Carriers of continuous measures in a Hilbertian norm

By

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Summary. From the standpoint of the theory of measures on the dual space of a nuclear space, we discuss the carrier of Wiener measure, regarding it as a measure on $(\mathcal{D}')$ (=Schwartz's space of distributions). This may be contrasted with the usual treatment which regards it as a measure on the space of paths.

It is shown that for $\alpha > \frac{1}{2}$, integral operator I_α is nuclear on $L^2((0, 1))$ $(\equiv H_0)$. Using this fact, we see that Wiener measure lies on the space $I_\beta(H_0)$ $\left(\beta < \frac{1}{2}\right)$ which consists of Hölder continuous functions of the β-th order in the sense of L^2. This result is true for any measure whose characteristic functional is continuous on $L^2((0, 1))$.

§ 1. Nuclear operators and carriers of measures

Let H_0 be a real Hilbert space with the scalar product $\langle \xi, \eta \rangle_0$, and L be its subspace which is dense and nuclear in H_0. It means that there exists a complete orthonormal system $\{\xi_k\}$ in H_0 and a sequence of positive numbers $\{a_k\}$ such that $\sum_{k=1}^{\infty} a_k^2 < \infty$ and the norm $\|\xi\|_1^2 = \sum_{k=1}^{\infty} \frac{\langle \xi, \xi_k \rangle_0^2}{a_k^2}$ is continuous in the proper topology of L. R. A. Minlos proved that for any positive definite and continuous functional $\chi(\xi)$ on H_0, there exists a measure μ on L^* such that for any $\xi \in L$,

$$\chi(\xi) = \int \exp[i\xi(x)]\, d\mu(x), \qquad (1)$$

$\chi(\xi)$ is called the characteristic functional of μ.

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Suppose that both L_1 and L_2 are dense and nuclear in H_0, and that $L_1 \subset L_2$. Then, for given $\chi(\xi)$, we can construct measures μ_1 on L_1^* and μ_2 on L_2^*. Identifying them, we can say that the measure μ_1 on L_1^* has the carrier in L_2^* $(\subset L_1^*)$.

Now, let L be a fixed dense and nuclear subspace of H_0, and $\chi(\xi)$ be a fixed positive definite and continuous functional on H_0. We shall discuss the carrier of the measure μ which is defined on L^*, corresponding to $\chi(\xi)$.

Consider an operator T which satisfies the following conditions;

(1) T is defined and nuclear on H_0.

(Nuclearity of T is defined by $\sum_{k=1}^{\infty} \|T\xi_k\|_0^2 < \infty$, where $\{\xi_k\}$ is a complete orthonormal system of H_0).

(2) T is one-to-one.

(3) The image $T(H_0)$ includes L, and $T^{-1}(L)$ is dense in H_0.

(4) $T\xi_j \to 0$ in the topology of L implies that $\|\xi_j\|_0 \to 0$. In other words, the inverse operator T^{-1} maps L into H_0 continuously.

Proposition 1. *For any operator T which satisfies* (1)~(4), *the measure* μ (*which corresponds to the given* $\chi(\xi)$) *has the carrier in* $T^{-1*}(H_0)$, *where* T^{-1*} *means the adjoint operator of* T^{-1}. (T^{-1*} *maps* $H_0^* \simeq H_0$ *into* L^* *continuously*).

Proof. For $\xi = T\eta \in T(H_0)$, define the norm $\|\xi\|_1$ by $\|\xi\|_1 = \|T^{-1}\xi\|_0$. By this norm, $T(H_0)$ becomes a Hilbert space which we denote by H_1. From the conditions (4) and (1), the topology of H_1 is weaker than that of L, but stronger than that of H_0. Hence, $H_0 \simeq H_0^* \subset H_1^* \subset L^*$.

From the condition (1), H_1 is nuclear in H_0, so that the measure μ has the carrier in H_1^*. Thus, only remained to prove is that $H_1^* = T^{-1*}(H_0)$.

If $x = T^{-1*}y \in T^{-1*}(H_0)$. then $(\xi, x) = (\xi, T^{-1*}y) = (T^{-1}\xi, y)$ is continuous on H_1 so that $x \in H_1^*$. Conversely, if $x \in H_1^*$, then the relation $(\xi, x) = (T^{-1}\xi, y)$ determines uniquely $y \in H_0$ and $x = T^{-1*}y \in T^{-1*}(H_0)$. (q.e.d.)

