Some extensions of Cartan-Behnke-Stein's theorem

By

Joji KAJIWARA

Introduction

Oka [15] proved that any domain of holomorphy in C^n is a Cousin-I domain, that is, a domain in which any additive Cousin's distribution has a solution. Conversely Cartan [5] had stated that any Cousin-I domain in C^2 is a domain of holomorphy and Behnke-Stein [2] gave its proof. From Cartan [6] $C^3-\{(0,0,0)\}$ is a Cousin-I domain which is not a domain of holomorphy and from Thullen [18] $C^2-\{(0,0)\}$ is not a domain of holomorphy but a Cousin-II domain, that is, a domain in which any multiple Cousin's distribution has a solution. These two facts suggest that Cartan-Behnke-Stein's theorem can not be generalized for Cousin-I domain in $C^n(n\geqq 3)$ and for Cousin-II domain in $C^n(n\geqq 2)$ directly. The main purpose of the present paper is to extend Cartan-Behnke-Stein's theorem in the following form:

Let L be an abelian complex Lie group and $\mathfrak{A}_L$ be the sheaf of all germs of holomorphic mappings in L. A domain (D,φ) over C^2 with $\mathrm{H}^1(D,\mathfrak{A}_L)=0$ is a domain of holomorphy. If D is a domain in C^n with continuous boundary such that $\mathrm{H}^1(D\cap P,\mathfrak{A}_L)=0$ for any simply connected and relatively compact polycylinder P in C^n, D is a domain of holomorphy. Moreover these two results hold also for $L=\mathrm{GL}(p,C)$.

Received April 28, 1966.
Communicated by S. Nakano.
Dedicated to Professor Wasao Sibagaki on his sixtieth birthday

§ 1. Relation between domains of holomorphy and domains D with ${}^{\#}\mathrm{H}^1(D, \mathfrak{A}_L)=0$.

Lemma. 1. *Let $\mathfrak{F}$ be a sheaf of groups over a topological space X and $\mathfrak{U}$ be an open covering of X. Then the canonical mapping* $\mathrm{H}^1(\mathfrak{U}, \mathfrak{F}) \rightarrow \mathrm{H}^1(X, \mathfrak{F})$ *is injective.*

Proof. Suppose we have two cocycles $\{f_{ij}\}$, $\{g_{ij}\} \in Z^1(\mathfrak{U}, \mathfrak{F})$ and suppose that there exist an open covering $\mathfrak{V}=\{V_k;\, k\in K\}$, a cocycle $\{f_k\}\in \mathrm{C}^0(\mathfrak{V}, \mathfrak{F})$ and a mapping $\rho: K \rightarrow I$ satisfying the following conditions:

(1) $V_k \subset U_{\rho(k)}$, for any $k\in K$.

(2) $f_k^{-1} f_{\rho(kl)} f_l = g_{\rho(kl)}$ in any $V_k \cap V_l \neq \phi$ where $\rho(kl)=\rho(k)\rho(l)$.

(3) ρ is surjective.

For any $i\in I$ we shall consider a fixed $k\in K$ with $\rho(k)=i$. For any $x\in U_i$ there exists $l\in K$ such that $x\in V_l \in \mathfrak{V}$. We put $\rho(l)=j$. If we put

$$F_i(x)=f_{ij}(x) f_l(x) (g_{ij}(x))^{-1}$$

in $U_i \cap V_l$, $\{F_i\}\in \mathrm{C}^0(\mathfrak{U}, \mathfrak{F})$ is well-defined and satisfies

$$F_i^{-1} f_{ij} F_j = g_{ij}$$

in $U_i \cap U_j \neq \phi$.

Lemma 2. *Let L be an abelian complex Lie group and L_e be the connected component of L containing the neutral element e. Then for a connected complex manifold D the canonical mapping* $\mathrm{H}^1(D, \mathfrak{A}_{L_e}) \rightarrow \mathrm{H}^1(D, \mathfrak{A}_L)$ *is injective.*

Proof. Let $\mathfrak{U}=\{U_i;\, i\in I\}$ be an open covering of D such that each $U_i\in\mathfrak{U}$ is connected. Suppose that a cocycle $\{f_{ij}\}\in Z^1(\mathfrak{U}, \mathfrak{A}_{L_e})$ is a coboundary of $\{f_i\}\in \mathrm{C}^0(\mathfrak{U}, \mathfrak{A}_L)$.

We take a fixed $i_0\in I$ and a fixed $x_0\in U_{i_0}$. If we put

$$g_i(x)=f_i(x)(f_{i_0}(x_0))^{-1}$$

in U_i, $\{f_{ij}\}\in Z^1(\mathfrak{U}, \mathfrak{A}_{L_e})$ is a coboundary of $\{g_i\}\in \mathrm{C}^0(\mathfrak{U}, \mathfrak{A}_{L_e})$.

If there exists a local biholomorphic mapping φ of a complex manifold D in a complex manifold M, (D, φ) is called an *open* *set over* M. Moreover, if D is a connected complex manifold, (D, φ) is called a *domain over* M. Let (D_1, φ_1) and (D_2, φ_2) be open sets over M. A holomorphic mapping λ of D_1 in D_2 is called a *mapping of* (D_1, φ_1) *in* (D_2, φ_2) if $\varphi_1 = \varphi_2 \circ \lambda$. A complex manifold with vanishing fundamental group is called *simply connected*. Let (D, φ) be a domain over M and $\mathfrak{F}$ be a family of holomorphic functions on D. A triple $(\lambda, \tilde{D}, \tilde{\varphi})$, or shortly a pair $(\tilde{D}, \tilde{\varphi})$, is called an *envelope of holomorphy of* (D, φ) *with respect to* $\mathfrak{F}$ if the following conditions are satisfied:

(1) λ is a mapping of (D, φ) in $(\tilde{D}, \tilde{\varphi})$.

(2) For any $f \in \mathfrak{F}$ there exists a holomorphic function $\tilde{f}$ on $\tilde{D}$ with $f = \tilde{f} \circ \lambda$. (In this case $\tilde{f}$ is called an *analytic continuation of* f *to* $(\lambda, \tilde{D}, \tilde{\varphi})$).

(3) For any (λ', D', φ') satisfying (1) and (2) (such is called an *analytic completion of* (D, φ) *with respect to* $\mathfrak{F}$) there exists a mapping ψ *of* (D', φ') in $(\tilde{D}, \tilde{\varphi})$ such that $(\psi, \tilde{D}, \tilde{\varphi})$ is an analytic completion of (D', φ') with respect to the family of all analytic continuations of functions of $\mathfrak{F}$.

Cartan [7] proved the unique existence of such envelope of holomorphy. Especially if $\mathfrak{F}$ consists of only one holomorphic function f on D, the envelope of holomorphy of (D, φ) with respect to $\mathfrak{F}$ is called a *domain of holomorphy of* f. A domain over M which is a domain of holomorphy of a holomorphic function on a domain over M is called shortly a *domain of holomorphy*. Moreover if $\mathfrak{F}$ is the family of all holomorphic functions on D, the envelope of holomorphy of (D, φ) with respect to $\mathfrak{F}$ is called shortly the *envelope of holomorphy of* (D, φ).

Let (D, φ) be a domain over M. A triple (D', φ', λ) is called a *covering domain of* (D, φ) if the following condition is satisfied:

λ is a mapping of (D', φ') on (D, φ) and for any point x of D there exists a neighbourhood U of x such that λ maps each connected component of $\lambda^{-1}(U)$ biholomorphically onto U.

Sometimes we also say that (D', λ) is a covering domain of D. A covering domain $(D^{\#}, \varphi^{\#}, \lambda)$ of (D, φ) satisfying the following condition is called a *universal covering domain of* (D, φ):

For any covering domain (D', φ', λ') of (D, φ) there exists a mapping μ such that $(D^{\#}, \varphi^{\#}, \mu)$ is a covering domain of (D', φ').

The universal covering domain of a domain over M exists uniquely and a covering domain (D', φ', λ) of (D, φ) is a universal covering domain of (D, φ) if and only if D' is simply connected.

Now let (D, φ) be a domain over M and $(D^{\#}, \varphi^{\#}, \lambda)$ be its universal covering domain. λ induces canonically a mapping $\lambda^{*}: \mathrm{H}^1(D, \mathfrak{A}_L) \rightarrow \mathrm{H}^1(D^{\#}, \mathfrak{A}_L)$ for a complex Lie group L.

$$^{\#}\mathrm{H}^1(D, \mathfrak{A}_L) = \lambda^{*}(\mathrm{H}^1(D, \mathfrak{A}_L))$$

is a subgroup of $\mathrm{H}^1(D^{\#}, \mathfrak{A}_L)$ if L is abelian. For $\alpha \in \mathrm{H}^1(D, \mathfrak{A}_L)$ we put

$$^{\#}\alpha = \lambda^{*}(\alpha) \in \mathrm{H}^1(D^{\#}, \mathfrak{A}_L)$$

and use these notations frequently hereafter.

