On the discrete eigenvalues of the many-particle system

By

Jun UCHIYAMA*

§ 1. Introduction

Let us consider a system in a static magnetic field which consists of N electrons and M infinitely heavy nuclei. Then the Schrödinger operator in the nonrelativistic quantum mechanics becomes,

$$(1.1)\qquad H=\sum_{k=1}^{N}\left\{\sum_{\nu=0}^{2}\frac{\hbar^2}{2\mu_k}\left(i\frac{\partial}{\partial x_{3k-\nu}}+\frac{e_k}{c}A(\boldsymbol{r}_k)\right)^2-\sum_{j=1}^{M}\frac{z_j e_k}{|\boldsymbol{r}_k-\boldsymbol{a}_j|}\right\}+\sum_{\substack{k,h=1\\k<h}}^{N}\frac{e_k e_h}{|\boldsymbol{r}_k-\boldsymbol{r}_h|}.$$

Recently Jörgens [1] has shown that the essential spectrum of H is $[\mu, \infty)$, where $\mu\leq 0$, and Žislin [2] has shown that the operator of the form

$$(1.2)\qquad -\sum_{k,h=1}^{N}\sum_{\nu=0}^{2}a_{kh}\frac{\partial^2}{\partial x_{3k-\nu}\partial x_{3k-\nu}}-\sum_{k=1}^{N}\sum_{j=1}^{M}\frac{b_{kj}}{|\boldsymbol{r}_k-\boldsymbol{a}_j|}+\sum_{\substack{k,h=1\\k<h}}^{N}\frac{c_{kh}}{|\boldsymbol{r}_k-\boldsymbol{r}_h|},$$

where (a_{kh}) is a constant positive matrix and $\sum_{j=1}^{M}b_{kj}>\sum_{\substack{h=1\\h\neq k}}^{N}c_{kh}$ for each k, has a countably infinite number of discrete eigenvalues.

Making use of Žislin's method, we get the same result for a many-particle system which composes a positive ion, a neutral atom or a neutral molecule in a static magnetic field. At the same time, it will be seen how the decreasing orders at infinity

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* Department of Mathematics, Kyoto University

of the attractive potentials and the vector potentials influence the number of the discrete eigenvalues. Our result is a kind of extension of the well-known fact that the operator in $L^2(R^3)$

$$-\Delta+c(x)\,, \tag{1.3}$$

where $c(x)\leqq -\frac{1}{4-\varepsilon}\cdot\frac{1}{|r|^2}$ for $|r|\geqq R_0$ and it converges uniformly to zero when $|r|$ tends to infinity, has countably infinite discrete eigenvalues.

§ 2. Statement of the theorem

We denote by R^m the m-dimensional Euclidean space, by R_k^{3N} the 3-dimensional Euclidean space with variables $r_k=(x_{3k-2}, x_{3k-1}, x_{3k})$ and by $R_{(k)}^{3N}$ the $(3N-3)$-dimensional Euclidean space with variables $r_1, \cdots, r_{k-1}, r_{k+1}, \cdots, r_N$.

Let us consider the Schrödinger operator of the form

$$H=\sum_{k=1}^{N}\left\{\sum_{\nu=0}^{2}\left(i\frac{\partial}{\partial x_{3k-\nu}}+b_{3k-\nu}(r_k)\right)^2+q_k(r_k)\right\}+\sum_{\substack{k,h=1\\ k<h}}^{N}P_{kh}(r_k, r_h)\,. \tag{2.1}$$

For each term of this operator, we assume that

(c–1) $b_{3k-\nu}$, q_k and P_{kh} are real-valued functions,

(c–2) $b_{3k-\nu}^2(r_k)$, $q_k(r_k)$ and $\frac{\partial b_{3k-\nu}}{\partial x_{3k-\nu}}(r_k)$ belong to $L_{loc}^2(R_k^{3N})$,

(c–3) there exist some $\beta(0<\beta<2)$, $\gamma\left(0<\gamma<\frac{3}{2}\right)$, $\beta'(\max(\beta,\gamma)<\beta'<3)$, $c_k>0$, $d_{kh}>0$, $\varepsilon>0$, $R_2(0<R_2<1)$ and sufficiently large $R_0>0$, $R_1>0$ such that

$$q_k(r_k)\leqq -\frac{c_k}{|r_k|^\beta} \qquad \text{for } |r_k|\geqq R_0, \tag{2.2}$$

[1]

$$|b_{3k-\nu}(r_k)|\leqq\frac{\text{const}}{|r_k|^{\frac{\beta}{2}+\varepsilon}} \qquad \text{for } |r_k|\geqq R_0, \tag{2.3}$$

$$0\leqq P_{kh}(r_k, r_h)\begin{cases}\leqq\dfrac{d_{kh}R_1^{\beta'-\beta}R_2^{\gamma-\beta'}}{|r_k-r_h|^\gamma} & \text{for } |r_k-r_k|\leqq R_2,\\[2ex] \leqq\dfrac{d_{kh}R_1^{\beta'-\beta}}{|r_k-r_h|^{\beta'}} & \text{for } R_2\leqq|r_k-r_h|\leqq R_1,\end{cases} \tag{2.4}$$

$$\Big| \leqq \frac{d_{kh}}{|\boldsymbol{r}_k - \boldsymbol{r}_h|^{\beta}} \qquad \text{for } |\boldsymbol{r}_k - \boldsymbol{r}_h| \geqq R_1.$$ [1]

(2.5) $q_k(\boldsymbol{r}_k)$ and $\frac{\partial b_{3k-\nu}}{\partial x_{3k-\nu}}(\boldsymbol{r}_k)$ converge uniformly to zero, when $|\boldsymbol{r}_k|$ tends to infinity. And for convenience, put $P_{kh}=P_{hk}$ for $k>h$.

(c–4) $c_k > \sum_{\substack{h=1 \\ k \neq h}}^{N} d_{kh}$.

Under these assumptions we can prove the following theorems.