Since μ is completely additive, we get the following corollary.

Corollary. *If each of operators* T_n $(n = 1, 2, \cdots)$ *satisfies the* *conditions* (1)~(4), *then the measure* μ *has the carrier in* $\bigcap_{n=1}^{\infty} T_n^{-1*}(H_0)$.

§ 2. Integral operators

We shall apply the result of § 1 to the case of $H_0=L^2((0, 1))$ and $L=\mathfrak{D}(0, 1))$. Namely, we shall discuss the carrier of a measure μ on $(\mathfrak{D}')$ whose characteristic functional is continuous in $L^2((0, 1))$.

At first, define the integral operator I_α as follows;

$$I_\alpha f(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds. & (\alpha<0) \\ \frac{d^m}{dt^m} I_{\alpha+m} f(t). & (\alpha\leq 0) \end{cases} \tag{2}$$

The main properties of this operator are;

a) For any real α, it is defined as a operator on $\mathfrak{D}((0, 1))$, and maps $\mathfrak{D}$ into $L^2((0, 1))$ continuously.

b) For any real α, it is defined as a operator on L^2 which maps L^2 into $(\mathfrak{D}')$ continuously.

c) For $\alpha>\frac{1}{2}$, it maps L^2 into L^2 continuously.

d) For any α and β, the relation: $I_\beta I_\alpha = I_{\alpha+\beta}$ holds on $\mathfrak{D}$. If $\alpha>\frac{1}{2}$, it holds on L^2 also.

Proposition 2. *For* $\alpha>\frac{1}{2}$, *the operator* I_α *satisfies the conditions* (1)~(4) *of* § 1.

Proof. It is easily seen that I_α satisfies (2)~(4). (Especially, $I_\alpha^{-1}=I_{-\alpha}$). So, we shall only prove the nuclearity of I_α.

Let $\{f_n(t)\}$ be a complete orthonormal system of $L^2((0,1))$. It is sufficient to show that $\sum_n \int_0^1 |I_\alpha f_n(t)|^2 dt<\infty$, namely,

$$\sum_n \int_0^1 \left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_n(s)ds \int_0^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)} \overline{f_n(r)}dr\right]dt < \infty \tag{3}$$

By Fubini's theorem, we can rewrite the left hand side of (3) into the form;

$$\sum_n \int_{s=0}^1 \int_{r=0}^1 F(s, r) f_n(s)\overline{f_n(r)}ds dr,$$

where $$F(s, r) = \frac{1}{\Gamma(\alpha)^2}\int_{\mathrm{Max}(s,r)}^{1}(t-s)^{\alpha-1}(t-r)^{\alpha-1}dt\,.$$

We see that $F(s, r)$ is a continuous function of (s, r) for $\alpha > \frac{1}{2}$. On the other hand, for instance, putting $f_n(s) = \exp(2\pi ins)$, we have

$$\sum_n f_n(s)\overline{f_n(r)} = \sum_n \exp(2\pi in(s-r)) \xrightarrow[(n\to\infty)]{} \delta(s-r)$$

in C' (=the dual of the space of continuous functions).

Hence, the left hand side of (3) is equal with

$$\int_0^1 F(s, s)ds = \frac{1}{\Gamma(\alpha)^2}\int_0^1\left[\int_s^1 (t-s)^{2\alpha-2}dt\right]ds$$

$$= \frac{1}{2\alpha(2\alpha-1)}\frac{1}{\Gamma(\alpha)^2} < \infty \qquad \text{(q.e.d.)}$$

From Prop. 1 and Prop. 2, we see that the carrier of the measure μ lies in $I^*_{-\alpha}(H_0)$. It is easy to see $I^*_{-\alpha} = PI_{-\alpha}P$, where $Pf(t) = f(1-t)$. However, if at the first step we change the variable t into $1-t$, we need not consider the effect of P. Thus, we can say that the carrier of μ lies in $I_{-\alpha}(H_0)$.

Remark that $I_{-\alpha} = \frac{d}{dt}I_{1-\alpha}$, Therefore, putting $\beta = 1-\alpha$, we get the following result.

