On difference equations containing a parameter

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0. Introduction

Recently the analytic theory of difference equations has received considerable attention. A classification of fixed singular points has been given for linear systems of difference equations and a general Fuchsian theory developed. Further, general solutions of systems of nonlinear difference equations have been constructed and general stability considerations have been made. Thus the fundamental structure of the local analytic theory of difference equations is becoming apparent.

However, the dependence of the solutions of a difference equation containing a parameter has received little attention. It is the purpose of this note to present a result concerning the dependence of solutions of a system of nonlinear difference equations containing a parameter.

This result is of intrinsic interest. Moreover, as shall be shown, it can be applied to effect a simplification in the study of linear difference equations containing a parameter.

1. Nonlinear difference equations

Consider the system of nonlinear difference equations

$$y(x+1)=\varepsilon f(x, y, \varepsilon) \tag{1.1}$$

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where x is a complex variable, y an n-dimensional vector, ε a complex parameter, and f an n-dimensional vector.

Each component of the n-dimensional vector f is assumed to be analytic in a region $R_0 \times Y_0 \times E_0$ where

$$R_0: \ |x| > r_0, \ l_1 < \arg x < l_2 ,$$
$$Y_0: \ \|y\| < \delta_0, \ (\|y\| = \sum |y_j|) ,$$
$$E_0: \ 0 < |\varepsilon| < \rho_0, \ |\arg \varepsilon| < \alpha_0 .$$

Let

$$f(x, y, \varepsilon) = f_0(x, \varepsilon) + A(x, \varepsilon)y + \sum_{|\mathfrak{p}| \geqq 2} f_{\mathfrak{p}}(x, \varepsilon) y^{\mathfrak{p}} \tag{1.2}$$

be the expansion of f in powers of $y_1, y_2, \cdots, y_n$ where $\mathfrak{p}$ is a set of nonnegative integers $p_1, p_2, \cdots, p_n$, $A(x, \varepsilon)$ an n by n matrix, $f_0(x, \varepsilon)$ and $f_{\mathfrak{p}}(x, \varepsilon)$ n-dimensional vectors and

$$y_{\mathfrak{p}} = y^{\mathfrak{p}} y_1^{p_1} y_2^{p_2} \cdots y_n^{p_n} ,$$
$$|\mathfrak{p}| = p_1 + p_2 + \cdots + p_n .$$

We shall suppose that $f_0(x, \varepsilon)$, $f_{\mathfrak{p}}(x, \varepsilon)$, $A(x, \varepsilon)$ are analytic in $R_0 \times E_0$ and admit for $x \in R_0$ the uniform asymptotic expansions

$$\begin{aligned} f_0(x, \varepsilon) &\cong \sum_{k=0}^{8} f_{0k}(x) \varepsilon^k , \\ f_{\mathfrak{p}}(x, \varepsilon) &\cong \sum_{k=0}^{\infty} f_{\mathfrak{p}k}(x) \varepsilon^k , \\ A(x, \varepsilon) &\cong \sum_{k=0}^{\infty} A_k(x) \varepsilon^k , \end{aligned} \tag{1.3}$$

as ε approaches zero through the sector E_0 where we shall suppose that f_{0k}, $f_{\mathfrak{p}k}$, A_k are analytic in R_0.

We may determine a formal solution of the nonlinear difference equation (1.1) in the form

$$p(x, \varepsilon) = \sum_{k=0}^{\infty} p_k(x) \varepsilon^k \tag{1.4}$$

by solving the sequence of equations

$$p_k(x) = h_k(x-1) , \tag{1.5}$$

where formally

$$\sum_{k=1}^{\infty} h_k(x)\varepsilon^k = \varepsilon f(x, \sum_{j=1}^{\infty} p_j(x)\varepsilon^j, \varepsilon)\,.$$

Thus $h_k(x)$ is a polynomial in the components of the vectors $p_j(x)$, $j<k$, with coefficients analytic in x.

Hence, to be able to determine a formal solution of this form with the components of the vectors $p_k(x)$ analytic, the region R_0 must contain a subregion R^- such that $x\in R^-$ implies $x-1\in R^-$.

Theorem 1.

