On nonlinear difference equations

By

L. J. GRIMM* and W. A. HARRIS, Jr.**

Introduction.

We consider a system of nonlinear difference equations of the form

$$(0.1)\qquad y(x+1)=\hat{f}(x, y(x)),$$

where x is a complex variable, y an n-dimensional vector, and $\hat{f}$ an n-dimensional vector.

Each component of the n-dimensional vector $\hat{f}$ is assumed to be holomorphic in a region $R=S_0\times U_0$, where

$$S_0: |\arg(x-a)|<\frac{\pi}{2}+\rho_0$$
$$U_0: \|y\|<\delta_0,$$

for some positive constants a, δ_0, ρ_0, where the norm of a vector u is given by $\|u\|=\sum_{j=1}^{n}|u_j|$.

Let

$$(0.2)\qquad \hat{f}(x, y)=\hat{f}_0(x)+B(x)y+\sum_{|\mathfrak{p}|\geq 2}\hat{f}_{\mathfrak{p}}(x)y^{\mathfrak{p}}$$

be the expansion of $\hat{f}$ in powers of $y_1, \cdots, y_n$, where $\mathfrak{p}$ is a set of non-negative integers $p_1, \cdots, p_n$, $B(x)$ an $n\times n$ matrix, $\hat{f}_0$ and $\hat{f}_{\mathfrak{p}}$ n-dimensional vectors, and

$$(0.3)\qquad \begin{cases} y^{\mathfrak{p}}=y_1^{p_1}y_2^{p_2}\cdots y_n^{p_n}, \\ |\mathfrak{p}|=p_1+p_2+\cdots+p_n. \end{cases}$$

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* Department of Mathematics, University of Utah, U. S. A.

** School of Mathematics, University of Minnesota, U. S. A. and Research Institute for Mathematical Sciences, Kyoto University, Japan. U. S.-Japan Cooperative Science Program, NSF Grant GF-214.

We shall suppose that $\hat{f}_0$, $\hat{f}_\mathfrak{p}$, B are holomorphic in S_0 and have the asymptotic expansions

$$(0.4)\qquad \begin{cases} \hat{f}_0(x) \cong \sum_{k=0} \hat{f}_{0k} x^{-k} \\ B(x) \cong \sum_{k=0} B_k x^{-k} \\ \hat{f}_\mathfrak{p}(x) \cong \sum_{k=0} \hat{f}_{\mathfrak{p}k} x^{-k} \end{cases}$$

as x approaches infinity through the sector S_0.

In order to construct solutions of (0.1) it is important to obtain their first approximation; in general this is difficult. However, if a solution $y(x)$ has a limit y_0 as x approaches infinity and $\hat{f}(\infty, y_0)$ is defined, then y_0 must satisfy the equation $y_0 = \hat{f}(\infty, y_0)$. On the other hand, if there exists a y_0 which satisfies this equation, $y_0 = \hat{f}(\infty, y_0)$, then, under suitable stability conditions, it may be expected that every solution in a neighborhood of y_0 will approach y_0 as x approaches infinity. The purpose of this paper is to show how, under suitable hypotheses, to construct the general solution of the system (0.1) in a region of the form

$$(0.5)\qquad l_1 < \arg(x-b) < l_2.$$

We shall assume with no loss of generality that $y_0 = 0$, and hence

$$(0.6)\qquad 0 = \hat{f}_{00} (= \hat{f}(\infty, 0)).$$

The following theorem is a special case of a result of W. A. Harris, Jr. and Y. Sibuya [8] and is the first step in the construction of the general solution:

Theorem 1. *Let the vector function $\hat{f}(x, y)$ be holomorphic in $R_1 = S_1 \times U_1$,*

$$S_1: |\arg(xe^{-i\theta} - a_1)| < \frac{\pi}{2} + \rho_1,$$
$$U_1: \|y\| < \delta_1$$

for positive constants a_1, δ_1, ρ_1. Suppose $\hat{f}$ has the representation (0.2) in powers of y, where $\hat{f}_0$, $\hat{f}_\mathfrak{p}$, B have asymptotic expansions

(0. 4) *as* x *approaches infinity through the sector* S_1. *Suppose that* B_0 and B_0-I *are nonsingular. Further assume* $\hat{f}_{00}=0$, *and also that if* λ_i, $i=1,\cdots,n$, *are the eigenvalues of* B_0, *then* $\theta\neq\arg(-\log\lambda_i)$. *Then if the positive constants* a_2 *and* ρ_2^{-1} *are sufficiently large, there exists a solution*

$$y=\phi(x) \tag{0.7}$$

of (0. 1) *such that* $\phi(x)$ *is analytic and admits the asymptotic expansion*

$$\phi(x)\cong\sum_{\nu=1}\phi_\nu x^{-\nu} \tag{0.8}$$

as x *tends to infinity in the domain,*

$$S_2:|\arg(xe^{-i\theta}-a_2)|<\frac{\pi}{2}+\rho_2.$$

Since this theorem will be used frequently in the course of this paper, we shall discuss it at greater length in the next section, with special attention to the region of validity.

By a transformation of the form

$$y(x)=z(x)+\phi(x), \tag{0.9}$$

the system (0. 1) is reduced to the form

$$z(x+1)=\hat{\hat{f}}(x,z(x)), \tag{0.10}$$

where the right member satisfies conditions similar to those satisfied by $\hat{f}$ and the expansion of $\hat{\hat{f}}$ in powers of z is given by

$$\hat{\hat{f}}(x,z)=A(x)z+f(x,z)=A(x)z+\sum_{|\mathfrak{p}|\geq2}f_{\mathfrak{p}}(x)z^{\mathfrak{p}} \tag{0.11}$$

and further

$$A(x)=B(x)+0(x^{-1}). \tag{0.12}$$

Next we shall prove that $A(x)$ may be assumed to have a convenient form:

Theorem 2. *Let the elements of the* $n\times n$ *matrix* $A(x)$ *be holomorphic in a sector* S_3,

$$S_3: |\arg(xe^{-i\theta}-a_3)|<\frac{\pi}{2}+\rho_3,\ a_3>0,\ \rho_3>0,$$

and let A possess the asymptotic expanion

$$A(x)\cong\sum_{k=0}^{\infty}A_kx^{-k}$$

as x approaches infinity through the sector S_3. Suppose that $\mu_1,\cdots,\mu_r$ are the distinct eigenvalues of A_0, that none of these is zero, and that $\theta\neq\arg(-\log(\mu_i/\mu_j))$, $i\neq j$. Suppose further that A_0 has the block-diagonal (Jordan) form $A_0=\mathrm{diag}(A_1^0,\cdots,A_r^0)$, where $A_j^0=\mu_jI_j+N_j$, with I_j the m_j-dimensional identity matrix and N_j a nilpotent matrix. Then there exists a matrix $T(x)$ with components holomorphic in some sector,

$$S_4: |\arg(xe^{-i\theta}-a_4)|<\frac{\pi}{2}+\rho_4,$$

ρ_4 sufficiently small, $0<\rho_4<\rho_3$, $a_4>a_3$, such that the transformation

$$y(x)=T(x)z(x) \tag{0.13}$$

transforms the linear difference equation

$$y(x+1)=A(x)z(x) \tag{0.14}$$

into the equation

$$z(x+1)=B(x)z(x) \tag{0.15}$$

where $B(x)=\mathrm{diag}(B_1(x),\cdots,B_r(x))$ is a block-diagonal matrix, the elements of $B(x)$ are holomorphic in S_4, and $B(x)$ has the asymptotic expansion

$$B(x)\cong\sum_{k=0}^{\infty}B_kx^{-k} \tag{0.16}$$

as x approaches infinity through the sector S_4; further $B_0=A_0$.

Theorem 1 is used to prove **Theorem 2**, and the form of S_4 will be specified in the next section by the remarks on **Theorem 1.** On the basis of **Theorem 2,** we can assume without loss of generality that the matrix $A(x)$ has the block-diagonal form of $B(x)$.

Let

$$z=P(x,u) \tag{0.17}$$

be a transformation of the vector z such that $P(x,u)$ can be represented by a uniformly convergent series of the form

$$P(x,u)=u+\sum_{|\mathfrak{p}|\geq 2} P_{\mathfrak{p}}(x)u^{\mathfrak{p}} \tag{0.18}$$

in a region $\widehat{R}_4=\widehat{S}_4\times\widehat{U}_4$ given by

$$\widehat{S}_4: |\arg(xe^{-i\theta}-\widehat{a}_4)|<\frac{\pi}{2}+\widehat{\rho}_4$$

$$\widehat{U}_4: \|u\|<\widehat{\delta}_4$$

for $\widehat{a}_4>0$, $\widehat{\rho}_4>0$, with $\widehat{\delta}_4>0$ sufficiently small, with coefficients $P_{\mathfrak{p}}(x)$ holomorphic for $x\in\widehat{S}_4$, and admitting asymptotic expansions

$$P_{\mathfrak{p}}(x)\cong\sum_{s=0}^{\infty} P_{s\mathfrak{p}}x^{-s}$$

as x approaches infinity through $\widehat{S}_4$.

We are now in a position to prove our main theorem.

Theorem 3. *Suppose that the matrix A_0 has eigenvalues satisfying*

$$0<|\lambda_j|<1, \tag{0.19}$$

and that θ satisfies the conditions

i) $|\theta|<\frac{\pi}{2}$

ii) $\theta\neq\arg(-\log\lambda_{j\mathfrak{p}})$ if $\lambda_{j\mathfrak{p}}\neq 1$,

for $j=1,\cdots,n$, $|\mathfrak{p}|\geq 2$, where

$$\lambda_{j\mathfrak{p}}=\frac{\lambda_j}{\lambda_1^{p_1}\lambda_2^{p_2}\cdots\lambda_n^{p_n}}, \tag{0.20}$$

with $\mathfrak{p}=(p_1,\cdots,p_n)$.

Let

$$\widehat{S}_5: |\arg(xe^{-i\theta}-\widehat{a}_5)|<\frac{\pi}{2}+\widehat{\rho}_5,$$

where $\hat{a}_5>0$ *and* $0<\hat{\rho}_5<\hat{\rho}_5+|\theta|<\frac{\pi}{2}$. *Consider the system*

$$(0.21)\qquad \begin{aligned} z(x+1) &= A(x)z(x)+f(x, z(x)) \\ &= A(x)z(x)+\sum_{|\mathfrak{p}|\geq 2} f_{\mathfrak{p}}(x)[z(x)]^{\mathfrak{p}}, \end{aligned}$$

where $f(x,z)$ *is holomorphic for* $x\in\widehat{S}_5$, $\|z\|$ *sufficiently small,* $A(x) = \mathrm{diag}(A^1(x), \cdots, A^r(x))$ *is a block-diagonal matrix of the form of* $B(x)$ *in* **Theorem 2**, *and* $A(x)$ *and* $f_{\mathfrak{p}}(x)$ *are holomorphic and admit the asymptotic expansions*

$$A(x)\cong A_0+\sum_{k=1}^{\infty} A_k x^{-k} \qquad f_{\mathfrak{p}}(x)\cong f_{\mathfrak{p}0}+\sum_{k=0}^{\infty} f_{\mathfrak{p}k}x^{-k}$$

as x *tends to infinity through the sector* $\widehat{S}_5$.

There exists a transformation of the form (0.18) *by which the system* (0.21) *is transformed into a system of the form*

$$(0.22)\qquad \begin{aligned} u(x+1) &= A(x)u(x)+g(x, u(x)) = A(x)u(x) \\ &\quad + \sum_{|\mathfrak{p}|\geq 2} g_{\mathfrak{p}}(x)[u(x)]^{\mathfrak{p}} \end{aligned}$$

where the coefficients $g_{\mathfrak{p}}(x)$ *have j'th component*

$$(0.23)\qquad g_{j\mathfrak{p}}(x)=0$$

if $\lambda_{j\mathfrak{p}}\neq 1$.

Assuming without loss of generality the ordering

$$\begin{aligned} 1 &> |\lambda_1| = |\lambda_2| = \cdots = |\lambda_{m_1}| \\ &> |\lambda_{m_1+1}| = \cdots = |\lambda_{m_2}| > \cdots \\ &> |\lambda_{m_k+1}| = \cdots = |\lambda_{m_{k+1}}| > 0 \qquad (m_{k+1}=n), \end{aligned}$$

using (0.23) and the block-diagonal form of A, we can show that the jth component of the vector $g(x, u)$ satisfies

$$(0.24)\qquad g_j(x, u) = \begin{cases} 0 & (j=1, \cdots, m_1) \\ \sum_{\lambda_{j\mathfrak{p}}=1} g_{j\mathfrak{p}}(x)u^{\mathfrak{p}} & (j>m_1), \end{cases}$$

and that $g_j(x, u)$ is a polynomial in $u_1, \cdots, u_{m_r}$ for $j=m_r+1, \cdots, m_{r+1}$, $r\geq 1$. Thus the general solution of (0.21) can be obtained by solving linear difference equations recursively.