Lemma 3. *Let L be a p-dimensional abelian complex Lie group, (D, φ) be a domain over C^n with $^{\#}\mathrm{H}^1(D, \mathfrak{A}_L) = 0$ and $(D^{\#}, \varphi^{\#}, \lambda)$ be the universal covering domain of (D, φ). Then for any $(n-1)$-dimensional analytic plane H and for any holomorphic function u on $\varphi^{-1}(H)$, there exists a holomorphic function F on $D^{\#}$ such that $F = u \circ \lambda$ in $\varphi^{\#-1}(H) = \lambda^{-1}(\varphi^{-1}(H))$.*

Proof. From Lemma 2 we may assume that L is connected. As well known, there exists a homomorphism χ of the additive group C^p on L such that (C^p, χ) is a covering domain of L. In this case the kernel N of χ is a discrete subgroup of C^p isomorphic with the fundamental group of L. There exists a holomorphic function u' on a neighbouhood V of $\varphi^{-1}(H)$ such that $u' = u$ in $\varphi^{-1}(H)$. If we put $W = D - \varphi^{-1}(H)$, $\{V, W\}$ is an open covering of D. We consider

$$\chi\left(\frac{u'}{z_1\circ\varphi}, 0, \cdots, 0\right)\in H^0(V\cap W, \mathfrak{A}_L).$$

As ${}^{\#}\mathrm{H}^1(D, \mathfrak{A}_L)=0$, from Lemma 1 there exist $A\in \mathrm{H}^0(\lambda^{-1}(V), \mathfrak{A}_L)$ and $B\in \mathrm{H}^0(\lambda^{-1}(W), \mathfrak{A}_L)$ such that

$$\chi\left(\frac{u'\circ\lambda}{z_1\circ\varphi^{\#}}, 0, \cdots, 0\right)=AB^{-1}$$

in $\lambda^{-1}(V\cap W)$. Since (C^p, χ) is a covering domain of L, there exist, respectively, vector-valued holomorphic functions $a=(a_1, a_2, \cdots, a_p)$ and $b=(b_1, b_2, \cdots, b_p)$ on a simply connected subdomain of $\lambda^{-1}(V\cap W)$ with $A=\chi(a)$ and $B=\chi(b)$. Since χ is a homomorphism, for a suitable choice of a and b there holds

$$(z_1\circ\varphi^{\#})a=(u'\circ\lambda, 0, \cdots, 0)+(z_1\circ\varphi^{\#})b\,.$$

Moreover a and b can be analytically continued along any curve in $\lambda^{-1}(V)$ and $\lambda^{-1}(W)$ respectively. If we denote their analytic continuations by the same symbols a and b, for any simply connected subdomain E of $\lambda^{-1}(V\cap W)$ there exists a constant $c(E)\in N$ such that

$$(z_1\circ\varphi^{\#})a=(u'\circ\varphi^{\#}, 0, \cdots, 0)+(z_1\circ\varphi^{\#})b+(z_1\circ\varphi^{\#})c(E)$$

in E. This means that $(z_1\circ\varphi^{\#})a_1$ can be analytically continued along any curve in $D^{\#}$. Since $D^{\#}$ is simply connected, from the principle of monodromy this gives a holomorphic function F on $D^{\#}$. Again from the above equation we have

$$F=u\circ\lambda$$

in $\varphi^{\#-1}(H)$.

Proposition 1. *Let* (D, φ) *be a domain over* C^2. *If* ${}^{\#}\mathrm{H}^1(D, \mathfrak{A}_L)=0$ *for an abelian complex Lie group* L, (D, φ) *is a domain of holomorphy.*

Proof. The set E of all points of ∂D which is a boundary point of $\varphi^{-1}(H)$ for some analytic plane H is dense in ∂D. Since $\varphi^{-1}(H)$ is holomorphically convex from Behnke-Stein [3], there exists a holomorphic function u on $\varphi^{-1}(H)$ which is unbounded at x for any boundary point x of $\varphi^{-1}(H)$. From Lemma 3 there exists a holomorphic function F on D which is unbounded at boundary points of $\partial D^{\#}$ over x. Since E is dense in ∂D, each boundary point of $D^{\#}$ has the frontier property in the sense of Bochner-Martin [4]. There exists a holomorphic function G on $D^{\#}$ which is unbounded at each point of $\partial D^{\#}$. Since $(D^{\#}, \varphi^{\#})$ is a covering domain of the domain (D', φ') of holomorphy of G, (D', φ') is a covering domain of (D, φ). Making use of Oka [16] or Stein [17] we can prove that a domain over C^n is a domain of holomorphy if and only if its covering domain is a domain of holomorphy. Hence (D, φ) is a domain of holomorphy.

Corollary to Proposition 1. *Let L be a p-dimensional connected abelian complex Lie group, $\pi(L)$ be its fundamental group and (D, φ) be a domain over C^2. Then $\mathrm{H}^1(D, \mathfrak{A}_L)=0$ if and only if (D, φ) is a domain of holomorphy with $\mathrm{H}^2(D, \pi(L))=0$.*

Proof. If $\mathrm{H}^1(D, \mathfrak{A}_L)=0$, (D, φ) is a domain of holomorphy from Proposition 1. Since $\mathrm{H}^1(D, \mathfrak{O}^p)=\mathrm{H}^2(D, \mathfrak{O}^p)=0$ in the canonical exact sequence

$$\mathrm{H}^1(D, \mathfrak{O}^p)\to\mathrm{H}^1(D, \mathfrak{A}_L)\to\mathrm{H}^2(D, \pi(L))\to\mathrm{H}^2(D, \mathfrak{O}_p)$$

where $\mathfrak{O}^p=\mathfrak{A}_{C^p}$, we have $\mathrm{H}^2(D, \pi(L))=0$. Conversely if (D, φ) is a domain of holomorphy with $\mathrm{H}^2(D, \pi(L))=0$, we have $\mathrm{H}^1(D, \mathfrak{A}_L)=0$ from the above exact sequence.

Making use of Lemma 3 we can prove the following proposition by induction with respect to n similarly to Prosition 1.

Proposition 2. *A domain (D, φ) over C^n which satisfies ${}^{\#}\mathrm{H}^1(\varphi^{-1}(H), \mathfrak{A}_L)=0$ for any analytic plane H represented by $H=\{z\,;\, z_{q_1}=a_1, z_{q_2}=a_2, \cdots, z_{q_{n-m}}=a_{n-m}\}$ $(2\leqq m\leqq n)$ is a domain of holomorphy.*

§ 2. Approximation of holomorphic functions

A collection $\mathfrak{S}=\{(D_n, \varphi_n)\,;\, \tau_m^n\}$ is called a *monotonously increasing sequence of domains over* a complex manifold M if the following conditions are satisfied:

(1) Each (D_n, φ_n) is a domain over M.

(2) Each $\tau_m^n (n \leqq m)$ is a mapping of (D_n, φ_n) in (D_m, φ_m) with $\tau_l^n = \tau_l^m \circ \tau_m^n$ for $n \leqq m \leqq l$.

A triple (τ_n, D, φ), or shortly a pair (D, φ), is called a *limit of* $\mathfrak{S}$ if the following conditions are satisfied:

(1) (D, φ) is a domain over M.

(2) Each τ_n is a mapping of (D_n, φ_n) in (D, φ) with $\tau_n = \tau_m \circ \tau_m^n$ for $n \leqq m$.

(3) If (τ_n', D', φ') satisfies (1) and (2), there exists a mapping ψ of (D, φ) in (D', φ') *with* $\tau_n' = \psi \circ \tau_n$ for any n.

As we stated in [11] the limit of $\mathfrak{S}$ exists uniquely and has the following property:

Lemma 4. *Let* (τ_n, D, φ) *be the limit of a monotonously increasing sequence* $\{(D_n, \varphi_n); \tau_m^n\}$ *of domains over a complex manifold M. Then for any compact set K in D there exist an integer m and a compact set K′ in* D_m *such that* τ_m *maps K′ biholomorphically on K.*

Let τ be a mapping of a domain (D_1, φ_1) over a Stein manifold S in a domain (D_2, φ_2) over S, $(\lambda_1, \tilde{D}_1, \tilde{\varphi}_1)$ and $(\lambda_2, \tilde{D}_2, \tilde{\varphi}_2)$ be, respectively, the envelopes of holomorphy of (D_1, φ_1) and (D_2, φ_2). As we remarked in [11], there exists a mapping $\tilde{\tau}$ of $(\tilde{D}_1, \tilde{\varphi}_1)$ in $(\tilde{D}_2, \tilde{\varphi}_2)$ with $\tilde{\tau} \circ \lambda_1 = \lambda_2 \circ \tilde{\tau}$. $\tilde{\tau}$ is called an *analytic continuation of* τ *to* $(\lambda, \tilde{D}_1, \tilde{\tau}_1)$.