Let R^- be a subregion of R_0 such that $x\in R^-$ implies $x-1\in R^-$. Then there exists a unique analytic solution $y(x, \varepsilon)$ of $y(x+1)=\varepsilon f(x, y, \varepsilon)$ in the region $R^-\times Y_1\times E_1$, where

$$Y_1:\ \ ||y||<\delta_1\leqq\delta_0\,,$$
$$E_1:\ \ 0<|\varepsilon|<\rho_1\leqq\rho_0,\ |\arg\ \varepsilon|<\alpha_0\,.$$

Further, $y(x, \varepsilon)$ admits for $x\in R^-$ the uniform asymptotic expansion

$$y(x, \varepsilon)\cong\sum_{k=1}^{\infty} p_k(x)\varepsilon^k$$

as ε approaches zero in the sector E_1.

In a similar manner, for the system of nonlinear difference equations

$$y(x-1)=\varepsilon f(x, y, \varepsilon) \tag{1.6}$$

we obtain the following theorem.

Theorem 2.

Let R^+ be a subregion of R_0 such that $x\in R^+$ implies $x+1\in R^+$. Then there exists a unique analytic solution $y(x, \varepsilon)$ of $y(x-1)=\varepsilon f(x, y, \varepsilon)$ in the region $R^+\times Y_2\times E_2$, where

$$Y_2:\ \ ||y||<\delta_2\leqq\delta_0\,,$$
$$E_2:\ \ 0<|\varepsilon|<\rho_2\leqq\rho_0,\ |\arg\ \varepsilon|<\alpha_0\,.$$

Further, $y(x, \varepsilon)$ admits for $x\in R^+$ the uniform asymptotic *expansion*

$$y(x, \varepsilon) \cong \sum_{k=1}^{\infty} q_k(x)\varepsilon^k$$

as ε *approaches zero through* E_2, *where* $q_k(x) = \bar{h}_k(x+1)$ *and* $\sum_{k=1}^{\infty} \bar{h}_k(x)\varepsilon^k = \varepsilon f(x, \sum_{j=1}^{\infty} q_j(x)\varepsilon^j, \varepsilon)$ *formally.*

Remark.

If we assume that $f_{0k}(x)$, $f_{\mathfrak{p}k}(x)$, $A_k(x)$ admit the asymptotic expansions

$$f_{0k}(x) \cong \sum_{j=0}^{\infty} f_{0kj} x^{-j}\,,$$

$$f_{\mathfrak{p}k}(x) \cong \sum_{j=0}^{\infty} f_{\mathfrak{p}kj} x^{-j}\,,$$

$$A_k(x) \cong \sum_{j=0}^{\infty} A_{kj} x^{-j}\,,$$

as x approaches infinity through R_0, then $p_k(x)$ and $q_k(x)$ admit asymptotic expansions as x approaches infinity through R^- and R^+ respectively. This is important for the applications of these theorems.

2. Linear difference equations

Consider the linear system of difference equations

$$\varepsilon^{\sigma} y(x+1) = B(x, \varepsilon) y(x) \tag{2.1}$$

where σ is an integer, x a complex variable, ε a complex parameter, $B(x, \varepsilon)$ an n by n matrix, and y an n-dimensional vector.

Each element of the n by n matrix $B(x, \varepsilon)$ is analytic in the region $R_3 \times E_3$ where

$$R_3:\ |x| > r_3,\ l_3 < \arg x < l_4\,,$$
$$E_3:\ 0 < |\varepsilon| < \rho_3,\ |\arg \varepsilon| < \alpha_3\,.$$

We shall suppose that $B(x, \varepsilon)$ admits for $x \in R_3$ the uniform asymptotic expansion

$$B(x, \varepsilon) \cong \sum_{k=0}^{\infty} B_k(x)\varepsilon^k \tag{2.2}$$

as ε approaches zero through the sector E_3. We shall suppose that the $B_k(x)$ are analytic for $x \in R_3$ and admit the asymptotic expansions

(2.3) $$B_k(x) \cong \sum_{j=0}^{\infty} B_{kj} x^{-j}$$

as x approaches infinity through the region R_3.