In particular, if the λ_j's are all distinct, the system (0. 21) becomes scalar equations and the system can be solved recursively. If, in addition, $\lambda_j \mathfrak{p} \neq 1$ for all indices j and $\mathfrak{p}$, the system (0. 22) has diagonal homogeneous form

$$u_j(x+1) = a_{jj}(x) u_j(x). \tag{0. 25}$$

After obtaining the general solution of the system (0. 22), we can construct the general solution of (0. 10) by substituting the solution of (0. 22) into the transformation (0. 17). In doing so it is necessary to estimate the magnitude of the solution of the system (0. 22).

If the reduced system (0. 22) is normal in an extended sense (we shall give a precise definition of this concept in Section 7) we shall find a region of the type

$$\alpha_1 < \arg(x - \hat{a}) < \alpha_2$$

in which the solution is uniformly bounded and approaches zero as x tends to infinity in this region. Choosing θ consistent with **Theorems 1-3**, the general solution of the original system (0. 1) is given in this region by

$$y(x) = \phi(x) + P(x, U(x, C(x))), \tag{0. 26}$$

where $U(x, C(x))$ is the general solution of the reduced equation (0. 22) and $C(x)$ is an arbitrary bounded periodic vector with period one. Thus we have attained our main objective.

The scalar case, $n=1$, has been treated by J. Horn [9] under the assumption that $\widehat{f}(x, y)$ is holomorphic for $|x| \geq R_0$, $|y| < \delta_0$ using Laplace transform techniques. The single n'th order equation

$$y(x+n) = f(x, y(x), y(x+1), \cdots, y(x+n-1))$$

has been treated by W. J. Trjitzinsky [13] under various hypotheses including $f(x, 0, \cdots, 0) = 0$, i. e., the existence of a particular solution $\phi(x) = 0$. He constructed formal series equivalent to our series $P(x, U(x, C(x)))$ which he proved asymptotic to actual solutions in an upper half-plane, while we have established the convergence of this series. General results of essentially the same nature as ours have been obtained by W. A. Harris, Jr. and Y. Sibuya [7] in half-planes of the form $|\mathrm{Im}\, x|>a$ under the conditions

$$|\lambda_i|\neq|\lambda_j|\,(i\neq j),\ |\lambda_i|\neq 1,\ \textit{and}\ \prod_{j=1}^{n}|\lambda_j|^{k_j}\neq|\lambda_i|$$

for all λ_i. Our results for nonlinear difference equations parallel similar results in the theory of ordinary differential equations for systems of the form

$$\frac{dy}{dx}=f(x, y), \qquad f(\infty, 0)=0$$

for which the corresponding linear system

$$\frac{dy}{dx}=f_y(x, 0)y$$

has an irregular singular point at infinity and the eigenvalues of $f_y(\infty, 0)$ have negative real parts; these results are due to M. Hukuhara [10], M. Iwano [11], J. Malmquist [12], and W. J. Trjitzinsky [14].

The authors gratefully acknowledge several stimulating discussions with Professor Y. Sibuya during the preparation of this paper and are indebted to him for the simple proof presented for **Lemma 3.**

1. Preliminaries.

a) *Removal of Nonhomogeneous Term.* First we obtain a holomorphic solution

$$y=\phi(x)\cong\sum_{k=1}^{\infty}x^{-k}\phi^{k} \tag{1.1}$$

of (0.1) in a region

$$S_2\colon|\arg(xe^{-i\theta}-a_2)|<\frac{\pi}{2}+\rho_2,\ a_2>a_0,\ 0<\rho_2<\rho_0$$

by use of **Theorem 1.** Let

(1. 2) $$y(x)=z(x)+\phi(x).$$

Then $y(x+1)=\widehat{f}(x, y(x))$ becomes

(1. 3) $$z(x+1)+\phi(x+1)=\widehat{f}(x, z(x)+\phi(x))$$

and, since $\phi(x)$ is a solution, (1. 3) becomes

$$z(x+1)=B(x)z(x)+\sum_{|\mathfrak{p}|\geq 2}\widehat{f}_{\mathfrak{p}}(x)[z(x)+\phi(x)]^{\mathfrak{p}} - \sum_{|\mathfrak{p}|\geq 2}\widehat{f}_{\mathfrak{p}}(x)[\phi(x)]^{\mathfrak{p}},$$

which can be written as

(1. 4) $$z(x+1)=A(x)z(x)+f(x, z(x))=A(x)z(x) + \sum_{|\mathfrak{p}|\geq 2} f_{\mathfrak{p}}(x)[z(x)]^{\mathfrak{p}}$$

where the series is convergent for $x\in S_2$, $\|z\|<\delta_2$ for some $\delta_2>0$, δ_2 sufficiently small, and where the coefficients $A(x)$ and $f_{\mathfrak{p}}(x)$ are holomorphic for $x\in S_2$, and have asymptotic expansions

(1. 5) $$A(x)\cong\sum_{s=0}^{\infty}A_s x^{-s}$$ $$f_{\mathfrak{p}}(x)\cong\sum_{s=0}^{\infty}f_{\mathfrak{p}s}x^{-s}$$

as x tends to infinity through the sector S_2. Equate the linear terms in (1. 3) and (1. 4), and using the fact that $\phi(x)=0(x^{-1})$ we obtain

(1. 6) $$A_0=B_0(=f_y(\infty, 0)).$$

b) *Remarks on* **Theorem 1.** The region of validity of solutions obtained in **Theorem 1** is the sector S_2 of the form

(1. 7) $$|\arg(xe^{-i\theta}-a)|<\frac{\pi}{2}+\rho,$$

where θ and ρ may be described as follows: In the complex ζ-plane, draw all the points

$$-\log\lambda_i=-\log|\lambda_i|-i\arg\lambda_i,$$

$i=1, \cdots, n$. Then draw rays through each of these points extending from the origin to infinity. These rays will divide the plane into a countable number of sectors, $\sum_1, \sum_2, \cdots$. The conculsion of **Theorem 1** holds for all choices of θ and ρ, $\rho>0$, such that the sector

$$\theta-\rho<\arg\zeta<\theta+\rho$$

is contained in some sector $\sum_j$.

Exactly one of the sectors $\sum_j$, call it $\sum^0$, will have one of the following properties:

i) The positive real axis will be interior to $\sum^0$.

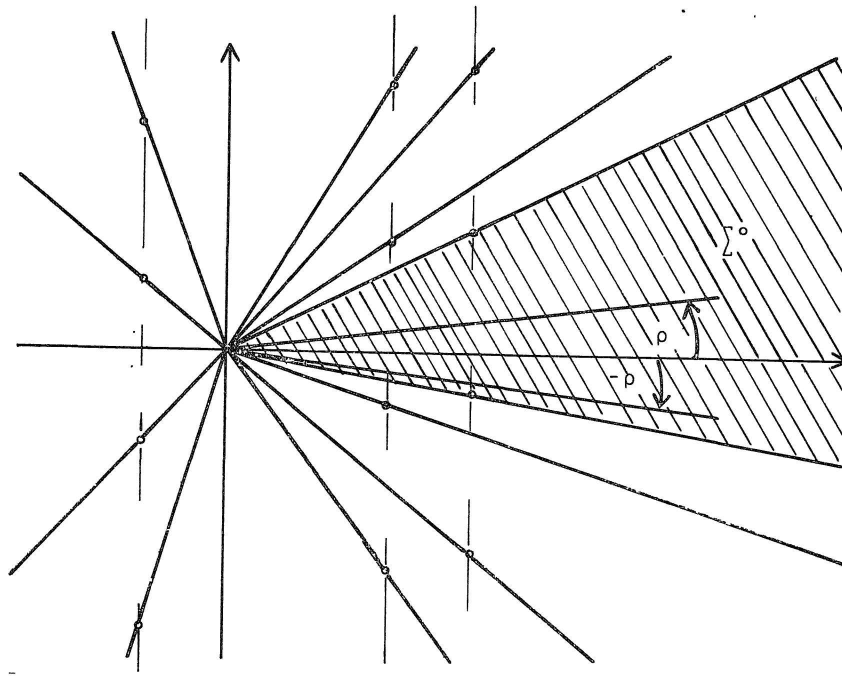
ii) The positive real axis will be the lower boundary of $\sum^0$, i.e., $\sum^0$ will be a sector of the form $0<\arg\zeta<\zeta_0$ for some $\zeta_0>0$. It is clear that case ii) will hold if, and only if, at least one of the eigenvalues λ_i satisfies $0<\lambda_i<1$. See Fiqure 1.

We shall apply **Theorem 1** again in the proofs of **Theorem 2** and **Theorem 3.** In the proof of **Theorem 2,** the numbers μ_i/μ_j assume the roles of the λ_i in determining sectors of validity; we obtain in this case the sectors $\sum_1', \sum_2', \cdots$. Choose $\sum'^0$ from this set in the same way as $\sum^0$ was choosen. In **Theorem 3,** we shall apply **Theorem 1** a finite number, N_0, of times; in these cases, the numbers $\lambda_{j\mathfrak{p}}, |\mathfrak{p}|\leq N_0$ determine the sectors $\sum_1'', \sum_2'', \cdots$. Choose a $\sum''^0$ from these in the manner in which $\sum^0$ was chosen. We now take the intersection of the three sectors $\sum^0$, $\sum'^0$, $\sum''^0$, and call it $\sum$. It is clear that $\sum$ will also have property i) or property ii). We will restrict θ and ρ so that the sector

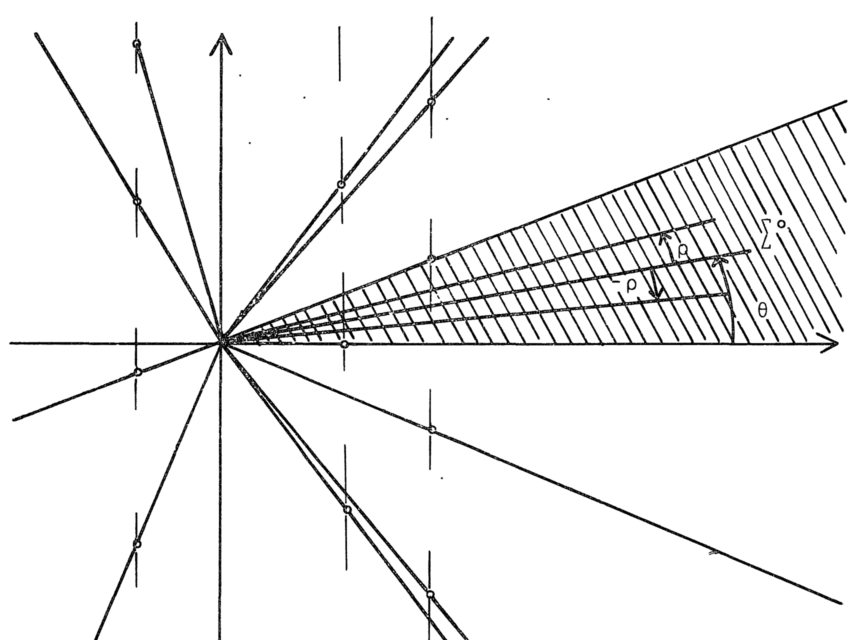
$$\theta-\rho<\arg\zeta<\theta+\rho$$

lies in $\sum$, in the final step of constructing the general solution of the original equation (0.1) in the form (0.26).

We note that if $\widehat{f}(x, y)$ is analytic in a full neighborhood of $x=\infty$, $\|y\|$ sufficiently small, the solutions of (0.1) obtained in **Theorem 1** will exist with asymptotic representations in sectors covering a full neighborhood of infinity. Similarly, the results of **Theorem 2** will hold in sectors covering a full neighborhood of



Case i) $\theta = 0$.



Case ii) $\theta \neq 0$. Figure 1.

infinity. However, the restrictions in **Theorem 3,** $|\theta|+\rho<\frac{\pi}{2}$, restrict the validity to sectors which cover a region of the form $|\arg(x-a)|<\pi$, but this is to be expected due to the form of our stability hypothesis, $0<|\lambda_i|<1$.

2. Proof of Theorem 2.

By hypothesis, the matrix $A(x)$ has the asymptotic representation

$$A(x)\cong\sum_{s=0}^{\infty}A_s x^{-s}, \tag{2.1}$$

where A_0 has block diagonal (Jordan) form, $A_0=\operatorname{diag}(A_1^0, \cdots, A_r^0)$ and with no loss of generality

$$A_j^0=\begin{pmatrix} \mu_j & \delta_{j1} & & 0 \\ & \ddots & \ddots & \\ & & \cdot & \delta_{jm_j-1} \\ 0 & & & \mu_j \end{pmatrix}, \quad j=1, \cdots, r, \tag{2.2}$$

with δ_{jk} arbitrarily small.

Let

$$\begin{cases} T(x)=I+Q(x) \\ A(x)=A_0+\widehat{A}(x) \\ B(x)=A_0+\widehat{B}(x). \end{cases} \tag{2.3}$$

We wish to show that the equation

$$T^{-1}(x+1)A(x)\,T(x)=B(x) \tag{2.4}$$

has a solution of the desired form.

Write (2.4) in the form

$$A(x)\,T(x)=T(x+1)B(x) \tag{2.5}$$

and substitute the representations for T, A and B given by (2.3) to obtain

$$\begin{aligned} \varDelta Q(x)A_0=A_0Q(x)-Q(x)A_0+\widehat{A}(x)Q(x)-Q(x)\widehat{B}(x) \\ +\widehat{A}(x)-\widehat{B}(x)-\varDelta Q(x)\widehat{B}(x), \end{aligned} \tag{2.6}$$

where $\varDelta Q(x)=Q(x+1)-Q(x)$.