We have the following Lemma as we remarked in [11].

Lemma 5. *Let* (τ_n, D, φ) *be a limit of a monotonously increasing sequence* $\{(D_n, \varphi_n); \tau_m^n\}$ *of domains over a Stein manifold S,* $(\lambda_n, \tilde{D}_n, \tilde{\varphi}_n)$ *and* $(\lambda, \tilde{D}, \tilde{\varphi})$ *be, respectively, the envelopes of holomorphy of* (D_n, φ_n) *and* (D, φ), $\tilde{\tau}_m^n$ *and* $\tilde{\tau}_n$ *be, respectively, the analytic continuations of* τ_m^n *and* τ_n *to* $(\lambda_n, \tilde{D}_n, \tilde{\varphi}_n)$. *Then* $\{(\tilde{D}_n, \tilde{\varphi}_n); \tilde{\tau}_m^n\}$ *is a monotonously increasing sequence of domains over S and* $(\tilde{\tau}_n, \tilde{D}, \tilde{\varphi})$ *is its limit.*

Let τ be a mapping of a domain (D_1, φ_1) over a complex manifold M in a domain (D_2, φ_2) *over* M, $(D_1^\#, \varphi_1^\#, \lambda_1)$ and $(D_2^\#, \varphi_2^\#, \lambda_2)$ be, respectively, the universal covering domains of (D_1, φ_1) and (D_2, φ_2). Then there exists a mapping $\tau^\#$ of $(D_1^\#, \varphi_1^\#)$ in $(D_2^\#, \varphi_2^\#)$ with $\tau\circ\lambda_1=\lambda_2\circ\tau^\#$. $\tau^\#$ is called a *mapping of* $(D_1^\#, \varphi_1^\#)$ *in* $(D_2^\#, \varphi_2^\#)$ *associated to the mapping* τ. We have the following Lemma.

Lemma 6. *Let* (τ_n, D, φ) *be the limit of a monotonously increasing sequence* $\{(D_n, \varphi_n); \tau_m^n\}$ *of domains over a complex mainfold* M, $(D_n^\#, \varphi_n^\#, \lambda_n)$ *and* $(D^\#, \varphi^\#, \lambda)$ *be, respectively, the universal covering domains of* (D_n, φ_n) *and* (D, φ), $\tau_m^{n\#}$ *and* $\tau_n^\#$ *be, respectively, the mappings of* $(D_n^\#, \varphi_n^\#)$ *in* $(D_m^\#, \varphi_m^\#)$ *and* $(D^\#, \varphi^\#)$ *associated to the mappings* τ_m^n *and* τ_n. *Then* $\{(D_n^\#, \varphi_n^\#); \tau_m^{n\#}\}$ *is a monotonously increasing sequence of domains over* M *and* $(\tau_n^\#, D^\#, \varphi^\#)$ *is its limit.*

Under these preperations we shall give the following Lemma.

Lemma 7. *Let* $\{(D_n, \varphi_n), \tau_m^n\}$ *be a monotonously increasing sequence of domains over a Stein manifold* S *and* (τ_n, D_n, φ_n) *be its limit. Let* $\{K_n\}$ *be a sequence of compact subsets of* D *such that* $K_n\subset K_{n+1}$ *and* $D=\bigcup_{n=1}^{\infty} K_n$. *Then there exists a subsequence* $\{\nu_n\}$ *of* $\{1, 2, 3, \cdots\}$ *with the following properties*:

(1) *There exists a compact subset* K_n' *of* D_{ν_n} *such that the restriction* $\tau_{\nu_n}|K_n'$ *of* τ_{ν_n} *to* K_n *is a biholomorphic mapping of* K_n' *onto* K_n.

(2) *For any* $f_n\in \mathrm{H}_0(D_{\nu_n}, \mathfrak{O})$ *and* $\varepsilon_n>0$ *there exists* $F_n\in \mathrm{H}^0(D, \mathfrak{O})$ *such that* $|F_n\circ\tau_{\nu_n}-f_n|<\varepsilon_n$ *in* K_{ν_n}'.

Proof. Let $(\lambda_n, D_n, \varphi_n)$ and (λ, D, φ) be, respectively, the envelopes of holomorphy of (D_n, φ_n) and (D, φ), $\tilde{\tau}_m^n$ and $\tilde{\tau}_n$ be, respectively, the analytic continuations of τ_m^n and τ_n to $(\lambda_n, \tilde{D}_n, \tilde{\varphi}_n)$ and $(\lambda, \tilde{D}, \tilde{\varphi})$. From Lemma 5 $\{(\tilde{D}_n, \tilde{\varphi}_n); \tau_m^n\}$ is the limit of a monotonously increasing sequence of domains over S and $(\tilde{\tau}_n, \tilde{D}, \tilde{\varphi})$ is its limit. Since $\tilde{D}$ is holomorphically convex, there exists a sequence of analytic polycylinders P_n in $\tilde{D}$ such that

$$\lambda(K_n) \subset P_n \subset P_{n+1}$$

for any $n \geqq 1$. Since (τ_n, D, φ) and $(\tilde{\tau}_n, \tilde{D}, \tilde{\varphi})$ are, respectively, limits of $\{(D_n, \varphi_n); \tau_m^n\}$ and $\{(\tilde{D}_n, \tilde{\varphi}_n); \tilde{\tau}_m^n\}$, from Lemma 4 there exists a subsequence $\{\nu_n\}$ of $\{1, 2, 3, \cdots\}$ such that τ_{ν_n} and $\tilde{\tau}_{\nu_n}$ map, respectively, a compact subset K_n' of D_{ν_n} and an open subset P_n' of $\tilde{D}_{\nu_n}$ onto K_n and P_n and that

$$\tau_{\nu_{n+1}}^{\nu_n}(P_n') \subset P_{n+1}', \ \lambda_{\nu_n}(K_n') \subset P_n'.$$

Now let f_n be any holomorphic function on D_{ν_n}. Since $(\lambda_{\nu_n}, \tilde{D}_{\nu_n}, \tilde{\varphi}_{\nu_n})$ is the envelope of holomorphy of $(D_{\nu_n}, \varphi_{\nu_n})$, there exists $\tilde{f}_n \in \mathrm{H}^0(\tilde{D}_{\nu_n}, \mathfrak{O})$ such that $\tilde{f}_n \circ \lambda_{\nu_n} = f_n$.

Then

$$\tilde{f}_n \circ (\tilde{\tau}_{\nu_n} | P_n')^{-1} \in \mathrm{H}^0(P_n, \mathfrak{O}).$$

Since P_n is holomorphically convex with respect to $\tilde{D}$, there exists $\tilde{F}_n \in \mathrm{H}^0(\tilde{D}, \mathfrak{O})$ such that

$$|F_n - f_n \circ (\tilde{\tau}_{\nu_n} | P_n')^{-1}| < \varepsilon_n$$

in K_n. Then $F_n = \tilde{F}_n \circ \lambda$ satisfies

$$|F_n \circ \tau_{\nu_n} - f_n| < \varepsilon_n$$

in K_n'.

