On some integral formulae containing Bessel functions

By

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So called Sounine's integral formula can be interpreted as follows [1]. Let G_n be n-dimensional Euclidean motion group and T_g be an irreducible unitary representaion of class 1. The group G_n contains the subgroup which is isomorphic to G_{n-1}. Being restricted to G_{n-1}, T_g is decomposed into irreducible representations of G_{n-1}. Using this decomposition, we can express the zonal spherical function of G_n by that of G_{n-1}. This expression is nothing but Sonnine's first integral formula. Sonnine's second integral formula can be proved in the same way.

On the other hand, the motion group of n-dimensional Lovachvsky space G (the Lorentz group) contains, as a subgroup, G_{n-1}. As in the previous case, restricting the irreducible representation of G to G_{n-1} and decomposing it into irreducible factors, we can obtain an integral formula (6) involving Legendre and Bessel functions. In this sense, we may call the formula (6) an analogue to Sonnine's formula.

If we consider imaginary Lovachevsky space, we obtain analogous formula (7). (8).

§1. Let G be the $(n+2)$-dimensional Lorentz group, that is, the connected component of the orthogonal group of the quadratic form $-x_0^2+x_1^2+\cdots x_{n+1}^2$.
We put,

$$K=\left\{k=\begin{pmatrix}1 & 0\\ 0 & k\end{pmatrix} \qquad k\in SO(n+1)\right\}$$

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$$A = \left\{ a_t = \begin{pmatrix} \mathrm{ch}\, t & 0 & \mathrm{ch}\, t \\ 0 & E_n & 0 \\ \mathrm{sh}\, t & 0 & \mathrm{sh}\, t \end{pmatrix} \quad t \in R \right\}$$

$$N_+ = \left\{ x = \begin{pmatrix} 1+\xi, & x_1 \cdots x_n & -\xi \\ x_1 & & -x \\ \vdots & E_n & \vdots \\ x_n & & -x_n \\ \xi & x_1 \cdots x_n & 1-\xi \end{pmatrix}; \ x_k \in R, \ \xi = \frac{1}{2} \sum_{k=1}^{n} x_k^2 \right\}$$

$$N_- = \left\{ y = \begin{pmatrix} 1-\eta, & y_1 \cdots y_n & \eta \\ y_1 & & y \\ \vdots & E_n & \vdots \\ y_n & & y_n \\ -\eta, & -y_1 \cdots -y_n & 1-\eta \end{pmatrix}; \ y_k \in R, \ \eta = \frac{1}{2} \sum_{k=1}^{n} y_k^2 \right\}$$

Then, $G = N_- AK$ (Iwasawa decomposition)
Further,

$$M = \left\{ m = \begin{pmatrix} 1 & & 0 \\ & m & \\ 0 & & 1 \end{pmatrix}; \ m \in so(n) \right\}$$

is the normalizer of N_- in K and $M_- N = N_- M$ (semi-direct product) is isomorphic to G_n. We denote the Lie algebras corresponding to these subgroups by the letters, $\mathfrak{n}_-$, $\mathfrak{a}$, $\mathfrak{k}$, $\mathfrak{m}$, respectively.

The homogeneous space $X = G/K$ is $(n+1)$-dimensional Lovachevsky space. It is homeomorphic to $N_- A$, so we can adopt the canonical coordinate of A and N_- as a global coordinate in X.

At first, we compute the invariant metric on X (which is unique up to a constant factor) in terms of this coordinate,

Lemma. *The invariant metric on X is given by*

$$ds^2 = dt^2 + e^{2t} \sum_{k=1}^{n} dx_k^2 \tag{1}$$

By the definition of the invariant metric, we must determine the positive definite quadratic form on the tangent space at $x (x \in X)$ $\varphi_x(Y)$, such that

$$\varphi_x(Y) = \varphi_{gx}(gY)$$

where we denote the transformation $x \to g \cdot x$ and its differential by the same letter.

For this, it is sufficient to construct the quadratic form on $\mathfrak{p}$, $\varphi(Y)$, invariant by Adk ($\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to killing form, which can be identified with the tangent space at the origine o).