Let $\widehat{A}, \widehat{B}, Q$ have the partitioning

$$A=\begin{pmatrix} \widehat{A}_{11} & \cdots & \widehat{A}_{1r} \\ \vdots & & \\ \widehat{A}_{r1} & \cdots & \widehat{A}_{rr} \end{pmatrix}, \quad B=\begin{pmatrix} \widehat{B}_{11} & & 0 \\ & \ddots & \\ 0 & & B_{rr} \end{pmatrix}, \quad Q=\begin{pmatrix} 0 & Q_{12} & \cdots & Q_{1r} \\ Q_{21} & 0 & & \\ \vdots & & \ddots & \vdots \\ Q_{r1} & & & 0 \end{pmatrix},$$

induced by the partitioning of A_0. If there is a solution of the desired form then,

$$\widehat{B}_{jj}=\sum_{k\neq j}\widehat{A}_{jk}Q_{kj}+\widehat{A}_{jj}$$

and the equation for determining Q becomes

$$\begin{aligned} \Delta Q_{ij}A_j^0 = {} & A_i^0 Q_{ij} - Q_{ij}A_j^0 + \sum_{k\neq j}\widehat{A}_{ik}Q_{kj} \\ & - (\Delta Q_{ij}+Q_{ij})(\widehat{A}_{jj}+\sum_{k\neq i}\widehat{A}_{jk}Q_{kj}) + \widehat{A}_{ij}. \end{aligned} \tag{2.7}$$

If Q is determined in this manner, then B and T are also determined. Equation (2. 7) is a system of nonlinear difference equations of the form

$$\Delta y(x)=\varphi(x, y, \Delta y)=\varphi_0^*(x)+C^*(x)y+h^*(x, y, \Delta y) \tag{2.8}$$

where the components of the vector $h^*(x, y, \Delta y)$ are polynomials in y and Δy with coefficients that are $O(x^{-1})$. Hence, for $|x|$ sufficiently large, $x\in S_3$, i. e., in some sector

$$\widehat{S}_6: |\arg(xe^{-i\theta}-\hat{a}_6)|<\frac{\pi}{2}+\rho_3$$

for $\hat{a}_6>a_3$, we may rewrite the system (2. 8) in the form

$$\Delta y=\varphi_0(x)+C(x)y+h(x, y) \tag{2.9}$$

where $\varphi_0(x)$ and $C(x)$ are holomorphic for $x\in S_6$, $\|y\|$ sufficiently small, and these functions have appropriate asymptotic representations and

$$C(x)=C^*(x)+O(x^{-1}). \tag{2.10}$$

It is easy to show that the eigenvalues of $I+C_0(C_0=\lim_{x\to\infty}C(x))$

are μ_i/μ_j, $i, j=1, \cdots, r$. Thus the problem of block-diagonalization has been reduced to that of finding a solution of a system of nonlinear difference equations of a form to which **Theorem 1** is applicable. Applying **Theorem 1** we obtain a solution $Q(x)$ of equation (2.7) in a sector $S_4 \subset \widehat{S}_6$,

$$S_4: |\arg(xe^{-i\theta}-a_4)|<\frac{\pi}{2}+\rho_4, \quad \hat{a}_6<a_4, \quad \hat{\rho}_6>\rho_4>0.$$

Hence the transformation $T(x)$ is analytic in S_4 and admits the asymptotic expansion

$$T(x)\cong I+\sum_{k=1}^{\infty} x^{-k} T_k .$$

To complete the proof, it remains only to note that for a_4 sufficiently large $T(x)$ is nonsingular. Hence under the transformation

$$y(x)=T(x)z(x),$$

the linear difference equation

$$y(x+1)=A(x)y(x)$$

becomes

$$z(x+1)=B(x)z(x).$$

3. A Lemma on Linear Nonhomogeneous Systems.

Lemma 1. *Consider the linear nonhomogeneous system*

$$A(x)y(x+1)=B(x)y(x)+f(x), \tag{3.1}$$

where the $m\times m$ matrices $A(x)$, $B(x)$ and the m-vector $f(x)$ are holomorphic for x in the sector

$$S_5: |\arg(xe^{-i\theta}-a_5)|<\frac{\pi}{2}+\rho_5$$

for some $a_5>0$, $0<\rho_5<\frac{\pi}{2}$, $-\frac{\pi}{2}<\theta-\rho_5<\theta+\rho_5<\frac{\pi}{2}$, and admit asymptotic expansions

(3. 2)
$$\begin{cases} A(x)\cong\sum_{i=0}^{\infty}A_i x^{-i} \\ B(x)\cong\sum_{i=0}^{\infty}B_i x^{-i} \\ f(x)\cong\sum_{i=0}^{\infty}f_i x^{-i} \end{cases}$$

as x approaches infinity through the sector S_5. *Suppose further that* B_0 *is nonsingular and that the eigenvalues of* $B_0^{-1}A_0$ *have absolute value less than* 1. *Then there exists a unique bounded holomorphic solution y of* (3. 1) *in some sector*

$$S_6: |\arg(xe^{-i\theta}-a_6)|<\frac{\pi}{2}+\rho_5$$

for some $a_6\geq a_5$, *and possessing there the asymptotic expansion*

$$y(x)\cong\sum_{i=0}^{\infty}y_i x^{-i}$$

as x approaches infinity through the sector S_6. *Further, there exists a constant C depending only upon the matrices* $B(x)$ *and* $A(x)$, *such that for* $x\in S_6$,

(3. 3)
$$\|y(x)\|\leq C\sup_{x\in S_6}\|f(x)\|.$$

Proof: Since B_0 is nonsingular, for $x\in S_5$, $|x|$ sufficiently large, $B^{-1}(x)$ will exist. Write (3. 1) as

(3. 4)
$$B^{-1}(x)A(x)y(x+1)=y(x)+B^{-1}(x)f(x).$$

Since by hypothesis the eigenvalues of $B_0^{-1}A_0$ have modulus less than 1, there exists a nonsingular constant matrix P such that

(3. 5)
$$\|P^{-1}B_0^{-1}A_0P\|<1.$$

[If $\|B_0^{-1}A_0\|<1$, we choose $P=I$.] Since (3. 5) holds, for $|x|$ sufficiently large, $x\in S_5$,

(3. 6)
$$\|P^{-1}B^{-1}(x)A(x)P\|<r<1.$$

In fact, there will be a sector S_6 as above where (3. 6) will hold.

Define

$$R(x)=P^{-1}B^{-1}(x)A(x)P,$$

and let $y=Pz$. Then (3. 4) becomes

$$z(x)=R(x)z(x+1)-P^{-1}B^{-1}(x)f(x). \tag{3.7}$$

Let

$$L=\sup_{x\in S_6}\|f(x)\|,$$
$$K=\sup_{x\in S_6}\|P^{-1}B^{-1}(x)\|,$$

and

$$M=\frac{KL}{1-r}.$$

Let $\mathfrak{F}$ be the family of all m-dimensional vector functions $\varphi(x)$ holomorphic for $x\in S_6$, such that $\|\varphi(x)\|\leq M$. Define the mapping T as follows: for $z\in\mathfrak{F}$, let

$$T[z](x)=R(x)z(x+1)-P^{-1}B^{-1}(x)f(x).$$

A solution of (3. 7) is equivalent to a fixed point of the mapping T. $\mathfrak{F}$ is closed, compact, and convex with respect to the topology of uniform convergence on each compact subset of the region S_6. Since the mapping is continuous, we need only show that $z\in\mathfrak{F}$ implies $T[z]\in\mathfrak{F}$. Since

$$\begin{aligned}\|T[z](x)\| &\leq\|R(x)z(x+1)\|+\|P^{-1}B^{-1}(x)f(x)\|\\ &\leq\|R(x)\|M+\sup_{x\in S_6}[\|P^{-1}B^{-1}(x)\|\|f(x)\|]\\ &<rM+KL=M,\end{aligned}$$

there is a fixed point of the mapping T which is the desired solution.

To prove uniqueness, suppose $y(x)$ and $z(x)$ are two bounded solutions of (3. 7). Subtraction yields

$$y(x)-z(x)=R(x)[y(x+1)-z(x+1)].$$

Hence assuming $\sup_{x\in S_6}\|y(x)-z(x)\|\neq 0$,

$$\sup_{x\in S_6}\| y(x)-z(x)\|\leq\sup_{x\in S_6}\{\|R(x)\|\,\|y(x+1)-z(x+1)\|\}$$
$$\leq r\sup_{x\in S_6}\|y(x)-z(x)\|,$$

which is a contradiction, since $r<1$. Thus the uniqueness follows.

Since $B_0^{-1}A_0-I$ is nonsingular, there exists a unique formal solution $\sum_{i=0}^{\infty}y_ix^{-i}$. The proof that this is the asymptotic representation of the solution y that we have constructed follows as in Harris and Sibuya [5].

Since $y=Pz$, setting $C=\dfrac{K\|P\|\|P^{-1}\|}{1-r}$, yields (3. 3), and the lemma is proved.

If $\|B_0^{-1}A_0\|<1$, a_6 will be any number not less than a_5 such that $\|B^{-1}(x)A(x)\|<r<1$ in S_6. In this case the constant C will depend only on $\sup_{x\in S_6}\|B^{-1}(x)\|$. If $\|B_0^{-1}A_0\|\geq 1$, a corresponding result holds with $\|B^{-1}(x)A(x)\|$ replaced by $\|P^{-1}B^{-1}(x)A(x)P\|$, with the P chosen above, and the constant C will depend upon $\|P^{-1}\|\cdot\|P\|$ and $\sup_{x\in S_6}\|B^{-1}(x)\|$. We note that if $B^{-1}(x)$ exists and $\|B^{-1}(x)A(x)\|<r<1$ in the region S_5, we may choose $S_6=S_5$. Hence we have proved

Lemma 2. *Let $B^{-1}(x)$ exist and $\|B^{-1}(x)A(x)\|<r<1$ in S_5. Then the solution $y(x)$ obtained in Lemma* 1 *exists and the estimate* (3. 3) *is valid for $x\in S_5$.*

4. Formal Transformation: Preliminaries.

a) *A Preliminary Estimate.* Consider the expression

$$\sum_{|\mathfrak{p}|=k}P_{\mathfrak{p}}(x+1)\,[A(x)u]^{\mathfrak{p}} \tag{4. 1}$$

where $P_{\mathfrak{p}}$ is an n-dimensional vector defined as in (0. 18), $A(x)$ is an $n\times n$ matrix assumed to have all the properties, including the block-diagonal form, of $B(x)$ in **Theorem 2**, with eigenvalues $\lambda_1,\cdots,\lambda_n$, $0<|\lambda_i|<1$. We can write (4. 1) in the form

$$\sum_{|\mathfrak{p}|=k}r_{\mathfrak{p}}u^{\mathfrak{p}} \tag{4. 2}$$

where $r_{\mathfrak{p}}=r_{\mathfrak{p}}(x)$ is an n-dimensional vector. We want an estimate of the magnitude of the $r_{\mathfrak{p}}$. Write

$$\sum_{|\mathfrak{q}|=k} r_{\mathfrak{q}} u^{\mathfrak{q}} = \sum_{|\mathfrak{p}|=k} P_{\mathfrak{p}}(x+1)(Au)^{\mathfrak{p}}.$$

Then

$$\text{(4.3)} \quad \|\sum_{|\mathfrak{q}|=k} r_{\mathfrak{q}} u^{\mathfrak{q}}\| \leq \sum_{|\mathfrak{p}|=k} \|P_{\mathfrak{p}}(x+1)(Au)^{\mathfrak{p}}\| \leq \sum_{|\mathfrak{p}|=k} \|P_{\mathfrak{p}}(x+1)\| \|Au\|^k$$
$$\leq \sum_{|\mathfrak{p}|=k} \|P_{\mathfrak{p}}(x+1)\| \ \|A\|^k \|u\|^k.$$

If $\|A\|<\sigma$ and $\|u\|<\delta$ for some positive numbers σ, δ, we obtain

$$\|\sum_{|\mathfrak{q}|=k} r_{\mathfrak{q}} u^{\mathfrak{q}}\| \leq \sum_{|\mathfrak{p}|=k} \|P_{\mathfrak{p}}(x+1)\| \sigma^k \delta^k.$$

Notice that each component of the vector $\sum_{|\mathfrak{q}|=k} r_{\mathfrak{q}} u^{\mathfrak{q}}$ is a polynomial in the $u_1, \cdots, u_n$ and as such is a multiple power series. Consider the multiple power series $\sum \alpha_{j\mathfrak{q}} u^{\mathfrak{q}}$, and suppose $|\sum \alpha_{j\mathfrak{q}} u^{\mathfrak{q}}| \leq M$ for $\|u\| \leq \delta$. Then $|\alpha_{j\mathfrak{q}}| \leq M\delta^{-|\mathfrak{q}|}$. By (4.3) we can take $M = \sum_{|\mathfrak{p}|=k} \|P_{\mathfrak{p}}(x+1)\| \sigma^k \delta^k$, and hence

$$\|r_{\mathfrak{q}}\| \leq n \sum_{|\mathfrak{p}|=k} \|P_{\mathfrak{p}}(x+1)\| \sigma^k.$$