We assume that $B_0(x)$ has the following form

$$B_0(x) = \begin{pmatrix} 0 & 0 \\ B_{21}^0(x) & B_{22}^0(x) \end{pmatrix}$$

where $B_{22}^0(x)$ is an s by s nonsingular matrix. Since $B_{22}^0(x) \cong \sum_{k=0}^{\infty} B_{22k}^0 x^{-k}$, this is equivalent to assuming r_3 sufficiently large and B_{220}^0 nonsingular.

Under the assumption that $B_{22}^0(x)$ is nonsingular we may assume without loss of generality that $B_{21}^0(x) \equiv 0$. Indeed, the matrix $P(x)$,

$$P(x) = \begin{pmatrix} I & 0 \\ B_{22}^0(x)^{-1} B_{21}^0(x) & I \end{pmatrix}$$

is a well defined nonsingular matrix for $x \in R_3$. The transformation $z = Py$ will yield the system $\varepsilon^{\sigma} z(x+1) = \overline{B}(x, \varepsilon)\, z(x)$ where $\overline{B}(x, \varepsilon)$ has similar properties to $B(x, \varepsilon)$. It is easily seen that

$$\overline{B}_0(x) = \begin{pmatrix} 0 & 0 \\ 0 & B_{22}^0(x) \end{pmatrix}.$$

Hence we shall assume

(2.4) $$B_0(x) = \begin{pmatrix} 0 & 0 \\ 0 & B_{22}^0(x) \end{pmatrix}, \quad B_{22}^0(x) \text{ nonsingular.}$$

Remark.

It is clear that the assumption

$$B_0(x) = \begin{pmatrix} 0 & B_{12}^0(x) \\ 0 & B_{22}^0(x) \end{pmatrix}, \quad B_{22}^0(x) \text{ nonsingular,}$$

is also equivalent to (2.4) for $|x|$ sufficiently large.

Theorem 3.

Let R_3^+ be a subregion of R_3 such that $x \in R_3^+$ implies $x+1 \in R_3^+$. Then there exists a nonsingular transformation $z = Q(x, \varepsilon) y$ which transforms the difference equation $\varepsilon^\sigma y(x+1) = B(x, \varepsilon) y(x)$ into $\varepsilon^\sigma z(x+1) = C(x, \varepsilon) z(x)$, where $C(x, \varepsilon)$ is a block triangular matrix of the form

$$C(x, \varepsilon) = \begin{pmatrix} C_{11}(x, \varepsilon) & C_{12}(x, \varepsilon) \\ 0 & C_{22}(x, \varepsilon) \end{pmatrix}.$$

The matrix $Q(x, \varepsilon)$ is analytic for $x \in R_3^+$, $\varepsilon \in E_4$: $0 < |\varepsilon| < \rho_4$, $|\arg \varepsilon| < \alpha_3$, and admits for $x \in R_3^+$ the uniform asymptotic expansion

$$Q(x, \varepsilon) \cong I + \sum_{k=1}^{\infty} Q_k(x) \varepsilon^k$$

as ε approaches zero through the sector E_4. Further, det $Q(x, \varepsilon) \equiv 1$, *$Q_k(x)$ are analytic for $x \in R_3^+$ and admit the asymptotic expansions*

$$Q_k(x) \cong \sum_{j=0}^{\infty} Q_{kj} x^{-j}$$

as x approaches infinity through the region R_3^+. Thus $C(x, \varepsilon)$ has similar properties to $B(x, \varepsilon)$.

In a similar manner we can obtain the following theorem.

Theorem 4.

Let R_3^- be a subregion of R_3 such that $x \in R_3^-$ implies $x-1 \in R_3^-$. Then there exists a nonsingular transformation $z = \overline{Q}(x, \varepsilon) y$ which transforms the difference equation $\varepsilon^\sigma y(x+1) = B(x, \varepsilon) y(x)$ into $\varepsilon^\sigma z(x+1) = \overline{C}(x, \varepsilon) z(x)$, where $\overline{C}(x, \varepsilon)$ is a block triangular matrix of the form

$$\overline{C}(x, \varepsilon) = \begin{pmatrix} \overline{C}_{11}(x, \varepsilon) & 0 \\ \overline{C}_{21}(x, \varepsilon) & \overline{C}_{22}(x, \varepsilon) \end{pmatrix},$$

with properties similar to $B(x, \varepsilon)$.