Theorem 1. *The Schrödinger operator H of the form* (2.1) *has the following properties;*

i) *the essential spectrum of H is $[\mu, \infty)$, where $\mu=0$ for $N=1$ or $\mu<0$ for $N \geqq 2$.*

ii) *there exist a countably infinite number of discrete eigenvalues and they have the only limit point at μ.*

Theorem 2. *When the conditions* (2.2) *and* (2.3) *are satisfied only in a cone C_k whose vertex is the origin of R_k^{3N}, then the statement of Theorem* 1 *is still true. In this case, however, we still assume that outside of the cone $b_{3k-\nu}(\boldsymbol{r}_k)$ converges uniformly to zero, when $|\boldsymbol{r}_k|$ tends to infinity.*[2]

Theorem 3. *If the condition* (c–3) *is satisfied by $\beta=2$, we have the same assertion as that of Theorem* 1 *by replacing the condition* (c–4) *by* the following one

(c–4′) $c_k - \sum_{\substack{h=1 \\ h \neq k}}^{N} d_{kh} > \frac{1}{4}$.

Remark 1. The condition (2.4) is satisfied if one takes P_{kh} to be the following:

$$P_{kh}(\boldsymbol{r}_k, \boldsymbol{r}_h) = f(\boldsymbol{r}_k - \boldsymbol{r}_h),$$

where

i) $f(\boldsymbol{r}) \geqq 0 \ (\boldsymbol{r} \in R^3)$

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1) We write $|\boldsymbol{r}_k| = \left(\sum_{\nu=0}^{2} x_{3k-\nu}^2\right)^{\frac{1}{2}}$ and $|\boldsymbol{r}_k - \boldsymbol{r}_h| = \left(\sum_{\nu=0}^{2} (x_{3k-\nu} - x_{3h-\nu})^2\right)^{\frac{1}{2}}$.

2) This supplementary condition is imposed so that Lemma 2 may hold.

ii) $f(r)=O\left(\frac{1}{|r|^{1.5-\varepsilon}}\right)$ in the neighborhood of the origin,

iii) $f(r)$ is bounded except for the neighborhood of the origin of R^3,

iv) $f(r)\leqq\frac{d_{kh}}{|r|^{\beta}}$ for $|r|\geqq R_0$.

§ 3. Some lemmas

In the first place let us introduce the spaces of functions, $C_0^\infty(R^m)$ and $\mathfrak{D}^2_{L^2}(R^m)$. The former is the space of all C^∞ functions with compact support, and the latter the completion of the space $C_0^\infty(R^m)$ with the norm

$$||f||_{2,L^2(R^m)}=\left(\sum_{i,j=1}^{m}\left\|\frac{\partial^2 f}{\partial x_i\partial x_j}\right\|^2_{L^2(R^m)}+\sum_{i=1}^{m}\left\|\frac{\partial f}{\partial x_i}\right\|^2_{L^2(R^m)}+||f||^2_{L^2(R^m)}\right)^{\frac{1}{2}}.\ [3]$$

Under the conditions (c-1)~(c-3) the following two lemmas hold. (see Jörgens [1]).[4]

Lemma 1. *If the domain of H is $\mathfrak{D}^2_{L^2}(R^{3N})$, then H is a lower-bounded selfadjoint operator in $L^2(R^{3N})$.*

Lemma 2. *The essential spectrum of H is $[\mu, \infty]$, where*

$$(3.1)\qquad \mu=\begin{cases}0, & \text{when } N=1,\\ \min\limits_{1\leqq i\leqq N}\inf\{(H_{(i)}\varphi,\varphi)_{L^2(R^{3N}_{(i)})};\ \varphi\in\mathfrak{D}^2_{L^2}(R^{3N}_{(i)}),\ ||\varphi||_{L^2(R^{3N}_{(i)})}=1\}, & \text{when } N\geqq 2,\end{cases}$$

$$(3.2)\qquad H_{(i)}=\sum_{\substack{k=1\\k\neq i}}^{N}\left\{\sum_{\nu=0}^{2}\left(i\frac{\partial}{\partial x_{3k-\nu}}+b_{3k-\nu}(r_k)\right)^2+q_k(r_k)\right\}+\sum_{\substack{k,h=1\\k<h,k,h\neq i}}^{N}P_{kh}(r_k,r_h).$$

Let us prove the following lemma.

Lemma 3. *Let $N\geqq 2$. If $\varphi\in L^2(R^{3N}_{(i)})$, $g\in C_0^\infty(R^{3N}_i)$ and $0<\gamma<3$ are given, so we have*

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3) We write $\int_{R^m}f(x)\overline{g(x)}dx=(f,g)_{L^2(R^m)}$ and $\|f\|_{L^2(R^m)}=(f,f)^{\frac{1}{2}}_{L^2(R^m)}$.

4) As for $\frac{\partial b_{3k-\nu}}{\partial x_{3k-\nu}}$ Jörgens has assumed that $\sum_{\nu=0}^{2}\frac{\partial b_{3k-\nu}}{\partial x_{3k-\nu}}=0$ in distribution sense in place of the conditions (C-2) and (2-5). But following his proof, we can easily get the same results as his under our conditions.

$$\int \frac{|\varphi g|^2}{|lr_j - r_i|^\gamma} dx \leqq \text{const.}, \quad (j \neq i).$$

where l is a real parameter and const. is some constant independent of l.[5]

PROOF. Putting $dr_i = dx_{3i-2}\, dx_{3i-1}\, dx_{3i}$ and

$$d\hat{\mathbf{r}}_i = dr_1 \cdots dr_{i-1}\, dr_{i+1} \cdots dr_N,$$

we have

$$\begin{aligned}\int \frac{|\varphi g|^2}{|lr_j - r_i|^\gamma} dx &= \int_{\mathrm{R}^{3N}_{(i)}} |\varphi|^2 d\hat{\mathbf{r}}_i \int_{|l\boldsymbol{r}_j - \boldsymbol{r}_i| \leqq 1} \frac{|g|^2}{|lr_j - r_i|^\gamma} dr_i \\ &\quad + \int_{\mathrm{R}^{3N}_{(i)}} |\varphi|^2 d\hat{\mathbf{r}}_i \int_{|l\boldsymbol{r}_j - \boldsymbol{r}_i| \geqq 1} \frac{|g|^2}{|lr_j - r_i|^\gamma} dr_i \\ &\leqq \frac{4\pi}{3-\gamma} \cdot \max_{\boldsymbol{r}_i \in \mathrm{R}^{3N}_i} |g|^2 \cdot \|\varphi\|^2_{\mathrm{L}^2(\mathrm{R}^{3N}_{(i)})} + \|g\varphi\|^2.\end{aligned}$$

Now we shall prove a lemma which is an extension of Lemma 7.1 of Žislin [2]. It plays an important role in showing the existence of the discrete eigenvalues of H.