Notice that for $|\mathfrak{q}|=k$, the number of terms in the sum is no greater than $(k+1)^n$. Hence we have

$$\text{(4.4)} \quad \sum_{|\mathfrak{q}|=k} \|r_{\mathfrak{q}}\| \leq (k+1)^n \cdot n\sigma^k \{ \sum_{|\mathfrak{p}|=k} \|P_{\mathfrak{q}}(x+1)\| \}.$$

Define a linear ordering of the $\mathfrak{p}=(p_1, \cdots, p_n)$ as follows: $\mathfrak{p}'=(p_1', \cdots, p_n') < \mathfrak{p}^2 = (p_1^2, \cdots, p_n^2)$ if $|\mathfrak{p}'| < |\mathfrak{p}^2|$ or if $|\mathfrak{p}'| = |\mathfrak{p}^2|$ and the first nonzero element of $\mathfrak{p}^2 - \mathfrak{p}'$ is positive. Order the $\mathfrak{p}$'s for $|\mathfrak{p}|=k$ in increasing order and call them $\mathfrak{p}^1, \cdots, \mathfrak{p}^{r_k}$. Write $(Au)^{\mathfrak{p}} = \sum_{|\mathfrak{q}|=|\mathfrak{p}|} c_{\mathfrak{p}\mathfrak{q}} u^{\mathfrak{q}}$. Then

$$\sum_{|\mathfrak{p}|=k} P_{\mathfrak{p}}(x+1)(Au)^{\mathfrak{p}} = \sum_{|\mathfrak{p}|=k} P_{\mathfrak{p}}(x+1) \sum_{|\mathfrak{q}|=|\mathfrak{p}|} c_{\mathfrak{p}\mathfrak{q}} u^{\mathfrak{q}} = \sum_{|\mathfrak{q}|=k} r_{\mathfrak{q}} u^{\mathfrak{q}}.$$

Equate coefficients of $u^{\mathfrak{q}}$ to obtain

$$\text{(4.5)} \quad \sum_{|\mathfrak{p}|=k} P_{\mathfrak{p}}(x+1) c_{\mathfrak{p}\mathfrak{q}} = \sum_{|\mathfrak{p}|=k} c_{\mathfrak{p}\mathfrak{q}} P_{\mathfrak{p}}(x+1) = r_{\mathfrak{q}}.$$

Each $\mathfrak{q}$ will be a $\mathfrak{p}^i$ for some i; thus we have $r_{\mathfrak{p}^1}, \cdots, r_{\mathfrak{p}^{r_k}}$. Write these as a single column vector

$$\Gamma=\begin{pmatrix} \gamma_{\mathfrak{p}^1} \\ \vdots \\ \gamma_{\mathfrak{p}^{r_k}} \end{pmatrix}.$$

Similarly define the column vector

$$P(k, x+1)=\begin{pmatrix} P_{\mathfrak{p}^1}(x+1) \\ \vdots \\ P_{\mathfrak{p}^{r_k}}(x+1) \end{pmatrix}. \tag{4.6}$$

Then (4.5) becomes

$$C(k)P(k, x+1)=\Gamma \tag{4.7}$$

where $C(k)$ is an $n\cdot r_k\times n\cdot r_k$ matrix. We shall first estimate the norm of $C(k)$, and later be more specific about its structure. Observe that, from (4.4),

$$\begin{aligned}\|\Gamma\|=\sum_{|\mathfrak{q}|=k}\|\gamma_{\mathfrak{q}}\|&\leq(k+1)^n\cdot n\cdot\sigma^k\sum_{|\mathfrak{p}|=k}\|P_{\mathfrak{p}}(x+1)\|\\ &=(k+1)^n\cdot n\sigma^k\|P(k, x+1)\|.\end{aligned}$$

Thus, since

$$\|C(k)\|=\sup_{\|v\|=1}\|C(k)v\|,$$

$$\|C(k)\|\leq(k+1)^n\cdot n\sigma^k, \tag{4.8}$$

since the vector P was arbitrary. We summarize this in the following lemma:

Lemma 3. *Let the* $\mathfrak{p}$ *for* $|\mathfrak{p}|=k$ *be given the linear ordering specified above, so that* $\mathfrak{p}^1<\mathfrak{p}^2<\cdots<\mathfrak{p}^{r_k}$. *Then the coefficient* $\gamma_{\mathfrak{p}^j}$ *of* $u^{\mathfrak{p}^j}$ *in the expansion*

$$\sum_{|\mathfrak{p}|=k}P_{\mathfrak{p}}[A(x)u]^{\mathfrak{p}}$$

is given by the $n(j-1)+1$*st through* $n\cdot j$*th components of the* $n\cdot r_k$ *vector* $C(k)P(k, x+1)$, *where* $P(k, x+1)$ *is given by* (4.6), *and, if* σ *is an upper bound for* $\|A(x)\|$, *we have the estimate* (4.8).

b) *Further Remarks on* $C(k)$. In the preceding section we used none of the hypotheses on the form of A to obtain **Lemma 3.** We shall now employ them to discuss the structure of $C(k)$ more explicitly. First of all, it is clear that the elements of $C(k)$ are polynomials in the elements of A. Hence $C(k)=\bar{C}(k, x)$ is holomorphic in the same region as $A(x)$, and has the asymptotic expansion

$$\bar{C}(k, x) \cong \bar{C}^0(k)+\sum_{s=1}^{\infty} C_s(k) x^{-s},$$

as x approaches infinity through the sector S_6. Further, it is clear that the coefficient of $u^{\mathfrak{p}^j}$ in the expansion

$$\sum_{|\mathfrak{p}|=k} P_{\mathfrak{p}}(x+1)(A_0 u)^{\mathfrak{p}} \tag{4.9}$$

is given by the corresponding components (as in **Lemma 3**) of $\bar{C}^0(k) P(k, x+1)$. By hypothesis,

$$A_0=\begin{pmatrix} \lambda_1 & \delta_1 & & & 0 \\ & \lambda_2 & \delta_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \delta_{n-1} \\ 0 & & & & \lambda_n \end{pmatrix}$$

and hence

$$A_0 u=(\lambda_1 u_1+\delta_1 u_2, \lambda_2 u_2+\delta_2 u_3, \cdots, \lambda_n u_n)^T,$$

and therefore

$$\begin{aligned}(A_0 u)^{\mathfrak{p}} &= (\lambda_1 u_1+\delta_1 u_2)^{p_1}(\lambda_2 u_2+\delta_2 u_3)^{p_2} \cdots(\lambda_n u_n)^{p_n}, \\ &= [\lambda_1^{p_1} \lambda_2^{p_2} \cdots \lambda_n^{p_n}] u^{\mathfrak{p}}+\text{polynomial in } u^{\mathfrak{q}} \text{ for } \mathfrak{q}<\mathfrak{p},\ (|\mathfrak{q}|=|\mathfrak{p}|),\end{aligned}$$

according to the linear ordering defined above. Define

$$\Lambda=(\lambda_1, \cdots, \lambda_n)^T.$$

Then we can write

$$(A_0 u)^{\mathfrak{p}}=\Lambda^{\mathfrak{p}} u^{\mathfrak{p}}+\mathfrak{P}(u^{\mathfrak{q}}), \quad \mathfrak{q}<\mathfrak{p} \tag{4.10}$$

where $\mathfrak{P}(u^{\mathfrak{q}})$ is the polynomial in the u's as above. Hence the coefficient of $P_{\mathfrak{p}}$ in (4.9) is given by (4.10), or, equating coefficients of $u^{\mathfrak{p}^j}$,

$$\overline{C}^0(k)P(k,x+1)=\overline{C}_k^0\begin{pmatrix}P_{\mathfrak{p}^1}\\ \vdots\\ P_{\mathfrak{p}^{r_k}}\end{pmatrix} \tag{4.11}$$

$$=\begin{pmatrix}\Lambda^{\mathfrak{p}^1}P_{\mathfrak{p}^1}+C_{12}^0P_{\mathfrak{p}^2}+\cdots+C_{1r_k}^0P_{\mathfrak{p}^{r_k}}\\ \Lambda^{\mathfrak{p}^2}P_{\mathfrak{p}^2}+C_{23}^0P_{\mathfrak{p}^3}+\cdots+C_{2r_k}^0P_{\mathfrak{p}^{r_k}}\\ \vdots\\ \Lambda^{\mathfrak{p}^{r_k}}P_{\mathfrak{p}^{r_k}}\end{pmatrix}$$

Hence $\overline{C}_k^0$ has the block triangular (actually triangular) form

$$\overline{C}_k^0=\begin{pmatrix}\Lambda^{\mathfrak{p}^1}I & C_{12}^0I & \cdots & C_{1r_k}^0I\\ & \Lambda^{\mathfrak{p}^2}I & & \\ & & \ddots & \\ 0 & & & \Lambda^{\mathfrak{p}^{r_k}}I\end{pmatrix} \tag{4.12}$$

where the components indicated are scalars times identity matrices, and hence $\overline{C}_k^0$ is triangular and has eigenvalues $\Lambda^{\mathfrak{p}}$, $|\mathfrak{p}|=k$. We remark that if A_0 is diagonal, then $\overline{C}_k^0$ will be diagonal also.

5. Formal Transformation.

Consider the system

$$z(x+1)=A(x)z(x)+f(x,z(x))\equiv A(x)z(x)+\sum_{|\mathfrak{p}|\geq 2}f_{\mathfrak{p}}(x)[z(x)]^{\mathfrak{p}} \tag{0.13}$$

under a formal transformation of the form

$$z(x)=u(x)+\sum_{|\mathfrak{p}|\geq 2}P_{\mathfrak{p}}(x)[u(x)]^{\mathfrak{p}}. \tag{0.18}$$

Formally,

$$\begin{aligned}u(x+1)+\sum_{|\mathfrak{p}|\geq 2}P_{\mathfrak{p}}(x+1)[u(x+1)]^{\mathfrak{p}}&=A(x)u(x)\\ &+\sum_{|\mathfrak{p}|\geq 2}A(x)P_{\mathfrak{p}}(x)[u(x)]^{\mathfrak{p}}\\ &+\sum_{|\mathfrak{p}|\geq 2}f_{\mathfrak{p}}(x)\{u(x)+\sum_{|\mathfrak{q}|\geq 2}P_{\mathfrak{q}}(x)[u(x)]^{\mathfrak{q}}\}^{\mathfrak{p}},\end{aligned} \tag{5.1}$$

which can be written in the form

$$u(x+1)=A(x)u(x)+g(x,u(x))=A(x)u(x)+\sum_{|\mathfrak{p}|\geq 2}g_{\mathfrak{p}}(x)[u(x)]^{\mathfrak{p}}. \tag{5.2}$$

We shall determine the transformation (0. 18) in such a way that the resulting equation (5. 2) will have a form as simple as possible; we shall show that in this case $g(x, u(x))$ will be a polynomial in the u's. Substitute (5. 2) into (5. 1) to obtain, suppressing the argument of u (which is now always x),

(5. 3) $$\sum_{|\mathfrak{p}|\geq 2} g_{\mathfrak{p}}(x)u^{\mathfrak{p}}+\sum_{|\mathfrak{p}|\geq 2} P_{\mathfrak{p}}(x+1)\{A(x)u+\sum_{|\mathfrak{q}|\geq 2} g_{\mathfrak{q}}(x)u^{\mathfrak{q}}\}^{\mathfrak{p}}$$
$$=\sum_{|\mathfrak{p}|\geq 2} A(x)P_{\mathfrak{p}}(x)u^{\mathfrak{p}}+\sum_{|\mathfrak{p}|\geq 2} f_{\mathfrak{p}}(x)\{u+\sum_{|\mathfrak{q}|\geq 2} P_{\mathfrak{q}}(x)u^{\mathfrak{q}}\}^{\mathfrak{p}}.$$

Notice that

(5. 4) $$\{A(x)u+\sum_{|\mathfrak{q}|\geq 2} g_{\mathfrak{q}}(x)u^{\mathfrak{q}}\}^{\mathfrak{p}}=[A(x)u]^{\mathfrak{p}}+\text{terms in } u^{\alpha} \text{ for } |\alpha|>|\mathfrak{p}|.$$

Since the P and g are to be chosen so that (5. 3) is a formal identity, we may equate the coefficients of $u^{\mathfrak{p}}$ to obtain, for $|\mathfrak{p}|=k$,

