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On the integro-differential equation satisfied by the p-adic $\log\Gamma$-function

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Abstract. Diamond's p-adic analogue $\mathrm{Log}\Gamma_{\mathrm{D}}(x)$ of the classical function $\log\Gamma(x)$ has recently been shown to satisfy the integro-differential equation

$$(*)\qquad \int_{\mathbb{Z}_p} f(x+t)\,dt = (x-1)f'(x) - x + 1/2 \quad (x \in \mathbb{Q}_p - \mathbb{Z}_p),$$

where $\int_{\mathbb{Z}_p}$ is a Volkenborn integral and f' is the derivative of f. We show that this equation characterizes $\mathrm{Log}\Gamma_{\mathrm{D}}(x)$ up to a function with everywhere vanishing second derivative. Namely, every solution f of $(*)$ is infinitely differentiable and satisfies $f'' = \mathrm{Log}\Gamma_{\mathrm{D}}''$.

We show that the set of solutions of the homogeneous equation

$$\int_{\mathbb{Z}_p} y(x+t)\,dt = (x-1)y'(x)$$

associated to $(*)$ is an infinite-dimensional commutative and associative p-adic algebra under the product law

$$(y_1 \diamond y_2)(x) := y_2'(x)y_1(x) + y_1'(x)y_2(x) - (x-1/2)y_1'(x)y_2'(x),$$

the unit being $y(x) = x - 1/2$. We also study Morita's alternate p-adic analogue $\mathrm{Log}\Gamma_{\mathrm{M}}$ of $\log\Gamma(x)$ and prove similar results.

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1. Introduction

Diamond's [Di] p-adic analogue of Euler's $\log\Gamma$-function was recently shown [CF] [Co, p. 335] to be the unique strictly differentiable function $f\colon \mathbb{Q}_p - \mathbb{Z}_p \to \mathbb{C}_p$ simultaneously satisfying the difference equation

$$f(x+1) - f(x) = \log_p x \tag{1}$$

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and the “Raabe formula”

$$\int_{\mathbb{Z}_p} f(x+t)\,dt = (x-1)f'(x) - x + \tfrac{1}{2}. \tag{2}$$

Here f' is the derivative of f, $\mathbb{C}_p$ is the completion of the algebraic closure of the p-adic field $\mathbb{Q}_p$, $\mathbb{Z}_p \subset \mathbb{Q}_p$ denotes the ring of p-adic integers and $\log_p$ is the Iwasawa p-adic logarithm, so $\log_p p = 0$ [Sc, p. 132]. Recall that the Volkenborn integral of a function $g\colon \mathbb{Z}_p \to \mathbb{C}_p$ is defined by

$$\int_{\mathbb{Z}_p} g(t)\,dt := \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} g(j), \tag{3}$$

and that this limits exists if g is strictly differentiable on $\mathbb{Z}_p$ [Ro, p. 264]. A function $g\colon X \to \mathbb{C}_p$ is strictly differentiable on a subset $X \subset \mathbb{C}_p$ if for all $a \in X$,

$$\lim_{(x,y)\to(a,a)} \frac{g(x)-g(y)}{x-y} \tag{4}$$

exists, the limit being restricted to $x, y \in X,\ x \neq y$ [Ro, p. 221]. Diamond’s function $\mathrm{Log}\Gamma_{\mathrm{D}}(x)$ can be defined by the Volkenborn integral formula [Sc, p. 182]

$$\mathrm{Log}\Gamma_{\mathrm{D}}(x) := \int_{\mathbb{Z}_p} (x+t)\cdot(\log_p(x+t)-1)\,dt \quad (x \in \mathbb{Q}_p - \mathbb{Z}_p). \tag{5}$$

Since the combined difference equation (1) and Raabe formula (2) uniquely determine Diamond’s function $\mathrm{Log}\Gamma_{\mathrm{D}}$, it is natural to wonder to what extent $\mathrm{Log}\Gamma_{\mathrm{D}}$ is determined by one of these equations alone. For the difference equation the answer is trivial: f is a continuous solution of the difference equation (1) if and only if $y := f - \mathrm{Log}\Gamma_{\mathrm{D}}$ is $\mathbb{Z}_p$-periodic, *i.e.*, $y(x+t) = y(x)$ for all $t \in \mathbb{Z}_p$. (Note that there are many non-constant $\mathbb{Z}_p$-periodic functions on $\mathbb{Q}_p - \mathbb{Z}_p$, the quotient group $\mathbb{Q}_p/\mathbb{Z}_p$ being discrete and infinite.) As the derivative of any $\mathbb{Z}_p$-periodic function vanishes identically, we conclude that $f' = (\mathrm{Log}\Gamma_{\mathrm{D}})'$.

Such a simple result cannot hold for the Raabe formula (2) since $f(x) := \mathrm{Log}\Gamma_{\mathrm{D}}(x) + x - \frac{1}{2}$ satisfies (2). We show that one more derivative does the trick.

Theorem 1.1. *Let* $\mathrm{Log}\Gamma_{\mathrm{D}}\colon \mathbb{Q}_p - \mathbb{Z}_p \to \mathbb{C}_p$ *be Diamond’s p-adic analogue* (5) *of the classical* $\log\Gamma$*-function and let* $f\colon \mathbb{Q}_p - \mathbb{Z}_p \to \mathbb{C}_p$ *be strictly differentiable and satisfy the Raabe formula*

$$\int_{\mathbb{Z}_p} f(x+t)\,dt = (x-1)f'(x) - x + \tfrac{1}{2} \quad (x \in \mathbb{Q}_p - \mathbb{Z}_p). \tag{6}$$

Then f is infinitely differentiable, $(f - \mathrm{Log}\Gamma_{\mathrm{D}})'$ is $\mathbb{Z}_p$-periodic, $f'' = (\mathrm{Log}\Gamma_{\mathrm{D}})''$ and

$$f(x+1) - f(x) = \log_p x + g(x), \tag{7}$$

where $g\colon \mathbb{Q}_p - \mathbb{Z}_p \to \mathbb{C}_p$ is strictly differentiable and g' vanishes identically. Conversely, given any such g, then

$$f(x) := \int_{\mathbb{Z}_p} (x+t)\cdot\bigl(g(x+t) + \log_p(x+t) - 1\bigr)\,dt \quad (x \in \mathbb{Q}_p - \mathbb{Z}_p) \tag{8}$$

is the unique strictly differentiable $f\colon \mathbb{Q}_p - \mathbb{Z}_p \to \mathbb{C}_p$ satisfying (6) *and* (7).