Then, $\varphi_x(Y) = \varphi(g^{-1}Y)$, where $g \cdot o = x$. The right hand side is independent of the choise of g.

In our case

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & y \cdots y_{n+1} \\ y & \\ \vdots & 0 \\ y_{n+1} & \end{pmatrix} y_k \in R \right\}$$

It is easy to see that $\varphi(Y) = \sum_{k=1}^{n+1} y_k^2$ (up to a constant factor)

Now, $\mathfrak{a} + \mathfrak{n}_- \cong \mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$ (as vector spaces)

By this isomorphism, $\varphi(Y)$ can be regarded as a quadratic form on $\mathfrak{a} + \mathfrak{n}_-$. That is,

for
$$Y = \begin{pmatrix} 0 & y \cdots y_n & t \\ y & & y \\ \vdots & 0 & \vdots \\ y_n & & y_n \\ t & -y \cdots -y_n & 0 \end{pmatrix} \in \mathfrak{a} + \mathfrak{n}_-$$

$$\varphi(Y) = t^2 + \sum_{k=1}^{n} y_k^2$$

We identify all tangent space of X with $\mathfrak{a} + \mathfrak{n}_-$. Then the differential of the transformation $x \to gx$ is the linear transformation on $\mathfrak{a} + \mathfrak{n}_-$ which is given by the Jacobian matrix.

$$J_g = \left. \frac{\partial(y' t')}{\partial(y, t)} \right|_{y=t=0}$$

where (y', t') is the coordinate of gx.

In particular, for $g = y_0 a_{t_0}$, $t' = t + t_0$, $y' = y_0 + e^{-t_0} y$

and
$$J_g = \begin{pmatrix} 1 & & & \\ & e^{-t_0} & & \\ & & \ddots & \\ & & & e^{-t_0} \end{pmatrix}$$

Consequently, the form at $x_0=(y_0, t_0)$ is given by

$$\varphi_x(Y) = t^2+e^{2t_0}\sum_{k=1}^{n} y_k^2, \text{ which proves} \tag{1}$$

Corollary. *Let* Δ *be the Laplace-Beltrami operator on* X. *Then*

$$\Delta = \frac{\partial^2}{\partial t^2}+n\frac{\partial}{\partial t}+e^{2t}\Delta_0, \quad \Delta_0 = \sum_{k=1}^{n}\frac{\partial^2}{\partial y_k^2} \tag{2}$$

§ 2. Irreducible unitary representations of class 1 of G are usually constructed in the function space on a maximal compact subgroup ([2], [3]). But for our purpose, the following realization is more convenient.

According to the decomposition.

$G = N_+MAN_-$ (Gauss decomposition. see Желобенко Трулы. Моск. Матем. о-Ва. 12 (1963))

, for $g\in G$ and $x\in N_-$, xg is uniquely expressed in the form

$xg = ymax'$ (if xg is a "regular" element).

We denote x', a by $x_{\bar{g}}$, $a_t(x, g)$ respectively. Then we have

Lemma. $$(x_{\bar{g}})_k = \frac{\frac{1}{2}g_{0,k}(1+|x|^2)+\sum_{j=1}^{n} x_j g_{ik}+\frac{1}{2}2g_{n+1,k}(|x|^2-1)}{\Delta(x, g)}$$

where

$$e^{t(x,g)} = \Delta(x, g) = \frac{1}{2}(g_{00}-g_{0,n+1})(|x|^2+1)+\sum_{j=1}^{n} x_j(g_{j0}-g_{j,n+1})$$
$$+\frac{1}{2}(g_{n+1,0}-g_{n+1,n+1})(|x|^2-1).$$

We prove this for $n=2m$ (for $n=2m+1$, the proof is similar)

put $$\tilde{G} = \left\{\tilde{g} = SgS^{-1}, \quad g\in G, \quad S = \begin{pmatrix} 1 & & & & & 1 \\ & \ddots & & & i & \\ & & \ddots & \cdot^{\cdot^{\cdot}} & & \\ & & 1 & i & & \\ & & 1 & -i & & \\ & & \cdot^{\cdot^{\cdot}} & & \ddots & \\ & \cdot^{\cdot^{\cdot}} & & & & -i \\ 1 & & & & & -1 \end{pmatrix}\right\}$$

we denote by $\tilde{A}$, $\tilde{N}_+$, $\tilde{N}_-$ the subgroups corresponding to A, N_+, N_-.