(5. 5) $$g_{\mathfrak{p}}(x)+\gamma_{\mathfrak{p}}=A(x)P_{\mathfrak{p}}(x)+h_{\mathfrak{p}}(x),$$

where $\gamma_{\mathfrak{p}}$ is defined by (4. 2), and $h_{\mathfrak{p}}(x)=h_{\mathfrak{p}}^{(2)}(x)-h_{\mathfrak{p}}^{(1)}(x)$, where

$$\sum_{|\mathfrak{p}|\geq 2} h_{\mathfrak{p}}^{(1)}(x)u^{\mathfrak{p}}=\sum_{|\mathfrak{p}|\geq 2} P_{\mathfrak{p}}(x+1)\{(A(x)u+\sum_{|\mathfrak{q}|\geq 2} g_{\mathfrak{q}}(x)u^{\mathfrak{q}})^{\mathfrak{p}}-(A(x)u)^{\mathfrak{p}}\}$$

and

$$\sum_{|\mathfrak{p}|\geq 2} h_{\mathfrak{p}}^{(2)}(x)u^{\mathfrak{p}}=\sum_{|\mathfrak{p}|\geq 2} f_{\mathfrak{p}}(x)\{u+\sum_{|\mathfrak{q}|\geq 2} P_{\mathfrak{q}}(x)u^{\mathfrak{q}}\}^{\mathfrak{p}}.$$

It is clear that $h_{\mathfrak{p}}(x)$ is a polynomial in the components of the P_{ζ}, f_{ζ}, and g_{ζ} for $|\zeta|<k=|\mathfrak{p}|$. Order the $\mathfrak{p}$ for $|\mathfrak{p}|=k$ in increasing order as in **Lemma 3**, and write all the equations (5. 5) for $|\mathfrak{p}|=k$ as a single vector equation as in **Lemma 3**. Then (5. 5) becomes

(5. 6) $$G(k, x)+\overline{C}(k, x)P(k, x+1)=A(k, x)P(k, x)+H(k, x),$$

where $\overline{C}(k, x)$ is the matrix defined in **Lemma 3**,

$$G(k, x)=\begin{pmatrix} g_{\mathfrak{p}^1}(x) \\ \vdots \\ g_{\mathfrak{p}^{r_k}}(x) \end{pmatrix}, \quad H(k, x)=\begin{pmatrix} h_{\mathfrak{p}^1}(x) \\ \vdots \\ h_{\mathfrak{p}^{r_k}}(x) \end{pmatrix},$$

$P(k, x)$ is as in **Lemma 3**, and $A(k, x)$ is a block-diagonal $n\cdot r_k\times n\cdot r_k$ matrix of the form

$$A(k, x)=\begin{pmatrix} A(x) & & & 0 \\ & A(x) & & \\ & & \ddots & \\ 0 & & & A(x) \end{pmatrix}.$$

Notice that $[A(k, x)]^{-1}$ has the same form as $A(k, x)$; in fact, $[A(k, x)]^{-1}=A^{-1}(k, x)$. Further,

$$\|A^{-1}(k, x)\|=\|A^{-1}(x)\|. \tag{5.7}$$

We also observe that because of the hypotheses on $A(x)$, $A^{-1}(k, x)$ will have an asymptotic representation in S_6 of the form

$$A^{-1}(k, x)\cong A_0^{-1}(k)+\sum_{s=1}^{\infty}A_s(k)x^{-s},$$

as x approaches infinity through the sector S_6, where $A_0^{-1}(k)$ is the block-diagonal matrix $\operatorname{diag}(A_0^{-1}, \cdots, A_0^{-1})$.

The eigenvalues of $A_0^{-1}(k)$ are thus the numbers λ_j^{-1}, $j=1, \cdots, n$, while $\overline{C}_0(k)$ has eigenvalues $\Lambda^{\mathfrak{p}}$ and is hence nonsingular. Further since $\overline{C}_0(k)$ is upper triangular, $\overline{C}_0^{-1}(k)$ will be triangular also, and further, $\overline{C}_0^{-1}(k, x)$ will exist in some sector

$$S_7(k): |\arg(xe^{-i\theta}-a_7(k))|<\frac{\pi}{2}+\rho_5,$$

$a_7(k)\geqq a_6$, $k=2, \cdots, N_0$, and there possess an asymptotic expansion

$$\overline{C}^{-1}(k, x)\cong\overline{C}^{-1}(k)+\sum_{s=1}^{\infty}C_s(k)x^{-s}.$$

Hence we can write (5.6) in the form

$$\begin{aligned} P(k, x+1)=&\overline{C}^{-1}(k, x)A(k, x)P(k, x)\\ &+\overline{C}^{-1}(k, x)[H(k, x)-G(k, x)] \end{aligned} \tag{5.8}$$

where the elements of $H(k, x)$ are polynomials in the elements of $P(j, x)$ for $j<k$. We shall determine the vectors $P(k, x)$ and $G(k, x)$ recursively by equation (5.8). The solvability of the difference equation (5.8) depends on the eigenvalues of the matrix $\overline{C}^{-1}(k)A_0(k)$ which are $\lambda_{j\mathfrak{p}}$, $j=1, \cdots, n$, $|\mathfrak{p}|=k$.

If $\lambda_{j\mathfrak{p}}\neq 1$, $j=1, \cdots, n$, $|\mathfrak{p}|=k$, we may choose and $G(k, x)=0$ and apply **Theorem 1** to the difference equation (5. 8) to determine $P(k, x)$.

If $\lambda_{j\mathfrak{p}}=1$ for some $j, \mathfrak{p}, j=1, \cdots, n$, $|\mathfrak{p}|=k$, we choose the corresponding components of $P(k, x)$ equal to zero and those of $G(k, x)$ equal to those of $H(k, x)$. In this manner we obtain a system of difference equations of lower order similar to (5. 8) whose eigenvalues differ from 1 which we solve as in the preceding case. Each time we use **Theorem 1**, $P(k, x)$ will be determined in a sector $S_8(k)\subset S_8(k-1)\subset\cdots\subset S_8(2)\subset S_8$.

Since $|\lambda_{j\mathfrak{p}}|\rightarrow\infty$ as $|\mathfrak{p}|\rightarrow\infty$, $G(k, x)=0$ for k sufficiently large. Let N_0-1 be the smallest positive integer k such that if $\lambda_{j\mathfrak{p}}\neq 1$, $|\lambda_{j\mathfrak{p}}|>1, j=1, \cdots, n$, $|\mathfrak{p}|=k$. We apply **Theorem 1** as above to obtain $P(k, x)$ and $G(k, x)$ for $k\leq N_0$. These solutions will be valid in the sector $S_8(N_0)$ of the form

$$S_8(N_0): |\arg(xe^{-i\theta}-a_8(N_0)|)<\frac{\pi}{2}+\rho_8(N_0).$$

From this point onward we shall apply **Lemma 1** to solve the system (5. 8), (deleting if necessary the components corresponding to $\lambda_{j\mathfrak{p}}=1$) as above. Hence, there exists a solution to the system (5. 6) for $P(k, x)$, $k>N_0$ in a sector

$$S_9(k): |\arg(xe^{-i\theta}-a_9(k))|<\frac{\pi}{2}+\rho_9$$

for some constants $a_9(k)$ and $\rho_9>0$.

It is important to show that there is a single region of this form in which all the $P(k, x)$ exist. This will be the case if we can show that $\|A^{-1}(k, x)\overline{C}(k,x)\|<r<1$, $x\in S_9(k)$, $k>N_2$. Let N_2 be the smallest positive integer greater than N_0 such that

$$BN_2^n n\sigma^{N_2-1}<1, \text{ where}$$

$\|A(x)\|<\sigma<1$, $\|A^{-1}(x)\|<B$ for $x\in S_9(N_0)$ and using **Lemma 3** the hypothesis of **Lemma 2** are satisfied and $P(k, x)$ may be determined

in the uniform region $S_9(N_2)$ which we write as

$$\mathcal{S}: |\arg(xe^{-i\theta}-b)|<\frac{\pi}{2}+\rho.$$

Further, there exists a constant depending only on $A(x)$ (with $0<|\lambda_i|<1$) such that

$$\sum_{|\mathfrak{p}|=k}\|P_{\mathfrak{p}}(x)\|<C\sup_{x\in\mathcal{S}}\sum_{|\mathfrak{p}|=k}\|h_{\mathfrak{p}}-g_{\mathfrak{p}}\|.$$

6. Convergence.

a) *Preliminary Transformation.* We have shown that all the $P_{\mathfrak{p}}(x)$ can be determined as holomorphic functions for $x\in\mathcal{S}$. It remains to be shown that the series $\sum_{|\mathfrak{p}|\geq 2} P_{\mathfrak{p}}(x)u^{\mathfrak{p}}$ converges for $x\in\widehat{S}\subset\mathcal{S}$, $\|u\|$ sufficiently small.

Choose N so large that i) $N>N_2$ and ii) $2C\sigma^N<1$, where $\sigma<1$ is an upper bound for $\|A(x)\|$ in $\mathcal{S}$. Let us first make the polynomial transformation $\mathcal{T}_N$;

$$z(x)=u(x)+\sum_{|\mathfrak{p}|=2}^{N-1}P_{\mathfrak{p}}(x)[u(x)]^{\mathfrak{p}}. \tag{6.1}$$

Then the system

$$z(x+1)=A(x)z(x)+f(x, z(x))\equiv A(x)z(x)+\sum_{|\mathfrak{p}|\geq 2}f_{\mathfrak{p}}(x)[z(x)]^{\mathfrak{p}} \tag{0.21}$$

becomes

$$\begin{aligned}u(x+1)+\sum_{|\mathfrak{p}|=2}^{N-1}P_{\mathfrak{p}}(x+1)[u(x+1)]^{\mathfrak{p}}&=A(x)u(x)\\ &+A(x)\sum_{|\mathfrak{p}|=2}^{N-1}P_{\mathfrak{p}}(x)[u(x)]^{\mathfrak{p}}\\ &+\sum_{|\mathfrak{p}|\geq 2}f_{\mathfrak{p}}(x)\{u(x)+\sum_{|\mathfrak{q}|=2}^{N-1}P_{\mathfrak{p}}(x)[u(x)]^{\mathfrak{q}}\}^{\mathfrak{p}},\end{aligned} \tag{6.2}$$

where the $P_{\mathfrak{p}}(x)$ have been determined in the preceding section. Notice that $\mathcal{T}_N$ is an analytic transformation in $\mathcal{S}$, $\|u\|$ sufficiently small, and hence (6.2) can be solved for $u(x+1)$ in some sector $\mathcal{S}_1\subset\mathcal{S}$ of the form

$$\mathcal{S}_1: |\arg(xe^{-i\theta}-b_1)|<\frac{\pi}{2}+\rho' \quad (b_1\geq b)$$

to yield

$$(6.3) \qquad u(x+1)=A(x)u(x)+g(x,u(x))+h(x,u(x))$$
$$\equiv A(x)u(x)+\sum_{|\mathfrak{p}|=2}^{N-1} g_{\mathfrak{p}}(x)[u(x)]^{\mathfrak{p}}+\sum_{|\mathfrak{p}|\geq N} h_{\mathfrak{p}}(x)[u(x)]^{\mathfrak{p}},$$

where $g(x, y)$ and $h(x, y)$ are analytic for $x\in\mathcal{S}_1$, $\|y\|<\bar{\delta}$ for some $\bar{\delta}>0$. Indeed, the $g_{\mathfrak{p}}(x)$ are the functions defined in the previous section as components of the vector $G(k, x)$ as we see from (5.5), since specifying the $P_{\mathfrak{p}}$ for $|\mathfrak{p}|=2, \cdots, (N-1)$ determines each $g_{\mathfrak{p}}$ uniquely for $|\mathfrak{p}|=2, \cdots, N-1$.

