On a theorem of S. Tanaka

By

W. A. HARRIS, Jr.

During the Symposium on Functional Equations held at Osaka University, January 19–20, 1966 Sen-ichiro Tanaka presented some interesting results concerning solutions of a class of nonlinear difference equations. In the ensuing discussion, the question of a possible extension of these results was raised. It is the purpose of this note to answer this question in the negative through the construction of a counter example.

For the scalar case, Tanaka's results can be phrased in the following manner.

Consider the nonlinear difference equation

$$y(x+1)=y^n f(x, y)\ ,$$

where n is an integer greater than one and $f(x, y)$ is an analytic function of x and y for $|x|>r$, $|\arg\ x|<\beta$, $|y|<\delta$.

Let

$$f(x, y)=\sum_{k=0}^{\infty} f_k(x) y^k$$

be the expansion of $f(x, y)$ in powers of y. We assume that the $f_k(x)$ are analytic for $|x|>r$, $|\arg\ x|<\beta$ and admit the asymptotic expansions

$$f_k(x) \cong \sum_{j=0}^{\infty} f_{kj} x^{-j}$$

as x approaches infinity through the region $|\arg\ x|<\beta$.

Further, we assume

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School of Mathematics, University of Minnesota and Research Institute for Mathematical Sciences, Kyoto University. U.S.-Japan Cooperative Science Program, NSF Grant GF-214.

$$f_{00} \neq 0.$$

Tanaka has shown that there exists a nonlinear transformation

(1) $$y = u[1 + p(x, u)]$$

which transforms the difference equation

$$y(x+1) = y^n f(x, y)$$

into the difference equation

$$u(x+1) = u^n f_{00}\,.$$

The transformation (1) is analytic for $x \in R$, $|u| < \delta'$, where R is a suitable subregion of $|x| > r$, $|\arg x| < \beta$ extending to infinity. Further, we have the asymptotic expansion

$$p(x,u) \cong \sum_{j \geq 1} \sum_{k \geq 0} p_{jk} u^j x^{-k}$$

as $|u| + |x|^{-1}$ approaches zero, $x \in R$, $|u| < \delta'$.

We are concerned with the following question.

Question. *Can the same type of results be obtained when n is a rational number by allowing the nonlinear transformation* (1) *to contain fractional powers?*

Counter example.

There is no formal transformation of the form

(2) $$y = u[1 + p(u)]$$

where

$$p(u) = \sum_{k=1}^{\infty} p_k u^{k\alpha}$$

with α a positive rational number, which transforms the diffrence equation

$$y(x+1) = y^{3/2}(1+y)$$

into the difference equation

$$u(x+1) = u^{3/2}\,.$$

Proof.

The existence of the transformation (2) is equivalent to a solution of the equation

$$(3) \qquad 1+p(u^{3/2})=[1+p(u)]^{3/2}[1+u+up(u)].$$

Write

$$[1+p(u)]^{3/2}=1+\frac{3}{2}p(u)+h(p(u)).$$

Then equation (3) becomes

$$(4) \qquad p(u^{3/2})=\frac{3}{2}p(u)+u+\frac{5}{2}up(u)+h(p(u))+\frac{3}{2}up(u)^2 + uh(p(u))+up(u)h(p(u)).$$

Case 1: $\alpha>1$.

Suppose there exists a formal solution $p(u)=\sum_{k=1}^{\infty} p_k u^{k\alpha}$ with $\alpha>1$. Then the following order relations hold: $p(u^{3/2})=O(u^{3\alpha/2})=o(u)$; $p(u)=O(u^{\alpha})=o(u)$; and $h(p(u))=O(p(u)^2)=O(u^{2\alpha})=o(u)$. Thus every term in equation (4) is of order $o(u)$ except the term u. Hence there can be no formal solution of this form in this case.

Case 2: $0<\alpha\leqq 1$.

If there is a solution of equation (4) of the form $\sum p_k u^{k\alpha}$ with rational α, $0<\alpha\leqq 1$, there exists an integer $q\geqq 1$ and a formal solution of the form

$$(5) \qquad p(u)=\sum_{j=1}^{\infty}\bar{p}_j u^{j/q}.$$

We shall now establish three properties of the coefficients of the formal series (5) under the assumption that this series is a formal solution of equation (4).

Property 1.

$$\bar{p}_1=\bar{p}_2=\cdots=\bar{p}_{q-1}=0,\ \bar{p}_q\neq 0.$$

This is a simple computation using the order relations for the various terms. In fact, $\bar{p}_q=-2/3$.

Property 2.

If the equation (4) *has a solution of the form* $\sum_{k=1}^{\infty} \bar{p}_k u^{k/q}$, *then this solution can be written in the form* $\sum_{k=1}^{\infty} \bar{p}_{2k} u^{2k/q}$.

Consider k such that $\bar{p}_k \neq 0$. The term $p(u^{3/2})$ will contribute the nonzero term $\bar{p}_k u^{3k/2q}$ to the left side of equation (4). The terms on the right side of equation (4) are of the form $u^{i/q}$. Hence, if $\bar{p}_k \neq 0$, then $k=2i$ for some i.

Property 3.

q must be an even integer.

From properties 1 and 2, $\bar{p}_q \neq 0$ and hence $q=2i$ for some i.

Since q is an integer, we may write

$$q=2^m q'$$

where q' is an odd integer. By using property 2 m-times, we are led to the conclusion that equation (4) has a solution of the form

$$p(u)=\sum_{k=1}^{\infty} \hat{p}_k u^{k/q'}.$$

If $q'=1$, then using property 2 again shows that there is a solution of the form $p(u)=\sum \hat{p}_{2k} u^{2k}$ which is impossible by Case 1 with $\alpha=2$.

If $q'>1$, then property 3 shows that q' must be even which is a contradiction.

Hence, there is no solution of equation (4) of the indicated form and the counter example has been established.

REFERENCE

[1] S. Tanaka, On the general solution of nonlinear difference equations, Publ. Res. Inst. Math. Sci. Kyoto Univ. Ser. A to appear.