2\}/(d^n - 1)$, and encode f group-theoretically. This means we let

$$A = \{a_0 = a_n = 0,\ a_1 = c,\ a_2 = c^d + c, \dots\}$$

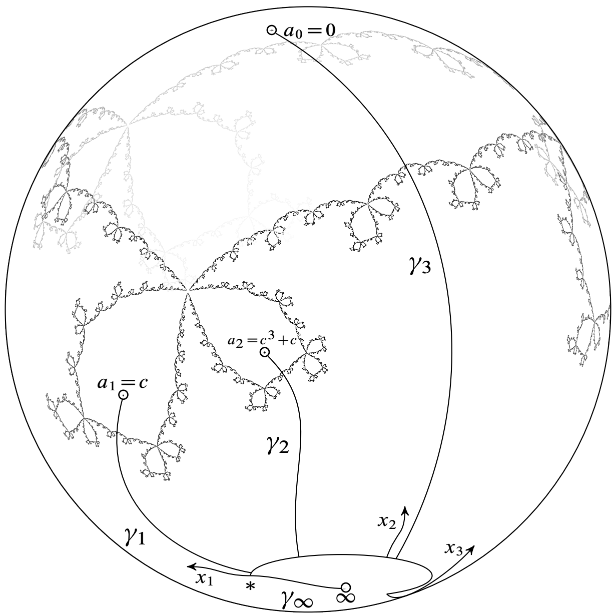
be the length-n critical cycle of f; set $B = C = A \cup \{\infty\}$; fix a basepoint $*$ in $\mathbb{C} \setminus A$ near ∞; let $\pi = \pi_1(\mathbb{C} \setminus A, *)$ be the fundamental group; and choose "lollipop" generators $\gamma_1, \dots, \gamma_n, \gamma_\infty$ of π: the generator γ_i turns a bit clockwise (on the sphere) around ∞, follows the external ray with angle $d^{i-1}/(d^n - 1)$ towards a_i, encircles it counterclockwise, and returns back to $*$, while the generator γ_∞ is a small (on the sphere) counterclockwise loop around ∞. Note that one of the generators of π is redundant, and we have

$$\pi = \langle \gamma_1, \dots, \gamma_n, \gamma_\infty \mid \gamma_\infty \gamma_n \cdots \gamma_1 = 1 \rangle.$$

The "iterated monodromy group" theory [10] encodes f as a *biset*: a set $\mathfrak{B}(f)$ with two commuting π-actions

$$\mathfrak{B}(f) = \{\alpha\colon [0, 1] \to \mathbb{C} \setminus B : \alpha(0) = f(\alpha(1)) = *\}/\sim,$$

with the left action by pre-concatenation with a loop and the right action by post-concatenation with the unique lift of a loop that starts where α ends. The left action on $\mathfrak{B}(f)$ is free with d orbits; so $\mathfrak{B}(f)$ may be written as $\pi \times \{x_1, \dots, x_d\}$ by choosing a system of orbit representatives. A natural choice consists of short paths x_i that turn angle $(i-1)/d$ counterclockwise around ∞ from $*$ and then reach a preimage of $*$, as illustrated in this figure for the map $f(z) = z^3 + c$ with $c \approx 0.55757 + 0.54035i$ satisfying $f^3(0) = 0$:



It is then straightforward to trace paths and their lifts so as to express the right action of π on $\mathfrak{B}(f)$; that is,

$$\begin{aligned} x_1 \cdot \gamma_1 &= \gamma_\infty \gamma_n \cdot x_2, & x_i \cdot \gamma_1 &= x_{i+1} && \text{if } 1 < i < d,\ x_d \cdot \gamma_1 = \gamma_\infty^{-1} \cdot x_1, \\ x_1 \cdot \gamma_{j+1} &= \gamma_j \cdot x_1, & x_i \cdot \gamma_{j+1} &= x_i && \text{if } 1 < i \le d \text{ and } 1 \le j \le n, \\ x_1 \cdot \gamma_\infty &= \gamma_\infty \cdot x_d, & x_{i+1} \cdot \gamma_\infty &= x_i && \text{if } 1 \le i < d. \end{aligned}$$

This is the natural degree-d generalization of the recursion defining the group $\mathfrak{K}(0^{n-1})$ from [2]; see also [11].

Note that a different choice of orbit representatives $\{x'_1, \dots, x'_d\}$ would give different formulas for the right action; the object $\mathfrak{B}(f)$, considered up to isomorphism of π-π-bisets, is a complete invariant of f, but its presentation relies on choices.