We will also show (see Proposition 4.2) that any f satisfying (6) can be uniquely written in the form

$$f(x) = \mathrm{Log}\Gamma_{\mathrm{D}}(x) + q(x)\bigl(x - \tfrac{1}{2}\bigr) + r(x) \quad (x \in \mathbb{Q}_p - \mathbb{Z}_p), \tag{9}$$

where q is $\mathbb{Z}_p$-periodic and r satisfies $\int_{\mathbb{Z}_p} r(x+t)\,dt = 0$ for all $x \in \mathbb{Q}_p - \mathbb{Z}_p$. Moreover, any f of the above form satisfies (6).

We note that functions $r\colon \mathbb{Q}_p - \mathbb{Z}_p \to \mathbb{C}_p$ satisfying $\int_{\mathbb{Z}_p} r(x+t)\,dt = 0$ abound. A strictly differentiable r has this property if and only if $r(x) = h(x+1) - h(x)$ for some strictly differentiable h satisfying $h' = 0$ everywhere (see Lemma 4.1). Such r are "trivial" solutions to $\int_{\mathbb{Z}_p} r(x+t)\,dt = (x-1)r'(x)$, in the sense that the equation holds because both sides vanish identically.

Morita [Mo], [Sc, §35] defined a different analogue $\mathrm{Log}\Gamma_{\mathrm{M}}\colon \mathbb{Z}_p \to \mathbb{C}_p$ of the classical log Γ-function. Morita's $\mathrm{Log}\Gamma_{\mathrm{M}}$ is by definition the Iwasawa logarithm of his p-adic Γ-function [Mo]. (We note in passing that $\mathrm{Log}\Gamma_{\mathrm{D}}$ is not the logarithm of any function. Our notation is only meant to recall the kinship with $\log\Gamma$.) Its domain is complementary to that of Diamond's $\mathrm{Log}\Gamma_{\mathrm{D}}$ and satisfies the modified difference equation

$$f(x+1) - f(x) = \chi(x)\log_p x,$$

where χ is the characteristic function of the units $\mathbb{Z}_p^*$ of $\mathbb{Z}_p$. Morita's $\mathrm{Log}\Gamma_{\mathrm{M}}$ satisfies the integral formula [Sc, p. 176]

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x) = \int_{\mathbb{Z}_p} (x+t)\cdot\chi(x+t)\cdot(\log_p(x+t) - 1)\,dt \quad (x \in \mathbb{Z}_p),$$

and the Raabe formula [CF], [Co, p. 344]

$$\int_{\mathbb{Z}_p} \mathrm{Log}\Gamma_{\mathrm{M}}(x+t)\,dt = (x-1)(\mathrm{Log}\Gamma_{\mathrm{M}})'(x) - x + \left\lceil \tfrac{x}{p} \right\rceil, \tag{10}$$

with $\left\lceil \tfrac{x}{p} \right\rceil$ as defined after (11) below.

Theorem 1.2. *Let* $\mathrm{Log}\Gamma_{\mathrm{M}}\colon \mathbb{Z}_p \to \mathbb{C}_p$ *be Morita's* p*-adic analogue of the classical* $\log\Gamma$*-function and let* $f : \mathbb{Z}_p \to \mathbb{C}_p$ *be strictly differentiable and satisfy*

$$\int_{\mathbb{Z}_p} f(x+t)\,dt = (x-1)f'(x) - x + \left\lceil \tfrac{x}{p} \right\rceil \quad (x \in \mathbb{Z}_p), \tag{11}$$

where $\left\lceil \frac{x}{p} \right\rceil$ *is the* p*-adic limit of the usual integer ceiling function* $\left\lceil \frac{x_n}{p} \right\rceil$ *as* $x_n \to x$ *through* $x_n \in \mathbb{Z}$*. Then* $(f - \mathrm{Log}\Gamma_{\mathrm{M}})'$ *is constant and*

$$f(x+1) - f(x) = \chi(x)\log_p x + g(x), \tag{12}$$

where $g\colon \mathbb{Z}_p \to \mathbb{C}_p$ *is strictly differentiable and* g' *vanishes identically. Conversely, given any such* g*, then*

$$f(x) = \int_{\mathbb{Z}_p} (x+t)\cdot\big(g(x+t) + \chi(x+t)(\log_p(x+t) - 1)\big)\,dt \quad (x \in \mathbb{Z}_p) \tag{13}$$

is the unique strictly differentiable $f : \mathbb{Z}_p \to \mathbb{C}_p$ *satisfying* (11) *and* (12).

Again, $f(x) := \mathrm{Log}\Gamma_{\mathrm{M}}(x) + x - \frac{1}{2}$ satisfies (11), and we will show in §4 (see Proposition 4.2) that any f satisfying (11) can be uniquely written in the form

$$f(x) = \mathrm{Log}\Gamma_{\mathrm{M}}(x) + c\cdot\big(x - \tfrac{1}{2}\big) + r(x) \quad (x \in \mathbb{Z}_p), \tag{14}$$

where $c \in \mathbb{C}_p$ and r satisfies $\int_{\mathbb{Z}_p} r(t)\,dt = 0$, as well as $r'(x) = 0$ for all $x \in \mathbb{Z}_p$. Moreover, any f of the above form satisfies (11).

On setting $y := f - \mathrm{Log}\Gamma_{\mathrm{D}}$ (or $y := f - \mathrm{Log}\Gamma_{\mathrm{M}}$ in the Morita case), we will obtain the two theorems above from

Theorem 1.3. *Fix* $\mathcal{D}$ *as either* $\mathbb{Z}_p$ *or* $\mathbb{Q}_p - \mathbb{Z}_p$*, and let* $y\colon \mathcal{D} \to \mathbb{C}_p$ *be strictly differentiable on* $\mathcal{D}$ *and satisfy*

$$\int_{\mathbb{Z}_p} y(x+t)\,dt = (x-1)y'(x) \quad (x \in \mathcal{D}). \tag{15}$$

Then y *is infinitely differentiable,* y' *is* $\mathbb{Z}_p$*-periodic,* $g(x) := y(x+1) - y(x)$ *is strictly differentiable,* $g'(x) = 0 = y''(x)$ *for all* $x \in \mathcal{D}$*, and*

$$y(x) = \int_{\mathbb{Z}_p} (x+t)g(x+t)\,dt. \tag{16}$$

Conversely, given any strictly differentiable function $g\colon \mathcal{D} \to \mathbb{C}_p$ *with everywhere vanishing derivative, then* (16) *defines the unique strictly differentiable function* $y\colon \mathcal{D} \to \mathbb{C}_p$ *satisfying* (15) *and* $y(x+1) - y(x) = g(x)$.