3. Proof of Theorem 3

Substituting $y = Q(x, \varepsilon) z$ into the difference equation

$\varepsilon^{\sigma}y(x+1)=B(x,\varepsilon)y(x)$ we obtain

$$\varepsilon^{\sigma}Q(x+1,\varepsilon)z(x+1)=B(x,\varepsilon)Q(x,\varepsilon)z(x)\,.$$

If the resulting difference equation is $\varepsilon^{\sigma}z(x+1)=C(x,\varepsilon)z(x)$, then we must have

$$Q(x+1,\varepsilon)C(x,\varepsilon)=B(x,\varepsilon)Q(x,\varepsilon)\,. \tag{3.1}$$

Let us seek a solution Q in the form

$$Q(x,\varepsilon)=\begin{pmatrix} I & 0 \\ Q_{21}(x,\varepsilon) & I \end{pmatrix}$$

with a partitioning compatible with that of $B_0(x)$ given in equation (2.4). Partitioning $B(x,\varepsilon)$, $C(x,\varepsilon)$ in a similar manner, equation (3.1) is equivalent to the following four equations.

$$\begin{aligned}
&C_{11}(x,\varepsilon)=B_{11}(x,\varepsilon)+B_{12}(x,\varepsilon)Q_{21}(x,\varepsilon)\,,\\
&C_{12}(x,\varepsilon)=B_{12}(x,\varepsilon)\,,\\
&C_{22}(x,\varepsilon)=B_{22}(x,\varepsilon)-Q_{21}(x,\varepsilon)C_{12}(x,\varepsilon)\,,\\
&Q_{21}(x+1,\varepsilon)C_{11}(x,\varepsilon)=B_{21}(x,\varepsilon)+B_{22}(x,\varepsilon)Q_{21}(x,\varepsilon)\,.
\end{aligned}$$

If $Q_{21}(x,\varepsilon)$ is known, then the first three equations will define $C(x,\varepsilon)$. Hence, equation (3.1) will be satisfied if $Q_{21}(x,\varepsilon)$ satisfies the following equation

$$\begin{aligned}
B_{22}(x,\varepsilon)Q_{21}(x,\varepsilon)=&-B_{21}(x,\varepsilon)+Q_{21}(x+1,\varepsilon)B_{11}(x,\varepsilon)\\
&+Q_{21}(x+1,\varepsilon)B_{12}(x,\varepsilon)Q_{21}(x,\varepsilon)\,.
\end{aligned} \tag{3.3}$$

We have that $B_{22}(x,\varepsilon)=B_{22}^{0}(x)+O(\varepsilon)$ where $B_{22}^{0}(x)$ is nonsingular, $B_{11}(x,\varepsilon)=O(\varepsilon)$, $B_{12}(x,\varepsilon)=O(\varepsilon)$, and $B_{21}(x,\varepsilon)=O(\varepsilon)$. Hence, equation (3.3) is equivalent to the system of nonlinear difference equations

$$u(x-1)=\varepsilon f(x,u,\varepsilon)\,. \tag{3.4}$$

Thus, we may apply Theorem 2 to solve the difference equation (3.4).

This completes the proof of Theorem 3. The proof of Theorem 4 is similarly reduced to Theorem 1 and is omitted.

4. Proof of Theorem 1

In Section 1 we constructed a formal solution of the difference equation (1.1) in the form

$$p(x,\varepsilon)=\sum_{k=1}^{\infty} p_k(x)\varepsilon^k ,$$

where the $p_k(x)$ are analytic in a region R^- which has the property that $x\in R^-$ implies $x-1\in R^-$. We may assume without loss of generality that $p_k(x+1)$ is also defined in the region R^-.

If we set

$$p(x,\varepsilon,m)=\sum_{k=1}^{m} p_k(x)\varepsilon^k ,$$
$$y(x,\varepsilon)=z(x,\varepsilon,m)+p(x,\varepsilon,m) ,$$

then equation (1.1) may be written

$$z(x+1,\varepsilon,m)=g(x,z,\varepsilon,m) , \tag{4.1}$$

where

$$\begin{aligned} g(x,z,\varepsilon,m) &= \varepsilon f(x,z+p(x,\varepsilon,m),\varepsilon)-p(x+1,\varepsilon,m) \\ &= [\varepsilon f(x,p(x,\varepsilon,m),\varepsilon)-p(x+1,\varepsilon,m)] \\ &\quad +\varepsilon[f(x,z+p(x,\varepsilon,m),\varepsilon)-f(x,p(x,\varepsilon,m),\varepsilon)] . \end{aligned}$$