§ 3. Limit of cohomology groups

Let (τ_n, D, φ) be a limit of a monotonously increasing sequence $\{(D_n, \varphi_n); \tau_m^n\}$ of domains over a complex manifold M and $\mathfrak{A}_L$ be the sheaf of all germs of holomorphic mappings in a complex Lie group L. τ_m^n induces canonically the mapping $\pi_n^m: \mathrm{H}^1(D_m, \mathfrak{A}_L) \to \mathrm{H}^1(D_n, \mathfrak{A}_L)$ for $n \leqq m$ such that $\pi_n^l = \pi_n^m \circ \pi_m^l$ for $n \leqq m \leqq l$. Hence $\{\mathrm{H}^1(D_n, \mathfrak{A}_L); \pi_n^m\}$ forms an inverse system over a directed set $\{1, 2, 3, \cdots\}$. Its inverse limit is denoted by lim $\mathrm{H}^1(D_n, \mathfrak{A}_L)$. The canonical mappings π_n; lim $\mathrm{H}^1(D_n, \mathfrak{A}_L) \to \mathrm{H}^1(D_n, \mathfrak{A}_L)$ and $\pi: \mathrm{H}^1(D, \mathfrak{A}_L) \to \lim \mathrm{H}^1(D_n, \mathfrak{A}_L)$ are also considered. If an element α of $\mathrm{H}^1(D, \mathfrak{A}_L)$ is the canonical image of an element of $B^1(\mathfrak{U}, \mathfrak{A}_L)$ for an open covering $\mathfrak{U}$ of D, we say that $\alpha=0$. We say that $H^1(D, \mathfrak{A}_L)=0$ if $\alpha=0$ for any $\alpha \in H^1(D, \mathfrak{A}_L)$. If $\alpha \in \lim H^1(D_n, \mathfrak{A}_L)$ satisfies $\pi_n(\alpha)=0$ for any $n \geqq 1$, we say that $\alpha=0$. If $\pi(\alpha)=0$ implies $\alpha=0$ for $\alpha \in H^1(D, \mathfrak{A}_L)$, we say that π is *quasi-injective*. If L is abelian and π is quasi-injective, π is injective. Let L be a complex Lie group and L_a be the connected component of L containing a. For $a, b \in L$ the mapping τ defined by $\tau(x)=ba^{-1}x$ for $x \in L$ maps L_a biholomorphically onto L_b. Hence, if (C^p, χ) is a covering domain of L_e, (C^p, χ_a) is a covering domain of L_a for any $a \in L$, where e is the neutral element of L and $\chi_a(z)=a\chi(z)$ for $z \in C^p$. If L is a p-dimensional abelian or soluble complex Lie group, (C^p, χ) is a covering domain of L_e for suitable χ.

Lemma 8. *Let (C^p, χ) be a covering manifold of L_e for a complex Lie group L, $\{(D_n, \varphi_n); \tau_m^n\}$ be a monotonously increasing sequence of simply connected domains over a Stein manifold S and (τ_n, D, φ) be its limit. Then the canonical mapping* $H^1(D, \mathfrak{A}_L) \rightarrow \lim H^1(D_n, \mathfrak{A}_L)$ *is quasi-injective.*

Proof. In this proof we shall denote a point of C^n by a gothic type as $\mathfrak{a}$ and the inverse element of an element a of the group L by a^{-1}. If we put

$$\chi_a(z)=a\chi(z)$$

for $z \in C^p$, (C^p, χ_a) is a covering domain of L_a for $a \in L$. Let $\{K_n\}$ be a sequence of compact subsets of D such that K_n is contained in the open kernel of K_{n+1} and $D=\bigcup_{n=1}^{\infty} K_n$. If we apply Lemma 7 to this $\{K_n\}$, we obtain a subsequence $\{\nu_n\}$ of $\{1, 2, 3, \cdots\}$ and a sequence $\{K_n'\}$ satisfying the conditions (1) and (2) in Lemma 7. Without loss of generality we may suppose that $\nu_n=n$.

There exists $\mathfrak{e} \in C^p$ with $\chi(\mathfrak{e})=e$ for the neutral element e of L. We consider the Euclidean distance in C^p. Let W' be an open sphere with centre $\mathfrak{e}$ and semiradius r such that χ

maps W' biholomorphically onto an after neighbourhood W' of e. We put

$$\operatorname{dist}(a, b) = \operatorname{dist}((\chi|W')^{-1}a, (\chi|W')^{-1}b) \qquad \|a\| = \operatorname{dist}(e, a),$$
$$W = \left\{z \,;\, \operatorname{dist}(e, z) < \frac{r}{2}\right\}, \quad W = \chi(W)$$

where $a, b \in W'$ and $\chi|W'$ is the restriction of χ to W'. For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ satisfying the following condition:

$0 < \delta(\varepsilon) < \varepsilon$. If $a, b \in W$ satisfy $\|b\| < \delta(\varepsilon)$, $ab \in W'$ and $\operatorname{dist}(a, ab) < \varepsilon$.

From Lemma 1 it suffices to prove that $f_{ij} \in \mathrm{B}^1(\mathfrak{U}, \mathfrak{A}_L)$ for an open covering $\mathfrak{U} = \{U_i \,;\, i \in I\}$ of D if $\{f_{ij} \circ \tau_n\} \in \mathrm{B}^1(\tau_n^{-1}(\mathfrak{U}), \mathfrak{A}_L)$ for $n \geqq 1$. The condition means that $\{f_{ij} \circ \tau_n\}$ satisfies

$$f_{ij} \circ \tau_n = h^n (h_j^n)^{-1}$$

in $\tau_n^{-1}(U_i) \cap \tau_n^{-1}(U_j)$ for a suitable $\{h^n\} \in \mathrm{C}^0(\tau_n^{-1}(U), \mathfrak{A}_L)$, where $\{\tau_n^{-1}(\mathfrak{U}) = \tau_n^{-1}(U_i) \,;\, i \in I\}$ an open covering of D_n. If we put

$$f^1 = (h^1)^{-1} (h^2 \circ \tau_2^1)$$

in $\tau_1^{-1}(U_i)$, $f^1 \in \mathrm{H}^0(D_1, \mathfrak{A}_L)$ is well-defined. Since D_1 is connected, $f^1(D_1) \subset L_{a_1}$ for some $a_1 \in L$. We put $\chi_1 = \chi_{a_1}$. Since D_1 is simply connected and (C^p, χ_1) is a universal covering domain of L_{a_1}, from the principle of monodromy there exists $F^1 \in \mathrm{H}^0(D_1, \mathfrak{O}^p)$ such that $f^1 = \chi_1 \circ F^1$. From Lemma 7 there exists $H^1 \in \mathrm{H}^0(D, \mathfrak{O}^p)$ such that the image of K_1' by the mapping $f^1(\chi_1 \circ H^1 \circ \tau_1)^{-1}$ is contained in W and

$$\|f^1(\chi_1 \circ H^1 \circ \tau_1)^{-1}\| < \delta(r 2^{-3})$$

in K_1'. If we put

$$\{g_i^1\} = \{h_i^1\} \in \mathrm{C}^0(\tau_2^{-1}(\mathfrak{U}), \mathfrak{A}_L)$$

and

$$g_i^2 = h_i^2 (\chi_1 \circ H^1 \circ \tau_2)^{-1}$$

in $\tau_2^{-1}(U_i)$, the coboundary of $\{g_i^2\} \in \mathrm{C}^0(\tau_2^{-1}(\mathfrak{U}), \mathfrak{A}_L)$ is $\{f_{ij} \circ \tau_2\}$. If we put

$$g^{1,2}=(g_i^1)^{-1}(g_i^2\circ\tau_2^1)$$

in $\tau_1^{-1}(U_i)$, $g^{1,2}\in \mathrm{H}^0(D_1, \mathfrak{A}_L)$ is well-defined, the image of K_1' by the mapping $g^{1,2}$ is contained in W and

$$||g^{1,2}||<\delta(r2^{-3})$$

in K_1'. Suppose that there exist $\{g_i^1\}\in C^0(\tau_1^{-1}(\mathfrak{U}), \mathfrak{A}_L)$, $\{g_i^2\}\in C^0(\tau_2^{-1}(\mathfrak{U}), \mathfrak{A}_L),\cdots$ and $\{g_i^m\}\in C^0(\tau_m^{-1}(\mathfrak{U}), \mathfrak{A}_L)$ satisfying the following conditions:

(1) The coboundary of $\{g_i^n\}$ is $\{f_{ij}\circ\tau_n\}$ for n with $1\leqq n\leqq m$.

(2) If we put

$$g^{n,l}=(g_i^n)^{-1}(g_i^l\circ\tau_l^n)$$

in $\tau_n^{-1}(U_i)$ for any $1\leqq n<l\leqq m$, $g^{n,l}\in \mathrm{H}^0(D_n, \mathfrak{A}_L)$ is well-defined, the image of K_n' by the mapping $g^{n,n+1}$ is contained in W and

$$||g^{n,n+1}||<\delta(r2^{-n-2})$$

in K_n'.