In §3 we prove a generalization of Theorems 1.1 and 1.3 where $\mathcal{D}$ is allowed to be any subset $\mathcal{D} \subset \mathbb{C}_p$ invariant under translation by $\mathbb{Z}_p$. This is relevant since the integral formula (5) allows the domain of Diamond's $\mathrm{Log}\Gamma_{\mathrm{D}}$ to be extended from $\mathbb{Q}_p - \mathbb{Z}_p$ to $\mathbb{C}_p - \mathbb{Z}_p$ [Sc, §60].

In §4 we prove a rather different kind of consequence of Theorem 1.3.

Proposition 1.4. *Fix $\mathcal{D}$ as either $\mathbb{Z}_p$ or $\mathbb{Q}_p - \mathbb{Z}_p$, and let $V_{\mathcal{D}}$ be the set of strictly differentiable functions $y\colon \mathcal{D} \to \mathbb{C}_p$ satisfying $\int_{\mathbb{Z}_p} y(x+t)\,dt = (x-1)y'(x)$. Then the binary operation*

$$(y_1 \diamond y_2)(x) := y_2'(x)y_1(x) + y_1'(x)y_2(x) - \left(x - \tfrac{1}{2}\right)y_1'(x)y_2'(x) \tag{17}$$

makes $V_{\mathcal{D}}$ into an infinite-dimensional, commutative, associative $\mathbb{C}_p$-algebra with unit $x - \frac{1}{2}$.

Let $W_{\mathcal{D}}$ be the space of all strictly differentiable $\mathbb{C}_p$-valued functions on $\mathcal{D}$ with everywhere vanishing derivative, a ring under the usual point-wise product of functions. Theorem 1.3 gives a $\mathbb{C}_p$-vector space isomorphism $\Delta\colon V_{\mathcal{D}} \to W_{\mathcal{D}}$ taking a solution y of (15) to a $g = \Delta y$ having vanishing derivative. Trivially, $V_{\mathcal{D}}$ can be made into a ring using Δ to transport to $V_{\mathcal{D}}$ the ring structure on $W_{\mathcal{D}}$. We shall show in §4 that the ring $V_{\mathcal{D}}$ defined in Proposition 1.4 is not isomorphic to $W_{\mathcal{D}}$, so the $\diamond$-product is a genuinely different ring structure on $V_{\mathcal{D}}$. In fact, $V_{\mathcal{D}}$ is isomorphic to a certain subring of the ring of 2×2 upper-triangular matrices with coefficients in $W_{\mathcal{D}}$.

In the p-adic content it is not surprising to see the space $W_{\mathcal{D}}$ parametrizing the space of solutions $V_{\mathcal{D}}$. For functions on $\mathcal{D} = \mathbb{Z}_p$, Schikhof [Sc, §65] showed that the general solution of a first order differential equation

$$y' = T(y)$$

is parametrized by the space $W_{\mathbb{Z}_p}$, which is infinite-dimensional over $\mathbb{C}_p$ [Sc, §63]. Here $T\colon C \to C$ is a Lipschitz map on the space $C = C^0(\mathbb{Z}_p, \mathbb{C}_p)$ of continuous functions from $\mathbb{Z}_p$ to $\mathbb{C}_p$, endowed with the supremum norm. In other words, in the p-adic domain one expects functions with everywhere vanishing derivative to play the role of constants in the archimedean theory of differential equations.

Although there does not seem to be a theory of Volkenborn integro-differential equations in the literature, it is reasonable to expect $W_{\mathcal{D}}$ to play an important role in some equations of the form

$$\int_{\mathbb{Z}_p} y(x+t)\,dt = T(y)(x). \tag{18}$$

Indeed, on applying the difference operator Δ to both sides we find

$$y' = \Delta T(y),$$

which is of Schikhof's type if T is. In Theorem 1.3, $T(y)(x) = (x-1)y'(x)$ is not of Schikhof's type. Still, the first step in our proof of Theorem 1.3 will be to apply the difference operator.

Another explanation of the importance of $W_{\mathfrak{D}}$ lies in the decomposition

$$W_{\mathfrak{D}} = A_{\mathfrak{D}} \oplus X_{\mathfrak{D}}, \tag{19}$$

where $A_{\mathfrak{D}} \subset W_{\mathfrak{D}}$ is the $\mathbb{C}_p$-algebra of $\mathbb{Z}_p$-periodic functions on $\mathfrak{D}$, and $X_{\mathfrak{D}}$ consists of all $y \in W_{\mathfrak{D}}$ satisfying $\int_{\mathbb{Z}_p} y(x+t)\,dt = 0$ for all $x \in \mathfrak{D}$ (see §4). We trivially have that any $y \in X_{\mathfrak{D}} \cap T^{-1}(0)$ is a solution of (18). In particular, if $X_{\mathfrak{D}} \subset T^{-1}(0)$, then all $y \in X_{\mathfrak{D}}$ are solutions of (18). This is indeed the case in Theorem 1.3.

The summand $A_{\mathfrak{D}}$ in (19) can also play a role in (18). If $h \in A_{\mathfrak{D}}$, then

$$\int_{\mathbb{Z}_p} h(x+t)y(x+t)\,dt = h(x)\int_{\mathbb{Z}_p} y(x+t)\,dt.$$

Hence, if T is an $A_{\mathfrak{D}}$-module map, then the set of solutions of (18) is an $A_{\mathfrak{D}}$-module. A rather general example of this kind of equation is

$$\int_{\mathbb{Z}_p} y(x+t)\,dt = T(y)(x) = \sum_{0\le j,k\le n} a_{j,k}(x)\cdot(D^j\Delta^k y)(x),$$

where D and Δ denote the differential and difference operators, and the $a_{j,k}$ are arbitrary functions on $\mathfrak{D}$.