Lemma 4. *Let $N \geqq 2$. For any functions $\varphi \in \mathrm{L}^2(\mathrm{R}^{3N}_{(i)})$, $g \in \mathrm{C}^\infty_0(\mathrm{R}^{3N}_i)$ and for $0 < \gamma < 3$, we have*

$$\lim_{l \to 0} \int \frac{|\varphi g|^2}{|lr_j - r_i|^\gamma} dx = \int \frac{|\varphi g|^2}{|r_i|^\gamma} dx \qquad (i, j = 1, \cdots, N;\ i \neq j).$$[6]

PROOF. We put $\gamma = 3 - 3\varepsilon (0 < \varepsilon < 1)$, choose M lagre enough to satisfy $\varepsilon M^{-1} < 1 - \varepsilon$ and $M \geqq \frac{1}{2}$, and then put $\theta = \varepsilon M^{-1}$.

Let $p' = (3 - \varepsilon) \cdot (3 - 3\varepsilon - \theta)^{-1}$ and $p = (3 - \varepsilon) \cdot (2\varepsilon + \theta)^{-1}$. Then p and p' satisfy the equality $\frac{1}{p} + \frac{1}{p'} = 1$ and the inequalities, $p'(3 - 3\varepsilon - \theta) < 3$, $2p\theta < 3$.

Thus by virtue of Lemma 3 and the inequalities $a^\alpha + b^\alpha \leqq \text{const}\ (a+b)^\alpha \leqq \text{const}\ (a^\alpha + b^\alpha) (\alpha > 0)$; $|a^\alpha - b^\alpha| \leqq |a - b|^\alpha (0 \leqq \alpha \leqq 1)$,

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5) Hereafter we write simply $\int f(x)dx$ in place of $\int_{\mathrm{R}^{3N}} f(x)dx$, and denote by (f, g) the integral $\int_{\mathrm{R}^{3N}} f(x)\overline{g(x)}dx$, finally $\|f\| = (f, f)^{\frac{1}{2}}$.

6) In the case $\gamma = 1$, Zislin [2] has shown this result.

we have

$$(3.3)\quad \left|\int\left\{\frac{1}{|lr_j-r_i|^{3-3\varepsilon}}-\frac{1}{|r_i|^{3-3\varepsilon}}\right\}|\varphi g|^2dx\right|$$

$$\leqq\int\left|\frac{1}{|lr_j-r_i|^{1-\varepsilon}}-\frac{1}{|r_i|^{1-\varepsilon}}\right|\left\{\frac{1}{|lr_j-r_i|^{2-2\varepsilon}}\right.$$

$$\left.+\frac{1}{|lr_j-r_i|^{1-\varepsilon}}\cdot\frac{1}{|r_i|^{1-\varepsilon}}+\frac{1}{|r_i|^{2-2\varepsilon}}\right\}|\varphi g|^2dx$$

$$\leqq\text{const}\int\left|\frac{1}{|lr_j-r_i|}-\frac{1}{|r_i|}\right|^{1-\varepsilon}\left\{\frac{1}{|lr_j-r_i|}\right.$$

$$\left.+\frac{1}{|r_i|}\right\}^{2-2\varepsilon}|\varphi g|^2dx$$

$$\leqq\text{const}\int\left|\frac{1}{|lr_j-r_i|}-\frac{1}{|r_i|}\right|^{\theta}\left\{\frac{1}{|lr_j-r_i|}\right.$$

$$\left.+\frac{1}{|r_i|}\right\}^{1-\varepsilon-\theta}\left\{\frac{1}{|lr_j-r_i|}+\frac{1}{|r_i|}\right\}^{2-2\varepsilon}|\varphi g|^2dx$$

$$\leqq\text{const}\left(\int\left|\frac{1}{|lr_j-r_i|}-\frac{1}{|r_i|}\right|^{p\theta}|\varphi g|^2dx\right)^{\frac{1}{p}}\cdot$$

$$\left(\int\left(\frac{1}{|lr_j-r_i|}+\frac{1}{|r_i|}\right)^{p'(3-3\varepsilon-\theta)}|\varphi g|^2dx\right)^{\frac{1}{p'}}$$

$$\leqq\text{const}\left(\int\left|\frac{1}{|lr_j-r_i|}-\frac{1}{|r_i|}\right|^{p\theta}|\varphi g|^2dx\right)^{\frac{1}{p}}\cdot$$

$$\left(\int\frac{|\varphi g|^2}{|lr_j-r_i|^{p'(3-3\varepsilon-\theta)}}dx+\int\frac{|\varphi g|^2}{|r_i|^{p'(3-3\varepsilon-\theta)}}dx\right)^{\frac{1}{p'}}$$

$$\leqq\text{const}\left(\int\left|\frac{1}{|lr_j-r_i|}-\frac{1}{|r_i|}\right|^{p\theta}|\varphi g|^2dx\right)^{\frac{1}{p}}.\text{[7]}$$

Now for any $\eta>0$, there exist positive constants N_η and l_η such that

$$\int_{r\geqq N_\eta}|\varphi g|^2dx<\eta^2,\quad\text{where}\quad r=\left(\sum_{k=1}^{N}|r_k|^2\right)^{\frac{1}{2}},$$

and for all $r\leqq N_\eta$ and for any l, satisfing $|l|<l_\eta$,

$$\big||r_i|-|lr_j-r_i|\big|^{p\theta}\leqq|lr_j|^{p\theta}<\eta$$

hold. Then the right member of the inequality (3.3) is estimated as follows,

7) In the sequel, const. signifies some constant independent of a parameter l.