If we put

$$f^m=(g_i^m)^{-1}(h^{m+1}\circ\tau_{m+1}^m)$$

in $\tau_m^{-1}(U_i)$, $f^m\in \mathrm{H}^0(D_m, \mathfrak{A}_L)$ is well-defined. Since D_m is connected, there exists $a_m\in L$ such that $f^m(D_m)\subset L_{a_m}$. We put $\chi_m=\chi_{a_m}$. Since D_m is simply connected and (C^p, χ_m) is a covering domain of L_{a_m}, there exists $F^m\in \mathrm{H}^0(D_m, \mathfrak{O}^p)$ such that $f^m=\chi_m\circ F^m$. From the condition (2) of Lemma 7 F_m can be uniformly approximated in K_m' by functions in $\mathrm{H}^0(D, \mathfrak{O}^p)$ so that the image of K_m' by the mapping $f^m(\chi_m\circ H^m\circ\tau_m)^{-1}$ is contained in W and

$$||f^m(\chi_m\circ H^m\circ\tau_m)^{-1}||<\delta(r2^{-m-2})$$

in K_m'. If we put

$$g_i^{m+1}=h_i^{m+1}(\chi_m\circ H^m\circ\tau_{m+1})^{-1}$$

in $\tau_{m+1}^{-1}(U_i)$, the coboundary of $\{g_i^{m+1}\}\in C^0(\tau_{m+1}^{-1}(\mathfrak{U}), \mathfrak{A}_L)$ is $\{f_{ij}\circ\tau_{m+1}\}$ $\in Z^1(\tau_{m+1}^{-1}(\mathfrak{U}), \mathfrak{A}_L)$. If we put

$$g^{m,m+1}=(g_i^m)^{-1}(g_i^{m+1}\circ\tau_{m+1}^m)$$

in $\tau_m^{-1}(U_i)$, $g^{m,m+1}\in \mathrm{H}^0(D_m, \mathfrak{A}_L)$ is well-defined and the image of K'_m by the mapping $g^{m,m+1}$ is contained in W and

$$||g^{m,m+1}||<\delta(r2^{-m-2})$$

in K'_m. In this way we can construct $\{g_i^m\}\in \mathrm{C}^0(\tau_m^{-1}(\mathfrak{U}), \mathfrak{A}_L)$ satisfying the above conditions (1) and (2) inductively for $m=1, 2, 3, \cdots$.

The image of K'_n by the mapping $g^{n,m}$ is contained in W,

$$||g^{n,m}||<r2^{-n-1}$$

in K'_n for $1\leqq n\leqq m$ and

$$\mathrm{dist}(g^{n,m}, g^{n,m+1})<r2^{-m-2}$$

in K'_n for $n\leqq m$. Therefore the subsequence $\{(\chi|W)^{-1}g^{n,m}; m=n, n+1, \cdots\}$ of $\mathrm{H}^0(D_n, \mathfrak{O}^p)$ tends uniformly to $G^n\in \mathrm{H}^0(\tau_n^{n-1}(K'_{n-1}), \mathfrak{O}^p)$ in $\tau_n^{n-1}(K'_{n-1})$ for $n=2, 3, 4, \cdots$. Since

$$\mathrm{dist}(e, G^n)<r2^{-n}$$

in $\tau_n^{n-1}(K'_{n-1})$ for $n=2, 3, 4, \cdots$, $\{g^{n,m}; m=n, n+1, \cdots\}$ tends uniformly to

$$g^n=\chi\circ G^n\in \mathrm{H}^0(\tau_n^{n-1}(K'_{n-1}), \mathfrak{A}_L)$$

in $\tau_n^{n-1}(K'_{n-1})$. If we put

$$f_i=(g_i^n g^n)\circ(\tau_n|K'_n)^{-1}$$

in K_{n-1} for n with $U_i\subset K_{n-1}$, $\{f_i\}\in \mathrm{C}^0(\mathfrak{U}, \mathfrak{A}_L)$ is well-defined and its coboundary is the original cocycle $\{f_{ij}\}\in Z^1(\mathfrak{U}, \mathfrak{A}_L)$.

The above proof of Lemma 8 gives the following Corollary.

Corollary to Lemma 8. *Let* (D, φ) *be a limit of a monotonously increasing sequence* $\{(D_n, \varphi_n)\}$ *of domains over a Stein manifold. Then the canonical homomorphism* $\mathrm{H}^1(D, \mathfrak{O}^p)\rightarrow \lim \mathrm{H}^1(D_n, \mathfrak{O}^p)$ *is injective for* $p\geqq 1$.

Let (τ_n, D, φ) be a limit of a monotonously increasing sequence $\{(D_n, \varphi_n), \tau_m^n\}$ of domains over M. Let $\tau_n^\sharp: (D_n^\sharp, \varphi_n^\sharp)\rightarrow (D^\sharp, \varphi^\sharp)$ and $\tau_m^{n\sharp}: (D_n^\sharp, \varphi_n^\sharp)\rightarrow(D_m^\sharp, \varphi_m^\sharp)$ be, respectively, the mappings associated to τ_n and τ_m^n for $1 \leqq n \leqq m$. Then $(\tau_n^\sharp, D^\sharp, \varphi^\sharp)$ is a limit of a monotonously increasing sequence $\{(D_n^\sharp, \varphi_n^\sharp), \tau_m^{n\sharp}\}$ of domains over M. $\tau_n \circ \lambda_n$'s induce a canonical mapping $\pi^\sharp : \mathrm{H}^1(D, \mathfrak{A}_L) \rightarrow \lim \mathrm{H}^1(D_n^\sharp, \mathfrak{A}_L)$. Under these notations we have the following Lemma.

Lemma 9. *Let (C^p, χ) be a covering domain of a connected component of a complex Lie group L, $\{(D_n, \varphi_n), \tau_m^n\}$ be a monotonously increasing sequence of domains over a Stein manifold S and (τ_n, D, φ) be its limit. For any $\alpha \in \mathrm{H}^1(D, \mathfrak{A}_L)$ with ${}^\sharp\pi(\alpha)=0$, we have ${}^\sharp\alpha = 0$.*

Proof. From Lemma 4 it suffices to prove that any $\{f_{ij}\} \in Z^1(\mathfrak{U}, \mathfrak{A}_L)$ for an open covering $\mathfrak{U}$ of D with $\{f_{ij} \circ \tau_n \circ \lambda_n\} \in \mathrm{B}^1(\lambda_n^{-1} \tau_n^{-1}(\mathfrak{U})), \mathfrak{A}_L)$ $(n \geqq 1)$ satisfies $\{f_{ij} \circ \lambda\} \in \mathrm{B}^1(\lambda^{-1}(\mathfrak{U}), \mathfrak{A}_L)$. Since $\tau_n \circ \lambda_n = \lambda \circ \tau_n^\sharp$ for any n,

$$\{f_{ij} \circ \lambda \circ \tau_n^\sharp\} \in \mathrm{B}^1(\tau_n^{\sharp -1}(\lambda^{-1}(\mathfrak{U})), \mathfrak{A}_L)$$

for any n. As each $D_n^\sharp$ is simply connected, we have

$$\{f_{ij} \circ \lambda\} \in \mathrm{B}^1(\lambda^{-1}(\mathfrak{U}), \mathfrak{A}_L)$$

from Lemma 8.

As a corollary to Lemma 9 we have the following Proposition.

Proposition 3. *Let L be an abelian or soluble complex Lie group, $\{(D_n, \varphi_n)\}$ be a monotonously increasing sequence of domains over a Stein manifold and (D, φ) be its limit. If D is simply connected, the canonical mapping $\mathrm{H}^1(D, \mathfrak{A}_L) \rightarrow \lim \mathrm{H}^1(D_n, \mathfrak{A}_L)$ is quasi-injective.*

§ 4. Cousin-I and Cousin-II distributions

Let L be an abelian or soluble complex Lie group and D be a domain in a Stein manifold S. Let $\mathfrak{F}$ be a subset of $\mathrm{H}^1(D, \mathfrak{A}_L)$. For a subdomain E of D we consider the inclusion mapping $i_E : E \rightarrow D$. Let $(E^\sharp, \lambda_E)$ be the universal covering domain of E. i_E and $i_E \circ \lambda_E$ induce canonically mappings

$i_E^*: \mathrm{H}^1(D, \mathfrak{A}_L) \to \mathrm{H}^1(E, \mathfrak{A}_L)$ and $\lambda_E^* \circ i_E^*: \mathrm{H}^1(D, \mathfrak{A}_L) \to \mathrm{H}^1(E^\sharp, \mathfrak{A}_L)$. For $\alpha \in \mathrm{H}^1(D, \mathfrak{A}_L)$ we put $\alpha|E = i_E^*(\alpha)$ and ${}^\sharp(\alpha|E) = \lambda_E^*(i_E^*(\alpha))$. We consider the set $E_{\mathfrak{F}}$ of all subdomains E of D such that ${}^\sharp(\alpha|E) = 0$ for any $\alpha \in \mathfrak{F}$. Then $(E_{\mathfrak{F}}, \subset)$ forms a partially ordered set. Let $\mathfrak{C} = \{C_t\,; t \in T\}$ be a totally ordered subset of $(E_{\mathfrak{F}}, \subset)$. We put

$$C = \bigcup_{t \in T} C_t\,.$$

Then C is a subdomain of D. There exists a sequence $\{K_n\,; n = 1, 2, 3, \cdots\}$ of compact subsets of C such that

$$K_n \subset K_{n+1},\ C = \bigcup_{n=1}^{\infty} K_n\,.$$

Since $\mathfrak{C}$ is an open covering of a compact set K_1 and $(\mathfrak{C}, \subset)$ is totally ordered, there exists $C_1 \in \mathfrak{C}$ such that $K_1 \subset C_{\mu_1}$. Suppose that there exists $C_{\mu_1}, C_{\mu_2}, \cdots, C_{\mu_n} \in \mathfrak{C}$ such that

$$C_{\mu_1} \subset C_{\mu_2} \subset \cdots \subset C_{\mu_n},\ K_m \subset C_{\mu_m}\ \ (1 \leqq m \leqq n).$$