Proposition 3. *Suppose that the characteristic functional of a measure μ is continuous in $L^2((0, 1))$, then the carrier of μ lies in the whole of derivatives of $I_\beta(H_0)$ for any $\beta < \frac{1}{2}$, hence in the whole of derivatives of $\bigcap_n I_{\beta_n}(H_0)$ where $\beta_n \uparrow \frac{1}{2}$.*

Here, we consider derivatives in the sense of distributions.

§ 3. Hölder continuity

Proposition 4. *Even for $0 < \beta \leqq \frac{1}{2}$, I_β is a continuous operator from $L^2 = H_0$ into itself, though $I_\beta f(t)$ can be defined only for almost all t.*

Proof. If $0 < \beta \leqq \frac{1}{2}$ and $f(t) \in H_0 = L^2$, $I_\beta f(t)$ can not be defined in the pointwise way, but it can be defined for almost all t, because the function $\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}f(s)$ $(t\geqq s)$ is integrable with respect to two variables (s, t). Moreover for any $g(t)\in H_0$, we have

$$\int_0^1 |g(t)| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |f(s)|\, ds dt \leq \frac{1}{\Gamma(\beta)} \iint_{s+t\leqq 1} |g(t+s)|\, |f(s)|\, t^{\beta-1} dt ds$$

$$\leqq \frac{1}{\Gamma(\beta)} \|g\|_0 \|f\|_0 \frac{1}{\beta}. \tag{4}$$

Hence, $\|I_\beta f(t)\|_0 \leqq \frac{1}{\beta\Gamma(\beta)} \|f\|_0$. (q.e.d.)

In general, for $g(t)\in L^2((0, 1))$, the shift operator $\tau_h g(t)=g(t+h)$ has no meaning, since $g(t+h)$ belongs to $L^2((-h, 1-h))$, but not to $L^2((0, 1))$. However, if $g(t)\in I_\beta(H_0)$, we can define the shift operator, because the definition (2) of I_α can be applied for $t>1$. (For $t<0$, we put $I_\alpha f(t)=0$).

Proposition 5. *If* $g(t)\in \bigcap_n I_{\beta_n}(H_0)$ *where* $\beta_n \uparrow \frac{1}{2}$, *then for any* $\beta<\frac{1}{2}$, $g(t)$ *is Hölder continuous in the sense of* L^2. *Namely*;

$$\frac{\tau_h g(t)-g(t)}{h^\beta} \xrightarrow[(h\to 0)]{} 0 \ \textit{in}\ L^2((0, 1)).$$

Proof. Since $g(t)=I_{\beta_n} f_n(t)$ where $f_n(t)\in L^2=H_0$, in a similar way with (4) we have for any $\varphi(t)\in L^2$,

$$\int_0^1 |\varphi(t)|\, |g(t+h)-g(t)|\, dt$$

$$\leqq \frac{1}{\Gamma(\beta_n)} \|f_n\|_0 \|\varphi\|_0 \int_{-|h|}^1 |t_+^{\beta_n-1}-(t+|h|)^{\beta_n-1}|\, dt \leqq \frac{2|h|^{\beta_n}}{\beta_n\Gamma(\beta_n)} \|f_n\|_0 \|\varphi\|_0 .$$ [1]

Thus, for given $\beta<\frac{1}{2}$, choose $\beta_n>\beta$, then

$$\left\| \frac{\tau_h g(t)-g(t)}{h^\beta} \right\|_0 \leqq \text{const.}\times |h|^{\beta_n-\beta} \xrightarrow[(h\to 0)]{} 0 .$$ (q.e.d.)

Since $g(t)$ is defined only for almost all t, the concept of pointwise Hölder continuity loses its meaning. This fault can not be removed as long as we regard $g(t)$ as a distribution. Along this line, we get only the following proposition.

1) $t_+^\beta=t^\beta$ for $t>0$, $=0$ for $t<0$.

Proposition 6. *If a sequence* $\{h_k\}$ *satisfies* 1) $h_k \to 0$, *and* 2) $\overline{\lim}\left|\frac{h_{k+1}}{h_k}\right| < 1$, *then for any* $g(t) \in \bigcap_n I_{\beta_n}(H_0)$ *and any* $\beta < \frac{1}{2}$, we have

$$\lim_k \frac{g(t+h_k)-g(t)}{h_k^\beta} = 0 \textit{ for almost all } t.$$

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