Now make the transformation $\mathcal{U}_N$:

$$(6.4) \qquad u(x)=R(x, w(x))\equiv w(x)+Q(x, w)\equiv w(x)$$
$$+\sum_{|\mathfrak{p}|\geq N} Q_{\mathfrak{p}}(x)[w(x)]^{\mathfrak{p}}.$$

Under this transformation (6.3) becomes

$$(6.5) \quad w(x+1)+\sum_{|\mathfrak{p}|\geq N} Q_{\mathfrak{p}}(x+1)[w(x+1)]^{\mathfrak{p}}=A(x)w(x)$$
$$+A(x)\sum_{|\mathfrak{p}|\geq N} Q_{\mathfrak{p}}(x)[w(x)]^{\mathfrak{p}}$$
$$+\sum_{|\mathfrak{p}|=2}^{N-1} g_{\mathfrak{p}}(x)(w(x)+\sum_{|\mathfrak{q}|\geq N} Q_{\mathfrak{q}}(x)w(x)^{\mathfrak{q}})^{\mathfrak{p}}$$
$$+\sum_{|\mathfrak{p}|\geq N} h_{\mathfrak{p}}(x)(w(x)+\sum_{|\mathfrak{q}|\geq N} Q_{\mathfrak{q}}(x)w(x)^{\mathfrak{q}})^{\mathfrak{p}}.$$

Since the formal transformation $\mathcal{T}$ reduced the original equation to the form

$$(0.22) \qquad u(x+1)=A(x)u(x)+g(x, u(x)),$$

it is clear that the transformation $\mathcal{U}_N$ can be chosen so that (6.5) becomes

$$(6.6) \qquad w(x+1)=A(x)w(x)+\sum_{|\mathfrak{p}|=2}^{N-1} g_{\mathfrak{p}}(x)[w(x)]^{\mathfrak{p}}.$$

In particular, the $Q_{\mathfrak{p}}$'s may be chosen so that

$$(6.7)\quad \sum_{|\mathfrak{p}|=2}^{N-1} g_{\mathfrak{p}}(x)[w(x)]^{\mathfrak{p}}+\sum_{|\mathfrak{p}|\geq N} Q_{\mathfrak{p}}(x+1)(A(x)w(x)$$
$$+\sum_{|\mathfrak{q}|=2}^{N-1} g_{\mathfrak{q}}(x)w(x)^{\mathfrak{q}})^{\mathfrak{p}}=A(x)\sum_{|\mathfrak{p}|\geq N} Q_{\mathfrak{p}}(x)[w(x)]^{\mathfrak{p}}$$
$$+\sum_{|\mathfrak{p}|=2}^{N-1} g_{\mathfrak{p}}(x)(w(x)+\sum_{|\mathfrak{q}|\geq N} Q_{\mathfrak{q}}(x)[w(x)]^{\mathfrak{q}})^{\mathfrak{p}}$$
$$+\sum_{|\mathfrak{p}|\geq N} h_{\mathfrak{p}}(x)(w(x)+\sum_{|\mathfrak{q}|\geq N} Q_{\mathfrak{q}}(x)[w(x)]^{\mathfrak{q}})^{\mathfrak{p}}.$$

Suppress the argument of w and rearrange to obtain

$$(6.8)\quad \sum_{|\mathfrak{p}|\geq N} Q_{\mathfrak{p}}(x+1)(A(x)w)^{\mathfrak{p}}-A(x)(\sum_{|\mathfrak{p}|\geq N} Q_{\mathfrak{p}}(x)w^{\mathfrak{p}})$$
$$=g(x,R(x,w))-g(x,w)+h(x,R(x,w))$$
$$-[\sum_{|\mathfrak{p}|\geq N} Q_{\mathfrak{p}}(x+1)(A(x)w+g(x,w))^{\mathfrak{p}}$$
$$-\sum_{|\mathfrak{p}|\geq N} Q_{\mathfrak{p}}(x+1)(A(x)w)^{\mathfrak{p}}].$$

After substituting the representation for $R(x,w)$ given in (6.4), we obtain the following formal representations:

$$g(x,R(x,w))=\sum_{|\mathfrak{p}|\geq 2} m_{\mathfrak{p}}(x)w^{\mathfrak{p}}$$
$$h(x,R(x,w))=\sum_{|\mathfrak{p}|\geq N} l_{\mathfrak{p}}(x)w^{\mathfrak{p}}$$
$$Q(x+1,Aw+g(x,w))=\sum_{|\mathfrak{p}|\geq N} B_{\mathfrak{p}}(x)w^{\mathfrak{p}}$$
$$Q(x+1,Aw)=\sum_{|\mathfrak{p}|\geq N} D_{\mathfrak{p}}(x)w^{\mathfrak{p}}.$$

We notice in particular that the system of equations for the $Q_{\mathfrak{p}}$ obtained by equating coefficiencts of $w^{\mathfrak{p}}$ is precisely the same as the system for the $P_{\mathfrak{p}}$ except that the nonhomogenous term is different. Since $|\mathfrak{p}|\geq N>N_2$, the $Q_{\mathfrak{p}}$'s can be determined in a uniform region

$$(6.9)\quad \mathcal{S}_2: |\arg(xe^{-i\theta}-b_2)|<\frac{\pi}{2}+\rho'$$

and will have asymptotic expansions as x approaches infinity through $\mathcal{S}_2$. Further, by **Lemma 2**, we have the fundamental estimate

(6.10)
$$\sum_{|\mathfrak{p}|=k} \|Q_{\mathfrak{p}}(x)\| < C \sup_{x\in S_2} \sum_{|\mathfrak{q}|=k} \|m_{\mathfrak{q}}(x) - g_{\mathfrak{q}}(x) + l_{\mathfrak{q}}(x) + B_{\mathfrak{q}}(x) - D_{\mathfrak{q}}(x)\|,$$

and hence the convergence of the series for Q implies the convergence of the series for P.

b) *Majorant Functions.* The following operations will be convenient: Let $\varphi(x, w) = \sum_{|\mathfrak{p}|\geq m} \varphi_{\mathfrak{p}}(x) w^{\mathfrak{p}}$ be an n-vector function. Then the jth component of φ will be given by

$$\varphi_j(x, w) = \sum_{|\mathfrak{p}|\geq m} \varphi_{j\mathfrak{p}}(x) w^{\mathfrak{p}}.$$

Define

$$\varphi_j^*(x, w) = \sum_{|\mathfrak{p}|\geq m} |\varphi_{j\mathfrak{p}}(x)| w^{\mathfrak{p}}$$

and

$$\bar{\varphi}(x, w) = \sum_{|\mathfrak{p}|\geq m} \|\varphi_{\mathfrak{p}}(x)\| w^{\mathfrak{p}} = \sum_{j=1}^{n} \varphi_j^*(x, w).$$

Also define

$$\hat{\varphi}(x, v) = \bar{\varphi}(x, \tilde{v})$$

when the n-vector $\tilde{v}$ is given by $\tilde{v} = (v,\cdots,v)^T$. Then

$$\hat{\varphi}(x, v) = \sum_{\alpha=2}^{\infty} \Big(\sum_{|\mathfrak{p}|=\alpha} \|\varphi_{\mathfrak{p}}(x)\| \Big) v^{\alpha}.$$

Hence we have, from

(6.11)
$$\begin{cases} g(x, w) = \sum_{|\mathfrak{p}|=2}^{N-1} g_{\mathfrak{p}}(x) w^{\mathfrak{p}}, \text{ defined the functions} \\ g_i(x, w) = \sum_{|\mathfrak{p}|=2}^{N-1} g_{i\mathfrak{p}}(x) w^{\mathfrak{p}}, \\ g_i^*(x, w) = \sum_{|\mathfrak{p}|=2}^{N-1} |g_{i\mathfrak{p}}(x)| w^{\mathfrak{p}}, \\ \bar{g}(x, w) = \sum_{|\mathfrak{p}|=2}^{N-1} \|g_{\mathfrak{p}}(x)\| w^{\mathfrak{p}}, \text{ and} \\ \hat{g}(x, v) = \sum_{\alpha=2}^{N-1} \Big(\sum_{|\mathfrak{p}|=\alpha} \|g_{\mathfrak{p}}(x)\| \Big) v^{\alpha}; \end{cases}$$

and from

$$(6.12)\quad \begin{cases} R(x,w)=w+\sum_{|\mathfrak{p}|\geq N} Q_{\mathfrak{p}}(x)w^{\mathfrak{p}}, \text{ defined the functions} \\ R_j(x,w)=w_j+\sum_{|\mathfrak{p}|\geq N} Q_{j\mathfrak{p}}(x)w^{\mathfrak{p}}, \\ R_j^*(x,w)=w_j+\sum_{|\mathfrak{p}|\geq N} |Q_{j\mathfrak{p}}(x)|w^{\mathfrak{p}}, \text{ and} \\ \overline{R}(x,w)=w_1+\cdots+w_n+\sum_{|\mathfrak{p}|\geq N} \|Q_{\mathfrak{p}}(x)\|w^{\mathfrak{p}} \\ \qquad\qquad =w_1+\cdots+w_n+\sum_{i=1}^{n} R_i^*(x,w). \end{cases}$$

Define $\widehat{p}(x,v)=\overline{R}(x,\tilde{v})$ with $\tilde{v}$ as above. Then

$$\widehat{p}(x,v)=nv+\sum_{\alpha=N}^{\infty}\Big(\sum_{|\mathfrak{p}|=\alpha} \|Q_{\mathfrak{p}}(x)\|\Big)v^{\alpha}.$$

Then the vector

$$(6.13)\qquad g(x,R(x,w))\equiv m(x,w)=\sum_{|\mathfrak{p}|\geq 2} m_{\mathfrak{p}}(x)w^{\mathfrak{p}}$$

has i'th component

$$g_i(x,R(x,w))=m_i(x,w)=\sum_{|\mathfrak{p}|\geq 2} m_{i\mathfrak{p}}(x)w^{\mathfrak{p}}.$$

Then

$$(6.14)\qquad m_i^*(x,w)=\sum_{|\mathfrak{p}|\geq 2} |m_{i\mathfrak{p}}(x)|w^{\mathfrak{p}}.$$

By definition

$$g_j(x,z)\ll g_j^*(x,z) \quad \text{for all } j,$$

i. e., the coefficients of z^k in the multiple power series for g_j^* are positive and not less than the absolute values of the corresponding coefficients of the series for g_j. Similarly, for all j,

$$R_j(x,z)\ll\overline{R}(x,z).$$

Thus

$$m_j(x,w)\equiv g_i(x,R(x,w))\ll g_j^*(x,\tilde{R}(x,w)),$$

where $\bar{\bar{R}}(x,w)=(\overline{R},\cdots,\overline{R})^T$. Since all terms of g_j^* are positive,

$$m_j^*(x,w)\ll g_j^*(x,\overline{R}(x,w)).$$

Sum over j to obtain

(6. 15) $$\overline{m}(x, w) \ll \bar{g}(x, \tilde{\bar{P}}(x, w)) \equiv \hat{g}(x, \tilde{\bar{P}}(x, w)).$$

Let $w_i = v$, $i = 1, \cdots, n$. Then (6. 15) becomes

$$\hat{m}(x, v) \ll \hat{g}(x, \hat{p}(x, v)) \equiv \sum_{\alpha=2}^{\infty} \tilde{m}_\alpha(x) v^\alpha$$

where

$$\hat{m}(x, v) = \sum_{\alpha=2}^{\infty} \Big(\sum_{|\mathfrak{p}|=\alpha} \|m_{\mathfrak{p}}(x)\| \Big) v^\alpha.$$

Let $M_\alpha(x) = \sum_{|\mathfrak{p}|=\alpha} \|m_{\mathfrak{p}}(x)\|$. Then

(6. 16) $$M_\alpha(x) \leq \tilde{m}_\alpha(x).$$

In a similar way define $\widehat{h}(x, \widehat{p}(x, v))$. Then if $l(x, w)$ is defined by

$$l(x, w) \equiv \sum_{|\mathfrak{p}| \geq N} l_{\mathfrak{p}}(x) w^{\mathfrak{p}} = h(x, R(x, w)),$$

and $L_\alpha(x)$ by

$$L_\alpha(x) \equiv \sum_{|\mathfrak{p}|=\alpha} \|l_{\mathfrak{p}}(x)\|,$$

it follows similarly that

(6. 17) $$L_\alpha(x) \leq \tilde{l}_\alpha(x),$$

where $\sum_{\alpha=N}^{\infty} \tilde{l}_\alpha(x) v^\alpha \equiv \widehat{h}(x, \widehat{p}(x, v))$.

Notice that, because of the form of $R(x, w)$, $\tilde{m}_\alpha(x) = \sum_{|\mathfrak{p}|=\alpha} \|g_{\mathfrak{p}}(x)\|$ for $\alpha = 2, \cdots, N-1$, and that $\tilde{m}_N(x) = 0$. Hence $\sum_{\alpha=N+1}^{\infty} \tilde{m}_\alpha(x) v^\alpha$ is a majorant for $\hat{m}(x, v) - \hat{g}(x, v)$, since $\hat{g}(x, v)$ is a polynomial of degree at most $N-1$ in v, and the terms in $\hat{m}$ of degree less than $N+1$ are independent of the $Q_{\mathfrak{p}}$, and are hence equal to the corresponding g's.