Since $\mathfrak{C}$ is an open covering of a compact set K_{n+1} and $(\mathfrak{C}, \subset)$ is totally ordered, there exists $C'_{\mu_{n+1}} \in \mathfrak{C}$ such that $K_{n+1} \subset C'_{\mu_{n+1}}$. Then we have

$$C_{\mu_{n+1}} = C_{\mu_n} \cup C'_{\mu_{n+1}} \in \mathfrak{C}$$

and

$$C_{\mu_n} \subset C_{\mu_{n+1}},\ K_{n+1} \subset C_{\mu_{n+1}}\,.$$

Thus we have proved the existence of $\{C_{\mu_n}\,; n = 1, 2, 3, \cdots\} \subset \mathfrak{C}$ such that

$$C_{\mu_1} \subset C_{\mu_2} \subset \cdots \subset C_{\mu_n} \subset \cdots,\ C = \bigcup_{n=1}^{\infty} C_{\mu_n}\,.$$

Since ${}^\sharp(\alpha|C_{\mu_n}) = 0$ for any n and any $\alpha \in \mathfrak{F}$, we have ${}^\sharp(\alpha|C) = 0$ for any $\alpha \in \mathfrak{F}$ from Lemma 9. Therefore $C \in E_{\mathfrak{F}}$ and C is an upper bound of $\mathfrak{C}$. From Zorn's Lemma there exists a maximal element $D_{\mathfrak{F}}$ of $(E_{\mathfrak{F}}, \subset)$ which is an upper bound of $\mathfrak{C}$.

A collection $\mathfrak{C}=\{(m_i, U_i); i\in I\}$ of pairs of an open subset U_i of D and a meromorphic function m_i in U_i is called a *Cousin-II distribution in D* if the following conditions are satisfied:

(1) $\mathfrak{U}=\{U_i; i\in I\}$ is an open covering of D.

(2) $\rho(\mathfrak{C})=\{m_i m_j^{-1}\}\in Z^1(\mathfrak{U}, \mathfrak{O}^*)$ where $\mathfrak{O}^*=\mathfrak{A}_{GL(1,C)}$.

Let E be a subdomain of D and $(E^\sharp, \lambda)$ be the universal covering domain of E. A meromorphic function M in $E^\sharp$ is called a *multiform solution* of $\mathfrak{C}$ in E if

$$(m_i\circ\lambda)M^{-1}\in H^0(\lambda^{-1}(U_i), \mathfrak{O}^*)$$

for any $i\in I$. $\mathfrak{C}$ has a multiform solution in E if and only if ${}^\sharp(\rho(\mathfrak{C})|E)=0$. ρ gives a homomorphism of the multiplicative group of all Cousin-II distributions in D into $H^1(D, \mathfrak{O}^*)$. Let $\mathfrak{F}$ be a set of Cousin-II distributions in D. Let $D_{\mathfrak{F}}$ be the set of all subdomain E of D in which any distribution in $\mathfrak{F}$ has a multiform solution. Then we have $(D_{\mathfrak{F}}, \subset)=(E_{\rho(\mathfrak{F})}, \subset)$. Hence we have the following Proposition.

Proposition 4. *Let $\mathfrak{F}$ be a set of Cousin-II distributions in a domain D in a Stein manifold, $D_{\mathfrak{F}}$ be the set of all subdomains E of D in which any distribution in $\mathfrak{F}$ has a multiform solution and $\mathfrak{C}$ be a totally ordered subset of $(D_{\mathfrak{F}}, \subset)$. Then there exists a maximal element of $(D_{\mathfrak{F}}, \subset)$ which is an upper bound of $\mathfrak{C}$.*

Making use of Corollary of Lemma 8 we can also prove the following Proposition similarly.

Proposition 5. *Let $\mathfrak{F}$ be a set of Cousin-I distributions in a domain D in a Stein manifold, $D_{\mathfrak{F}}$ be the set of all subdomains E of D in which any distribution of $\mathfrak{F}$ has a solution and $\mathfrak{C}$ be a totally ordered subset of $(D_{\mathfrak{F}}, \subset)$. Then there exists a maximal element of $(D_{\mathfrak{F}}, \subset)$ which is an upper bound of $\mathfrak{C}$.*

§ 5. Intersection of Cousin's domains

In this paragraph we denote by L a fixed, but arbitrary, abelian complex Lie group exclusively. A direct product P of n simply connected domains in a complex plane is called *simply* *connected polycylinder* in C^n. Of course we have $\mathrm{H}^1(P, \mathfrak{A}_L)=0$. In this paragraph we shall investigate intersections of domains D with $\mathrm{H}^1(D, \mathfrak{A}_L)=0$ and extend Cartan-Behnke-Stein's theorem.

An open set G in C^n with $\mathrm{H}^1(G\cap P, \mathfrak{A}_L)=0$ for any relatively compact and simply connected polycylinder P in C^n is called *L-regular.*

A domain G in C^n is said to be *exhausted by L-regular domains* G_p if G_p's are L-regular domains in C^n such that

$$G_p \Subset G_{p+1} (p=1, 2, 3, \cdots) \text{ and } G=\bigcup_{p=1}^{\infty} G_p .$$

Lemma 10. *Let G be a domain in C^n exhausted by L-regular domains G_p. Then* ${}^{\#}\mathrm{H}^1(G, \mathfrak{A}_L)=0$. *Moreover for any integers* $1 \leqq m < n$, $1=s_1<s_2<\cdots<s_{n-m} \leqq n$ *and for any complex numbers* $c_j(j=s_1, s_2, \cdots s_{n-m})$ *the intersection* $G\cap H$ *of G and* $H=\{z=z_1, z_2, \cdots, z_n) ; z_j=c_j(j=s_1, s_2, \cdots, s_{n-m})\}$ *satisfies* ${}^{\#}\mathrm{H}^1(G\cap H, \mathfrak{A}_L)=0$.

Proof. Since G_p is a relatively compact L-regular domain, we have $\mathrm{H}^1(G_p, \mathfrak{A}_L)=0$ for any p. From Lemma 9 we have ${}^{\#}\mathrm{H}^1(G, \mathfrak{A}_L)=0$.

Next we shall prove ${}^{\#}\mathrm{H}^1(G\cap H, \mathfrak{A}_L)=0$. We may assume that

$$H=\{(z, w)=(z_1, z_2, \cdots, z_m, w_1, w_2, \cdots, w_{n-m}) ; w_j =0(j=1, 2, \cdots, n-m)\} .$$

There exist $\varepsilon_p>0$ and $a_p>0$ such that

$$\begin{aligned} E_p=G_p\cap\{(z, w) ; |z_j|<a_p, |w_k|<\varepsilon_p(j=1, 2, \cdots, m ; k=1, 2, \cdots, n-m)\} \\ \subset \{(z, w) ; |z_j|<a_p, |w_k|<\varepsilon_p, (z, 0)\in G\cap H, \\ (j=1, 2, \cdots, m ; k=1, 2, \cdots, n-m)\} , \\ a_p<a_{p+1}(p\geqq 1) \text{ and } a_p\to\infty(p\to\infty) , \\ \varepsilon_p>\varepsilon_{p+1}(p\geqq 1) \text{ and } \varepsilon_p\to 0(p\to\infty) . \end{aligned}$$