Since $\|p(x,\varepsilon,m)\|=O(\varepsilon)$, for sufficiently small ρ', $g(x,z,\varepsilon,m)$ is analytic in $R^-\times Z\times E'$, where

$$Z:\ \|z\|<\delta'=\delta_0/2 ;\quad \text{and } E':\ 0<|\varepsilon|<\rho'\leqq\rho_0,\ |\arg\varepsilon|<\alpha_0 .$$

$p(x,\varepsilon,m)$ is the sum of the first m terms of a formal solution, hence

$$\varepsilon f(x,p(x,\varepsilon,m),\varepsilon)-p(x+1,\varepsilon,m)=\varepsilon^{m+1}b(x,\varepsilon,m) .$$

Thus the difference equation (4.1) can be written

$$z(x+1,\varepsilon,m)=\varepsilon^{m+1}b(x,\varepsilon,m)+\varepsilon h(x,z,\varepsilon,m) , \tag{4.2}$$

where $b(x,\varepsilon,m)$ is analytic in $R^-\times E'$ and $h(x,z,\varepsilon,m)$ is analytic in $R^-\times Z\times E'$. Further, there exist constants L_{0m} and L_{1m} such that

$$\begin{aligned} &\|b(x, \varepsilon, m)\| \leqq L_{0m}, \\ &\|h(x, z, \varepsilon, m)\| \leqq L_{1m}\|z\|. \end{aligned} \tag{4. 3}$$

We shall show, by the method of fixed points that there exists a unique analytic solution of equation (4. 2) satisfying

$$\|z(x, \varepsilon, m)\| \leqq M_m|\varepsilon|^{m+1}.$$

Let $\mathfrak{F}$ be the family of vector-valued functions $w(x, \varepsilon)$ whose components are analytic for $x \in R^-$, $\varepsilon \in E'_m$ and satisfy the inequality

$$\|w(x, \varepsilon)\| \leqq M_m|\varepsilon|^{m+1}, \tag{4. 4}$$

where E'_m: $0<|\varepsilon|<\rho'_m$, $|\arg\varepsilon|<\alpha_0$, and M_m is an arbitrary but fixed constant (not depending on w) satisfying $M_m > L_{0m}$.

$\mathfrak{F}$ is closed, compact, and convex with respect to the topology of uniform convergence on each compact subset of the region $R^- \times E'_m$. For functions $w \in \mathfrak{F}$ define the mapping T_m by

$$T_m[w](x, \varepsilon) = \varepsilon^{m+1} b(x-1, \varepsilon, m) + \varepsilon h(x-1, w(x-1, \varepsilon), \varepsilon, m). \tag{4. 5}$$

If $M_m[\rho'_m]^{m+1} \leqq \delta'$, the mapping (4. 5) is well defined. Since the mapping is continuous, if we show that $T_m[\mathfrak{F}] \subset \mathfrak{F}$ there is a fixed point which is the desired solution.

From the inequalities (4.3), for $w \in \mathfrak{F}$, we have

$$\|T_m[w]\| \leqq |\varepsilon|^{m+1}(L_{0m} + |\varepsilon| L_{1m}) \leqq M_m|\varepsilon|^{m+1}$$

if $$\rho'_m \leqq \min[(\delta'/M_m)^{1/(m+1)},\ (M_m - L_{0m})/L_{1m},\ \rho'],$$

and $T_m[\mathfrak{F}] \subset \mathfrak{F}$.

Thus there exists a solution of equation (4. 1) or (4. 2).