$$
\begin{aligned}
(\text{right member})^p \leqq & \operatorname{const}\left\{\int_{r\leqq N_\eta} \frac{\big||r_i|-|lr_j-r_i|\big|^{p\theta}}{|lr_j-r_i|^p\,|r_i|^{p\theta}}|\varphi g|^2 dx \right. \\
& \left. +\int_{r\geqq N_\eta} \frac{|\varphi g|^2}{|lr_j-r_i|^{p\theta}}dx+\int_{r\geqq N_\eta}\frac{|\varphi g|^2}{|r_i|^{p\theta}}dx\right\} \\
\leqq & \operatorname{const}\left\{\eta\left(\int_{r\leqq N_\eta}\frac{|\varphi g|^2}{|lr_j-r_i|^{2p\theta}}dx\right)^{\frac{1}{2}}\left(\int_{r\leqq N_\eta}\frac{|\varphi g|^2}{|r_i|^{2p\theta}}dx\right)^{\frac{1}{2}}\right. \\
& +\left(\int_{r\geqq N_\eta}\frac{|\varphi g|^2}{|lr_j-r_i|^{2p\theta}}dx\right)^{\frac{1}{2}}\left(\int_{r\geqq N_\eta}|\varphi g|^2 dx\right)^{\frac{1}{2}} \\
& \left.+\left(\int_{r\geqq N_\eta}\frac{|\varphi g|^2}{|r_i|^{2p\theta}}dx\right)^{\frac{1}{2}}\left(\int_{r\geqq N_\eta}|\varphi g|^2 dx\right)^{\frac{1}{2}}\right\} \\
\leqq & \operatorname{const}\eta\,,
\end{aligned}
$$

which proves the assertion.

§ 4. Proof of the theorems

At first we can assume $\gamma>\frac{\beta}{2}$ in the condition (2.4), for β is smaller than 2 and γ can be chosen as close to $\frac{3}{2}$ as we require. Now we divide the proof of Theorem 1. in several steps.

1–*st step.* We shall show that the lower limit of the spectrum is a discrete eigenvalue in the case $N=1$. For this purpose we have only to show $(H\varphi,\varphi)<0$ for some function $\varphi\in\mathfrak{D}^2_{L^2}(\mathrm{R}^3)$, since the essential spectrum of H is $[0,\ \infty)$.

Now there exists some function with the properties;

(4.1) $$g_1(x_1,x_2,x_3)\in C_0(\mathrm{R}^3),\quad \|g_1\|_{L^2(\mathrm{R}^3)}=1,$$

(4.2) $$g_1(x_1,x_2,x_3)\equiv 0 \qquad \text{for } |r|=\sqrt{x_1^2+x_2^2+x_3^2}\leqq R_0\,.$$

Let $g_l(x)=l^{\frac{3}{2}}g_1(lx_1,lx_2,lx_3)$ $(1\geqq l>0)$, then we have $\|g_l\|_{L^2(\mathrm{R}^3)}=1$. Thus

(4.3) $$
\begin{aligned}
(Hg_l,g_l)\leqq & \|\,|\operatorname{grad} g_l|\,\|^2+2\sum_{\nu=1}^{3}\int\left|b_\nu g_l\frac{\partial g_l}{\partial x_\nu}\right|dx \\
& +\sum_{\nu=1}^{3}\int b_\nu^2|g_l|^2dx+(q_1g_l,g_l) \\
\leqq & \operatorname{const}(l^2+l^{1+\frac{\beta}{2}+\varepsilon}+l^{\beta+2\varepsilon})-\operatorname{const}l^\beta\,.\ ^{8)}
\end{aligned}
$$

8) $|\operatorname{grad} f|=\left(\sum_{\nu=1}^{3}\left|\frac{\partial f}{\partial x_\nu}\right|^2\right)^{\frac{1}{2}}$.

Therefore there exists some $l_0(1\geqq l_0>0)$ such that $(Hg_{l_0}, g_{l_0})<0$, which shows the assertion.

2-nd step. Let us show that the assertion in the 1-st step is also true in the case $N\geqq 2$. First ws assume that there exists at least one discrete eigenvalue for the system composed of $N-1$ particles. Then by virtue of Lemma 2 there exists an operator $H_{(i_0)}$ in $L^2(R^{3N}_{(i_0)})$ such that

$$(4.4)\quad \mu=\inf\{(H_{(i_0)}\varphi, \varphi)_{L^2(R^{3N}_{(i_0)})};\ \varphi\in\mathfrak{D}^2_{L^2}(R^{3N}_{(i_0)}),\ \|\varphi\|_{L^2(R^{3N}_{(i_0)})}=1\}\,.$$

By assumption, μ is a discrete eigenvalue of $H_{(i_0)}$. Thus there exists some function φ_0 such that

$$(4.5)\qquad \varphi_0\in\mathfrak{D}^2_{L^2}(R^{3N}_{(i_0)}),\quad \|\varphi_0\|_{L^2(R^{3N}_{(i_0)})}=1 \text{ and } H_{(i_0)}\varphi_0=\mu\varphi_0\,.$$