Since G_p is L-regular, we have $\mathrm{H}^1(E_p, \mathfrak{A}_L)=0$ for any p. We put

$$H_p=G_p\cap H\cap\{(z, 0) ; |z_j|<a_p \ (j=1, 2, \cdots, m)\} .$$

Then $G \cap H$ is the limit of a monotonously increasing sequence of open sets H_p in H. Let $\mathfrak{V}=\{V_s\,;s\in S\}$ be an open covering of $G \cap H$. We put $V_s^p=V_s \cap H_p$ for $s\in S$. Then $V_p=\{V_s^p\,;s\in S\}$ is an open covering of H_p. We put

$$U_s^p=E_p \cap \{(z, w)\,;(z, 0)\in V_s\}$$

for $s\in S$. Then $\mathfrak{U}_p=\{U_s^p\,;s\in S\}$ is an open covering of E_p. Let $\{f_{st}(z)\}$ be an element of $Z^1(\mathfrak{V}, \mathfrak{A}_L)$. We put

$$F_{st}^p(z, w)=f_{st}(z)$$

in $U_s^p \cap U_t^p \neq \phi$. Then $\{F_{st}^p\}\in \mathrm{Z}^1(\mathfrak{U}_p, \mathfrak{A}_L)=\mathrm{B}^1(\mathfrak{U}_p, \mathfrak{A}_L)$ from Lemma 1. There exists $F_s^p\in \mathrm{H}^0(U_s^p, \mathfrak{A}_L)$ for any $s\in S$ such that

$$F_{st}^p=F_s^p(F_t^p)^{-1}$$

in $U_s^p \cap U_t^p \neq \phi$. If we put

$$f_s^p(z)=F_s^p(z, 0)$$

in V_s^p for any $s\in S$, then we have

$$f_{st}=f_s^p(f_t^p)^{-1}$$

in $V_s^p \cap V_t^p \neq \phi$. Therefore the restriction of $\{f_{st}\}$ in any H_p is a coboundary of $\{f_s^p\}\in \mathrm{C}^0(\mathfrak{V}_p, \mathfrak{A}_L)$ for any p. From Lemma 1 $\{f_{st}\circ\lambda\}\in \mathrm{B}^1(\lambda^{-1}(\mathfrak{V}), \mathfrak{A}_L)$, (G, λ) being the universal covering domain of G. Thus we have ${}^{\#}\mathrm{H}^1(G \cap H, \mathfrak{A}_L)=0$.

From Proposition 2 and Lemma 10 we have

Proposition 6. *A domain in C^n exhausted by L-regular domains is a domain of holomorphy.*

A boundary point x^0 of an open set G in R^n is called a *continuous boundary point of G* if there exists a real-valued continuous function g of variables $x_1, x_2, \cdots, \hat{x}_j, \cdots, x_n$ in a neighbounhood V of x^0 such that

$$\partial G \cap V=\{x=(x_1, x_2, \cdots, x_n)\,;x_j=g(x_1, x_2, \cdots, \hat{x}_j, \cdots, x_n), x\in V\}$$

for some j. Moreover if g is continuously differentiable in U,

x^0 is called a *smooth boundary point of G.* An open set G in C^p is called *pseudoconvex at a boundary point* x^0 of G if there exists an open neighbourhood U of x^0 such that $G \cap U$ is an open set of holomorphy.

Proposition 7. *An L-regular open set G in C^n is pseudoconvex at a continuous boundary point z^0 of G.*

Proof. We put $z^0=(z_1^0, z_2^0, \cdots, z_n^0)$. We may assume that there exists $\varepsilon>0$ and a real-valued continuous function g of variables $z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, \cdots, z_n$ in a neighbourhood $V=\{z=(z_1, z_2, \cdots, z_n)\,;\,|z_k-z_k^0|<\varepsilon(k=1, 2, \cdots, n)\}$ such that

$$\partial G \cap V=\{z\,;\,x_j=g(z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, \cdots, z_n), z\in V\}$$

for some j where $z_j=x_j+\sqrt{-1}\,y_j$. Then three cases (1), (2) and (3) may occur.

(1) $G\cap V=\{z\,;\,x_j<g(z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, \cdots, z_n), z\in V\}$.

For $0\leqq t<1$ we put

$$V_t=\left\{z\,;\,|z_k-z_k^0|<\frac{(1-t)\varepsilon}{2}(k=1, 2, \cdots, m)\right\}.$$

Then we have

$$\left\{z\,;\left(z_1, z_2, \cdots, z_{j-1}, z_j-\frac{t\varepsilon}{2}, z_{j+1}, \cdots, z_n\right)\in V_t\right\}\subset V$$

for $0<t<1$. We put

$$E_t=\left\{z\,;\,x_j<g(z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, \cdots, z_n)-\frac{t\varepsilon}{2}, z\in V_t\right\}.$$

Let P be a relatively compact and simply connected polycylinder in C_n. $E_t\cap P$ is mapped onto

$$\begin{aligned}&\Big\{w\,;\,u_j<g(w_1, w_2, \cdots, w_{j-1}, v_j, w_{j+1}, \cdots, w_n),\\ &\qquad\left(w_1, w_2, \cdots, w_{j-1}, w_j-\frac{t\varepsilon}{2}, w_{j+1}, \cdots, w_n\right)\in V_t\cap P\Big\}\\ &=G\cap V\cap\left\{z\,;\left(z_1, z_2, \cdots, z_{j-1}, z_j-\frac{t\varepsilon}{2}, z_{j+1}, \cdots, z_n\right)\in V_t\cap P\right\}\end{aligned}$$

by a biholomorphic mapping $w=(w_1, w_2, \cdots, w_n)=\gamma(z)$ defined by

$w_k = z_k (k \neq j)$, $w_j = z_j + t\varepsilon/2$. Since $\gamma(E_t \cap P)$ is the intersection of G and relatively compact and simply connected polycylinders, we have

$$\mathrm{H}^1(E_t \cap P, \mathfrak{A}_L) = \mathrm{H}^1(\gamma(E_t \cap P), \mathfrak{A}_L) = 0 .$$

Therefore E_t is an L-regular open set for $0 \leqq t < 1$. Since E_0 is exhausted by L-regular domains $\{E_t ; 0 < t < 1\}$, $E_0 = G \cap V_0$ is a domain of holomorphy from Proposition 1. Hence G is pseudoconvex at z^0.

(2) $G \cap V = \{z ; x_j > g(z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, \cdots, z_n), z \in V\}$.

In this case the situation is quite similar to the case (1).

(3) $G \cap V = \{z ; x_j \neq g(z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, \cdots, z_n), z \in V\}$.

Let

$$G_1 = \{z ; x_j < g(z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, \cdots, z_n), z \in V\}$$

and

$$G_2 = \{z ; x_j > g(z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, \cdots, z_n), z \in V\} .$$

From the cases (1) and (2) G_1 and G_2 are pseudoconvex at z^0. Hence $G \cap V = G_1 \cup G_2$ is pseudoconvex at z^0.

Theorem 1. *Let L be an abelian complex Lie group. An L-regular domain D in C^n with a continuous boundary is a domain of holomorphy.*

Proof. From Oka [16] and Proposition 7 D is a domain of holomorphy.

Theorem 2. *Let L be an abelian complex Lie group and D be a domain with a smooth boundary in a Stein manifold S. If $\mathrm{H}^1(D \cap P, \mathfrak{A}_L) = 0$ for any holomorphically convex subdomain P of S analytically contractible to each point of it, D is holomorphically convex.*

Proof. Let x_0 be a boundary point of D. There exists a biholomorphic mapping τ of an open neighbourhood U' of x_0 onto $Z' = \{z ; |z_1| < 2, |z_2| < 2, \cdots, |z_n| < 2\}$ and $\tau(x_0) = 0$ where n is the dimension of S. We put

$$Z=\{z\,;\,|z_1|<1,\ |z_2|<1, \cdots, |z_n|<1\}$$

and $U=\tau^{-1}(Z)$. Let Q be any relatively compact and simply connected polycylinder in C^n. Then each connected component of $Q\cap Z$ is relatively compact and simply connected polycylinder in C^n. From Riemann's mapping theorem $Q\cap Z$ and $\tau^{-1}(Q\cap Z)$ are relatively compact Stein manifolds analytically contractible to each point of it. Since

$$\mathrm{H}^1(Q\cap\tau(U\cap D), \mathfrak{A}_L)=\mathrm{H}^1(\tau^{-1}(Q\cap Z)\cap D, \mathfrak{A}_L)=0\,,$$

$\tau(U\cap D)$ is L-regular and pseudoconvex at its smooth boundary point x_0 from Proposition 7. Therefore D is pseudoconvex and is holomorphically convex from Docquier-Grauert [8].

Lemma 11. *For any $p\geqq 1$ there exists homomorphism α of* $\mathrm{GL}(1, C)$ *in* $\mathrm{GL}(p, C)$ *and β of* $\mathrm{GL}(p, C)$ *in* $\mathrm{GL}(1, C)$ *such that $\beta\circ\alpha$ is the identity of* $\mathrm{GL}(1, C)$.

Proof. For $a\in\mathrm{GL}(1, C)$ we define the p-p matrix $\alpha(a)=(a_{ij})$ by putting

$$a_{11}=a,\ a_{ii}=1 \qquad (i\geqq 2), \quad a_{ij}=0 \qquad (i\neq j).$$

For $a\in\mathrm{GL}(p, C)$ we put $\beta(a)=\det(a)$. Then α and β are desired homomorphisms.