Then

$$\tilde{A} = \left\{ \tilde{a}_t = \begin{pmatrix} e^t & & 0 \\ & E_n & \\ 0 & & e^{-t} \end{pmatrix} \quad t \in R \right\}$$

$$\tilde{N}_- = \left\{ x = \begin{pmatrix} 1 & 0 & 0 \\ z_1 & & \\ \vdots & & \\ z_m & E_n & 0 \\ \bar{z}_m & & \\ \vdots & & \\ \bar{z}_1 & & \\ |z|^2 & \bar{z}_1 \cdots \bar{z}_m \quad z_m \cdots z_1 & 1 \end{pmatrix} \begin{array}{l} z_k = x_k + i x_{2m-k+1} \\ 1 \leqq k \leqq m \end{array} \right\}$$

$$\tilde{N}_+ = \left\{ \tilde{y} = \begin{pmatrix} 1 & w_1 \cdots w_m \quad \bar{w}_m \cdots \bar{w}_1 & |w|^2 \\ & & \bar{w}_1 \\ & & \vdots \\ 0 & E_n & \bar{w}_m \\ & & w_m \\ & & \vdots \\ & & w_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{l} w_k = y_k + i y_{2m-k+1} \\ 1 \leqq k \leqq m \end{array} \right\}$$

From the equality $\tilde{x}\tilde{g} = \tilde{y}\tilde{m}\tilde{a}\tilde{x}'$, we thve

$e^t =$ the $(n+2)$th component of $(|z|^2, \bar{z}_1 \cdots \bar{z}_m, z_m \cdots z_1, 1)\tilde{g}$

$e^t(|z'|^2, \bar{z}'_1 \cdots \bar{z}'_m\ z'_m \cdots z'_1, 1) = (|z|^2, \bar{z}_1 \cdots \bar{z}_m, z_m \cdots z_1, 1)\tilde{g}$

But $(|z'|^2, \bar{z}'_1 \cdots \bar{z}'_m, z'_m \cdots z'_1, 1) = (1+|x'|^2, 2x'_1 \cdots 2x'_n, |x|^2-1)S^{-1}$

$(|z|^2, \bar{z}_1 \cdots \bar{z}_m, z_m \cdots z_1, 1)\tilde{g} = (1+|x|^2, 2x_1 \cdots 2x_n, |x|^2-1)gS^{-1}$

Therefore

$$e^t = \frac{|x|^2+1}{2}(g_{00} - g_{0,n+1}) + \sum_{k=1}^{n} x_k(g_{k_0} - g_{k;n+1}) + \frac{|x|^2-1}{2}(g_{n+1,0} - g_{n+1,n+1})$$

$$= \Delta(x, g)$$

$$x_k = \frac{|x|^2+1}{2} g_0 k + \sum_{j=1}^{n} x_j g_{jk} + \frac{|x|^2-1}{2} g_{n+1,k} / \Delta(x, g) \qquad (1 \leqq k \leqq n).$$

It is to be noticed that $\dfrac{\partial(x \cdots x_n)}{\partial(x \cdots x_n)} = \Delta(x, g)^n$

and $\Delta(x, g_1 g_2) = \Delta(x_{\bar{g}_1}, g_2)\Delta(x, g)$

Using these, we can construct the unitary representation $(\mathfrak{H}, U_g^\rho)$ *of G in the following way.*

$$\mathfrak{H} = L^2(R^n)$$

$U_g^\rho f(x) = \Delta(x, g)^{-(n/2)-i\rho} f(x_{\bar{g}})$ (ρ is a real number parametrizing the representation.)