2. Proof of the theorems

We begin by showing that Theorem 1.1 follows from Theorem 1.3. Diamond's function $\mathrm{Log}\Gamma_{\mathrm{D}}$ defined by

$$\mathrm{Log}\Gamma_{\mathrm{D}}(x) := \int_{\mathbb{Z}_p} (x+t)\cdot(\log_p(x+t)-1)\,dt \quad (x \in \mathbb{Q}_p - \mathbb{Z}_p)$$

is locally analytic on $\mathbb{Q}_p - \mathbb{Z}_p$, and is therefore strictly differentiable and infinitely differentiable [Sc, Theorem 60.2 and Corollary 29.11]. It satisfies the difference equation

$$\mathrm{Log}\Gamma_{\mathrm{D}}(x+1) - \mathrm{Log}\Gamma_{\mathrm{D}}(x) = \log_p(x) \quad (x \in \mathbb{Q}_p - \mathbb{Z}_p)$$

[Sc, Theorem 60.2] and the Raabe formula [CF]

$$\int_{\mathbb{Z}_p} f(x+t)\,dt = (x-1)f'(x) - x + \tfrac{1}{2} \quad (x \in \mathbb{Q}_p - \mathbb{Z}_p). \tag{20}$$

If $f : \mathbb{Q}_p - \mathbb{Z}_p \to \mathbb{C}_p$ is any other strictly differentiable function satisfying (20), then $y(x) := f(x) - \mathrm{Log}\Gamma_{\mathrm{D}}(x)$ is a strictly differentiable function on $\mathbb{Q}_p - \mathbb{Z}_p$ which satisfies

$$\int_{\mathbb{Z}_p} y(x+t)\,dt = (x-1)y'(x).$$

Bearing in mind the above properties of $\mathrm{Log}\Gamma_{\mathrm{D}}$, every assertion about f in Theorem 1.1 is a direct translation of the corresponding statement for y in Theorem1.3 for $\mathcal{D} = \mathbb{Q}_p - \mathbb{Z}_p$.

The proof that Theorem 1.2 follows from Theorem 1.3 is similar, but we can make better use of the difference equation on $\mathcal{D} = \mathbb{Z}_p$ than we could on $\mathcal{D} = \mathbb{Q}_p - \mathbb{Z}_p$. Indeed, Morita's $\mathrm{Log}\Gamma_{\mathrm{M}}(x)$ satisfies for $x \in \mathbb{Z}_p$ the difference equation

$$f(x+1) - f(x) = \chi(x)\log_p x, \tag{21}$$

where χ is the characteristic function of the units $\mathbb{Z}_p^*$ of $\mathbb{Z}_p$ [Sc, p. 176]. It also satisfies the integral formula

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x) = \int_{\mathbb{Z}_p} (x+t)\cdot\chi(x+t)\cdot(\log_p(x+t)-1)\,dt \quad (x \in \mathbb{Z}_p)$$

[Sc, p. 176] and the Raabe formula

$$\int_{\mathbb{Z}_p} \mathrm{Log}\Gamma_{\mathrm{M}}(x+t)\,dt = (x-1)(\mathrm{Log}\Gamma_{\mathrm{M}})'(x) - x + \left\lceil \frac{x}{p} \right\rceil, \tag{22}$$

where $\left\lceil \frac{x}{p} \right\rceil$ is defined in Theorem 1.2 [CF], [Co, p. 344]. Since $\mathrm{Log}\Gamma_{\mathrm{M}}$ is analytic on $p\mathbb{Z}_p$ [Sc, Lemma 58.1] and the Iwasawa logarithm is locally analytic on $\mathbb{C}_p - \{0\}$ [Sc, Theorem 45.12], the difference equation (21) implies that $\mathrm{Log}\Gamma_{\mathrm{M}}$ is locally analytic on $\mathbb{Z}_p$.

Now suppose $f : \mathbb{Z}_p \to \mathbb{C}_p$ is strictly differentiable and satisfies the Raabe formula (22). As before, $y(x) := f(x) - \mathrm{Log}\Gamma_{\mathrm{M}}(x)$ is then a strictly differentiable function on $\mathbb{Z}_p$ which satisfies

$$\int_{\mathbb{Z}_p} y(x+t)\,dt = (x-1)y'(x) \quad (x \in \mathbb{Z}_p).$$

By Theorem 1.3, $g(x) := y(x+1) - y(x)$ has an everywhere vanishing derivative. Thus $y'(x+1) = y'(x)$, and y' is continuous since y is strictly differentiable [Sc, §27], [Ro, p. 221]. It follows that y' is constant on $\mathbb{Z}_p$, as claimed in Theorem 1.2. The rest of Theorem 1.2 follows from Theorem 1.3 just as Theorem 1.1 did.

Turning now to the proof of Theorem 1.3, we begin by recalling some general properties of Volkenborn integrals. Let $\mathcal{D} = \mathbb{Z}_p$, or $\mathcal{D} = \mathbb{Q}_p - \mathbb{Z}_p$, and let

$$H(x) := \int_{\mathbb{Z}_p} q(x+t)\,dt \quad (x \in \mathcal{D}), \tag{23}$$

where $q\colon \mathcal{D} \to \mathbb{C}_p$ is assumed strictly differentiable. Then [Ro, p. 265]

$$H(x+1) - H(x) = q'(x), \tag{24}$$

and [CF, Lemma 2.1], [Co, p. 396]

$$H(x) = q(x) - \int_{\mathbb{Z}_p} (t+1)\Delta q(x+t)\,dt. \tag{25}$$

Here $\Delta q(x) := q(x+1) - q(x)$.

Let y be as in Theorem 1.3. Thus y is strictly differentiable on $\mathcal{D}$ and

$$H(x) := \int_{\mathbb{Z}_p} y(x+t)\,dt = (x-1)y'(x) \quad (x \in \mathcal{D}). \tag{26}$$

Set

$$g(x) := \Delta y(x). \tag{27}$$

Using (24), (26) and (27) we find

$$y'(x) = H(x+1) - H(x) = xy'(x+1) - (x-1)y'(x) = xg'(x) + y'(x). \tag{28}$$

Hence, $g'(x) = 0$ for all $x \in \mathcal{D}$, except possibly at $x = 0$ when $\mathcal{D} = \mathbb{Z}_p$. As g is strictly differentiable (because y is), g' is continuously differentiable [Ro, p. 221]. Hence $g'(x) = 0$ for all $x \in \mathcal{D}$.