Lemma 12. *Let L and L' be complex Lie groups with the following properties: There exist homomorphisms α of L in L' and β of L' in L such that $\beta\circ\alpha$ is the identity of L. Then the canonical mapping* $\mathrm{H}^1(X, \mathfrak{A}_L)\to\mathrm{H}^1(X, \mathfrak{A}_{L'})$ *induced by α is injective for any complex space X.*

Proof. Suppose that

$$f_i^{-1}(\alpha\circ f_{ij})f_j=\alpha\circ g_{ij}$$

in $U_i\cap U_j\neq\phi$ for $\{f_{ij}\}, \{g_{ij}\}\in\mathrm{Z}^1(\mathfrak{U}, \mathfrak{A}_L)$ and $\{f_i\}\in\mathrm{C}^0(\mathfrak{U}, \mathfrak{A}_{L'})$ where $\mathfrak{U}=\{U_i\,;\,i\in I\}$ is an open covering of X. Then $\{\beta\circ f_i\}\in\mathrm{C}^0(\mathfrak{U}, \mathfrak{A}_L)$ satisfies

$$(\beta\circ f_i)^{-1}f_{ij}(\beta\circ f_j)=g_{ij}$$

in $U_i \cap U_j \neq \phi$.

From Lemmas 11 and 12 we have the following Theorem.

Theorem 3. *For any positive integer p Propositions* 1, 2, 6 *and* 7 *Theorems* 1 *and* 2 *are valid if we replace* L *by* $\mathrm{GL}(p, C)$.

§ 6. The sheaf $\mathfrak{M}$ of all germs of meromorphic functions

In the previous paper [10] we remarked that $\mathrm{H}^1(D, \mathfrak{M})=0$ for any 1-dimensional Stein manifold D, that is, for any non-compact Riemann surface D where $\mathfrak{M}$ is the sheaf of all germs of meromorphic functions. But we have the following Proposition for a domain in C^n $(n \geqq 2)$.

Proposition 8. *Let D be a domain in C^n $(n \geqq 2)$. Then* $\mathrm{H}^1(D, \mathfrak{M}) \neq 0$.

Proof. Suppose that $\mathrm{H}^1(D, \mathfrak{M})=0$. For the sake of brevity of notations any point of C^n is denoted by $x=(z, w, w_3, \cdots, w_n)$. Then there exist complex numbers a and b with $0 \neq a \neq b \neq 0$ such that

$$D_1 = D - \{x\,;\, z=a\} \subsetneqq D,\ D_2 = D - \{x\,;\, z=b\} \subsetneqq D\,.$$

From Lemma 1 we have $\mathrm{H}^1(\mathfrak{U}, \mathfrak{M})=0$ for the open covering $\mathfrak{U}=\{D_1, D_2\}$ of D. We put

$$m' = \frac{1}{w-\exp((z-a)^{-1}(z-b)^{-1})} \in \mathrm{H}^0(D_1 \cap D_2, \mathfrak{M})\,.$$

There exists $m_1' \in \mathrm{H}^0(D_1, \mathfrak{M})$ and $m_2' \in \mathrm{H}^0(D_2, \mathfrak{M})$ such that

$$m' = m_1' - m_2'$$

in $D_1 \cap D_2$. The pole surfaces A', A_1' and A_2' of m', m_1' and m_2' are, respectively, analytic sets in $D_1 \cap D_2, D_1$ and D_2. There holds

$$A' \subset (A_1' \cup A_2') \cap D_1 \cap D_2\,.$$

Since A' is irreducible at each point of it, we have $A' \subset A_1 \cap D_1 \cap D_2$ or $A' \subset A_2 \cap D_1 \cap D_2$. Without loss of generality we may suppose that $A' \subset A_2' \cap D_1 \cap D_2$. Since $D_1 \subsetneqq D$, there exists $x^0=$

$(a, a_2, a_3, \cdots, a_n)\in D$ for our a. There exists $\delta>0$ such that

$$U=\{x=(z, w, w_3, \cdots, w_n)\,;\, |z-a|<\delta,\ |w-a_2|<\delta,\ |w_3-a_3|<\delta,\ \cdots,\ |w_n-a_n|<\delta\}\subset D\,.$$

There exists an integer $p\geqq 0$ such that the germ of the pole surfaces of

$$m_2=(z-a)^p m_2'\in \mathrm{H}^0(D_2, \mathfrak{M})$$

at x^0 does not contain that of $\{x\,;\,z=a\}$ at x^0. We put

$$m=(z-a)^p m'\in \mathrm{H}^0(D_1\cap D_2, \mathfrak{M}).$$

The pole surfaces of A and A_2 of m and m_2 are, respectively, analytic sets in $D_1\cap D_2$ and D_2 satisfying

$$A\subset A_2\cap D_1\cap D_2\,.$$

Let $x=(a, b_2, b_3, \cdots, b_n)$ be any point of $U\cap\{x\,;\,z=a\}$ with $b_2\neq 0$. There exists an integer $q_0'>0$ such that

$$\left|\frac{4}{\log b_2+i(\arg b_2+2q\pi)}\right|<|a-b|^2$$

for $q\geqq q_0'$. We put $x^q=(a^{(q)}, b_2, \cdots, b_n)$ $(q\geqq q_0')$ for

$$a^{(q)}=\frac{a+b+\sqrt{(a-b)^2+\dfrac{4}{\log b_2+i(\arg b_2+2q\pi)}}}{2}$$

where the function $\sqrt{z}$ is defined in $\{z\,;\,|z-(a-b)^2|<|a-b|^2\}$ so that $\sqrt{(a-b)^2}=a-b$. There exists $q_0(>q_0')$ such that

$$x^q\in U$$

for $q>q_0'$. Then we have

$$x^q\in A(q>q_0),\ x^q\to x(q\to\infty)\,.$$

Since $A\subset A_2$, A_2 is an analytic set in D_2 and D_2 contains $x^{(q)}$ $(q<q_0)$ and x, we have $x\in A_2$. This means that

$$U\cap\{x\,;\,z=a\}\subset A_2\,.$$

But this contradicts to the fact that the germ of A_2 at x^0 does not contain that of $\{x; z=a\}$ at x^0. Thus we have proved that $H^1(D, \mathfrak{M}) \neq 0$.

REFERENCES

[1] Behnke, H., Généralisation du théorème de Runge pour les fonctions multiformes des variables complexes. Coll. sur les fonct. des plus. var. Bruxelles (1953).

[2] Behnke, H., und K. Stein, Analytische Funktionen mehrer Veränderlichen zur vorgegebenen Null- und Pollstellenflächen. Jber. Deut. Math. Verein. **47** (1937), 177–192.

[3] ———, Entwicklung analytischer Funktionen auf Riemannschen Flächen. Math. Ann. **120** (1948), 430–461.

[4] Bochner, S., and W. T. Martin, Several complex variables. Princeton Univ. Press (1948).

[5] Cartan, H., Les problèmes de Poincaré et de Cousin pour les fonctions de plusieurs variables complexes. C. R. Paris **199** (1934), 1284–1287.

[6] ———, Sur les premières problèmes de Cousin. C. R. Paris **207** (1938), 558–560.

[7] Cartan, H., et al., Théorie des fonctions de plusieurs variables: Sem. de Cartan, Ecole Norm., Sup. (1951–52).

[8] Docquier, F., und H. Grauert, Levisches Problem und Rungescher Satz fur Teilgebiete Steinscher Mannigfaltigkeit. Math. Ann. **140** (1960), 94–123.

[9] Eilenberg, S., and N. Steenrod, Foundations of algebraic topology. Princeton Univ. Press.

[10] Kajiwara, J., On an application of L. Ehrenpreis's method to ordinary differential equations. Kōdai. Math. Sem. Rep. **15** (1963), 94–105.

[11] ———, On the limit of a monotonous sequence of Cousin's domains. J. Math. Soc. Japan **17** (1965), 36–46.

[12] ———, Some characterizations of Stein manifold through the notion of locally regular boundary points. Kōdai Math. Sem. Rep. **16** (1964), 36–46.

[13] ———, Note on a Cousin-II domain over C^2. Ibid. **17** (1965), 44–47.

[14] ———, Relations between domains of holomorphy and multiple Cousin's problems. Ibid., 261–272.

[15] Oka, K., Sur les fonctions analytiques de plusieurs variables: II Domaines d'holomorphie. J. Sci. Hiroshima Univ. **7** (1937), 115–130.

[16] ———, ———: IX Domaines finis sans point critique intérieur. Jap. J. Math. **23** (1953), 97–155.

[17] Stein, K., Überlagerungen holomorph-vollständiger komplexer Räume. Arch. Math, **7** (1956), 354–361.

[18] Thullen, P., Sur les deusième problème de Cousin. C. R. Paris **200** (1935), 720–721.

Mathematical Institute Kyushu University