Putting $\psi_l=\varphi_0(\hat{\mathbf{r}}_{i_0})g_l(r_{i_0})$, we have $\psi_l\in\mathfrak{D}^2_{L^2}(R^{3N})$, $\|\psi_l\|=1$ and

$$\begin{aligned}(4.6)\qquad (H\psi_l, \psi_l)&=(H_{(i_0)}\varphi_0 g_l, \varphi_0 g_l)+\sum_{\nu=0}^{2}\Big(\Big(i\frac{\partial}{\partial x_{3i_0-\nu}}\\ &\quad+b_{3i_0-\nu}(r_{i_0})\Big)^2\varphi_0 g_l, \varphi_0 g_l\Big)+(q_{i_0}\varphi_0 g_l, \varphi_0 g_l)\\ &\quad+\sum_{\substack{k=1\\k\neq i_0}}^{N}(P_{i_0k}\varphi_0 g_l, \varphi_0 g_l)\\ &\leqq\mu+\text{const}\,(l^2+l^{1+\frac{\beta}{2}+\varepsilon}+l^{\beta+2\varepsilon})\\ &\quad-c_{i_0}l^\beta\int_{R^{3N}_{i_0}}\frac{|g_1|^2}{|r_{i_0}|^\beta}dr_{i_0}+\sum_{\substack{k=1\\k\neq i_0}}^{N}(P_{i_0k}\varphi_0 g_l, \varphi_0 g_l)\,.\end{aligned}$$

On the other hand, we have the following inequality

$$(4.7)\qquad P_{i_0k}(r_{i_0}, r_k)\leqq\frac{d_{i_0k}}{|r_{i_0}-r_k|^\beta}+\frac{d_{i_0k}R_1^{\beta'-\beta}}{|r_{i_0}-r_k|^{\beta'}}\,.$$

Therefore taking account of Lemma 4, for any $\eta>0$, there exists some $l_1(1\geqq l_1>0)$ such that

$$(4.8)\qquad (P_{i_0k}\varphi_0 g_l, \varphi_0 g_l)\leqq\text{const}\,l^{\beta'}+d_{i_0k}l^\beta\Big(\int_{R^{3N}_{i_0}}\frac{|g_1|^2}{|r_{i_0}|^\beta}dr_{i_0}+\eta\Big)$$

for any l $(l_1\geqq l>0)$.

Choosing $\eta>0$ in such a way that

$$\eta\sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k}\leqq\frac{1}{2}(c_{i_0}-\sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k})\int_{R^{3N}_{i_0}}\frac{|g_1|^2}{|r_{i_0}|^\beta}dr_{i_0}\,,$$

we have

$$(4.9)\qquad (H\psi_l,\psi_l)\leqq\mu+\text{const}(l^2+l^{1+\frac{\beta}{2}+\varepsilon}+l^{\beta+2\varepsilon}+l^{\beta'}) \\ -\frac{1}{2}\Big(c_{i_0}-\sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k}\Big)l^\beta\int_{R^{3N}_{i_0}}\frac{|g_1|^2}{|r_{i_0}|^\beta}dr_{i_0}\,.$$

Consequently we have $(H\psi_{l_2},\psi_{l_2})<\mu$ for some l_2 $(l_1\geqq l_2>0)$.

3–rd step. In the case $N\geqq 2$, taking account of Lemma 2 and the 1–st step of this proof, we have $\mu<0$.

4–th step. We shall show that there exist a countably infinite number of discrete eigenvalues. Let $N\geqq 2$. On account of the first and second steps, there exists some function φ_0 satisfying (4.5). We put $\psi_l=\varphi_0(\hat{r}_{i_0})g_l(r_{i_0})$ as before. Now we assume that there exist s discrete eigenvalues of H. Let the discrete eigenvalues be $\{\lambda_p\}_{p=1,\cdots,s}$, and the eigenfunctions $\{u_p\}_{p=1,\cdots,s}$, where they form an orthonormal system. Putting

$$(4.10)\qquad v_l=\psi_l+\sum_{p=1}^{s}\beta_l^{(p)}u_p\,,\qquad \text{where } \beta_l^{(p)}=-(\psi_l,u_p)\,,$$

we have

$$(4.11)\qquad (v_l,u_p)=0\ (p=1,\cdots,s)\ \text{and}\ \|v_l\|^2=1-\sum_{p=1}^{s}|\beta_l^{(p)}|^2\leqq 1\,.$$

Taking account of the orthonormal relation of $\{u_p\}_{p=1,\cdots,s}$, we also get

$$(4.12)\qquad (Hv_l,v_l)=(H\psi_l,\psi_l)-\sum_{p=1}^{s}\lambda_p|\beta_l^{(p)}|^2\,.$$

On the other hand, we have

$$(4.13)\qquad |\lambda_p-\mu||\beta_l^{(p)}|=|(Hu_p,\psi_l)-(u_p,H_{(i_0)}\psi_l)| \\ =\Big|\Big(u_p,\Big\{\sum_{\nu=0}^{2}\Big(i\frac{\partial}{\partial x_{3i_0-\nu}}+b_{3i_0-\nu}\Big)^2\varphi_0g_l+q_{i_0}\varphi_0g_l\Big\}\Big) \\ +\sum_{\substack{k=1\\k\neq i_0}}^{N}(u_p,P_{i_0k}\varphi_0g_l)\Big|$$

$$\leqq \sum_{\nu=0}^{?} \Big| \Big(u_p, -\varphi_0 \frac{\partial^2 g_l}{\partial x_{3i_0-\nu}^2}\Big) + \Big(i\frac{\partial u_p}{\partial x_{3i_0-\nu}}, \varphi_0 b_{3i_0-} g_l\Big)$$
$$+ \Big(b_{3i_0-} u_p, \varphi_0 i\frac{\partial g_l}{\partial x_{3i_0-\nu}}\Big) + (b_{3i_0-\nu} u_p, \varphi_0 b_{3i_0-\nu} g_l) \Big|$$
$$+ \|q_{i_0} g_l\|_{L^2(R_{i_0}^{3N})} + \sum_{\substack{k=1\\k\neq i_0}}^{N} \|P_{i_0k}\varphi_0 g_l\|$$
$$\leqq \mathrm{const}(l^2 + l^{1+\frac{\beta}{2}+\varepsilon} + l^{\beta+2\cdot} + l^{\frac{\beta}{2}+\varepsilon}) + \|q_{i_0} g_l\|_{L^2(R_{i_0}^{3N})}$$
$$+ \sum_{\substack{k=1\\k\neq i_0}}^{N} \|P_{i_0k}\varphi_0 g_l\| .$$