Let us make this a bit more precise. A *presentation* of a biset $\mathfrak{B}(f')$, for a map $f' \in W$, is a $d \times (n+1)$ matrix with in entry (i, j) a pair $(g, k) \in \pi \times \{1, \dots, d\}$, describing the relation

$$x_i \cdot \gamma_j = g \cdot x_k;$$

here, we use the convention $n+1=\infty$. A permutation of the i and k defines an isomorphic presentation (this amounts to reordering the orbit representatives); and for every choice of $g_1,\dots,g_d\in\pi$, the replacement of every (g,k) at position (i,j) with $(g_i g g_k^{-1},k)$ also defines an isomorphic presentation (this amounts to replacing the orbit representatives $(x_1,\dots,x_d)$ with $(g_1x_1,\dots,g_dx_d)$.)

Furthermore, the left and right actions on W may be expressed in terms of these presentations: pre-composition by ψ amounts to replacing each (g,k) by $(\psi(g),k)$, while post-composition by ϕ amounts to using the table to rewrite $x_i\cdot\phi(\gamma_j)$ in the form $g\cdot x_k$ and recording the result in a new table.

Now, $\mathrm{Mod}(B)$ is generated by a collection of full Dehn twists between elements of B and acts on π by outer automorphisms. We choose as generators the $\tau_{i,j}$ for $1\le i<j\le\infty$ which, geometrically, push a_i and a_j closer while avoiding all other paths γ_k, and twist them fully around each other. The action of $\tau_{i,j}$ on π may be written concretely as follows: set $\alpha=\gamma_{j-1}\gamma_{j-2}\cdots\gamma_i$; then

$$\tau_{i,j}(\gamma_i)=\gamma_i^{\alpha^{-1}\gamma_j\alpha},\quad \tau_{i,j}(\gamma_j)=\gamma_j^{\alpha\gamma_i\alpha^{-1}},\quad \tau_{i,j}(\gamma_k)=\gamma_k.$$

The subgroup of point pushes is

$$P=\langle\tau_{i,\infty}:1\le i\le n\rangle,$$

and its liftable elements include $\tau_{i,\infty}^d$ as well as $\tau_{1,\infty}$ and its conjugates.

Once all these choices are made, it is straightforward to check the following identities.

Lemma 2.1. *In W, we have the identities*

$$\tau_{1,\infty}\cdot f=f\cdot\tau_{n,\infty}, \tag{1}$$

$$\tau_{1,\infty}^{\tau_{i+1,\infty}}\cdot f=f\cdot\tau_{i,\infty}\tau_{i,n}\tau_{n,\infty}, \tag{2}$$

$$\tau_{i+1,\infty}^d\cdot f=f\cdot\tau_{i,\infty}^{\tau_{i,n}}, \tag{3}$$

$$\tau_{i+1,j+1}\cdot f=f\cdot\tau_{i,j} \tag{4}$$

for all $1\le i,\ j\le n-1$.

Proof. We only consider (2), the other ones being checked in a similar but easier manner. Set

$$\phi=\tau_{1,\infty}^{\tau_{i+1,\infty}};$$

then, writing $\delta=\gamma_i\gamma_{i-1}\cdots\gamma_1$ and $\beta=\gamma_{i+1}^{\delta}\gamma_\infty$, we have

$$\phi(\gamma_1)=\gamma_1^{\gamma_\infty^{\beta}},$$

$$\phi(\gamma_{i+1})=\gamma_{i+1}^{\delta\gamma_\infty^{-\gamma_1^{\beta^{-1}}}\gamma_\infty\delta^{-1}},$$

$$\phi(\gamma_\infty)=\gamma_\infty^{\beta\gamma_1\gamma_\infty^{-\gamma_1^{\beta^{-1}}}\gamma_\infty\beta},$$

with ϕ fixing all other generators. The biset $B(\phi \circ f)$ of the post-composition of f with ϕ is then presented as follows in a basis $(y_1, \dots, y_d)$, with $\varepsilon = \gamma_{i-1} \cdots \gamma_1$ and $\zeta = \gamma_{n-1} \cdots \gamma_{i+1}$, so $\gamma_\infty \gamma_n \zeta \gamma_i \varepsilon = 1$:

$$\mathfrak{B}(\phi \circ f) : \begin{cases} y_1 \cdot \gamma_1 = \gamma_i^{-\zeta^{-1}} \cdot y_2, \\ y_{d-1} \cdot \gamma_1 = \gamma_i^{-\zeta^{-1}\gamma_\infty} \cdot y_d, \\ y_d \cdot \gamma_1 = \gamma_\infty^{-1} \gamma_i^{\zeta^{-1}} \varepsilon^{-1} \zeta^{-1} \cdot y_1, \\ y_1 \cdot \gamma_{i+1} = \gamma_i^{\zeta^{-1}\varepsilon^{-1}} \cdot y_1, \\ y_1 \cdot \gamma_\infty = \zeta \varepsilon \gamma_i^{-\zeta^{-1}} \gamma_\infty \cdot y_d, \\ y_2 \cdot \gamma_\infty = \varepsilon^{-1} \zeta^{-1} \cdot y_1, \\ y_d \cdot \gamma_\infty = \gamma_i^{\zeta^{-1}\gamma_\infty} \cdot y_2; \end{cases}$$