But y' being continuous and $\Delta(y') = (\Delta y)' = g' = 0$ show that y' is $\mathbb{Z}_p$-periodic. Hence y is infinitely differentiable and $y'' = 0$ identically, as claimed in Theorem 1.3.

Next we show

$$y(x) = \int_{\mathbb{Z}_p} (x+t)g(x+t)\,dt \quad (x \in \mathcal{D}). \tag{29}$$

Note that the Volkenborn integral above is well-defined because g is strictly differentiable on $\mathcal{D}$. From (25)–(27) we find

$$(x-1)y'(x) = \int_{\mathbb{Z}_p} y(x+t)\,dt = y(x) - \int_{\mathbb{Z}_p} (t+1)g(x+t)\,dt,$$

or

$$y(x) = (x-1)y'(x) + \int_{\mathbb{Z}_p} (t+1)g(x+t)\,dt. \tag{30}$$

But, by (24),

$$y'(x) = \Delta\Big(\int_{\mathbb{Z}_p} y(x+t)\,dt\Big) = \int_{\mathbb{Z}_p} (\Delta y)(x+t)\,dt = \int_{\mathbb{Z}_p} g(x+t)\,dt. \tag{31}$$

On substituting (31) in (30), we obtain (29), proving the first part of Theorem 1.3.

We now prove the converse claim in Theorem 1.3. The uniqueness claim is a special case of [CF, Proposition, p. 365], but we give the proof here for completeness. Suppose both y_1 and y_2 are solutions of

$$y(x+1) - y(x) = g(x) \quad \text{and} \quad \int_{\mathbb{Z}_p} y(x+t)\,dt = (x-1)y'(x), \tag{32}$$

where g is given. Then $h := y_1 - y_2$ satisfies

$$h(x+1) - h(x) = 0 \quad \text{and} \quad \int_{\mathbb{Z}_p} h(x+t)\,dt = (x-1)h'(x).$$

But $h(x+1) = h(x)$ implies that $h'(x) = 0$ identically. It also implies, by the very definition (3) of the Volkenborn integral, that $h(x) = \int_{\mathbb{Z}_p} h(x+t)\,dt$. Hence $h(x) = 0$ and uniqueness is proved.

We now turn to rest of the converse claim in Theorem 1.3. Suppose then that $g\colon \mathcal{D} \to \mathbb{C}_p$ is strictly differentiable and $g'(x) = 0$ for all $x \in \mathcal{D}$. We wish to show that there is a strictly differentiable solution y to (32) and that y is given by

$$y(x) = \int_{\mathbb{Z}_p} (x+t)g(x+t)\,dt \quad (x \in \mathcal{D}). \tag{33}$$

To this end, define y by (33). Then, from (24),

$$y(x+1) - y(x) = \bigl(xg(x)\bigr)' = g(x), \tag{34}$$

since $g' = 0$. Thus

$$\Delta y = g. \tag{35}$$

Now assume $\mathcal{D} = \mathbb{Z}_p$ (we will deal with the case $\mathcal{D} = \mathbb{Q}_p - \mathbb{Z}_p$ in a bit). Then (35) shows

$$y(x) = Sg(x) + c, \tag{36}$$

where $Sg\colon \mathbb{Z}_p \to \mathbb{C}_p$ is the indefinite sum of g [Ro, p. 177], [Sc, pp. 105–106] and c is a constant. But Sg is strictly differentiable because g is [Ro, p. 232], [Sc, p. 162]. Hence y is strictly differentiable, as claimed in Theorem 1.3.

We now show that y satisfies $\int_{\mathbb{Z}_p} y(x+t)\,dt = (x-1)y'(x)$. Note that

$$\int_{\mathbb{Z}_p} g(x+t)\,dt = (Sg)'(x) = y'(x), \tag{37}$$

where the first equality amounts to the proof of existence of the Volkenborn integral [Ro, p. 264], [Sc, p. 167], and the second equality uses (36). Now (25) and $\Delta y = g$

yield

$$\begin{aligned}\int_{\mathbb{Z}_p} y(x+t)\,dt &= y(x) - \int_{\mathbb{Z}_p} (t+1)g(x+t)\,dt \\ &= y(x) + (x-1)\int_{\mathbb{Z}_p} g(x+t)\,dt - \int_{\mathbb{Z}_p} (x+t)g(x+t)\,dt \\ &= (x-1)y'(x),\end{aligned} \tag{38}$$

where we also used (37) and (33). This completes the proof of Theorem 1.3 in the Morita case $\mathcal{D} = \mathbb{Z}_p$.

When $\mathcal{D} = \mathbb{Q}_p - \mathbb{Z}_p$ we meet a nuisance in (36) because the indefinite sum operator S is only defined for (continuous) functions with domain $\mathbb{Z}_p$. To get around this, for $q\colon \mathcal{D} \to \mathbb{C}_p$ and $x \in \mathcal{D}$, define $q_x\colon \mathbb{Z}_p \to \mathbb{C}_p$ by $q_x(t) := q(x+t)$. Then $\Delta y = g$ in (34) becomes

$$\Delta y_x(t) = y_x(t+1) - y_x(t) = g_x(t) \quad (x \in \mathcal{D},\ t \in \mathbb{Z}_p).$$

Hence

$$y_x(t) = S(g_x)(t) + c_x \quad (x \in \mathcal{D},\ t \in \mathbb{Z}_p), \tag{39}$$

and $t \to y_x(t)$ is strictly differentiable on $\mathbb{Z}_p$. Thus, for any fixed $a \in \mathcal{D}$,

$$\lim_{(r,s)\to(0,0)} \frac{y_a(r) - y_a(s)}{r-s} = \lim_{(r,s)\to(0,0)} \frac{y(a+r) - y(a+s)}{r-s} \tag{40}$$

exists, the limit being restricted to $r, s \in \mathbb{Z}_p,\ r \neq s$. As $a \in \mathcal{D} \subset \mathbb{Q}_p$, the existence of (40) is equivalent to the existence of

$$\lim_{(r',s')\to(a,a)} \frac{y(r') - y(s')}{r'-s'},$$

the limit being restricted to $r', s' \in \mathcal{D},\ r' \neq s'$. Hence y is again strictly differentiable on $\mathcal{D} = \mathbb{Q}_p - \mathbb{Z}_p$.