We shall now show that the solution is unique if ρ'_m is chosen sufficiently small. Suppose that there are two solutions of the equation (4. 2), z_1 and z_2 which satisfy

$$\|z_i(x, \varepsilon)\| \leqq M_m|\varepsilon|^{m+1},\ i = 1, 2.$$

Then

$$z_2(x+1, \varepsilon) - z_1(x+1, \varepsilon) = \varepsilon[h(x, z_2, \varepsilon) - h(x, z_1, \varepsilon)].$$

Since $h(x, z, \varepsilon)$ is analytic in $R^- \times Z \times E'_m$, if $||z_i|| < \delta'/2$, then

$$||h(x, z_2, \varepsilon) - h(x, z_1, \varepsilon)|| \leqq L_{2m}||z_2 - z_1|| .$$

Hence if $M_m[\rho'_m]^{m+1} \leqq \delta'/2$ and $\rho'_m L_{2m} \leqq \beta < 1$, then

$$||z_2(x, \varepsilon) - z_1(x, \varepsilon)|| \leqq \beta ||z_2(x-1, \varepsilon) - z_1(x-1, \varepsilon)|| , \text{ and}$$
$$\sup_{R^- \times E'_m} ||z_2(x, \varepsilon) - z_1(x, \varepsilon)|| = 0 .$$

Hence, $z_2(x, \varepsilon) \equiv z_1(x, \varepsilon)$ in $R^- \times E'_m$ and the uniqueness is established.

The existence of a solution with the desired asymptotic properties follows immediately.

5. Remarks

(1) It is clear from the proof of Theorem 1 that the more general system

$$y(x+1) = f(x, y, \varepsilon)$$

can be treated in a similar fashion if we assume only that

$$f_{00}(x) \equiv 0 , \quad A_0(x) \equiv 0 ,$$

which is, roughly speaking, equivalent to the conditions

$$f(x, 0, 0) \equiv 0 , \quad f_y(x, 0, 0) \equiv 0.$$

(2) If $f(x, y, \varepsilon)$ is analytic for $|\varepsilon| < \rho_0$, then the solutions constructed in Theorems 1–2 are analytic in ε for $|\varepsilon| < \rho_1$ and we may obtain an estimate for ρ_1 in the following manner.

Since $f(x, y, \varepsilon)$ is analytic for $x \in R_0$, $||y|| < \delta_0$, $|\varepsilon| < \rho_0$ in this case, for a suitable subregion $\hat{R} \subset R^- \subset R_0$, $\rho < \rho_0$, $\delta < \delta_0$, there exists a constant M such that each component of $f(x, y, \varepsilon)$ is majorized by $\varphi(y, \varepsilon)$,

$$f_i(x, y, \varepsilon) \ll \varphi(y, \varepsilon) \equiv M(1 - \varepsilon/\rho)^{-1}(1 - (y_1 + \cdots + y_n)/\delta)^{-1} .$$

Hence, the system of equations

$$y_i = \varepsilon \varphi(y, \varepsilon) , \quad i = 1, \cdots, n ,$$

has a formal solution $p(\varepsilon) = \sum_{k=1}^{\infty} p_k \varepsilon^k$, where the components of the vector p_k are all the same, say $\bar{p}_k$. Further, we have for the formal solution $p(x,\varepsilon)=\sum_{k=1}^{\infty} p_k(x)\ \varepsilon^k$

$$||p_k(x)|| \leqq n\bar{p}_k$$

and hence the formal solution $p(x,\varepsilon)$ will be uniformly and absolutely convergent for $|\varepsilon|<\gamma$ if the formal solution $p(\varepsilon)$ is convergent for $|\varepsilon|<\gamma$.

The equation

$$y=\varepsilon M(1-\varepsilon/\rho)^{-1}(1-ny/\delta)^{-1}$$

has a unique solution $y(\varepsilon)$ with $y(0)=0$ valid for

$$|\varepsilon|<\rho\delta(\delta+4\rho nM)^{-1}.$$

Hence, in this case

$$\rho_1 \geqq \rho(1+4nM\rho/\delta)^{-1}.$$

(3) It is clear that the parameter ε may be a periodic function of x of period one, $\varepsilon(x+1)=\varepsilon(x)$, without essential change in the results of this note.

(4) The linear system of difference equations considered in Section 3 contains as a special case the system

$$\varepsilon^{\sigma_i} y_i(x+1)=\sum_{j=1}^{n} a_{ij}(x,\varepsilon)y_j(x),\ i=1,\ \cdots,\ n\,,$$

where σ_i are integers, not all equal, and

$$\sigma_1 \leqq \sigma_2 \leqq \cdots \leqq \sigma_n\,.$$

The assumptions made are reminiscent of those made in singular perturbation non-turning point problems for differential equations.

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