all other entries are as in the presentation of $\mathfrak{B}(f)$. On the other hand, set

$$\psi = \tau_{i,\infty} \tau_{i,n} \tau_{n,\infty};$$

we have

$$\begin{aligned} \psi(\gamma_i) &= \gamma_i^{\zeta^{-1}\varepsilon^{-1}}, \\ \psi(\gamma_n) &= \gamma_n^{\varepsilon^{-1}\zeta^{-1}}, \\ \psi(\gamma_\infty) &= \gamma_\infty^{\varepsilon^{-1}\zeta^{-1}}; \end{aligned}$$

all other generators are fixed. The biset $\mathfrak{B}(f \circ \psi)$ is presented as follows in a basis $(z_1, \dots, z_d)$:

$$\mathfrak{B}(f \circ \psi) : \begin{cases} x_1 \cdot \gamma_1 = (\gamma_\infty \gamma_n)^{\varepsilon^{-1}\zeta^{-1}} \cdot x_2, \\ x_d \cdot \gamma_1 = \gamma_\infty^{-\varepsilon^{-1}\zeta^{-1}} \cdot x_1, \\ x_1 \cdot \gamma_{i+1} = \gamma_i^{\zeta^{-1}\varepsilon^{-1}} \cdot x_1, \\ x_1 \cdot \gamma_\infty = \gamma_\infty^{\varepsilon^{-1}\zeta^{-1}} \cdot x_d; \end{cases}$$

all other entries are as in the presentation of $\mathfrak{B}(f)$. Now, to prove that $\mathfrak{B}(\phi \circ f)$ and $\mathfrak{B}(f \circ \psi)$ are isomorphic, it suffices to map the basis of the former into the latter, as follows:

$$\begin{aligned} & y_1 \mapsto z_1, \\ & y_k \mapsto \varepsilon^{-1} \zeta^{-1} \cdot z_k \text{ for } 1 < k < d, \\ & y_d \mapsto \gamma_i^{\zeta^{-1}\gamma_\infty} \varepsilon^{-1} \zeta^{-1} \cdot z_d \end{aligned}$$

and to check that the right actions of π on $\mathfrak{B}(\phi \circ f)$ and $\mathfrak{B}(f \circ \psi)$ are intertwined by this map. ∎

Corollary 2.2. *We have $W = Pf^{\mathrm{Mod}(B)}$, where P is the group of point pushes and* $\mathrm{Mod}(B)$ *acts by conjugation.*

Proof. The solution to the Hurwitz problem implies $W = \mathrm{Mod}(B)f^{\mathrm{Mod}(B)}$. Now, consider the set $M \subseteq \mathrm{Mod}(B)$ of all $m \in \mathrm{Mod}(B)$ with $mf^{\mathrm{Mod}(B)} \subseteq Pf^{\mathrm{Mod}(B)}$. Clearly, M is closed under conjugation, since P is normal in $\mathrm{Mod}(B)$. Also, M is a subgroup: if $m, n \in M$ and $mf^{\mathrm{Mod}(B)}, nf^{\mathrm{Mod}(B)} \subseteq Pf^{\mathrm{Mod}(B)}$, then

$$mnf^{\mathrm{Mod}(B)} \subseteq m(Pf^{\mathrm{Mod}(B)}) = mf^{\mathrm{Mod}(B)}P \subseteq Pf^{\mathrm{Mod}(B)}P = P^2 f^{\mathrm{Mod}(B)} = Pf^{\mathrm{Mod}(B)}.$$

Consider $m \in M$, and assume that we have an identity $m \cdot f = f \cdot h$ in W; then

$$h \cdot f = (f \cdot h)^{h^{-1}} = (m \cdot f)^{h^{-1}} = m^{h^{-1}} \cdot f^{h^{-1}} \in Pf^{\mathrm{Mod}(B)},$$

so $h \in M$.

Obviously, we have $P \subseteq M$, namely, $\tau_{i,\infty} \in M$ for all i. Then, by Lemma 2.1 (2), we have $\tau_{i,n} \in M$. Then, Lemma 2.1 (4) gives $\tau_{i-1,n-1} \in M$, etc. So, finally, all $\tau_{i,j} \in M$. Therefore, $M = \mathrm{Mod}(B)$. ■

3. Proof of Theorem A

We expand on the sketch presented in Section 1.3. Consider first the map

$$\tau : \mathcal{P}_\Pi \ni (f, b, c) \mapsto (b, c)$$

sending a branched covering to its marked sets b, c on S^2. This map is evidently a fibration, with fiber $\mathcal{W}$ consisting of all branched coverings f with fixed b, c. We treat, from now on, A, B, C as fixed subsets of S^2 with $A \subset B \cap C$, so $\mathcal{W}$ is the set of branched coverings

$$(S^2, C) \to (S^2, B)$$

that map C to B with combinatorics and degree prescribed by Π.