The proof of Theorem 1.3 in the Diamond case $\mathcal{D} = \mathbb{Q}_p - \mathbb{Z}_p$ now mimics the Morita case. For example, (37) becomes

$$\int_{\mathbb{Z}_p} g(x+t)\,dt = \int_{\mathbb{Z}_p} g_x(t)\,dt = \frac{d}{dt}(Sg_x)\big|_{t=0} = \frac{d}{dt}\big(y(x+t) - c_x\big)\big|_{t=0} = y'(x),$$

where we used (39).

3. Other domains

Diamond's function $\mathrm{Log}\Gamma_{\mathrm{D}}$ can be defined for $x \in \mathbb{C}_p - \mathbb{Z}_p$ by the Volkenborn integral formula (5), and is actually locally analytic on $\mathbb{C}_p - \mathbb{Z}_p$ [Sc, Theorem 60.2 (iv)]. The difference equation (1) and the Raabe formula (2) still hold for $x \in \mathbb{C}_p - \mathbb{Z}_p$ and they uniquely characterize $\mathrm{Log}\Gamma_{\mathrm{D}}$ among all strictly differentiable functions on $\mathbb{C}_p - \mathbb{Z}_p$. More generally, let $\mathcal{D} \subset \mathbb{C}_p - \mathbb{Z}_p$ be such that $x \in \mathcal{D}$ and $t \in \mathbb{Z}_p$ imply $x + t \in \mathcal{D}$. Then $\mathrm{Log}\Gamma_{\mathrm{D}}$ restricted to $\mathcal{D}$ is again the unique strictly differentiable function on $\mathcal{D}$ satisfying (1) and (2), [CF].

Part of Theorem 1.3 holds for any $\mathcal{D} \subset \mathbb{C}_p$, assumed invariant under translation by $\mathbb{Z}_p$. Namely, if y is strictly differentiable on $\mathcal{D}$ and satisfies (15), then $g = \Delta y$ is strictly differentiable on $\mathcal{D}$ and $g' = 0$ everywhere on $\mathcal{D}$. The proof given in §2 goes through without change.

Other parts of Theorem 1.3, however, do not seem to generalize. The problem is that if we begin with g strictly differentiable on $\mathcal{D}$ and satisfying $g' = 0$, the differentiability (strict or not) of the function $t \to y_x(t) = y(x + t)$ for $t \in \mathbb{Z}_p$ (see (39)) does not imply differentiability of $x \to y(x)$ for $x \in \mathcal{D}$ (unless $\mathcal{D} \subset \mathbb{Q}_p$).

To account for this we shall say that $y\colon \mathcal{D} \to \mathbb{C}_p$ is strictly $\mathbb{Z}_p$-differentiable on $\mathcal{D}$ if for each fixed $x \in \mathcal{D}$, the map $t \to y(x + t)$ for $t \in \mathbb{Z}_p$ is strictly differentiable as a function with domain $\mathbb{Z}_p$. We denote the corresponding derivative by

$$y'(x) := \lim_{t \to 0,\, t \in \mathbb{Z}_p} \frac{y(x+t) - y(x)}{t},$$

and call it the $\mathbb{Z}_p$-derivative of y. We note that for strictly $\mathbb{Z}_p$-differentiable y, the $\mathbb{Z}_p$-derivative y' is $\mathbb{Z}_p$-continuous, *i.e.*, $t \to y'(x + t)$ is continuous for $t \in \mathbb{Z}_p$. Although strict $\mathbb{Z}_p$-differentiability is a very weak notion if $\mathcal{D} \not\subset \mathbb{Q}_p$, it is exactly what is needed to define the Volkenborn integral $\int_{\mathbb{Z}_p} y(x + t)\, dt$. An example of a strictly $\mathbb{Z}_p$-differentiable function on $\mathcal{D} = \mathbb{C}_p$ which is not even continuous on $\mathcal{D}$ is $y(x) := 1$ if $x \in \mathbb{Q}_p$, $y(x) := 0$ if $x \in \mathbb{C}_p - \mathbb{Q}_p$.

Examination of the proof given in [CF] shows that $f = \mathrm{Log}\Gamma_{\mathrm{D}}$ restricted to $\mathcal{D}$ is the unique strictly $\mathbb{Z}_p$-differentiable function on $\mathcal{D}$ satisfying the difference equation (1) and the Raabe formula (2). With these adaptations we have

Theorem 3.1. *Let $\mathcal{D} \subset \mathbb{C}_p$ be invariant under translation by $\mathbb{Z}_p$ and let $y\colon \mathcal{D} \to \mathbb{C}_p$ be strictly $\mathbb{Z}_p$-differentiable and satisfy*

$$\int_{\mathbb{Z}_p} y(x+t)\, dt = (x-1)y'(x) \quad (x \in \mathcal{D}), \tag{41}$$

where y' denotes the $\mathbb{Z}_p$-derivative of y. Then y is infinitely $\mathbb{Z}_p$-differentiable, y' is $\mathbb{Z}_p$-periodic, $g(x) := y(x + 1) - y(x)$ is a strictly $\mathbb{Z}_p$-differentiable function on $\mathcal{D}$ whose $\mathbb{Z}_p$-derivative vanishes everywhere on $\mathcal{D}$, and

$$y(x) = \int_{\mathbb{Z}_p} (x+t)g(x+t)\, dt. \tag{42}$$

Conversely, given any strictly $\mathbb{Z}_p$-differentiable function $g\colon \mathcal{D} \to \mathbb{C}_p$ with everywhere vanishing $\mathbb{Z}_p$-derivative, then (42) *defines the unique strictly $\mathbb{Z}_p$-differentiable function $y\colon \mathcal{D} \to \mathbb{C}_p$ satisfying* (41) *and $g(x) = y(x+1) - y(x)$.*

Proof. The proof of Theorem 1.3 for $\mathcal{D} = \mathbb{Q}_p - \mathbb{Z}_p$ goes through after replacing "strictly differentiable" everywhere by "strictly $\mathbb{Z}_p$-differentiable" and understanding all derivatives to be $\mathbb{Z}_p$-derivatives. We should also note, regarding equation (25), that its proof in [CF] only requires strict $\mathbb{Z}_p$-differentiability. □

Using the above result, it is clear that Theorem 1.1 can be extended from the domain $\mathbb{Q}_p - \mathbb{Z}_p$ to any $\mathcal{D} \subset \mathbb{C}_p - \mathbb{Z}_p$ invariant under translation by $\mathbb{Z}_p$, provided differentiability (strict or not) is understood in the $\mathbb{Z}_p$ sense defined above.