Recall that

$$R(x, w) = w + Q(x, w),$$

and that

$$Q(x+1, Aw + g(x, w)) = \sum_{|\mathfrak{p}| \geq N} B_{\mathfrak{p}}(x) w^{\mathfrak{p}}.$$

Since $\|A(x)\|<\sigma$ for $x\in S_2$, the i'th component of $Aw+g(x,w)$, call it $[Aw+g(x,w)]_i$, satisfies

$$[Aw+g(x,w)]_i\ll\sigma(w_1+\cdots+w_n)+\bar{g}(x,w),$$

and hence

$$Q_i^*(x+1,Aw+g(x,w))\ll Q_i^*(x+1,\widetilde{\sigma(w_1+\cdots+w_n)+\bar{g}(x,w)}).$$

Sum on i from 1 to n to obtain

$$\begin{aligned}\overline{Q}(x+1,Aw+g(x,w))&=\sum_{|\mathfrak{p}|\geq N}\|B_{\mathfrak{p}}(x)\|w^{\mathfrak{p}}\\ &\ll\hat{q}(x+1,\sigma(w_1+\cdots+w_n)+\bar{g}(x,w)),\end{aligned}$$

where $\hat{q}(x,\eta)=\overline{Q}(x,\tilde{\eta})$, with $\tilde{\eta}=(\eta,\cdots,\eta)^T$. Set $w=(v,\cdots,v)^T$ to obtain

$$\sum_{\alpha=N}^{\infty}\sum_{|\mathfrak{p}|=\alpha}\|B_{\mathfrak{p}}(x)\|v^{\alpha}\ll\hat{q}(x+1,\sigma nv+\hat{g}(x,v)).$$

Write

$$\hat{q}(x+1,\sigma nv+\hat{g}(x,v))=\sum_{\alpha=N}^{\infty}\hat{b}_{\alpha}(x)v^{\alpha}.$$

Note also that

$$\begin{aligned}Q_j^*(x+1,Aw)&\ll Q_j^*(x+1,\widetilde{\sigma(w_1+\cdots+w_n)})\\ &\ll Q_j^*(x+1,\widetilde{\sigma(w_1+\cdots+w_n)+\bar{g}(x,w)}).\end{aligned}$$

Now define the majorant functions $\widehat{G}(t)$ and $\widehat{H}(t)$ by

$$\text{(6.18)}\qquad\begin{cases}\hat{g}(x,t)\ll\widehat{G}(t)=\sum_{k=2}^{N-1}\widehat{G}_k t^k\\ \hat{h}(x,t)\ll\widehat{H}(t)=\sum_{k=N}\widehat{H}_k t^k.\end{cases}$$

We notice that $\widehat{G}(t)$ is analytic because it is just a polynomial, and that $\widehat{H}(t)$ can be assumed to be analytic, since by construction $\hat{h}(x,t)$ is analytic in t.

c) *Majorant Equation.* Consider the following functional equation:

$$\text{(6.19)}\qquad\begin{aligned}&\xi=C\{\widehat{G}(nv+\xi)-\widehat{G}(nv)+\widehat{H}(nv+\xi)+\xi(\sigma nv+\widehat{G}(v))\},\\ &\xi=\xi(nv).\end{aligned}$$

We shall show that (6. 19) has a unique formal solution of the form

(6. 20) $$\xi=\sum_{k=N}\xi_k(nv)^k.$$

Then define $\tilde{\tilde{g}}_k$ and $\tilde{\tilde{h}}_k$ by

(6. 21) $$\begin{cases}\widehat{G}(nv+\xi)=\sum_{k=2}^{\infty}n^k\tilde{\tilde{g}}_kv^k\\ \widehat{H}(nv+\xi)=\sum_{k=N}^{\infty}n^k\tilde{\tilde{h}}_kv^k.\end{cases}$$

Substitute (6. 20) into (6. 19) to obtain the formal equation

$$\sum_{k=N}^{\infty}\xi_kn^kv^k=C\,[\sum_{k=2}^{N-1}\widehat{G}_k(nv+\sum_{m=N}^{\infty}\xi_m(nv)^m)^k-\sum_{k=2}^{N-1}\widehat{G}_k(nv)^k$$
$$+\sum_{k=N}^{\infty}\widehat{H}_k(nv+\sum_{m=N}^{\infty}\xi_m(nv)^m)^k+\sum_{k=N}^{\infty}\xi_k(\sigma nv+\sum_{m=2}^{N-1}\widehat{G}_mv^m)^k].$$

Since this is to be a formal identity in v, equate coefficients of v^k to obtain

(6. 22) $$\xi_kn^k=C\,[\widehat{G}_kn^k+n^kR_k(\xi_\alpha)-\widehat{G}_kn^k+n^kS_k(\xi_\alpha)+\sigma^kn^k\xi_k+n^kT_k(\xi_\alpha)],$$
$$\alpha<k,\ N\leq k,$$

where R_k, S_k, and T_k are defined by

$$\sum_{k=N}^{\infty}n^kR_kv^k=\sum_{k=2}^{N-1}\widehat{G}_k(nv+\sum_{m=N}^{\infty}\xi_m(nv)^m)^k-\sum_{k=2}^{N-1}\widehat{G}_k(nv)^k,\quad S_k=\tilde{\tilde{h}}_k,$$
$$\sum_{k=N}^{\infty}n^kT_kv^k=\sum_{k=N}^{\infty}\xi_k(\sigma nv+\sum_{m=2}^{N-1}\widehat{G}_mv^m)^k-\sum_{k=N}^{\infty}\xi_k\sigma^kn^kv^k.$$

Notice that R_k, S_k, and T_k are all polynomials in the $\xi_\alpha(\alpha<k)$ with positive coefficients. Hence we can solve (6. 22) to obtain the coefficients ξ_k of the formal solution of (6. 19):

(6. 23) $$n^k\xi_k=\frac{n^kC}{1-C\sigma^k}[R_k(\xi_\alpha)+S_k(\xi_\alpha)+T_k(\xi_\alpha)].$$

Clearly all of the coefficients $n^k\xi_k$ are nonnegative, since $k\geq N$ and N was chosen so large that $k\geq N$ implies $0<1-C\sigma^k<1$. Hence also

$$n^k\xi_k>n^kC\,[R_k(\xi_\alpha)+S_k(\xi_\alpha)+T_k(\xi_\alpha)].$$

Therefore (6. 19) has a formal solution (6. 20) with all ξ_k non-negative.

We now show that (6. 20) is a majorant for Q, i. e., that

$$n^k\xi_k \geq \sup_{x\in\mathcal{S}_2} \sum_{|\mathfrak{p}|=k} \|Q_{\mathfrak{p}}(x)\|, \quad (k\geq N).$$

The proof proceeds by induction. First, notice that for $k=N$, $R_N=0$, $S_N=\tilde{\tilde{h}}_N$, $T_N=0$. Since $m_N(x)=0$, $n^N R_N \geq \tilde{m}_N$. Also $n^N\tilde{\tilde{h}}_N \geq \tilde{l}_N(x)$, since $\sum_{k=N}^{\infty} \widehat{H}_k t^k \gg h(x, t)$, and the term $\tilde{\tilde{h}}_N$ is independent of the ξ's, while $\tilde{l}_N$ is independent of $\widehat{p}$. Since the Q's were determined in $\mathcal{S}_2$, and $\mathcal{S}_2$ is such that for $x\in\mathcal{S}_2$, $(x+1)\in\mathcal{S}_2$, we have in addition to (6. 10), the estimate

$$(6.24) \qquad \sum_{|\mathfrak{p}|=k} \|Q_{\mathfrak{p}}(x+1)\| \leq C \sup_{x\in\mathcal{S}_2} \sum_{|\mathfrak{p}|=k} \|m_{\mathfrak{p}}(x) - g_{\mathfrak{p}}(x) + l_{\mathfrak{p}}(x) + B_{\mathfrak{p}}(x) - D_{\mathfrak{p}}(x)\|.$$

But since $g_{\mathfrak{p}}(x)=0$ for $|\mathfrak{p}|\geq N$, we obtain from (6. 10)

$$\sum_{|\mathfrak{p}|=N} \|Q_{\mathfrak{p}}(x)\| \leq C \sup \sum_{|\mathfrak{p}|=N} \|m_{\mathfrak{p}}(x) + l_{\mathfrak{p}}(x) + B_{\mathfrak{p}}(x) - D_{\mathfrak{p}}(x)\|,$$

and the same estimate follows for $\sum_{|\mathfrak{p}|=k} \|Q_{\mathfrak{p}}(x+1)\|$, from (6. 24). For $|\mathfrak{p}|=N$, $m_{\mathfrak{p}}(x)=0$, and $B_{\mathfrak{p}}(x)\equiv D_{\mathfrak{p}}(x)$, hence $\sup_{x\in\mathcal{S}_2} \sum_{|\mathfrak{p}|=N} \|Q_{\mathfrak{p}}(x)\| \leq C \sup_{x\in\mathcal{S}_2} \sum_{|\mathfrak{p}|=N} \|l_{\mathfrak{p}}(x)\| \leq C \sup_{x\in\mathcal{S}_2} \tilde{l}_N(x) \leq C n^N \tilde{\tilde{h}}_N < \xi_N n^N$. Now suppose as induction hypothesis that

$$\sup_{x\in\mathcal{S}_2} \sum_{|\mathfrak{p}|=k} \|Q_{\mathfrak{p}}(x)\| \leq \xi_k n^k \quad \text{for} \quad k=N, N+1, \cdots, (m-1).$$

From the estimate (6. 10) we have

$$\sup_{x\in\mathcal{S}_2} \sum_{|\mathfrak{p}|=m} \|Q_{\mathfrak{p}}(x)\| < C \sup_{x\in\mathcal{S}_2} \sum_{|\mathfrak{p}|=m} (\|m_{\mathfrak{p}}(x)\| + \|l_{\mathfrak{p}}(x)\| + \|B_{\mathfrak{p}}(x) - D_{\mathfrak{p}}(x)\|)$$

and notice that since

$$\sum_{|\mathfrak{p}|=N}^{\infty} [B_{\mathfrak{p}}(x) - D_{\mathfrak{p}}(x)]\, w^{\mathfrak{p}} = \sum_{|\mathfrak{p}|=N}^{\infty} Q_{\mathfrak{p}}(x+1)\, [(Aw + g(x, w))^{\mathfrak{p}} - (Aw)^{\mathfrak{p}}],$$

$B_{\mathfrak{p}}(x) - D_{\mathfrak{p}}(x)$ is a vector which is a polynomial in the $Q_\alpha(x+1)$ for $|\alpha|<|\mathfrak{p}|$. Thus $\sum_{|k|=m} \|B_k(x) - D_k(x)\| \leq \widehat{\widehat{b}}_m(x)$, where $\widehat{\widehat{b}}_m(x)$ is obtained by subtracting the terms in the components of Q_α, $|\alpha|=m$ from $\hat{b}_m(x)$. Recall that

$$\sum_{k=N}^{\infty} n^k T_k v^k \equiv \sum_{k=N}^{\infty} \xi_k [(\sigma n v+\sum_{\alpha=2}^{N-1} \hat{G}_\alpha v^\alpha)^k-(\sigma n v)^k].$$

If we expand $\sum_{k=N}^{\infty} \xi_k(\sigma n v+\sum_{\alpha=2}^{N-1} \hat{G}_\alpha v^\alpha)^k$ in powers of v, we obtain $\sum_{k=N}^{\infty} \approx{b}_k n^k v^k$, where $\approx{b}_k$ is a polynomial in ξ_α for $\alpha \leq k$. Subtract all the terms in ξ_k from $\approx{b}_k$ to obtain $\check{b}_k$, where

$$\sum_{k=N}^{\infty} n^k T_k v^k=\sum_{k=N}^{\infty} n^k \check{b}_k v^k.$$

We shall now show that the terms in $\hat{b}_m(x)$ independent of Q_α for $|\alpha|=m$ are dominated by the terms in $\approx{b}_m$ independent of ξ_m. Recall that

$$\sum_{k=N}^{\infty} \hat{b}(x) v^k=\hat{q}(x+1, \sigma n v+\hat{g}(x, v)).$$

Consider $Q_i^*(x+1, \overline{\sigma(w_1+\cdots+w_n)+\bar{g}(x,w)})$. This is the ith component of

$$\sum_{k=N}^{\infty}(\sum_{|\mathfrak{p}|=k}|Q_\mathfrak{p}(x+1)|)(\sigma(w_1+\cdots+w_n)+\bar{g}(x, w))^k,$$

where $|Q_\mathfrak{p}(x+1)|$ is the vector $Q_\mathfrak{p}(x+1)$ with all its components replaced by their absolute values.

Replace w_i by $v(i=1, \cdots, n)$ to obtain

$$\sum_{k=N}^{\infty}(\sum_{|\mathfrak{p}|=k}|Q_\mathfrak{p}(x+1)|)(\sigma n v+\hat{g}(x, v))^k. \tag{6.25}$$

The sum of the components of this expression is $\hat{q}(x+1, \sigma n v+\hat{g}(x, v))$. The expression (6.25) is majorized by

$$\sum_{k=N}^{\infty}(\sum_{|\mathfrak{p}|=k}|Q_\mathfrak{p}(x+1)|)(\sigma n v+\hat{G}(v))^k.$$

Sum on i to get

$$\sum_{k=N}^{\infty}(\sum_{|\mathfrak{p}|=k}\|Q_\mathfrak{p}(x+1)\|)(\sigma n v+\hat{G}(v))^k.$$

However, since by the induction hypothesis,

$$\sum_{|\mathfrak{p}|=k}\|Q_{\mathfrak{p}}(x+1)\|\leq\xi_k n^k,\ k=N,\quad N+1,\cdots,(m-1),$$

and

$$\sum_{k=N}^{\infty}\tilde{b}_k n^k v^k=\sum_{k=N}^{\infty}\xi_k(\sigma nv+\widehat{G}(v))^k,$$

the result follows.