In particular.

$$\begin{array}{llll} \text{for} & g = y \in N_-, & U_g^\rho f(x) = f(x+y) & \text{(a)} \\ \text{for} & g = a_t \in A, & U_g^\rho f(x) = e^{(n/2+i\rho)t} f(e^t x) & \text{(b)} \\ \text{for} & g = m \in M, & U_g^\rho f(x) = f(xm) & \text{(c)} \end{array}$$

From these, the irreducibility of U_g^ρ can be easily seen. For, if $\tilde{U}_g^\rho$ is the Fourier transform of U_g^ρ and A is a bounded operator in $L^2(R^n)$ which commutes with U_g^ρ $(g \in G)$, then,

by (a), A is the muliplication operator by a bounded function $\varphi(x)$

by (b), $\varphi(x)$ is independent of $|x|$

by (c), $\varphi(x)$ is constant on sphere. if $n \geqq 2$.

Therefore, U_g^ρ is irreducible for $n \geqq 2$. For $n=1$ the above consideration is not sufficient for the irreducibility. We must prove that $\tilde{\mathfrak{H}}_t = \{\tilde{f} \in \tilde{\mathfrak{H}}, \tilde{f}(x) = 0 \ x \gtrless 0\}$ are not invariant subspaces.

If they are K-invariant, the K-invariant vector f_0 (see below) is contained in one of them. Then, being an even function, we have $\tilde{f}_0 \equiv 0$ which is a contradiction.

Next, we show the existence of K-invariant vector.

If $f \in \mathfrak{H}$ is a K-invariant vector,

$$f(o) = (T_k f)(o) = \left| \frac{1+g_{n+1,n+1}}{2} \right|^{-(n/2)-i\rho} f\left(-\frac{g_{k,n+1}}{1+g_{n+1,n+1}} \right)$$

We put $-g_{k,n+1}/1+g_{n+1,n+1} = x_k$ (which take arbitrary real value)

then $\quad f(x) = c \dfrac{1}{(1+|x|^2)^{(n/2)+i\rho}} \qquad$ (c is a constant)

Conversely it is easy to see that such an f is K-invariant (it is sufficient to verify this property for $g = \begin{pmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & 1 & & \\ & 0 & & \cos\theta & \sin\theta \\ & & & -\sin\theta & \cos\theta \end{pmatrix} = b_\theta$)

Therefore, U_g^ρ is of class 1 and the normalized K-invariant vector is

$$f_0(x) = \left[\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)\pi^{n/2}}\right]^{1/2} \frac{1}{(1+|x|^2)^{(n/2)+i\rho}}.$$

$\varphi_\rho(g) = (U_g^\rho f_0, f_0)$ is by definition, the zonal spherical function of G.

For $$g = ka_t k' \qquad (k, k' \in K)$$

$$\varphi_\rho(g) = \varphi_\rho(t) = \frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)\pi^{n/2}} \int_{R_n} \frac{e^{((n/2)+i\rho)t}}{(1+e^{2t}|x|^2)^{(n/2)+i\rho}} \frac{1}{(1+|x|^2)^{(n/2)+i\rho}} dx$$

$$= \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)2\pi^{n/2}} \int_0^\infty \frac{e^{((n/2)+i\rho)t}}{(1+e^{2t}r^2)^{(n/2)+i\rho}} \frac{r^{n-1}}{(1+r^2)^{(n/2)+i\rho}} dx$$

$$= \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)\pi^{n/2}} \frac{1}{2^n} \int_{-1}^1 \frac{(1-y^2)^{(n/2)-1}}{(\operatorname{ch} t + y \operatorname{sh} t)^{(n/2)+i\rho}} dy$$

(Here, we put $y = 1-r^2/1+r^2$)

$$= \frac{1}{B\left(\frac{n}{2}, \frac{1}{2}\right)} \int_0^\pi \frac{\sin^{n-1}\theta}{(\operatorname{ch} t + \cos\theta \operatorname{sh} t)^{(n/2)+i\rho}} d\theta$$

$$= \frac{2^{(n+1)/2}\Gamma\left(\frac{n+1}{2}\right)}{\operatorname{sh}^{(n-1)/2} t} \mathfrak{P}^{(1/2)-(n/2)}_{-(1/2)+\rho i} (\operatorname{ch} t) \tag{3}$$