4. Decomposition of the solution space

We again fix $\mathcal{D} \subset \mathbb{C}_p$, assumed invariant under translation by elements of $\mathbb{Z}_p$, and use the notions of $\mathbb{Z}_p$-continuity, $\mathbb{Z}_p$-derivative and strict $\mathbb{Z}_p$-differentiability defined in §3. When $\mathcal{D} \subset \mathbb{Q}_p$ these coincide with the usual definitions of continuity, derivative and strict differentiability. All derivatives in this section are to be understood as $\mathbb{Z}_p$-derivatives.

Let $W_\mathcal{D}$ be the $\mathbb{C}_p$-algebra (under the usual point-wise product of functions) of all strictly $\mathbb{Z}_p$-differentiable $\mathbb{C}_p$-valued functions on $\mathcal{D}$ with everywhere vanishing $\mathbb{Z}_p$-derivative, and let $A_\mathcal{D} \subset W_\mathcal{D}$ be the sub-algebra of $\mathbb{Z}_p$-periodic functions. If we let $\Delta\colon W_\mathcal{D} \to W_\mathcal{D}$ be the usual difference operator $\Delta(y)(x) := y(x+1) - y(x)$, we have $A_\mathcal{D} = \ker(\Delta)$.

Lemma 4.1. *For $y \in W_\mathcal{D}$, let $T(y)\colon \mathcal{D} \to \mathbb{C}_p$ be defined by*

$$T(y)(x) := \int_{\mathbb{Z}_p} y(x+t)\,dt,$$

and let $X_\mathcal{D} = \ker(T) \subset W_\mathcal{D}$. Then

(1) *T maps $W_\mathcal{D}$ onto $A_\mathcal{D}$, T restricted to $A_\mathcal{D}$ is the identity map on $A_\mathcal{D}$, and so $T^2 = T$;*

(2) *T and Δ are $A_\mathcal{D}$-module maps and $T\Delta = \Delta T = 0$;*

(3) *$W_\mathcal{D} = A_\mathcal{D} \oplus X_\mathcal{D}$, the internal direct sum being as $\mathbb{C}_p$-vector spaces and as $A_\mathcal{D}$-modules;*

(4) *Δ maps $W_\mathcal{D}$ onto $X_\mathcal{D}$, and Δ restricted to $X_\mathcal{D}$ is an isomorphism of $X_\mathcal{D}$ onto itself.*

Proof. For any strictly $\mathbb{Z}_p$-differentiable y, the function $x \to \int_{\mathbb{Z}_p} y(x+t)\,dt$ is $\mathbb{Z}_p$-continuous, as the Volkenborn integral is the $\mathbb{Z}_p$-derivative of a strictly $\mathbb{Z}_p$-differentiable function, namely of the indefinite sum of $t \to y(x+t)$ [Sc, p. 167], [Ro, p. 267]. Also [Ro, p. 265]

$$\Delta\big(T(y)\big)(x) = T\big(\Delta(y)\big)(x) = \int_{\mathbb{Z}_p} \big(y(x+1+t)-y(x+t)\big)\,dt = y'(x) = 0, \quad (43)$$

where the last step uses $y \in W_{\mathfrak{D}}$. Since $x \to T(y)(x)$ is $\mathbb{Z}_p$-continuous, it follows that $T(y)$ is $\mathbb{Z}_p$-periodic. Hence $T(W_{\mathfrak{D}}) \subset A_{\mathfrak{D}}$. For $f \in A_{\mathfrak{D}}$,

$$T(f)(x) = \int_{\mathbb{Z}_p} f(x+t)\,dt = \int_{\mathbb{Z}_p} f(x)\,dt = f(x). \quad (44)$$

Claim (1) in the lemma is now clear (it essentially amounts to [Sc, Corollary 55.6]).

To prove claim (2) take $h \in A_{\mathfrak{D}}$ and any strictly $\mathbb{Z}_p$-differentiable y. Then

$$\int_{\mathbb{Z}_p} h(x+t)y(x+t)\,dt = h(x)\int_{\mathbb{Z}_p} y(x+t)\,dt,$$

and so T is an $A_{\mathfrak{D}}$-module map. For Δ the verification of this fact is equally simple. We have already seen in (43) that $T\Delta = \Delta T = 0$, as claimed in (2).

To prove (3), take $y \in W_{\mathfrak{D}}$ and write $y = T(y) + \big(y - T(y)\big)$. By claim (1) in the lemma, $T\big(y - T(y)\big) = T(y) - T^2(y) = 0$, so $y - T(y) \in X_{\mathfrak{D}}$. Since we have already shown that $T(y) \in A_{\mathfrak{D}}$, we have $W_{\mathfrak{D}} = A_{\mathfrak{D}} + X_{\mathfrak{D}}$. If $f \in X_{\mathfrak{D}} \cap A_{\mathfrak{D}}$, then $T(f) = 0$ as $f \in \ker(T) = X_{\mathfrak{D}}$. But (44) shows $f = T(f)$, since $f \in A_{\mathfrak{D}}$. Hence $X_{\mathfrak{D}} \cap A_{\mathfrak{D}} = \{0\}$ and claim (3) is proved.