However we have the following inequalities

$$(4.14)\qquad 0 \leqq P_{i_0k}(r_{i_0}, r_k)$$
$$\begin{cases} \leqq \dfrac{d_{i_0k}}{|r_{i_0}-r_k|^3} + \dfrac{d_{i_0k} R_1^{\beta'-\beta} R_2^{\gamma-\beta'}}{|r_{i_0}-r_k|^{\gamma}}, & \text{when } 0<\beta<\gamma<\dfrac{3}{2}, \\ \leqq \dfrac{d_{i_0k} R_1^{\beta-\beta} R_2^{\gamma-\beta'}}{|r_{i_0}-r_k|^{\gamma}}, & \text{when } 0<\gamma\leqq\beta<2. \end{cases}$$

Then on account of Lemma 3, we have

$$(4.15)\qquad \|P_{i_0k}\varphi_0 g_l\| \leqq \mathrm{const}(l^{\gamma} + l^{\cdot}) .$$

Therefore by means of (4.6), (4.8), (4.12), (4.13) and (4.15), we have for any $l(l_1 \geqq l > 0)$

$$(4.16)\qquad (Hv_l, v_l) \leqq \mu + \mathrm{const}\{l^2 + l^{1+\frac{\beta}{2}+\varepsilon} + l^{\beta+2\varepsilon} + l^{\beta'}$$
$$+ (l^2 + l^{1+\frac{\beta}{2}+\varepsilon} + l^{\cdot+2\varepsilon} + l^{\frac{\beta}{2}+\varepsilon} + l^{\cdot} + l^{\beta})^2\}$$
$$+ (q_{i_0} g_l, g_l)_{L^2(R_{i_0}^{3N})} + K\|q_{i_0} g_l\|^2_{L^2(R_{i_0}^{3N})}$$
$$+ \sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k} l^3 \Big(\int_{R_{i_0}^{3N}} \frac{|g_1|^2}{|r_{i_0}|^{\beta}} dr_{i_0} + \eta\Big) ,$$

where K is a constant independent of l.

Since $q_{i_0}(r_{i_0}) < 0$ for $|r_{i_0}| \geqq R_0$ and $g_l(r_{i_0}) \equiv 0$ for $|r_{i_0}| \leqq \dfrac{R_0}{l}$, according to (2.5) there exists for any $\eta' > 0$ some $l_3(1 \geqq l_3 > 0)$ such that for any $l(l_3 \geqq l > 0)$

$$(4.17)\quad (q_{i_0} g_l, g_l)_{L^2(R_{i_0}^{3N})} + K\|q_{i_0} g_l\|^2_{L^2(R_{i_0}^{3N})} \leqq -c_{i_0}(1-\eta') l^{\beta} \int_{R_{i_0}^{3N}} \frac{|g_1|^2}{|r_{i_0}|^{\beta}} dr_{i_0}$$

holds. Now let us choose, at first, $\eta'>0$ in such a way that

$$c_{i_0}\eta' \leqq \frac{1}{2}(c_{i_0}-\sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k}),$$

and in the next $\eta>0$ small enough to satisfy

$$\eta(\sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k}) \leqq \frac{1}{4}(c_{i_0}-\sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k})\int_{R_{i_0}^{3N}} \frac{|g_1|^2}{|r_{i_0}|^{\beta}} dr_{i_0}.$$

Then there exists some $l_4(\min(l_1, l_3)\geqq l_4>0)$ such that

$$(4.18)\qquad (Hv_l, v_l) \leqq \mu + \text{const}\{l^2 + l^{1+\frac{\beta}{2}+\varepsilon} + l^{\beta+2\varepsilon} + l^{\beta'} + (l^2 + l^{1+\frac{\beta}{2}+\varepsilon} + l^{\beta+2\varepsilon} + l^{\frac{\beta}{2}+\varepsilon} + l^{\gamma} + l^{\beta'})^2\} - \frac{1}{4}(c_{i_0}-\sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k}) l^{\beta} \int_{R_{i_0}^{3N}} \frac{|g_1|^2}{|r_{i_0}|^{\beta}} dr_{i_0}$$

holds for any $l(l_4\geqq l>0)$. Because of $2\gamma>\beta$, there exists some $l_5(l_4\geqq l_5>0)$ such that $(Hv_{l_5}, v_{l_5})<\mu$. According to $\mu<0$ and $||v_{l_5}||^2\leqq 1$, we have $(Hv_{l_5}, v_{l_5})<u\leqq\mu||v_{l_5}||^2$. Now we put $\tilde{v}=v_{l_5}||v_{l_5}||^{-1}$. Then $\tilde{v}$ satisfies the following conditions

$$(4.19)\qquad \tilde{v}\in\mathfrak{D}^2_{L^2}(R^{3N}),\ ||\tilde{v}||=1,\ (\tilde{v}, u_p)=0\ (p=1, \cdots, s) \text{ and } (H\tilde{v}, \tilde{v})<\mu,$$

which proves our assertion.

In the case $N=1$, we have only to make use of g_l and follow the above method.

PROOF OF THEOREM 2. In the above reasoning, we have only to take as g_1 such a function that its support is contained in the cone.

Before proving Theorem 3, we shall show the following lemma.