(See also [2], [3])

§ **3.** Let $\varphi(g)$ be a spherical function of G. As is known, (g) is a function on X and satisfies the differential equation

$$\Delta\varphi = -\left\{\rho^2 + \frac{n^2}{4}\right\}\varphi \tag{4}$$

By (2), $\tilde{\varphi}(k, t) = \int \varphi(y, t) e^{i<y,k>} dy$ satisfies

$$\left[\frac{d^2}{du^2} - \frac{n-1}{u}\frac{d}{du} - |k|^2 + \frac{\rho^2 + \frac{n^2}{4}}{u}\right]\tilde{\varphi} = 0$$

(Here, we put $u = e^t$)

put $\tilde{\varphi}(n) = u^{n/2} v(u)$, then $\dfrac{d^2 v}{du^2} + \dfrac{1}{u}\dfrac{dv}{du} - \left(|k|^2 - \dfrac{\rho^2}{u^2}\right)v = 0$

Let $f(x)$ be an even integrable function on R^n. Fourier transform of which does not vanish. We put $\varphi(g) = (U_g^\rho f_0, f)$, then

$$\varphi(xa) = (U_a^\rho f_0, U_{-n}^\rho f) = U_a^\rho f_0 * f$$

and $$\tilde{\varphi}(k, t) = \widetilde{T_a f_0} \cdot f$$

Therefore

$$\widetilde{T_a f_0} = u^{i\rho + (r/2)} \int_{R_n} \frac{e^{i<x,k>}}{(u^2 + |x|^{\alpha})^{(n/2)+i\rho}} dx = \frac{\tilde{\varphi}(k, u)}{\tilde{f}}$$

satisfies (4). Taking into account the behavior at infinity, we conclude $\widetilde{T_a f_0}(k, u) = c(|k|) u^{n/2} K_{i\rho}(|k|, u)$ where $c(t)$ is a homogeneous function of degree $v = i\rho$.

From $$\int_{R_n} \frac{dx}{(1 + |x|^2)^{(n/2)+\nu}} = \frac{\pi^{n/2}\Gamma(\nu)}{\Gamma\left(\frac{n}{2} + \nu\right)}$$

and $$\lim_{z \to 0} z^\nu K_\nu(z) = \frac{\Gamma(\nu)}{2^{1-\nu}},$$

we have $$c(t) = \frac{|t|^\nu 2^{1-\nu} \pi^{n/2}}{\Gamma\left(\nu + \frac{u}{2}\right)}$$

Therefore

$$\int_{R_n} \frac{e^{i<x,k>}}{(u^2 + |x|^2)^{(n/2)+\nu}} dx = 2\pi^{n/2} \left(\frac{|k|}{2u}\right)^\nu \frac{1}{\Gamma\left(\nu + \frac{n}{2}\right)} K_\nu(|k| u) \qquad (5)$$

As is seen from §2 (a) and (c), if we restrict U_g^ρ to the subgroup $G_n = MN_-$, we obtain the quasi-regular representatiou V_g of G_n (the regular representation of R^1, for $n=1$). Fourier transform of which are decomposed as follows.

$$\tilde{V}_g = d_n \int_0^\infty V_g^R R^{n-1} dR, \qquad d_n^2 = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$$

where $(V_g^R, \mathfrak{H}_R)$ is the irreducible unitary representation of class 1 of G [1]. This means that to $f \in \mathfrak{H}$, there corresponds $f_R \in \tilde{\mathfrak{H}}_R$, such that

$$||f||^2 = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \langle f_R, f_R \rangle R^{n-1} dR$$

$$(V_g f)_R = V_g^R f_R$$

($\langle f_R, f_R \rangle$ is the inner product in $\mathfrak{H}_R$).