Similarly, we can show that $n^m S_m\geq\tilde{l}_m(x)$: Recall that $n^m S_m=n^m\tilde{\tilde{h}}_m$ where

$$\sum_{k=N}^{\infty}n^k\tilde{h}_k v^k=\sum_{k=N}^{\infty}\widehat{H}_k(nv+\sum_{\alpha=N}^{\infty}\xi_\alpha(nv)^\alpha)^k, \tag{6.26}$$

and

$$\sum_{k=N}^{\infty}\tilde{l}_k(x)v^k\equiv\widehat{h}(x,\widehat{p}(x,v)).$$

We begin by considering $h_j^*(x,\tilde{\tilde{P}}(x,w))$. This is the j'th component of

$$\sum_{k=N}^{\infty}(\sum_{|\mathfrak{p}|=k}|h_{\mathfrak{p}}(x)|)(\tilde{\tilde{P}}(x,w))^k.$$

Replace w by $(v,\cdots v)^T$ to get

$$\sum_{k=N}^{\infty}(\sum_{|\mathfrak{p}|=k}|h_{\mathfrak{p}}(x)|)(\widehat{p}(x,v))^k. \tag{6.27}$$

The sum of the components of (6.27) is

$$\sum_{k=N}^{\infty}(\sum_{|\mathfrak{p}|=k}\|h_{\mathfrak{p}}(x)\|)(\widehat{p}(x,v))^k\equiv\widehat{h}(x,\widehat{p}(x,v)).$$

But

$$\widehat{p}(x,v)=nv+\sum_{k=N}^{\infty}(\sum_{|\mathfrak{p}|=k}\|Q_{\mathfrak{p}}(x)\|)v^k,$$

and thus

$$\widehat{h}(x,\widehat{p}(x,v))=\sum_{k=N}^{\infty}(\sum_{|\mathfrak{p}|=k}\|h_{\mathfrak{p}}(x)\|)(nv+\sum_{\alpha=N}^{\infty}(\sum_{|\mathfrak{q}|=\alpha}\|Q_{\mathfrak{q}}(x)\|)v^\alpha)^k \tag{6.28}$$

and writing this as $\sum_{k=N}\tilde{\tilde{l}}_k v^k$, using the induction hypothesis together with (6.26) and (6.28), the result follows. The g's can be treated in an essentially similar fashion to yield $n^m R_m \geq m_m(x)$. Thus the estimate (6. 10) yields for all $x \in \mathcal{S}_2$,

$$\sum_{|\mathfrak{p}|=m} \|Q_{\mathfrak{p}}(x)\| \leq C(n^m R_m + n^m S_m + n^m T_m) < n^m \xi_m.$$

Hence for all $k \geq N$, $x \in \mathcal{S}_2$,

$$\sum_{|\mathfrak{p}|=k} \|Q_{\mathfrak{p}}(x)\| \leq n^k \xi_k.$$

Thus $\sum_{k=N}^{\infty} \xi_k (nv)^k$ is a majorant for the formal series $\sum_{|\mathfrak{p}| \geq N} Q_{\mathfrak{p}}(x) v^k$, ($k = |\mathfrak{p}|$). It remains to be shown that the series $\sum_{k=N}^{\infty} \xi_k (nv)^k$ converges.

d) *Convergence of the Majorant*. Recall that $\xi = \sum_{k=N}^{\infty} \xi_k (nv)^k$ is a formal solution of the functional equation (6. 19). Let $nv = z$ and let $\hat{G}\left(\frac{z}{n}\right) = \hat{\hat{G}}(z)$. Then (6. 19) becomes

$$\xi(z) = C[\hat{G}(z + \xi(z)) - \hat{G}(z) + \hat{H}(z + \xi(z)) + \xi(\sigma z + \hat{\hat{G}}(z))]. \tag{6. 29}$$

We shall prove

Lemma 4. *The equation* (6. 29) *has an analytic solution* ξ *of the form* $\sum_{k=N} \xi_k z^k$ *for* $|z|$ *sufficiently small, which is unique in the class of analytic functions of this form.*

Proof: For $|z| < 2\delta_1$, $\hat{G}$, $\hat{H}$, $\hat{G}$ are holomorphic and $\hat{G}(z) = O(z^2)$, $\hat{\hat{G}}(z) = O(z^2)$, $\hat{H}(z) = O(z^N)$. Hence for $|z| < \delta_1$, $|\varphi(z)| < \delta_1$, $\varphi(z)$ analytic for $|z| < \delta_1$, there exist constants G and H such that

$$|\hat{G}(z + \varphi(z)) - \hat{G}(z)| \leq G|z| |\varphi(z)|$$

$$|\hat{H}(z + \varphi(z))| \leq G|z| |\varphi(z)| + H|z|^N.$$

Further, without loss of generality, we may assume that δ_1 is so small that for $|z| < \delta_1$, $|\sigma z + \hat{\hat{G}}(z)| < r|z|$, $\sigma < r < 1$, $r \leq \sqrt[N]{2}\,\sigma$.

Let $\mathcal{T}$ be the family of functions $\varphi(z)$ analytic for $|z| < \delta$ such that $|\varphi(z)| \leq K|z|^N$, where K is a constant which will be specified later. For functions $\varphi \in \mathcal{T}$ define the mapping T by

$$T[\varphi](z)=C[\widehat{G}(z+\varphi(z))-\widehat{G}(z)+\widehat{H}(z+\varphi(z))+\varphi(\sigma z+\widehat{\widehat{G}}(z))].$$

Clearly the mapping is well defined if $\delta\leq\delta_1$, $K\delta^N\leq\delta$ and a solution of (6. 29) is equivalent to a fixed point of the mapping T. $\mathcal{T}$ is closed, compact, and convex with respect to the topology of uniform convergence on each compact subset of the region $|z|<\delta$. Since the mapping is continuous, we need only show that it is into. Recalling that $1-2C\sigma^N>0$, $r\leq\sqrt[N]{2}\,\sigma$, choose

$$K=\frac{2CH}{1-2C\sigma^N}, \qquad \delta=\min\left\{\delta_1,\ \sqrt[N-1]{K},\ \frac{1-2C\sigma^N}{4GC}\right\}.$$

Then, $|T[\varphi](z)|\leq[(2CG\delta+Cr^N)K+CH]|z|^N\leq K|z|^N$, and there is a fixed point of the mapping T which is the desired solution. Since the coefficients of the formal solution are unique, the solution of equation (6. 29) is unique in this class.

7. Estimates of Solutions of the Reduced Equation.

Consider the reduced equation (0. 22). This system is equivalent to r systems of linear equations. Let the distinct eigenvalues μ_i of A_0 satisfy

$$\begin{aligned}1>|\mu_1|&=|\mu_2|=\cdots=|\mu_{k_1}|\\ >|\mu_{k_1}+1|&\cdots\qquad\ =|\mu_{k_2}|\\ >&\cdots\\ >|\mu_{k_t+1}|&=\cdots\qquad\ =|\mu_{k_{t+1}}|>0,\ (k_{t+1}=r),\end{aligned}$$

and let

$$u=\begin{pmatrix}u^1\\ u^2\\ \vdots\\ u^r\end{pmatrix}$$

be the partitioning of u compatible with the μ_i. Then the first k_1 systems are linear homogeneous systems of the form

(7. 1) $$u^j(x+1)=A_{jj}(x)u^j(x),\quad j=1,\cdots,k_1,$$

where

$$A_{jj}(x) \cong \mu_j I + N_j + \sum_{s=1} A^s_{jj} x^{-s},$$

with N_i the nilpotent matrix defined in (2. 2). The next $k_2 - k_1$ systems, corresponding to the indices $j = k_1 + 1, k_1 + 2, \cdots, k_2$, are nonhomogeneous systems of the form

$$u^j(x+1) = A_{jj}(x) u^j(x) + g^j(x, u^1, u^2, \cdots, u^{k_1}), \tag{7. 2}$$

where the components of g^j are polynomials in the components of $u^1, \cdots, u^{k_1}$. Hence the general solution of (7. 2) can be obtained by obtaining the general solutions of all of the systems (7. 1) and utilizing these to evaluate the functions g^j. Let $\bar{g}^j(x)$ be the g^j evaluted in this way. The remaining systems for $j = k_2 + 1, \cdots$ are of form analogous to that of (7. 2), and we proceed in the manner described above to find the general solution of the reduced system (0. 22).

Thus the problem of solving (0. 22) falls naturally into two parts, the solving of linear homogeneous equations and of linear nonhomogeneous of the forms

$$u(x+1) = A(x) u(x), \quad \text{and} \tag{7. 3}$$

$$u(x+1) = A(x) u(x) + \bar{g}(x), \tag{7. 4}$$

respectively, where

$$A(x) \cong \mu I + N + \sum_{s=1}^{\infty} A^s x^{-s},$$

where N has the form of N_i above.

We consider the homogenous case (7. 3) first: A system of the form (7. 3) is called *normal* if there exist a formal fundamental matrix of the form

$$U(x) \cong \mu^x x^R \left(I + \frac{U_1}{x} + \frac{U_2}{x^2} + \cdots \right),$$

where R is a constant matrix. Otherwise the system (7. 3) is called *anormal*.

If all of the corresponding linear homogeneous systems are normal, we may assume that all of the nilpotent matrices N are zero, since Harris [4] has shown that this may be effected by a linear transformation which is a polynomial in x^{-1} with determinant not identically zero. Further, it is known [1], [5] that in the normal case there exist analytic fundamental matrices which have the formal fundamental matrices as asymptotic representations in right half-planes. Hence the behavior of the fundamental matrix as x tends to infinity is essentially determined by μ^x, but since $0<|\mu|<1$, μ^x is bounded in a half-plane which contains a portion of the positive real axis in its interior. Hence there exists a sector of the form

$$-\frac{\pi}{2}<l_1<\arg(x-a)<l_2<\frac{\pi}{2}, \tag{7.5}$$

in which the fundamental matrix exists, is bounded, and approaches zero uniformly as x tends to infinity through this sector.

Now consider the anormal case. Birkhoff and Trjitzinsky [2] have shown that in this case there exist sectors of the form (7.5) in which there exists a fundamental matrix for (7.3) which is of the form

$$U(x)=\mu^x e^{Q(x)} x^R (I+U_1 x^{-1/p}+\cdots),$$

where $Q(x)$ is a diagonal matrix with elements of the form

$$q_k=\delta_k x^{\frac{p-1}{p}}+\cdots+v_k x^{\frac{1}{p}}.$$

It is clear that again μ^x is the dominant term. Thus, we may infer, in case the reduced equation (0.22) is linear, the existence of a sector of the form (7.5) in which the solutions of the systems (7.3) are bounded and approach zero uniformly as x approaches infinity in this sector.

Now, we consider the remaining problem, the case when (0.22) is nonlinear. Then we have to find particular solutions of the nonhomogeneous systems (7.4). First, we shall make the following definition: the system (0.22) will be called *normal in the extended sense if*

i) all the systems

$$u^j(x+1)=A_{jj}(x)u^j(x), \quad j=1, \cdots, r$$

are normal;

ii) if $\mu_j^{-x}x^{-r_j}(\log x)^{-\beta_j}\bar{g}^j(x)\cong g_{j0}+g_{j1}x^{-1}+\cdots$ for some r_j and integer β_j for all j;

iii) there exists a formal particular solution of the form $\mu_j^x x^\nu (\log x)^\gamma \bar{h}(x)$, where

$$\bar{h}(x)=h_0+h_1x^{-1}+\cdots.$$

Hence by the results of Harris and Sibuya [5] there exists an analytic solution asymptotic to this formal solution in a sector of the form (7.5). This particular solution has the same rate of growth as the solution of the corresponding homogeneous equation. By induction the general solution of the reduced equation (0.22) can be thus constructed in a region of the form (7.5), if the reduced equation is normal in the extended sense, and will have properties similar to those of the general solutions when the reduced equation was linear.

We may summarize our results in the following:

Theorem 4. *Let the reduced equation* (0.22) *be either linear or normal in the extended sense. Then the general solution of* (0.22) *can be written in the form*

$$u(x)=U(x, C(x)) \tag{7.6}$$

in a sector R of the form

$$-\frac{\pi}{2}<l_1<\arg(x-\hat{a})<l_2<\frac{\pi}{2}, \qquad \hat{a}>0,$$

where $U(x, C(x))$ is holomorphic in R, tends uniformly to zero as x approaches infinity through R, and possesses an asymptotic expansion in this region, and $C(x)$ is an arbitrary bounded periodic vector of period 1.

Hence, if θ is chosen sufficiently small $\theta\geq 0$, and compatible with the hypotheses of **Theorem 1, 2,** and **3,** (using, for instance, the sector $\sum$ defined in Section 1 to choose θ) we can combine **Theorems**

1, 2, 3, and **4** to obtain in this case the general solution of (0. 1) in the form

$$y(x)=\phi(x)+P(x, U(x, C(x))).$$

8. General remarks.

If the eigenvalues λ_j of the matrix A_0 satisfy $1<|\lambda_j|$, similar results corresponding to **Theorem 3** are available in sectors which cover a region of the form $0<\arg(x+a)<2\pi,\ a>0$.

If we assume the existence of a particular solution, or $f(x,0)=0$, and

$$|\lambda_1|\geq\cdots\geq|\lambda_k|>1=|\lambda_{k+1}|=\cdots=|\lambda_p|>|\lambda_{p+1}|\geq\cdots\geq|\lambda_n|,$$

then by choosing either $u_1=\cdots=u_p=0$, or $u_{k+1}=\cdots=u_n=0$, similar results are available where now $C(x)$ will be either an $n-p$ or k dimensional arbitrary periodic vector.

The possibility of obtaining the uniform asymptotic expansion

$$P(x, u)\cong\sum_{k=0} x^{-k}P_k(u)$$

for the transformation $P(x, u)$ has been demonstrated by Harris and Sibuya [7] under more restrictive hypothesis including the uniform asymptotic expansion

$$\hat{f}(x, z)\cong\sum_{k=0} x^{-k}f_k(z).$$

We shall treat this question in a subsequent paper.

One would expect that the results embodied in **Theorm 4** are valid without the restriction: normal in the extended sense.

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