To prove claim (4), note that $T\Delta = 0$ implies $\Delta W_{\mathfrak{D}} \subset \ker(T) = X_{\mathfrak{D}}$. To prove $\Delta W_{\mathfrak{D}} = \Delta X_{\mathfrak{D}} = X_{\mathfrak{D}}$, take $y \in X_{\mathfrak{D}}$ and let

$$h(x) := \int_{\mathbb{Z}_p} (x+t)y(x+t)\,dt. \quad (45)$$

Then $\Delta h(x) = \big(xy(x)\big)' = y(x)$, since $y' = 0$ for $y \in W_{\mathfrak{D}}$. As in §2 (see the paragraph containing (39)), we conclude that h is strictly $\mathbb{Z}_p$-differentiable. Since $\Delta h = y$, we have [Ro, p. 264], [Sc, p. 167]

$$h'(x) = \int_{\mathbb{Z}_p} y(x+t)\,dt = T(y)(x) = 0,$$

where the last step uses $y \in X_{\mathfrak{D}}$. Thus $h \in W_{\mathfrak{D}}$. To prove $h \in X_{\mathfrak{D}}$ we use $\Delta h = y$ and (25) to compute

$$\int_{\mathbb{Z}_p} h(x+t)\,dt = h(x) - \int_{\mathbb{Z}_p} (t+1)y(x+t)\,dt = h(x) - \int_{\mathbb{Z}_p} (t+x)y(x+t)\,dt = 0,$$

where we also used $\int_{\mathbb{Z}_p} y(x+t)\,dt = 0$ and (45). Thus Δ maps $X_{\mathfrak{D}}$ onto itself. The injectivity of Δ on $X_{\mathfrak{D}}$ is clear from $X_{\mathfrak{D}} \cap A_{\mathfrak{D}} = \{0\}$, which we have already shown. □

We can now prove

Proposition 4.2. *Let $y\colon \mathfrak{D} \to \mathbb{C}_p$ be strictly $\mathbb{Z}_p$-differentiable and satisfy*

$$\int_{\mathbb{Z}_p} y(x+t)\,dt = (x-1)y'(x). \tag{46}$$

Then

$$y(x) = q(x)\big(x - \tfrac{1}{2}\big) + r(x), \tag{47}$$

for a unique $q \in A_{\mathfrak{D}}$ and a unique $r \in X_{\mathfrak{D}}$. Conversely, given $q \in A_{\mathfrak{D}}$ and $r \in X_{\mathfrak{D}}$, (47) *defines a strictly $\mathbb{Z}_p$-differentiable function y satisfying* (46).

Proof. We first prove the converse claim. If $r \in X_{\mathfrak{D}}$, then $\int_{\mathbb{Z}_p} r(x+t)\,dt = 0 = r'(x)$, and so r is trivially a solution of (46). One checks directly that $y(x) = x - \frac{1}{2}$ is a solution of (46) and that the set of solutions $V_{\mathfrak{D}}$ is an $A_{\mathfrak{D}}$-module. Thus the converse statement is clear.

To prove the main statement, note that by Theorem 3.1 the map $\Delta\colon V_{\mathfrak{D}} \to W_{\mathfrak{D}}$ is a $\mathbb{C}_p$-isomorphism (we again denote the difference operator by Δ, despite the change of domain with respect to Lemma 4.1). One easily checks that Δ is an $A_{\mathfrak{D}}$-module isomorphism. Under it $x - \frac{1}{2}$ maps to the constant function 1. Hence $A_{\mathfrak{D}} \cdot (x - \frac{1}{2})$ maps isomorphically onto $A_{\mathfrak{D}} \subset W_{\mathfrak{D}}$. But we have just seen that $X_{\mathfrak{D}} \subset V_{\mathfrak{D}}$. By Lemma 4.1 (4), Δ restricts to an isomorphism of $X_{\mathfrak{D}}$ onto itself. Since $W_{\mathfrak{D}} = A_{\mathfrak{D}} \oplus X_{\mathfrak{D}}$ (see Lemma 4.1 (3)), we conclude that $V_{\mathfrak{D}} = A_{\mathfrak{D}} \cdot (x - \frac{1}{2}) \oplus X_{\mathfrak{D}}$. □

Lastly, we turn to the $\diamond$-product structure on $V_{\mathfrak{D}}$ defined in Proposition 1.4. Using Proposition 4.2, we can write $y_i \in V_{\mathfrak{D}}$ as

$$y_i(x) = q_i(x)\big(x - \tfrac{1}{2}\big) + r_i(x) \quad (r_i \in X_{\mathfrak{D}},\ q_i \in A_{\mathfrak{D}},\ i = 1 \text{ or } 2).$$

Then

$$\begin{aligned}(y_1 \diamond y_2)(x) &:= y_2'(x)y_1(x) + y_1'(x)y_2(x) - \big(x - \tfrac{1}{2}\big)y_1'(x)y_2'(x) \\ &= q_2(x)y_1(x) + q_1(x)y_2(x) - \big(x - \tfrac{1}{2}\big)q_1(x)q_2(x) \\ &= \big(q_1(x)q_2(x)\big)\big(x - \tfrac{1}{2}\big) + \big(q_1(x)r_2(x) + q_2(x)r_1(x)\big),\end{aligned} \tag{48}$$

which is again in the form of Proposition 4.2. Hence $V_{\mathfrak{D}}$ with the $\diamond$-product is isomorphic to the ring R of all upper triangular matrices of the form $\begin{pmatrix} q(x) & r(x) \\ 0 & q(x) \end{pmatrix}$, where $q \in A_{\mathfrak{D}}$ and $r \in X_{\mathfrak{D}}$. The ring R is clearly an associative and commutative $\mathbb{C}_p$-algebra.

The nilpotent elements of R are exactly the matrices of the form $\begin{pmatrix} 0 & r(x) \\ 0 & 0 \end{pmatrix}$. Since $W_{\mathfrak{D}}$, under the usual point-wise product of functions, has no non-zero nilpotent elements, we see that $V_{\mathfrak{D}}$ is not ring-isomorphic to $W_{\mathfrak{D}}$.

To see that $W_{\mathfrak{D}}$ is an infinite-dimensional $\mathbb{C}_p$-vector space, note the isomorphism $W_{\mathfrak{D}} \cong \prod_{x \in \mathfrak{D}/\mathbb{Z}_p} W_{x+\mathbb{Z}_p}$ and $W_{x+\mathbb{Z}_p} \cong W_{\mathbb{Z}_p}$. The latter space is known to be infinite-dimensional over $\mathbb{C}_p$ as it can be explicitly described in van der Put's base [Sc, Theorem 63.3].

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