We denote this correspondence by $f=d_n\int f_R R^{n-1}dR$. Then by (5)

$$\tilde{f}_0=\int_0^\infty c(R)K_{i\rho}(R)\varphi_0 R^{n-1}dR$$

$$\widetilde{T_{a_t}f_0}=\int_0^\infty c(R)a^{n/2}K_{i\rho}(aR)\varphi_0\cdot R^{n-1}dR\,.$$

(here, φ_0 is the nhrmalized M-invariant vector in $\mathfrak{H}_R$ and $a=e^t$)
Therefore, for $g=a_{t_1}^{-1}xa_{t_2}$

$$\varphi_\rho(g)=(T_xT_{a_{t_2}}f_0,\ T_{a_{t_1}}f_0)=\frac{1}{(2\pi)^n}(\tilde{T}_x\widetilde{T_{a_t}f_0},\ \widetilde{T_{a_t}f_0})$$

$$=\frac{1}{(2\pi)^n}\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\int_0^\infty|c(R)|^2(ab)^{n/2}K_{i\rho}(aR)K_{i\rho}(bR)\langle T_x^R\varphi_0,\ \varphi_0\rangle R^{n-1}dR$$

As is known [1], $\langle T_x^R\varphi_0,\ \varphi_0\rangle=2^{(n/2)-1}\Gamma\left(\frac{n}{2}\right)(|x|R)^{1-(n/2)}J_{(n/2)-1}(|x|R)$
(the zonal spherical function of G_n)
On the other hand, $\varphi_\rho(g)$ is given by (5) for $g=ka_tk'$.
By an easy computation, from $a_{t_1}xa_{t_2}=ka_tk$, we obtain

$$\operatorname{ch}t=\frac{r^2+a^2+b^2}{2ab}\qquad(a=e^{t_1},\ b=e^{t_2})$$

(compare the (1, 1)-component of the matrices on both sides.)
thus we have proved the following formula

$$(6)\qquad \int_0^\infty t^{n/2}K_{i\rho}(as)K_{i\rho}(bs)J_{(n/2)-1}(us)ds$$

$$=\frac{\pi^{1/2}u^{(n/2)-1}}{2^{1/3}(ab)^{n/2}}\Gamma\left(\frac{n}{2}+i\rho\right)\Gamma\left(\frac{n}{2}-i\rho\right)\frac{\mathfrak{P}_{-(1/2)+i\rho}^{(1/2)-(n/2)}(t)}{\operatorname{sh}^{(n-1)/2}t}\qquad\text{for } n\geqq 2\,,$$

and

$$\int_0^\infty 4\sqrt{ab}\,K_{i\rho}(as)K_{i\rho}(bs)\cos ut\,dt$$

$$=\pi\Gamma\left(\frac{1}{2}+i\rho\right)\Gamma\left(\frac{1}{2}-i\rho\right)\mathfrak{P}_{-(1/2)+i\rho}(\operatorname{ch}t)\qquad\text{for } n=1\,.$$

§4. Instead of $K=SO(n+1)$, we consider the subgroup $H=\left\{h=\begin{pmatrix}h&0\\0&1\end{pmatrix}\in G\right\}$ which is isomorphic to $(n+1)$-dimensional Lorentz group.

The the unitary representation $(\mathfrak{H}, U_g^\rho)$ is of "class 1" with respect to this subgroup. H-invariant vectors are (up to constant factors) $f_0(x)=(1-|x|^2)_+^{-((n/2)+i\rho)}$ and $f_1(x)=(|x|^2-1)_+^{-((n/2)+i\rho)}=U_\varepsilon^\rho f_0(x)$ (for ε, see below)

where, by definition, $F(x)_+^\lambda=\begin{cases} F(x)^\lambda & \text{if } F(x)>0 \\ 0 & \text{otherwise}\end{cases}$

$$\varphi_\rho^{(1)}(g)=(U_g^\rho f_0, f_0) \quad \text{and} \quad \varphi_\rho^{(2)}(g)=(U_g^\rho f_0, f_1)$$

are "zonal spherical functious"