Lemma 5. *For any* $\tilde{\varepsilon}>0$, *there exists some function* $\varphi(x)\in C_0^{\infty}(R^3)$ $(\varphi(x)\not\equiv 0)$ *such that it is identically zero in the unit ball having its center at the origin and satisfies the inequality*

$$\int_{R^3} \frac{|\varphi(x)|^2}{|r|^2} dx \geqq (4-\tilde{\varepsilon}) \int_{R^3} |\text{grad } \varphi|^2 dx \text{ .}^{9)}$$

PROOF. Take a real-valued function $\rho(r)$ with the following properties,

$$\rho(r) = \begin{cases} 0, & \left(\frac{1}{2} \leqq r \leqq 1\right) \\ 1, & \left(2 \leqq r \leqq \frac{5}{2}\right) \end{cases}, \quad \rho(r) \in C^\infty\left(\frac{1}{2} < r < \frac{5}{2}\right).$$

Now we define $\zeta_n(r) \in C_0^\infty(0 \leqq r < \infty)$ and $\varphi_n(x) \in C_0^\infty(R^3)$ as follows,

$$\zeta_n(r) = \begin{cases} 0, & (0 \leqq r \leqq 1) \\ \rho(r), & (1 \leqq r \leqq 2) \\ 1, & (2 \leqq r \leqq 2n), \\ \rho\left(4-\frac{r}{n}\right), & (2n \leqq r \leqq 3n) \\ 0, & (r \geqq 3n) \end{cases}$$

$$\varphi_n(x) = \frac{\zeta_n(r)}{\sqrt{r}}, \text{ where } r = |r| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Then we have

$$|\text{grad } \varphi_n|^2 = \left|\frac{d\varphi_n}{dr}\right|^2 = \frac{1}{4} \cdot \frac{\zeta_n(r)^2}{r^3} - \frac{\zeta_n(r) \cdot \zeta_n'(r)}{r^2} + \frac{\zeta_n'(r)^2}{r} .$$

Thus

$$\int_{R^3} \frac{|\varphi_n(x)|^2}{r^2} dx = \int_{R^3} \frac{\zeta_n(r)^2}{r^3} dx = 4 \int_{R^3} |\text{grad } \varphi_n|^2 dx - 16\pi \int_1^{3n} \zeta_n'(r)^2 \cdot r dr .$$

On the other hand, we have

$$\int_1^{3n} \zeta_n'(r)^2 r dr = \int_1^2 \rho'(r)^2 r dr + \int_{2n}^{3n} \rho'\left(4-\frac{r}{n}\right)^2 \cdot \frac{r}{n^2} dr$$
$$= \int_1^2 \rho'(r)^2 r dr + \int_2^3 \rho'(4-r)^2 r dr ,$$

and

9) Taking account of the well-known inequality

$$\int_{R^3} \frac{|\varphi(x)|^2}{|r|^2} dx \leqq 4 \int_{R^3} |\text{grad } \varphi|^2 dx \quad (\varphi \in C_0^\infty(R^3)),$$

Lemma 4 shows that in the above estimate the constant 4 is best.

$$\int_{R^3}\frac{|\varphi_n(x)|^2}{r^2}dx=4\pi\int_0^\infty\frac{\zeta_n(r)^2}{r}dr\geqq 4\pi\int_2^{2n}\frac{dr}{r}=4\pi\log n\,.$$

Then choosing n_0 sufficiently large, we have

$$\int_{R^3}\frac{|\varphi_{n_0}(x)|^2}{r^2}dx$$
$$=4\left\{1+16\pi\int_1^{3n_0}\zeta'_{n_0}(r)^2 r dr\left(\int_{R^3}\frac{|\varphi_{n_0}(x)|^2}{r^2}dx\right)^{-1}\right\}^{-1}\int_{R^3}|\text{grad}\,\varphi_{n_0}(x)|^2dx$$
$$\geqq(4-\tilde{\varepsilon})\int_{R^3}|\text{grad}\,\varphi_{n_0}|^2dx,$$

which shows that $\varphi_{n_0}(x)\in C_0^\infty(R^3)$ satisfies the property required.

Now we shall prove Theorem 3.

Proof. We can choose $4>\tilde{\varepsilon}>0$ small enough to satisfy

$$c_{i_0}-\sum_{\substack{k=1\\k\neq i_0}}^{N}d_{i_0k}-\frac{1}{4-\tilde{\varepsilon}}>0\,.$$

Then we take $\varphi(x)\in C_0^\infty(R^3)$ satisfying Lemma 5 for $\tilde{\varepsilon}>0$ and put

$$g_1(x_1,x_2,x_3)=R_0^{-\frac{3}{2}}\cdot\|\varphi\|_{L^2(R^3)}^{-1}\cdot\varphi\left(\frac{x_1}{R_0},\frac{x_2}{R_0},\frac{x_3}{R_0}\right),$$

which satisfies (4. 1) and (4. 2). Let $g_l(x)=l^{\frac{3}{2}}\,g_1(lx_1,lx_2,lx_3)$ as before. Now we shall follow the proof of Theorem 1. By means of Lemma 5, we have the following inequality in place of (4. 3)

$$(4.3')\quad (Hg_l,g_l)\leqq\text{const}(l^{2+\varepsilon}+l^{2+2\varepsilon})-c_{i_0}l^2\int_{R^3}\frac{|g_1(x)|^2}{|r|^2}dx$$
$$+l^2\int_{R^3}|\text{grad}\,g_1|^2dx$$
$$\leqq\text{const}(l^{2+\varepsilon}+l^{2+2\varepsilon})-\left(c_{i_0}-\frac{1}{4-\tilde{\varepsilon}}\right)l^2\int_{R^3}\frac{|g_1(x)|^2}{|r|^2}dx,$$

which shows that the assertion of the 1-st step in the proof of Theorem 1 is also true in this case.

Secondly by virtue of Lemma 5 and (4. 8), we have the following estimate in place of (4. 6).

$$(4.6') \quad (H\psi_l, \psi_l) \leqq \mu + \mathrm{const}(l^{2+\varepsilon} + l^{2+2\varepsilon} + l^{\beta'}) + \Big\{\int_{\mathrm{R}_{i_0}^{3N}} |\mathrm{grad}\, g_1|^2 dr_{i_0} - c_{i_0}\int_{\mathrm{R}_{i_0}^{3N}} \frac{|g_1|^2}{|r_{i_0}|^2} dr_{i_0} + \sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k}\Big(\int_{\mathrm{R}_{i_0}^{3N}} \frac{|g_1|^2}{|r_{i_0}|^2} dr_{i_0} + \eta\Big)\Big\} l^2$$

$$\leqq \mu + \mathrm{const}(l^{2+\varepsilon} + l^{2+2\varepsilon} + l^{\beta'}) - \Big\{c_{i_0} - \sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k} - \frac{1}{4-\tilde{\varepsilon}}\Big\} l^2 \int_{\mathrm{R}_{i_0}^{3N}} \frac{|g_1|^2}{|r_{i_0}|^2} dr_{i_0} + \eta\Big(\sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k}\Big) l^2 .$$