We let W denote the quotient of $\mathcal{W}$ under isotopy. For any two branched coverings $f, f' \in \mathcal{W}$ that are isotopic, there is a unique homeomorphism $h \in \mathrm{Homeo}_0(S^2, C)$ with $f' = f \circ h$; in other words, W is the quotient of $\mathcal{W}$ by the free action of $\mathrm{Homeo}_0(S^2, C)$. Now, the group $\mathrm{Homeo}_0(S^2, C)$ is contractible, assuming $\#C \geq 3$. It follows from the long exact sequence of a fibration and Whitehead's theorem that $\mathcal{W}$ has the homotopy type of a discrete set, and (except in dimension ≤ 1) the homotopy type of $\mathcal{P}_\Pi$ is that of the base $\mathcal{Q}$ of the fibration

$$\mathcal{Q} := \{(b : B \hookrightarrow S^2, c : C \hookrightarrow S^2) : b \restriction A = c \restriction A\}.$$

We have thus proven Proposition D modulo the main connectedness claim.

There are two commuting actions on W by the pure mapping class groups $\mathrm{Mod}(B)$ and $\mathrm{Mod}(C)$, respectively, by post-composition and pre-composition. The action of $\mathrm{Mod}(C)$ is free with finitely many orbits. More precisely, for every $f \in W$, there is a finite-index subgroup $L_f \leq \mathrm{Mod}(B)$ consisting of "liftable" classes: L_f is the set of all $\ell \in \mathrm{Mod}(B)$ for which there exists $m \in \mathrm{Mod}(B)$ with $\ell \circ f \cong f \circ m$. We denote this (necessarily unique) $m \in \mathrm{Mod}(C)$ by $\sigma_f(\ell)$, defining thus a homomorphism

$$\sigma_f : L_f \to \mathrm{Mod}(C).$$

The long exact sequence of homotopy groups gives

$$\pi_1(\mathcal{Q}) \to \pi_0(\mathcal{W}) \to \pi_0(\mathcal{P}_\Pi) \to \pi_0(\mathcal{Q}).$$

There is a natural map $\varepsilon_B : \mathrm{Mod}(B) \twoheadrightarrow \mathrm{Mod}(A)$ induced by the inclusion $A \subset B$; and similarly, $\varepsilon_C : \mathrm{Mod}(C) \twoheadrightarrow \mathrm{Mod}(A)$. The equalizer of these two maps is the subgroup

$$E = \{(g, h) \in \mathrm{Mod}(B) \times \mathrm{Mod}(C) : \varepsilon_B(g) = \varepsilon_C(h)\},$$

and we naturally have $E \cong \pi_1(\mathcal{Q})$. We thus have:

Lemma 3.1. *The space $\mathcal{P}_\Pi$ is path connected if and only if the action of E on W is transitive.* ∎

3.1. Specifics for degree-d bicritical maps with an n-cycle

All the above considerations applied to a general portrait Π. We now consider the specific choice of a portrait Π specifying an n-cycle marked A that contains a single order-d critical point and another order-d critical point marked in C, with its image marked in B.

Lemma 3.2. *The action of* $\mathrm{Mod}(B) \times \mathrm{Mod}(C)$ *on W is transitive.*

Proof. In considering the action of $\mathrm{Mod}(B) \times \mathrm{Mod}(C)$, we are in effect ignoring the condition that A is an invariant n-cycle; that is, we are considering the space of coverings of (S^2, B) with two order-d critical values. Hurwitz's theorem 1.1 shows that there is a single equivalence class, since all d-cycles are conjugate. ∎

(As mentioned in the sketch, this fact holds more generally for any portrait with a critical value of maximal order by [14, Théorème (1)]. Imposing the fact that this critical value be fixed defines a polynomial slice.)

Let us now fix a polynomial map $f \in \mathcal{P}_{d,n}$, assuming $B = C$. We consider two subgroups of E: the diagonal $\Delta = \{(g, g) : g \in \mathrm{Mod}(B)\}$, which acts on W by conjugation, and $P = \ker(\varepsilon_B)$, the "point pushes" of the critical value, which act on W by post-composition. We have $P = \pi_1(S^2 \setminus A, b)$, and Birman's exact sequence [3] gives

$$1 \to P \to \mathrm{Mod}(B) \xrightarrow{\varepsilon_B} \mathrm{Mod}(A) \to 1.$$

It suffices to show that every element of W may be written as $pgfg^{-1}$ with $g \in \mathrm{Mod}(B)$ and $p \in P$. Now, this is precisely Corollary 2.2.

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