1) for $g=a_t$ $(t>0)$

$$\varphi_\rho^{(1)}(g^{-1})=\varphi_\rho^{(1)}(-t)=\int_{|x|<1}\frac{e^{-((n/2)+i\rho)t}}{(1-e^{-2t}|x|^2)^{(n/2)+i\rho}}\frac{1}{(1-|x|^2)^{(n/2)-i\rho}}dx$$

$$=\omega_n\int_0^1\frac{e^{-((n/2)+i\rho)t}}{(1-e^{-2t}r^2)^{(n/2)+i\rho}}\frac{r^{n-1}}{(1-r^2)^{(n/2)-i\rho}}dr$$

$$=\frac{\omega_n}{2^n}\int_1^\infty\frac{(y^2-1)^{(n/2)-1}}{(\mathrm{ch}\,t+y\,\mathrm{sh}\,t)^{(n/2)+i\rho}}dy \qquad \left(y=\frac{1+r^2}{1-r^2}\right)$$

$$=\frac{\pi^{n/2}}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}\int_0^\infty\frac{\mathrm{sh}^{n-1}s}{(\mathrm{ch}\,t+\mathrm{ch}\,s\,\mathrm{sh}\,t)^{(n/2)+i\rho}}ds \qquad (y=\mathrm{ch}\,s)$$

$$\varphi_\rho^{(1)}(t)=\varphi_{-\rho}^{(1)}(-t)$$

2) for $g=b_\theta$

$$\varphi_\rho^{(1)}(g)=\varphi_\rho^{(1)}(\theta)=\int_{|x|<1}\{(1-|x|^2)\cos\theta+2x_n\sin\theta\}_+^{-((n/2)+i\rho)}$$

$$\times(1-|x|^2)^{-(n/2)+i\rho}dx$$

$$=\int_{|x|<1}\left\{\cos\theta+\frac{2x_n}{1-|x|^2}\sin\theta\right\}_+^{-((n/2)+i\rho)}\frac{dx}{(1-|x|^2)^n}$$

$$=\frac{1}{2^n}\int_{R^n}(\cos\theta+z_n\sin\theta)_+^{-((n/2)+i\rho)}(1+|z|^2)^{-(1/2)}dz$$

$(z_k=2x_k/1-|x|^2,\ 1\leqq k\leqq n)$

$$=\frac{1}{2^n}\frac{1}{\sin^{n-1}\theta}\int_{u_n<0}\frac{u_n^{-((n/2)+i\rho)}}{(1+|u|^2-2u_n\cos\theta)}du$$

$(u_n=\cos\theta+{}_{zn}\sin\theta,\quad u_k=z_k\sin\theta\ \ 1\leqq k\leqq n-1$

$$=\frac{\pi^{-(n/2)1}\Gamma\left(1-\frac{n}{2}\right)}{2^n\sin^{n-1}\theta}\int_0^\infty u^{((n/2)+i\rho)}(u^2-2u\cos\theta+1)^{(n/2)-1}du$$

3) for $g=a_t$ $(t>0)$

$$\varphi_\rho^{(2)}(g) = \varphi_{-\rho}^{(2)}(-t) = \int_{1<|x|<e^t} \frac{e^{((n/2)+i\rho)t}}{(1-e^{-2t}|x|^2)^{(n/2)+i\rho}} \frac{1}{(|x|^2-1)^{(n/2)-i\rho}} dx$$

$$= \frac{\omega_n}{2^n}\int_{\coth t}^{\infty} \frac{(y^2-1)^{(n/2)-1}}{(y\,\mathrm{sh}\,t-\mathrm{ch}\,t)^{(n/2)+i\rho}} dy \qquad \left(y=\frac{r^2+1}{r^2-1}\right)$$

$$= \frac{\pi^{n/2}}{2^{n-1}\Gamma\left(\frac{n}{2}\right)} \frac{1}{\mathrm{sh}^{n-1}t}\int_0^\infty y^{((n/2)+i\rho)}(y^2+2y\,\mathrm{ch}\,t+1)^{(n/2)-1}dy$$

$$\varphi_\rho^{(2)}(t) = 0$$

So far as f_0, f_1 do not belong to the Hibert space $\mathfrak{H}$, these integral are, in general, divergent.
But, for $n=1$, they converge almost everyhere and can be expressed by special functions:

$$\varphi_\rho^{(1)}(t) = \int_0^\infty (\mathrm{ch}\,t+\mathrm{ch}\,s\,\mathrm{sh}\,t)^{-(1/2)+i\rho}ds = Q_{(1/2)+i\rho}(\mathrm{ch}\,t)$$

$$\varphi_\rho^{(1)}(\theta) = \frac{1}{2}\int_0^\infty u^{-(1/2)-i\rho}(1+u^2-2u\cos\theta)^{1/2}du$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}+i\rho\right)\Gamma\left(\frac{1}{2}-i\rho\right)P_{-(1/2)-i\rho}(-\cos\theta)$$

for $t<0$, $$\varphi_\rho^{(2)}(t) = \int_0^\infty y^{-(1/2)-i\rho}(1+y^2+2y\,\mathrm{ch}\,t)dy$$

$$= \Gamma\left(\frac{1}{2}+i\rho\right)\Gamma\left(\frac{1}{2}-i\rho\right)\mathfrak{P}_{-(1/2)i\rho}(\mathrm{ch}\,t)$$

The following lemma can be proved easily.

Lemma.

(1) *if* $r<|a-b|$, $a_{t_2}^{-1}xa_{t_1} = ha_th'$ $\quad (h, h'\in H)$

and $\mathrm{ch}\,t = \dfrac{a^2+b^2-r^2}{2ab}$ $\quad (a=e^{t_1}, b=e^{t_2}, r=|x|)$

$t \gtrless 0$ *according as* $b \gtrless a$

(2) *if* $|a-b|<r<a+b$, $a_{t_2}^{-1}xa_{t_1} = hh_\theta h'$

and $\cos\theta = \dfrac{a^2+b^2-r^2}{2ab}$

(3) *if* $r>a+b$, $a_{t_2}^{-1}xa_{t_1} = h\varepsilon a_th'$, $\varepsilon = \begin{pmatrix}1 & & \\ & \ddots & \\ & & 1 \\ & & & -1\end{pmatrix}$

$$\text{and} \quad \operatorname{ch} t = \frac{r^2-a^2-b^2}{2ab}, \qquad t > 0$$

From the above results, we obtain the following.

$$(7) \quad \sqrt{ab}\int_0^\infty J_{i\rho}(au)J_{i-\rho}(bu)\cos ut\,du = \frac{\operatorname{ch}\pi\rho}{\pi}Q_{-(1/2)+i\rho}\frac{a^2+b^2-t^2}{2ab} \quad \text{for } 0<t<|a-b|$$

$$= P_{-(1/2)-i\rho}\left(\frac{t^2-a^2-b^2}{2ab}\right) \quad \text{for } |a-b|<t<a+b$$

$$= 0 \quad \text{for } t>a+b$$

$$(8) \quad \sqrt{ab}\int_0^\infty Y_{i\rho}(au)Y_{-i\rho}(bu)\cos ut\,du = \frac{\operatorname{ch}\pi\rho}{\pi}Q_{-(1/2)+i\rho}\left(\frac{a^2+b^2-t^2}{2ab}\right) \quad \text{for } 0<t<|a-b|$$

$$= P_{-(1/2)-i\rho}\left(\frac{t^2-a^2-b^2}{2ab}\right) \quad \text{for } |a-b|<t<a+b$$

$$= 2\mathfrak{P}_{-(1/2)-i\rho}\left(\frac{t^2-a^2-b^2}{2ab}\right) \quad \text{for } t>a+b$$

The proof is the same as in §3. Instead of (5) we use the following.

$$\int_{|t|<a}\frac{e^{itu}}{(a^2-t^2)^{(1/2)+\nu}}dt = \frac{1}{2}\pi^{1/2}\Gamma\left(\frac{1}{2}-\nu\right)\left(\frac{u}{2a}\right)^\nu J_\nu(au)$$

$$\int_{|t|>a}\frac{e^{itu}}{(t^2-a^2)^{(1/2)+\nu}}dt = \frac{-1}{2}\pi^{1/2}\Gamma\left(\frac{1}{2}-\nu\right)\left(\frac{u}{2a}\right)^\nu Y_\nu(au)$$

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