Then choosing $\eta > 0$ in such a way that

$$\eta \sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k} \leqq \frac{1}{2}\Big(c_{i_0} - \sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k} - \frac{1}{4-\tilde{\varepsilon}}\Big)\int_{\mathrm{R}_{i_0}^{3N}} \frac{|g_1|^2}{|r_{i_0}|^2} dr_{i_0} ,$$

we have

$$(4.9') \quad (H\psi_l, \psi_l) \leqq \mu + \mathrm{const}(l^{2+\varepsilon} + l^{2+2\varepsilon} + l^{\beta'}) - \frac{1}{2}\Big(c_{i_0} - \sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k} - \frac{1}{4-\tilde{\varepsilon}}\Big)\int_{\mathrm{R}_{i_0}^{3N}} \frac{|g_1|^2}{|r_{i_0}|^2} dr_{i_0} ,$$

which shows the assertion of the second step in the proof of Theorem 1 is also true in this case.

Last of all, taking account of Lemma 5, (4.8) and (4.17), we have the following inequality in place of (4.16).

$$(4.16') \quad (Hv_l, v_l) \leqq \mu + \mathrm{const}\{l^{2+\varepsilon} + l^{2+2\varepsilon} + l^{\beta'} + (l^2 + l^{2+\varepsilon} + l^{2+2\varepsilon} + l^{1+\varepsilon} + l^{\gamma})^2\} + \Big\{\int_{\mathrm{R}_{i_0}^{3N}} |\mathrm{grad}\, g_1|^2 dr_{i_0} - c_{i_0}(1-\eta')\int_{\mathrm{R}_{i_0}^{3N}} \frac{|g_1|^2}{|r_{i_0}|^2} dr_{i_0} + \sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k}\Big(\int_{\mathrm{R}_{i_0}^{3N}} \frac{|g_1|^2}{|r_{i_0}|^2} dr_{i_0} + \eta\Big)\Big\} l^2$$

$$\leqq \mu + \mathrm{const}\{l^{2+\varepsilon} + l^{2+2\varepsilon} + l^{\beta'} + (l^2 + l^{2+\varepsilon} + l^{2+2\varepsilon} + l^{1+\varepsilon} + l^{\gamma})^2\} - \Big(c_{i_0} - \sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k} - \frac{1}{4-\tilde{\varepsilon}}\Big) l^2 \int_{\mathrm{R}_{i_0}^{3N}} \frac{|g_1|^2}{|r_{i_0}|^2} dr_{i_0} + c_{i_0}\eta' l^2 \int_{\mathrm{R}_{i_0}^{3N}} \frac{|g_1|^2}{|r_{i_0}|^2} dr_{i_0} + \Big(\sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k}\Big)\eta l^2 .$$

Then choosing $\eta' > 0$ and $\eta > 0$ to satisfy

$$c_{i_0}\eta' \leqq \frac{1}{2}\Big(c_{i_0} - \sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k} - \frac{1}{4-\tilde{\varepsilon}}\Big) \text{ and}$$

$$\eta \sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k} \leqq \frac{1}{4}\Big(c_{i_0} - \sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k} - \frac{1}{4-\tilde{\varepsilon}}\Big)\int_{R^{3N}_{0l}} \frac{|g_1|^2}{|r_{i_0}|^2} dr_{i_0},$$

we have

$$(4.18')\quad (Hv_l, v_l) \leqq \mu + \text{const}\{l^{2+\varepsilon} + l^{2+2\varepsilon} + l^{\beta'} + (l^2 + l^{2+\varepsilon} + l^{2+2\varepsilon} + l^{1+\varepsilon} + l^{\gamma})^2\}$$
$$- \frac{1}{4}\Big(c_{i_0} - \sum_{\substack{k=1\\k\neq i_0}}^{N} d_{i_0k} - \frac{1}{4-\tilde{\varepsilon}}\Big) l^2 \int_{R^{3N}_{l0}} \frac{|g_1|^2}{|r_{i_0}|^2} dr_{i_0},$$

which shows that the same assertion as that of the 4-th step in the proof of Theorem 1 is true in our case.

Remark 2. In the case when the masses of the particles of the system are not all the same, applying a suitable linear transformation such that $r_k' = \frac{1}{\omega_k} r_k (\omega_k > 0)$, we see easily that we have only to consider the system under the following assumption. Namely, we replace the conditions (2.2) and (2.4) by the conditions

(C-3′)

$$(2.2')\qquad q_k(r_k) \leqq -\frac{c_k}{|\omega_k r_k|^{\beta}} \qquad \text{for } |r_k| \geqq R_0,$$

$$(2.4')\quad 0 \leqq P_{kh}(r_k, r_h) \begin{cases} \leqq \dfrac{d_{kh} R_1^{\beta'-\beta} R_2^{\gamma-\beta'}}{|\omega_k r_k - \omega_h r_h|^{\gamma}} & \text{for } |\omega_k r_k - \omega_h r_h| \leqq R_2, \\ \leqq \dfrac{d_{kh} R_1^{\beta'-\beta}}{|\omega_k r_k - \omega_h r_h|^{\beta'}} & \text{for } R_2 \leqq |\omega_k r_k - \omega_h r_h| \leqq R_1, \\ \leqq \dfrac{d_{kh}}{|\omega_k r_k - \omega_h r_h|^{\beta}} & \text{for } |\omega_k r_k - \omega_h r_h| \geqq R_1. \end{cases}$$

Then we can get the same results as that of theorem 1. And in this case replacing (C-4′) by the condition

$$(\text{C-4}'')\qquad c_k - \sum_{\substack{h=1\\h\neq k}}^{N} d_{kh} > \frac{1}{4}\cdot\omega_k^2,$$

we can easily obtain the same result as that of Theorem 3.

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